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The bounded derived categories of the Tamari lattices are fractionally Calabi-Yau

Baptiste Rognerud

Abstract

We prove that the bounded derived category of the incidence algebra of the Tamari lattice is fractionally Calabi-Yau, giving a positive answer to a conjecture of Chapoton. The proof involves a combinatorial description of the Serre functor of this derived category on a sufficiently nice family of indecomposable objects.

1 Introduction

Tamari lattices are partial orders on Catalan sets introduced by Dov Tamari in his thesis. They are now classical and well studied objects in combinatorics and they appear in an incredible number of recent developments in mathematics such as, algebra, computer science, category theory, topology and many others.

In representation theory, they have two classical interpretations: as posets of *tilting modules* of an equioriented quiver of type A ([BK04, HU05]) and as posets of *cluster-tiling* modules for the same quiver. In particular, the Tamari lattices are part of the *Cambrian lattices* of type A in the sense of Reading ([Rea06]).

Chapoton was one of the first to realize that the representation theory of these lattices is also extremely fascinating. At first he used the Auslander-Reiten translation of the Tamari lattices to describe an anti-cyclic structure on the Dendriform operad ([Cha05, Cha07b]). Then, using this anti-cyclic structure he proved that the so-called *Coxeter transformation* of the Tamari lattices is *periodic* ([Cha07b]).

A couple of years later he published a beautiful article ([Cha12]) which contains three important conjectures on the derived category of the incidence algebras of these lattices. The main conjecture can be seen as a 'categorification' of his result on the periodicity of the Coxeter transformation.

Conjecture 1.1 (Chapoton)**.** *Let $n \in \mathbb{N}$ and Tam_n be the Tamari lattice of binary trees with n inner vertices. Let $\mathbf{k}$ be a field. Then, the bounded derived category of the incidence algebra of Tam_n over $\mathbf{k}$ is fractionally Calabi-Yau of dimension $\big(n(n-1), 2n+2\big)$.*

The notion of *Calabi-Yau* category has its roots in algebraic geometry: it generalizes elliptic curves, abelian varieties and $K3$ surfaces. One can translate the geometric properties of Calabi-Yau varieties into algebraic properties of the derived categories of coherent sheaves on them, and this leads to the notion of Calabi-Yau category that are now well studied. *Fractionally Calabi-Yau* categories satisfies weaker axioms and there are more subtle geometric examples of such categories.

They also appear in relation with singularity theory, mirror symmetry and Fukaya categories (See Section 6 of [Cha12] for a conjecture relating Tamari lattices and *Fukaya categories*).

In representation theory, they appear naturally as the bounded derived categories of the path algebra of an orientation of a Dynkin diagram or as derived categories of coherent sheaves on a weighted projective line of tubular type ([vR12]). Conjecture 1.1 provides an algebraic and combinatorial example with *very different* homological properties: the incidence algebras of the Tamari lattices are not hereditary in general, and when $n \geqslant 4$, they have a wild representation type ([CR19]).

Our main objective in this article is to prove Conjecture 1.1. The first part of the article is mostly algebraic and it reduces the conjecture to a much easier problem only involving a finite family of objects in the derived category of the Tamari lattice. The second part of the article is mostly combinatorial and is devoted to the understanding of this finite family.

In Section 2, we define the category of $\mathbf{k}$-linear representations of an arbitrary finite poset as the category of finitely generated right-modules over its incidence algebra. We recall that this category has a finite global dimension (bounded by the length of the largest chain in the poset). This implies that its bounded derived category has a so-called *Serre functor*. This is a universal endo-functor of the derived category which induces, up to a sign, the Coxeter transformation on the Grothendieck group. This leads to the definition of a fractionally Calabi-Yau category (Definition 2.5). A bounded derived category is *fractionally Calabi-Yau* if some power of the Serre functor is isomorphic to a shift.

The classical algebraic example is the bounded derived category of the path algebra of an orientation of a Dynkin diagram. Here we illustrate it in Example 2.6 with the slightly more complicated case of an algebra derived equivalent to an orientation of D_5. This example is of course trivial from an algebraic point of view, but it illuminates the difficulty of understanding the images of indecomposable modules under the Serre functor.

In Section 3, we prove the first main result of the article which reduces the property of being fractionally Calabi-Yau to a property involving only finitely many objects of the derived category.

Theorem 1.2. *Let X be a finite poset with a unique minimal element or a unique maximal element. Let $A_{\mathbf{k}}(X)$ be the incidence algebra of X. If there are integers m and n such that $\mathbb{S}^n(P) \cong P[m]$ in $D^b(X)$ for all projective indecomposable $A_{\mathbf{k}}(X)$-module P, then the category $D^b\big(A_{\mathbf{k}}(X)\big)$ is (m,n)-fractionally Calabi-Yau.*

In the rest of this article we check the hypothesis of Theorem 1.2 for the Tamari lattices. As it is suggested by Example 2.6, this is the difficult part of the proof. In general for a finite poset, the orbits of the projective indecomposable modules under the action of the Serre functor contains indecomposable complexes (not isomorphic to a shift of a module). However, for the Tamari lattices, it turns out that these orbits only contain shifts of indecomposable modules. We actually prove this on a larger family of modules that we call *exceptional*. We show that the Serre functor has a nice combinatorial description on this family.

Exceptional modules come from Chapoton's work on the Dendriform operad **Dend**. More precisely, in [Cha07a] he introduced the operad **NCP** of noncrossing plants and the operad **Mould** of functions with a 'variable number of variables'. Then, he proved that there is a commutative diagram of operads

$$\begin{tikzcd} \mathbf{NCP} \arrow[rr, "\Theta"] \arrow[dr, "\phi"'] & & \mathbf{Dend} \arrow[dl, "\psi"] \\ & \mathbf{Mould} & \end{tikzcd}$$

where all the morphisms are injective.

The family of noncrossing plants contains the more classical family of noncrossing trees (See Section 4.2 for a precise definition). Since the elements of the Dendriform operad are linear combinations of binary trees, we can identify them with the elements of the Tamari lattices. It was proved in [CHNT08] that the image of a noncrossing tree by Θ is of the form $\sum_{t\in I} t$ where I is an interval in a Tamari lattice. Such an interval is called *exceptional*. For our purpose, an interval in the Tamari lattice is seen as an indecomposable module over its incidence algebra. As a consequence, in the derived category of the incidence algebra of the Tamari lattice there is a family of indecomposable objects in bijection with the noncrossing trees.

Since not all the intervals of the Tamari lattices are exceptional, the first task was to characterize the exceptional intervals. This was done in a previous article ([Rog20]). They have a simple characterization in terms of the so-called *interval-posets* of the Tamari lattices recently introduced by Châtel and Pons ([CP15]). This characterization is recalled in Section 4, where we also introduce a new bijection between noncrossing trees and exceptional interval-posets which may be of independent interest.

In Section 5, we show that the family of exceptional intervals contains the simple modules, the projective indecomposable modules and the injective indecomposable modules. We also show that they have particularly nice 'boolean' projective resolutions. In Section 7, we use these projective resolutions in order to obtain a first description of the action of the Serre functor on these objects in terms of the two bijections introduced in Section 4.2. Finally, in Section 6.2, we investigate a *planar duality* of noncrossing trees and we show that the Serre functor acts as a planar dual up to a shift on this family. In conclusion, we prove the following result.

Theorem 1.3. *Let T be a noncrossing tree of size n. Let I_T be the corresponding indecomposable object in $D^b(\mathrm{Tam}_n)$. Then, there is an integer n_T such that*

$$\mathbb{S}(I_T) \cong I_{T^*}[n_T]$$

where I_{T^*} is the indecomposable object in $D^b(\mathrm{Tam}_n)$ corresponding to the planar dual of T.

Since the square of the planar duality is nothing but a *rotation*, we see that up to a shift the Serre functor has a finite order on these objects. The proof of Conjecture 1.1 follows from Theorems 1.2 and 1.3 and a combinatorial interpretation of the integers n_T in terms of certain edges in the noncrossing trees T^*.

The image of the Serre functor on the Grothendieck group of $D^b(\mathrm{Tam}_n)$ induces up to shift the so-called Coxeter transformation of the incidence algebra of Tam_n. As a consequence, we recover the result of Chapoton saying that the Coxeter transformation is periodic of period $2n+2$. Our approach is also interesting at this level since it gives a generating set of the Grothendieck group on which we clearly see why the Coxeter transformation is periodic. In particular, we are able to deduce new informations about the Coxeter matrix (See Proposition 8.5).

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2 Representations of finite posets

In the literature, there are two main points of view on the representations of finite posets. Here we consider representations of the incidence algebra of the poset. This is very different from the other kind of "representations of posets", that was considered by Nazarova, Kleiner and others [Kle72, Naz75, Sim92] which involves a non-abelian sub-category of the category of modules over a one-point extension of the Hasse diagram.

The *incidence algebra* $A_{\mathbf{k}}(P)$ of a finite poset P over a field $\mathbf{k}$ has a basis (a, b) indexed by pairs of comparable elements $a \leq b$ in P and its associative product is defined on this basis by

$$(a, b)(c, d) = \begin{cases} 0 & \text{if} \quad b \neq c, \\ (a, d) & \text{if} \quad b = c. \end{cases}$$

As a consequence, $A_{\mathbf{k}}(P)$ is a finite dimensional algebra whose dimension is the number of intervals in the poset P. In order to avoid confusion, we reserve the notation (a, b) for the basis elements of the incidence algebra. The intervals of the poset P will be denoted by $[a, b]$.

In this article, we define the category of $\mathbf{k}$-linear representations of the finite poset P as the category of finite dimensional *right-modules* over the incidence algebra $A_{\mathbf{k}}(P)$.

Remark 2.1. Alternatively, we can define the category of representations of a finite poset as the category of functors from the poset to the category of vector spaces. It is classical that this category is equivalent to the category of right modules over the incidence algebra. We think that both points of view are useful, but in order to simplify the exposition we restrict ourself to the category of modules over the incidence algebra.

The category $A_{\mathbf{k}}(P)$-mod is an *abelian* category with *enough projective* objects and *enough injective* objects.

For $x \in P$, let $P_x = (x, x) \cdot A_{\mathbf{k}}(P)$. Similarly we let $I_x = \mathrm{Hom}_{\mathbf{k}}(A_{\mathbf{k}}(P) \cdot (x, x), \mathbf{k})$. Let S_x be the $A_{\mathbf{k}}(P)$-module of dimension 1 on which every basis element acts by 0 except for (x, x) which acts by 1.

Proposition 2.2. *Let P be a finite poset and $\mathbf{k}$ be a field. Let $x, y \in P$.*

1. *The module S_x is simple.*
2. *The module P_x is projective and the module I_x is injective. They are both indecomposable. P_x is a projective cover of S_x and I_x is an injective envelope of S_x.*

3.

$$\mathrm{Hom}_{A_{\mathbf{k}}(X)}\left(P_x, P_y\right) \cong \mathrm{Hom}_{A_{\mathbf{k}}(X)}\left(I_x, I_y\right) \cong \begin{cases} \mathbf{k} \text{ if } y \leqslant x, \\ 0 \text{ otherwise.} \end{cases}$$

4. A nonzero morphism between two projective indecomposable modules is a monomorphism.

5. A nonzero morphism between two injective indecomposable modules is an epimorphism.

6. $A_{\mathbf{k}}(P)$-mod *has a finite global dimension bounded by the largest chain in* P.

