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A uniformization theorem for closed convex polyhedra in Euclidean 3-space.

Georg Grütznér

ABSTRACT. We introduce a notion of discrete-conformal equivalence of closed convex polyhedra in Euclidean 3-space. Using this notion, we prove a uniformization theorem for closed convex polyhedra in Euclidean 3-space.

INTRODUCTION

In this paper, we introduce an equivalence relation on the class of closed convex polyhedra in the Euclidean 3-space \mathbb{E}^3 . This equivalence relation has the property that, if P and Q are two convex polyhedra inscribed in the unit sphere, then P is equivalent to Q if and only if there exists a Möbius transformation on the sphere that maps the vertex set of P to the vertex set of Q . This property suggests this equivalence relation as a concept of discrete conformality.

Inspired by Riemann's mapping theorem and the more general uniformization theorem of Poincaré and Koebe, we prove a uniformization theorem for closed convex polyhedra in Euclidean 3-space in the following sense.

Theorem 4 (Uniformization). *Every closed convex polyhedron in \mathbb{E}^3 is discrete-conformally equivalent to a closed convex polyhedron inscribed in the unit sphere. This polyhedron is unique up to Möbius transformations on the sphere.*

In a special case, we further characterize the equivalence relation by simple transformations on the vertices of the polyhedra. More specifically, if two polyhedra P and Q share a common Delaunay triangulation \mathcal{T} (to be defined below), then P and Q are conformally equivalent if and only if there exists a real valued function $u_{\mathcal{T}}$ on the vertices of P such that, for every edge ij in the Delaunay triangulation between vertices i and j , its length in Q is related to its length in P by

$$l_Q(ij) = l_P(ij) e^{\frac{1}{2}(u_{\mathcal{T}}(i) + u_{\mathcal{T}}(j))}.$$

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We conjecture more generally that P and Q are discrete-conformally equivalent if and only if there exists a finite sequence of closed convex polyhedra $P = P_1, P_2, \dots, P_{n-1}, P_n = Q$ such that, for $k = 1, \dots, n - 1$ the polyhedra P_k and P_{k+1} share a common Delaunay triangulation \mathcal{T}_k and there exists a real valued function $u_{\mathcal{T}_k}$ on the vertices of P_k with the following property. For every edge ij in the Delaunay triangulation between vertices i and j , its length in P_{k+1} is related to its length in P_k by

$$l_{P_{k+1}}(ij) = l_{P_k}(ij) e^{\frac{1}{2}(u_{\mathcal{T}_k}(i) + u_{\mathcal{T}_k}(j))}.$$

This work arose out of a general interest in understanding the relationship between different concepts of discrete conformality that have been developed in the last decades.

Vertex scalings. The concept of discrete conformality by a vertex scaling as above, first appeared in a paper by Luo in 2004 [13]. Luo introduces a discrete scalar curvature on piecewise flat surfaces and describes a discrete analog of Yamabe flow in this setting. Luo works with a triple (S, \mathcal{T}, ρ) of a surface S and a triangulation \mathcal{T} of S , together with a positive real valued function ρ on the set of edges of \mathcal{T} such that the edge lengths of any triangle in \mathcal{T} define an isometric Euclidean triangle. Luo calls the function ρ a *polyhedral metric* on (S, \mathcal{T}) .

Given a polyhedral metric ρ on (S, \mathcal{T}) , let u be a real valued function defined on the vertex set of (S, \mathcal{T}) , Luo defines a discrete-conformal change of ρ by the vertex scaling

$$u * \rho(vv') = \rho(vv') e^{\frac{1}{2}(u(v) + u(v'))}$$

on edges of \mathcal{T} . If $u * \rho$ defines a polyhedral metric, we say that ρ and $u * \rho$ are discrete-conformally equivalent.

Circle packings. A hint that the concept of conformality could make sense also in a discrete setting appeared in the theory of circle packings in the 1930's. A *circle packing* is a connected collection of circles in the plane whose interiors are disjoint. A classical result in this area is Koebe's circle packing theorem [12].

Theorem (Koebe). *For every connected simple planar graph G there is a circle packing in the plane whose intersection graph is G .*

The *intersection graph* of a circle packing is the graph having a vertex for each circle, and an edge for every pair of circles that are tangent. Let S be an oriented surface, i.e. a connected topological 2-manifold, with a metric. Given a collection $C = \{c_v\}$ of circles (e.g. metric spheres) in S and a simplicial 2-complex K triangulating S , the pair (C, K) is said to be a *circle packing for a simplicial 2-complex K* , denoted C_K , if

1. for each vertex v in K there exists exactly one circle c_v in C with center v and vice versa,
2. if $\langle u, v \rangle$ is an edge of K , then the two circles c_u and c_v form a tangent pair and

3. if $\langle u, v, w \rangle$ forms a positively oriented face of K , then the three circles c_u , c_v and c_w form a positively oriented tangent triple in S .

We say that an abstract simplicial 2-complex K is a combinatorial sphere if it triangulates a topological sphere.

For circle packings for a combinatorial sphere K , Thurston observed the following rigidity property (see Proposition 6.1, p. 72 in [18]). This constitutes a uniqueness statement, completing Koebe's existence theorem.

Theorem (Thurston). *Let K be a combinatorial sphere. Then there exists a univalent circle packing C_K , i.e. the interior of the circles are disjoint, for K on the sphere. This circle packing is unique up to Möbius transformations on the sphere.*

Circle patterns. Closely related to circle packings is the concept of circle patterns. Let G^* be the dual of a connected planar graph G viewed as a graph embedded in the sphere, and let $\alpha : E(G^*) \rightarrow (0, \pi)$ be a weight on the edges $E(G^*)$. A *spherical circle pattern* on the sphere with adjacency graph G^* and intersection angles α is a collection of circles for each vertex, such that the following conditions hold.

1. For each edge uv in $E(G^*)$, the two circles associated to u, v in $V(G^*)$ intersect with exterior intersection angle $\alpha(uv)$.
2. The circles corresponding to the vertices adjacent to the same face of G^* intersect in a single point.
3. Consider a counterclockwise cyclic order of the intersection points from (2) on the circle corresponding to a vertex v of G^* . This order agrees with the counterclockwise cyclic order of the cycle of faces of G^* adjacent to v .

Theorem (Rivin [17][15]). *Let G^* be the dual graph of a connected planar graph G . Let $w : G^* \rightarrow (0, \pi)$ be a weight on the edges of G^* such that for all edges incident to a face f of G^* we have*

$$\sum_{e \text{ incident to } f} \pi - \alpha(e) = 2\pi,$$

and for every simple circuit e_1, \dots, e_k of edges in G^ that does not bound a single face of G^* we have*

$$\sum_i \pi - \alpha(e_i) > 2\pi.$$

Then there exists a spherical circle pattern C_{G^} in the sphere with adjacency graph G^* and intersection angles α . This circle pattern is unique up to Möbius transformations on the sphere.*

Bobenko and Springborn give an alternative proof of Rivin's theorem in [11] which is applicable to higher genus surfaces. Rivin formulates in [17] the above theorem in terms of ideal convex polyhedra. Circle patterns on the sphere are closely related to ideal convex polyhedra. We may interpret the sphere as the ideal boundary of the hyperbolic space \mathbb{H}^3 in the Poincaré model. If we carve out all hyperbolic half-planes defined by the circles in

C_{G^*} on the ideal boundary of \mathbb{H}^3 , we obtain an ideal convex polyhedron $P_{C_{G^*}}$ in \mathbb{H}^3 with the dihedral angle at an edge e of $P_{C_{G^*}}$ given by $\alpha(e)$.

From circle packings to vertex scalings. A hint that the concept of discrete conformality by vertex scaling and the concept of discrete conformality associated to circle packings are related, appears in a paper by Bobenko, Pinkall and Springborn [5]. In their paper they address the following question: Given a polyhedral surface (S, \mathcal{T}, ρ) with N vertices and a set of complete angles $(\theta_1, \dots, \theta_N)$ (i.e. the sum of angles around vertices), satisfying some necessary conditions, does there exist a conformal factor u such that $u * \rho$ is a polyhedral metric and has complete angle θ_i at each vertex? Bobenko, Pinkall and Springborn give a partial answer using a variational principle. Their functional is closely related to a family of functionals developed within the theory of circle packings and circle patterns. To this family belongs for example the functional of Rivin introduced in his paper on “Euclidean structures on simplicial surfaces and hyperbolic volume” [15] and the functional of Colin de Verdière that gives an existence and uniqueness proof of circle packings [7].

Structure of the paper. In section 1 we define the notion of *closed convex polyhedra* and *ideal polyhedra*, state a rigidity property for closed polyhedra, introduce the notions of *polyhedral surfaces*, *ideal polyhedral surfaces* and *development* and outline the proof of an isometric embedding theorem of polyhedral surfaces used in section 3.

In section 2 we outline the proof of an isometric embedding theorem of ideal polyhedral surfaces used in section 3.

In section 3 we introduce the notion of *Delaunay triangulation*, define a notion of *discrete conformality of polyhedra* and prove the uniformization theorem of polyhedra mentioned in the introduction. We further characterize discrete conformality in special cases by elementary transformations on the vertices of polyhedra. Finally we relate the notion of discrete conformality of this paper with Thurston’s notion of discrete conformality based on circle packings.

