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L^1 –Theory for Incompressible Limit of Reaction-Diffusion Porous Medium Flow with Linear Drift

Noureddine Igbida *

December 13, 2021

Abstract

Our aim is to study the limit of the solution of reaction-diffusion porous medium equation with linear drift $\partial_t u - \Delta u^m + \nabla \cdot (uV) = g(t, x, u)$, as $m \rightarrow \infty$. We study the problem in bounded domain Ω with Dirichlet boundary condition, compatible initial data ; i.e. $|u_0| \leq 1$, and an outpointing vector field V on the boundary $\partial\Omega$. In particular, by means of new BV_{loc} estimates, we show uniform L^1 –convergence towards the solution of reaction-diffusion Hele-Shaw flow with linear drift.

1 Introduction and main results

1.1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with regular boundary $\partial\Omega =: \Gamma$. Our aim here is to study the limit, as $m \rightarrow \infty$, of the equation

$$(1.1) \quad \frac{\partial u}{\partial t} - \Delta u^m + \nabla \cdot (uV) = g(t, x, u) \quad \text{in } Q := (0, T) \times \Omega,$$

where the expression r^m points out $|r|^{m-1}r$, for any $r \in \mathbb{R}$, $1 < m < \infty$, $V : \Omega \rightarrow \mathbb{R}^N$ is a given vector field and $g : Q \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory application.

There is a huge literature on qualitative and quantitative studies of (1.1) in the case where $V \equiv 0$. We refer the reader to the book [39] for a thoroughgoing survey of results as well as corresponding literature. The case $V \neq 0$ arise mainly in the theory of population dynamics, where u represents density of a population trying to exit a finite habitat Ω following the vector field V (see for instance [34, 35, 36, 36, 38] and the refs therein). Indeed, the exponent $m > 1$, particularly for large m , describes the anti-crowd leaning of the density motion. Under reasonable assumptions on g and V (let say for instance $g = g(t, x, u)$ Lipschitz continuous in u and $V = V(x)$ regular enough), existence and uniqueness of weak solution, as well as L^1 –comparison principal are more or less well known by now for (1.1) subject to initial data and boundary conditions of Dirichlet or Neumann type(cf. [15, 13, 23]). One can see also [30] for the study of (1.1) in the framework of viscosity solutions. Asymptotic convergence to equilibrium is shown in [13] and [17] when V is the gradient of a convex

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potential. Our main focus here lies in the limit as $m \rightarrow \infty$ of weak solutions in the case where (1.1) is subject to Dirichlet boundary condition and arbitrary compatible initial data $|u_0| \leq 1$, a.e. in Ω . We give proof of the convergence process to the so called Hele-Shaw flow with linear drift in general context of L^1 -theory for nonlinear PDE (cf. [5] and [6]). This approach enables to give answers and evidence to many questions left open in some papers dedicated to this subject. Moreover, it offers many supply for the traitment of the challenging case of non compatible initial data ; i.e. the case where $\|u_0\| > 1$. This will be treated separately in the forthcoming work [29].

1.2 Historical notes

The study of the incompressible limit of (1.3) receives a lot of attention since its interest for the applications and for the description of constrained nonlinear flow. The problem is well understood by now in the case where $V \equiv 0$. Actually, it is well know that the solution of the problem

$$\frac{\partial u}{\partial t} - \Delta u^m = g(\cdot, u) \quad \text{in } Q := (0, T) \times \Omega,$$

converges, as $m \rightarrow \infty$, to the solution of the so called Hele-Shaw problem

$$(1.2) \quad \left. \begin{array}{l} \frac{\partial u}{\partial t} - \Delta p = g(\cdot, u) \\ u \in \text{sign}(p) \end{array} \right\} \quad \text{in } Q.$$

The convergence holds to be true in $\mathcal{C}([0, T], L^1(\Omega))$ in the case where $|u_0| \leq 1$, a.e. in Ω , otherwise it holds in $\mathcal{C}((0, T), L^1(\Omega))$ and a boundary layer appears for $t = 0$. This boundary layer is given by some kind of plateau-like function refereed to as ‘mesa’, and it is given by the limit, as $m \rightarrow \infty$, of the solution of homogeneous porous medium equation (see for instance [8, 7, 10, 11, 28, 25] and the references therein)

$$(1.3) \quad \frac{\partial u}{\partial t} = \Delta u^m \quad \text{in } Q.$$

The problem has been well scouted in the case of homogeneous and nonhomogeneous, Dirichlet and Neumann boundary conditions. Yet, one needs to be careful with the special case of Neumann boundary condition since, in this case the limiting problem (1.2) could be ill posed. With respect to the assumptions on g , the limiting problem exhibits an extra phase to be mixed with the Hele-Shaw phase (see [11] for more details). Other variations of reaction term have been proposed in recent years together with the analysis of their incompressible limit (see for instance [38, 21, 37, 31] and the references therein). The recent work [26] treats once again a particular reaction term $g = g(u)$, with a special focus on the limit of the so called associated pressure $p := \frac{m}{m+1} u^{m-1}$, furthermore the authors seem to be altogether not aware of the general works [10, 11].

The treatment of the case where $V \not\equiv 0$, leads to the formal reaction-diffusion dynamic of Hele-Shaw type with a linear drift ; i.e.

$$(1.4) \quad \left. \begin{array}{l} \frac{\partial u}{\partial t} - \Delta p + \nabla \cdot (u V) = g(t, x, u) \\ u \in \text{sign}(p) \end{array} \right\} \quad \text{in } (0, T) \times \Omega.$$

The problem was studied first in [12] when $g \equiv 0$ and the drift term is of the type $\nabla \cdot F(u)$, with $F : \mathbb{R} \rightarrow \mathbb{R}^N$ a Lipschitz continuous function (this corresponds particularly to space-independent drift). In [12], it is proven that $L^1(\Omega)$ -compactness result remains to be true uniformly in t . Moreover, the limiting problem here is simply the transport equation

$$(1.5) \quad \partial_t u + \nabla \cdot F(u) = 0.$$

The Hele-Shaw flow wear off since the nature of the transport term (incompressible) in (1.5) compel the solution to be less than 1, and then $p \equiv 0$. Then, in [32] the authors studied the case of space dependent drift and reaction terms both linear and regular in $\Omega = \mathbb{R}^N$. Assuming some "strong" conditions on V , which insure some kind of monotonicity properties, and using the notion of viscosity solutions, they study the limit, as $m \rightarrow \infty$, in the case where u_0 is a nonnegative (compatible) initial data. The benefit of their approach is its ability to cover accurately the free boundary view of the limiting problem (particularly the dynamic of the so called congestion region [$p > 0$]), as well as the rate of convergence. Using a weak (distributional) interpretation of the solution the same problem was studied recently in [20] with a variant of reaction term g in $\Omega = \mathbb{R}^N$. Using a blend of recently developed tools on Aronson-Bénilan regularizing effect as well as sophisticated L^p -regularity of the pressure gradient the authors studied the incompressible limit again in the case of nonnegative compatible initial data and regular drift (one can see also [22] for some convergence rate in a negative Sobolev norm).

Here, we study the incompressible limit of (1.3) subject to Dirichlet boundary condition and compatible initial data (even changing sign data). The reaction term satisfies general conditions, including Lipschitz continuous assumptions, and the given velocity field enjoys Sobolev regularity and an outpointing condition on the boundary that we'll precise below. To this aim we use L^1 -nonlinear semi-group theory, which consist in performing first the L^1 -strong compactness for the stationary problem and work with the general theory of nonlinear semi group to pass to the limit in the evolution problem.

At last, let us mention that for other application non-local drift could be concerned as well by the incompressible about porous medium equation. One can see for instance [16] and the references therein on this topic. In particular, the authors of [16] adopt techniques relying on the gradient flow structure of the equation. Our approach here is general and may be adopted for this type of drift and more general one even nonlinear.

1.3 Main results

We assume that $\Omega \subset \mathbb{R}^N$ is a bounded open set, with regular boundary $\partial\Omega$ (say, piecewise \mathcal{C}^2). Throughout the paper, we assume that $V \in W^{1,2}(\Omega)$, $\nabla \cdot V \in L^\infty(\Omega)$ and satisfies the following outward pointing condition on the boundary :

$$(1.1) \quad V \cdot \nu \geq 0 \quad \text{on } \partial\Omega,$$

where ν represents the outward unitary normal to the boundary $\partial\Omega$.

We consider the evolution problem

$$(1.2) \quad \begin{cases} \frac{\partial u}{\partial t} - \Delta u^m + \nabla \cdot (u V) = f & \text{in } Q := (0, T) \times \Omega \\ u = 0 & \text{on } \Sigma := (0, T) \times \partial\Omega \\ u(0) = u_0 & \text{in } \Omega. \end{cases}$$

We denote by $H_0^1(\Omega)$ the Sobolev space

$$H_0^1(\Omega) = \left\{ u \in H^1(\Omega) : u = 0, \mathcal{L}^{N-1}\text{-a.e. in } \partial\Omega \right\}.$$

Definition 1.1 (Notion of solution). *A function u is said to be a weak solution of (1.2) if $u \in L^2(Q)$, $p := u^m \in L^2(0, T; H_0^1(\Omega))$ and*

$$\frac{d}{dt} \int_{\Omega} u \xi + \int_{\Omega} (\nabla p - u V) \cdot \nabla \xi = \int_{\Omega} f \xi, \quad \text{in } \mathcal{D}'(0, T), \quad \forall \xi \in H_0^1(\Omega).$$

We'll say plainly that u is a solution of (1.2) if $u \in \mathcal{C}([0, T], L^1(\Omega))$, $u(0) = u_0$ and u is a weak solution of (1.2).

