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Asymptotic behavior for a time-inhomogeneous stochastic differential equation driven by an α-stable Lévy process

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Abstract: We study a one-dimensional kinetic stochastic model driven by a Lévy process, with a non-linear time-inhomogeneous drift. More precisely, the process (V, X) is considered, where X is the position of the particle and its velocity V is the solution of a stochastic differential equation with a drift of the form $t^{-\beta}F(v)$. The driving noise can be a stable Lévy process of index α or a general Lévy process under appropriate assumptions. The function F satisfies a homogeneity condition and β is non-negative. The behavior in large time of the process (V, X) is proved and the precise rate of convergence is pointed out by using stochastic analysis tools. To this end, we compute the moment estimates of the velocity process.
Keywords: kinetic stochastic equation; time-inhomogeneous stochastic differential equation; Lévy process; explosion times; scaling transformations; asymptotic distributions; ergodicity; tightness.
MSC2020 Subject Classification: Primary 60H10; Secondary 60F05; 60G52; 60G18; 60J65.

1 Introduction

In this paper, we consider a one-dimensional stochastic kinetic model driven by a Lévy process.

$$\mathrm{d}V_t = \mathrm{d}L_t - F(V_t)t^{-\beta}\,\mathrm{d}t \quad \text{and} \quad X_t = X_0 + \int_0^t V_s\,\mathrm{d}s. \tag{1}$$

The process $(V_t, X_t)_{t\geq 0}$ may be thought of as the velocity and position processes of a particle evolving in a force field $F(v)t^{-\beta}$ and interacting with its environment.

The study of Stochastic Differential Equation (SDE) driven by a Lévy process is a new topic of great interest (see [Bas03] for a survey). So far, most of the papers present results about existence and uniqueness of solution, see for instance [AS09], [Don18], [Kur07], [Pil13], [CSZ18] and [CZZ17]. Furthermore, the coefficients of the studied SDE are often supposed to be time-homogeneous (see for instance [AS09] and [Don18]). Accordingly, the case of time-dependent coefficients is scarcely studied (see [CZZ17], [Kur07] and [Zha13]). In this situation, the usual tools associated with time-homogeneous equation may no longer be invoked.
Otherwise, few papers (see [AS09], [PSXZ12] and [Rek20]) state results about the asymptotic behavior of such SDEs. For instance, in [AS09] the authors give conditions for asymptotic stability for a SDE driven by a Brownian motion and a compensated Poisson process, with coefficients that are supposed to satisfy usual global Lipschitz and growth assumptions. In [PSXZ12], the authors establish the exponential ergodicity of the solutions of a SDE driven by an α-stable process, where the drift coefficient is supposed to be the sum of two components one linear and the other bounded. In these papers, coefficients are time-homogeneous.

The purpose of the present paper is to study the long-time behavior of solutions of the SDE (1), driven by an α-stable (non-symmetric) Lévy process. More precisely, we look for the convergence in distribution of the process $(V_{t/\varepsilon}, \varepsilon X_{t/\varepsilon})_{t>0}$, as $\varepsilon \to 0$, with an appropriate rate. Consequently, we extend the results obtained in [GL21], where the driving process is a Brownian motion ($\alpha = 2$).

It is a simple observation, when $F = 0$, to see that the rescaled process $(\varepsilon^{1/\alpha} V_{t/\varepsilon}, \varepsilon^{1+1/\alpha} X_{t/\varepsilon})_{t>0}$ converges in distribution towards the Kolmogorov process $(\mathcal{S}_t, \int^t \mathcal{S}_s \, \mathrm{d}s)_{t>0}$, where $\mathcal{S}$ has the same distribution as the driving process.
And more generally, in long-time regime, we observe two schemes, depending on the balance between the space and time coefficients of the drift function with respect to α, the parameter of stability of the driving noise. When the potential is sufficiently "small at infinity", the convergence towards the Kolmogorov process $(\mathcal{S}, \int \mathcal{S})$ still holds. The other scheme, called critical, is when the two terms in the stochastic equation of the velocity process offset. We still get a kinetic process of the form $(\mathcal{V}, \int \mathcal{V})$, as limiting process. Though the process $\mathcal{V}$ no longer has the same distribution as the driving Lévy process.

Proofs are mainly based on moment estimates and on the self-similarity of the driving process. By their scaling property, Lévy stable processes are natural extensions of the Brownian motion. However, the jump component of the Lévy noise brings difficulties. Indeed, by contrast with a Brownian motion, an α-stable Lévy process can only have moments of order $\kappa \in [0, \alpha)$. Thus, moment estimation of the velocity process stands as a significant part of our study (see Section 4). Moment estimates of Lévy and Lévy-type processes were studied in [LP06], [Küh17] and [DS15]. Nevertheless, the methods used can not be easily adapted to the solutions of a SDE.
In addition, an extension for the solution of SDE driven by a general Lévy process is stated in Theorems 6.1 and 6.2. To this end, we first study the asymptotic behavior of the rescaled Lévy driving process $(v_\varepsilon L_{t/\varepsilon})_{t\geq 0}$, where v_ε is some rate of convergence. Under appropriate assumptions on its Lévy measure, it converges in distribution, as ε goes to zero (see Proposition 6.3).

The paper is organized as follows. In Section 2, we introduce some notations and state our main results. Theorems 2.2 and 2.4 correspond to the main results stated in [GL21] in case of a Brownian driving noise. We study the existence of the solution up to explosion of (1) in Section 3. In Section 4, we give estimates of the moment, which also ensure the non-explosion of the velocity process. The proofs of our main results are presented in Section 5. Finally, Section 6 is devoted to the study of the convergence of a rescaled Lévy process and to the extension of our main result.

2 Notations and main results

Let $(L_t)_{t\geq 0}$ be a Lévy process, β a non-negative real number and F a continuous function satisfying

$$\text{for some } \gamma \in \mathbb{R},\ \forall v \in \mathbb{R},\ \lambda > 0,\ F(\lambda v) = \lambda^\gamma F(v). \tag{H_γ}$$

We introduce another assumption on F, which will sometimes be imposed in the sequel.

$$\begin{gathered}\text{When (i) } \alpha \in (0,1) \text{ or (ii) } \alpha \in (1,2) \text{ and } \gamma \geq 1,\\ \text{we suppose furthermore that for all } v \in \mathbb{R},\ vF(v) \geq 0.\end{gathered} \tag{H_{sgn}}$$

We take an interest into the following one-dimensional stochastic kinetic model defined, for $t \geq t_0 > 0$, by

$$\mathrm{d}V_t = \mathrm{d}L_t - t^{-\beta} F(V_t)\,\mathrm{d}t, \text{ with } V_{t_0} = v_0 > 0, \quad \text{and} \quad \mathrm{d}X_t = V_t\,\mathrm{d}t, \text{ with } X_{t_0} = x_0 \in \mathbb{R}. \tag{SKE}$$

Throughout the paper, we deal with L as an α-stable Lévy process with $\alpha \in (0,2) \setminus \{1\}$. We call ν its Lévy measure, given by $\nu(\mathrm{d}z) = |z|^{-1-\alpha}\left[a_+ \mathbb{1}_{z>0} + a_- \mathbb{1}_{z<0}\right] \mathrm{d}z$ with $a_+, a_- \geq 0$. As a Lévy measure, it satisfies $\int_{\mathbb{R}^*} (1 \wedge z^2)\nu(\mathrm{d}z) < +\infty$. By Lévy-Itô's decomposition, there exists a Poisson point process N and its compensated Poisson measure $\widetilde{N}$ such that, for all $t \geq 0$,

$$L_t = \int_0^t \int_{\{0<|z|<1\}} z\widetilde{N}(\mathrm{d}s, \mathrm{d}z) + \int_0^t \int_{\{|z|\geq 1\}} zN(\mathrm{d}s, \mathrm{d}z). \tag{2}$$

In Section 6, the case of a generalized Lévy driving noise will be discussed.
In the following, sgn is the sign function with the convention that $\mathrm{sgn}(0) = 0$. The abbreviation a.s. stands for almost surely. We denote by C some positive constants, which may change from line to line. We indicate the dependencies in index. For instance $C_{t_0,\alpha}$ denotes a constant depending on the parameters t_0 and α.

Remark 2.1. *If a function π satisfies (H_γ), then for all $x \in \mathbb{R}$, $\pi(x) = \pi(\mathrm{sgn}(x))\,|x|^\gamma$.*

As an example of a function satisfying (H_γ) one can keep in mind $F : v \mapsto \mathrm{sgn}(v)\,|v|^\gamma$ (see also [GO13]).
The space of continuous functions $\mathcal{C}((0,+\infty),\mathbb{R})$ is endowed with the uniform topology

$$\mathrm{d}_u : f, g \in \mathcal{C}((0,+\infty)) \mapsto \sum_{n=1}^{+\infty} \frac{1}{2^n} \min\Big(1, \sup_{[\frac{1}{n},n]} |f-g|\Big).$$

Set $\Lambda := \{\lambda : \mathbb{R}^+ \to \mathbb{R}^+, \text{continuous and increasing function such that } \lambda(0) = 0, \lim_{t\to+\infty} \lambda(t) = +\infty\}$ and

$$k_n(t) = \begin{cases} 1, & \text{if } \frac{1}{n} \leq t \leq n, \\ n+1-t, & \text{if } n < t < n+1, \\ 0, & \text{if } n+1 \leq t. \end{cases}$$

The space of right-continuous with left limits (càdlàg) functions $\mathcal{D}((0,+\infty))$ is endowed with the Skorokhod topology d_s defined for $f, g \in \mathcal{D}((0,+\infty))$ by

$$\sum_{n=1}^{+\infty} \frac{1}{2^n} \min\left(1, \inf\left\{a \text{ such that } \exists \lambda \in \Lambda,\ \sup_{s\neq t}\left|\log\frac{\lambda(t)-\lambda(s)}{t-s}\right| \leq a,\ \sup_{t>0} |k_n(t)\,(f\circ\lambda(t) - g(t))| \leq a\right\}\right).$$

For simplicity, we shall write $\mathcal{C}$ and $\mathcal{D}$ for $\mathcal{C}((0,+\infty),\mathbb{R})$ and $\mathcal{D}((0,+\infty))$, respectively.
Let us state our main results.

Theorem 2.2. *Consider $\gamma \in (1-\frac{\alpha}{2}, \alpha)$. Define*

$$p(\gamma) := \begin{cases} \frac{\gamma}{\alpha}, & \text{if } \gamma \in [0,1), \\ \gamma, & \text{if } \gamma = 1, \\ \frac{\gamma}{\alpha} + \frac{\gamma}{2}, & \text{if } \gamma \in (1,\alpha). \end{cases}$$

Assume that (H_γ) and (H_{sgn}) are satisfied, and $\beta > 1 + p(\gamma) - \frac{1}{\alpha}$. Let $(V_t, X_t)_{t\geq t_0}$ be the solution to (SKE) *and $(\mathcal{S}_t)_{t\geq 0}$ be an α-stable process.*
Then, as $\varepsilon \to 0$, in the space $\mathcal{D}$,

$$(\varepsilon^{1/\alpha} V_{t/\varepsilon}, \varepsilon^{1+1/\alpha} X_{t/\varepsilon})_{t\geq \varepsilon t_0} \quad \Longrightarrow \quad \left(\mathcal{S}_t, \int_0^t \mathcal{S}_s\,\mathrm{d}s\right)_{t>0}.$$

Remark 2.3. *Theorem 2.2 is also true when the following hypothesis holds instead of* (H_γ).

$$\begin{gathered} F \text{ is such that (SKE) has a unique solution up to explosion and} \\ |F| \leq G \text{ where } G \text{ is a positive function satisfying } (H_\gamma). \end{gathered} \tag{H^γ_{bis}}$$

For instance, the function $F : v \mapsto \frac{v}{(1+v^2)}$ *(see also [FT21]) satisfies* (H^γ_{bis}) *(with* $\gamma = 0$*).*

Theorem 2.4. *Consider* $\gamma \in (1 - \frac{\alpha}{2}, \alpha)$ *and* $\beta = 1 + \frac{\gamma-1}{\alpha}$. *Assume that* (H_γ) *and* (H_{sgn}) *are satisfied. Let* $(V_t, X_t)_{t\geq t_0}$ *be the solution to* (SKE).
Call $\widetilde{H}$ *the ergodic process solving the following SDE driven by an* α*-stable process* $(R_t)_{t\geq 0}$ *and starting at its invariant measure,*

$$\mathrm{d}H_s = \mathrm{d}R_s - \frac{H_s}{\alpha}\,\mathrm{d}s - F\big(H_s\big)\,\mathrm{d}s.$$

We denote $\Lambda_{F,t_1,\cdots,t_d}$ *the finite-dimensional distributions of* $\widetilde{H}$. *We call* $(\mathcal{V}_t)_{t\geq 0}$ *the process having as finite-dimensional distributions the pushforward measure of* $\Lambda_{F,\log(t_1/t_0),\cdots,\log(t_d/t_0)}$ *by the linear map* $T(u_1, \cdots, u_d) := (t_1^{1/\alpha}u_1, \cdots, t_d^{1/\alpha}u_d)$.
Then, under (H_γ), *as* $\varepsilon \to 0$, *the following convergence of finite-dimensional distributions (f.d.d.), in the space* $\mathcal{D}$, *holds,*

$$(\varepsilon^{1/\alpha}V_{t/\varepsilon}, \varepsilon^{1+1/\alpha}X_{t/\varepsilon})_{t\geq \varepsilon t_0} \quad \overset{f.d.d.}{\Longrightarrow} \quad \left(\mathcal{V}_t, \int_0^t \mathcal{V}_s\,\mathrm{d}s\right)_{t>0}.$$

There is convergence as a process, provided that $\gamma < 1$.

Remark 2.5. *As we will see in Section 3, the assumption* $\gamma > 1 - \frac{\alpha}{2}$ *is needed in order to obtain the existence up to explosion of the solution under hypothesis* (H_γ).

Remark 2.6. *Let us point out that, during the proof of Theorem 2.2 and Theorem 2.4, we employ some moment estimates for the solution* V. *We state the estimates below.*
Assume that (H_{sgn}) *is satisfied. Define*
for $\alpha \in (0, 1)$,

$$p_\alpha(\gamma, \kappa) := \frac{\kappa}{\alpha};$$

and for $\alpha \in (1, 2)$,

$$p_\alpha(\gamma, \kappa) := \begin{cases} \frac{\kappa}{\alpha}, & \text{when } \gamma \in [0, 1) \text{ and } \kappa \in [0, 1], \\ \frac{3\kappa}{2\alpha}, & \text{when } \gamma \in [0, 1),\ \kappa \in (1, \alpha) \text{ and } vF(v) \geq 0, \\ \kappa, & \text{when } \gamma \geq 1 \text{ and } \kappa \in [0, 1], \\ \frac{\kappa}{2} + \frac{\kappa}{\alpha}, & \text{when } \gamma \geq 1 \text{ and } \kappa \in (1, \alpha). \end{cases}$$

Then, for any $\alpha \in (0, 2) \setminus \{1\}$, $\gamma \in \mathbb{R}$, $\beta \in \mathbb{R}$ *and* $\kappa \in [0, \alpha)$, *there exists a constant* $C_{\gamma,\kappa,\beta,t_0}$ *such that,*

$$\forall t \geq t_0,\ \mathbb{E}\left[|V_t|^\kappa\right] \leq C_{\gamma,\kappa,\beta,t_0} t^{p_\alpha(\gamma,\kappa)}.$$

3 Existence up to explosion of a solution to (SKE)

In this section, we study the existence of the solution to (SKE) up to explosion time.

