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Firm non-expansive mappings in weak metric spaces

Armando W. Gutiérrez and Cormac Walsh

Abstract. We introduce the notion of firm non-expansive mapping in weak metric spaces, extending previous work for Banach spaces and certain geodesic spaces. We prove that, for firm non-expansive mappings, the minimal displacement, the linear rate of escape, and the asymptotic step size are all equal. This generalises a theorem by Reich and Shafrir.

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1. Introduction

A fundamental question in the theory of metric spaces is the long term behaviour of iterates of non-expansive mappings. Recall that a mapping T of a metric space (X, δ) into itself is said to be *non-expansive* if, for every $x, y \in X$,

$$\delta(Tx, Ty) \leq \delta(x, y).$$

Particularly interesting is the case where the mapping has no fixed point, because here the iterates have the possibility of escaping to infinity.

In the setting of Banach spaces, Bruck [3] introduced a special class of non-expansive mappings, which he called *firmly* non-expansive. These mappings were further studied in [7, p.41, p.129] and [26]. In particular, it was shown in [26] that for firmly non-expansive mappings the minimal displacement, the linear rate of escape, and the asymptotic step size are all equal. The definition of firmly non-expansive and the equality of these three quantities were extended by Ariza-Ruiz et al. [1, Theorem 5.1] to a broader class of spaces, namely, the W-hyperbolic spaces. These are geodesic metric spaces with a certain "negative curvature"-type condition.

In this paper, we broaden the idea further. We introduce the notion of *firm non-expansive* mapping in arbitrary (weak) metric spaces. Our definition does not assume the existence of geodesics. This is significant since in modern

optimisation applications one often deals with discrete spaces or has access to a collection of points of a space whose geometric structure is unknown. We show, in Section 2, that our class of firm non-expansive mappings contains all the firmly non-expansive ones in the setting of Banach spaces or W-hyperbolic spaces. In Section 3, we provide non-trivial examples by characterising the firm non-expansive mappings of 1-dimensional asymmetric normed spaces. We then prove in Section 4 our main result, which is as follows.

Theorem 1.1. *Let T be a firm non-expansive mapping of a weak metric space (X, δ) into itself. Then, for every $x \in X$ and every integer $k \geq 1$,*

$$\inf_{w \in X} \delta(w, Tw) = \lim_{n \to \infty} \frac{\delta(x, T^n x)}{n} = \lim_{n \to \infty} \delta(T^n x, T^{n+1} x) = \frac{1}{k} \lim_{n \to \infty} \delta(T^n x, T^{n+k} x).$$

The first three quantities in this equation are, respectively, the minimal displacement, the linear rate of escape, and the asymptotic step size, mentioned earlier. Our theorem thus generalises the results by Reich and Shafrir and by Ariza-Ruiz et al., referred to above.

We conclude this note by proving in Section 5 a couple of corollaries of our main result, concerning the behaviour of the iterates of firm non-expansive mappings in terms of *metric functionals*.

The first enhances [5, Theorem 16] for firm non-expansive mappings in arbitrary metric spaces when the linear rate of escape, denoted by $\rho(T)$, equals zero (compare also with [12, Proposition 3]).

Corollary 1.2. *Let T be a firm non-expansive mapping of a metric space (X, δ) into itself such that $\rho(T) = 0$. Then, there exists a metric functional h on X such that, for every $x \in X$,*

$$h(Tx) \leq h(x).$$

We point out that the metric functional h in Corollary 1.2 is defined in terms of the iterates $T^n x_0$ for some $x_0 \in X$, whereas the construction of the metric functional appearing in [5, Theorem 16] relies strongly on the non-positive curvature condition assumed there.

A theorem by Lins [22, Theorem 2.1] states that, for every fixed-point-free non-expansive mapping in a finite-dimensional normed space, there is a metric functional along which the orbits are seen to escape to infinity. It is not known in general whether this behaviour of escaping to infinity is monotone. We use Corollary 1.2 to show that, in the case of firm non-expansive mappings, it is.