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1. ALEXANDROV’S THEORY ON CLOSED CONVEX POLYHEDRA

We will consider closed convex polyhedra in Euclidean 3-space \mathbb{E}^3 and hyperbolic 3-space \mathbb{H}^3 . A *closed convex polyhedron* in \mathbb{E}^3 or \mathbb{H}^3 is the convex hull of a finite set of points in \mathbb{E}^3 or \mathbb{H}^3 . This definition includes doubly-covered closed convex polygons. By a *closed polygon* we mean any domain in \mathbb{E}^2 or \mathbb{H}^2 that is bounded by finitely many geodesic line segments.

The boundary of a closed convex polyhedron is composed of finitely many closed convex polygons in the respective 2-dimensional space. In the follow-

ing, we will not explicitly stipulate that the polyhedron under consideration is closed and convex.

The polygons bounding a polyhedron are the *faces* of the polyhedron. The sides and vertices of the faces of a polyhedron are the *edges* and *vertices* of the polyhedron.

In the same manner one could define the vertices of a polyhedron P as the minimal number of points, whose convex hull agrees with P .

A *convex polyhedron with vertices at infinity* in \mathbb{H}^3 is the convex hull of a finite set of points, some of them lying on the ideal boundary of \mathbb{H}^3 . A convex polyhedron with all vertices on the ideal boundary is called an *ideal convex polyhedron*.

A rigidity property of convex polyhedra. It is a fundamental result of rigidity theory that convex polyhedra in \mathbb{E}^3 or \mathbb{H}^3 with congruent corresponding faces must be congruent to each other. This result is attributed to Augustin Cauchy who published this result in 1813 [6]. Cauchy's Theorem may be formulated as follows (Theorem 1, p. 171 in [1]).

Theorem 1 (Cauchy, Aleksandrov). *Every isometry φ from the boundary of a closed convex polyhedron P in \mathbb{R}^3 or \mathbb{H}^3 onto the boundary of another closed convex polyhedron Q can be realized as a motion or a motion and a reflection, i.e. there is a motion, or a motion followed by a reflection, which takes each point of the boundary of P to its image under the mapping φ .*

In fact, this is a slightly stronger form of Cauchy's Theorem that resulted from work of Aleksandrov and was published in the 1940's.

Polyhedral surface. A *polyhedral surface* (S, d) is a surface S together with a flat cone metric d on S that has finitely many cone points. A *cone point* is a point v in S that admits a circle centered at v with circumference different from $2\pi r$, where r is its radius.

Given two points x and y on the boundary of a Euclidean or hyperbolic polyhedron P , there exists a polygonal path from x to y on the boundary of P . The infimum of the lengths of polygonal paths from x to y defines a *distance* on the boundary of P , we denote this polyhedral surface by (S, d_P) . This construction associates with every Euclidean polyhedron P a Euclidean polyhedral surface (S, d_P) homeomorphic to the sphere.

An *ideal polyhedral surface* is a complete hyperbolic surface of finite area, homeomorphic to the N times punctured sphere. We denote a surface homeomorphic to the N times punctured sphere by X . Analogously, every ideal polyhedron P gives rise to an ideal polyhedral surface (X, \hat{d}_P) .

The *complete angle* at a point x in a polyhedral surface S is the number

$$\lim_{\epsilon \rightarrow 0} \frac{C_\epsilon(x)}{\epsilon},$$

where C_ϵ is the circumference of a circle of radius ϵ at x . The notion of a complete angle is an intrinsic property of the polyhedral surface.

Let θ be the complete angle at a point x , the difference $2\pi - \theta$ is the *curvature* at x . A polyhedral surface that has a non-negative curvature at every point is said to be a *polyhedral surface of non-negative curvature*.

A polyhedral surface arising as the boundary of a convex polyhedron has a non-negative curvature everywhere. Conversely, does every polyhedral surface of non-negative curvature arise from a convex polyhedron in \mathbb{E}^3 or \mathbb{H}^3 ?

An affirmative answer was given by Alexandrov in the 1940's. In fact, Alexandrov showed that every polyhedral surface of non-negative curvature defines a unique polyhedron in \mathbb{E}^3 or \mathbb{H}^3 up to congruence [1].

Development. A *development* is a finite collection of closed polygons in \mathbb{E}^2 or \mathbb{H}^2 together with a set of rules for “gluing” them together along their edges. The rule for gluing satisfies the following conditions:

1. The correspondence of “gluing” two segments is an isometry.
2. It is possible to pass from each polygon to any other polygon by traversing polygons with glued sides.
3. Each side of every polygon is glued to exactly one side of another polygon.

The sides and vertices of the polygons within a development are the *edges* and *vertices* of the development, where identified sides and vertices are considered the same. We denote a development by R .

Every development R defines an underlying polyhedral surface, which we denote by (S, d_R) . In other words, a development is a polyhedral surface plus a subdivision into geodesic polygons.

Several developments can define the same polyhedral surface. One may think of a development as a “coordinate representation” of a polyhedral surface. Different cuttings of a polyhedral surface into polygons correspond to different coordinate representations of the same polyhedral surface.

Two developments R and R' can be obtained from each other by *cutting and gluing* if the polygons in R can be cut into polygons and glued along edges such that we obtain the development R' . One observes:

Proposition 1. *Two developments R and R' are related by cutting and gluing if and only if (S, d_R) and $(S, d_{R'})$ are isometric.*

We will use the above ideas to turn the space of closed polyhedral surfaces into a manifold by “cutting” polyhedral surfaces into triangles. Those representations will turn out to be convenient coordinate charts for our space.

Every convex polyhedron is naturally associated with a development. The *face development* of a polyhedron P is the development R_P whose polygons are the faces of the polyhedron P .

Isometric embedding of polyhedral surfaces. We now return to the question by Alexandrov: Does every polyhedral surface of positive curvature in \mathbb{E}^3 arise as the boundary of a convex polyhedron?

We suggested to think of developments of polyhedral surfaces as coordinate representations of the polyhedral surface. In this section we will make this more precise by cutting polyhedral surfaces into triangles. We then outline Alexandrov's proof that the map assigning its boundary to a polyhedron achieves a homeomorphism from the space of polyhedra with N vertices up to congruence to the space of polyhedral surfaces of non-negative curvature with N cone points.

We say that two polyhedral surfaces (S, d) and (S', d') are equivalent, if there exists an isometry $f : (S, d) \rightarrow (S', d')$. Let $M_{PL}^{\text{con}}(N)$ be the space of equivalence classes of closed simply-connected polyhedral surfaces with N cone points of strictly positive curvature.

When dealing with coordinates, it is more convenient to work with marked polyhedral surfaces. A *marked polyhedral surface* is a polyhedral surface together with a homeomorphism μ from the standard sphere \mathbb{S} with N distinct points $\{p_1, \dots, p_N\}$ to (S, d) , where the N distinct points map to cone points in (S, d) . Two marked polyhedral surfaces are said to be equivalent, if there exists an isometry between them, whose pullback on \mathbb{S} is homotopic to the identity by an homotopy that fixes the points $\{p_1, \dots, p_N\}$. A *triangulation* of $(S, d)_\mu$ is a triangulation \mathcal{T} of S with vertices the N distinct points of S . A *geodesic triangulation* is a triangulation of $(S, d)_\mu$ whose edges are minimizing geodesics in (S, d) . We denote the space of marked polyhedral surfaces of strict positive curvature by $\widetilde{M}_{PL}^{\text{con}}(N)$. The map from $\widetilde{M}_{PL}^{\text{con}}(N)$ to $M_{PL}^{\text{con}}(N)$ that forgets the marking is a covering map.

Let \mathcal{T} be a geodesic triangulation of a marked polyhedral surface $(S, d)_\mu$ in $\widetilde{M}_{PL}^{\text{con}}(N)$. The map $\vartheta_{\mathcal{T}}$ that associates to every edge of \mathcal{T} its length, is a coordinate chart of $\widetilde{M}_{PL}^{\text{con}}(N)$ around $(S, d)_\mu$. The corresponding atlas turns $\widetilde{M}_{PL}^{\text{con}}(N)$ into a $3N - 6$ dimensional manifold (section 2 in [9]).

We may now sketch the proof of Alexandrov's embedding theorem (see p. 210 in [1]).

Theorem 2 (Alexandrov). *Let (S, d) be a polyhedral surface with N cone points of strictly positive curvature, homeomorphic to the sphere. Then (S, d) can be realized as the boundary of a closed convex polyhedron P with N vertices. This polyhedron is unique up to congruence.*

Outline of the proof. Let \mathcal{P}^N be the space of closed convex polyhedra with N vertices. Let $\widetilde{\mathcal{P}}^N$ be the space of marked closed convex polyhedra with N vertices in \mathbb{R}^3 , parametrized by the positions of their vertices. A *marked closed polyhedron* is a polyhedron P together with a homeomorphism μ from the standard sphere \mathbb{S} with N distinct points $\{p_1, \dots, p_N\}$ to the boundary of P , where the N distinct points map to the vertices of P . Two polyhedra are said to be equivalent, if there exists an isometry between them, whose pullback on \mathbb{S} is homotopic to the identity by an homotopy that fixes the points $\{p_1, \dots, p_N\}$. $\widetilde{\mathcal{P}}^N$ is a $3N - 6$ dimensional manifold. Indeed, three vertices are sent by an isometry to the origin, the positive x-axis and the half-plane $y > 0$ of the xy -plane, respectively. If the polyhedron does not

degenerate into a doubly-covered polygon, then a fourth point not contained in the xy -plane is mapped into the half-space $z > 0$ by reflecting along the $z = 0$ plane if needed. This eliminates the action of the isometry group of \mathbb{R}^3 . There are $3N$ variable coordinates, however three vertices are constant in three, two and one coordinates respectively. Therefore, we have $3N - 6$ variable coordinates.