Proof. For simplicity, we denote $A_{\mathbf{k}}(X)$ by A. The module S_x is one-dimensional, so it is simple. Since (x, x) is an idempotent, the module P_x is projective. The module I_x is the dual of a projective left A-module, so it is injective.

We have $\mathrm{Hom}_A\left(P_x, P_y\right) \cong (y, y)A(x, x)$. It is a one-dimensional vector space with basis (y, x) when $y \leqslant x$ and it is zero otherwise. The left multiplication by (y, x) induces an injective morphism from P_x to P_y. Every morphism between P_x and P_y is a scalar multiple of this injective morphism, so it is either zero or injective. The proof for the injective modules is dual. This also imply that P_x and I_x are indecomposable.

It is easy to see that the radical of P_x has basis the set of (x, z) such that $x \leqslant z$ and $z \neq x$. In particular P_x surjects onto S_x so it is a projective cover. The first term of a minimal projective resolution of S_x is P_x. The composition factors of the kernel are given by the S_y for $x \leqslant y$ and $x \neq y$. Then, either the kernel is projective or it has a projective cover of the form $\bigoplus_{z'} P_z$ where $x \leqslant z'$ is a cover relation. By induction, we see that S_x has a finite minimal projective resolution. Moreover, the global dimension of the category is bounded by the largest chain in P. □

Remark 2.3. In the proof above, we see that the global dimension of P is bounded by the largest chain in P. However, it is not true that the global dimension has an easy combinatorial description. There are finite posets P such that the global dimension of their incidence algebra depends on the characteristic of the ground field (See Proposition 2.3 of [IZ90] for more details).

Since the category $A_{\mathbf{k}}(P)$-mod is abelian, one can consider its bounded derived category. Note that the ground field does not play an important role in our work, so by a slight abuse of notation we denote by $D^b(P)$ the bounded derived category of the category $A_{\mathbf{k}}(P)$-mod. If the category $A_{\mathbf{k}}(P)$-mod is abelian, the bounded derived category $D^b(P)$ is a more subtle object. It has a structure of *triangulated category* (for more details about triangulated categories and derived category we refer to Chapters 3 and 6 of [Zim14]).

Informally, the objects of the derived category are left and right bounded chain complexes of $A_{\mathbf{k}}(P)$-modules (See below in the next Paragraph). The morphisms are more complicated, but by construction if two complexes are quasi-isomorphic, they become isomorphic in the derived category. There is a fully faithful embedding of the category $A_{\mathbf{k}}(P)$-mod into the derived category $D^b(P)$. It is obtained by sending a right $A_{\mathbf{k}}(P)$-module to the complex consisting of this module in degree zero.

In the rest of this article we will *always* work in the derived category, so when we speak about the $A_{\mathbf{k}}(P)$-module M, we have in mind the image of this module in the derived category by the fully faithful functor. Since any $A_{\mathbf{k}}(P)$-module is quasi-isomorphic to a projective resolution, in the derived category, *we can identify a module with one of its projective resolutions.*

The derived category has a *suspension functor*. Here we call it the *shift* and denote it by [1]. Let (C_i, d_i) be a chain complex of $A_{\mathbf{k}}(P)$-modules. That is C_i is a $A_{\mathbf{k}}(P)$-module and $d_i : C_i \to C_{i-1}$ is a morphism of $A_{\mathbf{k}}(P)$-modules such that $d_{i-1} \circ d_i = 0$ for every i. Then [1] denotes the usual left shift. More generally, for $n \in \mathbb{Z}$, we let $(C[n])_k = C_{k-n}$ with ith differential $(-1)^n d_{i-n}$.

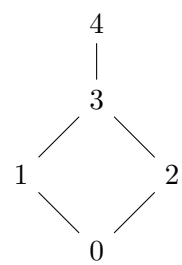
A Serre functor of the derived category $D^b(P)$ is a triangulated autoequivalence $\mathbb{S}$ of $D^b(P)$ such that there is a bi-functorial isomorphism

$$\mathrm{Hom}(X, \mathbb{S}(Y)) \cong \mathrm{Hom}(Y, X)^*, \forall X, Y \in D^b(P),$$

where $-^*$ denotes the $\mathbf{k}$-linear dual. Since the global dimension of the incidence algebra $A_{\mathbf{k}}(P)$ is finite, it is classical that the derived Nakayama functor $- \otimes^L_{A_{\mathbf{k}}(P)} A_{\mathbf{k}}(P)^*$ gives a Serre functor for $D^b(A)$ (See Theorem 3.1 [Kel08] for example). Note that Serre functors are unique up to unique isomorphism, so sometimes we will abusively speak about 'the' Serre functor.

Let $x \in P$. It is easy to see that $\mathbb{S}(P_x) \cong I_x$. Let $y \leqslant x$. Let $i : P_x \to P_y$ be the embedding. Then $\mathbb{S}(i) : I_x \to I_y$ is the canonical epimorphism. So, in fact, when we want to compute the value of $\mathbb{S}$ on a module M, we start by finding a projective resolution $\mathcal{P}_M$ of M. Then, we replace every projective indecomposable module by the corresponding injective module and the embeddings are replaced by the epimorphisms.

Example 2.4. Even in the very simple case of representation of finite poset, the image of a module by a Serre functor is not always a module. For example, let P be the poset $\{0, 1, 2, 3, 4\}$ where the cover relations are $0 < 1$, $0 < 2$, $1 < 3$, $2 < 3$ and $3 < 4$. In other words, it is the poset with Hasse diagram:



It is easy to see that $\mathbb{S}(P_3) \cong I_3$. In the derived category $I_3 \cong \big(P_4 \hookrightarrow P_0\big)$. So $\mathbb{S}^2(P_3) \cong \big(I_4 \to I_0\big) \cong \mathrm{Rad}(P_0)[1]$. In the derived category we have $\mathrm{Rad}(P_0) \cong \big(P_3 \xrightarrow{f} P_1 \oplus P_2\big)$ where f is the diagonal embedding of P_3 in $P_1 \oplus P_2$. So $\mathbb{S}^3(P_3) \cong (I_3 \xrightarrow{g} I_1 \oplus I_2)[1]$ where g is the diagonal projection of I_3 onto $I_1 \oplus I_2$. The kernel of g is isomorphic to S_3 and the cokernel is isomorphic to S_0. So $\mathbb{S}^3(P_3)$ has homology concentrated in two degrees. Since it is the image of an indecomposable object by an equivalence, it is an indecomposable object. This implies that $\mathbb{S}^3(P_3)$ is an indecomposable object of $D^b(P)$ which is not isomorphic to a shift of a module.

Definition 2.5. Let A be a finite dimensional $\mathbf{k}$-algebra with finite global dimension. Then $D^b(A)$ is (m, n)-fractionally Calabi-Yau if there is an isomorphism of functors $\mathbb{S}^n \cong [m]$.

This means that for every object X in $D^b(A)$ there is an isomorphism $\phi_X : \mathbb{S}^n(X) \to X[m]$ and

for every morphism $f : X \to Y$ in $D^b(A)$, the following diagram is commutative

$$\begin{tikzcd} \mathbb{S}^n(X) \arrow[r, "\phi_X"] \arrow[d, "\mathbb{S}^n(f)"] & X[m] \arrow[d, "f[m]"] \\ \mathbb{S}^n(Y) \arrow[r, "\phi_Y"] & Y[m]. \end{tikzcd}$$

Example 2.6. Let Q be a finite connected acyclic quiver. It is well-known that $\mathbf{k}Q$ is Fractionally Calabi-Yau if and only if Q is an orientation of a Dynkin diagram of type A, D, E (see for example [vR12]).

The particular case of incidence algebras of finite posets has recently received attention in the work of several authors (See for example [Cha12, DPW17, KLM13]). If P is a finite poset, then it is easy to see that, up to a sign, the linear map induced by the Serre functor on the Grothendieck group of $D^b(P)$ is the so-called *Coxeter* transformation. Its matrix in the basis of the simple modules is called the Coxeter matrix. It has a simple description as $C = -I(I^{-1})^t$ where I is the incidence matrix of P. There is also an abundance of studies on Coxeter transformations in the literature (See for example [Cha07b, Lad08, Yil19]).

In general, it is not easy to check that the derived category is fractionally Calabi-Yau, or that the Coxeter matrix has finite order because the classical families of modules (or classical basis of the Grothendieck groups) are not preserved by the Serre functor (see Example 2.4).

Looking one last time at Example 2.4, we see that even in a simple case, understanding 'by brute force' the Serre functor is not so easy. Nevertheless, it is easy to check that $D^b(P)$ is triangulated equivalent to $D^b(\mathbf{k}D_5)$ and it is well-known that this category is $(6, 8)$-fractionally Calabi-Yau. In other terms, finding a 'good' description of our triangulated category as bounded derived category of a well chosen finite dimensional algebra, or abelian category, can simplify a lot the understanding of the Serre functors.

The situation is particularly well understood when one can find a hereditary such category: for a finite dimensional path algebra, or more generally for an abelian hereditary category, we know precisely when their bounded derived category is fractionally Calabi-Yau (See [vR12, Len07]). However, this cannot be used in order to solve Conjecture 1.1.

Proposition 2.7. *If $n \geqslant 4$, then $D^b(\mathrm{Tam}_n)$ is not triangulated equivalent to $D^b(\mathcal{H})$ for an abelian hereditary category $\mathcal{H}$.*

Proof. If there is a hereditary abelian category $\mathcal{H}$ such that $D^b(\mathrm{Tam}_n) \cong D^b(\mathcal{H})$, then the representation type of $A_{\mathbf{k}}(\mathrm{Tam}_n)$ is dominated[1] by the representation type of $\mathcal{H}$ (see Section 2 of [HZ10] or Proposition 2.3 of [CR19]). Looking at the classification, we see that if $D^b(\mathcal{H})$ is fractionally Calabi-Yau derived category, then $\mathcal{H}$ has a finite or a tame representation type. But, when $n \geqslant 4$, the incidence algebras of the Tamari lattices have a wild representation type (Section 3 of [CR19]). □

[1]Here dominated means smaller in the order where (finite) $\leq$ (tame) $\leq$ (wild).

3 Posets with a unique maximal or a unique minimal element

We consider a finite poset X and a field $\mathbf{k}$. The aim of this section is to prove the following result.

Theorem 3.1. *Let X be a finite poset with a unique minimal element or a unique maximal element. If there are integers m and n such that $\mathbb{S}(P)^n \cong P[m]$ in $D^b(X)$ for all projective indecomposable modules of X, then the category $D^b(X)$ is (m,n)-fractionally Calabi-Yau.*

For simplicity, in all this section we denote by A the incidence algebra of X over the field $\mathbf{k}$.

Let ϕ be a $\mathbf{k}$-algebra endomorphism of A. Then, one can twist the regular left action of A on itself by $\phi : a \cdot x := \phi(a)x$ for $a, x \in A$. We denote by ${}_\phi A_1$ the A-A-bimodule A where the left action is twisted by ϕ and the right action is given by the right multiplication.