Corollary 1.3. *Let T be a firm non-expansive mapping of a finite-dimensional normed space $(V, \|\cdot\|)$ into itself. Suppose that $\rho(T) = 0$. Then, either T has bounded orbits, or there is a metric functional h on V such that, for every $x \in V$, the sequence $(h(T^n x))_n$ converges monotonically to $-\infty$. Moreover, the metric functional h is a limit point of the orbit $(T^n 0)_{n \geq 0}$.*

2. Firm non-expansive mappings

A *weak metric* on a set X is a mapping $\delta\colon X \times X \to [0, +\infty[$ satisfying $\delta(x, x) = 0$, for every $x \in X$, and the triangle inequality

$$\delta(x, z) \leq \delta(x, y) + \delta(y, z),$$

for every $x, y, z \in X$. Examples of weak metrics include the *Funk weak metric*, the *Apollonian weak metric*, and *Thurston's metric*; see the papers [23, 24] by Papadopoulos and Troyanov for more details. Weak metrics are sometimes called *quasi-pseudometrics*.

Motivated by Ćirić's work [4] on a generalisation of Banach's contraction principle, we introduce the notion of firm non-expansive mapping in an arbitrary weak metric space.

Definition 2.1. A mapping T of a weak metric space (X, δ) into itself is *firm* if there exist mappings q, r, s, and t of $X \times X$ into the interval $[0, +\infty[$ satisfying the following three conditions:

$$\inf_{(x,y)\in X\times X} t(x, y) \;>\; 0; \tag{2.1}$$

$$\sup_{(x,y)\in X\times X} q(x, y) + r(x, y) + s(x, y) + 2t(x, y) \;\leq\; 1; \tag{2.2}$$

and, for every $x, y \in X$,

$$\begin{aligned}\delta(Tx, Ty) \leq\; & q(x, y)\delta(x, y) + r(x, y)\delta(x, Tx) + s(x, y)\delta(y, Ty) \\ & +\; t(x, y)\Big[\delta(x, Ty) + \delta(Tx, y)\Big].\end{aligned} \tag{2.3}$$

We say that T is *firm non-expansive* if it is both firm and non-expansive.

For purposes of comparison, we recall Bruck's notion of firmly non-expansive mapping. A mapping T of a Banach space $(V, \|\cdot\|)$ into itself is said to be firmly non-expansive if, for every $x, y \in V$ and every $0 < \lambda \leq 1$,

$$\|Tx - Ty\| \leq \|(1 - \lambda)(Tx - Ty) + \lambda(x - y)\|. \tag{2.4}$$

A simple example of a firm non-expansive mapping that is not firmly non-expansive is the mapping $x \mapsto Tx = |x| + 1$ on the real line endowed with its usual metric. Indeed, for $x < 0$ and $y = 0$, there is no $\lambda \in]0, 1[$ such that (2.4) holds. One can readily verify that this mapping is firm non-expansive. Another example of a firm non-expansive mapping on the real line is

$$Tx = \begin{cases} -x + 1 & \text{if } x < 0, \\ x + e^{-x} & \text{if } x \geq 0. \end{cases}$$

By setting $q(x, y) = r(x, y) = s(x, y) = 0$ and $t(x, y) = 1/2$ for every x, y, we see that T is firm non-expansive.

Every firmly non-expansive mapping in a Banach space is firm non-expansive. Indeed, if T satisfies inequality (2.4) with $\lambda \in]0, 1[$, then, for every $x, y \in V$,

$$\|Tx - Ty\| \leq \frac{\lambda}{2 - \lambda}\|x - y\| + \frac{1 - \lambda}{2 - \lambda}\left(\|x - Ty\| + \|Tx - y\|\right).$$

Our claim follows from this by fixing $\lambda \in]0,1[$ and setting

$$q(x,y) = \frac{\lambda}{2-\lambda}, \qquad r(x,y) = 0, \qquad s(x,y) = 0, \qquad \text{and} \quad t(x,y) = \frac{1-\lambda}{2-\lambda},$$

for every $x, y \in V$. More generally, every firmly non-expansive mapping in a W-hyperbolic space is firm non-expansive. This follows immediately from [1, Lemma 5.6].

Our definition of firm non-expansive is inspired by Ćirić's definition [4] of a *generalised contraction*. This is a mapping T from a metric space (X, δ) to itself such that there exist mappings q, r, s, and t from $X \times X$ to $[0, +\infty[$ such that (2.3) above holds, and the supremum in (2.2) is strictly less than 1. Note that when this is the case, it is possible to slightly increase t everywhere so that (2.1) also holds. The relation between generalised contractions and our firm non-expansive mappings may be considered similar to the relation between strict contractions and non-expansive mappings. Ćirić showed that generalised contractions satisfy the conclusion of Banach's Contraction Theorem.

Other authors have focused on the existence of a unique fixed point of mappings that satisfy conditions similar to those considered in our definition of firm mapping; see [6, 13, 25, 27, 32]. In contrast, we are interested in the asymptotic behaviour of fixed-point-free non-expansive mappings.