The boundary of every marked closed convex polyhedron with N vertices can be viewed as a marked polyhedral surface with N cone points of strictly positive curvature homeomorphic to the sphere. Formally this gives a map $\tilde{\mathfrak{g}} : \tilde{\mathcal{P}}^N \rightarrow \tilde{M}_{PL}^{\text{con}}(N)$. Alexandrov shows that $\tilde{\mathfrak{g}}$ is a (1) continuous, (2) injective and (3) closed map and (4) that every connected component of $\tilde{M}_{PL}^{\text{con}}(N)$ admits a preimage in $\tilde{\mathcal{P}}^N$.

$\tilde{\mathcal{P}}^N$ and $\tilde{M}_{PL}^{\text{con}}(N)$ are manifolds of equal dimension, by (1) and (2) and the invariance of domain principle of Brouwer, $\tilde{\mathfrak{g}}$ is an open map. Since $\tilde{\mathfrak{g}}$ is also closed, we conclude together with (4) that $\tilde{\mathfrak{g}}$ is a homeomorphism from $\tilde{\mathcal{P}}^N$ onto $\tilde{M}_{PL}^{\text{con}}(N)$.

$$\begin{array}{ccc} \tilde{\mathcal{P}}^N & \xrightarrow{\tilde{\mathfrak{g}}} & \tilde{M}_{PL}^{\text{con}}(N) \\ \downarrow & & \downarrow \\ \mathcal{P}^N & \xrightarrow{\mathfrak{g}} & M_{PL}^{\text{con}}(N) \end{array}$$

Hence, \mathfrak{g} is surjective and by the Theorem of Cauchy and Alexandrov it is also injective. \blacksquare

Remark: The fact that a polyhedron in \mathbb{R}^3 is determined by the geometry of its surface, is particular to polyhedra in three dimensional space. A polygon is not at all determined by the length of its edges. Also, in higher dimensions such a correspondence does not hold in general. The dependence of the theory on the dimension reveals itself in the usage of Brouwer's invariance of domain principle. It is particular to \mathbb{R}^3 , that the space of closed convex polyhedra with N vertices has the same dimension as the space of polyhedral surfaces with N cone points of strictly positive curvature.

2. RIVIN'S THEORY ON IDEAL CONVEX POLYHEDRA

Isometric embedding of ideal polyhedral surfaces. Does every ideal polyhedral surface arise from the boundary of an ideal hyperbolic polyhedron?

Analogously to the Euclidean case, we can use triangulations to give ideal polyhedral surfaces a manifold structure. We will need a few concepts from classical hyperbolic geometry to do so.

Let ABC be an ideal triangle in \mathbb{H}^2 . Let h_A be a horocycle centered at A , define $\mathcal{D}_{ABC}(h_A)$ to be the length of the arc of h_A cut out by the triangle ABC . The difference in size between arcs of two horocycles h_A and h'_A cut out by ABC gives information on the distance between the arcs. More precisely:

Lemma 1. *Let h_A and h'_A be two horocycles at A . The hyperbolic distance between h_A and h'_A is equal to $|\log(\mathcal{D}_{ABC}(h_A)/\mathcal{D}_{ABC}(h'_A))|$.*

Proof: Let ABC be the triangle $A = \infty$, $B = 0$ and $C = 1$ in the upper half-space model. The horocycles h_A and h'_A are horizontal lines through i/y and i/y' , respectively. Hence, the length of the arcs of h_A and h'_A cut out by ABC is $1/y$ and $1/y'$ respectively and the distance between h_A and h'_A is $|\log(y/y')|$. ■

Two ideal triangles ABC and ADC can slide with respect to each other along the common side AC . For any choice of horocycle h_A , the number $f_{AC} := \log(\mathcal{D}_{ABC}(h_A)/\mathcal{D}_{ADC}(h_A))$ measures the *shear* between the triangles ABC and ADC along AC . The shear f_{AC} does not depend on which of the vertices A or C is taken as the center of the horocycles.

Intuitively, two triangles ABC and ADC are joined along AC without a shear, if for any horocycle at A the arcs cut out by ABC and ADC have the same “distance” to A .

The *cross-ratio* of four points z_1, z_2, z_3, z_4 in the complex plane is the number

$$[z_1, z_2, z_3, z_4] := \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_2)(z_3 - z_4)}.$$

The notions of cross-ratio of four points and shear between two triangles are related.

Lemma 2. *The shear between two triangles ABC and ABD is equal to the log of the absolute value of the cross-ratio $[C, B, D, A]$.*

Proof: Let ABC be the triangle $A = \infty$, $B = 1$ and $C = 0$. In this case, the shear between ABC and ABD is $\log|D|$. ■

A *marked ideal polyhedral surface* is a polyhedral surface (X, \hat{d}) together with a homeomorphism μ from a standard ideal polyhedral surface (\mathbb{X}, \hat{d}) to (X, \hat{d}) . Two marked polyhedral surfaces are equivalent if there exists an isometry between them, whose pullback on (\mathbb{X}, \hat{d}) is isotopic to the identity. Let $\widetilde{M}(N)$ be the set of equivalence classes of marked ideal polyhedral surfaces.

A *triangulation* of a marked ideal polyhedral surface is a triangulation whose vertices are at the cusps of the hyperbolic surface. A *geometric triangulation* of a marked ideal polyhedral surface is a triangulation of the ideal polyhedral surface whose edges are geodesics.

The set $\widetilde{M}(N)$ is parametrized by *shears* along the edges of a geodesic triangulation of marked ideal polyhedral surfaces. Let \mathcal{T} be a geometric triangulation of a marked ideal polyhedral surface with N cusps. To each edge of \mathcal{T} , associate the shear of the two abutting triangles of \mathcal{T} . This information determines the geometry completely. Conversely, an assignment of real numbers to the edges of \mathcal{T} specifies a complete hyperbolic structure if and only if the shears around any cusp add up to zero. Hence the set $\widetilde{M}(N)$ is naturally parametrized by $\mathbb{R}^{|E(\mathcal{T})| - N}$. According to the Euler formula, $|E(\mathcal{T})| - N = 2N - 6$, so the dimension of this space depends only on the number of cusps.

Lemma 3. *Any triangulation of a complete hyperbolic surface with cusps can be straightened to a geodesic triangulation.*

Proof: We need to show that, if A, B, C and D are cusps of a complete hyperbolic surface such that A and B are connected by a path γ_1 in \mathcal{T} and C and D are connected by a path γ_2 in \mathcal{T} , then the corresponding geodesics also do not intersect.

The path γ_1 and γ_2 do not intersect in (X, \hat{d}) if and only if their lifts to the universal cover \mathbb{H}^2 of S_N do not intersect. The lifts of γ_1 and γ_2 hit the boundary of \mathbb{H}^2 in a single point. Indeed, one end of γ_1 will be completely inside a circular neighborhood of A . The corresponding end of the lift of γ_1 will be inside a horosphere. Hence, this end of the lift of γ_1 touches the boundary in a single point. If two paths between the ideal boundary of \mathbb{H}^2 do not intersect, then the corresponding minimizing geodesics do not intersect either. \blacksquare

The *shear coordinate system* on $\widetilde{M}(N)$ corresponding to the triangulation \mathcal{T} of a marked polyhedral surface is given as follows. With a particular metric in $\widetilde{M}(N)$, one can associate its shears along the straightened edges of \mathcal{T} . This embeds $\widetilde{M}(N)$ into \mathbb{R}^{3N-6} as a linear subspace. The subspace is given by the N conditions that shears add up to zero along vertices. Let $\eta_{\mathcal{T}} : \widetilde{M}(N) \rightarrow \mathbb{R}^{2N-6}$ be the orthogonal projection to this $2N-6$ dimensional linear subspace. Notice that $\eta_{\mathcal{T}} : \widetilde{M}(N) \rightarrow \mathbb{R}^{2N-6}$ is a homeomorphism. Given a point x in \mathbb{R}^{2N-6} , we can compute the remaining N shears from the condition that shears must add up to zero around vertices. Hence, $\widetilde{M}(N)$ is connected.

Theorem 3 (Rivin [16]). *Let (X, \hat{d}) be an ideal polyhedral surface. Then (X, \hat{d}) can be isometrically embedded in \mathbb{H}^3 as the boundary of a convex polyhedron P with all vertices on the sphere at infinity.*

The proof needs some specific techniques related to the fact that we are dealing with geodesics between ideal points. Nevertheless, the proof follows essentially the same philosophy as Alexandrov's.