Proposition 3.2. *Let X be a finite poset and A its incidence algebra over a field $\mathbf{k}$. Then, the following are equivalent:*

1. *There are integers m and n such that $\mathbb{S}^m(P_x) \cong P_x[n]$ in $D^b(A)$ for all $x \in X$*
2. *There is a $\mathbf{k}$-algebra automorphism ϕ of A such that $\phi(e_x) = e_x$ for all $x \in X$ and such that*

$$\mathbb{S}^m \cong - \otimes^L_A {}_\phi A_1 \circ [n].$$

Proof. Let e be an idempotent in A. The A-module morphism $\theta : eA \otimes_A {}_\phi A \to \phi(e)A$ defined by $\theta(\sum_{i=0}^n ea_i \otimes b_i) = \sum_{i=0}^n \phi(ea_i)b_i$ is an isomorphism of right A-modules with inverse the morphism ψ that sends $\phi(e)b$ to $e \otimes b$. As consequence, we see that 2 implies 1.

Conversely, we start by using Lemmas 4.1, 4.2 and Proposition 4.3 of [HI11]. For the convenience of the reader we recall the main arguments.

By additivity of the derived Nakayama functor, hypothesis 1 implies that

$$\mathbb{S}^m(A) \cong A[n] \text{ in } D^b(A).$$

In other terms, the complex of A-A-bimodules $D(A)^{\otimes^L m} := \overbrace{D(A) \otimes^L \cdots \otimes^L D(A)}^{m}$ is isomorphic in $D^b(A)$ to $A[n]$. Let $M = D(A)^{\otimes^L m}[-n]$. Then M is isomorphic to A in $D^b(A)$. It means that M has its homology concentrated in degree 0 and that the A-A-bimodule $H^0(M)$ is isomorphic to A as a right-module. It is classical that the second condition is equivalent to the existence of a $\mathbf{k}$-algebra endomorphism ϕ such that $H^0(M) \cong {}_\phi A_1$ as A-A-bimodules (See e.g. Lemma 4.1 of [HI11]). In other terms, the complex of bimodules M has its homology concentrated in degree 0 and its homology is isomorphic to ${}_\phi A_1$. So, we have that $M \cong {}_\phi A_1$ in $D^b(A \otimes_{\mathbf{k}} A^{op})$ (this is the statement of Lemma 4.2 of [HI11]). This isomorphism of complexes of bimodules induces an isomorphism of functors

$$\mathbb{S}^m \cong - \otimes^L_A {}_\phi A_1 \circ [n].$$

Since $\mathbb{S}^m$ is an autofunctor, we see that ϕ is a $\mathbf{k}$-algebra automorphism. In other terms, the $\mathbf{k}$-algebra A is a twisted fractionally Calabi-Yau algebra in the sense of Herschend and Iyama (this it the statement of Proposition 4.3 of [HI11]).

By hypothesis, if $x \in X$, we have $\mathbb{S}^m(e_x A) \cong e_x A[n]$. Since ${}_\phi A \otimes_A e_x A \cong \phi(e_x)A[n]$, we see that $e_x A \cong \phi(e_x)A$ as right A-modules. In other words, the idempotents e_x and $\phi(e_x)$ are conjugated. We let $J(A)$ be the Jacobson radical of A. Since $A/J(A)$ is commutative, the idempotents e_x and $\phi(e_x)$ have the same image in the quotient. It follows that $e_x - \phi(e_x) \in J(A)$.

The element $g = \sum_{x \in X} e_x \phi(e_x)$ is invertible since $1 - g \in J(A)$. Moreover, $g\phi(e_x)g^{-1} = e_x$ for all $x \in X$. Let $\phi' = c_g \circ \phi$ where c_g denotes the conjugation by g. Then, ϕ' is a $\mathbf{k}$-algebra automorphism of A such that $\phi'(e_x) = e_x$ for all $x \in X$. Finally, since ${}_{c_g}A_1 \cong {}_1A_1$ as bimodules, there is an isomorphism of functors between $- \otimes^L_A {}_\phi A_1$ and $- \otimes^L_A {}_{\phi'} A_1$.

□

Lemma 3.3 (Stanley [Sta70])**.** *Let X be a finite poset with a unique minimal or a unique maximal element. Let ϕ be a $\mathbf{k}$-algebra automorphism of A such that $\phi(e_x) = e_x$ for all $x \in X$. Then, ϕ is an inner automorphism.*

Proof. We only give a proof when X has a unique minimal element, the other case is similar.

We denote the minimal element of X by 0. The vector space $e_x A e_y$ is $\{0\}$ when $x \nleqslant y$ and is generated by (x, y) when $x \leqslant y$. For every basis element (x, y) there is a scalar $k_{x,y} \in \mathbf{k}^*$ such that $\phi(x, y) = k_{x,y}(x, y)$.

Let $u = e_0 + \sum_{x \neq 0} k_{0,x} e_x$. Then u is an invertible element with inverse $u^{-1} = e_0 + \sum_{x \neq 0} k_{0,x}^{-1} e_x$. It is easy to see that $u(0, x)u^{-1} = k_{0,x}^{-1}(0, x)$. In other words, $\phi \circ c_u(0, x) = (0, x)$ and $\phi \circ c_u(e_x) = e_x$ for all $x \in X$.

For $0 \neq x < y$, we have $(0, y) = (0, x) \cdot (x, y)$. This implies that an automorphism that fixes all the basis elements $(0, x)$ and all the idempotents e_x is the identity. □

Proof of Theorem 3.1. By Proposition 3.2, under the hypothesis we have $\mathbb{S}^m \cong - \otimes^L_A {}_\phi A_1 \circ [n]$ where ϕ is a $\mathbf{k}$-algebra automorphism that fixes the idempotents e_x. By Lemma 3.3 any such automorphism is inner, so the induced bimodule ${}_\phi A_1$ is isomorphic to the regular bimodule. It shows that $\mathbb{S}^m \cong [n]$. □

4 Tamari lattices, interval-posets and noncrossing trees

4.1 Tamari lattices and interval-posets

Let $n \in \mathbb{N}$. A (planar) binary tree of size n is a graph embedded in the plane which is a tree, has n vertices with valence 3, $n + 2$ vertices with valence 1 and a distinguished univalent vertex called the *root*. The other vertices of valence 1 are called the *leaves* of the tree. For the rest of the paper, when we speak about vertices of the tree, we have in mind the trivalent vertices. The planar binary trees are pictured with their root at the bottom and their leaves at the top.

Let Tam_n be the set of all binary trees with n vertices. It is well-known that the cardinality of this set is the Catalan number $c_n = \frac{1}{n+1}\binom{2n}{n}$.

There is a partial order relation on Tam_n which was introduced by Tamari in [Tam62]. It is defined as the transitive closure of the following covering relations. A tree T is covered by a tree S if they only differ in some neighborhood of an edge by replacing the configuration in T by the configuration in S. The poset Tam_n is known to be a lattice (See e.g. [HT72, Tam62]).

Our convention is maybe not the most classical: in general we use the left rotation of trees in order to describe the partial order of the Tamari lattices. We make this choice since it yields to a more natural description of some bijections.

A binary *search tree* is a binary tree labeled by integers such that if a vertex x is labeled by k, then the vertices of the left subtree (resp. right subtree) of x are labeled by integers less than or equal (resp. superior) to k.

If T is a binary tree with n vertices, there is a unique labeling of the vertices by each of the integers $1, 2, \cdots, n$ that makes it a binary search tree. This procedure is sometimes called the *in-order traversal* of the tree (visit left subtree, root and then right subtree).

Using this labeling, a binary tree T with n vertices induces a partial order relation $\lhd_T$ on the set $\{1, \cdots, n\}$ by setting $i \lhd_T j$ if and only if the vertex labeled by i is in the subtree with root j.

When $(P, \lhd)$ is a partial order on the set $\{1, \cdots, n\}$, one can use the natural total ordering of the integers $1, \cdots, n$ that we denote by $<$ to split the relations $\lhd$ in two families. Let $1 \leqslant a < b \leqslant n$ be two integers. If $a \lhd b$ we say that the relation is *increasing*. On the other hand, if $b \lhd a$, we say that the relation is *decreasing*. We denote by $\mathrm{Dec}(P)$ and $\mathrm{Inc}(P)$ the set of decreasing and increasing relations of P. We denote by $\mathrm{Hasse}(P)$ the set of cover relations of P.

If T is a binary tree, we will implicitly see it as a binary search tree and we define $\mathrm{Dec}(T)$ and $\mathrm{Inc}(T)$ as the respective set of decreasing and increasing relations of the poset induced by T. Then, we have a useful characterization due to Châtel, Pilaud and Pons [CPP19] of the partial order of the Tamari lattice in terms of increasing or decreasing relations.

Proposition 4.1. *Let T_1 and T_2 be two binary trees. Then $T_1 \leqslant T_2$ in the Tamari lattice if and only if $\mathrm{Dec}(T_2) \subseteq \mathrm{Dec}(T_1)$ if and only if $\mathrm{Inc}(T_1) \subseteq \mathrm{Inc}(T_2)$.*

Proof. See Proposition 40 and Remark 52 of [CPP19]. Note that our convention differs from theirs. □

Let us recall the definition of interval-posets.

Definition 4.2. An interval-poset $(P, \lhd)$ is a poset over the integers $1, \cdots, n$ such that

1. If $a \lhd c$ and $a < c$, then for all integers b such that $a < b < c$, we have $b \lhd c$.
2. If $c \lhd a$ and $a < c$, then for all integers b such that $a < b < c$, we have $b \lhd a$.

The conditions (1) and (2) of this definition will be referred as the *interval-poset condition*. The integer n in the definition is called the *size* of the interval-poset.

Theorem 4.3 (Châtel, Pons)**.** *Let $n \in \mathbb{N}$. There is a bijection between the set of intervals in Tam_n and the set of interval-posets of size n.*

Proof. This is Theorem 2.8 of [CP15]. We will only need the bijection associating an interval-poset to an interval of the Tamari lattice. If $[S, T]$ is an interval in Tam_n, we can construct an interval-poset as follows. The trees S and T can be seen as binary search trees and they induce two partial order relations $\lhd_S$ and $\lhd_T$. Let $P = \{1, 2, \cdots, n\}$. Let $\lhd$ be the binary relation on P given by the disjoint union of the increasing relations of S and the decreasing relations of T. Then, it is proved in [CP15] that $(P, \lhd)$ is an interval-poset. For the inverse bijection we refer to Theorem 2.8 of [CP15]

□

4.2 Noncrossing trees and exceptional interval-posets

Using three operads, it was proved in [Cha07a, CHNT08], that one can associate an interval in the Tamari lattice to any noncrossing tree. In [Rog20] we obtained a description of these intervals in terms of interval-posets avoiding certain configurations that we called *exceptional*. For the purpose of this article, we can forget about the operads behind this description.

We start by recalling the definitions of noncrossing trees and exceptional interval-posets. Then, we recall the bijections obtained in [Rog20] between the noncrossing trees and the exceptional interval-posets. Finally, we give a new bijection between these two families. Both of these bijections will be used in the rest of the article.

Let $\mathcal{C}_n$ be a circle with $n+1$ equidistant points labeled in the clockwise order from 1 to $n+1$. When we draw $\mathcal{C}_n$, we will always put the vertex with label 1 on the bottom.

A *noncrossing tree* in $\mathcal{C}_n$ is a set of edges between the marked points with the following properties

- edges do not cross pairwise,
- any two points are connected by a sequence of edges,
- There is no loop made of edges.