Lemma 3.1. *Assume that* (H_γ) *holds. Pick* $\beta \geq 0$*. If* $0 < \gamma < 1$*, the function* $x \mapsto F(x)t^{-\beta}$ *is* γ*-Hölder and if* $\gamma \geq 1$*, it is locally Lipschitz.*

Proof. Suppose that $\beta \geq 0$. Since (H_γ) holds, the function F is of the form given in Remark 2.1. If $\gamma > 1$ then F is of class $\mathcal{C}^1$ and hence is locally Lipschitz.
Suppose now that $\gamma \leq 1$. It is enough to prove that F is γ-Hölder, that is there exists a positive constant C, such that

$$\text{for all } x, y \in \mathbb{R},\ |F(x) - F(y)| \leq C\,|x-y|^\gamma\,.$$

Let first sketch the proof when $F(x) = \rho\,\mathrm{sgn}(x)\,|x|^\gamma$, with $\rho \in \mathbb{R}$. Pick $x, y \in \mathbb{R}$. Assume that $\mathrm{sgn}(x) = \mathrm{sgn}(y)$ thereby, since $x \mapsto |x|^\gamma$ is γ-Hölder,

$$|\mathrm{sgn}(x)\,|x|^\gamma - \mathrm{sgn}(y)\,|y|^\gamma| = ||x|^\gamma - |y|^\gamma| \leq ||x| - |y||^\gamma \leq |x-y|^\gamma\,.$$

Suppose now that $x \leq 0 \leq y$. Thus, $|\mathrm{sgn}(x)\,|x|^\gamma - \mathrm{sgn}(y)\,|y|^\gamma| = |x|^\gamma + |y|^\gamma$. Noting that $|x-y|^\gamma = (|x| + |y|)^\gamma$, the conclusion follows from the concavity of the function $x \mapsto |x|^\gamma$ for $0 < \gamma \leq 1$. This ends up the proof for $F(x) = \rho\,\mathrm{sgn}(x)\,|x|^\gamma$.
If F is an arbitrary function satisfying (H_γ) then, it is of the form given in Remark 2.1. As previously, F is γ-Hölder on $(0, +\infty)$ and $(-\infty, 0)$. Let take $x < 0 < y$, we have

$$|F(x) - F(y)| = |F(-1)\,|x|^\gamma - F(1)\,|y|^\gamma|\,.$$

We split the end of the proof in three cases.

- If $\mathrm{sgn}(F(1)) = \mathrm{sgn}(F(-1))$, we use again the fact that $x \mapsto |x|^\gamma$ is Hölder on $\mathbb{R}$.

$$\begin{aligned}\left|\left|F(-1)|^{1/\gamma}\,x\right|^\gamma - \left|F(1)|^{1/\gamma}\,y\right|^\gamma\right| &\leq C\left|F(-1)|^{1/\gamma}\,x - |F(1)|^{1/\gamma}\,y\right|^\gamma \\ &= C\left|F(-1)|^{1/\gamma}\,|x| + |F(1)|^{1/\gamma}\,|y|\right|^\gamma \\ &\leq C\,(|x| + |y|)^\gamma = C\,|x-y|^\gamma\,.\end{aligned}$$

- If $\mathrm{sgn}(F(1)) = -\,\mathrm{sgn}(F(-1))$, we use the fact that $x \mapsto |x|^\gamma$ is concave on $[0, +\infty)$.

$$\begin{aligned}\left|\left|F(-1)|^{1/\gamma}\,|x|\right|^\gamma + \left|F(1)|^{1/\gamma}\,|y|\right|^\gamma\right| &\leq C\left|F(-1)|^{1/\gamma}\,|x| + |F(1)|^{1/\gamma}\,|y|\right|^\gamma \\ &\leq C\,||x| + |y||^\gamma = C\,|x-y|^\gamma\,.\end{aligned}$$

- If $0 \in \{F(1), F(-1)\}$, say for instance $F(1) = 0$ then, we get

$$|F(-1)\,|x|^\gamma| \leq C\,|x|^\gamma \leq C(|x| + |y|)^\gamma = C\,|x-y|^\gamma\,.$$

□

Proposition 3.2. *Assume that* (H_γ) *is satisfied. There exists a pathwise unique strong solution to* (SKE)*, defined up to the explosion time, provided that*

(i) $1 - \frac{\alpha}{2} < \gamma < 1$ *and* $\beta \geq 0$ *when* $\alpha \in (0, 2)$*.*

(ii) $\gamma \geq 1$ *when* $\alpha > 1$*.*

Proof. If $0 < \gamma < 1$, the drift coefficient is γ-Hölder (see Lemma 3.1) and locally bounded, thereby the conclusion of the first point follows from Remark 1.3 in [CZZ17].
Assume now that $\alpha > 1$ and $\gamma \geq 1$. The drift coefficient is locally Lipschitz (see Lemma 3.1) and locally bounded, so we can apply Lemma 115 p. 78 in [Sit05] to get the pathwise uniqueness. Thanks to Theorem 137 p. 104 in [Sit05], it suffices to prove that there exists a weak solution. The drift coefficient is continuous with respect to its two variables, hence it is a locally bounded and measurable function. By a standard localization argument, using Theorem 9.1 p. 231 in [IW81], since the drift coefficient is locally Lipschitz, there is a unique solution defined up to explosion. □

4 Moment estimates and non-explosion of the velocity process

In this section, we present estimates on moments of the velocity process V solution to (SKE). This will be useful to conclude of the non-explosion of solution to (SKE) with Lemma 4.1, and to control some terms appearing along the proofs of Theorem 2.2 and 2.4 in Section 5.
Let V be the unique solution up to explosion time to (SKE). For all $r \geq 0$, define the stopping time

$$\tau_r := \inf\{t \geq t_0,\ |V_t| \geq r\}. \tag{3}$$

Set $\tau_\infty := \lim_{r\to+\infty} \tau_r$ the explosion time of V. Note that the "pre-stopped" process, defined below, is bounded by r:

$$V_{t\wedge\tau_r-} := \begin{cases} V_t, & \text{if } t < \tau_r, \\ V_{\tau_r-} := \lim\limits_{s\downarrow 0} V_{\tau_r - s}, & \text{otherwise.} \end{cases}$$

We give first a sufficient condition for the non-explosion of a general process.

Lemma 4.1. *Let $(Y_t)_{t\geq t_0}$ be a process and τ_∞ its explosion time. Assume that there exist two measurable functions ϕ and b such that*

(i) $\phi(+\infty) = +\infty$,

(ii) *b is finite-valued,*

(iii) *and for all $t \geq t_0$,*

$$\mathbb{E}\left[\phi(|Y_{t\wedge\tau_\infty}|)\right] \leq b(t). \tag{4}$$

Then $\tau_\infty = +\infty$ a.s.

Proof. Assume by contradiction that $\mathbb{P}(\tau_\infty < +\infty) > 0$. Thus, there exists $T \geq t_0$ such that $\mathbb{P}(\tau_\infty \leq T) > 0$. From (4), we get

$$\mathbb{E}\left[\phi(|V_{\tau_\infty}|)\mathbb{1}_{\{\tau_\infty \leq T\}}\right] \leq b(T). \tag{5}$$

Since $\phi(|V_{\tau_\infty}|) = \phi(+\infty) = +\infty$, on a set of positive measure, the left-hand side of (5) should be infinite. This is a contradiction with *(ii)*. □

We will show that there exists a constant $C_{\gamma,\kappa,\beta,t_0}$ such that

$$\forall t \geq t_0,\ \mathbb{E}\left[|V_t|^\kappa\right] \leq C_{\gamma,\kappa,\beta,t_0} t^{p_\alpha(\gamma,\kappa)}, \tag{6}$$

where $p_\alpha(\gamma,\kappa)$ has to be detailed.

Proposition 4.2. *Pick $\alpha \in (0,1)$. Assume that (H_{sgn}) holds. Recall that $(V_t)_{t\geq t_0}$ is the solution of* (SKE)*. For any γ, β, the explosion time τ_∞ is a.s. infinite and for all $\kappa \in [0,\alpha)$, there exists a constant C_{κ,t_0} such that, we have*

$$\forall t \geq t_0,\ \mathbb{E}\left[|V_t|^\kappa\right] \leq C_{\kappa,t_0} t^{\frac{\kappa}{\alpha}},\ \textit{or equivalently } p_\alpha(\gamma,\kappa) = \frac{\kappa}{\alpha}. \tag{7}$$

Proof. Fix $t \geq t_0$. Since $\alpha < 1$, the stable process can be written as

$$L_t = \int_0^t \int_{\mathbb{R}^*} zN(\mathrm{d}s,\mathrm{d}z) = \sum_{s\leq t} \Delta L_s.$$

Fix $\kappa \in [0,\alpha)$. Pick the sequence of $\mathcal{C}^2$-functions $f_n : x \mapsto \sqrt{x^2+\frac{1}{n}}$, which converges uniformly to $x \mapsto |x|$ on $\mathbb{R}$. Then, for all $n \geq 1$, we apply Itô's formula (see Theorem 32 p. 78 in [Pro05]) to get

$$\begin{aligned} f_n(V_{t\wedge\tau_\infty}) &= f_n(v_0) - \int_{t_0}^{t\wedge\tau_\infty} f_n'(V_s)F(V_s)s^{-\beta}\,\mathrm{d}s + \int_{t_0}^{t\wedge\tau_\infty}\int_{\mathbb{R}^*} \left(f_n(V_{s-}+z) - f_n(V_{s-})\right) N(\mathrm{d}s,\mathrm{d}z) \\ &\leq f_n(v_0) + \sum_{s\leq t\wedge\tau_\infty} \left(f_n(V_{s-}+\Delta L_s) - f_n(V_{s-})\right). \end{aligned}$$

The term $\int_{t_0}^{t\wedge\tau_\infty} f_n'(V_s)F(V_s)s^{-\beta}\,\mathrm{d}s$ is non-negative, since (H_{sgn}) holds.
Hence, the previous equation can be written as

$$f_n(V_{t\wedge\tau_\infty}) \leq f_n(v_0) + \sum_{s\leq t\wedge\tau_\infty} \left(f_n(V_s) - f_n(V_{s-})\right).$$

Since $\|f_n'\|_\infty \leq 1$, we deduce that $\left(f_n(V_s) - f_n(V_{s-})\right) \leq |\Delta V_s| = |\Delta L_s|$, hence,

$$|V_{t\wedge\tau_\infty}| \leq f_n(V_{t\wedge\tau_\infty}) \leq f_n(v_0) + \sum_{s\leq t\wedge\tau_\infty} |\Delta L_s|.$$

Furthermore, since $\kappa < \alpha < 1$, we have

$$|V_{t\wedge\tau_\infty}|^\kappa \leq f_n(v_0)^\kappa + \left(\sum_{s\leq t\wedge\tau_\infty} |\Delta L_s|\right)^\kappa.$$

Taking the expectation, we get

$$\mathbb{E}\left[|V_{t\wedge\tau_\infty}|^\kappa\right] \leq \mathbb{E}\left[f_n(v_0)^\kappa\right] + \mathbb{E}\left[\left(\sum_{s\leq t\wedge\tau_\infty} |\Delta L_s|\right)^\kappa\right] \leq \mathbb{E}\left[f_n(v_0)^\kappa\right] + \mathbb{E}\left[\left(\sum_{s\leq t} |\Delta L_s|\right)^\kappa\right].$$

Notice that the process $L_t^+ := \sum_{s\leq t} |\Delta L_s|$ is an α-stable process. Then, since $\kappa < \alpha$, letting $n \to +\infty$, we obtain

$$\mathbb{E}\left[|V_{t\wedge\tau_\infty}|^\kappa\right] \leq |v_0|^\kappa + \mathbb{E}\left[\left|L_t^+\right|^\kappa\right] \leq C_{t_0,\kappa} t^{\kappa/\alpha}.$$

Thanks to Lemma 4.1, we can conclude that the explosion time of V is a.s. infinite and we get (7). □

Proposition 4.3. *Pick $\alpha \in (1,2)$. Recall that $(V_t)_{t\geq t_0}$ is the solution of* (SKE)*. For any $\gamma \in [0,1)$ and any $\beta \in \mathbb{R}$ such that $\frac{\gamma-1}{\alpha}+1 \leq \beta$, the explosion time τ_∞ is a.s. infinite and for all $\kappa \in [0,1]$, there exists $C_{\gamma,\kappa,\beta,t_0}$ such that we have*

$$\forall t \geq t_0,\ \mathbb{E}\left[|V_{t-}|^\kappa\right] \leq C_{\gamma,\kappa,\beta,t_0} t^{\frac{\kappa}{\alpha}},\ \textit{or equivalently } p_\alpha(\gamma,\kappa) = \frac{\kappa}{\alpha}. \tag{8}$$

Remark 4.4. *If $\frac{\gamma-1}{\alpha}+1 > \beta$, then the moment estimate* (8) *becomes*

$$\forall t \geq t_0,\ \mathbb{E}\left[|V_t|\right] \leq C_{\gamma,\beta,t_0} t^{\frac{1-\beta}{1-\gamma}}.$$

Proof of Proposition 4.3. Assume that $\gamma \in [0,1)$ and fix $\kappa \in [0,1]$. Then Jensen's inequality yields, for all $t \geq t_0$, $\mathbb{E}\left[|V_t|^\kappa\right] \leq \mathbb{E}\left[|V_t|\right]^\kappa$, hence it suffices to verify (8) only for $\kappa = 1$.
Recall that under (H_γ), there exists a positive constant K, such that for all $v \in \mathbb{R}$, $|F(v)| \leq K|v|^\gamma$. Hence, we can write, for any $t \geq t_0$ and $r \geq 0$,

$$\begin{aligned}
|V_{t\wedge\tau_r-}| &\leq |v_0 - L_{t_0}| + |L_{t\wedge\tau_r-}| + \int_{t_0}^{t\wedge\tau_r} s^{-\beta}\left|F(V_{s\wedge\tau_r-})\right| \mathrm{d}s \\
&\leq |v_0 - L_{t_0}| + |L_{t\wedge\tau_r-}| + K\int_{t_0}^{t\wedge\tau_r} s^{-\beta}\left|V_{s\wedge\tau_r-}\right|^\gamma \mathrm{d}s.
\end{aligned}$$

Since L is an α-stable process, it has a finite first moment, which can be computed. Taking the expectation in the above inequality, we get, by choosing C_{t_0} big enough,

$$\begin{aligned}
\mathbb{E}\left[|V_{t\wedge\tau_r-}|\right] &\leq \mathbb{E}\left[|v_0 - L_{t_0}|\right] + \mathbb{E}\left[|L_{t\wedge\tau_r-}|\right] + K\int_{t_0}^{t} s^{-\beta}\mathbb{E}\left[|V_{s\wedge\tau_r-}|^\gamma\right] \mathrm{d}s \\
&\leq C_{t_0} t^{\frac{1}{\alpha}} + K\int_{t_0}^{t} s^{-\beta}\mathbb{E}\left[|V_{s\wedge\tau_r-}|\right]^\gamma \mathrm{d}s.
\end{aligned}$$