Outline of the proof. Let $\mathcal{P}_{\text{ideal}}^N$ be the space of convex ideal polyhedra with N vertices in \mathbb{H}^3 . Let $\widetilde{\mathcal{P}}_{\text{ideal}}^N$ be the space of marked convex ideal polyhedra with N vertices in \mathbb{H} , this space is parametrized by the positions of their vertices on the sphere at infinity. A *marked convex ideal polyhedron* is an ideal polyhedron P together with a homeomorphism μ from a standard ideal polyhedral surface (\mathbb{X}, \hat{d}) to the boundary of P . Two marked ideal polyhedra are equivalent if there exists an isometry between them, whose pullback on (\mathbb{X}, \hat{d}) is isotopic to the identity. $\widetilde{\mathcal{P}}_{\text{ideal}}^N$ is a $2N-6$ dimensional manifold. Indeed, three of the vertices of P are fixed at $0, 1$, and ∞ . This eliminates the action of the isometry group of \mathbb{H}^3 . There are $2N$ variable coordinates, however three vertices are fixed. Therefore, we have $2N-6$ variable coordinates.

The boundary of every marked convex ideal polyhedron with N vertices can be viewed as a complete marked hyperbolic surface of finite area, homeomorphic to the N times punctured sphere. Formally this gives a map $\tilde{\mathfrak{h}} : \widetilde{\mathcal{P}}_{\text{ideal}}^N \rightarrow \widetilde{M}(N)$. Rivin shows that $\tilde{\mathfrak{h}}$ is a (1) continuous, (2) injective and (3) closed map.

$\tilde{\mathcal{P}}_{\text{ideal}}^N$ and $\tilde{M}(N)$ are manifolds of equal dimension, by (1) and (2) and the invariance of domain principle of Brouwer, $\tilde{\mathfrak{h}}$ is an open map. Since $\tilde{\mathfrak{h}}$ is also closed, we conclude, together with the fact that $\tilde{M}(N)$ is connected, that $\tilde{\mathfrak{h}}$ is a homeomorphism from $\tilde{\mathcal{P}}_{\text{ideal}}^N$ onto $\tilde{M}(N)$.

$$\begin{array}{ccc} \tilde{\mathcal{P}}_{\text{ideal}}^N & \xrightarrow{\tilde{\mathfrak{h}}} & \tilde{M}(N) \\ \downarrow & & \downarrow \\ \mathcal{P}_{\text{ideal}}^N & \xrightarrow{\mathfrak{h}} & M(N) \end{array}$$

Since every conformal transformation on the sphere corresponds to an isometry of \mathbb{H}^3 , $\pi(\mathcal{P}_{\text{ideal}}^N) = \pi(M(N))$. The fundamental group of $M(N)$ is co-Hopfian [4], therefore $\mathfrak{h}_*\pi_1(\mathcal{P}_{\text{ideal}}^N) = \pi_1(M(N))$ and hence \mathfrak{h} is one to one. \blacksquare

3. A CONFORMAL EQUIVALENCE RELATION FOR CONVEX POLYHEDRA

Discrete conformality of convex polyhedra. A *Delaunay triangulation* of a finite set of points V in the Euclidean plane is a triangulation of the convex hull of V into triangles such that no point in V is inside the circumcircle of any other triangle. A *Delaunay triangulation of a development* R is a Delaunay triangulation of every polygon in R . The following lemma is a classical property of Delaunay triangulations in the plane. A proof can be found in Aurenhammer's book on Voronoi diagrams [3].

Lemma 4. *If a finite set of points in the plane admits two Delaunay triangulations, then there exists a sequence of Delaunay triangulations between them, such that each is related to the next by a diagonal switch.*

Hence, if a Euclidean development R admits two distinct Delaunay triangulations, then they differ by a finite number of diagonal switches between two abutting triangles within a polygon in R that share the same circumcircle.

Every Euclidean development R has a unique set of circumcircles attached to its vertices, by taking the circumcircles of a Delaunay triangulation of R . A Euclidean triangle with its circumcircle can be viewed as an ideal hyperbolic triangle in the Klein model. This construction does not depend on the chosen Delaunay triangulation and associates with every Euclidean development R with N vertices an ideal polyhedral surface (X, \hat{d}_R) with a cusp for each vertex of the development. Indeed, by the following theorem the associated hyperbolic surface with cusps is complete, since the shear coordinates add up to zero around vertices.

Proposition 2. *Let R be a Euclidean development with N vertices and \mathcal{T} a Delaunay triangulation of R . Let ijk and ilj be two triangles in \mathcal{T} abutting along the edge ij . The hyperbolic structure \hat{d}_R on X is the unique complete hyperbolic structure on X with shear*

$$\log \frac{\vartheta_{\mathcal{T}}((S, d_R))_{il}}{\vartheta_{\mathcal{T}}((S, d_R))_{ik}} : \frac{\vartheta_{\mathcal{T}}((S, d_R))_{jl}}{\vartheta_{\mathcal{T}}((S, d_R))_{jk}}$$

along the edge ij of \mathcal{T} .

Proof: The map associating every Euclidean development R with a hyperbolic surface with cusps (X, \hat{d}_R) , can be described in upper half-space model as follows. Consider \mathbb{C} as the sphere at infinity of the hyperbolic 3-space $\mathbb{H}^3 = \mathbb{C} \times \mathbb{R}_{>0}$. Let ijk and ijl be two abutting triangles in R . Embed $ijk \cup ijil$ into the sphere at infinity by an isometry f . The hyperbolic metric \hat{d}_R on $ijk \cup ijil$ is the hyperbolic metric of the ideal hyperbolic triangles in \mathbb{H}^3 , having the same vertices as ijk and $ijil$, glued by the same isometry f , considered as a hyperbolic motion of \mathbb{H}^3 .

The shear of (X, \hat{d}_R) along the edge ij is the logarithm of the absolute value of the complex cross-ratio of the four vertices z_i, z_j, z_k and z_l of the triangles ijk and $ijil$ in \mathbb{C} . Clearly,

$$\log \left| \frac{z_i - z_l}{z_i - z_k} : \frac{z_j - z_l}{z_j - z_k} \right| = \log \frac{\vartheta_{\mathcal{T}}((S, d_R))_{il}}{\vartheta_{\mathcal{T}}((S, d_R))_{ik}} : \frac{\vartheta_{\mathcal{T}}((S, d_R))_{jl}}{\vartheta_{\mathcal{T}}((S, d_R))_{jk}}.$$

■

A *Delaunay triangulation of a convex polyhedron* P is a triangulation of its boundary coming from a Delaunay triangulation of its face development R_P .

Note: A Delaunay triangulation of an ideal convex polyhedron P in \mathbb{H}^3 , is a Delaunay triangulation of a convex polyhedron if P is viewed as a Euclidean convex polyhedron inscribed in the sphere.

Given a convex Euclidean polyhedron P , we associate with P the ideal polyhedral surface (X, \hat{d}_{R_P}) coming from the face development of P . In the following, we denote (X, \hat{d}_{R_P}) by (X, \hat{d}_P) . Formally we obtain a function

$$\mathfrak{f} : \mathcal{P}^N \rightarrow M(N) \tag{1}$$

mapping P to (X, \hat{d}_P) .

Definition 1. *Two closed convex polyhedra P and Q with N vertices are discrete-conformally equivalent if and only if (X, \hat{d}_P) and (X', \hat{d}_Q) are isometric e.g. $\mathfrak{f}(P) = \mathfrak{f}(Q)$.*

The definition is inspired by a closely related concept of discrete conformality of polyhedral surfaces introduced by Bobenko, Pinkall and Springborn in [5]. Bobenko, Pinkall and Springborn work with the data of a polyhedral surface together with a triangulation and associate with it an ideal polyhedral surface like above. They define two polyhedral surfaces with the same triangulation as being discrete-conformally equivalent, if and only if their associated ideal polyhedral surfaces are isometric. This notion allows them to associate a triangulated polyhedral surface with a polyhedron inscribed in the sphere that is star-shaped with respect to one point and unique if it exists. The notion of discrete conformality in this paper is adapted to the setting of closed convex polyhedra. The use of Delaunay triangulations makes the construction canonical in the sense that the function \mathfrak{f} does not depend on a triangulation. This construction allows us to associate to every convex polyhedron a polyhedron inscribed in the sphere that is convex, unique and always exists.

Proposition 3. *Let P and Q be two convex polyhedra inscribed in the unit sphere that are discrete-conformally equivalent. Then there exists a Möbius transformation on the sphere that maps the vertex set of P to the vertex set of Q .*

Proof: If P is inscribed in the unit sphere, then the association $P \rightarrow (X, \hat{d}_P)$ defined above is nothing but interpreting P as a convex ideal polyhedron in the Klein model and moving to the boundary. Hence, if P and Q are discrete-conformally equivalent, there exists a hyperbolic isometry from the boundary of P to the boundary of Q . According to the rigidity theory of Cauchy, Alexandrov and Rivin, this isometry can be realized as a motion or a motion and a reflection in \mathbb{H}^3 . Equally, there exists a Möbius transformation on the sphere mapping the vertex set of P to the vertex set of Q . ■

The above rigidity theorem allows us to classify Euclidean polyhedra up to discrete conformality.