See Figure 1 for an example. We say that n is the size of the tree and we denote by $\mathbf{NCP}_n$ the set of all noncrossing trees of size n. It is classical that the number of noncrossing trees in C_n is $\frac{1}{2n+1}\binom{3n}{n}$ (See e.g. Theorem 3.10 of [DP93]).

Let T be a noncrossing tree in $\mathcal{C}_n$. An edge between two vertices x and y is denoted by $\mathrm{x} \text{ —— } \mathrm{y}$ with the convention that $x < y$. An edge between i and $i+1$ in T is called a *side* of T. If there is no edge from i to $i+1$, we say that $(i,(i+1))$ is an *open side* of T. The side $(1,(n+1))$ is called the base of T.

Definition 4.4. An interval-poset whose Hasse diagram does not contain any configuration of the form $y \to z$ and $y \to x$ where $x < y < z$ is called an *exceptional* interval-poset.

Let us recall a bijection between exceptional interval-posets and noncrossing trees. For more details we refer to Section 3 of [Rog20].

If $(P, \lhd)$ is an exceptional interval-poset over the integers $[1,n]$ we can construct a list of edges in $\mathcal{C}_n$ by:

for an integer v consider the poset $\{x \in [1,n]\ ;\ x \lhd v\}$. This poset has a minimal element (for the usual order relation $<$) v_1 and a maximal element v_2. We associate to v the edge $v_1 \text{ — } (v_2+1)$. We denote by $\psi(P)$ the set of these edges in $\mathcal{C}_n$. See Figure 1 for an illustration.

Conversely, let T be a noncrossing tree of size n.

1. We label the edges of the noncrossing T:

   if an edge is a boundary side $(i,i+1)$, then it is labeled by i. If not, the edge separates a unique open side $(i,i+1)$ from the basis and it is labeled by i.

2. We define a relation $\lhd_T$ on $\{1,2,\cdots,n\}$ by $i \lhd_T j$ if the edge labelled by i is separated from the base by the edge labelled by j. We let $\psi^{-1}(T) = \big(\{1,2,\cdots,n\}, \lhd_T\big)$.

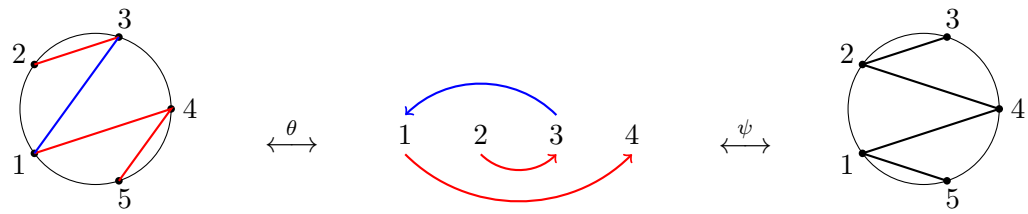


Figure 1: Illustration of the bijections θ and ψ between interval-posets and noncrossing trees.

Theorem 4.5. *Let $n \in \mathbb{N}$. The map ψ is a bijection between the set of exceptional interval-posets of size n and the set of noncrossing trees of size n with ψ^{-1} as inverse.*

Proof. See Section 3 of [Rog20]. □

Let us introduce another bijection between these two families.

Let $(P, \lhd)$ be an exceptional interval-poset of size n. Then, we consider $\widehat{P} = P \sqcup \{n+1\}$. We extend the relation $\lhd$ to $\widehat{P}$ by saying that $x \lhd n+1$ for $x \in P$. The new cover relations of $\widehat{P}$ are of the form $x \lhd (n+1)$ for x a maximal element of P. We can see that $\widehat{P}$ is still an exceptional interval-poset because a new forbidden configuration would be of the form $y \lhd n+1$ and $y \lhd x$ for $x < y < n+1$. If $y \lhd n+1$ is a cover relation, then y is a maximal element of P. This contradicts the relation $y \lhd x$.

From the Hasse diagram of $\widehat{P}$ we construct $\theta(P)$ a set of edges in the circle $\mathcal{C}_n$ by sending the cover relation $i \lhd j$ to the edge $\mathrm{i} \text{———} \mathrm{j}$.

Conversely, let T be a noncrossing tree in $\mathcal{C}_n$. Since T is a noncrossing tree, for each vertex i there is a unique path made of edges towards the vertex $n+1$. In particular, we can orient the edges of T towards the vertex $n+1$. Then, a *descent* is an edge from i to j where $i > j$ and a *rise* is an edge from i to j where $i < j$.

From the noncrossing tree T we construct the set H of relations $i \lhd j$ where $\mathrm{i} \longrightarrow \mathrm{j}$ is a rise from i to j or $\mathrm{i} \longleftarrow \mathrm{j}$ is a descent from i to j. See Figure 1 for an illustration.

Theorem 4.6.

1. *Let P be an exceptional interval-poset of size n. Then, $\theta(T)$ is a noncrossing tree of size n.*
2. *Let P be a noncrossing tree. Then, H is the Hasse diagram of an exceptional interval-poset denoted by $\theta^{-1}(P)$.*
3. *θ induces a bijection between the set of exceptional interval-posets of size n and noncrossing trees in $\mathcal{C}_n$ with inverse θ^{-1}.*

Proof. The proof is not difficult and is left to the reader. Alternatively, in Proposition 6.3 we show that θ is the composition of ψ and the planar duality for noncrossing trees. This will imply that it is a bijection. □

5 Interval-posets in the derived category of the Tamari lattices

If I is an interval in the Tamari lattice Tam_n, then $M_I := \bigoplus_{x\in I} \mathbf{k}e_x$ is an indecomposable $A_{\mathbf{k}}(\mathrm{Tam}_n)$-module for the following action. The basis element (y, z) has a non-zero action on M_I if and only if $y, z \in I$ and in this case it sends e_y to e_z. For simplicity, we also denote this module by I.

As explained in Theorem 4.3, the intervals of the Tamari lattices have a nice description in terms of special partial order relations. So, starting with an interval-poset P of size n, we implicitly use the bijection of Châtel and Pons in order to have an interval I_P of Tam_n. Then, we construct the corresponding module, and finally we look at its image in the derived category. From now on, *we will always identify the interval-posets with their corresponding indecomposable object in the derived category.* In this section we give a small dictionary that will allow us to work with them.

Let us start by a characterization of the interval-posets corresponding to simple modules.

Proposition 5.1. *Let I be an interval-poset of size n. Then, I is a simple $A_{\mathbf{k}}$-module if and only if for all $i < j$, there exists $i \leqslant z \leqslant j$ such that $i \lhd z$ and $j \lhd z$.*

Proof. This is Proposition 39 [CPP19]. □

We also have a simple description of the projective indecomposables and the injective indecomposables.

Proposition 5.2. *Let I be an interval-posets of size n.*

1. *I is projective if and only if $\mathrm{Dec}(I) = \emptyset$.*
2. *I is injective if and only if $\mathrm{Inc}(I) = \emptyset$.*
3. *Let P_1 and P_2 be two projective interval-posets. Then*
$$\mathrm{Hom}_{A_{\mathbf{k}}(\mathrm{Tam}_n)}(P_1, P_2) = \begin{cases} \mathbf{k} \text{ if } \mathrm{Inc}(P_2) \subseteq \mathrm{Inc}(P_1), \\ 0 \text{ otherwise.} \end{cases}$$
4. *Let I_1 and I_2 be two injective interval-posets. Then*
$$\mathrm{Hom}_{A_{\mathbf{k}}(\mathrm{Tam}_n)}(I_1, I_2) = \begin{cases} \mathbf{k} \text{ if } \mathrm{Dec}(P_1) \subseteq \mathrm{Dec}(P_2), \\ 0 \text{ otherwise.} \end{cases}$$

Proof. Let I be an interval-poset and $i = [S, T]$ be the corresponding interval of the Tamari lattice. Then i is projective if and only if T is the maximal element of the Tamari lattice. The maximal element of T is the left comb and it has no decreasing relation. Conversely, if A is a binary tree, then the vertices of its right subtree are labeled by integers larger than the label of the root. So if A has no decreasing relation, it is the left comb.

By construction, the decreasing relations of I are the decreasing relations of T. So I is the interval-poset of a projective interval if and only if it has no decreasing relations. The proof for the injective interval-posets is dual.

Let x, y be two binary trees. By Proposition 4.1, we have $x \leqslant y$ if and only if $\mathrm{Inc}(x) \subseteq \mathrm{Inc}(y)$ if and only if $\mathrm{Dec}(y) \subseteq \mathrm{Dec}(x)$. The result follows from the description of the morphisms between projective modules and between injective modules given in Proposition 2.2.

□

As immediate corollary, we have:

Corollary 5.3. *Let $n \in \mathbb{N}$. The projective indecomposable, the injective indecomposable and the simple $A_{\mathbf{k}}(\mathrm{Tam}_n)$-modules are exceptional interval-posets.*

We will also need a characterization of the trees belonging in an interval which involves the corresponding interval-posets.

Lemma 5.4. *Let I be an interval-poset of size n and $T \in \mathrm{Tam}_n$. We denote by $[T_1, T_2]$ the interval of Tam_n corresponding to I. Let $J = \mathrm{Dec}(I) \cap \mathrm{Hasse}(I)$. We set $\overline{J} := \{i \lhd j \ ; \ j \lhd i \in J\}$.*

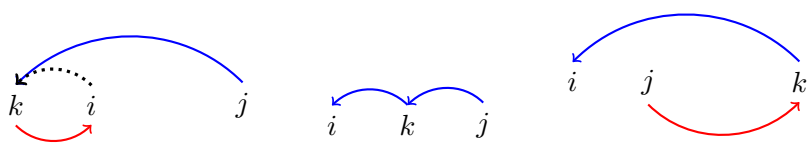
Then, we have $T \in [T_1, T_2]$ if and only if $\mathrm{Inc}(I) \subseteq \mathrm{Inc}(T)$ and $\overline{J} \cap \mathrm{Inc}(T) = \emptyset$.

Proof. By Proposition 4.1, the tree $T \in [T_1, T_2]$ if and only if $\mathrm{Inc}(T_1) \subseteq \mathrm{Inc}(T)$ and $\mathrm{Dec}(T_2) \subseteq \mathrm{Dec}(T)$. By the proof of Theorem 4.3, we have $\mathrm{Inc}(T_1) = \mathrm{Inc}(I)$ and $\mathrm{Dec}(T_2) = \mathrm{Dec}(I)$. Hence, if $T \in [T_1, T_2]$, we have $\mathrm{Inc}(I) \subseteq \mathrm{Inc}(T)$. Moreover, any relation in $\overline{J} \cap \mathrm{Inc}(T)$ contradicts the anti-symmetry of the poset induced by T, because $J \subseteq \mathrm{Dec}(I) \subseteq \mathrm{Dec}(T)$. Hence we have $\overline{J} \cap \mathrm{Inc}(T) = \emptyset$.