The definition of firmly non-expansive mappings of Ariza-Ruiz et al. was studied in the context of CAT(0) spaces in [21]. For a different approach to generalising firmly non-expansive mappings, see [2].

3. An example: asymmetric norms on $\mathbb{R}$.

The following characterisation of firmness for non-expansive mappings will prove useful in studying our example. Let T be a non-expansive self-mapping of a weak metric space (X, δ). For each $x, y \in X$, define

$$M(x,y) := \max\Big[\delta(x,y), \delta(x,Tx), \delta(y,Ty)\Big]$$

$$A(x,y) := \frac{1}{2}\Big(\delta(x,Ty) + \delta(Tx,y)\Big)$$

$$\text{and} \qquad \tau(x,y) := \frac{M(x,y) - \delta(Tx,Ty)}{2\big(M(x,y) - A(x,y)\big)}.$$

Then, T is firm if and only if

$$\inf\Big\{\tau(x,y) \mid x,y \in X,\ A(x,y) < \delta(Tx,Ty)\Big\} > 0.$$

3.1. Asymmetric norms on $\mathbb{R}$.

We consider $\mathbb{R}$ endowed with an asymmetric norm

$$||x|| := \max\big(-\alpha x, \beta x\big),$$

where α and β are positive real numbers.

For each distinct x and y in $\mathbb{R}$, define z_{xy} to be the unique element of $\mathbb{R}$ such that $||z_{xy} - y|| = ||y - x||$ and $(y - x)(z_{xy} - y) < 0$. If x and y are points of $\mathbb{R}$, and T is a mapping such that $Tx = y$ and $Ty = z_{xy}$, then one can calculate that $\tau(x, y) = 0$ in the criterion above with $\delta(x, y) := ||y - x||$, and hence the mapping T is not firm.

The following proposition shows that a non-expansive self-mapping of $(\mathbb{R}, ||\cdot||)$ fails to be firm precisely when one can find points arbitrarily close to this configuration.

Proposition 3.1. *Let $(\mathbb{R}, ||\cdot||)$ be the asymmetric normed space above. Then, a non-expansive mapping $T\colon \mathbb{R} \to \mathbb{R}$ is not firm if and only if, for every $\epsilon > 0$ there exist distinct points x and y in $\mathbb{R}$ such that $||Tx - y||$ and $||Ty - z_{xy}||$ are both less than $||y - x||\epsilon$.*

In particular, we see that every fixed-point-free non-expansive mapping on $(\mathbb{R}, ||\cdot||)$ is firm. This shows that firm non-expansive mappings are considerably more general than firmly non-expansive mappings, even in dimension one. Recall that, on $\mathbb{R}$, the latter mappings are precisely the 1-Lipschitz mappings that are non-decreasing.

Proof. Assume first that T satisfies the condition. So, we can find sequences of points x_n and y_n in $\mathbb{R}$ such that $||Tx_n - y_n||/d_n$ and $||Ty_n - z_{x_n,y_n}||/d_n$ converge to zero, where $d_n := ||y_n - x_n||$, for all n. It follows that M_n/d_n and δ_n/d_n both converge to 1, where $M_n := M(x_n, y_n)$ and $\delta_n := ||Ty_n - Tx_n||$, for all n. Define $A_n := A(x_n, y_n)$, for all n. Since $||z_{x_n,y_n} - x_n|| \leq ||y_n - x_n||$, for all n, we have

$$\limsup_n \frac{A_n}{d_n} \leq \frac{1}{2}.$$

This gives $A_n < \delta_n$ for n large enough, and that $\tau(x_n, y_n)$ converges to zero. By the criterion at the start of this section, T is not firm.

Now assume that T is not firm. So, there exists sequences of points x_n and y_n in $\mathbb{R}$ such that $A_n < \delta_n$ for all n, and τ_n converges to zero, where we have written $A_n := A(x_n, y_n)$ and $\delta_n := ||Ty_n - Tx_n||$, and $\tau_n := \tau(x_n, y_n)$. We assume without loss of generality that $x_n < y_n$, for all n. If there is a subsequence where the opposite inequality is true, it can be handled in a similar manner. Note that, for any n,

$$A_n \geq \frac{1}{2}\beta\big(Ty_n - x_n + y_n - Tx_n\big).$$

If $Ty_n \geq Tx_n$, then the non-expansiveness of T would imply that the quantity on the right-hand-side is greater than or equal to δ_n, which is not the case. We conclude that $Ty_n < Tx_n$, for all n.