We denote by sign^+ (resp. sign^-) the maximal monotone graph given by

$$\text{sign}^+(r) = \begin{cases} 1 & \text{for } r > 0 \\ [0, 1] & \text{for } r = 0 \\ 0 & \text{for } r < 0. \end{cases} \quad (\text{resp. } \text{sign}^-(r) = \text{sign}^+(-r), \text{ for } r \in \mathbb{R}).$$

Moreover, we denote by sign_0^\pm the discontinuous applications defined from \mathbb{R} to \mathbb{R} by

$$\text{sign}_0^+(r) = \begin{cases} 1 & \text{for } r > 0 \\ 0 & \text{for } r \leq 0 \end{cases} \quad \text{and } \text{sign}_0^-(r) = \text{sign}_0^+(-r), \text{ for } r \in \mathbb{R}.$$

Theorem 1.1. *If u_1 and u_2 are two weak solutions of (1.2) associated with $f_1, f_2 \in L^1(Q)$ respectively, then*

$$(1.3) \quad \frac{d}{dt} \int_{\Omega} (u_1 - u_2)^+ dx \leq \int_{\Omega} (f_1 - f_2) \text{sign}_0^+(u_1 - u_2) dx, \quad \text{in } \mathcal{D}'(0, T).$$

In particular, we have

$$\frac{d}{dt} \|u_1 - u_2\|_1 \leq \|f_1 - f_2\|_1, \quad \text{in } \mathcal{D}'(0, T),$$

and, if $f_1 \leq f_2$, a.e. in Q , and $u_1(0) \leq u_2(0)$ a.e. in Ω , then

$$u_1 \leq u_2, \quad \text{a.e. in } Q.$$

Theorem 1.2. *For any $u_0 \in L^2(\Omega)$ and $f \in L^2(Q)$, the problem (1.2) has a solution u . Moreover, u satisfies the following :*

1. For any $q \in [1, \infty]$, we have

$$(1.4) \quad \|u(t)\|_q \leq M_q := \begin{cases} e^{(q-1)\|(\nabla \cdot V)^-\|_\infty} \left(\|u_0\|_q + \int_0^T \|f(t)\|_q dt \right) & \text{if } q < \infty \\ e^{\|(\nabla \cdot V)^-\|_\infty} \left(\|u_0\|_\infty + \int_0^T \|f(t)\|_\infty dt \right) & \text{if } q = \infty \end{cases}.$$

2. For any $t \in [0, T)$, we have

$$(1.5) \quad \frac{1}{m+1} \frac{d}{dt} \int_\Omega |u|^{m+1} dx + \int_\Omega |\nabla p|^2 dx \leq \int_\Omega f p dx + \int p u (\nabla \cdot V)^- dx, \quad \text{in } \mathcal{D}'(0, T).$$

Remark 1. 1. See that $V \cdot \nu$ may be understood in a weak sense, like

$$\int_\Omega V \cdot \nabla \xi dx + \int_\Omega \nabla \cdot V \xi dx \geq 0, \quad \text{for any } 0 \leq \xi \in \mathcal{D}(\Omega).$$

2. For any $h > 0$, we denote by

$$(1.6) \quad \xi_h(t, x) = \frac{1}{h} \min \left\{ h, d(x, \partial\Omega) \right\} \quad \text{and} \quad \nu_h(x) = -\nabla \xi_h, \quad \text{for any } x \in \Omega,$$

where $d(\cdot, \partial\Omega)$ names the euclidean distance-to-the-boundary function. We see that $\xi_h \in H_0^1(\Omega)$ is a regular (as well as the boundary is) function, $0 \leq \xi_h \leq 1$ and

$$\nu_h(x) = -\frac{1}{h} \nabla d(\cdot, \partial\Omega), \quad \text{for any } x \in \Omega \setminus \Omega_h \text{ and } 0 < h \leq h_0 \text{ (small enough)}.$$

Here

$$\Omega_h = \left\{ x \in \Omega : d(x, \partial\Omega) > h \right\}, \quad \text{for small } h > 0.$$

In particular, for any $x \in \Omega_h$, we have $h\nu_h(x) = \nu(\pi(x))$, where $\pi(x)$ design the projection of x on the boundary $\partial\Omega$, and $\nu(y)$ represents the outward unitary normal to the boundary $\partial\Omega$ at y . Thanks to (1.1), we have

$$(1.7) \quad \liminf_{h \rightarrow 0} \int_{\Omega \setminus \Omega_h} \xi V(x) \cdot \nu_h(x) dx \geq 0, \quad \text{for any } 0 \leq \xi \in L^\infty(\Omega).$$

3. Thanks to the local \mathcal{C}^2 -boundary regularity assumption on Ω , we have

$$(1.8) \quad \liminf_{h \rightarrow 0} \int_\Omega \nabla w \cdot \nabla \xi_h dx \geq 0, \quad \text{for any } 0 \leq w \in H_0^1(\Omega).$$

This property is connected to smoothness of the boundary of Ω . For the case of Lipschitz boundary domain, one needs to work with more sophisticated test functions in the spirit of ξ_h to fill (1.8) like property (one can see Lemma 4.4 and Remark 6.5 of [?]).

4. Typical examples of vector fields V may be given by

$$V = -\nabla\Phi \quad \text{and} \quad 0 \leq \Phi \in H_0^1(\Omega) \cap W^{2,2}(\Omega).$$

For the limit, as $m \rightarrow \infty$, one sees formally that the problem (1.2) converges to so called Hele-Shaw problem

$$(1.9) \quad \left\{ \begin{array}{l} \frac{\partial u}{\partial t} - \Delta p + \nabla \cdot (u V) = f \\ u \in \text{sign}(p) \end{array} \right\} \quad \text{in } Q$$

$$\left\{ \begin{array}{l} p = 0 \end{array} \right\} \quad \text{on } \Sigma$$

Existence, L^1 -comparison and uniqueness of weak solution for the problem (1.9), with mixed boundary conditions, has been studied recently in [27]. Thanks to [27], we know that for any $f \in L^2(Q)$ and $u_0 \in L^\infty(\Omega)$, s.t. $0 \leq u_0 \leq 1$, a.e. in Ω , (1.9) has a unique weak solution (see the following Theorem for the precise sense) satisfying $u(0) = u_0$. To prove rigorously the convergence of u_m to the solution of (1.9), we assume moreover that V satisfies the following assumption : there exists $h_0 > 0$, such that

$$(1.10) \quad V \cdot \nu_h \geq 0, \quad \text{in } \Omega_h, \quad \text{for any } 0 < h < h_0;$$

that is $V(x) \cdot \nabla d(x, \partial\Omega) \geq 0$, for any $x \in \Omega$ with being such that $d(x, \partial\Omega) < h \leq h_0$.

Theorem 1.3. *Under the assumptions (1.10), for each $m = 1, 2, \dots$, let $u_{0m} \in L^\infty(\Omega)$, $f_m \in L^2(Q)$ and u_m be the corresponding solution of (1.2). If, as $m \rightarrow \infty$,*

$$f_m \rightarrow f, \quad \text{in } L^1(Q), \quad u_{0m} \rightarrow u_0, \quad \text{in } L^1(\Omega),$$

and $|u_0| \leq 1$, then

$$u_m \rightarrow u, \quad \text{in } \mathcal{C}([0, T]; L^1(\Omega)),$$

$$u_m^m \rightarrow p, \quad \text{in } L^2([0, T]; H^1(\Omega))\text{-weak},$$

and (u, p) is the unique solution of (1.9) satisfying $u(0) = u_0$. That is $u \in \mathcal{C}([0, T], L^1(\Omega))$, $u(0) = u_0$ and $u = \text{sign}(p)$, a.e. in Q , and

$$(1.11) \quad \frac{d}{dt} \int_{\Omega} u \xi + \int_{\Omega} \nabla p \cdot \nabla \xi \, dx - \int_{\Omega} u V \cdot \nabla \xi \, dx = \int_{\Omega} f \xi \, dx, \quad \text{in } \mathcal{D}'([0, T]), \text{ for any } \xi \in H_0^1(\Omega).$$

Remark 2. *Thanks to [27], we can deduce that u , the limit of u_m , satisfies the following :*

1. *If there exists $\omega_1 \in W^{1,1}(0, T)$ (resp. $\omega_2 \in W^{1,1}(0, T)$) such that $u_0 \leq \omega_2(0)$ (resp. $\omega_1(0) \leq u_0$) and, for any $t \in (0, T)$,*

$$\dot{\omega}_2(t) + \omega_2(t) \nabla \cdot V \geq f(t, \cdot) \quad \text{a.e. in } \Omega$$

(rep. $\dot{\omega}_1(t) + \omega_1(t) \nabla \cdot V \leq f(t, \cdot)$, a.e. in Ω), then we have

$$u \leq \omega_2 \quad (\text{resp. } \omega_1 \leq u) \quad \text{a.e. in } Q.$$

2. If f and V satisfies

$$0 \leq f(t, \cdot) \leq \nabla \cdot V, \quad \text{a.e. in } Q$$

then $p \equiv 0$, and u is the unique solution of the reaction-transport equation

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t} + \nabla \cdot (u V) = f(t, x) \\ 0 \leq u \leq 1 \\ u V \cdot \nu = 0 \\ u(0) = u_0 \end{array} \right\} \quad \begin{array}{l} \text{in } Q \\ \text{on } \Sigma_N \\ \text{in } \Omega, \end{array}$$

in the sense that $u \in \mathcal{C}([0, T], L^1(\Omega))$, $0 \leq u \leq 1$ a.e. in Q and

$$\frac{d}{dt} \int_{\Omega} u \xi - \int_{\Omega} u V \cdot \nabla \xi = \int_{\Omega} f \xi, \quad \text{in } \mathcal{D}'(0, T), \quad \forall \xi \in H_0^1(\Omega).$$

Remark 3. 1. The condition (1.10) is equivalent in some sense that the vector field V is outward pointing in a neighbor of the boundary $\partial\Omega$. In particular, this implies that for any $0 < h < h_0$, there exists $0 \leq \xi_h \in \mathcal{C}^2(\Omega_h)$ compactly supported in Ω such that $\xi_h \equiv 1$ in Ω_h and

$$\int_{\Omega \setminus \Omega_h} \varphi V \cdot \nabla \xi_h dx \leq 0, \quad \text{for any } 0 \leq \varphi \in L^1(\Omega).$$

2. One sees that the assumption (1.10) is fulfilled for instance in the following cases

(a) There exists $h_0 > 0$ such that, for any $0 < h < h_0$, we have

$$V(x) = V(\pi(x)), \quad \text{for any } x \in \Omega_h.$$

Indeed, since $\nu_h(x) = \nu(\pi(x))$, we have $V(x) \cdot \nabla \xi_h(x) = V(\pi(x)) \cdot \nabla \xi(\pi(x))$ which is nonnegative by the assumption (1.1).

(b) Strictly outpointing vector field V :

$$V \cdot \nu > 0, \quad \text{on } \partial\Omega.$$

Indeed, This follows from the fact that

$$\lim_{h \rightarrow 0} \int_{\Omega \setminus \Omega_h} \varphi V \cdot \nu_h dx = \int_{\partial\Omega} \varphi V \cdot \nu dx, \quad \text{for any } 0 \leq \varphi \in \mathcal{C}(\overline{\Omega}).$$

(c) V compactly supported ; i.e. V vanishes on a neighbor of the boundary $\partial\Omega$.

3. A typical choice for V is given by $V = -\nabla d(\cdot, \partial\Omega)$, the distance function up to the boundary.

4. As we will see in the proofs, it is possible to replace the assumption (1.10) by the following : there exists $h_0 > 0$, such that for any $0 < h < h_0$, there exists $0 \leq \omega_h \in \mathcal{C}^2(\Omega_h)$ compactly supported in Ω , such that $\omega_h \equiv 1$ in Ω_h and

$$(1.12) \quad \int_{\Omega \setminus \Omega_h} \varphi V \cdot \nabla \omega_h dx \leq 0, \quad \text{for any } 0 \leq \varphi \in L^1(\Omega),$$

See that the condition (1.10) corresponds to the selection $\omega_h = \xi_h$.