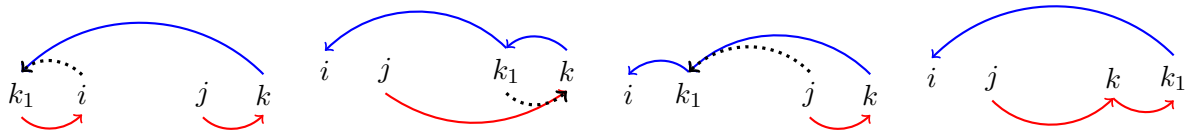
For the converse, using one more time Proposition 4.1, it is enough to show that the two conditions $\mathrm{Inc}(I) \subseteq \mathrm{Inc}(T)$ *and* $\overline{J} \cap \mathrm{Inc}(T) = \emptyset$ imply that $\mathrm{Dec}(I) \subseteq \mathrm{Dec}(T)$. The proof is rather technical but the idea is simple: the characterization of the interval-posets induced by a single tree of Proposition 5.1, implies that these posets must have a lot of relations. For example any two consecutive elements must be comparable. Hence, the fact that *the opposite* of the relations in $J = \mathrm{Dec}(I) \cap \mathrm{Hasse}(I)$ are not in $\mathrm{Inc}(T)$ will lead to $\mathrm{Dec}(I) \subset \mathrm{Dec}(T)$.

Let $j \lhd i \in \mathrm{Dec}(I)$. We assume that $j \lhd i$ cannot be written as the concatenation of two smaller decreasing relations (i.e. that $j \lhd i$ is a cover relation in the poset of decreasing relations of I). We start by showing that $j \lhd i = j \lhd l \lhd i$ and $j \lhd l \in \mathrm{Inc}(I)$ and $l \lhd i \in J = \mathrm{Dec}(I) \cap \mathrm{Hasse}(I)$.

Indeed, $j \lhd i$ is a cover relation in I or there is $k \in \{1, \cdots, n\}$ such that $j \lhd k \lhd i$. We have three possibilities for k. Either it is in $\{1, \cdots, i-1\}$, or in $\{i, \cdots, j-1\}$ or in $\{j, \cdots, n\}$. Here is a simple picture of the three possibilities.



In the first case, the interval-poset condition implies the existence in T of the relation $i \lhd k$ which contradicts the anti-symmetry of the poset. In the second case, the relation $j \lhd i$ is not a cover relation in the poset of decreasing relations. Hence, the only possibility is the third. If $k \lhd i$ is not a cover relation, there is k_1 such that $k \lhd k_1$ and $k_1 \lhd i$. There are four possibilities displayed below. The dotted arrows are the relations induced by the interval-poset condition.

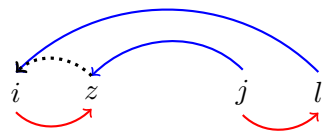


The two first cases contradict the anti-symmetry of the poset. The third case contradicts the fact that the relation $j \lhd i$ is a cover relation in the poset of decreasing relations. Hence, we are left with the fourth case. Since I is finite, we see that there is $l \geq j$ such that $j \lhd i = j \lhd l \lhd i$ and $j \lhd l \in \mathrm{Inc}(I)$ and $l \lhd i \in J = \mathrm{Dec}(I) \cap \mathrm{Hasse}(I)$.

Now, we prove by induction on the length of the relations that $\mathrm{Dec}(I) \subseteq \mathrm{Dec}(T)$.

Step 1. Relations of length 2. Let $i+1 \lhd i \in D$. Such a relation of length 2 is a cover relation in the poset of decreasing relations, hence there is $l \geq i+1$ such that $i+1 \lhd l \in \mathrm{Inc}(I) \subseteq \mathrm{Inc}(T)$ and $l \lhd i \in J$. By hypothesis, we have $i \lhd l \notin \mathrm{Inc}(T)$. By Proposition 5.1 we have $i \lhd i+1 \in \mathrm{Inc}(T)$ or $i+1 \lhd i \in \mathrm{Dec}(T)$. Only the second case is possible since $i \lhd i+1 \lhd l = i \lhd l \notin \mathrm{Inc}(T)$.

Step 2. The general case. Let $j \lhd i \in D$. By induction, we can assume that $j \lhd i$ is a cover relation in the poset of decreasing relations of I. Hence, we have $l \geq j$ such that $j \lhd l \in \mathrm{Inc}(I) \subseteq \mathrm{Inc}(T)$ and $l \lhd i \in J$. Hence $i \lhd l \notin \mathrm{Inc}(T)$. By Proposition 5.1, there is $z \in \{i, \cdots, j\}$ such that $i \lhd z \in \mathrm{Inc}(T)$ and $j \lhd z \in \mathrm{Dec}(T)$. If $z = j$, then $j \lhd i \in \mathrm{Dec}(T)$ and we are done. If $z = i$, then $i \lhd j \in \mathrm{Inc}(T)$, hence $i \lhd l = i \lhd j \lhd l \in \mathrm{Inc}(T)$, this is a contradiction. The last case leads to a contradiction as displayed below.



□

As explained in Section 2, the Nakayama functor sends the projective indecomposable module P_T to the injective indecomposable module I_T. If $T \in \mathrm{Tam}_n$, the interval-poset corresponding to the projective module P_T is obtained by taking the increasing relations of T and the interval-poset of the injective module I_T is obtained by taking the decreasing relations of T. However, for our computations, we want to be able to send the interval poset of P_T to the interval-poset of I_T without having to construct the tree T. In other terms, we need to understand how the decreasing relations of a simple interval-poset are characterized by its increasing relations.

Lemma 5.5. *Let I an interval-poset corresponding to a single tree $T \in \mathrm{Tam}_n$. Let C be the set of increasing relations of T. The set of decreasing relations of T is given by*

$$C^0 = \{j \lhd i \ ; \ \forall i < s \leqslant j, \ i \lhd s \notin C\}.$$

Proof. If $j \lhd i$ is a decreasing relation of x, then by the interval-poset condition for $i \leqslant s < j$ there is a relation $s \lhd i$ in x. This implies that $j \lhd i \in C^0$.

Conversely, let $j \lhd i \in C^0$. Since x is simple, there exists $i \leqslant t \leqslant j$ such that $i \lhd t$ and $j \lhd t$. The only possibility is to have $t = i$, so the decreasing relation $j \lhd i$ is in x. □

Proposition 5.6. *Let P_C be a projective interval-poset with set of increasing relations C. Then,*

$$\mathbb{S}(P_C) \cong I_{C^0},$$

where I_{C^0} is the injective interval-poset with set of decreasing relations C^0.

Proof. It is an immediate corollary of Lemma 5.5. □

We are now almost ready to produce projective resolutions for the intervals of the Tamari lattices in terms of interval-posets.

Let $(I, \lhd)$ be an interval-poset of size n. Let $C = \mathrm{Inc}(I)$. Let $D = \mathrm{Dec}(I)$. And let J be the set of decreasing relations that are in the Hasse diagram of I.

If $f = j \lhd i$ is a relation of I, we denote by $\overline{f}$ the relation $i \lhd j$.

If $R \subseteq J$, we denote by $C + \overline{R}$ the smallest interval-poset containing the relations of C and the relations $\overline{f}$ such that $f \in R$. In other words, $C + \overline{R}$ is the *interval-poset closure* of $C \sqcup \overline{R}$, which exists since the intersection of two interval-posets is an interval-poset. We denote by $P_{C+\overline{R}}$ the interval-poset consisting of the relations of $C + \overline{R}$. By Proposition 5.2 it is a projective interval-poset.

If $R_1 \subset R_2 \subset J$, then $C \subset C + \overline{R_1} \subset C + \overline{R_2}$. By Proposition 5.2, we have injective morphisms $P_{C+\overline{R_2}} \to P_{C+\overline{R_1}} \to P_C$.

Let $\Delta = \Delta^{|J|-1}$ be a standard $(|J| - 1) - simplex$. We identify the vertices of Δ with the elements in J. Let $\mathcal{P}(I)$ be the following complex of projective modules where the morphisms are given by the (signed) embedding of projective modules:

$$P_{C+\overline{J}} \to \bigoplus_{\substack{R \subset J \\ |R|=|J|-1}} P_{C+\overline{R}} \to \cdots \to \bigoplus_{\substack{R \subset J \\ |R|=1}} P_{C+\overline{R}} \to P_C,$$

where P_C is in degree 0. It is well known that choosing an orientation of Δ allows us to assign a sign to each of the monomorphisms in such a manner that $\mathcal{P}(I)$ is a chain complex. In the derived category, all the different choices lead to isomorphic complexes.

Lemma 5.7. *$\mathcal{P}(I)$ is a projective resolution of the interval $I = [S, T]$*

Proof. Let $x \in \mathrm{Tam}_n$. By evaluating the complex $\mathcal{P}(I)$ at x, we see that this is an exact complex except if is in the support of P_C and not in the support of the other modules. Hence, the homology is concentrated in degree zero and it is the largest quotient of P_C which is supported by the set of trees x such that $C \subseteq \mathrm{Inc}(x)$ and $\mathrm{Inc}(x) \cap \overline{J} = \emptyset$. So by Lemma 5.4, the homology of the complex $\mathcal{P}(I)$ is isomorphic to the interval $[S, T]$. The surjective morphism from P_C to I induces a quasi-isomorphism between $\mathcal{P}(I)$ and I. □

Remark 5.8. In Lemma 5.7 the interval I is not assumed to be exceptional. Dually we can produce an injective resolution by adding to the set of decreasing relations of I the opposite of the increasing covers of I.

6 Noncrossing trees

6.1 Duality of noncrossing trees

Let T be a noncrossing tree in $\mathcal{C}_n$. A noncrossing tree gives a partition of the disk $\mathcal{C}_n$ into $n + 1$ areas, that we call the *cells* of the noncrossing tree. Since there is no loop made of edges, each cell has exactly one open side. We label each cell by the leftmost vertex of the open side. The cell

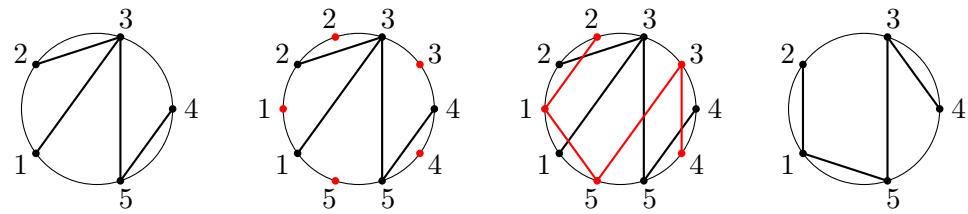


Figure 2: Illustration of the duality of noncrossing trees. The tree on the right is the dual of the tree on the left.

containing the vertices 1 and $n+1$ is called the base cell and is labeled by $n+1$. Another way of seing this labelling is by inserting $n+1$ new points of another color (say red) on the border of $\mathcal{C}_n$, one between each pair of successive vertices of T. The red point between the old points i and $i+1$ has label i for $1 \leq i \leq n$ and the red point between 1 and $n+1$ has label $n+1$. Each cell contains exactly one red point. Then we define the dual T^* of the non-crossing tree T by joining two red points if and only if they are separated by exactly one edge. We refer to Figure 2 for an illustration.

For $n \in \mathbb{N}$, we denote by ϑ the rotation by an angle of $\frac{2\pi}{n+1}$. Then, it turns out that the duality of non-crossing tree is a square root of the rotation ϑ.