For each n,

$$\tau_n \geq \frac{M_n - \delta_n}{2M_n} = \frac{1}{2}\Big(1 - \frac{\delta_n}{M_n}\Big) \qquad \text{and} \quad \delta_n \leq ||y_n - x_n|| \leq M_n,$$

where we have defined $M_n := M(x_n, y_n)$. We deduce that the ratios

$$\delta_n/||y_n - x_n|| \qquad \text{and} \quad M_n/||y_n - x_n||$$

both converge to 1, as n tends to infinity.

It follows from the convergence of the latter ratio that (u_n, v_n) stays within a bounded region of the plane, where

$$u_n := \frac{Tx_n - x_n}{||y_n - x_n||} \qquad \text{and} \quad v_n := \frac{Ty_n - y_n}{||y_n - x_n||}.$$

Let (u, v) be a limit point of this sequence. Again from the convergence of the ratio, both u and v lie in the interval $[-1/\alpha, 1/\beta]$. Since, for each n,

$$\begin{aligned}\frac{\delta_n}{||y_n - x_n||} = \frac{||Ty_n - Tx_n||}{||y_n - x_n||} &= -\alpha\Big(v_n - u_n + \frac{y_n - x_n}{||y_n - x_n||}\Big) \\ &= -\alpha\Big(v_n - u_n + \frac{1}{\beta}\Big),\end{aligned}$$

we get that $v - u = -1/\alpha - 1/\beta$. We deduce that $v = -1/\alpha$ and $u = 1/\beta$. Hence $||u_n||$ and $||v_n||$ both converge to 1. It follows that

$$||Tx_n - y_n||/||y_n - x_n|| \qquad \text{and} \quad ||Ty_n - z_{x_n,y_n}||/||y_n - x_n||$$

both converge to zero, as n tends to infinity. □

4. Minimal displacement, linear rate of escape, and asymptotic step size

Let T be a non-expansive mapping of a weak metric space (X, δ) into itself. We first recall the relations among the minimal displacement,

$$\overline{\rho}(T) := \inf_{w \in X} \delta(w, Tw),$$

the (linear) escape rate of the orbits of T,

$$\rho(T) := \lim_{n\to\infty} \frac{\delta(x, T^n x)}{n},$$

and the asymptotic step size of the orbit $(T^n x)_{n\geq 0}$,

$$\sigma_1(x, T) := \lim_{n\to\infty} \delta(T^n x, T^{n+1} x).$$

Non-expansiveness and the triangle inequality imply that the linear rate of escape $\rho(T)$ is well defined and does not depend on x; see the comment after [5, Definition 11]. Moreover, the following inequality holds (see [5, Lemma 12]):

$$\overline{\rho}(T) \geq \rho(T).$$

In general, this inequality may be strict, as shown for example in [5, Example 26]. Kohlberg and Neyman [19, Theorem 1.1] proved that $\rho(T) = \overline{\rho}(T)$ when (X, δ) is a Banach space. Gaubert and Vigeral [5, Theorem 1] showed that this equality also holds in a larger class of geodesic metric spaces, namely when (X, δ) is a complete *metrically star-shaped* space [5, Definition 5].

Another consequence of non-expansiveness is that, for every $x \in X$ and every integer $k \geq 1$, the following limit exists:

$$\sigma_k(x,T) := \lim_{n\to\infty} \delta(T^n x, T^{n+k} x). \tag{4.1}$$

Unlike the escape rate, this quantity may depend on x.

By using again the triangle inequality and non-expansiveness, one can verify that

$$\frac{\sigma_k(x,T)}{k} \leq \sigma_1(x,T), \qquad \text{for } k \geq 1, \tag{4.2}$$

and

$$\sigma_1(x,T) \geq \overline{\rho}(T). \tag{4.3}$$

Our main result is that for firm non-expansive mappings the minimal displacement, the escape rate, and the asymptotic step size are equal.