Theorem 4 (Uniformization). *Every closed convex polyhedron in \mathbb{E}^3 is discrete-conformally equivalent to a closed convex polyhedron inscribed in the unit sphere. This polyhedron is unique up to Möbius transformations on the sphere.*

The uniqueness part was proven above. The existence follows from Rivin's isometric embedding of ideal polyhedra in hyperbolic 3-space.

Proposition 4. *Given a convex polyhedron P in \mathbb{E}^3 , there exists a convex polyhedron Q inscribed in the unit sphere that is discrete-conformally equivalent to P .*

Proof: Let (X, \hat{d}_P) be the ideal polyhedral surface associated with P . According to Rivin's isometric embedding theorem, (X, \hat{d}_P) can be isometrically embedded in \mathbb{H}^3 as the boundary of a convex hyperbolic polyhedron Q with all vertices on the sphere at infinity. The polyhedron Q may be interpreted as a convex Euclidean polyhedron inscribed in the sphere if viewed in the Klein model. This interpretation is just the inverse of the map $Q \rightarrow (X', \hat{d}_Q)$. Hence, (X, \hat{d}_P) is isometric to (X', \hat{d}_Q) and P and Q are discrete-conformally equivalent. ■

Characterization of discrete conformality. The notion of discrete conformality passes through hyperbolic geometry. In the following we characterize discrete conformality of Euclidean polyhedra that share a common Delaunay triangulation by elementary transformations on vertices. To every Delaunay triangulation one can associate a lattice formed by its vertices, edges and triangles, which are ordered by inclusion. Two polyhedra *share a common Delaunay triangulation* if the associated lattices are isomorphic.

We will first construct a function

$$\tilde{\mathfrak{f}} : \tilde{\mathcal{P}}^N \rightarrow \tilde{\mathcal{M}}(N) \tag{2}$$

that is a lift of \mathfrak{f} using Penner's theory on decorated Teichmüller spaces [14].

A *decorated ideal triangle* is an ideal triangle ABC together with a choice of horocycles h_A , h_B and h_C . The *Penner distance* between two distinct horocycles h_A and h_B is

$$l_{AB}^P := l^P(h_A, h_B) := e^{\lambda_{AB}/2}, \quad (3)$$

where $\lambda_{AB} := \lambda(h_A, h_B)$ is the signed distance between two distinct horocycles.

Two decorated ideal triangles (ABC, h_A, h_B, h_C) and (ADC, h_A, h_D, h_C) can be glued along the edge AC by an isometry preserving the horocycles h_A and h_C .

A *decorated ideal polyhedral surface* $(X, \hat{d}, \{h_i\})_\mu$ is the data of a marked ideal polyhedral surface $(X, \hat{d})_\mu$ plus a horoball for every puncture i . Two decorated ideal polyhedral surfaces $(X, \hat{d}, \{h_i\})_\mu$ and $(X', \hat{d}', \{h'_i\})_{\mu'}$ are equivalent if there exists an isometric map $f : (X, \hat{d}) \rightarrow (X', \hat{d}')$ such that $\mu'^{-1} \circ f \circ \mu$ is homotopic to the identity in (\mathbb{X}, \hat{d}) and every horoball h_i is mapped to h'_i by f .

Let $\widetilde{M}_D(N)$ be the set of equivalence classes of decorated ideal polyhedral surfaces with N punctures. Let $\widetilde{M}(N)$ be the set of complete, finite volume hyperbolic structures on (X, \hat{d}) as introduced above. The mapping

$$\begin{aligned} \widetilde{M}_D(X) &\rightarrow \widetilde{M}(X) \times \mathbb{R}_{>0}^N \\ (X, \hat{d}, \{h_i\})_\mu &\mapsto ((X, \hat{d})_\mu, (w_1, \dots, w_N)) \end{aligned}$$

is a bijection, where w_i is the sum of the lengths $\mathcal{D}_{ijk}(h_i)$ of horoarcs cut out by the ideal triangles at i .

Let \mathcal{T} be a geodesic triangulation of $(X, \hat{d}, \{h_i\})_\mu$. The map $\varphi_{\mathcal{T}}$ that associates to every edge ij in \mathcal{T} the Penner distance l_{ij}^P , gives local coordinates to $(X, \hat{d}, \{h_i\})_\mu$ in $\widetilde{M}_D(N)$. Those *Penner coordinate charts* turn $\widetilde{M}_D(N)$ into a real analytic manifold [14].

Let $\tilde{\mathfrak{p}} : \widetilde{M}_D(N) \rightarrow \widetilde{M}(N)$ be the projection, mapping $(X, \hat{d}, \{h_i\})_\mu$ to $(X, \hat{d})_\mu$, and let $\tilde{\mathfrak{g}} : \widetilde{\mathcal{P}}^N \rightarrow \widetilde{M}_{PL}^{\text{con}}(N)$ be Alexandrov's homeomorphism. We aim to construct a function $F : \widetilde{M}_{PL}^{\text{con}}(N) \rightarrow \widetilde{M}_D(N)$ such that the following diagram commutes

$$\begin{array}{ccccccc} \widetilde{\mathcal{P}}^N & \xrightarrow{\tilde{\mathfrak{g}}} & \widetilde{M}_{PL}^{\text{con}}(N) & \xrightarrow{F} & \widetilde{M}_D(N) & \xrightarrow{\tilde{\mathfrak{p}}} & \widetilde{M}(N) \\ \downarrow & & & & & & \downarrow \\ \mathcal{P}^N & & & \xrightarrow{\mathfrak{f}} & & & M(N). \end{array}$$

We denote by \tilde{P} a preimage of a polyhedron P in $\widetilde{\mathcal{P}}^N$. We denote by $(S, d_{\tilde{P}})_\mu$ the image of the polyhedron \tilde{P} under $\tilde{\mathfrak{g}}$ in $\widetilde{M}_{PL}^{\text{con}}(N)$. Let $D_{PL}(\mathcal{T})$ be the set of elements $(S, d_{\tilde{P}})_\mu$ in $\widetilde{M}_{PL}^{\text{con}}(N)$ such that \mathcal{T} is isotopic to a Delaunay triangulation of the associated polyhedron P . The sets $D_{PL}(\mathcal{T})$ for different isotopy classes of triangulations of $(S, d_{\tilde{P}})_\mu$ form a covering of $\widetilde{M}_{PL}^{\text{con}}(N)$. Let $F_{\mathcal{T}} = \varphi_{\mathcal{T}}^{-1} \circ \vartheta_{\mathcal{T}}$, define a function F on $\widetilde{M}_{PL}^{\text{con}}(N)$ by setting $F((S, d_{\tilde{P}})_\mu) = F_{\mathcal{T}}((S, d_{\tilde{P}})_\mu)$ if $(S, d_{\tilde{P}})_\mu \in D_{PL}(\mathcal{T})$.

Lemma 5. *The function $F : \widetilde{M}_{PL}^{\text{con}}(N) \rightarrow \widetilde{M}_D(N)$ is well-defined.*

Proof: Suppose $(S, d_{\widetilde{P}})_\mu \in D_{PL}(\mathcal{T}) \cap D_{PL}(\mathcal{T}')$, i.e. both \mathcal{T} and \mathcal{T}' are Delaunay triangulations of \widetilde{P} . Then there exists a sequence of Delaunay triangulations $\mathcal{T} = \mathcal{T}_1, \dots, \mathcal{T}_n = \mathcal{T}'$ of \widetilde{P} such that \mathcal{T}_i is obtained from \mathcal{T}_{i+1} by a diagonal switch. In particular $F_{\mathcal{T}}((S, d_{\widetilde{P}})_\mu) = F_{\mathcal{T}'}((S, d_{\widetilde{P}})_\mu)$ follows from $F_{\mathcal{T}_i}((S, d_{\widetilde{P}})_\mu) = F_{\mathcal{T}_{i+1}}((S, d_{\widetilde{P}})_\mu)$ for $i = 1, 2, \dots, n-1$. Hence, assume that \mathcal{T}' is obtained from \mathcal{T} by a diagonal switch at an edge e .