Lemma 6.1. *Let T be a noncrossing tree.*

1. *The square of the duality is the rotation by an angle of $\frac{2\pi}{n+1}$. That is $(T^*)^* = \vartheta(T)$.*
2. *The duality commutes with the rotation. That is $\vartheta(T)^* = \vartheta(T^*)$.*

Proof. The first statement is a graphical evidence when drawing the tree and its dual on the same circle with vertices of alternating colors. The second statement is implied by the first. □

6.2 Descents and rises of noncrossing trees

Let T be a noncrossing tree in the circle $\mathcal{C}_n$. The edges of T are oriented towards the vertex $n+1$.

Let $f =$ a —— b be an edge in T, recall that by convention we have $a < b$. Since there is no loop made of edges, and since any pair of vertices are connected by a succession of edges, the edge f determines two open sides (i_f, i_f+1) and (j_f, j_f+1) of T where i_f is is connected to a in $T - \{$ a —— b $\}$ and j_f is is connected to b in $T - \{$ a —— b $\}$.

If the edge $f =$ a $\longleftarrow$ b is a descent, it means that a is after b in the path that goes from b to $n+1$. The path connecting 1 to $n+1$ cannot go through the vertices a or b. So, there must be an edge in T between $1 \leqslant a_1 \leqslant a$ and $b < b_1 \leqslant n+1$. If a_1 is maximal and b_1 is minimal for this property, the edge a_1 —— b_1 is the first (from the top to the bottom) edge that separates a $\longleftarrow$ b from the base. The situation is illustrated in the left case of Figure 3.

If the edge $f =$ a $\longrightarrow$ b is a rise, the situation is slightly more complicated. Since a is before b in the path from a to $n+1$, we have $j_f = n+1$ or $1 \leqslant j_f < a$. If $j_f = n+1$, then there is no

edge that separates f from the base. We say that f is a rise of *type* 2. Otherwise, there is such an edge and we say that f is a rise of *type* 1. The two situations are illustrated in Figure 3.

In other words an edge f determines two integers i_f and j_f. The integer i_f is the label of the edge (for the bijection ψ). If f is a descent or a rise of type 2, it determines a second edge which is labelled by j_f. If f is a rise of type 2, then $j_f = n+1$. In both cases i_f and j_f are the labels of the two adjacent cells with common boundary f. Hence we have an edge i_f —— j_f in T^* the dual of T.

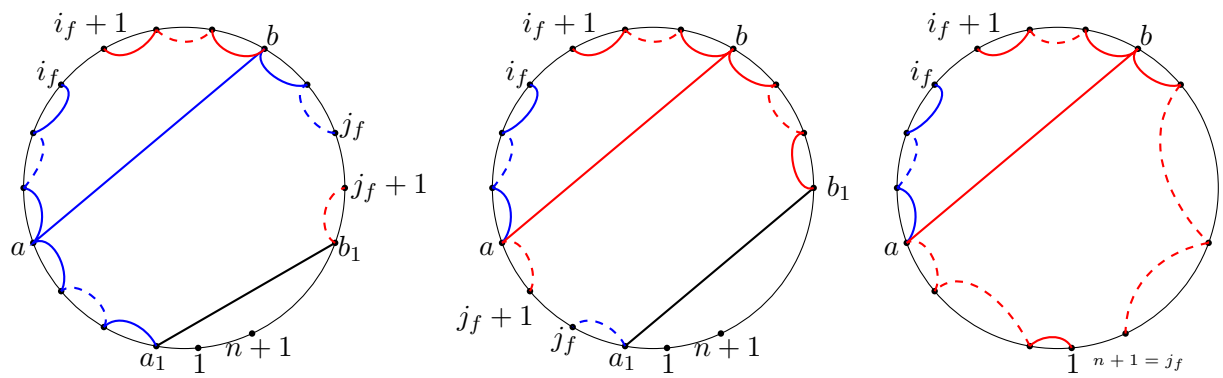


Figure 3: Shape of the subtrees determined by the edge between a and b. Edges in blue are descents, and edges in red are rises.

Lemma 6.2. *Let I be an exceptional interval-poset of size n. Let $x, y \in \{1, 2, \cdots, n\}$.*

1. *There is an increasing cover relation $x \lhd y$ in I if and only if x is the label of a descent f in $\psi(I)$. In this case $x = i_f$ and $y = j_f$.*
2. *There is a decreasing cover relation $x \lhd y$ in I if and only if x is the label of a rise f in $\psi(I)$ such that $j_f \neq n+1$. In this case $x = i_f$ and $y = j_f$.*
3. *x is a maximal element of I if and only if it is the label of a rise f in $\psi(I)$ such that $j_f = n+1$.*

Proof. If x is the label of a descent $f =$ a $\longleftarrow$ b in $\psi(I)$, looking at Figure 3, we see that $x = i_f$ and the first edge between f and the base is labeled by j_f, so in I we have $i_f \lhd j_f$. It is an increasing relation since $i_f < j_f$.

Conversely, if $x \lhd y$ is an increasing cover relation in I, then there are two edges $f =$ a —— b and $f_1 = a_1$ —— b_1 in $\psi(I)$ respectively labeled by x and y and such that f_1 is the first edge that separates f from the base. This implies that the tree has the shape of the leftmost or of the middle cases of Figure 3. Since $x < y$, the open side is between b and b_1. This implies that the path from b to $n+1$ goes through a. In other terms $f =$ a —— b is a descent.

The proofs of the second and third points are similar. □

Proposition 6.3. *Let I be an exceptional interval poset, then*

$$\psi(I)^* = \theta(I).$$

Proof. This is a consequence of Lemma 6.2. We do not assume $\theta(I)$ to be a noncrossing tree for this proof, because we haven't established it yet. First we show that every edge in $\theta(I)$ is in $\psi(I)^*$.

There is a descent $\mathrm{x} \longleftarrow \mathrm{y}$ in $\theta(I)$ if and only if there is a decreasing cover relation $x \lhd y$ in I. Then by Lemma 6.2 there is a rise f of type 1 in $\psi(I)$ such that $y = j_f$ and $i_f = x$. Looking at the middle case of Figure 3, we see that it separates a cell labeled by i_f and a cell labeled by j_f. So, the edge $\mathrm{x} \text{ —— } \mathrm{y}$ is in $\psi(I)^*$.

There is a rise $\mathrm{x} \longrightarrow \mathrm{y}$ in $\theta(I)$ if and only if there is an increasing cover relation $x \lhd y$ in I. Then by Lemma 6.2 there is a descent f in $\psi(I)$ such that $x = i_f$ and $y = j_f$. Looking at the left case of Figure 3 we see that the edge $\mathrm{x} \text{ —— } \mathrm{y}$ is in $\psi(I)^*$.

There is a rise $\mathrm{i} \longrightarrow \mathrm{n{+}1}$ in $\theta(I)$ if and only if i is maximal in I. Then, it labels a rise f of type 2. Looking at the right case of Figure 3, we see that the edge $\mathrm{i} \text{ —— } (\mathrm{n{+}1})$ is in $\psi(I)^*$.

All the noncrossing trees in $\mathcal{C}_n$ have exactly n edges. So, the tree $\psi(I)^*$ has as many edges as the tree $\psi(I)$. By Lemma 6.2, the set of edges of $\psi(I)$ is in bijection with the set consisting of the cover relations of I and its maximal elements. In other terms, the tree $\psi(I)^*$ has as many edges as $\theta(I)$. In conclusion, we have $\theta(I) = \psi(I)^*$.

□

7 Serre functor on noncrossing trees

In this section we compute the image of an exceptional interval-poset by the Serre functor. The main ingredient is the existence of boolean projective resolutions $\mathcal{P}(I)$ obtained in Lemma 5.7 and the description of the action of the Serre functor on a projective interval-poset obtained in Proposition 5.6. This description is obtained in terms of interval-posets and could be entirely proved in terms of interval-posets. However, the proof would be very technical and rather obscure. So, we use as much as possible the noncrossing trees and the bijections ϕ and θ described in Section 4.2. The strategy is to see that the Serre functor acts as the duality on the noncrossing tree of a projective interval-poset and then use it for understand the interval-posets appearing in the complex $\mathbb{S}(\mathcal{P}(I))$.

Lemma 7.1. *Let P_C be a projective interval-poset and $T = \psi(P_C)$ be its corresponding noncrossing tree. Then, we have $\psi\big(\mathbb{S}(P_C)\big) = T^*$.*

Proof. By Lemma 5.6 we have $\mathbb{S}(P_C) = I_{C^0}$ where $C^0 = \{j \lhd i \ ; \ \forall i < s \leqslant j, \ i \lhd s \notin C\}$. By Proposition 6.3, we have $\psi(P_C)^* = \theta(P_C)$, hence we have to prove that $\psi^{-1}(\theta(P_C)) = I_{C^0}$.

We only have increasing relations in P_C , so we only have rises in $\theta(P_C)$. This makes it relatively easy to understand the interval-poset $\psi^{-1}(\theta(P_C))$. Since there is a path from any vertex to $n+1$, each vertex is the start of a rise. If $f = \mathrm{i} \longrightarrow \mathrm{j}$ is a rise from i to j, then we have $i_f = i$ (see Figure 3), hence the rise f is labelled by i. Moreover, there is a unique rise starting at i. Indeed, if there is another rise $g = \mathrm{i} \longrightarrow \mathrm{x}$ starting at i, we may assume that $j < x$. Then by the interval-poset condition we have in P_C, $x \lhd j$. So, we have in $\theta(P_C)$ a loop made of edges involving i, x and j and this is a contradiction.

If $x \in [i+1, j-1]$, then x is the start of a rise $\mathrm{x} \longrightarrow \mathrm{y}$ with $y \leq j$ (otherwise, there is a crossing in $\theta(P_C)$), and these rises are exactly the rises that are separated from the base by f. In conclusion we see that in $\psi^{-1}(\theta(P_C))$ we have $x \lhd i$ if and only if $i \leq x < j$.

Let $x \lhd i \in \psi^{-1}(\theta(P_C))$ as above. Since the unique rise starting at i ends at j, there is no rise from i to the vertices in $[i+1, j-1]$. Hence in P_C we have no relation $i \lhd s$ for $i < s \leq x$. In other words we have $x \lhd i \in C^0$. This proves that $\psi^{-1}(\theta(P_C)) \subseteq C^0$. Conversely, let $x \lhd i \in C^0$. Let $f = \mathrm{i} \longrightarrow \mathrm{j}$ be the rise starting at i in $\theta(P_C)$. By hypothesis, $i \lhd s \notin C$ for each $i < s \leq x$. Hence, $j > x$ and we have $x \lhd i$ in $\psi^{-1}(\theta(P_C))$.

□

For the rest of this section I denotes an *exceptional* interval-poset of size n. We simply denote by C its set of increasing (resp. decreasing) relations. We denote by J the set of decreasing relations of I that are in the Hasse diagram of I. We identify the cover relations and maximal elements of I with the edges of $\theta(I)$. Our objective is to understand the interval-posets with relations $(C + \overline{J})^0$ and $(C + \overline{J \setminus R})^0$ for $R \subset J$. By Lemma 7.1, their corresponding non-crossing tree is the given by $\theta(P_{C+\overline{J}})$ and $\theta\big(P_{C+\overline{J\setminus R}}\big)$. Our next objective is to explain how these noncrossing trees can be constructed from the noncrossing tree $\theta(I)$ by a simple modification of the edges.