Recalling that τ_r is given by (3), the function $g_r : t \mapsto \mathbb{E}\left[|V_{t\wedge\tau_r-}|\right]$ is bounded by r. Applying a Gronwall-type lemma (see Lemma A.1) and Fatou's lemma, we end up, for $\beta \neq 1$, with

$$\begin{aligned}
\forall t \geq t_0,\ \mathbb{E}\left[|V_{t\wedge\tau_\infty}|\right] \leq \liminf_{r\to+\infty} \mathbb{E}\left[|V_{t\wedge\tau_r-}|\right] &\leq C_\gamma\left[C_{t_0} t^{\frac{1}{\alpha}} + \left(\frac{1-\gamma}{1-\beta}K(t^{1-\beta} - t_0^{1-\beta})\right)^{\frac{1}{1-\gamma}}\right] \\
&\leq C_{\gamma,\beta,t_0} t^{\frac{1}{\alpha}},
\end{aligned}$$

since $\frac{\gamma-1}{\alpha}+1-\beta \leq 0$. The case $\beta = 1$ can be treated similarly. Thanks to Lemma 4.1, we conclude that the explosion time of V is a.s. infinite, and we have (8). □

Proposition 4.5. *Pick $\alpha \in (1,2)$. Assume here that for all $v \in \mathbb{R}$, $vF(v) \geq 0$. Recall that $(V_t)_{t\geq t_0}$ is the solution of* (SKE)*. For any $\gamma \in \mathbb{R}$ and any $\beta \in \mathbb{R}$, the explosion time τ_∞ is a.s. infinite and there exists C_{κ,t_0} such that*

$$\textit{for } \kappa \in [0,1],\ \forall t \geq t_0,\ \mathbb{E}\left[|V_t|^\kappa\right] \leq C_{\kappa,t_0} t^\kappa,\ \textit{or equivalently } p_\alpha(\gamma,\kappa) = \kappa; \tag{9}$$

$$\textit{for } \kappa \in (1,\alpha),\ \textit{if } \gamma \in [0,1),\ \forall t \geq t_0,\ \mathbb{E}\left[|V_t|^\kappa\right] \leq C_{\kappa,t_0} t^{3\kappa/2\alpha},\ \textit{or equivalently } p_\alpha(\gamma,\kappa) = \frac{3\kappa}{2\alpha}; \tag{10}$$

$$\textit{for } \kappa \in (1,\alpha),\ \forall t \geq t_0,\ \mathbb{E}\left[|V_t|^\kappa\right] \leq C_{\kappa,t_0} t^{\kappa/\alpha+\kappa/2},\ \textit{or equivalently } p_\alpha(\gamma,\kappa) = \frac{\kappa}{\alpha} + \frac{\kappa}{2}. \tag{11}$$

Proof. Step 1. We first apply Itô's formula and estimate the expectation of each term for $\kappa \leq 1$, in order to get (9).
Fix $\eta > 0$ and define the $\mathcal{C}^2$-function $f : v \mapsto (\eta + v^2)^{\kappa/2}$. For all $t \geq t_0$, by Itô's formula, we have

$$f(V_{t\wedge\tau_r-}) = f(V_0) - \int_{t_0}^{t\wedge\tau_r-} f'(V_s)F(V_s)s^{-\beta}\,\mathrm{d}s + M_{t\wedge\tau_r-} + R_{t\wedge\tau_r-} + S_{t\wedge\tau_r-}, \tag{12}$$

where

$$M_t := \int_{t_0}^{t}\int_{0<|z|<1} (f(V_{s-}+z) - f(V_{s-}))\widetilde{N}(\mathrm{d}s,\mathrm{d}z); \tag{13}$$

$$R_t := \int_{t_0}^{t}\int_{|z|\geq 1} (f(V_{s-}+z) - f(V_{s-}))N(\mathrm{d}s,\mathrm{d}z); \tag{14}$$

$$S_t := \int_{t_0}^{t}\int_{0<|z|<1} (f(V_s+z) - f(V_s) - zf'(V_s))\nu(\mathrm{d}z)\,\mathrm{d}s. \tag{15}$$

Here ν is the Lévy measure of the driving process. Recall that it satisfies

$$\int_{\mathbb{R}^*} (1\wedge z^2)\nu(\mathrm{d}z) < +\infty. \tag{16}$$

The second term of (12) is non-negative, since for all $v \in \mathbb{R}$, $vF(v) \geq 0$. We estimate expectations of M, R and S.
To that end, we first show that the local martingale $(M_{t\wedge\tau_r-})_{t\geq t_0}$ is a martingale.

Fix $q \geq 2$ and $r \geq 0$. Set

$$I_t(q) := \int_{t_0}^{t\wedge\tau_r}\int_{0<|z|<1} |f(V_{s-}+z) - f(V_{s-})|^q\,\nu(\mathrm{d}z)\,\mathrm{d}s.$$

Notice that, since for all $|v| \leq r$ and $|z| \leq 1$, $|f(v+z) - f(v)| \leq \|f'\mathbb{1}_{[-r-1,r+1]}\|_\infty\,|z|$, so we have

$$I_t(q) \leq \|f'_{|[-(r+1),r+1]}\|_\infty^q \int_{t_0}^{t\wedge\tau_r}\int_{0<|z|<1} |z|^q\,\nu(\mathrm{d}z)\,\mathrm{d}s.$$

Hence, it is a finite quantity, since $q \geq 2$ and ν is a Lévy measure i.e. (16) holds. Therefore, for $q \geq 2$, by Kunita's inequality (see Theorem 4.4.23 p. 265 in [App09]), there exists D_q such that

$$\mathbb{E}\left[\sup_{t_0\leq s\leq t} |M_s|^q\right] \leq D_q\left(\mathbb{E}\left[I_t(2)^{q/2}\right] + \mathbb{E}\left[I_t(q)\right]\right) < +\infty.$$

Hence, by Theorem 51 p. 38 in [Pro05], M is a martingale.
We estimate now the finite variation part S defined in (15). We use a similar idea as in the proof of Theorem 3.1 p. 3863 in [DS15]. Assume that $|z| < 1$ and note that for all $v \in \mathbb{R}$,

$$\begin{aligned}
|f''(v)| &= \kappa(2-\kappa)v^2(v^2+\eta)^{\kappa/2-2} + \kappa(v^2+\eta)^{\kappa/2-1} \\
&= \kappa(2-\kappa)v^2(v^2+\eta)^{-1}(v^2+\eta)^{\kappa/2-1} + \kappa(v^2+\eta)^{\kappa/2-1} \\
&\leq \kappa(3-\kappa)(v^2+\eta)^{\kappa/2-1} \\
&\leq \kappa(3-\kappa)\eta^{\kappa/2-1}, \text{ since } \tfrac{\kappa}{2} - 1 < 0.
\end{aligned}$$

Using Taylor's formula, there exists a random number $\Theta_{V_s,z}$, with values in $[-1,1]$, depending on V_s and z, such that a.s.

$$|f(V_s+z)-f(V_s)-zf'(V_s)| = \frac{1}{2}\left|f''(V_s+\Theta_{V_s,z}z)\right| z^2 \leq \frac{1}{2}\left|\kappa(3-\kappa)\right| \eta^{\kappa/2-1} z^2.$$

Hence, we get the almost sure following bound

$$\left|\int_{0<|z|<1} \left(f(V_s+z)-f(V_s)-zf'(V_s)\right)\nu(\mathrm{d}z)\,\mathrm{d}s\right| \leq \frac{1}{2}\left|\kappa(3-\kappa)\right| \eta^{\kappa/2-1}\int_{0<|z|<1} z^2\nu(\mathrm{d}z). \quad (17)$$

It remains to study the Poisson integral R defined in (14). Pick $\kappa \leq 1$, by classical Hölder inequalities, we can write,

$$\begin{aligned} f(v+z)-f(v) &= \left(\eta+(v+z)^2\right)^{\kappa/2} - \left((v+z)^2\right)^{\kappa/2} + (v+z)^\kappa - v^\kappa + \left(v^2\right)^{\kappa/2} - \left(\eta+v^2\right)^{\kappa/2} \\ &\leq 2\eta^{\kappa/2} + z^\kappa. \end{aligned}$$

We deduce that

$$\int_{|z|\geq 1} f(V_s+z)-f(V_s)\nu(\mathrm{d}z)\,\mathrm{d}s \leq \eta^{\kappa/2}\nu(|z|\geq 1) + \int_{|z|\geq 1}|z|^\kappa\,\nu(\mathrm{d}z). \quad (18)$$

Note that $\int_{|z|\geq 1}|z|^\kappa\,\nu(\mathrm{d}z) < +\infty$, since (16) holds and $\kappa \leq 1$.
Gathering (18) and (17), we get

$$\begin{aligned} &\mathbb{E}\left[|V_{t\wedge\tau_r-}|^\kappa\right] \leq \mathbb{E}\left[f(V_{t\wedge\tau_r-})\right] \leq \mathbb{E}\left[f(V_{t_0})\right] + \mathbb{E}\left[t\wedge\tau_r\right]\times \\ &\left(\eta^{\kappa/2}\nu(|z|\geq 1) + \int_{|z|\geq 1}|z|^\kappa\,\nu(\mathrm{d}z) + \frac{1}{2}\kappa(3-\kappa)\eta^{\kappa/2-1}\int_{0<|z|<1} z^2\nu(\mathrm{d}z)\right) \leq C_{\kappa,t_0}t. \end{aligned} \quad (19)$$

Thanks to Lemma 4.1 and letting $r\to+\infty$, we can conclude that the explosion time of V is a.s. infinite and thus, for all $\kappa\in[0,1]$,

$$\mathbb{E}\left[|V_t|^\kappa\right] \leq C_{\kappa,t_0}t.$$

By Jensen's inequality, we can deduce (9).
Step 2. Pick $\kappa\in[0,\alpha)$. We estimate R in another way.
By Hölder's inequality, we get

$$\int_{|z|\geq 1}\left|f(V_s+z)-f(V_s)\right|\nu(\mathrm{d}z)\,\mathrm{d}s \leq \int_{|z|\geq 1}\left|2zV_s+z^2\right|^{\kappa/2}\nu(\mathrm{d}z) \leq C(1+|V_s|^{\kappa/2})\int_{|z|\geq 1}|z|^\kappa\,\nu(\mathrm{d}z). \quad (20)$$

The last integral is finite since $\alpha>\kappa$.
Gathering (17), (20) and then using (8) and (9), we have

$$\mathbb{E}\left[|V_{t\wedge\tau_r-}|^\kappa\right] \leq \mathbb{E}\left[f(V_{t\wedge\tau_r-})\right] \leq C_{\kappa,t_0}t + C\int_{t_0}^t (1+\mathbb{E}\,|V_s|^{\kappa/2})\,\mathrm{d}s \leq Ct^{1+p_\alpha(\gamma,\kappa/2)}. \quad (21)$$

Taking $r\to+\infty$, we can conclude that the explosion time is infinite and that the exponent p_α satisfies the recurrence relation $p_\alpha(\gamma,\kappa) = 1+p_\alpha(\gamma,\kappa/2)$.
Step 3. We refine the estimates obtained in the previous steps.
Fix $\kappa\in[0,\alpha)$. There exists ε_1 such that for all $\varepsilon\leq\varepsilon_1$, $\kappa\leq\alpha-\varepsilon$ and $\alpha-\varepsilon>1$. Applying (21) with $\alpha-\varepsilon$, we can write

$$\mathbb{E}\left[|V_t|^{\alpha-\varepsilon}\right] \leq C_{\kappa,t_0}t^{1+p_\alpha(\gamma,(\alpha-\varepsilon)/2)}.$$

Using Jensen's inequality, we get

$$\mathbb{E}\left[|V_t|^\kappa\right] \leq C_{\kappa,t_0} t^{\frac{\kappa}{\alpha-\varepsilon}(1+p_\alpha(\gamma,(\alpha-\varepsilon)/2))},$$

and it suffices to let $\varepsilon \to 0$ to conclude. □

Remark 4.6. *To study the asymptotic behavior, we will only need $p_\alpha(\gamma,\gamma)$ with $\gamma \in [0,\alpha)$. Hence, estimates for $\mathbb{E}\left[|V_t|^\gamma\right]$ are sharp except for $(\alpha,\gamma) \in (1,2) \times [1,\alpha)$.*

Example 4.1. Remark that the velocity process V is more explicit in the linear case ($\gamma = 1$). Choose $F(1) = \rho > 0$, $F(-1) = -\rho$. Pick $\beta \neq 1$, then

$$V_t = v_0 + \exp\left(-\rho\frac{t^{1-\beta}}{1-\beta}\right)\int_{t_0}^t \exp\left(\rho\frac{s^{1-\beta}}{1-\beta}\right)\mathrm{d}L_s$$

is solution of (SKE).
Hence, by an integration by parts, for $\beta \geq 1 + \dfrac{\gamma-1}{\alpha} = 1$,

$$V_t = v_0 + L_t - e^{\rho\frac{1}{1-\beta}(t_0^{1-\beta}-t^{1-\beta})}L_{t_0} + e^{-\rho\frac{t^{1-\beta}}{1-\beta}}\int_{t_0}^t \rho s^{-\beta}e^{\rho\frac{s^{1-\beta}}{1-\beta}}L_s\,\mathrm{d}s.$$

Thus,

$$\mathbb{E}\left[|V_t|\right] \leq C_{t_0}\left(t^{1/\alpha} + t^{1-\beta+1/\alpha}\right) \leq C_{t_0}t^{1/\alpha}.$$

The case $\beta = 1$ can be treated similarly. So, we conclude that $p_\alpha(1) = \frac{1}{\alpha}$ when $F(v) = \rho v$. We remark that the exponent is sharper than in Proposition 4.5.

5 Proof of the asymptotic behavior of the solution

This section is devoted to the proofs of our main results, Theorems 2.2 and 2.4.
Notice that it suffices to prove the convergence of the rescaled velocity process $(\varepsilon^{1/\alpha}V_{t/\varepsilon})_{t\geq\varepsilon t_0}$ in the space $\mathcal{D}$ endowed with the Skorokhod topology. Assume for a moment that this convergence is proved.
For $\varepsilon \in (0,1]$ and $t \geq \varepsilon t_0$ we can write

$$\varepsilon^{1+1/\alpha}X_{t/\varepsilon} = \varepsilon^{1+1/\alpha}x_0 + \int_{\varepsilon t_0}^t V_s^{(\varepsilon)}\,\mathrm{d}s.$$

Let us introduce the mapping $g_\varepsilon : V \mapsto \left(V_t, \int_{\varepsilon t_0}^t V_s\,\mathrm{d}s\right)_{t>0}$ defined and valued on $\mathcal{D}$. Clearly, the theorem will be proved once we show that $g_\varepsilon(V_\bullet^{(\varepsilon)})$ converges weakly in $\mathcal{D}$ endowed by the Skorokhod topology. This mapping is converging, as $\varepsilon \to 0$, to the continuous mapping $g : V \mapsto \left(V_t, \int_0^t V_s\,\mathrm{d}s\right)_{t>0}$.
In order to see $V^{(\varepsilon)}$ as a process of $\mathcal{D}([0,+\infty))$, let us state for all $s \in [0,\varepsilon t_0]$, $V_s^{(\varepsilon)} := V_{\varepsilon t_0}^{(\varepsilon)} = \varepsilon^{1/\alpha}v_0$. Call P_ε, P the distribution of $V^{(\varepsilon)}$, $\mathcal{S}$, respectively. Invoking the Portmanteau theorem (see Theorem 2.1 p. 16 in [Bil99]), it suffices to prove that for all bounded and uniformly continuous function $h : \mathcal{D}([0,+\infty)) \times \mathcal{D}([0,+\infty)) \to \mathbb{R}$,

$$\int_{\mathcal{D}([0,+\infty))^2} h(g_\varepsilon(\omega))\,\mathrm{d}P_\varepsilon(\mathrm{d}\omega) \underset{\varepsilon\to 0}{\longrightarrow} \int_{\mathcal{D}([0,+\infty))^2} h(g(\omega))\,\mathrm{d}P(\mathrm{d}\omega).$$

Pick such a function h. By assumption, the convergence $P_\varepsilon \Longrightarrow P$ holds, hence, using Lemma A.4 it suffices to prove that the uniformly bounded sequence $(h \circ g_\varepsilon)$ of continuous functions on $\mathcal{D}([0,+\infty))$ converges to the continuous function $h \circ g$ uniformly on compact subsets of $\mathcal{D}([0,+\infty))$. Let K be a compact set of $\mathcal{D}([0,+\infty))$. Then, for all $\omega \in K$, $\max_{[0,\varepsilon t_0]} |\omega|$ is uniformly bounded by a constant, say M. Fix $\eta > 0$. By the uniform continuity of h, there exists $\delta > 0$ such that for all $\omega \in K$,

$$\mathrm{d}_u(g_\varepsilon(\omega), g(\omega)) \leq \delta \quad \Longrightarrow \quad |h \circ g_\varepsilon(\omega) - h \circ g(\omega)| \leq \eta.$$

There exists $\varepsilon_1 > 0$ small enough, such that for all $\varepsilon \leq \varepsilon_1$, for all $\omega \in K$,

$$\mathrm{d}_u(g_\varepsilon(\omega), g(\omega)) \leq C \left| \int_0^{\varepsilon t_0} \omega(s)\, \mathrm{d}s \right| \leq C \varepsilon t_0 M \leq \delta.$$

Therefore, we proved that it suffices to prove the convergence of the rescaled velocity process $(\varepsilon^{1/\alpha} V_{t/\varepsilon})_{t \geq \varepsilon t_0}$ in order to prove Theorems 2.2 and 2.4.
In Sections 5.1 and 5.2, the aim is to prove the convergence of the velocity process.