Let $\vartheta_{\mathcal{T}}((S, d_{\widetilde{P}})_\mu) = (x_0, x_1, \dots, x_n)$. Since both \mathcal{T} and \mathcal{T}' are Delaunay triangulations of \widetilde{P} , the triangles abutting at e share a common circumcircle. In this case the transition function is of the form

$$\vartheta_{\mathcal{T}'} \vartheta_{\mathcal{T}}^{-1}(x_0, x_1, \dots, x_n) = \left(\frac{x_1 x_3 + x_2 x_4}{x_0}, x_1, x_2, \dots, x_n \right).$$

On the other hand, according to Penner [14] the λ -lengths satisfy the Ptolemy relation for decorated ideal triangles. Hence,

$$\varphi_{\mathcal{T}'} \varphi_{\mathcal{T}}^{-1}(x_0, x_1, \dots, x_n) = \left(\frac{x_1 x_3 + x_2 x_4}{x_0}, x_1, x_2, \dots, x_n \right).$$

This shows,

$$\vartheta_{\mathcal{T}'} \vartheta_{\mathcal{T}}^{-1}(x_0, x_1, \dots, x_n) = \varphi_{\mathcal{T}'} \varphi_{\mathcal{T}}^{-1}(x_0, x_1, \dots, x_n),$$

which is

$$F_{\mathcal{T}}((S, d_{\widetilde{P}})_\mu) = \varphi_{\mathcal{T}}^{-1} \circ \vartheta_{\mathcal{T}}((S, d_{\widetilde{P}})_\mu) = \varphi_{\mathcal{T}'}^{-1} \circ \vartheta_{\mathcal{T}'}((S, d_{\widetilde{P}})_\mu) = F_{\mathcal{T}'}((S, d_{\widetilde{P}})_\mu). \quad \blacksquare$$

Lemma 6. *Let $(X, \hat{d}, \{h_i\})_\mu \in \widetilde{M}_D(N)$ and let $\varphi_{\mathcal{T}}$ be a coordinate chart containing $(X, \hat{d}, \{h_i\})_\mu$, then the shear coordinate between two abutting triangles ilj and ikj in \mathcal{T} of $(X, \hat{d}, \{h_i\})_\mu$ is given by*

$$\log \frac{\varphi_{\mathcal{T}}((X, \hat{d}, \{h_i\})_\mu)_{il}}{\varphi_{\mathcal{T}}((X, \hat{d}, \{h_i\})_\mu)_{ik}} : \frac{\varphi_{\mathcal{T}}((X, \hat{d}, \{h_i\})_\mu)_{jl}}{\varphi_{\mathcal{T}}((X, \hat{d}, \{h_i\})_\mu)_{jk}}.$$

Proof: Recall that $\varphi_{\mathcal{T}}((X, \hat{d}, \{h_i\})_\mu)_{il} = e^{\lambda_{il}/2}$, where λ_{il} is the signed distance between the horospheres h_i and h_l (3). Hence,

$$\begin{aligned} \log \frac{\varphi_{\mathcal{T}}((X, \hat{d}, \{h_i\})_\mu)_{il}}{\varphi_{\mathcal{T}}((X, \hat{d}, \{h_i\})_\mu)_{ik}} : \frac{\varphi_{\mathcal{T}}((X, \hat{d}, \{h_i\})_\mu)_{jl}}{\varphi_{\mathcal{T}}((X, \hat{d}, \{h_i\})_\mu)_{jk}} &= \\ \frac{1}{2}(\lambda_{il} - \lambda_{lj} + \lambda_{jk} - \lambda_{ki}). \end{aligned}$$

Let us focus first only on the decorated triangle ijk . The axis of symmetry through the point i of the ideal triangle ijk splits the signed distance λ_{jk} between the horocycles h_i and h_l into the sum of two numbers p_{ij}^k and p_{ki}^j , being the signed distance between the base point of the axis of symmetry and the horocycle h_k and h_j , respectively. Doing the same for λ_{ij} and λ_{ki} gives $\lambda_{ij} = p_{jk}^i + p_{ki}^j$, $\lambda_{jk} = p_{ki}^j + p_{ij}^k$ and $\lambda_{ki} = p_{ij}^k + p_{jk}^i$. Solving for p_{ki}^j gives

$$p_{ki}^j = \frac{1}{2}(\lambda_{ij} + \lambda_{jk} - \lambda_{ki}).$$

Doing the same for the triangle ijl gives

$$p_{il}^j = \frac{1}{2}(\lambda_{ij} + \lambda_{jl} - \lambda_{il}).$$

Hence,

$$\log \frac{\varphi_{\mathcal{T}}((X, \hat{d}, \{h_i\})_{\mu})_{il}}{\varphi_{\mathcal{T}}((X, \hat{d}, \{h_i\})_{\mu})_{ik}} : \frac{\varphi_{\mathcal{T}}((X, \hat{d}, \{h_i\})_{\mu})_{jl}}{\varphi_{\mathcal{T}}((X, \hat{d}, \{h_i\})_{\mu})_{jk}} = p_{ki}^j - p_{il}^j.$$

But the right-hand-side is nothing but the shear between the two triangles ijk and ijl . \blacksquare

Proposition 5. *The following diagram commutes*

$$\begin{array}{ccccccc} \tilde{\mathcal{P}}^N & \xrightarrow{\tilde{\mathfrak{g}}} & \tilde{M}_{PL}^{\text{con}}(N) & \xrightarrow{F} & \tilde{M}_D(N) & \xrightarrow{\tilde{\mathfrak{p}}} & \tilde{M}(N) \\ \downarrow & & & & & & \downarrow \\ \mathcal{P}^N & & & \xrightarrow{\mathfrak{f}} & & & M(N). \end{array}$$

Proof: Let \mathcal{T} be a Delaunay triangulation of \tilde{P} . Let ijk and ijl be two triangles in \mathcal{T} abutting along the edge ij . Let $(X, \hat{d}_{\tilde{P}}, \{h_i\})_{\mu} = F \circ \mathfrak{g}(\tilde{P})$, by Lemma 6 the shear coordinates of the decorated ideal polyhedral surface $(X, \hat{d}_{\tilde{P}}, \{h_i\})_{\mu}$ along the edge ij is

$$\log \frac{\varphi_{\mathcal{T}}((X, \hat{d}_{\tilde{P}}, \{h_i\})_{\mu})_{il}}{\varphi_{\mathcal{T}}((X, \hat{d}_{\tilde{P}}, \{h_i\})_{\mu})_{ik}} : \frac{\varphi_{\mathcal{T}}((X, \hat{d}_{\tilde{P}}, \{h_i\})_{\mu})_{jl}}{\varphi_{\mathcal{T}}((X, \hat{d}_{\tilde{P}}, \{h_i\})_{\mu})_{jk}}.$$

But $\mathfrak{g}(\tilde{P})$ lies in $D_{PL}(\mathcal{T})$, hence

$$(X, \hat{d}_{\tilde{P}}, \{h_i\})_{\mu} = F_{\mathcal{T}}((S, d_{\tilde{P}})_{\mu}) = \varphi_{\mathcal{T}}^{-1} \vartheta_{\mathcal{T}}((S, d_{\tilde{P}})_{\mu}).$$

Hence,

$$\log \frac{\varphi_{\mathcal{T}}((X, \hat{d}_{\tilde{P}}, \{h_i\})_{\mu})_{il}}{\varphi_{\mathcal{T}}((X, \hat{d}_{\tilde{P}}, \{h_i\})_{\mu})_{ik}} : \frac{\varphi_{\mathcal{T}}((X, \hat{d}_{\tilde{P}}, \{h_i\})_{\mu})_{jl}}{\varphi_{\mathcal{T}}((X, \hat{d}_{\tilde{P}}, \{h_i\})_{\mu})_{jk}} = \quad (4)$$

$$\log \frac{\vartheta_{\mathcal{T}}((S, d_{\tilde{P}})_{\mu})_{il}}{\vartheta_{\mathcal{T}}((S, d_{\tilde{P}})_{\mu})_{ik}} : \frac{\vartheta_{\mathcal{T}}((S, d_{\tilde{P}})_{\mu})_{jl}}{\vartheta_{\mathcal{T}}((S, d_{\tilde{P}})_{\mu})_{jk}}. \quad (5)$$

By Proposition 2, (5) is exactly the coordinate description of the hyperbolic structure on $\mathfrak{f}(P)$ if we fix the triangulation \mathcal{T} on P and $\mathfrak{f}(P)$. \blacksquare

Let us return to the main theorem of this section.

Theorem 5. *Let P and Q be two polyhedra that share a common Delaunay triangulation \mathcal{T} , then P and Q are discrete-conformally equivalent if and only if there exist two lifts \tilde{P} and \tilde{Q} and a real valued function $u_{\mathcal{T}}$ on the vertices of \tilde{P} so that, if e is an edge in \mathcal{T} between the vertices i and j , then the length $l_{\tilde{P}}(e)$ and $l_{\tilde{Q}}(e)$ of e in \tilde{P} and \tilde{Q} are related by*

$$l_{\tilde{Q}}(e) = l_{\tilde{P}}(e) e^{\frac{1}{2}(u_{\mathcal{T}}(i) + u_{\mathcal{T}}(j))}.$$