1.4 Plan of the paper

The next section is devoted to the proof of L^1 -comparison principle for weak solutions of (1.2). To this aim, we use doubling and dedoubling variables techniques. This enables us to deduce the uniqueness and lay out the study plan of the equation in the framework of L^1 -nonlinear semi-group theory. Section 3 concerns the study of existence of a solution. To set the problem in the framework of nonlinear semi group theory, we begin with stationary problem to roll in Euler-implicit discretization and put up an ε -approximate solution. Then, using mainly a Crandall-Liggett theorem, $L^2(\Omega)$ and $H_0^1(\Omega)$ estimates on u and u^m respectively, we pass to the limit as $\varepsilon \rightarrow 0$, to built the solution of the evolution problem (1.2). Section 4 is devoted to the study of the limit as $m \rightarrow \infty$. Using the outpointing vector filed condition (1.10), we study first the limit for the stationary problem connecting it to the the Hele Shaw flow with linear drift. To this aim, we derive BV_{loc} new estimates on the solution. Then, using regular perturbation results for nonlinear semi group we establish the convergence results for the evolution problem. Section 6 is devoted to the study of the limit of the solution u and u^m in the of the presence of a reaction term with linear drift. We prove the convergence of reaction diffusion problem of a Hele-Shaw flow with linear drift At last, in Section 7 (Appendix), we provide for the unaccustomed reader a short recap on the main tools from L^1 -nonlinear semi-group theory.

2 L^1 -comparison principle and uniqueness proofs

As usual for parabolic-hyperbolic and elliptic-hyperbolic problems, the main tool to prove the uniqueness is doubling and de-doubling variables. To this aim, we prove first that a weak solution satisfies the following version of entropic inequality :

We assume throughout this section that $V \in W^{1,2}(\Omega)$, $(\nabla \cdot V) \in L^\infty(\Omega)$ and V satisfies the outpointing condition (1.1).

Proposition 2.1. *Let u be a weak solution of (1.2). Then, for any $k \in \mathbb{R}$, and $0 \leq \xi \in H_0^1(\Omega)$, we have*

$$(2.13) \quad \begin{aligned} & \frac{d}{dt} \int_{\Omega} (u - k)^+ \xi \, dx + \int_{\Omega} (\nabla(u^m - k^m)^+ - (u - k)^+ V) \cdot \nabla \xi \, dx \\ & + \int_{\Omega} (k \nabla \cdot V - f) \xi \, \text{sign}_0^+(u - k) \, dx \leq - \limsup_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{[0 \leq u^m - k^m \leq \varepsilon]} |\nabla u^m|^2 \xi \, dx, \end{aligned}$$

and

$$(2.14) \quad \begin{aligned} & \frac{d}{dt} \int_{\Omega} (k - u)^+ \xi \, dx + \int_{\Omega} \nabla(k^m - u^m)^+ - (k - u)^+ V) \cdot \nabla \xi \, dx \\ & + \int_{\Omega} (f - k \nabla \cdot V) \xi \, \text{sign}_0^+(k - u) \, dx \leq - \limsup_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{[0 \leq k^m - u^m \leq \varepsilon]} |\nabla u^m|^2 \xi \, dx, \end{aligned}$$

in $\mathcal{D}'(0, T)$.

Proof. We extend u onto $\mathbb{R} \times \Omega$ by 0 for any $t \notin (0, T)$. Then, for any $h > 0$ and nonnegative $\xi \in H_0^1(\Omega)$ and $\psi \in \mathcal{D}(\mathbb{R})$, we consider

$$\Phi^h(t) = \xi \frac{1}{h} \int_t^{t+h} \mathcal{H}_\varepsilon^+(u^m(s)) \psi(s) \, ds,$$

where $\mathcal{H}_\varepsilon^+$ is given by

$$\mathcal{H}_\varepsilon^+(r) = \min\left(\frac{(r - k^m)^+}{\varepsilon}, 1\right), \quad \text{for any } r \in \mathbb{R},$$

for arbitrary $\varepsilon > 0$. It is clear that $\Phi_h \in W^{1,2}(0, T; H_0^1(\Omega)) \cap L^\infty(Q)$ is an admissible test function for the weak formulation, so that

$$(2.15) \quad - \iint_Q u \partial_t \Phi^h dt dx + \iint_Q (\nabla u^m - V u) \cdot \nabla \Phi^h dt dx = \iint_Q f \Phi^h dt dx.$$

See that

$$\begin{aligned} \iint_Q u \partial_t \Phi^h dt dx &= \iint_Q \psi(t) \mathcal{H}_\varepsilon(u^m(t)) \frac{u(t-h) - u(t)}{h} \xi dt dx \\ &\leq \frac{1}{h} \iint_Q \psi(t) \left(\int_{u(t)}^{u(t-h)} \mathcal{H}_\varepsilon^+(r^m) dr \right) \xi dt dx \\ &\leq \frac{1}{h} \iint_Q \left(\int_k^{u(t)} \mathcal{H}_\varepsilon^+(r^m) dr \right) (\psi(t+h) - \psi(t)) dt dx. \end{aligned}$$

Letting $h \rightarrow 0$, we have

$$\limsup_{h \rightarrow 0} \iint_Q u \partial_t \Phi^h dt dx \leq \iint_Q \left(\int_k^{u(t)} \mathcal{H}_\varepsilon^+(r^m) dr \right) \partial_t \psi \xi dt dx.$$

So, by letting $h \rightarrow 0$ in (2.15), we get

$$(2.16) \quad \begin{aligned} & - \iint_Q \left\{ \left(\int_k^{u(t)} \mathcal{H}_\varepsilon^+(r^m) dr \right) \partial_t \psi \xi + \psi \nabla u^m \cdot \nabla \xi \mathcal{H}_\varepsilon(u^m) \xi - \mathcal{H}_\varepsilon(u^m) (u - k) V \cdot \nabla \xi \right\} dt dx \\ & \leq \iint_Q \left\{ \psi (f + k \nabla \cdot V) \mathcal{H}_\varepsilon(u^m) \xi + \psi \xi (u - k) V \cdot \nabla \mathcal{H}_\varepsilon(u^m) \right\} dt dx \\ & \quad - \frac{1}{\varepsilon} \iint_{[0 \leq u^m - k^m \leq \varepsilon]} |\nabla u^m|^2 \xi dt dx, \end{aligned}$$

where we use the fact that $|\nabla u^m| \mathcal{H}'_\varepsilon(u^m(t)) = \frac{1}{\varepsilon} |\nabla u^m|^2 \chi_{[0 \leq u^m - k^m \leq \varepsilon]}$ a.e. in Q . Setting

$$\Psi_\varepsilon := \frac{1}{\varepsilon} \int_{\min(u^m, k^m)}^{\min(u^m, k^m + \varepsilon)} (r^{1/m} - k) dr,$$

we see that

$$(u - k) \mathcal{H}'_\varepsilon(u^m - k^m) \cdot \nabla u^m = \nabla \Psi_\varepsilon.$$

This implies that the last term of (2.16) satisfies

$$\begin{aligned}
\iint_Q \psi \xi(u-k) V \cdot \nabla \mathcal{H}_\varepsilon(u^m - k^m) &= \iint_Q \xi(u-k) \mathcal{H}'_\varepsilon(u^m - k^m) V \cdot \nabla u^m \\
&= \iint_Q \psi \xi V \cdot \nabla \Psi_\varepsilon dx \\
&= - \iint_Q \psi \nabla \cdot (\xi V) \Psi_\varepsilon dx \\
&\rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0.
\end{aligned}$$

See also that, by using Lebesgue's dominated convergence Theorem, we have

$$\limsup_{\varepsilon \rightarrow 0} \iint_Q \left(\int_k^{u(t)} \mathcal{H}_\varepsilon^+(r^m) dr \right) \partial_t \psi \xi = \iint_Q (u(t) - k)^+ \partial_t \psi \xi.$$

Then, letting $\varepsilon \rightarrow 0$ in (2.16) and using the fact that $\text{sign}_0^+(u^m - k^m) = \text{sign}_0^+(u - k)$, for any $k \in \mathbb{R}$, we get (2.13). As to (2.14), it follows by using the fact that $-u$ is also a solution of (1.2) with f replaced by $-f$, and applying (2.13) to $-u$. \square

Proposition 2.2 (Kato's inequality). *If u_1 and u_2 satisfy (2.13) and (2.14) corresponding to $f_1 \in L^1(Q)$ and $f_2 \in L^1(Q)$ respectively, then*

$$(2.22) \quad \partial_t(u_1 - u_2)^+ - \Delta(u_1^m - u_2^m)^+ + \nabla \cdot ((u_1 - u_2)^+ V) \leq (f_1 - f_2) \text{sign}_0^+(u_1 - u_2) \text{ in } \mathcal{D}'(Q).$$

Proof. The proof of this lemma is based on doubling and de-doubling variable techniques. Let us give here briefly the arguments. To double the variables, we use first the fact that $u_1 = u_1(t, x)$ satisfies (2.13) with $k = u_1(s, y)$, we have

$$\begin{aligned}
&\frac{d}{dt} \int (u_1(t, x) - u_2(s, y))^+ \zeta dx + \int (\nabla_x(u_1^m(t, x) - u_2^m(s, y))^+ - (u_1(t, x) - u_2(s, y))^+ V(x) \cdot \nabla_x \zeta dx \\
&\quad + \int_\Omega \nabla_x \cdot V u_2(s, y) \zeta \text{sign}_0^+(u_1(t, x) - u_2(s, y)) dx \leq \int f_1(t, x) \text{sign}_0^+(u_1(t, x) - u_2(s, y)) \zeta dx \\
&\quad - \limsup_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{[0 \leq u_1^m - u_2^m \leq \varepsilon]} |\nabla_x u_1^m(t, x)|^2 \zeta dx.
\end{aligned}$$

Integrating with respect to y , we get

$$\begin{aligned}
&\frac{d}{dt} \iint (u_1(t, x) - u_2(s, y))^+ \zeta + \iint (\nabla_x(u_1^m(t, x) - u_2^m(s, y))^+ - (u_1(t, x) - u_2(s, y))^+ V(x) \cdot \nabla_x \zeta \\
&\quad + \iint \nabla_x \cdot V u_2(s, y) \zeta \text{sign}_0^+(u_1(t, x) - u_2(s, y)) \leq \iint f_1(t, x) \text{sign}_0^+(u_1(t, x) - u_2(s, y)) \zeta \\
&\quad - \limsup_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \iint_{[0 \leq u_1^m - u_2^m \leq \varepsilon]} |\nabla_x u_1^m(t, x)|^2 \zeta.
\end{aligned}$$