5.1 Asymptotic behavior above the critical line

In the remainder of this section, we assume that $\gamma \geq 0$ and $\beta > 1 + p(\gamma) - \frac{1}{\alpha}$.

Proof of Theorem 2.2. Thanks to a change of variables, we have, for all $\varepsilon \in (0,1]$ and $t \geq \varepsilon t_0$,

$$\begin{aligned}
\varepsilon^{1/\alpha} V_{t/\varepsilon} =& \varepsilon^{1/\alpha}(v_0 - L_{t_0}) + \varepsilon^{1/\alpha} L_{t/\varepsilon} - \varepsilon^{1/\alpha} \int_{t_0}^{t/\varepsilon} F(V_s) s^{-\beta}\, \mathrm{d}s \\
=& \varepsilon^{1/\alpha}(v_0 - L_{t_0}) + \varepsilon^{1/\alpha} L_{t/\varepsilon} - \varepsilon^{\beta - 1 + 1/\alpha} \int_{\varepsilon t_0}^{t} F(V_{u/\varepsilon}) u^{-\beta}\, \mathrm{d}u.
\end{aligned}$$

By self-similarity, $L^{(\varepsilon)} := (\varepsilon^{1/\alpha} L_{t/\varepsilon})_{t \geq 0}$ has the same distribution as an α-stable process. As a consequence, thanks to Theorem 3.1, p. 27, in [Bil99] and Lemma A.3, it suffices to prove

$$\forall T \geq t_0 \quad \sup_{\varepsilon t_0 \leq t \leq T} \left| V_t^{(\varepsilon)} - L_t^{(\varepsilon)} \right| \xrightarrow{\mathbb{P}} 0, \quad \text{as } \varepsilon \to 0. \tag{22}$$

Recall that under (H_γ), there exists a positive constant K, such that

$$\varepsilon^{\gamma/\alpha} \left| F\left(\frac{V_\bullet^{(\varepsilon)}}{\varepsilon^{1/\alpha}} \right) \right| \leq K \left| V_\bullet^{(\varepsilon)} \right|^\gamma. \tag{23}$$

Modifying the factor in front of the integral, we get

$$V_t^{(\varepsilon)} = \varepsilon^{1/\alpha}(v_0 - L_{t_0}) + L_t^{(\varepsilon)} - \varepsilon^{\beta - 1 + (1-\gamma)/\alpha} \int_{\varepsilon t_0}^{t} \varepsilon^{\gamma/\alpha} F\left(\frac{V_u^{(\varepsilon)}}{\varepsilon^{1/\alpha}} \right) u^{-\beta}\, \mathrm{d}u. \tag{24}$$

Gathering (24) and (23), for all $T \geq \varepsilon t_0$, we have,

$$\begin{aligned}
\sup_{\varepsilon t_0 \leq t \leq T} \left| V_t^{(\varepsilon)} - L_t^{(\varepsilon)} \right| \leq& \varepsilon^{1/\alpha}(v_0 - L_{t_0}) + \varepsilon^{\beta - 1 + (1-\gamma)/\alpha} \sup_{\varepsilon t_0 \leq t \leq T} \left| \int_{\varepsilon t_0}^{t} \varepsilon^{\gamma/\alpha} F\left(\frac{V_u^{(\varepsilon)}}{\varepsilon^{1/\alpha}} \right) u^{-\beta}\, \mathrm{d}u \right| \\
\leq& \varepsilon^{1/\alpha}(v_0 - L_{t_0}) + \varepsilon^{\beta - 1 + (1-\gamma)/\alpha} \int_{\varepsilon t_0}^{T} K \left| V_u^{(\varepsilon)} \right|^\gamma u^{-\beta}\, \mathrm{d}u.
\end{aligned}$$

Taking the expectation and using the moment estimates on V (ssee Remark 2.6), we obtain, when $\beta \neq p_\alpha(\gamma,\gamma)+1$,

$$\begin{aligned}
\varepsilon^{\beta-1+(1-\gamma)/\alpha}\mathbb{E}\left[\int_{\varepsilon t_0}^T K\left|V_u^{(\varepsilon)}\right|^\gamma u^{-\beta}\,\mathrm{d}u\right] &= \varepsilon^{\beta-1+(1-\gamma)/\alpha}\int_{\varepsilon t_0}^T K\mathbb{E}\left[\left|V_u^{(\varepsilon)}\right|^\gamma\right]u^{-\beta}\,\mathrm{d}u \\
&= \varepsilon^{\beta-1+1/\alpha}\int_{\varepsilon t_0}^T K\mathbb{E}\left[\left|V_{u/\varepsilon}\right|^\gamma\right]u^{-\beta}\,\mathrm{d}u \\
&\leq \varepsilon^{\beta-1-p_\alpha(\gamma,\gamma)+1/\alpha}\int_{\varepsilon t_0}^T KC_{\alpha,\beta,t_0}u^{p_\alpha(\gamma,\gamma)-\beta}\,\mathrm{d}u \\
&= C\left(\varepsilon^{\beta-1-p_\alpha(\gamma,\gamma)+1/\alpha}T^{p_\alpha(\gamma,\gamma)-\beta+1} - t_0^{p_\alpha(\gamma,\gamma)-\beta+1}\varepsilon^{1/\alpha}\right).
\end{aligned}$$

Hence, setting $q := \min(\beta-1+\frac{1}{\alpha}-p_\alpha(\gamma,\gamma),\frac{1}{\alpha})$ which is positive, since $\beta > 1+p_\alpha(\gamma,\gamma)-\frac{1}{\alpha}$, we get

$$\mathbb{E}\left[\sup_{\varepsilon t_0\leq t\leq T}\left|V_t^{(\varepsilon)}-L_t^{(\varepsilon)}\right|\right] \underset{\varepsilon\to 0}{=} O(\varepsilon^q).$$

The case $\beta = 1+p_\alpha(\gamma,\gamma)$ can be treated similarly to get

$$\mathbb{E}\left[\sup_{\varepsilon t_0\leq t\leq T}\left|V_t^{(\varepsilon)}-L_t^{(\varepsilon)}\right|\right] \underset{\varepsilon\to 0}{=} O(\varepsilon^{1/\alpha}\ln(\varepsilon)).$$

This concludes the proof, setting $p(\gamma) := p_\alpha(\gamma,\gamma)$. □

Remark 5.1. *Observe that we did not use the condition "$\gamma < 1$ or for all $v\in\mathbb{R},\ vF(v)\geq 0$" in this proof, except to get moment estimates.*

5.2 Asymptotic behavior on the critical line

We adapt the Proposition 2.1, p. 187 of [GO13] to the α-stable Lévy case.
Pick a $\mathcal{C}^2$-diffeomorphism $\varphi : [0,t_1)\to[t_0,+\infty)$. Let V be the solution to the equation (SKE). Thanks to Proposition 3.4.1 p. 124 in [ST94], the following process is also an α-stable process

$$(R_t)_{t\geq 0} := \left(\int_0^t \frac{\mathrm{d}L_{\varphi(s)}}{\varphi'(s)^{1/\alpha}}\right)_{t\geq 0}. \tag{25}$$

Then, by the change of variables $t=\varphi(s)$, we get

$$V_{\varphi(t)} - V_{\varphi(0)} = \int_0^t \varphi'(s)^{1/\alpha}\,\mathrm{d}R_s - \int_0^t \frac{F(V_{\varphi(s)})}{\varphi(s)^\beta}\varphi'(s)\,\mathrm{d}s.$$

Thanks to an integration by parts, we get

$$\mathrm{d}\left(\frac{V_{\varphi(s)}}{\varphi'(s)^{1/\alpha}}\right) = \mathrm{d}R_s - \frac{\varphi'(s)^{1-1/\alpha}}{\varphi(s)^\beta}F(V_{\varphi(s)})\,\mathrm{d}s - \frac{\varphi''(s)}{\alpha\varphi'(s)}\frac{V_{\varphi(s)}}{\varphi'(s)^{1/\alpha}}\,\mathrm{d}s.$$

Set $\Omega = \overline{\mathcal{D}}([t_0,\infty))$ the set of càdlàg functions, that equal ∞ after their (possibly infinite) explosion time. Introduce the scaling transformation Φ_φ defined, for $\omega\in\Omega$, by

$$\Phi_\varphi(\omega)(s) := \frac{\omega(\varphi(s))}{\sqrt{\varphi'(s)}}, \text{ with } s\in[0,t_1).$$

As a consequence, we obtain the following result.

Proposition 5.2. *If V is a solution to the equation* (SKE)*, then $V^{(\varphi)} := \Phi_\varphi(V)$ is a solution to*

$$\mathrm{d}V_s^{(\varphi)} = \mathrm{d}R_s - \frac{\varphi'(s)^{1-1/\alpha}}{\varphi(s)^\beta} F(\varphi'(s)^{1/\alpha} V_s^{(\varphi)})\,\mathrm{d}s - \frac{\varphi''(s)}{\varphi'(s)} \frac{V_s^{(\varphi)}}{\alpha}\,\mathrm{d}s, \text{ with } V_0^{(\varphi)} = \frac{V_{\varphi(0)}}{\varphi'(0)^{1/\alpha}}, \tag{26}$$

where R is given by (25)*.*
Conversely, if $V^{(\varphi)}$ is a solution to (26)*, then $\Phi_\varphi^{-1}(V^{(\varphi)})$ is a solution to the equation* (SKE)*, where*

$$L_t - L_{t_0} := \int_{t_0}^t (\varphi' \circ \varphi^{-1})^{1/\alpha}(s)\,\mathrm{d}R_{\varphi^{-1}(s)}.$$

Furthermore, uniqueness in law, pathwise uniqueness, strong existence hold for the equation (SKE) *if and only if they hold for the equation* (26)*.*

In the following, we will focus on an exponential change of time $\varphi_e : 0 \le t \mapsto t_0 e^t$. This scaling is convenient since it allows to produce a time-homogeneous term in (26). Thanks to Proposition 5.2, the process $V^{(e)} := \Phi_e(V)$ satisfies the SDE driven by an α-stable process $(R_t)_{t\ge 0}$,

$$\mathrm{d}V_s^{(e)} = \mathrm{d}R_s - \frac{V_s^{(e)}}{\alpha}\,\mathrm{d}s - t_0^{1-1/\alpha-\beta} e^{(1-1/\alpha-\beta)s} F\big(t_0^{1/\alpha} e^{s/\alpha} V_s^{(e)}\big)\,\mathrm{d}s. \tag{27}$$

Proof of Theorem 2.4. Assume in the sequel that $\beta = 1 + \frac{\gamma-1}{\alpha}$.
Step 1. Firstly we prove the finite-dimensional convergence.
Since (H_γ) holds, (27) becomes

$$\mathrm{d}V_s^{(e)} = \mathrm{d}R_s - \frac{V_s^{(e)}}{\alpha}\,\mathrm{d}s - F\big(V_s^{(e)}\big)\,\mathrm{d}s. \tag{28}$$

Using the bijection Φ_e induced by the exponential change of time (see Proposition 5.2), and the unique strong existence of the velocity process V (see Proposition 3.2 and Remark 2.6), there exists a pathwise unique strong solution H to the time-homogeneous equation (28). Hence, we have the equality

$$\left(\frac{V_{t_0 e^t}}{(t_0 e^t)^{1/\alpha}}\right)_{t\ge 0} = (H_t)_{t\ge 0},$$

as two solutions of the same SDE, starting from the same point. We can write the above equality as

$$\left(\frac{V_t}{t^{1/\alpha}}\right)_{t\ge t_0} = (H_{\log(t/t_0)})_{t\ge t_0}.$$

So, we have, for all $\varepsilon > 0$, $d \in \mathbb{N}^*$, and $(t_1, \cdots, t_d) \in [t_0, +\infty)^d$,

$$\left(\frac{V_{\varepsilon^{-1}t_1}}{(\varepsilon^{-1}t_1)^{1/\alpha}}, \cdots, \frac{V_{\varepsilon^{-1}t_d}}{(\varepsilon^{-1}t_d)^{1/\alpha}}\right) = \left(H_{\log(t_1/t_0)+\log(\varepsilon^{-1})}, \cdots, H_{\log(t_d/t_0)+\log(\varepsilon^{-1})}\right). \tag{29}$$

Since $\limsup_{|x|\to+\infty} \frac{-F(x)-x/\alpha}{x} < 0$, it follows from Proposition 0.1 in [Kul09] that the process $(H_t)_{t\ge 0}$ is exponentially ergodic. We denote its invariant measure by Λ_F. Call $\widetilde{H}$ the solution of the time homogeneous equation (28) starting from Λ_F. For $t_1, \cdots, t_d \in \mathbb{R}^d$, let $\Lambda_{F,t_1,\cdots,t_d} := \mathcal{L}(\widetilde{H}_{t_1}, \cdots, \widetilde{H}_{t_d})$ be the distribution of $(\widetilde{H}_{t_1}, \cdots, \widetilde{H}_{t_d})$. Then, for all $s \ge 0$, $\Lambda_{F,t_1,\cdots,t_d} = \Lambda_{F,t_1+s,\cdots,t_d+s}$. Indeed, thanks to the invariance property of Λ_F, $(\widetilde{H}_\bullet)$ and $(\widetilde{H}_{\bullet+s})$

satisfy the same SDE, starting from the same point. As a consequence, we get the stationary limit

$$\lim_{\varepsilon\to 0}\mathcal{L}\left(\widetilde{H}_{\log(t_1/t_0)+\log(\varepsilon^{-1})},\cdots,\widetilde{H}_{\log(t_d/t_0)+\log(\varepsilon^{-1})}\right) \quad = \quad \Lambda_{F,\log(t_1/t_0),\cdots,\log(t_d/t_0)}.$$

Moreover, by exponential ergodicity, setting $\|\cdot\|_{TV}$ for the total variation norm, we deduce

$$\left\|\mathcal{L}\left(H_{\log(t_1/t_0\varepsilon)},\cdots,H_{\log(t_d/t_0\varepsilon)}\right)-\mathcal{L}\left(\widetilde{H}_{\log(t_1/t_0\varepsilon)},\cdots,\widetilde{H}_{\log(t_d/t_0\varepsilon)}\right)\right\|_{TV} \quad \underset{\varepsilon\to 0}{\longrightarrow} \quad 0.$$

Hence, by (29), we get

$$\left(\frac{V_{\varepsilon^{-1}t_1}}{(\varepsilon^{-1}t_1)^{1/\alpha}},\cdots,\frac{V_{\varepsilon^{-1}t_d}}{(\varepsilon^{-1}t_d)^{1/\alpha}}\right) \quad \underset{\varepsilon\to 0}{\Longrightarrow} \quad \Lambda_{F,\log(t_1/t_0),\cdots,\log(t_d/t_0)}.$$

This can also be written as

$$\left(\varepsilon^{1/\alpha}V_{t_1/\varepsilon},\cdots,\varepsilon^{1/\alpha}V_{t_d/\varepsilon}\right) \quad \underset{\varepsilon\to 0}{\Longrightarrow} \quad T*\Lambda_{F,\log(t_1/t_0),\cdots,\log(t_d/t_0)},$$

where $T*\Lambda_{F,\log(t_1/t_0),\cdots,\log(t_d/t_0)}$ is the pushforward of the measure $\Lambda_{F,\log(t_1/t_0),\cdots,\log(t_d/t_0)}$ by the linear map $T(u_1,\cdots,u_d):=(t_1^{1/\alpha}u_1,\cdots,t_d^{1/\alpha}u_d)$.