Theorem 4.1. *Let T be a firm non-expansive mapping of a weak metric space (X,δ) into itself. Then, for every $x \in X$ and every integer $k \geq 1$,*

$$\overline{\rho}(T) = \rho(T) = \sigma_1(x,T) = \frac{\sigma_k(x,T)}{k}.$$

Proof. Let x be a point in X. First, we show by induction that for every integer $k \geq 1$,

$$\frac{\sigma_k(x,T)}{k} = \sigma_1(x,T).$$

Let k be a positive integer. Assume that the inductive hypothesis is true, that is,

$$\sigma_j(x,T) = j\sigma_1(x,T),$$

for every positive integer $j \leq k$. Let ϵ be a positive real number. It follows from (4.1) and the inductive hypothesis that there exists a positive integer N so that for every $n \geq N$ and every $j \leq k$,

$$j(\sigma_1(x,T) - \epsilon) \leq \delta(T^n x, T^{n+j} x) \leq j(\sigma_1(x,T) + \epsilon). \tag{4.4}$$

Since T is firm, there exist non-negative functions q, r, s, t on $X \times X$ satisfying the properties (2.1), (2.2) and (2.3). The following notation will be helpful:

$$q_a^b := q(T^a x, T^b x), \qquad r_a^b := r(T^a x, T^b x), \qquad s_a^b := s(T^a x, T^b x),$$
$$t_a^b := t(T^a x, T^b x), \qquad \delta_a^b := \delta(T^a x, T^b x),$$

for all positive integers a, b. Now, the property (2.3) implies that

$$\delta_n^{n+k+1} \geq \frac{1}{t_n^{n+k}}\delta_{n+1}^{n+k+1} - \frac{q_n^{n+k}}{t_n^{n+k}}\delta_n^{n+k} - \frac{r_n^{n+k}}{t_n^{n+k}}\delta_n^{n+1} - \frac{s_n^{n+k}}{t_n^{n+k}}\delta_{n+k}^{n+k+1} - \delta_{n+1}^{n+k}. \tag{4.5}$$

By applying (4.4) in (4.5), it follows that for every $n \geq N$,

$$\begin{aligned}\delta_n^{n+k+1} \geq{}& \frac{k}{t_n^{n+k}}(\sigma_1(x,T) - \epsilon) - \frac{q_n^{n+k}}{t_n^{n+k}}k(\sigma_1(x,T) + \epsilon) - \frac{r_n^{n+k}}{t_n^{n+k}}(\sigma_1(x,T) + \epsilon) \\ &- \frac{s_n^{n+k}}{t_n^{n+k}}(\sigma_1(x,T) + \epsilon) - (k-1)(\sigma_1(x,T) + \epsilon),\end{aligned}$$

or equivalently,

$$\delta_n^{n+k+1} \geq (A_{n,k} - k + 1)\, \sigma_1(x,T) - (B_{n,k} + k - 1)\, \epsilon, \tag{4.6}$$

where

$$A_{n,k} := \frac{k - kq_n^{n+k} - r_n^{n+k} - s_n^{n+k}}{t_n^{n+k}}$$

and

$$B_{n,k} := \frac{k + kq_n^{n+k} + r_n^{n+k} + s_n^{n+k}}{t_n^{n+k}}.$$

On the other hand, if we let α denote the positive real number given in (2.1), then the property (2.2) implies that

$$B_{n,k} \leq \frac{1+2k}{\alpha} \tag{4.7}$$

and also

$$A_{n,k} \geq 2k. \tag{4.8}$$

By applying the inequalities (4.7) and (4.8) in (4.6), we obtain for every $n \geq N$,

$$\delta_n^{n+k+1} \geq (k+1)\sigma_1(x,T) - \left[\frac{1+2k}{\alpha} + k - 1\right] \epsilon.$$

Taking the limit first as n tends to infinity, and then as ϵ tends to zero, we get

$$\sigma_{k+1}(x,T) \geq (k+1)\sigma_1(x,T).$$

This inequality together with (4.2) show our first claim.

To complete the proof of our theorem, we need only show that $\sigma_1(x,T)$ and $\rho(T)$ are equal. Indeed, since T is non-expansive, we have that, for every positive integer m,

$$\sigma_m(x,T) \leq \delta(x, T^m x).$$

So, by our first claim, $\sigma_1(x,T) \leq \delta(x,T^m x)/m$. By letting m tend to infinity, we obtain

$$\sigma_1(x,T) \leq \rho(T).$$

This inequality is finally combined with (4.3) and the inequality $\overline{\rho}(T) \geq \rho(T)$ to complete the proof. □

5. Metric functionals

Let (X,δ) be a metric space and let x_0 be an arbitrary base-point in X. We consider the mapping

$$\begin{aligned} \Phi : X &\to \mathbb{R}^X \\ w &\mapsto h_w(\cdot) := \delta(\cdot, w) - \delta(x_0, w). \end{aligned}$$

If we endow the target space $\mathbb{R}^X$ with the topology of point-wise convergence, then the mapping Φ is a continuous injection. Moreover, the closure $\overline{\Phi(X)}$ is compact and consists of mappings $h : X \to \mathbb{R}$ vanishing at x_0 and satisfying $|h(x) - h(y)| \leq \delta(x,y)$, for every $x, y \in X$. Each element of the compact

space $\overline{\Phi(X)}$ is called a *metric functional*. Metric functionals and the related notion of horofunctions are discussed in [9, 8, 17, 18]. These objects have been studied in spaces such as Hilbert and Thompson geometries [29, 20, 30], Teichmüller geometry [31], and normed spaces [28, 14, 10, 9, 11].