See that

$$\begin{aligned} \iint \nabla_y(u_1^m(t, x) - u_2^m(s, y))^+ \cdot \nabla_x \zeta \, dx dy &= - \lim_{\varepsilon \rightarrow 0} \iint \nabla_y u_2^m(s, y) \cdot \nabla_x \zeta H_\varepsilon(u_1^m(t, x) - u_2^m(s, y)) \, dx dy \\ &= - \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \iint_{[0 \leq u_1^m - u_2^m \leq \varepsilon]} \nabla_x u_1^m(t, x) \cdot \nabla_y u_2^m(s, y) \zeta \, dx dy, \end{aligned}$$

so that, denoting by

$$u(t, s, x, y) = u_1(t, x) - u_2(s, y), \quad \text{and} \quad p(t, s, x, y) = u_1^m(t, x) - u_2^m(s, y),$$

we obtain

$$\begin{aligned} (2.23) \quad & \frac{d}{dt} \iint u(t, s, x, y)^+ \zeta \, dx dy + \iint \left\{ (\nabla_x + \nabla_y)p(t, s, x, y) - u(t, s, x, y)^+ V(x) \right\} \cdot \nabla_x \zeta \, dx dy \\ & + \iint \nabla_x \cdot V u_2(s, y) \zeta \operatorname{sign}_0^+ u(t, s, x, y) \, dx dy \leq \iint f_1(t, x) \operatorname{sign}_0^+ u(t, s, x, y) \zeta \, dx dy \\ & - \lim_{\varepsilon \rightarrow 0} \iint \nabla_x u_1^m(t, x) \cdot \nabla_y u_2^m(s, y) \zeta H_\varepsilon(u_1^m(t, x) - u_2^m(s, y)) \, dx dy \\ & - \limsup_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \iint_{[0 \leq u_1^m - u_2^m \leq \varepsilon]} |\nabla_x u_1^m(t, x)|^2 \zeta \, dx dy. \end{aligned}$$

On the other hand, using the fact that $u_2 = u_2(s, y)$ satisfies (2.14) with $k = u_1(t, x)$, we have

$$\begin{aligned} & \frac{d}{ds} \int u(t, s, x, y)^+ \zeta \, dy + \int \nabla_y p(t, s, x, y) - u(t, s, x, y)^+ V(y) \cdot \nabla_y \zeta \\ & - \int_{\Omega} \nabla_y \cdot V u_1(t, x) \zeta \operatorname{sign}_0^+(u(t, s, x, y)) \, dy \leq - \int f_2(s, y) \operatorname{sign}_0^+(u(t, s, x, y)) \zeta \, dy \\ & - \limsup_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{[0 \leq u_1^m - u_2^m \leq \varepsilon]} \|\nabla_y u_2^m(s, y)\|^2 \zeta \, dy. \end{aligned}$$

Working in the same way for (2.23), we get

$$\begin{aligned} & \frac{d}{ds} \iint u(t, s, x, y)^+ \zeta \, dx dy + \iint \left\{ (\nabla_x + \nabla_y)p(t, s, x, y) - u(t, s, x, y)^+ V(y) \right\} \cdot \nabla_y \zeta \, dx dy \\ & - \iint \nabla_y \cdot V(y) u_1(t, x) \zeta \operatorname{sign}_0^+(u(t, s, x, y)) \, dx dy \leq - \iint f_2(s, y) \operatorname{sign}_0^+(u(t, s, x, y)) \zeta \, dx dy \\ & - \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \iint_{[0 \leq u_1^m - u_2^m \leq \varepsilon]} \nabla_x u_1^m(t, x) \cdot \nabla_y u_2^m(s, y) \zeta \, dx dy \\ & - \limsup_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \iint_{[0 \leq u_1^m - u_2^m \leq \varepsilon]} |\nabla_y u_2^m(s, y)|^2 \zeta \, dy \, dx dy. \end{aligned}$$

Adding both inequalities, and using the fact that

$$-(\|\nabla_x u_1(t, x)\|^2 + \|\nabla_y u_2(s, y)\|^2 + 2 \nabla_x u_1(t, x) \cdot \nabla_y u_2(s, y)) \chi_{[0 \leq u_1^m - u_2^m \leq \varepsilon]} \leq 0, \quad \text{a.e. in } Q^2,$$

we obtain

$$\begin{aligned}
(2.24) \quad & \left(\frac{d}{dt} + \frac{d}{ds} \right) \iint u(t, s, x, y)^+ \zeta \, dx dy + \iint (\nabla_x + \nabla_y) p(t, s, x, y) \cdot (\nabla_x + \nabla_y) \zeta \, dx dy \\
& - \iint u(t, s, x, y)^+ (V(x) \cdot \nabla_x \zeta + V(y) \cdot \nabla_y \zeta) \, dx dy \\
& + \iint (\nabla_x \cdot V(x) u_2(s, y) \, dx dy - \nabla_y \cdot V(y) u_1(t, x)) \zeta \, \text{sign}_0^+(u(t, s, x, y)) \, dx dy \\
& \leq \iint (f_1(t, x) - f_2(s, y)) \text{sign}_0^+(u(t, s, x, y)) \zeta \, dx dy
\end{aligned}$$

and then,

$$\begin{aligned}
(2.25) \quad & \left(\frac{d}{dt} + \frac{d}{ds} \right) \iint u(t, s, x, y)^+ \zeta \, dx dy + \iint (\nabla_x + \nabla_y) p(t, s, x, y) \cdot (\nabla_x + \nabla_y) \zeta \, dx dy \\
& - \iint u(t, s, x, y)^+ V(x) \cdot (\nabla_x \zeta + \nabla_y \zeta) \, dx dy + \iint u(t, s, x, y)^+ (V(x) - V(y) \cdot \nabla_y \zeta) \, dx dy \\
& + \iint (\nabla_x \cdot V(x) u_2(s, y) - \nabla_y \cdot V(y) u_1(t, x)) \zeta \, \text{sign}_0^+(u(t, s, x, y)) \, dx dy \\
& \leq \iint (f_1(t, x) - f_2(s, y)) \text{sign}_0^+(u(t, s, x, y)) \zeta \, dx dy.
\end{aligned}$$

Now, we can de-double the variables t and s , as well as x and y , by taking as usual the sequence of test functions

$$\psi_\varepsilon(t, s) = \psi \left(\frac{t+s}{2} \right) \rho_\varepsilon \left(\frac{t-s}{2} \right) \quad \text{and} \quad \zeta_\lambda(x, y) = \xi \left(\frac{x+y}{2} \right) \delta_\lambda \left(\frac{x-y}{2} \right),$$

for any $t, s \in (0, T)$ and $x, y \in \Omega$. Here $\psi \in \mathcal{D}(0, T)$, $\xi \in \mathcal{D}(\Omega)$, ρ_ε and δ_λ are sequences of usual mollifiers in \mathbb{R} and \mathbb{R}^N respectively. See that

$$\left(\frac{d}{dt} + \frac{d}{ds} \right) \psi_\varepsilon(t, s) = \rho_\varepsilon \left(\frac{t-s}{2} \right) \dot{\psi} \left(\frac{t+s}{2} \right)$$

and

$$(\nabla_x + \nabla_y) \zeta_\lambda(x, y) = \delta_\lambda \left(\frac{x-y}{2} \right) \nabla \xi \left(\frac{x+y}{2} \right)$$

Moreover, for any $h \in L^1((0, T)^2 \times \Omega^2)$ and $\Phi \in L^1((0, T)^2 \times \Omega^2)^N$, we know that

- $\lim_{\lambda \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \int_0^T \int_0^T \iint_{\Omega \times \Omega} h(t, s, x, y) \zeta_\lambda(x, y) \rho_\varepsilon(t, s) = \int_0^T \int_\Omega h(t, t, x, x) \xi(x) \psi(t).$
- $\lim_{\lambda \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \int_0^T \int_0^T \iint_{\Omega \times \Omega} h(t, s, x, y) \zeta_\lambda(x, y) \left(\frac{d}{dt} + \frac{d}{ds} \right) \rho_\varepsilon(t, s) = \int_0^T \int_\Omega h(t, t, x, x) \xi(x) \dot{\psi}(t).$
- $\lim_{\lambda \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \int_0^T \int_\Omega \iint_{\Omega \times \Omega} \Phi(t, s, x, y) \cdot (\nabla_x + \nabla_y) \zeta_\lambda(x, y) \rho_\varepsilon(t, s) = \int_0^T \int_\Omega \Phi(t, t, x, x) \cdot \nabla \xi(x) \psi(t) \, dt dx.$

Moreover, we also know that

$$\begin{aligned} & \lim_{\lambda \rightarrow 0} \int_0^T \int_{\Omega} \int_{\Omega} h(t, x, y) (V(x) - V(y)) \cdot \nabla_y \zeta_{\lambda}(x, y) \psi(t) dt dx dy \\ &= \int_0^T \int_{\Omega} h(x, x) \nabla \cdot V(x) \xi(x) \psi(t) dt dx. \end{aligned}$$

The proof of this result is more or less well known by now (one can see for instance a detailed proof in [27]). So replacing ζ and ψ in (2.25) by ζ_{λ} and ψ_{ε} resp., and, letting $\varepsilon, \lambda \rightarrow 0$, we get

$$\begin{aligned} & \int_0^T \int_{\Omega} \left\{ -(u_1 - u_2)^+ \xi \dot{\psi} + \nabla(p_1 - p_2)^+ \cdot \nabla \xi \psi - (u_1 - u_2)^+ V \cdot \nabla \xi \psi \right\} dt dx \\ & \leq \int_0^T \int_{\Omega} (f_1 - f_2) \text{sign}_0^+(u_1 - u_2) \xi \psi dt dx. \end{aligned}$$

Thus

$$\begin{aligned} & \frac{d}{dt} \int (u_1 - u_2)^+ \xi dx + \int \nabla(u_1^m(t, x) - u_2^m(t, x))^+ \cdot \nabla \xi dx - \int (u_1 - u_2)^+ V \cdot \nabla \xi dx \\ & \leq \int \kappa(x) (f_1 - f_2) \xi dx. \end{aligned}$$

Thus the result of the proposition. \square

The aim now is to proceed with the sequence of test function ξ_h given by (1.6) in Kato's inequality and let $h \rightarrow 0$, to cover (1.3).