Step 2. Let us prove now the tightness of the family of distributions of càdlàg process $\left(V^{(\varepsilon)}\right)_{t\geq\varepsilon t_0}=\left(\varepsilon^{1/\alpha}V_{t/\varepsilon}\right)_{t\geq\varepsilon t_0}$ on every compact interval $[m,M]$, $0<m\leq M$, provided that $p_\alpha(\gamma)\leq\frac{\gamma}{\alpha}$.

We check the Aldous's tightness criterion stated at (16.23) p. 176 in [Bil99]. Let a, η, T be positive reals. Let τ be a discrete stopping time with finite range $\mathcal{T}$, bounded by T. Choose $\delta>0$ and $\varepsilon>0$ small enough.

We have, by Jensen's inequality, for $r=\min(1,\alpha)$,

$$\mathbb{E}\left[\left|V^{(\varepsilon)}_{\tau+\delta}-V^{(\varepsilon)}_{\tau}\right|^r\right]\leq\mathbb{E}\left[\left|L^{(\varepsilon)}_{\tau+\delta}-L^{(\varepsilon)}_{\tau}\right|^r\right]+\mathbb{E}\left[\int_\tau^{\tau+\delta}K\left|V^{(\varepsilon)}_u\right|^\gamma u^{-\beta}\,\mathrm{d}u\right]^r.$$

Since $L^{(\varepsilon)}$ is an α-stable process, by the strong Markov property,

$$\mathbb{E}\left[\left|L^{(\varepsilon)}_{\tau+\delta}-L^{(\varepsilon)}_{\tau}\right|^r\right]=\mathbb{E}\left[\mathbb{E}_{L_\tau}\left[|L_\delta-L_0|^r\right]\right]\leq C\delta^{\frac{r}{\alpha}}.$$

The stopping time has a finite range $\mathcal{T}$. Hence, we can write

$$\begin{aligned}\mathbb{E}\left[\int_\tau^{\tau+\delta}K\left|V^{(\varepsilon)}_u\right|^\gamma u^{-\beta}\,\mathrm{d}u\right]&=\mathbb{E}\left[\mathbb{E}\left[\int_\tau^{\tau+\delta}K\left|V^{(\varepsilon)}_u\right|^\gamma u^{-\beta}\,\mathrm{d}u\middle|\tau\right]\right]\\&=\mathbb{E}\left[\sum_{\tau_i\in\tau}\frac{1}{\mathbb{P}(\tau=\tau_i)}\mathbb{E}\left[\mathbb{1}_{\{\tau=\tau_i\}}\int_{\tau_i}^{\tau_i+\delta}K\left|V^{(\varepsilon)}_u\right|^\gamma u^{-\beta}\,\mathrm{d}u\right]\mathbb{1}_{\{\tau=\tau_i\}}\right]\\&\leq\mathbb{E}\left[\sum_{\tau_i\in\tau}\frac{1}{\mathbb{P}(\tau=\tau_i)}\mathbb{E}\left[\int_{\tau_i}^{\tau_i+\delta}K\left|V^{(\varepsilon)}_u\right|^\gamma u^{-\beta}\,\mathrm{d}u\right]\mathbb{1}_{\{\tau=\tau_i\}}\right].\end{aligned}$$

For each $\tau_i\in\mathcal{T}$, using the relation $\beta=1+\frac{(\gamma-1)}{\alpha}$ and the moment estimates on V (see Re-

mark 2.6), we obtain

$$\begin{aligned}\mathbb{E}\left[\int_{\tau_i}^{\tau_i+\delta} K\left|V_u^{(\varepsilon)}\right|^{\gamma} u^{-\beta}\,\mathrm{d}u\right] &= \int_{\tau_i}^{\tau_i+\delta} K\mathbb{E}\left[\left|V_u^{(\varepsilon)}\right|^{\gamma}\right] u^{-\beta}\,\mathrm{d}u \\ &\leq K\varepsilon^{\frac{\gamma}{\alpha}-p_\alpha(\gamma,\gamma)}\int_{\tau_i}^{\tau_i+\delta} u^{p_\alpha(\gamma,\gamma)-\beta}\,\mathrm{d}u \\ &= K\varepsilon^{\frac{\gamma}{\alpha}-p_\alpha(\gamma,\gamma)}\left[(\tau_i+\delta)^{p_\alpha(\gamma,\gamma)+\frac{1-\gamma}{\alpha}} - \tau_i^{p_\alpha(\gamma,\gamma)+\frac{1-\gamma}{\alpha}}\right] \\ &\leq K\varepsilon^{\frac{\gamma}{\alpha}-p_\alpha(\gamma,\gamma)}\delta^{1,p_\alpha(\gamma,\gamma)+\frac{1-\gamma}{\alpha}}.\end{aligned}$$

The term $\delta^{1,p_\alpha(\gamma,\gamma)+\frac{1-\gamma}{\alpha}}$ has to be read as δ or $\delta^{p_\alpha(\gamma,\gamma)+\frac{1-\gamma}{\alpha}}$ depending on the fact that $x \mapsto x^{p_\alpha(\gamma)+\frac{1-\gamma}{\alpha}}$ is a Lipschitz continuous function on $[0, T+\delta]$, if $\alpha < 1$, or a $\left(p_\alpha(\gamma)+\frac{1-\gamma}{\alpha}\right)$-Hölder function, if $\alpha > 1$.
By Markov's inequality, for δ small enough and provided that $\frac{\gamma}{\alpha} - p_\alpha(\gamma,\gamma) \geq 0$, we have

$$\mathbb{P}\left(\left|V_{\tau+\delta}^{(\varepsilon)} - V_\tau^{(\varepsilon)}\right| \geq a\right) \leq \frac{K\varepsilon^{\frac{\gamma}{\alpha}-p_\alpha(\gamma,\gamma)}\delta^{1,p_\alpha(\gamma,\gamma)+\frac{1-\gamma}{\alpha}}}{a} \leq \eta.$$

This concludes the proof of the tightness of the velocity process and therefore the proof of Theorem 2.4. □

6 Extended results in the Lévy case

In this section, the driving noise of (SKE) is supposed to be a general Lévy process L. We denote by (A, ν, b) its generating triplet, with respect to the truncation function $h : z \mapsto -1 \vee (z \wedge 1)$. Here ν is the Lévy measure, $A > 0$ and $b \in \mathbb{R}$. This means that, in virtue of the Lévy-Khintchine formula (see Theorem 8.1 p. 37 in [Sat99]), the characteristic function of L_t is given by

$$\mathbb{E}\left[e^{iuL_t}\right] = e^{t\psi(u)}, \text{ where } \psi(u) := -\frac{Au^2}{2} + ibu + \int_{\mathbb{R}}(e^{iux} - 1 - iuh(x))\nu(\mathrm{d}x).$$

Pick $\alpha \in (0,2)\setminus\{1\}$. The Lévy measure ν of the driving noise is supposed to satisfy either

$\nu(z) = \dfrac{g(z)}{|z|^{1+\alpha}}$, where g is a measurable function admitting the following positive limits at $\pm\infty$:

$$c^+ := \lim_{z\to+\infty} g(z), \quad c^- := \lim_{z\to-\infty} g(z), \tag{$H_1^{\nu,\alpha}$}$$

or

$$\text{for some } \alpha_0 > 1, \int_{|z|\geq 1} |z|^{\alpha_0}\,\nu(\mathrm{d}z) < +\infty. \tag{H_2^{ν,α_0}}$$

Remark that if ν satisfies $(H_1^{\nu,\alpha})$ with $\alpha > 1$, then it satisfies (H_2^{ν,α_0}). Note that any truncated α-stable process and any tempered stable process satisfy (H_2^{ν,α_0}).
For clarity, we recall the stochastic kinetic model: for $t \geq t_0 > 0$,

$$\mathrm{d}V_t = \mathrm{d}L_t - t^{-\beta}F(V_t)\,\mathrm{d}t, \text{ with } V_{t_0} = v_0 > 0, \quad \text{and} \quad \mathrm{d}X_t = V_t\,\mathrm{d}t, \text{ with } X_{t_0} = x_0 \in \mathbb{R}. \tag{SKE}$$

We work under the assumption that there exists a unique solution to (SKE). We will show that Theorem 2.2 can be extended with this general Lévy driving noise. We suppose first the

Lévy process to be without a Brownian component. This case will be discussed, in Section 6.3. We obtain the two following theorems, depending on which hypothesis is satisfied by the Lévy measure ν.

Theorem 6.1. *Consider $\gamma \geq 0$ and $\beta \geq 0$. Assume that $(H_1^{\nu,\alpha})$ and (H_{sgn}) are satisfied and define*

$$p_\alpha(\gamma) := \begin{cases} \frac{\gamma}{\alpha}, & \text{if } \alpha \in (0,1); \\ \frac{\gamma}{\alpha}, & \text{if } \alpha \in (1,2),\ \gamma \in [0,1) \text{ and } b = 0; \\ \gamma, & \text{if } \alpha \in (1,2),\ \gamma \in [0,1) \text{ and } b \neq 0; \\ \gamma, & \text{if } \alpha \in (1,2),\ \gamma = 1; \\ \frac{\gamma}{\alpha} + \frac{\gamma}{2}, & \text{if } \alpha \in (1,2),\ \gamma \in (1,\alpha). \end{cases}$$

Let $(V_t, X_t)_{t \geq t_0}$ be the solution to (SKE).
Then there exist a rate of convergence ε^θ and a Lévy process $\mathcal{L}$, given in Proposition 6.3 (iii)-(v), such that, as $\varepsilon \to 0$, if $\beta > 1 + p_\alpha(\gamma) - \theta$, in the space $\mathcal{D}$,

$$(\varepsilon^\theta V_{t/\varepsilon}, \varepsilon^{1+\theta} X_{t/\varepsilon})_{t \geq \varepsilon t_0} \quad \Longrightarrow \quad \left(\mathcal{L}_t, \int_0^t \mathcal{L}_s \, \mathrm{d}s\right)_{t>0}.$$

Theorem 6.2. *Consider $\gamma \geq 0$ and $\beta \geq 0$. Assume that (H_2^{ν,α_0}) is satisfied and define*

$$p_{\alpha_0}(\gamma) := \begin{cases} \gamma, & \text{if } \gamma \in [0,1); \\ \min(\gamma, \frac{\gamma}{\alpha_0} + \frac{\gamma}{2}), & \text{if } \gamma \in [0,1) \text{ and } vF(v) \geq 0; \\ \frac{\gamma}{\alpha_0} + \frac{\gamma}{2}, & \text{if } \gamma \in [1,\alpha_0] \text{ and } vF(v) \geq 0. \end{cases}$$

Suppose that $\beta > 1 + p_{\alpha_0}(\gamma) - 1$. Let $(V_t, X_t)_{t \geq t_0}$ be the solution to (SKE).
Then there exists a Lévy process $\mathcal{L}$, given in Proposition 6.3 (iii), such that, as $\varepsilon \to 0$, in the space $\mathcal{D}$,

$$(\varepsilon V_{t/\varepsilon}, \varepsilon^2 X_{t/\varepsilon})_{t \geq \varepsilon t_0} \quad \Longrightarrow \quad \left(\mathcal{L}_t, \int_0^t \mathcal{L}_s \, \mathrm{d}s\right)_{t>0}.$$

6.1 Large time behavior of the Lévy driving noise

Since the Lévy noise is no longer self-similar, we need to study its large time behavior. We dedicate this subsection to the study of the convergence in distribution of the rescaled noise $(L_t^{(\varepsilon)})_{t \geq 0} := (v_\varepsilon L_{t/\varepsilon})_{t \geq 0}$, for a suitable rate v_ε, tending to 0.
We introduce another assumption on ν, which will sometimes be imposed in the sequel:

$$\nu(\mathrm{d}z) = \frac{g(z)}{|z|^{1+\alpha}} \, \mathrm{d}z, \text{ where } g \text{ is a measurable function with } \int_1^{+\infty} \frac{|g(z) - g(-z)|}{z^\alpha} \, \mathrm{d}x < +\infty. \tag{H_g}$$

Inspired from [RT11], we get the following result.