Lemma 7.2. *Let v be a vertex of $\theta(I)$ such that v is not the source of a rise of $\theta(I)$ and is not the target of a descent of $J \setminus R$.*

1. *There is an edge in $\theta(I) \setminus R$ which separates the side $(v, v+1)$ from the base. Let $f = i \text{———} j$ be the first edge (from top to bottom) with this property.*
2. *Then the rise $v \longrightarrow j$ is one of the edges of $\theta\big(P_{C+\overline{J\setminus R}}\big)$.*

Proof. There is a path from v to $n+1$ and by hypothesis it has to start with a descent. Hence, in this path, there is at least a rise. This rise gives an edge which separates the side $(v, v+1)$ from the base.

The edge f induces the relation $i \lhd j$ in the poset $P_{C+\overline{J\setminus R}}$. Hence by the interval-poset condition, we have $v \lhd j$. To prove the lemma, we have to see that it is a cover relation in this poset. If not, there is a vertex x such that $v \lhd x \lhd j$. We assume that $v \lhd x$ is a cover relation. Since the poset only contains increasing relations, we have $v < x \leq j$. The hypothesis on the vertex v implies that the relation $v \lhd x$ is obtained by applying the interval-poset condition to $k \longleftarrow x$ a descent of $\theta(I) \setminus R$ with $k \leq v < x$. Such an edge separates the side $(v, v+1)$ from the base, hence it is equal to f. So we have $x = j$ and we have the cover relation $v \lhd j$ in $\theta\big(P_{C+\overline{J\setminus R}}\big)$. □

Note that this lemma can be in particular applied to the vertex i_f determined by an edge f of $\theta(I)$.

Lemma 7.3. *Let $R \subseteq J$ and $f = i \text{———} j$ an edge of $\theta(I) \setminus R$. There is a path made of descents $i \longleftarrow i_1 \longleftarrow \cdots \longleftarrow i_n \longleftarrow i_f$ from i_f to i (see Figure 4).*

1. *If none of these descents are in R, then we have a rise $i_f \longrightarrow j$ in the noncrossing tree $\theta\big(P_{C+\overline{J\setminus R}}\big)$ and we set $s_f = i_f$ and $t_f = j$.*
2. *Otherwise, let $s \in \{1, \cdots, n\}$ be minimal such that $g = i_s \longleftarrow i_{s+1} \in R$. Let $s_f \in \{i_s, \cdots, i_g\}$ be maximal such that there is a path made of descents of $J \setminus R$ from s_f to i_s. Then, we have a rise $s_f \longrightarrow j$ in the noncrossing tree $\theta\big(P_{C+\overline{J\setminus R}}\big)$. We set $t_f = j$.*

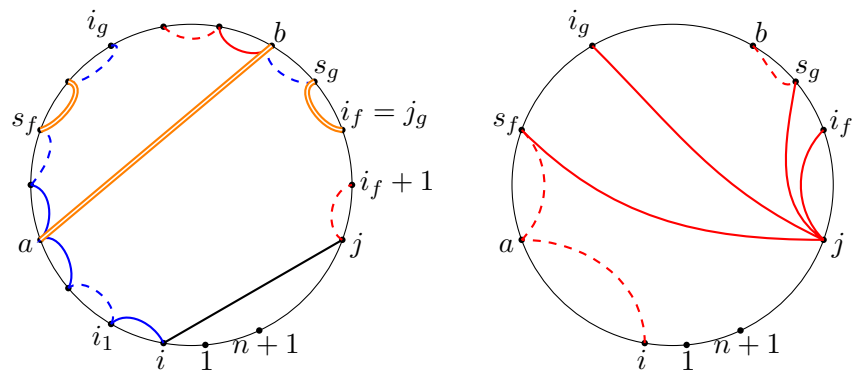


Figure 4: Illustration of Lemma 7.3. On the left, the noncrossing tree $\theta(I)$. The descents of R are the orange double edges. The edge f is between i and j and the edge g is between a and b. On the right the corresponding edges in $\theta\big(P_{C+\overline{J\setminus R}}\big)$.

3. *For $g = a \longleftarrow b \in R$, we let $f = i \text{——} j \in \theta(I) \setminus R$ be the first edge separating g from the base. We set $t_g = j$. Let $s_g \in [b, i_f]$ be maximal such that there is a path made of descents in $J \setminus R$ from s_g to b. Then, there is a rise $s_g \longrightarrow t_g$ in the noncrossing tree $\theta\big(P_{C+\overline{J\setminus R}}\big)$.*

4. *The map sending the edge $f = i \text{——} j$ of $\theta(I)$ to the rise $s_f \longrightarrow t_f$ of $\theta\big(P_{C+\overline{J\setminus R}}\big)$ is a bijection.*

Proof. The existence of the edge f in the third point comes from the fact that there must be a rise in the path from a to $n+1$. For the first point, we apply Lemma 7.2 to i_f and for the second and third points we apply it to s_f and s_g (we refer to Figure 4 left for an illustration). Since the noncrossing trees have the same number of edges, it is enough to see that the map is injective. Let $f = i \text{——} j$ and $g = k \text{——} l$ be two different edges of $\theta(I)$ having the same image. We do a case by case analysis.

1. If f and g are in $J \setminus R$, they have the same image if $j = l$ and $s_g = s_f$, and we may assume that $i < k$. Then there is a path in $\theta(I)$ from s_f to i which goes through k. Hence we have a loop made of edges involving k, i and j.

2. If one of the two edges is in R and the other one is not in R, say $f \in R$ and $g \notin R$, then by the first we may assume that f is the first edge separating g from the base. But the descent g is an obstruction for the edge f to go up. Hence we have $s_f \leq i_g < l \leq s_g$.

3. Le $f = a \longleftarrow b \in R$ and $g = a_1 \longleftarrow b_1 \in R$. Since two descents cannot have the same starting point (dual of the argument used in the proof of Lemma 7.1), we may assume $b < b_1$. If they have the same image, then $s_f = s_g$ and there is a path from b to b_1 (since $s_g \geq b_1$ and there is path from b_1 to s_g). Hence, g is one of the descents going from j_f to b (otherwise there is a loop made of edges). Since $g \in R$, the edge g is an obstruction for s_f and we have $s_f \leq i_g < b_1$.

□

Lemma 7.4. *There is a relation $y \lhd x \in (C+\overline{J})^0$ if and only if there is an edge $f = \ i \text{ —— } j \ \in \theta(I)$ such that $x = i_f$ and $i_f \leq y < j$.*

Proof. If there is such an edge f in $\theta(I)$, then in $\theta(C+\overline{J})$ it factorizes as $i \lhd i_f \lhd j$. By lemma 7.2, the relation $i_f \lhd j$ is a cover relation, hence we have a rise $\ i_f \longrightarrow j \ $ in the tree $\theta(C+\overline{J})$. The label of this rise is i_f and it separates all the edges with label in $[i_f, j-1]$ from the base. Hence we have $y \lhd i_f$ for $i_f \leq y < j$.

Conversely, if $y \lhd x \in (C+\overline{J})^0$, there is a rise $\ x \longrightarrow j \ $ in $\theta(C+\overline{J})$ and $y \in [x, j-1]$. By the bijection of Lemma 7.3, the rise is induced by an edge $f = \ \mathrm{i} \text{ —— } \mathrm{j} \ $ in $\theta(I)$ with $i_f = x$. □

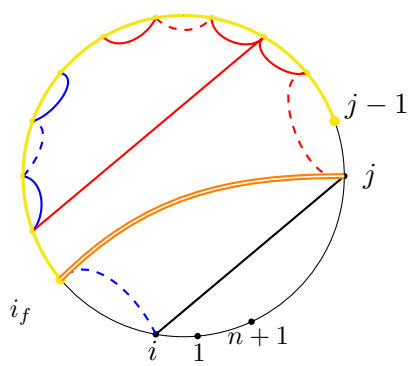


Figure 5: The relations of $(C+\overline{J})^0$ characterized by the edge $f = i - j \in \theta(I)$ are of the form $y \lhd i_f$ for y a vertex in the thick yellow part. The edge $i - j$ 'goes up' and become the orange double edge $i_f - j$ in $\theta\big(P_{(C+\overline{J})^0}\big)$

Lemma 7.5. *Let $R \subset J$. We identify the relations in R with the corresponding edges in $\theta(I)$. Then,*

$$(C+\overline{J-R})^0 = (C+\overline{J})^0 + \{j_f \lhd i_f \ ; \ f \in R\}.$$

Proof. We let $T = \theta\big(P_{C+\overline{J-R}}\big)$.

First we prove that the right side is a subset of the left side. If $g = \ \mathrm{a} \longleftarrow \mathrm{b} \ \in R$, let $f = \ \mathrm{i} \text{ —— } \mathrm{j} \ $ be the first edge in $\theta(I) \setminus R$ separating g from the base (see Figure 4 left for an illustration). By Lemma 7.2, we have a rise $r_1 = \ i_g \longrightarrow j \ $ and a rise $r_2 = \ i_f \longrightarrow j \ $ in T. The rise r_1 separates r_2 from the base so we have $j_g = i_f \lhd i_g \in (C+\overline{J-R})^0$. Going back to the definition of $(C+\overline{J})^0$, it is clear that $(C+\overline{J})^0 \subseteq (C+\overline{J-R})^0$ and since the left side is an interval poset, we obtain that the right side is a subset of the left side.

Conversely, let $y \lhd x \in (C+\overline{J-R})^0$. Since, there is no decreasing relation in $(C+\overline{J-R})$, in the tree T we have a rise $\ \mathrm{x} \longrightarrow \mathrm{j} \ $ and $y \in [x, j-1]$. By the bijection of Lemma 7.3, the rise $\ \mathrm{x} \longrightarrow \mathrm{j} \ $ comes from an edge $f = \ \mathrm{i} \text{ —— } \mathrm{j} \ $ in $\theta(I) \setminus R$ where $x = s_f$ or from a descent $g = \ \mathrm{a} \longleftarrow \mathrm{b} \ \in R$ with $s_g = x$ and $t_g = j$.

Let us start with the first case, which is more complicated. By Lemma 7.4, if the relation $y \lhd x \notin (C+\overline{J})^0$, we have $x < i_f$. Hence, we are in the second point of Lemma 7.3. So, in the tree $\theta(I)$, the path of descents from i_f to i contains at least one descent $g =$ a $\longleftarrow$ b which is in R and we choose a minimal for this property. The tree $\theta(I)$ has the shape of the left tree of Figure 4. By lemma 7.2 we have in T the rises $i_f \longrightarrow j$, $i_g \longrightarrow j$. The first rise is labelled by i_f and the second is labelled by i_g and the tree T has the shape of the right tree of Figure 4. Note that $i_f = j_g$, so the relation $j_g \lhd i_g$ is in $(C+\overline{J})^0 + \{j_f \lhd i_f \ ; \ f \in R\}$.

1. If $y \in [i_f, j-1]$, then we have $y \lhd i_f = j_g \in (C+\overline{J})^0$ and the relation $y \lhd x$ factorizes as $y \lhd j_g \lhd i_g \lhd x$. The relations $y \lhd j_g$ and $j_g \lhd i_g$ are in $(C+\overline{J})^0 + \{j_f \lhd i_f \ ; \ f \in R\}$.

2. If $y \in [i_g, j_g - 1]$, then we have the relation $y \lhd i_g \in (C+\overline{J})^0 + \{j_f \lhd i_f \ ; \ f \in R\}$ by the interval-poset condition and $y \lhd x$ factorizes as $y \lhd i_g \lhd x$.