Proof of Theorem 1.1. Let (u_1, p_1) and (u_2, p_2) be two couples of $L^\infty(Q) \times L^2(0, T; H_0^1(\Omega))$ satisfying (2.13) and (2.14) corresponding to $f_1 \in L^1(Q)$ and $f_2 \in L^1(Q)$ respectively, to prove (1.3) we see that

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} (u_1 - u_2)^+ dx - \int (f_1 - f_2) \text{sign}_0^+(u_1 - u_2) dx \\ &= \lim_{h \rightarrow 0} \frac{d}{dt} \int_{\Omega} (u_1 - u_2)^+ \xi_h dx - \int (f_1 - f_2) \text{sign}_0^+(u_1 - u_2) \xi_h dx =: I(h), \end{aligned}$$

in the sense of distribution in $[0, T)$. Taking ξ_h as a test function in (2.22) and using (1.8), we have

$$\begin{aligned} I(h) & \leq - \int (\nabla(u_1^m - u_2^m)^+ - (u_1 - u_2)^+ V) \cdot \nabla \xi_h dx \\ & \leq \int (u_1 - u_2)^+ V \cdot \nabla \xi_h dx. \end{aligned}$$

Then, using the outpointing velocity vector field assumption (1.7), we get

$$\begin{aligned} \lim_{h \rightarrow 0} I(h) & \leq - \lim_{h \rightarrow 0} \int (u_1 - u_2)^+ V \cdot \nu_h(x) dx \\ & \leq 0. \end{aligned}$$

Thus (1.3). The rest of the theorem is a straightforward consequence of (1.3). \square

3 Main estimates and existence proofs

3.1 Stationary problem

To prove Theorem 1.2, we consider the stationary problem associated with Euler-implicit discretization of (1.2). That is

$$(3.1) \quad \begin{cases} v - \lambda \Delta v^m + \lambda \nabla \cdot (v V) = f & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f \in L^2(\Omega)$ and $\lambda > 0$. Following Definition 1.1, a function $v \in L^1(\Omega)$ is said to be a weak solution of (3.1) if $v^m \in H_0^1(\Omega)$ and

$$(3.2) \quad \int_{\Omega} v \xi + \lambda \int_{\Omega} \nabla v^m \cdot \nabla \xi - \lambda \int_{\Omega} v V \cdot \nabla \xi = \int_{\Omega} f \xi, \quad \text{for all } \xi \in H_0^1(\Omega).$$

Theorem 3.4. *Assume $V \in W^{1,2}(\Omega)$ and $(\nabla \cdot V)^- \in L^\infty(\Omega)$. For $f \in L^2(\Omega)$ and λ satisfying*

$$(3.3) \quad 0 < \lambda < 1 / \|(\nabla \cdot V)^-\|_\infty,$$

the problem (3.2) has a solution v that we denote by v_m . Moreover, for any $1 \leq q \leq \infty$, we have

$$(3.4) \quad \|v_m\|_q \leq \begin{cases} \left(1 - (q-1)\lambda \|(\nabla \cdot V)^-\|_\infty\right)^{-1} \|f\|_q, & \text{if } 1 \leq q < \infty \\ \left(1 - \lambda \|(\nabla \cdot V)^-\|_\infty\right)^{-1} \|f\|_\infty, & \text{if } q = \infty \end{cases}$$

and

$$(3.5) \quad \left(1 - \lambda \|(\nabla \cdot V)^-\|_\infty\right) \int |v_m|^{m+1} dx + \lambda \int |\nabla v_m^m|^2 dx \leq \int f v_m^m dx.$$

Moreover, thanks to Theorem 1.1, we have

Corollary 3.1. *Under the assumption of Theorem 3.4, if moreover V satisfies the outpointing condition (1.1), the problem (3.1) has a unique solution. Moreover, if v_1 and v_2 are two solutions associated with $f_1 \in L^1(\Omega)$ and $f_2 \in L^1(\Omega)$ respectively, then*

$$\|(v_1 - v_2)^+\|_1 \leq \|(f_1 - f_2)^+\|_1$$

and

$$\|v_1 - v_2\|_1 \leq \|f_1 - f_2\|_1.$$

To prove Theorem 3.4, we proceed by regularization and compactness. For each $\varepsilon > 0$, we consider β_ε a regular Lipschitz continuous function strictly increasing satisfying $\beta_\varepsilon(0) = 0$ and, as $\varepsilon \rightarrow 0$,

$$\beta_\varepsilon(r) \rightarrow r^{1/m}, \quad \text{for any } r \in \mathbb{R}.$$

One can take, for instance, β_ε the regularization by convolution of the application $r \in \mathbb{R} \rightarrow r^{1/m}$. Then, we consider the problem

$$(3.6) \quad \begin{cases} \left. \begin{array}{l} v - \lambda \Delta p + \lambda \nabla \cdot (v V) = f \\ v = \beta_\varepsilon(p) \end{array} \right\} & \text{in } \Omega \\ p = 0 & \text{on } \partial\Omega. \end{cases}$$

Lemma 3.1. *For any $f \in L^2(\Omega)$ and $\varepsilon > 0$, the problem (3.6) has a solution v_ε , in the sense that $v_\varepsilon \in L^2(\Omega)$, $p_\varepsilon := \beta_\varepsilon^{-1}(u_\varepsilon) \in H_0^1(\Omega)$, and*

$$(3.7) \quad \int v_\varepsilon \xi \, dx + \lambda \int \nabla p_\varepsilon \cdot \nabla \xi \, dx - \lambda \int v_\varepsilon V \cdot \nabla \xi \, dx = \int f \xi \, dx,$$

for any $\xi \in H_0^1(\Omega)$. Moreover, for any λ satisfying (3.3) the solution v_ε satisfies the estimates

$$(3.8) \quad \|v_\varepsilon\|_q \leq \begin{cases} \left(1 - (q-1)\lambda\|(\nabla \cdot V)^-\|_\infty\right)^{-1} \|f\|_q, & \text{if } 1 \leq q < \infty \\ \left(1 - \lambda\|(\nabla \cdot V)^-\|_\infty\right)^{-1} \|f\|_\infty, & \text{if } q = \infty \end{cases}$$

and

$$(3.9) \quad \left(1 - \lambda\|(\nabla \cdot V)^-\|_\infty\right) \int v_\varepsilon p_\varepsilon \, dx + \lambda \int |\nabla p_\varepsilon|^2 \, dx \leq \int f p_\varepsilon \, dx.$$

Proof. We can assume without loose of generality throughout the proof that $\lambda = 1$ and remove the script ε in the notations of $(v_\varepsilon, p_\varepsilon)$ and β_ε , along the proof. We consider $H^{-1}(\Omega)$ the usual topological dual space of $H_0^1(\Omega)$ and $\langle \cdot, \cdot \rangle$ the associate dual bracket. See that the operator $A : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$, given by

$$\langle Ap, \xi \rangle = \int \beta(p) \xi \, dx + \int \nabla p \cdot \nabla \xi \, dx - \int \beta(p) V \cdot \nabla \xi \, dx, \quad \text{for any } \xi, p \in H_0^1(\Omega),$$

is a bounded weakly continuous operator. Moreover, A is coercive. Indeed, for any $p \in H_0^1(\Omega)$, we have

$$\begin{aligned} \langle Ap, p \rangle &= \int \beta(p) p \, dx + \int |\nabla p|^2 \, dx - \int \beta(p) V \cdot \nabla p \, dx \\ &= \int \beta(p) p \, dx + \int |\nabla p|^2 \, dx - \int V \cdot \nabla \left(\int_0^p \beta(r) dr \right) \, dx \\ &= \int \beta(p) p \, dx + \int |\nabla p|^2 \, dx + \int \nabla \cdot V \left(\int_0^p \beta(r) dr \right) \, dx \\ &\geq \int \beta(p) p \, dx + \int |\nabla p|^2 \, dx - \int (\nabla \cdot V)^{-2} p \beta(p) \, dx \\ &\geq \frac{1}{2} \int \beta(p) p \, dx + \int |\nabla p|^2 \, dx - \frac{1}{2} \int (\nabla \cdot V)^{-2} \, dx \\ &\geq \int |\nabla p|^2 \, dx - \frac{1}{2} \int (\nabla \cdot V)^{-2} \, dx, \end{aligned}$$

where we use Young inequality. So, for any $f \in H^{-1}(\Omega)$ the problem $Ap = f$ has a solution $p \in H_0^1(\Omega)$. Now, for each $1 < q < \infty$, taking v^{q-1} as a test function, and using the fact that

$$v\nabla(v^{q-1}) = \frac{q-1}{q} \nabla|v|^q, \quad \text{a.e. in } \Omega$$

and

$$\nabla p \cdot \nabla(v^{q-1}) \geq 0,$$

we get

$$\begin{aligned} \int |v|^q dx &\leq \int f v^{q-1} dx + \lambda \frac{q-1}{q} \int V \cdot \nabla |v|^q dx \\ &\leq \int f v^{q-1} dx - \lambda \frac{q-1}{q} \int \nabla \cdot V |v|^q dx \\ &\leq \int f v^{q-1} dx + \lambda \frac{q-1}{q} \int (\nabla \cdot V)^- |v|^q dx \\ &\leq \frac{1}{q} \int |f|^q dx + \frac{q-1}{q} \int |v|^q dx + \lambda \frac{q-1}{q} \|(\nabla \cdot V)^-\|_\infty \int |v|^q dx, \end{aligned}$$

where we use again Young inequality. This implies that

$$(1 - \lambda(q-1) \|(\nabla \cdot V)^-\|_\infty) \int |v|^q dx \leq \int |f|^q dx.$$

Thus (3.4). To prove (3.5), we take p as a test function, we obtain

$$\begin{aligned} \lambda \int |\nabla p|^2 dx &= \int f p dx - \int v p dx + \lambda \int \beta(p) V \cdot \nabla p dx \\ &= \int f p dx - \int v p dx + \lambda \int V \cdot \nabla \left(\int_0^p \beta(r) dr \right) dx \\ &= \int f p dx - \int v p dx - \lambda \int \nabla \cdot V \left(\int_0^p \beta(r) dr \right) dx \\ &\leq \int f p dx - \int v p dx + \lambda \int (\nabla \cdot V)^- \int_0^p \beta(r) dr dx \\ &\leq \int f p dx - \int v p dx + \|(\nabla \cdot V)^-\|_\infty \int u p dx \end{aligned}$$

where we use the fact that $\int_0^p \beta(r) dr \leq p\beta(p) = vp$. Thus (3.5) for $1 < q < \infty$. For the case $q \in \{1, \infty\}$, we take $H_\varepsilon(u - k) \in H_0^1(\Omega)$, for a given $k \geq 0$, as a test function in (3.6), where

$$H_\varepsilon(r) = \begin{cases} 1 & \text{if } r \geq 1 \\ r/\varepsilon & \text{if } |r| < \varepsilon \\ -1 & \text{if } r \leq -1. \end{cases}$$

Then, letting $\varepsilon \rightarrow 0$ and using the fact that $\nabla p \cdot \nabla H_\varepsilon(u - k) \geq 0$ a.e. in Ω , it is not difficult to see that

$$\begin{aligned} \int (v - k)^+ dx &\leq \int (f - k(1 + \lambda \nabla \cdot V)) \text{sign}^+(v - k) + \lambda \lim_{\varepsilon \rightarrow 0} \int (v - k) V \cdot \nabla H_\varepsilon(v - k) \\ &\leq \int (f - k(1 + \lambda \nabla \cdot V)) \text{sign}^+(v - k), \end{aligned}$$

where we use the fact that $\lim_{\varepsilon \rightarrow 0} \int (v - k) V \cdot \nabla H_\varepsilon(v - k) = \lim_{\varepsilon \rightarrow 0} \int (v - k) V \cdot \nabla(u - k) H'_\varepsilon(v - k) = 0$. In particular, this implies that

$$\int (v - k)^+ dx \leq \int (f - k(1 + \lambda \nabla \cdot V))^+.$$

So, taking

$$k = \frac{\|f\|_\infty}{1 - \lambda \|(\nabla \cdot V)^-\|_\infty},$$

we have $(f - k(1 + \lambda \nabla \cdot V))^+ \leq 0$, and then $v \leq k$. Working in the same way with $H_\varepsilon(-v + k)$ as a test function, we obtain

$$v \geq -\frac{\|f\|_\infty}{1 - \lambda \|(\nabla \cdot V)^-\|_\infty}.$$

Thus the result of the lemma for $q = \infty$. The case $q = 1$ follows by Corollary 3.1. \square

Lemma 3.2. *Under the assumption of Theorem 3.4, by taking a subsequence $\varepsilon \rightarrow 0$ if necessary, we have*

$$(3.25) \quad v_\varepsilon \rightarrow v, \quad \text{in } L^2(\Omega)\text{-weak}$$

and

$$(3.26) \quad p_\varepsilon \rightarrow v^m, \quad \text{in } H_0^1(\Omega).$$

Moreover, v is a weak solution of (3.1).