Proposition 6.3. *Let $(L_t)_{t \geq 0}$ be a Lévy process having generating triplet (A, ν, b), with respect to the truncation function $h : z \mapsto -1 \vee (z \wedge 1)$.*

(i) Assume that the Lévy measure satisfies the condition $(H_1^{\nu,\alpha})$ with $\alpha = 1$ and (H_g). Then the process $\left(\varepsilon L_{t/\varepsilon}\right)_{t \geq 0}$ converges in distribution to the strictly 1-stable Lévy process $\mathcal{L}$ generated by $(0, \nu^*, b^*)$, where $b^* := b + \int_1^{+\infty} \frac{g(z) - g(-z)}{z^{1+\alpha}} (z - h(z)) \, \mathrm{d}z$ and

$$\nu^*(\mathrm{d}z) := \frac{c^+ \mathbb{1}_{z>0} + c^- \mathbb{1}_{z<0}}{|z|^2} \, \mathrm{d}z.$$

(ii) Suppose that the Lévy measure satisfies the condition $(H_1^{\nu,\alpha})$ *with* $1 < \alpha < 2$ *and* (H_g)*. If*

$$b^* := b + \int_1^{+\infty} \frac{g(z) - g(-z)}{z^{1+\alpha}} (z - h(z)) \, \mathrm{d}z \neq 0,$$

then the process $\left(\varepsilon L_{t/\varepsilon}\right)_{t\geq 0}$ *converges in distribution to the deterministic Lévy process* $\mathcal{L}$ *generated by* $(0, 0, b^*)$*.*

(iii) If the Lévy measure satisfies (H_2^{ν,α_0}) *and* $b^* := b + \int_{|z|\geq 1} (z - h(z)) \, \nu(\mathrm{d}z) \neq 0$*, then the process* $\left(\varepsilon L_{t/\varepsilon}\right)_{t\geq 0}$ *converges in distribution to the deterministic Lévy process* L^* *generated by* $(0, 0, b^*)$*.*

(iv) Assuming that $0 < \alpha < 1$*, if the Lévy measure satisfies the condition* $(H_1^{\nu,\alpha})$*, then the process* $\left(\varepsilon^{1/\alpha} L_{t/\varepsilon}\right)_{t\geq 0}$ *converges in distribution to the* α*-stable Lévy process* $\mathcal{L}$ *with*

$$\nu^*(\mathrm{d}z) := \frac{c^+ \mathbb{1}_{z>0} + c^- \mathbb{1}_{z<0}}{|z|^{1+\alpha}} \, \mathrm{d}z.$$

(v) Assuming that $1 < \alpha < 2$ *and* $b = 0$*, if the Lévy measure satisfies the condition* $(H_1^{\nu,\alpha})$*, then the process* $\left(\varepsilon^{1/\alpha} L_{t/\varepsilon}\right)_{t\geq 0}$ *converges in distribution to the Lévy process* $\mathcal{L}$ *with measure and center*

$$\nu^*(\mathrm{d}z) := \frac{c^+ \mathbb{1}_{z>0} + c^- \mathbb{1}_{z<0}}{|z|^{1+\alpha}} \, \mathrm{d}z, \quad b^* := \int_{\mathbb{R}^*} z \nu^*(\mathrm{d}z).$$

Proof. From the Lévy-Khintchine's formula, the generating triplet of $(L^{(\varepsilon)})_{t\geq 0} := (v_\varepsilon L_{t/\varepsilon})_{t\geq 0}$ is given by

$$A^\varepsilon = \frac{v_\varepsilon^2}{\varepsilon} A; \tag{30}$$

$$\text{for all } B \in \mathcal{B}(\mathbb{R}), \ \nu^\varepsilon(B) = \varepsilon^{-1} \nu(\{z, z v_\varepsilon \in B\}); \tag{31}$$

$$b^\varepsilon = \frac{v_\varepsilon}{\varepsilon} \left[b + \int_{\mathbb{R}^*} \left(v_\varepsilon^{-1} h(v_\varepsilon z) - h(z) \right) \nu(\mathrm{d}z) \right]. \tag{32}$$

Call (A^*, ν^*, b^*) the generating triplet of the limiting process $\mathcal{L}$. By Corollary 3.6 p. 415 in [JS03] and Theorem 14.7 p. 81 in [Sat99], we have to check that, as $\varepsilon \to 0$,

$$b^\varepsilon \quad \longrightarrow \quad b^*, \tag{33}$$

$$A^\varepsilon + \int_{\mathbb{R}^*} h^2(z) \nu^\varepsilon(\mathrm{d}z) \quad \longrightarrow \quad A^* + \int_{\mathbb{R}^*} h^2(z) \nu^*(\mathrm{d}z), \tag{34}$$

and that for any continuous and bounded function f which is zero in a neighborhood of zero,

$$\int_{\mathbb{R}^*} f(z) \nu^\varepsilon(\mathrm{d}z) \quad \longrightarrow \quad \int_{\mathbb{R}^*} f(z) \nu^*(\mathrm{d}z). \tag{35}$$

(i) Recall that $\alpha = 1$ and assume that the Lévy measure ν satisfies the conditions $(H_1^{\nu,\alpha})$ and (H_g).
To prove (33), we write b^ε as

$$b^\varepsilon = b + \int_0^{+\infty} \frac{g(z) - g(-z)}{z^{1+\alpha}} \left(\varepsilon^{-1} h(\varepsilon z) - h(z) \right) \mathrm{d}z.$$

The dominated convergence theorem can be applied, and we show that it converges to

$$b^* = b + \int_1^{+\infty} \frac{g(x) - g(-x)}{x^2} (x - h(x)) \, \mathrm{d}x.$$

Observe that the condition (H_g) was only required for this step.
Afterwards, note that, using a change of variables,

$$\int_{\mathbb{R}^*} h^2(z)\nu^\varepsilon(\mathrm{d}z) = v_\varepsilon^\alpha \varepsilon^{-1} \int_{\mathbb{R}^*} h^2(z) \frac{g(zv_\varepsilon^{-1})}{|z|^{1+\alpha}} \, \mathrm{d}z,$$

and thus, we can apply the dominated convergence theorem to prove that the last integral converges to

$$\int_{\mathbb{R}^*} h^2(z)\nu^*(\mathrm{d}z).$$

Let f be a continuous and bounded function which is zero in a neighborhood of zero, then, using again a change of variables and applying the dominated convergence theorem,

$$\int_{\mathbb{R}^*} f(z)\nu^\varepsilon(\mathrm{d}z) \quad \underset{\varepsilon \to 0}{\longrightarrow} \quad \int_{\mathbb{R}^*} f(z)\nu^*(\mathrm{d}z).$$

(ii) The proof is similar to the previous one and thus is left to the reader.

(iii) In this point, we assume that (H_2^{ν,α_0}) holds.
The convergence (33) follows from the dominated convergence theorem.
Using the explicit form of the truncation function, we get, for $v_\varepsilon < 1$,

$$\begin{aligned} \int_{\mathbb{R}^*} h^2(z)\nu^\varepsilon(\mathrm{d}z) &= \int_{\mathbb{R}^*} \varepsilon^{-1} h^2(v_\varepsilon z)\nu(\mathrm{d}z) \\ &= \varepsilon \int_{0<|z|<1} z^2 \nu(\mathrm{d}z) + \int_{1 \le |z| \le \varepsilon^{-1}} \varepsilon z^2 \nu(\mathrm{d}z) + \int_{|x| > \varepsilon^{-1}} \varepsilon^{-1} \nu(\mathrm{d}z). \end{aligned} \tag{36}$$

Using the property of a Lévy measure, the first term in (36) converges to zero, as $\varepsilon \to 0$, by assumption. Then, the last two terms in (36) are lower than

$$\varepsilon^{\alpha_0 - 1} \int_{1<|z|<\varepsilon^{-1}} |z|^{\alpha_0} \, \nu(\mathrm{d}z) + \varepsilon^{\alpha_0 - 1} \int_{|z| \ge \varepsilon^{-1}} |z|^{\alpha_0} \, \nu(\mathrm{d}z),$$

which converges to zero, when ε goes to zero, since $\alpha_0 > 1$.
Let f be a continuous and bounded function and assume that there exists $\delta > 0$ such that $f(z) = 0$ for all $|z| \le \delta$, thus,

$$\int_{\mathbb{R}^*} f(z)\nu^\varepsilon(\mathrm{d}z) = \int_{|z| > \frac{\eta}{\varepsilon}} \varepsilon^{-1} f(v_\varepsilon z)\nu(\mathrm{d}z) \le C\varepsilon^{\alpha_0 - 1} \int_{|z| > \frac{\eta}{\varepsilon}} |z|^{\alpha_0} \, \nu(\mathrm{d}z).$$

This vanishes, as $\varepsilon \to 0$.

(iv) Take $0 < \alpha < 1$.
Using the explicit form of h, giving in Proposition 6.3, we have,

$$\begin{aligned} b^\varepsilon = \varepsilon^{\frac{1}{\alpha}-1} b &+ \int_{-\infty}^{-\varepsilon^{-\frac{1}{\alpha}}} [\varepsilon^{\frac{1}{\alpha}-1} - \varepsilon^{-1}]\nu(\mathrm{d}z) + \int_{\varepsilon^{-\frac{1}{\alpha}}}^{+\infty} [\varepsilon^{-1} - \varepsilon^{\frac{1}{\alpha}-1}]\nu(\mathrm{d}z) \\ &+ \varepsilon^{\frac{1}{\alpha}-1} \int_{-\varepsilon^{-\frac{1}{\alpha}}}^{-1} (z+1)\nu(\mathrm{d}z) + \varepsilon^{\frac{1}{\alpha}-1} \int_1^{\varepsilon^{-\frac{1}{\alpha}}} (z-1)\nu(\mathrm{d}z). \end{aligned}$$

Since g has finite limits at infinity, for any $\delta > 0$, we can choose $\eta > 1$ big enough so that $|g(z) - c^+| < \delta$ for $z \geq \eta$, and $|g(z) - c^-| < \delta$ for $z \leq -\eta$.
Hence,

$$\limsup_{\varepsilon \to 0} \varepsilon^{\frac{1}{\alpha}-1} \int_1^{\varepsilon^{-\frac{1}{\alpha}}} (z-1)\nu(\mathrm{d}z) = \limsup_{\varepsilon \to 0} \varepsilon^{\frac{1}{\alpha}-1} \int_\eta^{\varepsilon^{-\frac{1}{\alpha}}} (z-1)\nu(\mathrm{d}z)$$
$$\leq \limsup_{\varepsilon \to 0} \varepsilon^{\frac{1}{\alpha}-1}(c^+ + \delta) \int_\eta^{\varepsilon^{-\frac{1}{\alpha}}} \frac{x-1}{x^{1+\alpha}}\,\mathrm{d}x = \frac{c^+ + \delta}{1-\alpha}.$$

Similarly,

$$\liminf_{\varepsilon \to 0} \varepsilon^{\frac{1}{\alpha}-1} \int_1^{\varepsilon^{-\frac{1}{\alpha}}} (z-1)\nu(\mathrm{d}z) = \liminf_{\varepsilon \to 0} \varepsilon^{\frac{1}{\alpha}-1} \int_\eta^{\varepsilon^{-\frac{1}{\alpha}}} (z-1)\nu(\mathrm{d}z)$$
$$\geq \liminf_{\varepsilon \to 0} \varepsilon^{\frac{1}{\alpha}-1}(c^+ - \delta) \int_\eta^{\varepsilon^{-\frac{1}{\alpha}}} \frac{z-1}{z^{1+\alpha}}\,\mathrm{d}z = \frac{c^+ - \delta}{1-\alpha}.$$

The choice of δ being arbitrary, we get

$$\lim_{\varepsilon \to 0} \varepsilon^{\frac{1}{\alpha}-1} \int_1^{\varepsilon^{-\frac{1}{\alpha}}} (z-1)\nu(\mathrm{d}z) = \frac{c^+}{1-\alpha}.$$

If ε is small enough, then we can upper bound

$$\limsup_{\varepsilon \to 0} \int_{\varepsilon^{-\frac{1}{\alpha}}}^{+\infty} [\varepsilon^{-1} - \varepsilon^{\frac{1}{\alpha}-1}]\nu(\mathrm{d}z) \leq \limsup_{\varepsilon \to 0} (\varepsilon^{-1} - \varepsilon^{\frac{1}{\alpha}-1})(c^+ + \delta)\nu([\varepsilon^{-\frac{1}{\alpha}}, +\infty))$$
$$= \limsup_{\varepsilon \to 0} (1 - \varepsilon^{\frac{1}{\alpha}}) \frac{c^+ + \delta}{\alpha} = \frac{c^+ + \delta}{\alpha}.$$

Moreover,

$$\liminf_{\varepsilon \to 0} \int_{\varepsilon^{-\frac{1}{\alpha}}}^{+\infty} [\varepsilon^{-1} - \varepsilon^{\frac{1}{\alpha}-1}]\nu(\mathrm{d}z) \geq \liminf_{\varepsilon \to 0} (\varepsilon^{-1} - \varepsilon^{\frac{1}{\alpha}-1})(c^+ - \delta)\nu([\varepsilon^{-\frac{1}{\alpha}}, +\infty])$$
$$= \liminf_{\varepsilon \to 0} (1 - \varepsilon^{\frac{1}{\alpha}}) \frac{c^+ - \delta}{\alpha} = \frac{c^+ - \delta}{\alpha}.$$

Similarly, this leads to

$$\lim_{\varepsilon \to 0} \varepsilon^{\frac{1}{\alpha}-1} \int_{-\varepsilon^{-\frac{1}{\alpha}}}^{-1} (z-1)\nu(\mathrm{d}z) = -\frac{c^-}{1-\alpha},$$

and,

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{-\varepsilon^{-\frac{1}{\alpha}}} [\varepsilon^{\frac{1}{\alpha}-1} - \varepsilon^{-1}]\nu(\mathrm{d}z) = -\frac{c^-}{\alpha}.$$

Hence, we obtain

$$\lim_{\varepsilon \to 0} b^\varepsilon = \frac{c^+ - c^-}{\alpha(1-\alpha)}.$$

Since, $\int_{\mathbb{R}^*} h(z)\nu^*(\mathrm{d}z) = \frac{c^+ - c^-}{\alpha(1-\alpha)}$, the drift coefficient of the limiting process equals zero.
The proof of (34) and (35) are identical to the one done in *(i)*.

(v) Take $1 < \alpha < 2$ and assume that $b = 0$.
After a change of variables, we can apply the dominated convergence theorem to

$$b^\varepsilon = \int_{\mathbb{R}^*} \left(h(y) - \varepsilon^{1/\alpha} h(y\varepsilon^{-1/\alpha}) \right) \frac{g(y\varepsilon^{-1/\alpha})}{|y|^{1+\alpha}} \, \mathrm{d}y.$$

The proof of (34) and (35) are identical to the one done in *(i)*.

□

6.2 Proofs of Theorems 6.1 and 6.2

In this section, we suppose that $\beta > 1 + p_\alpha(\gamma) - \theta$, where θ is the exponent of the rate of convergence given in Proposition 6.3 and $p_\alpha(\gamma)$ is given in the statement of Theorems 6.1 and 6.2. Recall that $(V_t)_{t \geq t_0}$ is a solution of (SKE).

6.2.1 Moment estimates of the velocity process

As in Section 4, we will show that there exists a constant $C_{\gamma,\kappa,\beta,t_0}$ such that

$$\forall t \geq t_0, \ \mathbb{E}\left[|V_t|^\kappa\right] \leq C_{\gamma,\kappa,\beta,t_0,b} t^{p_\alpha(\gamma,\kappa)}, \tag{37}$$

where $p_\alpha(\gamma, \kappa)$ has to be detailed.