Proof: Suppose there exist two lifts \tilde{P} and \tilde{Q} and $u_{\mathcal{T}}$ such that $u_{\mathcal{T}} * \tilde{P} = \tilde{Q}$. By Proposition 2 the hyperbolic structure on $(X, \hat{d}_{\tilde{Q}})_{\mu}$ is the unique complete hyperbolic structure with shear

$$\log \frac{\vartheta_{\mathcal{T}}((S, d_{\tilde{Q}})_{\mu})_{il}}{\vartheta_{\mathcal{T}}((S, d_{\tilde{Q}})_{\mu})_{ik}} : \frac{\vartheta_{\mathcal{T}}((S, d_{\tilde{Q}})_{\mu})_{jl}}{\vartheta_{\mathcal{T}}((S, d_{\tilde{Q}})_{\mu})_{jk}}$$

along the edge ij of \mathcal{T} . If there exists a conformal factor $u_{\mathcal{T}}$ such that $\tilde{Q} = u_{\mathcal{T}} * \tilde{P}$, this number equals

$$\log \frac{e^{\frac{1}{2}(u_{\mathcal{T}}(i)+u_{\mathcal{T}}(l))} \vartheta_{\mathcal{T}}((S, d_{\tilde{P}})_{\mu})_{il}}{e^{\frac{1}{2}(u_{\mathcal{T}}(i)+u_{\mathcal{T}}(k))} \vartheta_{\mathcal{T}}((S, d_{\tilde{P}})_{\mu})_{ik}} : \frac{e^{\frac{1}{2}(u_{\mathcal{T}}(j)+u_{\mathcal{T}}(l))} \vartheta_{\mathcal{T}}((S, d_{\tilde{P}})_{\mu})_{jl}}{e^{\frac{1}{2}(u_{\mathcal{T}}(j)+u_{\mathcal{T}}(k))} \vartheta_{\mathcal{T}}((S, d_{\tilde{P}})_{\mu})_{jk}},$$

which equals

$$\log \frac{\vartheta_{\mathcal{T}}((S, d_{\tilde{P}})_{\mu})_{il}}{\vartheta_{\mathcal{T}}((S, d_{\tilde{P}})_{\mu})_{ik}} : \frac{\vartheta_{\mathcal{T}}((S, d_{\tilde{P}})_{\mu})_{jl}}{\vartheta_{\mathcal{T}}((S, d_{\tilde{P}})_{\mu})_{jk}}.$$

Hence, $\mathfrak{f}(P) = \mathfrak{f}(Q)$, that means P and Q are discrete-conformally equivalent. If P and Q are discrete-conformally equivalent, i.e. $\mathfrak{f}(P) = \mathfrak{f}(Q)$, then there exist lifts \tilde{P} and \tilde{Q} such that $\tilde{\mathfrak{f}}(\tilde{P}) = \tilde{\mathfrak{f}}(\tilde{Q})$. In particular, there exists an isometric map f from $(X, \hat{d}_{\tilde{P}})_{\mu}$ to $(X', \hat{d}_{\tilde{Q}})_{\mu}$ homotopic to the identity. Thus, we obtain a marked ideal polyhedral surface with two decorations $(X', \hat{d}_{\tilde{Q}}, \{h'_i\}, \{f(h_i)\})_{\mu}$. Notice that since f is homotopic to the identity, $f(h_i)$ is a horoball in $(X', \hat{d}_{\tilde{Q}})_{\mu}$ at the i -th cone point.

Let $\lambda_{\tilde{P} \rightarrow \tilde{Q}}^i$ be the signed distance between the horoballs h'_i and $f(h_i)$ at the i -th cone point in $(X', \hat{d}_{\tilde{Q}})_{\mu}$, which is negative if and only if the horoball $f(h_i)$ is smaller than the horoball h'_i . Given an edge ij of \mathcal{T} , the signed distances between horoballs $\lambda_{ij}^{\tilde{P}} = \lambda(h_i, h_j)$ and $\lambda_{ij}^{\tilde{Q}} = \lambda(h'_i, h'_j)$ are related by

$$\lambda_{ij}^{\tilde{Q}} = \lambda_{ij}^{\tilde{P}} + \lambda_{\tilde{P} \rightarrow \tilde{Q}}^i + \lambda_{\tilde{P} \rightarrow \tilde{Q}}^j.$$

In particular,

$$e^{\lambda_{ij}^{\tilde{Q}}/2} = e^{\lambda_{ij}^{\tilde{P}}/2} e^{\frac{1}{2}(\lambda_{\tilde{P} \rightarrow \tilde{Q}}^i + \lambda_{\tilde{P} \rightarrow \tilde{Q}}^j)}.$$

By definition $F \circ \mathfrak{g}(\tilde{P}) = \varphi_{\mathcal{T}}^{-1} \vartheta_{\mathcal{T}}((S, d_{\tilde{P}})_{\mu})$, thus

$$e^{\lambda_{ij}^{\tilde{P}}/2} = \varphi_{\mathcal{T}}((X, \hat{d}_{\tilde{P}}, \{h_i\})_{\mu})_{ij} = \vartheta_{\mathcal{T}}((S, d_{\tilde{P}})_{\mu})_{ij} = l_{\tilde{P}}(ij)$$

and likewise $e^{\lambda_{ij}^{\tilde{Q}}/2} = l_{\tilde{Q}}(ij)$. Hence, if we define

$$u_{\mathcal{T}}(i) := \lambda_{\tilde{P} \rightarrow \tilde{Q}}^i,$$

for every vertex $i = 1, \dots, N$ of the polyhedron \tilde{P} , then $u_{\mathcal{T}}$ is a conformal factor satisfying $u_{\mathcal{T}} * \tilde{P} = \tilde{Q}$. \blacksquare

Alternative proof of characterization theorem. It would be convenient to construct the function $u_{\mathcal{T}}$ directly, without passing through Penner's theory of decorated Teichmüller spaces. Given a triangle ijk in the triangulation \mathcal{T} and a path γ in \mathcal{T} that contains the edges ik and jk , one can transform

γ to a path γ' in \mathcal{T} by reflecting along the edge ij . Such a transformation is called a *discrete homotopy*. We say that a triangulation \mathcal{T} is simply connected, if every even loop can be transformed by discrete homotopies to the constant loop.

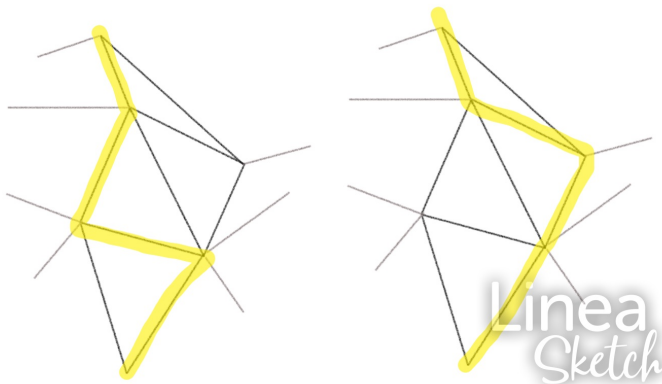


Figure 1: Illustration of a discrete homotopy

In the special case where two polyhedra P and Q share a common simply-connected Delaunay triangulation \mathcal{T} , we can prove the analogue weaker statement of Theorem 5.

Proof: If P and Q are discrete-conformally equivalent, then there exist two lifts \tilde{P} and \tilde{Q} that have equal shear along every edge of \mathcal{T} . In other words

$$\log \frac{l_{\tilde{P}}(il)}{l_{\tilde{P}}(ik)} : \frac{l_{\tilde{P}}(jl)}{l_{\tilde{P}}(jk)} = \log \frac{l_{\tilde{Q}}(il)}{l_{\tilde{Q}}(ik)} : \frac{l_{\tilde{Q}}(jl)}{l_{\tilde{Q}}(jk)}.$$

Rearrangement gives

$$\log \frac{l_{\tilde{P}}(il)}{l_{\tilde{Q}}(il)} - \log \frac{l_{\tilde{P}}(ik)}{l_{\tilde{Q}}(ik)} + \log \frac{l_{\tilde{P}}(jk)}{l_{\tilde{Q}}(jk)} - \log \frac{l_{\tilde{P}}(jl)}{l_{\tilde{Q}}(jl)} = 0.$$

If we assign to every edge in \mathcal{T} the value

$$y_{ij} := \log \frac{l_{\tilde{P}}(ij)}{l_{\tilde{Q}}(ij)},$$

then their alternating sum around two abutting triangles vanishes. Notice that for any even loop in \mathcal{T} , the alternating sum of y_{ij} 's along the loop does not change by discrete homotopies. Since every even loop in \mathcal{T} may be changed by a discrete homotopy to a path around two abutting triangles, the alternating sum of y_{ij} 's vanishes around every even loop in \mathcal{T} .

Fix a vertex 0 and an adjacent vertex 1 in \mathcal{T} . Define a function $u_{\mathcal{T}}^0$ by the following system of equations

$$u_{\mathcal{T}}^0(i) - u_{\mathcal{T}}^0(0) = \sum_{n=1} (-1)^n y_{k_n l_n} \quad (6)$$

$$u_{\mathcal{T}}^0(0) + u_{\mathcal{T}}^0(1) = y_{01}, \quad (7)$$

where the alternating sum in the first equation is along an even path from 0 to i . Analogously, define an function $u_{\mathcal{T}}^1$ by the following system of equations

$$u_{\mathcal{T}}^1(i) - u_{\mathcal{T}}^1(1) = \sum_{n=1} (-1)^n y_{k_n l_n} \quad (8)$$

$$u_{\mathcal{T}}^1(0) + u_{\mathcal{T}}^1(1) = y_{01}, \quad (9)$$

where the alternating sum in the first equation is along an even path from 1 to i .