Among other applications, metric functionals are useful for understanding the behaviour of the iterates of non-expansive mappings. Karlsson [15] proved the following result, which he calls the Metric Spectral Principle [16].

Theorem 5.1. *Let T be a non-expansive mapping of a metric space (X, δ) into itself. Then, there exists a metric functional h such that*

$$h(T^n(x_0)) \leq -\rho(T)n$$

for every $n \geq 1$. Moreover, for every $x \in X$,

$$\lim_{n\to\infty} -\frac{1}{n} h(T^n x) = \rho(T).$$

Complementing this, Gaubert and Vigeral [5] proved the following.

Theorem 5.2. *If (X, δ) is a Banach space or, more generally, a complete metrically star-shaped space, then $\rho(T) = \overline{\rho}(T)$ and there exists a metric functional h such that for every $x \in X$,*

$$h(Tx) \leq h(x) - \rho(T).$$

It should be noted that the metric functional appearing in Theorem 5.1 is a limit point of the orbit $(T^n x_0)_{n\geq 0}$ in the compact space $\overline{\Phi(X)}$. On the other hand, in Theorem 5.2, the metric functional is constructed by composing T with a retraction along geodesics to get a strict contraction, having of course a fixed point, and then taking a limit point of the sequence of fixed points as the retraction approaches the identity.

Proof of Corollary 1.2. Recall that $\overline{\Phi(X)}$ is a compact space. This space is sequentially compact whenever X is separable. In this case, there is a subsequence n_i such that $\Phi(T^{n_i} x_0)$ converges to some $h \in \overline{\Phi(X)}$ as $i \to \infty$. Since T is a firm non-expansive mapping and δ is a metric, it follows from Theorem 1.1 that for every $x \in X$,

$$\begin{aligned} h(Tx) &= \lim_{i\to\infty} \left[\delta(Tx, T^{n_i} x_0) - \delta(x_0, T^{n_i} x_0)\right] \\ &\leq \liminf_{i\to\infty} \left[\delta(Tx, T^{n_i+1} x_0) + \delta(T^{n_i+1} x_0, T^{n_i} x_0) - \delta(x_0, T^{n_i} x_0)\right] \\ &\leq \liminf_{i\to\infty} \left[\delta(x, T^{n_i} x_0) - \delta(x_0, T^{n_i} x_0) + \delta(T^{n_i} x_0, T^{n_i+1} x_0)\right] \\ &= h(x) + \rho(T). \end{aligned}$$

Our claim follows immediately from the assumption $\rho(T) = 0$.

In general, $\overline{\Phi(X)}$ is not sequentially compact. However, we can argue as follows. For every $n \geq 1$ define the set

$$W_n = \{h \in \overline{\Phi(X)} : h(Tx) \leq h(x) + \delta(T^n x_0, T^{n+1} x_0), \text{ for all } x \in X\}.$$

Each W_n is non-empty since $\Phi(T^n x_0)$ belongs to W_n. Moreover, $(W_n)_{n\geq 1}$ is a non-increasing sequence of compact subsets of $\overline{\Phi(X)}$. Therefore, there exists $h \in \bigcap_{n\geq 1} W_n$ satisfying our claim. □

Proof of Corollary 1.3. We fix here $x_0 = 0 \in V$. If the orbit $(T^n 0)_{n\geq 0}$ is unbounded, it follows from [22, p. 2390] that there exists a subsequence n_i such that $\Phi(T^{n_i} 0)$ converges to some $h \in \overline{\Phi(V)}$ as $i \to \infty$. Moreover, for every $x \in V$,

$$\lim_{n\to\infty} h(T^n x) = -\infty.$$

Since T is firm non-expansive and $\rho(T) = 0$, it follows from Corollary 1.2 that

$$h(T^{n+1}x) \leq h(T^n x),$$

for every $n \geq 0$. This completes the proof. □

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