Proposition 6.4. *Pick $\alpha \in (0,1)$. Assume $(H_1^{\nu,\alpha})$ and (H_{sgn}). For any γ, β, the explosion time τ_∞ is a.s. infinite and for all $\kappa \in [0, \alpha)$, there exists $C_{\kappa,t_0,b}$ such that, we have*

$$\forall t \geq t_0, \ \mathbb{E}\left[|V_t|^\kappa\right] \leq C_{\kappa,t_0,b} t^{\frac{\kappa}{\alpha}}, \ \textit{or equivalently } p_\alpha(\gamma,\kappa) = \frac{\kappa}{\alpha}.$$

Proof. The proof is analogous to the proof of Proposition 4.2. The term $\int_{t_0}^t b f_n'(V_s) \, \mathrm{d}s$ is bounded by $|b|\, t$ since $\|f_n'\|_\infty \leq 1$. Moreover, since ν satisfies $(H_1^{\nu,\alpha})$ with $\alpha < 1$, the process $L_t^+ := \sum_{s \leq t} |\Delta L_s|$ satisfies the conditions of Theorem 3.1 c) in [DS15]. Thus, for all $\kappa \in [0, \alpha)$, for all $t \geq t_0$,

$$\mathbb{E}\left[\left|L_t^+\right|^\kappa\right] \leq C_{t_0,\kappa} t^{\frac{\kappa}{\alpha}}.$$

The estimates for V follows. □

Proposition 6.5. *Assume either (H_2^{ν,α_0}) or $(H_1^{\nu,\alpha})$ with $\alpha \in (1,2)$. For any $\gamma \in [0,1)$ and any $\beta \in \mathbb{R}$ the explosion time τ_∞ is a.s. infinite and for all $\kappa \in [0,1]$, there exists $C_{\gamma,\kappa,\beta,t_0}$ and C_{γ,t_0} such that under $(H_1^{\nu,\alpha})$, we have*

$$\forall t \geq t_0, \ \mathbb{E}\left[|V_t|^\kappa\right] \leq C_{\gamma,\kappa,\beta,t_0} t^{\frac{\kappa}{\alpha}} + C_{\gamma,t_0} |b|^\kappa t^\kappa. \tag{38}$$

Or equivalently $p_\alpha(\gamma,\kappa) = \frac{\kappa}{\alpha}$ if $b = 0$, and $p_\alpha(\gamma,\kappa) = \kappa$, else.
And under (H_2^{ν,α_0}), there exists $C_{\gamma,\kappa,\beta,t_0,b}$ such that we have

$$\forall t \geq t_0, \ \mathbb{E}\left[|V_t|^\kappa\right] \leq C_{\gamma,\kappa,\beta,t_0,b} t^\kappa.$$

Proof. We explain the differences with respect to the proof of Proposition 4.3. Under each hypothesis on the Lévy measure ν, the Lévy process has a first finite moment. We write L_t as the sum $bt + \widehat{L}_t$, where $\widehat{L}$ is the Lévy process without the drift part.
We get,

$$\mathbb{E}\left[|V_{t\wedge\tau_r-}|\right] \leq C_{t_0}\,|b|\,t + \mathbb{E}\left[\left|\widehat{L}_{t\wedge\tau_r-}\right|\right] + K\int_{t_0}^{t} s^{-\beta}\mathbb{E}\left[|V_{s\wedge\tau_r-}|\right]^{\gamma}\mathrm{d}s.$$

The proof of Theorem 3.1 (a) p. 3861 in [DS15] can be adapted to estimate the moment of the Lévy process stopped at the stopping time τ_r, given by (3).

(i) If the Lévy measure ν satisfies $(H_1^{\nu,\alpha})$ with $1 < \alpha < 2$, then it satisfies the conditions of Theorem 3.1 (a) and (c) in [DS15]. Thus, for all $\kappa \in [0,1]$, there exists $C_{t_0,\kappa}$ such that, for all $t \geq t_0$,

$$\mathbb{E}\left[\left|\widehat{L}_{t\wedge\tau_r-}\right|^{\kappa}\right] \leq C_{t_0,\kappa}t,$$

and,

$$\mathbb{E}\left[\left|\widehat{L}_{t}\right|^{\kappa}\right] \leq C_{t_0,\kappa}t^{\frac{\kappa}{\alpha}}.$$

(ii) If the Lévy measure ν satisfies (H_2^{ν,α_0}), then it satisfies the conditions of Theorem 3.1 (a) in [DS15]. Thus, for all $\kappa \in [0,1]$, there exists $C_{t_0,\kappa}$ such that, for all $t \geq t_0$,

$$\mathbb{E}\left[\left|\widehat{L}_{t\wedge\tau_r-}\right|^{\kappa}\right] \leq C_{t_0,\kappa}t.$$

Applying again the Gronwall-type lemma (see Lemma A.1) and Fatou's lemma, for $\beta \neq 1$, we end up with

$$\forall t \geq t_0,\ \mathbb{E}\left[|V_{t\wedge\tau_\infty}|\right] \leq \liminf_{r\to+\infty}\mathbb{E}\left[|V_{t\wedge\tau_r-}|\right] \leq C_\gamma\left[C_{t_0,b}t + \left(\frac{1-\gamma}{1-\beta}K(t^{1-\beta}-t_0^{1-\beta})\right)^{\frac{1}{1-\gamma}}\right] < +\infty.$$

The case $\beta = 1$ can be done in a similar manner.
We conclude that the explosion time of V is a.s. infinite. To refine the estimates under the hypothesis $(H_1^{\nu,\alpha})$, we apply again the Gronwall-type lemma to

$$\mathbb{E}\left[|V_t|\right] \leq C_{t_0}\,|b|\,t + \mathbb{E}\left[\left|\widehat{L}_t\right|\right] + K\int_{t_0}^{t} s^{-\beta}\mathbb{E}\left[|V_s|\right]^{\gamma}\mathrm{d}s.$$

This proves (38) since $\frac{\gamma-1}{\alpha} + 1 - \beta \leq 0$. □

Proposition 6.6. *Assume that for all $v \in \mathbb{R}$, $vF(v) \geq 0$. Pick $\gamma \in \mathbb{R}$ and $\beta \in \mathbb{R}$. For each of the following cases, the explosion time τ_∞ is a.s. infinite.*

(i) Assume $(H_1^{\nu,\alpha})$ and $\alpha \in (1,2)$. Then,
for $\kappa \in [0,1]$, there exists C_{κ,t_0} such that

$$\forall t \geq t_0,\ \mathbb{E}\left[|V_t|^{\kappa}\right] \leq C_{\kappa,t_0}t^{\kappa},\ \textit{or equivalently } p_\alpha(\gamma,\kappa) = \kappa;$$

for $\kappa \in (1,\alpha)$, $\gamma \in [0,1)$ and $b = 0$, there exists C_{κ,t_0} such that

$$\forall t \geq t_0,\ \mathbb{E}\left[|V_t|^{\kappa}\right] \leq C_{\kappa,t_0}t^{\frac{3\kappa}{2\alpha}},\ \textit{or equivalently } p_\alpha(\gamma,\kappa) = \frac{3\kappa}{2\alpha};$$

for $\kappa \in (1,\alpha)$, there exists C_{κ,t_0} such that

$$\forall t \geq t_0,\ \mathbb{E}\left[|V_t|^{\kappa}\right] \leq C_{\kappa,t_0}t^{\frac{\kappa}{\alpha}+\frac{\kappa}{2}},\ \textit{or equivalently } p_\alpha(\gamma,\kappa) = \frac{\kappa}{\alpha} + \frac{\kappa}{2}.$$

(ii) Assume (H_2^{ν,α_0})*. Then,*
for $\kappa \in [0,1]$*, there exists* C_{κ,t_0} *such that,*

$$\forall t \geq t_0,\ \mathbb{E}\left[|V_t|^\kappa\right] \leq C_{\kappa,t_0} t^\kappa,\ \textit{or equivalently } p(\gamma,\kappa) = \kappa;$$

for $\kappa \in [0,\alpha_0]$*, there exists* C_{κ,t_0} *such that*

$$\forall t \geq t_0,\ \mathbb{E}\left[|V_t|^\kappa\right] \leq C_{\kappa,t_0} t^{\frac{\kappa}{\alpha_0}+\frac{\kappa}{2}},\ \textit{or equivalently } p(\gamma,\kappa) = \frac{\kappa}{\alpha_0} + \frac{\kappa}{2}.$$

Proof. We highlight only the differences with respect to the proof of Proposition 4.5. When working under (H_2^{ν,α_0}), we will pick $\kappa \leq \alpha_0$ during the proof.
Step 1. We adapt the estimates of the Itô's formula's terms.
There is an additional term in (12), given, for $t \geq t_0$, by

$$\int_{t_0}^{t\wedge\tau_r-} f'(V_s) b \,\mathrm{d}s.$$

The other terms of (12) can be estimated in the same way. This leads to the same bound as in (19):

$$\mathbb{E}\left[|V_{t\wedge\tau_r-}|^\kappa\right] \leq \mathbb{E}\left[f(V_{t\wedge\tau_r-})\right] \leq \mathbb{E}\left[f(V_{t_0})\right] + \int_{t_0}^t |b|\,\mathbb{E}\left[|f'\left(V_{s\wedge\tau_r-}\right)|\right]\mathrm{d}s + \mathbb{E}\left[t\wedge\tau_r\right] \times$$
$$\left(\eta^{\kappa/2}\nu(|z|\geq 1) + \int_{|z|\geq 1} |z|^\kappa\,\nu(\mathrm{d}z) + \frac{1}{2}\kappa(3-\kappa)\eta^{\kappa/2-1}\int_{0<|z|<1} z^2\nu(\mathrm{d}z)\right) \leq C_{\kappa,t_0} t.$$

Hence, this step can be concluded identically as in the proof of Proposition 4.5.
Step 2. We estimate R, given by (14) in another way.
The inequality (21) becomes

$$\mathbb{E}\left[|V_{t\wedge\tau_r-}|^\kappa\right] \leq \mathbb{E}\left[f(V_{t\wedge\tau_r-})\right] \leq C_{\kappa,t_0} t + C\int_{t_0}^t \mathbb{E}\left[|V_s|^{\kappa/2}\right]\mathrm{d}s + \int_{t_0}^t |b|\,\mathbb{E}\left[|V_s|^{\kappa-1}\right]\mathrm{d}s.$$

Hence, since $\kappa - 1 \leq \frac{\kappa}{2}$, using the Step 1, we conclude. The last step is identical under $(H_1^{\nu,\alpha})$ and can be done with α_0 instead of $\alpha - \varepsilon$, under (H_2^{ν,α_0}). □

6.2.2 Proofs of Theorems 6.1 and 6.2

We are now in position to give the proofs of Theorems 6.1 and 6.2. Assume that either $(H_1^{\nu,\alpha})$ or (H_2^{ν,α_0}) is satisfied.
As in the α-stable case (see Section 5), it suffices to prove the convergence of the rescaled velocity process $(V_t^{(\varepsilon)})_{t>0} := (v_\varepsilon V_{t/\varepsilon})_{t>0}$.
Thanks to Proposition 6.3, there exists $\theta \in \{1, \frac{1}{\alpha}\}$ such that $L^{(\varepsilon)} := (\varepsilon^\theta L_{t/\varepsilon})_{t\geq 0}$ converges in distribution.
For sake of simplicity, we omit the dependencies of p with respect to α and α_0. Assume that $\gamma \geq 0$ and $\beta > 1 + p(\gamma) - \theta$.
We can write, for $T \geq \varepsilon t_0$,

$$\sup_{\varepsilon t_0 \leq t \leq T} \left|V_t^{(\varepsilon)} - L_t^{(\varepsilon)}\right| \leq v_\varepsilon(v_0 - L_{t_0}) + v_\varepsilon^{1-\gamma}\varepsilon^{\beta-1}\int_{\varepsilon t_0}^T K\left|V_u^{(\varepsilon)}\right|^\gamma u^{-\beta}\,\mathrm{d}u.$$

Using the moment estimates on V (see Section 6.2.1), this leads, with $\beta \neq p(\gamma)+1$, to

$$v_\varepsilon^{1-\gamma}\varepsilon^{\beta-1}\mathbb{E}\left[\int_{\varepsilon t_0}^{T} K\left|V_u^{(\varepsilon)}\right|^\gamma u^{-\beta}\,\mathrm{d}u\right] \leq C\left(v_\varepsilon \varepsilon^{\beta-1-p(\gamma)}T^{p(\gamma)-\beta+1} - t_0^{p(\gamma)-\beta+1}v_\varepsilon\right).$$

Hence, setting $q := \min(\beta-1+\theta-p(\gamma),\theta)$, which is a positive number since $\beta > 1+p(\gamma)-\theta$, we obtain

$$\mathbb{E}\left[\sup_{\varepsilon t_0 \leq t \leq T}\left|V_t^{(\varepsilon)} - L_t^{(\varepsilon)}\right|\right] \underset{\varepsilon\to 0}{=} O(\varepsilon^q).$$

The case $\beta = 1+p(\gamma)$ can be treated similarly. This concludes both of the proofs.

6.3 Lévy driving noise with a Brownian component

We extend the results given in Theorems 6.1 and 6.2 to a driving noise having a Brownian component. To this end, we decompose the Lévy noise as the sum of a Brownian part $(B_t)_{t\geq 0}$ and a Lévy part without a Brownian component $(L_t)_{t\geq 0}$.

Remark 6.7. *The reasoning is the same with a Lévy process $\widehat{L}$ instead of the Brownian driving process. Indeed, let $\widehat{v}_\varepsilon$ be the rate such that $(\widehat{v}_\varepsilon \widehat{L}_{t/\varepsilon})_{t\geq 0}$ converges. The conclusion follows, provided that $\widehat{v}_\varepsilon^{-1}v_\varepsilon$ tends to 0.*

Pick $\rho \geq 0$. We consider the following one-dimensional stochastic kinetic model, for $t \geq t_0 > 0$,

$$\begin{aligned} \mathrm{d}V_t^{(2)} &= \mathrm{d}L_t + \mathrm{d}B_t - t^{-\beta}\rho\,\mathrm{sgn}(V_t^{(2)})\left|V_t^{(2)}\right|^\gamma \mathrm{d}t,\ V_{t_0}^{(2)} = v_0 \in \mathbb{R}, \\ \text{and} \quad \mathrm{d}X_t^{(2)} &= V_t^{(2)}\,\mathrm{d}t,\ X_{t_0}^{(2)} = x_0 \in \mathbb{R}. \end{aligned} \tag{SKE2}$$

We compare this solution with the solution of the equation driven by a Lévy process L without a Brownian component. From now on, for sake of convenience, the solutions of (SKE) with $F(v) = \rho\,\mathrm{sgn}(v)\,|v|^\gamma$, will be denoted by $(V_t^{(1)}, X_t^{(1)})_{t\geq t_0}$ instead of $(V_t, X_t)_{t\geq t_0}$.
The asymptotic behavior of the latter SDE is given by Theorems 6.1 and 6.2. We will show that the Brownian part has no contribution.
Let us first point out some results about existence up to explosion of (SKE2).

Proposition 6.8.

(i) When $\gamma \geq 1$, (SKE2) *admits a unique solution up to explosion.*

(ii) Pick $\alpha \in (1,2)$. If (SKE2) *is driven by the sum of a Brownian motion and an α-stable process, then, if $0 \leq \gamma < 1$, there exists a solution of* (SKE2).

Proof.

(i) Since the drift coefficient is locally Lipschitz, by a standard localization argument, using Theorem 3.1 and Remark 2 p. 338-339 in [Kun04], there is a unique solution defined up to explosion.

(ii) Since the drift coefficient is a continuous function, using a standard localization argument, we can apply Theorem 3.1 p. 866 in [Kur07] to conclude.

□

Assume in the following that (SKE2) admits a unique solution up to explosion.