We like to know how $u_{\mathcal{T}}^0$ and $u_{\mathcal{T}}^1$ differ at a point i . Pick a point k adjacent to 0 and 1, then

$$(u_{\mathcal{T}}^0(i) - u_{\mathcal{T}}^0(0)) - (u_{\mathcal{T}}^1(i) - u_{\mathcal{T}}^1(1)) + y_{0k} - y_{k1}$$

is an alternating sum along an even loop in \mathcal{T} . Hence, it vanishes and

$$(u_{\mathcal{T}}^0(i) - u_{\mathcal{T}}^0(0)) - (u_{\mathcal{T}}^1(i) - u_{\mathcal{T}}^1(1)) = -y_{0k} + y_{k1}.$$

The right-hand-side is an alternating sum along an even path from 0 to 1, hence

$$(u_{\mathcal{T}}^0(i) - u_{\mathcal{T}}^0(0)) - (u_{\mathcal{T}}^1(i) - u_{\mathcal{T}}^1(1)) = u_{\mathcal{T}}^0(1) - u_{\mathcal{T}}^0(0).$$

Rearrangement gives

$$u_{\mathcal{T}}^1(i) - u_{\mathcal{T}}^1(1) = u_{\mathcal{T}}^0(i) - u_{\mathcal{T}}^0(1). \quad (10)$$

Let i and j be two adjacent vertices in \mathcal{T} ,

$$(u_{\mathcal{T}}^0(i) - u_{\mathcal{T}}^0(0)) + (u_{\mathcal{T}}^1(j) - u_{\mathcal{T}}^1(1)) + y_{01} - y_{ij}$$

is an alternating sum along an even loop in \mathcal{T} . Hence, it vanishes. Using equality (10) we obtain

$$u_{\mathcal{T}}^0(i) + u_{\mathcal{T}}^0(j) + y_{01} - u_{\mathcal{T}}^0(0) - u_{\mathcal{T}}^0(1) = y_{ij}$$

and since

$$u_{\mathcal{T}}^0(0) + u_{\mathcal{T}}^0(1) = y_{01},$$

we obtain

$$u_{\mathcal{T}}^0(i) + u_{\mathcal{T}}^0(j) = y_{ij}.$$

Hence, $u_{\mathcal{T}}$ is a function on the vertices of \mathcal{T} such that for every edge ij of \mathcal{T} ,

$$l_{\bar{P}}(ij) = l_{\bar{Q}}(ij) e^{u_{\mathcal{T}}^0(i) + u_{\mathcal{T}}^0(j)}.$$

■

Concepts of discrete conformality. Let $\mathcal{P}_{\text{ideal}}$ be the space of ideal convex polyhedra in \mathbb{H}^3 . Using the Klein model of the hyperbolic 3-space, let us interpret $\mathcal{P}_{\text{ideal}}$ as the space of convex Euclidean polyhedra inscribed in the unit sphere.

There exists a beautiful correspondence between convex Euclidean polyhedra inscribed in the unit sphere and circles covering the unit sphere.

Let \mathcal{C} be the set of circle patterns covering the unit sphere. To every circle pattern C in \mathcal{C} corresponds a unique convex Euclidean polyhedron P_C in $\mathcal{P}_{\text{ideal}}$ by cutting off all half-planes defined by the circles in C . Conversely, every convex polyhedron inscribed in the unit sphere corresponds to a unique circle pattern covering the unit sphere.

This allows us to relate the equivalence of circle patterns with the discrete-conformal equivalence of convex Euclidean polyhedra inscribed in the sphere by using purely terms from Euclidean geometry.

Proposition 6. *Let C_1 and C_2 be two circle patterns in \mathcal{C} and let P_{C_1} and P_{C_2} be the corresponding convex polyhedra inscribed in the unit sphere. There exists a Möbius transformation f on the sphere mapping the circle pattern C_1 onto the circle pattern C_2 if and only if P_{C_1} and P_{C_2} share a common Delaunay triangulation \mathcal{T} and there exists a function $u_{\mathcal{T}}$ defined on the vertex set of P_{C_1} such that for every edge ij in the Delaunay triangulation between vertices i and j , its length in P_{C_2} is related to its length in P_{C_1} by*

$$l_{P_{C_2}}(ij) = l_{P_{C_1}}(ij) e^{\frac{1}{2}(u_{\mathcal{T}}(i)+u_{\mathcal{T}}(j))}.$$

Moreover, f and $u_{\mathcal{T}}$ are related by $u_{\mathcal{T}} = \log |df|_V$, where V is the vertex set of P_{C_1} .

Proof: Thanks to the Uniformization-Theorem 4, it only remains to prove the relation of $u_{\mathcal{T}}$ and the Möbius transformation f . Let x and y be two distinct vertices of P_{C_1} . Let $\{x_n\}$ and $\{y_n\}$ be sequences on the unit sphere converging to x and y , respectively, but not containing the points x and y . The Euclidean length cross-ratio is invariant under Möbius transformations on the sphere. Hence

$$\frac{|x - x_n|}{|x - y_n|} \cdot \frac{|y - x_n|}{|y - y_n|} = \frac{|f(x) - f(x_n)|}{|f(x) - f(y_n)|} \cdot \frac{|f(y) - f(x_n)|}{|f(y) - f(y_n)|}.$$

A rearrangement gives

$$\frac{|f(x) - f(y_n)|}{|x - y_n|} \frac{|f(y) - f(x_n)|}{|y - x_n|} = \frac{|f(x) - f(x_n)|}{|x - x_n|} \frac{|f(y) - f(y_n)|}{|y - y_n|}.$$

Taking the limit $n \rightarrow \infty$ results in

$$\frac{|f(x) - f(y)|^2}{|x - y|^2} = |df(x)||df(y)|.$$

This shows that P_{C_1} and P_{C_2} are discrete-conformally equivalent with $u_{\mathcal{T}} = \log |df|_V$. \blacksquare

4. DIRECTIONS OF FURTHER RESEARCH

Characterization of discrete conformality. It would be more elegant to have a definition of discrete-conformal equivalence of convex polyhedra by elementary transformations on vertices. We conjecture that P and Q are discrete-conformally equivalent if and only if there exists a finite sequence of closed convex polyhedra $P = P_1, P_2, \dots, P_{n-1}, P_n = Q$ such that, for

$k = 1, \dots, n - 1$ the polyhedra P_k and P_{k+1} share a common Delaunay triangulation \mathcal{T}_k and there exists a real valued function $u_{\mathcal{T}_k}$ on the vertices of P_k with the following property. For every edge ij in the Delaunay triangulation between vertices i and j , its length in P_{k+1} is related to its length in P_k by

$$l_{P_{k+1}}(ij) = l_{P_k}(ij) e^{\frac{1}{2}(u_{\mathcal{T}_k}(i) + u_{\mathcal{T}_k}(j))}.$$

The statement is difficult to prove because the function \tilde{f} can be discontinuous when passing from a “cell” $D_{PL}(\mathcal{T})$ to another. This discontinuity arises because a Delaunay triangulation of a Euclidean convex polyhedron P is not a Delaunay triangulation of the associated marked polyhedral surface $(S, d_P)_\mu$.

Delaunay triangulations were originally introduced to the topic of polyhedral surfaces by Luo in [13]. Using Delaunay triangulations of polyhedral surfaces, Luo proves that the variational principal of Bobenko, Pinkall and Springborn always has a solution [9]. In this paper we are using a different triangulation than Luo to construct an ideal polyhedral surface from a polyhedron. This allows us to give a uniformization theorem for polyhedra. Its disadvantage is, that it becomes more difficult to completely characterize this concept of discrete conformality by elementary transformations.

Variational principles. The uniformization theory of convex polyhedra may shed some light on the relationships between the different variational principles developed in the context of discrete conformality. Glickenstein suggested a formal framework in [8]. The natural appearance of real analytic cell decompositions in the work of Gu, Luo, Sun and Wu [9], may suggest the theory of moment maps as a general setting. According to Atiyah, Guillemin and Sternberg, the image of the moment map of a hamiltonian torus action on a compact connected symplectic manifold is always a polytope [2] [10].

Möbius geometry. Theorem 6 suggests a third variant of discrete conformality, namely *discrete Möbius geometry*.

Roughly speaking, a Möbius structure on a set X is an equivalence class of metrics on X , where two metrics are equivalent if they define the same crossratio. Let \mathcal{M} be a Möbius structure on a set X . The pair (X, \mathcal{M}) is called a Möbius space.

If X is a strongly hyperbolic metric space, then its ideal boundary carries a natural Möbius structure as observed by Nica and Spakula.

Let X be a finite set, let d_f be the pull-back metric of the Euclidean distance on X induced by an embedding f of X into the sphere. One could ask the following question.

Question 1. The metric spaces (X, d_{f_1}) and (X, d_{f_2}) are Möbius equivalent if and only if the associated convex polyhedra P_{f_1} and P_{f_2} are discrete-conformally equivalent.

Notice that if (X, d_{f_1}) and (X, d_{f_2}) are Möbius equivalent, then $\mathfrak{f}((X, d_{f_1}))$ and $\mathfrak{f}((X, d_{f_2}))$ are not obviously isometric since \mathfrak{f} depends on a choice of a triangulation which is initially not given. However, if P_{f_1} and P_{f_2} are discrete-conformally equivalent, then there exists a Möbius transformation on the sphere mapping the vertex set of P_{f_1} onto the vertex set of P_{f_2} , hence (X, d_{f_1}) and (X, d_{f_2}) are Möbius equivalent.

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