3. If $y \in [x, i_g - 1]$, then the relation $y \lhd x$ will be obtained by the interval-poset condition applied to $i_g \lhd x$.

Hence, it remains to prove that $i_g \lhd x \in (C+\overline{J})^0 + \{j_f \lhd i_f \ ; \ f \in R\}$. If $i_g \neq x = s_f$, then by Lemma 7.3 the vertex x is the target of a descent $g_1 \in R$ with $x = i_{g_1}$. But, then $i_g = j_{g_1}$ and the relation $i_g \lhd x = j_{g_1} \lhd i_{g_1} \in (C+\overline{J})^0 + \{j_f \lhd i_f \ ; \ f \in R\}$.

If the rise x $\longrightarrow$ j comes from a descent $g =$ a $\longleftarrow$ b with $s_g = x$, we let $f =$ i —— j be the first edge of $\theta(I) \setminus R$ separating g from the base. This situation is also illustrated in Figure 4. If $s_g \neq i_f$, then s_g is the target of a descent $g_1 \in R$ with $s_g = i_{g_1}$ and $j_{g_1} = i_f$. So the relation $i_f \lhd s_g \in \{j_f \lhd i_f \ ; \ f \in R\}$. If $y \in [i_f, j-1]$, the relation $y \lhd i_f \in (C+\overline{J})^0$. If $y \in [s_g, i_f - 1]$, the relation $y \lhd s_g$ is obtained by the interval-poset condition applied to $i_f \lhd s_g$. Hence, if $y \in [s_g, j-1]$, the relation $x \lhd s_g \in (C+\overline{J})^0 + \{j_f \lhd i_f \ ; \ f \in R\}$. □

Let $\Delta = (C+\overline{J})^0$. Let Γ be the smallest interval-poset containing $\{i_f \lhd j_f \ ; \ f \in J\}$. We want to construct an interval-poset having for increasing relations the set Γ and for decreasing relations the set Δ. This is possible if the relations of Δ are compatible with the relations of Γ in the sense that if $y \lhd x \in \Delta$ and $x \lhd y \in \Gamma$, then $x = y$.

Lemma 7.6. *The relations of Δ are compatible with the relations of Γ.*

Proof. Let $x \lhd y \in \Gamma$ such that $y \lhd x \in \Delta$. If $x \lhd y = x \lhd z \lhd y \in \Gamma$, then by the interval-poset condition, we have $z \lhd x \in \Delta$. Hence we can assume that $x \lhd y$ is a cover relation in Γ. Hence, there is $f =$ a $\longleftarrow$ b $\in J$ such that $i_f \leq x < j_f = y$ and we refer to the leftmost case of Figure 3 for an illustration and we keep the same notation. By Lemma 7.4, there is a relation $j_f \lhd x \in \Delta$ if and only if there is an edge $h =$ i —— j $\in \theta(I)$ such that $i_h = x$ and $i_f \leq x \leq j_f < j$. To prove the lemma, we have to see that x is not the open side determined by such an edge. If $x \in [i_f, b-1]$, then the edge h cannot cross $f =$ $a \longleftarrow b$, hence $j \leq b \leq j_f$. If $x \in [b, j_f - 1]$, the side $(x, x+1)$ is separated from the base by one of the descents going from j_f to b. The edge h cannot cross this descent, so we have $j \leq j_f$.

□

As a consequence, one can consider the interval-poset I_s having Δ as set of decreasing relations and Γ as set of increasing relations. It turns out that $\psi(I_s) = \theta(I)$. So the non-crossing tree $\psi(I_s)$ is the dual of the noncrossing tree $\psi(I)$.

Proposition 7.7. *Let I be an exceptional interval-poset of size n. Then $I_s = \psi^{-1}\big(\theta(I)\big)$.*

Proof. We denote the exceptional interval-poset $\psi^{-1}\big(\theta(I)\big)$ by I'.

By Lemma 6.2, the increasing cover relations of I' are of the form $i_f \lhd j_f$ for $f =$ a $\longleftarrow$ b a descent of $\theta(I)$. Hence we have $i_f \lhd j_f \in I_s$.

The decreasing cover relations of I' are of the form $i_f \lhd j_f$ for $f =$ a $\longrightarrow$ b $\in \theta(I)$ a rise of type 1. Then, looking at the middle case of Figure 3 and using Lemma 7.3, we see that the edge a_1 —— b_1 becomes $j_f \longrightarrow b_1$ in $\theta(P_{C+\overline{J}})$. By Lemma 7.4, we have $i_f \lhd j_f \in (C+\overline{J})^0 \subset I_s$.

Conversely, we already saw that the relations $i_f \lhd j_f$ where $f \in J$ are in I'. By Lemma 7.4, if $y \lhd x \in (C+\overline{J})^0$, there is an edge $f =$ i —— j in $\theta(I)$ such that $x = i_f$ and $y \in [i_f, j-1]$. In $\theta(I)$, this edge is labelled by i_f, and since $y \in [i_f, j-1]$ it labels an edge which is separated from the base by f, hence we have $y \lhd i_f$ in I'.

□

Proposition 7.8. *Let I be an exceptional interval-poset with set of decreasing relations in its Hasse diagram denoted by J. Let $n_I = |J|$.*

Then, in $D^b(\mathrm{Tam}_n)$ we have

$$\mathbb{S}(I) \cong \psi^{-1}\big(\theta(I)\big)[n_I].$$

Proof. By Lemma 6.2 the increasing cover relations of $I_s = \psi^{-1}\big(\theta(I)\big)$ are of the form $i_f \lhd j_f$ for f a descent in $\theta(I)$. In other terms, they are of the form $i_f \lhd j_f$ for $f \in J$.

Using a dual version of Lemma 5.7, we see that the following complex $\mathcal{I}$ is an injective resolution of I_s:

$$I_\Delta \to \bigoplus_{f\in J} I_{\Delta+\{j_f \lhd i_f\}} \to \bigoplus_{\substack{R\subset J\\ |R|=2}} I_{\Delta+\{j_f\lhd i_f\ ;\ f\in R\}} \to \cdots \to \bigoplus_{\substack{R\subset J\\ |R|=|J|-1}} I_{\Delta+\{j_f\lhd i_f\ ;\ f\in R\}} \to I_{\Delta+\{j_f\lhd i_f\ ;\ f\in J\}}.$$

On the other hand, by Lemma 5.7, I is isomorphic to $\mathcal{P}(I)$ in $D^b(\mathrm{Tam}_n)$. By Proposition 5.6, the Nakayama functor sends the projective interval-poset P_C to the injective interval-poset I_{C^0}. So, by Lemma 7.5, we see that $\mathbb{S}(I)$ is isomorphic to $\mathcal{I}[n_I]$. □

8 The bounded derived category of the Tamari lattice is fractionally Calabi-Yau

Combining the results of Sections 7 and 6 we have the following result.

Theorem 8.1. *Let T be a noncrossing tree of size n. Then, there is an integer n_T such that*

$$\mathbb{S}(\psi^{-1}(T)) \cong \psi^{-1}(T^*)[n_T]$$

in $D^b(\mathrm{Tam}_n)$.

Proof. By Proposition 7.8 we have that $\mathbb{S}(\psi^{-1}T) \cong \psi^{-1}(\theta\psi^{-1}T)[n_T]$ for an integer n_T. By Proposition 6.3, we have $\theta \circ \psi^{-1}T = T^*$. □

In other terms, the Serre functor acts - up to a shift - as the planar duality of the noncrossing trees.

It remains to understand the shifts that appear here.

Proposition 8.2. *Let I be an exceptional interval-poset of size n. Then,*

$$\mathbb{S}^{2n+2}(I) \cong I[n(n-1)]$$

in $D^b(\mathrm{Tam}_n)$.

Proof. We denote by T the noncrossing tree $\psi(I)$. By Theorem 8.1, the interval-poset obtained after $2n+2$ applications of the Serre functor corresponds via ψ to the tree obtained by successively taking $2n+2$ times the planar duality of T. By Lemma 6.1, this tree is nothing but T.

We denote by $m_T = n_T + n_{T^*}$. This is the shift obtained after two applications of the Serre functor on I. The shift obtained after $2n+2$ applications of the Serre functor is $\sum_{i=0}^{n} m_{\vartheta^i T}$.

By Proposition 7.8, the number n_T is the number of decreasing relations of I which is also the number of descents of $\theta I = T^*$. By Lemma 6.2 the number n_{T^*} is the number of rises f type 1 in T^*. Then, the number m_T is the number of edges of T^* minus the number of rise of type 2. If we denote the number of rise of type 2 by F_{T^*}, we have

$$\sum_{i=0}^{n} m_{\vartheta^i T} = n(n+1) - \sum_{i=0}^{n} F_{\vartheta^i T^*}.$$

Let f be an edge in T. Then it becomes a rise of type 2 exactly twice during all the rotations of T. This is when i_f or j_f is rotated to $n+1$.

So, we have

$$\sum_{i=0}^{n} m_{\vartheta^i T} = n(n+1) - 2n = n(n-1).$$

□

Theorem 8.3. *Let $n \in \mathbb{N}$. The bounded derived category $D^b(\mathrm{Tam}_n)$ is $\frac{n(n-1)}{2n+2}$ fractionally Calabi-Yau.*

Proof. The result follows from Proposition 8.2 and Theorem 3.1 and the fact that the projective indecomposable modules are in the family of exceptional interval-posets. □

Remark 8.4. Since the square of the Serre functor acts as the rotation of noncrossing trees, we see that $\frac{n(n-1)}{2n+2}$ is the 'effective' dimension of the derived category in the sense that it cannot be simplified when $n \geqslant 3$. The fraction $\frac{n(n-1)}{2n+2}$ is larger than 1 when $n \geqslant 4$, this shows that the incidence algebra of the Tamari lattices are not piecewise hereditary. Similarly to the examples of Section 7 of [KLM13] the Tamari lattices are wild when their Calabi-Yau dimension is larger than 1.

For the Coxeter matrix of the Tamari lattices, we recover Chapton's Theorem ([Cha07b]) and we deduce some new properties.

Proposition 8.5. *Let $n \in \mathbb{N}$. Let $C = -I \cdot (I^{-1})^t$ be the Coxeter matrix of* Tam_n.

1. $C^{2n+2} = \mathrm{Id}$.
2. *The elements of the matrix C are $0, 1$ and -1.*
3. *The nonzero elements of a column have the same sign.*

Proof. Since C is up to sign the matrix in the bases of the simple induced by the Serre functor on the Grothendieck group of $D^b(\mathrm{Tam}_n)$ the first point is clear. The image of a simple module by the Serre functor is up to a shift an interval. This implies the two last statements. □

The Tamari lattice Tam_n is isomorphic to the poset of tilting modules by Theorem C of [BK04] (resp. Cluster tilting modules) over an equioriented quiver of type A_n (resp. A_{n-1}). Using results of Ladkani, our result can be generalized to these two families of posets.

Corollary 8.6. *The Cambrian lattices of type A and the lattices of tilting modules over a quiver of type A have fractionally Calabi-Yau derived categories.*

Proof. It has been proved by Ladkani in [Lad07a, Lad07b] that these posets are derived equivalent to the Tamari lattices. □

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