Theorem 6.9. *Consider $\gamma \geq 0$ and $\beta \geq 0$. Let $(V_t^{(2)}, X_t^{(2)})_{t\geq t_0}$ be a solution to* (SKE2)*. The Lévy noise is supposed to be the sum of a Brownian noise $(B_t)_{t\geq 0}$ and a Lévy process $(L_t)_{t\geq 0}$ without Brownian component. Suppose also that* (SKE) *satisfies the conditions of either Theorems 6.1 or 6.2. Then $(V^{(2)}, X^{(2)})$ has the same asymptotic behavior as $(V^{(1)}, X^{(1)})$, which is given in Theorems 6.1 and 6.2.*

Proof. For $i \in \{1, 2\}$, let us introduce the following rescaled processes:

$$(V_t^{(i,\varepsilon)})_{t\geq \varepsilon t_0} := (v_\varepsilon V_{t/\varepsilon}^{(i)})_{t\geq t_0},\ (X_t^{(i,\varepsilon)})_{t\geq \varepsilon t_0} := (\varepsilon v_\varepsilon X_{t/\varepsilon}^{(i)})_{t\geq t_0},\ \text{and } (B_t^{(\varepsilon)})_{t\geq 0} := (\sqrt{\varepsilon} B_{t/\varepsilon})_{t\geq 0}.$$

We write, for all $t \geq \varepsilon t_0$,

$$V_t^{(1,\varepsilon)} - V_t^{(2,\varepsilon)} = v_\varepsilon B_{t_0} - v_\varepsilon \varepsilon^{-1/2} B_t^{(\varepsilon)} - v_\varepsilon^{1-\gamma}\varepsilon^{\beta-1}\int_{\varepsilon t_0}^t v_\varepsilon^\gamma \left(F\left(\frac{V_u^{(1,\varepsilon)}}{v_\varepsilon^\gamma}\right) - F\left(\frac{V_u^{(2,\varepsilon)}}{v_\varepsilon^\gamma}\right)\right) u^{-\beta}\,\mathrm{d}u.$$

Hence, by Itô's formula, we get

$$\begin{aligned}\left(V_t^{(1,\varepsilon)} - V_t^{(2,\varepsilon)}\right)^2 &= v_\varepsilon^2 B_{t_0}^2 - 2\int_{\varepsilon t_0}^t \left(V_s^{(1,\varepsilon)} - V_s^{(2,\varepsilon)}\right) v_\varepsilon \varepsilon^{-1/2}\,\mathrm{d}B_s^{(\varepsilon)} + v_\varepsilon^2\varepsilon^{-1}(t-\varepsilon t_0)\\ &\quad - 2v_\varepsilon^{1-\gamma}\varepsilon^{\beta-1}\int_{\varepsilon t_0}^t \left(V_s^{(1,\varepsilon)} - V_s^{(2,\varepsilon)}\right) v_\varepsilon^\gamma \left(F\left(\frac{V_u^{(1,\varepsilon)}}{v_\varepsilon^\gamma}\right) - F\left(\frac{V_u^{(2,\varepsilon)}}{v_\varepsilon^\gamma}\right)\right) u^{-\beta}\,\mathrm{d}u.\end{aligned}$$

The last term on the right-hand side of the upper equality is positive since F is an increasing function.
Moreover, we can apply BDG inequalities (see Theorem 73 p. 222 in [Pro05]) to the local martingale, defined by

$$M_t := -2\int_{\varepsilon t_0}^t \left(V_s^{(1,\varepsilon)} - V_s^{(2,\varepsilon)}\right) v_\varepsilon \varepsilon^{-1/2}\,\mathrm{d}B_s^{(\varepsilon)}.$$

This yields, for each $T > 0$,

$$\mathbb{E}\left[\sup_{\varepsilon t_0\leq t\leq T} M_t^2\right] \leq \mathbb{E}\left[4v_\varepsilon^2\varepsilon^{-1}\int_{\varepsilon t_0}^T \left(V_s^{(1,\varepsilon)} - V_s^{(2,\varepsilon)}\right)^2 \mathrm{d}s\right].$$

For the sake of clarity, given x and y, we introduce $x \lesssim y$ to mean that there exists a positive constant C such that $x \leq Cy$. We get

$$\begin{aligned}\mathbb{E}\left[\sup_{\varepsilon t_0\leq t\leq T}\left(V_t^{(1,\varepsilon)} - V_t^{(2,\varepsilon)}\right)^4\right] &\lesssim v_\varepsilon^4\mathbb{E}\left[B_{t_0}^4\right] + v_\varepsilon^4\varepsilon^{-2}T^2 + \mathbb{E}\left[\sup_{\varepsilon t_0\leq t\leq T} M_t^2\right]\\ &\lesssim v_\varepsilon^4\mathbb{E}\left[B_{t_0}^4\right] + v_\varepsilon^4\varepsilon^{-2}T^2 + 4v_\varepsilon^2\varepsilon^{-1}\int_{\varepsilon t_0}^T \mathbb{E}\left[\left(V_s^{(1,\varepsilon)} - V_s^{(2,\varepsilon)}\right)^2\right]\mathrm{d}s.\end{aligned}$$

Hence, using that for all real a, $a^2 \leq a^4 + 1$, we deduce that

$$\begin{aligned}&\mathbb{E}\left[\sup_{\varepsilon t_0\leq t\leq T}\left(V_t^{(1,\varepsilon)} - V_t^{(2,\varepsilon)}\right)^4\right]\\ &\qquad\lesssim v_\varepsilon^4\mathbb{E}\left[B_{t_0}^4\right] + v_\varepsilon^4\varepsilon^{-2}T^2 + 4v_\varepsilon^2\varepsilon^{-1}T + 4v_\varepsilon^2\varepsilon^{-1}\int_{\varepsilon t_0}^T \mathbb{E}\left[\sup_{\varepsilon t_0\leq t\leq s}\left(V_t^{(1,\varepsilon)} - V_t^{(2,\varepsilon)}\right)^4\right]\mathrm{d}s.\end{aligned}$$

Applying Gronwall's lemma, up to stopping times, we get

$$\mathbb{E}\left[\sup_{\varepsilon t_0\le t\le T}\left(V_t^{(1,\varepsilon)}-V_t^{(2,\varepsilon)}\right)^4\right]\lesssim\left(v_\varepsilon^4\mathbb{E}\left[B_{t_0}^4\right]+v_\varepsilon^4\varepsilon^{-2}T^2+4v_\varepsilon^2\varepsilon^{-1}T\right)\exp\left(4v_\varepsilon^2\varepsilon^{-1}(T-\varepsilon t_0)\right).$$

We deduce also that

$$\begin{aligned}\mathbb{E}\left[\sup_{\varepsilon t_0\le t\le T}\left(X_t^{(1,\varepsilon)}-X_t^{(2,\varepsilon)}\right)^4\right]&\le T^4\mathbb{E}\left[\sup_{\varepsilon t_0\le t\le T}\left(V_t^{(1,\varepsilon)}-V_t^{(2,\varepsilon)}\right)^4\right]\\&\lesssim T^4\left(v_\varepsilon^4\mathbb{E}\left[B_{t_0}^4\right]+v_\varepsilon^4\varepsilon^{-2}T^2+4v_\varepsilon^2\varepsilon^{-1}T\right)\exp\left(4v_\varepsilon^2\varepsilon^{-1}(T-\varepsilon t_0)\right).\end{aligned}$$

Hence, $\left(V_t^{(1,\varepsilon)}-V_t^{(2,\varepsilon)},X_t^{(1,\varepsilon)}-X_t^{(2,\varepsilon)}\right)_{t\ge\varepsilon t_0}$ converges uniformly towards 0 in probability on compacts, as ε goes to zero.
The conclusion follows from Theorems 6.1, 6.2, Theorem 3.1, p. 27, in [Bil99] and Lemma A.3. □

A Some technical results

Let us state first a Gronwall-type lemma which has been used to get moment estimates.

Lemma A.1 (Gronwall-type lemma)**.** *Fix $r\in[0,1)$ and $t_0\in\mathbb{R}$. Assume that g is a non-negative real-valued function, b is a positive function and a is a differentiable real-valued function. Moreover, suppose that the function bg^r is a continuous function.*
Assume that

$$\forall t\ge t_0,\ g(t)\le a(t)+\int_{t_0}^t b(s)g(s)^r\,\mathrm{d}s. \tag{39}$$

Then, setting $C_r:=2^{\frac{1}{1-r}}$,

$$\forall t\ge t_0,\ g(t)\le C_r\left[a(t)+\left((1-r)\int_{t_0}^t b(s)\,\mathrm{d}s\right)^{\frac{1}{1-r}}\right].$$

Remark A.2. *Call $H(t)$ the right-hand side of* (39)*. If g is not continuous, note that the function H is still continuous and satisfies* (39) *(since b is positive and $g\le H$). So, one can apply the lemma to H and thereafter use the inequality $g\le H$.*

We sketch the proof for the sake of completeness.

Proof of Lemma A.1. For $t\ge t_0$, since $r\ge 0$, and $b(t)>0$,

$$a'(t)+b(t)g(t)^r\le a'(t)+b(t)\left(a(t)+\int_{t_0}^t b(s)g(s)^r\,\mathrm{d}s\right)^r.$$

That is, as $\int_{t_0}^t b(s)g(s)^r\,\mathrm{d}s$ is non-negative,

$$\frac{a'(t)+b(t)g(t)^r}{\left(a(t)+\int_{t_0}^t b(s)g(s)^r\,\mathrm{d}s\right)^r}\le b(t)+\frac{a'(t)}{a(t)^r}.$$

Integrating, since $r\ne 1$, we get

$$(1-r)^{-1}\left[\left(a(t)+\int_{t_0}^t b(s)g(s)^r\,\mathrm{d}s\right)^{1-r}-a(t_0)^{1-r}\right]\le(1-r)^{-1}\left[a(t)^{1-r}-a(t_0)^{1-r}\right]+\int_{t_0}^t b(s)\,\mathrm{d}s,$$

or equivalently, using that $r < 1$, we have

$$H(t)^{1-r} \leq a(t)^{1-r} + (1-r)\int_{t_0}^{t} b(s)\,\mathrm{d}s.$$

Since $\frac{1}{1-r} > 0$ and using (39)

$$g(t) \leq \left(a(t)^{1-r} + (1-r)\int_{t_0}^{t} b(s)\,\mathrm{d}s\right)^{\frac{1}{1-r}} \leq C_r\left[a(t) + \left((1-r)\int_{t_0}^{t} b(s)\,\mathrm{d}s\right)^{\frac{1}{1-r}}\right].$$

This concludes the proof of the lemma. □

We state now a technical lemma about the convergence in the spaces $\mathcal{C}$ and $\mathcal{D}$. We recall that the space of continuous functions $\mathcal{C}$ is endowed with the uniform topology

$$\mathrm{d}_u : f, g \in \mathcal{C}((0,+\infty)) \mapsto \sum_{n=1}^{+\infty} \frac{1}{2^n} \min\Big(1, \sup_{[\frac{1}{n},n]} |f-g|\Big).$$

Set $\Lambda := \{\lambda : \mathbb{R}^+ \to \mathbb{R}^+, \text{continuous and increasing function such that } \lambda(0) = 0, \lim_{t\to+\infty} \lambda(t) = +\infty\}$ and

$$k_n(t) = \begin{cases} 1, & \text{if } \frac{1}{n} \leq t \leq n, \\ n+1-t, & \text{if } n < t < n+1, \\ 0, & \text{if } n+1 \leq t. \end{cases}$$

The space of càdlàg functions $\mathcal{D}$ is endowed with the Skorokhod topology d_s defined for $f, g \in \mathcal{D}((0,+\infty))$ by

$$\sum_{n=1}^{+\infty} \frac{1}{2^n} \min\left(1, \inf\left\{a \text{ s.t. } \exists\lambda \in \Lambda,\ \sup_{s\neq t}\left|\log\frac{\lambda(t)-\lambda(s)}{t-s}\right| \leq a,\ \sup_{t>0}|k_n(t)\,(f\circ\lambda(t) - g(t))| \leq a\right\}\right).$$

Lemma A.3.

(i) The uniform distance is finer than the Skorokhod one i.e. $d_s \leq d_u$.

(ii) Let $(f_\varepsilon)_{\varepsilon\geq 0}, (h_\varepsilon)_{\varepsilon\geq 0}$ be two sequences of functions of $\mathcal{D}$. If for all $n \geq 1$,

$$\sup_{t\in[\frac{1}{n},n]} |f_\varepsilon(t) - h_\varepsilon(t)| \xrightarrow{\mathbb{P}} 0 \text{ as } \varepsilon \to 0,$$

then $\mathrm{d}(f_\varepsilon, h_\varepsilon) \xrightarrow{\mathbb{P}} 0$, where $\mathrm{d} \in \{\mathrm{d}_u, \mathrm{d}_s\}$.

Proof. Let f, g be two càdlàg functions. The first point is true using the definition of the metrics d_s and d_u and noting that

$$\inf\left\{a \text{ s.t. } \exists\lambda \in \Lambda,\ \sup_{s\neq t}\left|\log\frac{\lambda(t)-\lambda(s)}{t-s}\right| \leq a,\ \sup_{t>0}|k_n(t)\,(f(\lambda(t) - g(t))| \leq a\right\} \leq \sup_{[\frac{1}{n+1},n+1]} |f-g|.$$

Let us now prove the second part. Assume that for all $n \geq 1$, $\sup_{[\frac{1}{n},n]} |f_\varepsilon - h_\varepsilon| \xrightarrow{\mathbb{P}} 0$, as $\varepsilon \to 0$. Fix $\eta > 0$ and choose $N > 0$ such that $\sum_{n=N+1}^{+\infty} \frac{1}{2^n} \leq \frac{\eta}{2}$. Then,

$$\mathrm{d}_s(f_\varepsilon, h_\varepsilon) \leq \mathrm{d}_u(f_\varepsilon, h_\varepsilon) \leq \frac{\eta}{2} + \sum_{n=1}^{N} \frac{1}{2^n} \sup_{[\frac{1}{n},n]} |f_\varepsilon - h_\varepsilon|.$$

It follows, by setting $\eta' := \frac{\eta}{2}\left(\sum_{n=1}^{+\infty}\frac{1}{2^n}\right)^{-1}$, that

$$\mathbb{P}\left(\mathrm{d}\left(f_\varepsilon, h_\varepsilon\right) > \eta\right) \leq \sum_{n=1}^{N} \mathbb{P}\left(\sup_{[\frac{1}{n},n]} |f_\varepsilon - h_\varepsilon| > \eta'\right) \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

□

For the sake of completeness, we state and improve the result of Problem 4.12 p. 64 in [KS98], on a general metric space.

Lemma A.4. *Let S be a Polish metric space endowed with a Borel σ-field $\mathcal{S}$. Suppose that $(P_n)_{n\geq 1}$ is a sequence of probability measures on $(S, \mathcal{S})$ which converges weakly to a probability measure P. Suppose, in addition, that the sequence $(f_n)_{n\geq 1}$ of real-valued continuous functions on S is uniformly bounded and converges to a continuous function f, the convergence being uniform on compact subsets of S. Then, we have*

$$\lim_{n\to+\infty} \int_S f_n(\omega)\,\mathrm{d}P_n(\omega) = \int_S f(\omega)\,\mathrm{d}P(\omega).$$

Proof. Notice that, since $(P_n)_{n\geq 1}$ converges weakly thus, it is tight. So, for each $\varepsilon > 0$, there exists a compact subset K of S such that for any $n \geq 1$, $P_n(K) \geq 1 - \varepsilon$.
Let us decompose

$$\int_S f_n\,\mathrm{d}P_n - \int_S f\,\mathrm{d}P = A + B + C + D,$$

where

$$A := \int_{S\setminus K} f_n\,\mathrm{d}P_n, \quad B := \int_K (f_n - f)\,\mathrm{d}P_n, \quad C := \int_{S\setminus K} f\,\mathrm{d}P_n, \quad \text{and} \quad D := \int_S f\,\mathrm{d}P_n - \int_S f\,\mathrm{d}P.$$

Let M be a bound for the sequence (f_n). Thus, by the choice of K,

$$|A| \leq M P_n(S \setminus K) \leq M\varepsilon.$$

The third integral can be treated analogously. Besides, since the sequence (f_n) converges uniformly on K to f, there exists n_ε such that for all $n \geq n_\varepsilon$, $\sup_K |f_n - f| \leq \varepsilon$. Thereby, we get

$$|B| \leq \varepsilon P_n(K).$$

The last integral is smaller than ε for n large enough, since P_n converges weakly to P, and this concludes the proof. □

Remark A.5. *Lemma A.4 could be applied with $S = \mathcal{C}([0, +\infty))$ or $\mathcal{D}([0, +\infty))$. However, the result for $S = \mathcal{C}([0, +\infty))$ is already contained in Problem 4.12 p 64. in [KS98].*

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