Proof. Using Lemma 3.1 as well as Young and Poincaré inequalities, we see that the sequences v_ε and p_ε are bounded in $L^2(\Omega)$ and $H_0^1(\Omega)$, respectively. So, there exists a subsequence that we denote again by v_ε and p_ε such that (3.25) is fulfilled and

$$(3.27) \quad p_\varepsilon \rightarrow v^m, \quad \text{in } H_0^1(\Omega)\text{-weak.}$$

Letting $\varepsilon \rightarrow 0$ in (3.7), we obtain that v is a weak solution of (3.1). Let us prove that actually (3.27) holds to be true strongly in $H_0^1(\Omega)$. Indeed, taking p_ε as a test function, we have

$$\begin{aligned} \lambda \int |\nabla p_\varepsilon|^2 dx &= \int (f - v_\varepsilon) p_\varepsilon dx + \lambda \int V \cdot \nabla \left(\int_0^{p_\varepsilon} \beta_\varepsilon(r) dr \right) dx \\ &= \int (f - v_\varepsilon) p_\varepsilon dx - \lambda \int \nabla \cdot V \int_0^{p_\varepsilon} \beta_\varepsilon(r) dr dx. \end{aligned}$$

Since $\int_0^r \beta_\varepsilon(s) ds$ converges to $\int_0^r \beta(s) ds$, for any $r \in \mathbb{R}$, $p_\varepsilon \rightarrow v^m$ a.e. in Ω and $\left| \int_0^{p_\varepsilon} \beta_\varepsilon(s) ds \right| \leq v_\varepsilon p_\varepsilon$ which is bounded in $L^1(\Omega)$ by (3.9), we have

$$\int_0^{p_\varepsilon} \beta_\varepsilon(s) ds \rightarrow \int_0^p s^{1/m} ds = \frac{m}{m+1} |v|^{m+1}, \quad \text{in } L^1(\Omega).$$

So, in one hand we have

$$\lim_{\varepsilon \rightarrow 0} \lambda \int |\nabla p_\varepsilon|^2 dx = \int (f - v) p dx - \lambda \frac{m}{m+1} \int \nabla \cdot V |v|^{m+1} dx.$$

On the other, since v is a weak solution of (3.1), one sees easily that

$$\lambda \int |\nabla p|^2 dx = \int (f - v) p dx - \lambda \frac{m}{m+1} \int \nabla \cdot V |v|^{m+1} dx;$$

which implies that $\lim_{\varepsilon \rightarrow 0} \int |\nabla p_\varepsilon|^2 dx = \int |\nabla p|^2 dx$. Combing this with the weak convergence of ∇p_ε , we deduce the strong convergence (3.26). \square

Remark 4. One sees in the proof that the results of Lemma 3.2 remain to be true if one replace f in (3.6) by a sequence of $f_\varepsilon \in L^2(\Omega)$ and assumes that, as $\varepsilon \rightarrow 0$,

$$f_\varepsilon \rightarrow f, \quad \text{in } L^2(\Omega).$$

Proof of Theorem 3.4. The proof follows by Lemma 3.2. Moreover, the estimates hold to be true by letting $\varepsilon \rightarrow 0$, in the estimate (3.8) and (3.9) for v_ε and p_ε . \square

3.2 Existence for the evolution problem

To study the evolution problem, we use Euler-implicit discretization scheme. For an arbitrary $0 < \varepsilon \leq \varepsilon_0$, and $n \in \mathbb{N}^*$ being such that $(n+1)\varepsilon = T$, we consider the sequence $(u_i, p_i)_{i=0, \dots, n}$ given by the ε -Euler implicit scheme associated with (1.2) :

$$(3.28) \quad \left\{ \begin{array}{l} u_{i+1} - \varepsilon \Delta p_{i+1} + \varepsilon \nabla \cdot (u_{i+1} V) = u_i + \varepsilon f_i \\ p_{i+1} = u_{i+1}^m \end{array} \right\} \quad \begin{array}{l} \text{in } \Omega \\ \\ \text{on } \partial\Omega, \end{array} \quad i = 0, 1, \dots, n-1,$$

where, for each $i = 0, \dots, n-1$, f_i is given by

$$f_i = \frac{1}{\varepsilon} \int_{i\varepsilon}^{(i+1)\varepsilon} f(s) ds, \quad \text{a.e. in } \Omega.$$

Now, for a given ε -time discretization $0 = t_0 < t_1 < t_2 < \dots < t_n < t_{n+1} = T$, satisfying $t_{i+1} - t_i \leq \varepsilon$, we define the ε -approximate solution by

$$(3.29) \quad u_\varepsilon := \sum_{i=0}^{n-1} u_i \chi_{[t_i, t_{i+1})}, \quad \text{and} \quad p_\varepsilon := \sum_{i=1}^n p_i \chi_{[t_i, t_{i+1})}.$$

In order to use the results of the previous section and the general theory of evolution problem governed by accretive operator (see for instance [6, 5]), we define the operator \mathcal{A}_m in $L^1(\Omega)$, by $\mu \in \mathcal{A}_m(z)$ if and only if $\mu, z \in L^1(\Omega)$ and z is a solution of the problem

$$\begin{cases} -\Delta z^m + \nabla \cdot (z V) = \mu & \text{in } \Omega \\ z = 0 & \text{on } \partial\Omega, \end{cases}$$

in the sense that $z \in L^2(\Omega)$, $z^m \in H_0^1(\Omega)$ and

$$\int_{\Omega} \nabla z^m \cdot \nabla \xi - \int_{\Omega} z V \cdot \nabla \xi = \int_{\Omega} \mu \xi, \quad \forall \xi \in H_0^1(\Omega) \cap L^\infty(\Omega).$$

As a consequence of Theorem 1.1, we see that the operator \mathcal{A}_m is accretive in $L^1(\Omega)$; i.e. $(I + \lambda \mathcal{A}_m)^{-1}$ is a contraction in $L^1(\Omega)$, for small $\lambda > 0$ (cf. Appendix section). Moreover, thanks to Theorem 3.4, $R(I + \lambda \mathcal{A}_m) \supseteq L^2(\Omega)$, for small $0 < \lambda < \lambda_0 := 1/\|(\nabla \cdot V)^-\|_\infty$, so that $\overline{\mathcal{A}_m}$, the closure of \mathcal{A}_m in $L^1(\Omega)$, is m -accretive in $L^1(\Omega)$; i.e. $R(I + \lambda \overline{\mathcal{A}_m}) = L^1(\Omega)$, for any $\lambda > 0$. One sees easily that $\overline{\mathcal{D}(\mathcal{A}_m)} = L^1(\Omega)$. Furthermore, for any $u_0 \in L^1(\Omega)$, thanks to the general theory of nonlinear semi-group governed by accretive operator, as $\varepsilon \rightarrow 0$, we have (see Appendix)

$$(3.30) \quad u_\varepsilon \rightarrow u, \quad \text{in } \mathcal{C}([0, T], L^1(\Omega)),$$

and u is the so called "mild solution" of the evolution problem

$$(3.31) \quad \begin{cases} u_t + \mathcal{A}_m u \ni f & \text{in } (0, T) \\ u(0) = u_0. \end{cases}$$

To accomplish the proof of existence for the problem (1.2), we prove that the mild solution u satisfies all the conditions of Definition 1.1. More precisely, we prove the following result.

Proposition 3.3. *Assume $V \in W^{1,2}(\Omega)$, $(\nabla \cdot V)^- \in L^\infty(\Omega)$ and V satisfies the outpointing condition (1.1). For any $u_0 \in L^2(\Omega)$ and $f \in L^2(Q)$, the mild solution u of the problem (3.31) is the unique solution of (1.2).*

To prove this result, thanks to (3.30), it is enough to study moreover the limit of sequence p_ε given by the ε -approximate solution.

Lemma 3.3. *Let u_ε and p_ε be the ε -approximate solution given by (3.29). We have*

1. *For any $q \in [1, \infty]$, we have*

$$(3.32) \quad \|u_\varepsilon(t)\|_q \leq M_q^\varepsilon, \quad \text{for any } t \geq 0,$$

$$M_q^\varepsilon := \begin{cases} \left(\|u_0\|_q + \int_0^T \|f_\varepsilon(t)\|_q dt \right) \exp((q-1) \|(\nabla \cdot V)^-\|_\infty) & \text{if } 1 \leq q < \infty \\ \left(\|u_0\|_\infty + \int_0^T \|f_\varepsilon(t)\|_\infty dt \right) \exp(\|(\nabla \cdot V)^-\|_\infty) & \text{if } q = \infty. \end{cases}$$

2. For each $\varepsilon > 0$, we have

$$(3.33) \quad \begin{aligned} \frac{1}{m+1} \int_\Omega |u_\varepsilon(t)|^{m+1} + \int_0^t \int_\Omega |\nabla p_\varepsilon|^2 &\leq \int_0^t \int_\Omega f_\varepsilon p_\varepsilon dx + \int_0^t \int_\Omega (\nabla \cdot V)^- p_\varepsilon u_\varepsilon dx \\ &+ \frac{1}{m+1} \int_\Omega |u_0|^{m+1}. \end{aligned}$$

Proof. Thanks to Theorem 3.4, the sequence $(u_i)_{i=1, \dots, n}$ of solutions of (3.28) is well defined in $L^2(\Omega)$ and satisfies

$$\int_\Omega u_{i+1} \xi + \varepsilon \int_\Omega \nabla p_{i+1} \cdot \nabla \xi - \varepsilon \int_\Omega u_{i+1} V \cdot \nabla \xi = \int_{i\varepsilon}^{(i+1)\varepsilon} \int_\Omega f_i \xi, \quad \text{for } i = 1, \dots, n-1,$$

for any $\xi \in H_0^1(\Omega)$. Thanks to (3.4), for any $1 \leq q \leq \infty$, we have

$$\|u_i\|_q \leq \|u_{i-1}\|_q + \varepsilon \|f_i\|_q + \varepsilon (q-1) \|(\nabla \cdot V)^-\|_\infty \|u_i\|_q.$$

By induction, this implies that, for any $t \in [0, T)$, we have

$$\|u_\varepsilon(t)\|_q \leq \|u_0\|_q + \int_0^t \|f_\varepsilon(t)\|_q dt + (q-1) \|(\nabla \cdot V)^-\|_\infty \int_0^t \|u_\varepsilon(t)\|_q dt.$$

Using Gronwall Lemma, we deduce (3.32), for any $1 \leq q < \infty$. The proof for the case $q = \infty$ follows in the same way by using (3.4) with $q = \infty$. Now, using the fact that

$$(u_i - u_{i-1}) p_i = (u_i - u_{i-1}) u_i^m \geq \frac{1}{m+1} (u_i^{m+1} - u_{i-1}^{m+1})$$

and

$$\int u_i V \cdot \nabla p_i \leq \int \int (\nabla \cdot V)^- p_i u_i,$$

we get

$$\begin{aligned} \frac{1}{m+1} \int_\Omega |u_i|^{m+1} + \varepsilon \int_\Omega |\nabla p_i|^2 &\leq \varepsilon \int_\Omega f_i p_i dx + \varepsilon \int (\nabla \cdot V)^- p_i u_i dx \\ &+ \frac{1}{m+1} \int_\Omega |u_{i-1}|^{m+1}. \end{aligned}$$

Summing this identity for $i = 1, \dots$, and using the definition of u_ε , p_ε and f_ε , we get (3.33). \square

Proof of Proposition 3.3. Recall that we already know that $u_\varepsilon \rightarrow u$ in $\mathcal{C}([0, T]; L^1(\Omega))$, as $\varepsilon \rightarrow 0$. Now, combining (3.32) and (3.33) with Poincaré and Young inequalities, one sees that

$$\frac{1}{m+1} \frac{d}{dt} \int_{\Omega} |u_\varepsilon|^{m+1} dx + \int_{\Omega} |\nabla p_\varepsilon|^2 dx \leq C(N, \Omega) \left(\int_{\Omega} |f_\varepsilon|^2 dx + \|(\nabla \cdot V)^-\|_\infty (M_2^\varepsilon)^2 \right), \quad \text{in } \mathcal{D}'(0, T).$$

This implies that p_ε is bounded in $L^2(0, T; H_0^1(\Omega))$. This implies that

$$p_\varepsilon \rightarrow u^m, \quad \text{in } L^2(0, T; H_0^1(\Omega)) - \text{weak}, \quad \text{as } \varepsilon \rightarrow 0.$$

Recall that taking

$$\tilde{u}_\varepsilon(t) = \frac{(t - t_i)u_{i+1} - (t - t_{i+1})u_i}{\varepsilon}, \quad \text{for any } t \in [t_i, t_{i+1}), \quad i = 1, \dots, n,$$

we have

$$(3.37) \quad \partial_t \tilde{u}_\varepsilon - \Delta p_\varepsilon + \nabla \cdot (u_\varepsilon V) = f_\varepsilon, \quad \text{in } \mathcal{D}'(Q).$$

Moreover, we know that $\tilde{u}_\varepsilon \rightarrow u$, in $\mathcal{C}([0, T], L^1(\Omega))$. So letting $\varepsilon \rightarrow 0$ in (3.37), we deduce that u is a solution of (1.2). Letting $\varepsilon \rightarrow 0$ in (3.32) and (3.33), we get respectively (1.4) and (1.5). \square

Proof of Theorem 1.2. The proof follows by Proposition 3.3. \square

4 The limit as $m \rightarrow \infty$.

Since the solution of the problem (1.2) is the mild solution associated with the operator \mathcal{A}_m , we begin by studying the L^1 -limit, as $m \rightarrow \infty$, of the solution of the stationary problem 3.1. Formally, this limiting problem is given by

$$(4.1) \quad \left\{ \begin{array}{l} v - \Delta p + \nabla \cdot (v V) = f \\ v \in \text{Sign}(p) \end{array} \right\} \quad \text{in } \Omega$$

$$\left\{ \begin{array}{l} p = 0 \end{array} \right\} \quad \text{on } \partial\Omega.$$

This is the stationary problem associated with the so called Hele-Shaw problem. Thanks to [28], for any $f \in L^2(\Omega)$, (4.1) has a unique solution (u, p) in the sense that $(v, p) \in L^\infty(\Omega) \times H_0^1(\Omega)$, $v \in \text{sign}(p)$ a.e. in Ω , and

$$(4.2) \quad \int_{\Omega} v \xi + \int_{\Omega} \nabla p \cdot \nabla \xi - \int_{\Omega} v V \cdot \nabla \xi = \int_{\Omega} f \xi, \quad \text{for any } \xi \in H_0^1(\Omega).$$

First, by using the results of the previous section, we have

Proposition 4.4. *Under the assumptions of Theorem 3.4, let us consider v_m the solution of (3.1). As $m \rightarrow \infty$, we have*

$$(4.3) \quad v_m \rightarrow v, \quad \text{in } L^2(\Omega)\text{-weak,}$$

$$(4.4) \quad v_m^m \rightarrow p, \quad \text{in } H_0^1(\Omega),$$

and (v, p) is the solution of (4.1).

Proof. Thanks to (3.4), there exists $v \in H_0^1(\Omega)$, such that (4.3) is fulfilled. Thanks to (3.5), we see that the sequences p_m is bounded in $H_0^1(\Omega)$, which implies that

$$(4.5) \quad v_m^m \rightarrow p, \quad \text{in } H_0^1(\Omega)\text{-weak.}$$

Using monotonicity arguments we see that $v \in \text{Sign}(p)$ a.e. in Ω , and letting $m \rightarrow \infty$ in (3.7), we obtain that (u, p) satisfies (4.2). To prove the strong convergence of p_m , we use the same argument of the proof of Lemma 3.2. Indeed, taking p_m as a test function in (3.2), we have

$$\begin{aligned} \lambda \int |\nabla p_m|^2 dx &= \int (f - v_m) p_m dx + \lambda \int \nabla \cdot V \left(\int_0^{p_m} r^{\frac{1}{m}} dr \right) dx \\ &= \int (f - v_m) p_m dx + \lambda \frac{m}{m+1} \int \nabla \cdot V v_m p_m dx. \end{aligned}$$

Letting $m \rightarrow \infty$, and using (4.4) and (4.5), we see that

$$\begin{aligned} \lim_{m \rightarrow \infty} \lambda \int |\nabla p_m|^2 dx &= \int (f - u) p dx + \lambda \int \nabla \cdot V u p \\ &= \int (f - u) p dx + \lambda \int \nabla \cdot V |p|. \end{aligned}$$

We know that (u, p) is a solution of (4.1), so one sees easily that

$$\lambda \int |\nabla p|^2 dx = \int (f - u) p dx + \lambda \int \nabla \cdot V |p|,$$

so that

$$\lim_{m \rightarrow \infty} \lambda \int |\nabla p_m|^2 dx = \lambda \int |\nabla p|^2 dx.$$

Thus the strong convergence of ∇p_m .

□

For the strong convergence of v_m , we need to use the assumption (1.10).

Theorem 4.5. *Under the assumptions of Theorem 4.6 ; i.e. $V \in W^{1,2}(\Omega)$, $\nabla \cdot V \in L^\infty(\Omega)$ and satisfies (1.10), for any $0 < \lambda < \lambda_V$, the convergence (4.3) holds to be true strongly in $L^1(\Omega)$. Here*

$$\lambda_V := \sum_{i,k} \|\partial_{x_i} V_k\|_\infty.$$

Corollary 4.2. *Under the assumptions of Theorem 4.5, the operator \mathcal{A}_m converges to \mathcal{A} in the sense of resolvent in $L^1(\Omega)$, where \mathcal{A} is defined by : $\mu \in \mathcal{A}(z)$ if and only if $\mu, z \in L^1(\Omega)$ and z is a solution of the problem*

$$\begin{cases} -\Delta p + \nabla \cdot (z V) = \mu & \text{in } \Omega \\ z \in \text{sign}(p) \\ p = 0 & \text{on } \partial\Omega, \end{cases}$$

in the sense that $z \in L^\infty(\Omega)$, $\exists p \in H_0^1(\Omega)$ such that $p \in H_0^1(\Omega)$, $u \in \text{sign}(p)$ a.e. in Ω and

$$\int_{\Omega} \nabla p \cdot \nabla \xi - \int_{\Omega} z V \cdot \nabla \xi = \int_{\Omega} \mu \xi, \quad \forall \xi \in H_0^1(\Omega) \cap L^\infty(\Omega).$$

Moreover, we have

$$\overline{\mathcal{D}(\mathcal{A})} = \left\{ z \in L^\infty(\Omega) : |z| \leq 1 \text{ a.e. in } \Omega \right\}.$$

The main element to prove Theorem 4.5 is BV_{loc} -estimates on v_m . Recall that a given function $u \in L^1(\Omega)$ is said to be of bounded variation if and only if, for each $i = 1, \dots, N$,

$$TV_i(u, \Omega) := \sup \left\{ \int_{\Omega} u \partial_{x_i} \xi \, dx : \xi \in \mathcal{C}_c^1(\Omega) \text{ and } \|\xi\|_\infty \leq 1 \right\} < \infty,$$

here $\mathcal{C}_c^1(\Omega)$ denotes the set of \mathcal{C}^1 -function compactly supported in Ω . More generally a function is locally of bounded variation in a domain Ω if and only if for any open set $\omega \subset\subset \Omega$, $TV_i(u, \omega) < \infty$ for any $i = 1, \dots, N$. In general a function locally of bounded variation (as well as function of bounded variation) in Ω , may not be differentiable, but by the Riesz representation theorem, their partial derivatives in the sense of distributions are Borel measure in Ω . This gives rise to the definition of the vector space of functions of bounded variation in Ω , usually denoted by $BV(\Omega)$, as the set of $u \in L^1(\Omega)$ for which there are Radon measures μ_1, \dots, μ_N with finite total mass in Ω such that

$$\int_{\Omega} u \partial_{x_i} \xi \, dx = - \int_{\Omega} \xi \, d\mu_i, \quad \text{for any } \xi \in \mathcal{C}_c(\Omega), \quad \text{for } i = 1, \dots, N.$$

Without abusing we'll continue to point out the measures μ_i by $\partial_{x_i} v$ anyway, and by $|\partial_{x_i} v|$ the total variation of μ_i . Moreover, we'll use as usual $Du = (\partial_{x_1} u, \dots, \partial_{x_N} u)$ the vector valued Radon measure pointing out the gradient of any function $u \in BV(\Omega)$, and $|Du|$ indicates the total variation measure of u . In particular, for any open set $\omega \subset\subset \Omega$, $TV_i(u, \omega) = |\partial_{x_i} v|(\omega) < \infty$, and the total variation of the function u in ω is finite too ; i.e.

$$\|Du\|(\omega) = \sup \left\{ \int_{\Omega} u \nabla \xi \, dx : \xi \in \mathcal{C}_c^1(\omega) \text{ and } \|\xi\|_\infty \leq 1 \right\} < \infty.$$

At last, let us remind the reader here the well known compactness result for functions of bounded variation : given a sequence u_n of functions in $BV_{loc}(\Omega)$ such that, for any open set $\omega \subset\subset \Omega$, we have

$$\sup_n \left\{ \int_{\omega} |u_n| \, dx + |Du_n|(\omega) \right\} < \infty,$$

there exists a subsequence that we denote again by u_n which converges in $L^1_{loc}(\Omega)$ to a function $u \in BV_{loc}(\Omega)$. Moreover, for any compactly supported continuous function $0 \leq \xi$, the limit u satisfies

$$\int \xi |\partial_{x_i} u| \leq \liminf_{n \rightarrow \infty} \int \xi |\partial_{x_i} u_n|,$$

for any $i = 1, \dots, N$, and

$$\int \xi |Du| \leq \liminf_{n \rightarrow \infty} \int \xi |Du_n|.$$

Theorem 4.6. *Assume $f \in BV_{loc}(\Omega)$, $V \in W^{1,\infty}(\Omega)^N$, $\nabla \cdot V \in W^{1,2}_{loc}(\Omega)$ and satisfies (1.10). Let v_m be the solution of (3.1). For any $0 < \lambda < 1/\lambda_V$, $v_m \in BV_{loc}(\Omega)$. Moreover, for any $0 < h < h_0$, we have*

$$(4.10) \quad \begin{aligned} (1 - \lambda\lambda_V) \sum_{i=1}^N \int \xi_h d |\partial_{x_i} v| &\leq \lambda \sum_{i=1}^N \int (\Delta \xi_h)^+ |\partial_{x_i} p| dx + \sum_{i=1}^N \int \xi_h d |\partial_{x_i} f| \\ &+ \lambda \sum_{i=1}^N \int \xi_h |v| |\partial_{x_i} (\nabla \cdot V)| dx. \end{aligned}$$

To prove this result we use again the regularized problem (3.6) and we let $\varepsilon \rightarrow 0$.

Lemma 4.4. *Under the assumptions of Theorem 4.6, let us consider v_ε be the solution of (3.6) given by Lemma 3.1. For each $i = 1, \dots, N$, we have*

$$(4.11) \quad \begin{aligned} |\partial_{x_i} v_\varepsilon| - \lambda \sum_{k=1}^N |\partial_{x_k} v_\varepsilon| \sum_{k=1}^N |\partial_{x_i} V_k| - \lambda \Delta |\partial_{x_i} p_\varepsilon| + \lambda \nabla \cdot (|\partial_{x_i} v_\varepsilon| V) \\ \leq |\partial_{x_i} f_\varepsilon| + \lambda |v_\varepsilon| |\partial_{x_i} (\nabla \cdot V)| \quad \text{in } \mathcal{D}'(\Omega). \end{aligned}$$

Proof. Assume right away without loose of generality that $\lambda = 1$ and remove the script ε in the notations of v_ε , p_ε and β_ε , throughout the proof. Recall that (u, p) satisfies

$$v - \Delta p = f - (\nabla v \cdot V + v \nabla \cdot V) \quad \text{in } \mathcal{D}'(\Omega).$$

Since $p \in H^1(\Omega)$, $v = \beta(p) \in H^1(\Omega)$, $V \in L^\infty(\Omega)$, $\nabla \cdot V \in L^\infty(\Omega)$ and $f \in L^2(\Omega)$, $\Delta p \in L^2(\Omega)$ and then $u, p \in H^2(\Omega)$. So, for each $i = 1, \dots, N$, we see that the partial derivatives $\partial_{x_i} v$ and $\partial_{x_i} p$ satisfy the following equation

$$(4.12) \quad \partial_{x_i} v - \Delta \partial_{x_i} p + \nabla \cdot (\partial_{x_i} v V) = \partial_{x_i} f - (\nabla v \cdot \partial_{x_i} V + v \partial_{x_i} (\nabla \cdot V)), \quad \text{in } \mathcal{D}'(\Omega).$$

For a given $\xi \in \mathcal{C}^2_c(\Omega)$, taking $\xi H_\varepsilon(\partial_{x_i} v)$ as a test function in (4.12), we obtain

$$(4.13) \quad \begin{aligned} \int \left(\partial_{x_i} v \xi H_\varepsilon(\partial_{x_i} v) + \nabla \partial_{x_i} p \cdot \nabla (\xi H_\varepsilon(\partial_{x_i} v)) \right) dx - \int \partial_{x_i} v V \cdot \nabla (\xi H_\varepsilon(\partial_{x_i} v)) dx \\ = \int \partial_{x_i} f \xi H_\varepsilon(\partial_{x_i} v) dx - \int (\nabla v \cdot \partial_{x_i} V + v \partial_{x_i} (\nabla \cdot V)) \xi H_\varepsilon(\partial_{x_i} v) dx. \end{aligned}$$

To pass to the limit as $\varepsilon \rightarrow 0$, we see first that

$$(4.14) \quad H'_\varepsilon(\partial_{x_i} v) \partial_{x_i} v = \frac{1}{\varepsilon} \partial_{x_i} v \chi_{[|\partial_{x_i} v| \leq \varepsilon]} \rightarrow 0, \quad \text{in } L^q(\Omega), \text{ for any } q \geq 1.$$

So, the last term of the first part of (4.13) satisfies

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int \partial_{x_i} v V \cdot \nabla(\xi H_\varepsilon(\partial_{x_i} v)) dx &= \int |\partial_{x_i} v| V \cdot \nabla \xi dx + \lim_{\varepsilon \rightarrow 0} \int \partial_{x_i} v \nabla \partial_{x_i} v \cdot V H'_\varepsilon(\partial_{x_i} v) \xi dx \\ &= \int |\partial_{x_i} v| V \cdot \nabla \xi, \end{aligned}$$

On the other hand, we see that

$$\int \nabla \partial_{x_i} p \cdot \nabla(\xi H_\varepsilon(\partial_{x_i} v)) dx = \int H_\varepsilon(\partial_{x_i} v) \nabla \partial_{x_i} p \cdot \nabla \xi dx + \int \xi \nabla \partial_{x_i} p \cdot \nabla H_\varepsilon(\partial_{x_i} v) dx.$$

Since $\text{sign}_0(\partial_{x_i} v) = \text{sign}_0(\partial_{x_i} p)$, the first term satisfies

$$\lim_{\varepsilon \rightarrow 0} \int H_\varepsilon(\partial_{x_i} v) \nabla \partial_{x_i} p \cdot \nabla \xi dx = \int |\partial_{x_i} p| \Delta \xi dx.$$

As to the second term, we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int \xi \nabla \partial_{x_i} p \cdot \nabla H_\varepsilon(\partial_{x_i} v) dx &= \lim_{\varepsilon \rightarrow 0} \int \xi H'_\varepsilon(\partial_{x_i} v) \nabla \partial_{x_i} p \cdot \nabla \partial_{x_i} v dx \\ &= \lim_{\varepsilon \rightarrow 0} \int \xi H'_\varepsilon(\partial_{x_i} v) \nabla(\beta'(u) \partial_{x_i} v) \cdot \nabla \partial_{x_i} v dx \\ &= \lim_{\varepsilon \rightarrow 0} \int \xi H'_\varepsilon(\partial_{x_i} v) \beta'(u) \|\nabla \partial_{x_i} v\|^2 dx \\ &\quad + \lim_{\varepsilon \rightarrow 0} \int \xi H'_\varepsilon(\partial_{x_i} v) \partial_{x_i} v \beta''(v) \nabla v \cdot \nabla \partial_{x_i} v dx \\ &\geq \lim_{\varepsilon \rightarrow 0} \int \xi H'_\varepsilon(\partial_{x_i} v) \partial_{x_i} v \beta''(v) \nabla v \cdot \nabla \partial_{x_i} v dx \\ &\geq 0, \end{aligned}$$

where we use again (4.14). So, letting $\varepsilon \rightarrow 0$ in (4.13) and using again the fact that $\text{sign}_0(\partial_{x_i} v) = \text{sign}_0(\partial_{x_i} p)$, we get

$$\begin{aligned} |\partial_{x_i} v| - \Delta |\partial_{x_i} p| + \nabla \cdot (|\partial_{x_i} v| V) &\leq \text{sign}_0(\partial_{x_i} v) \partial_{x_i} f - (\nabla v \cdot \partial_{x_i} V \\ &\quad + v \partial_{x_i} (\nabla \cdot V)) \text{sign}_0(\partial_{x_i} v) \quad \text{in } \mathcal{D}'(\Omega) \end{aligned}$$

Coming back with $\lambda > 0$ in the formula and using the fact that

$$|\nabla v \cdot \partial_{x_i} V| \leq \sum_k |\partial_{x_k} v| \sum_k |\partial_{x_i} V_k|,$$

the result of the lemma follows. \square

Proof of Theorem 4.6. Under the assumptions of Theorem 4.6, for any $\epsilon > 0$, let us consider f_ϵ a regularization of f satisfying $f_\epsilon \rightarrow f$ in $L^1(\Omega)$ and

$$\int \xi |\partial_{x_i} f_\epsilon| dx \rightarrow \int \xi d|\partial_{x_i} f|, \quad \text{for any } \xi \in \mathcal{C}_c(\Omega) \text{ and } i = 1, \dots, N.$$

Then, let us consider v_ϵ be the solution of the problem (3.1), where we replace f by the regularization f_ϵ . Thanks to Lemma 4.4, let us remind that for any $0 \leq \xi \in \mathcal{C}_c^2(\Omega)$, we have

$$\begin{aligned} \int |\partial_{x_i} v_\epsilon| \xi dx - \lambda \int \sum_{k=1}^N |\partial_{x_i} V_k| \sum_{k=1}^N |\partial_{x_k} v_\epsilon| \xi dx - \lambda \int |v_\epsilon| |\partial_{x_i}(\nabla \cdot V)| \xi dx &\leq \lambda \sum_{k=1}^N \int |\partial_{x_i} p_\epsilon| (\Delta \xi)^+ dx \\ &+ \int |\partial_{x_i} f_\epsilon| \xi dx - \lambda \int |\partial_{x_i} v_\epsilon| V \cdot \nabla \xi dx, \quad \text{for any } i = 1, \dots, N. \end{aligned}$$

Now, taking $\xi = \xi_h$ and using (1.10), we see that the last term satisfies

$$\begin{aligned} \int |\partial_{x_i} v_\epsilon| V \cdot \nabla \xi_h dx &= \int_{\Omega \setminus \Omega_h} |\partial_{x_i} v_\epsilon| V \cdot \nabla \xi_h dx \\ &= - \int_{\Omega \setminus \Omega_h} |\partial_{x_i} v_\epsilon| V \cdot \nu_h dx \\ &\leq 0 \end{aligned}$$

so that

$$\begin{aligned} \int |\partial_{x_i} v| \xi_h dx - \lambda \sum_k |\partial_{x_i} V_k| \int \sum_k |\partial_{x_k} v| \xi_h dx &\leq \lambda \sum_k \int |\partial_{x_i} p| (\Delta \xi_h)^+ dx \\ &+ \int |\partial_{x_i} f| \xi_h dx + \lambda \int |v| |\partial_{x_i}(\nabla \cdot V)| \xi_h dx, \quad \text{for any } i = 1, \dots, N. \end{aligned}$$

Summing up, for $i = 1, \dots, N$, and using the definition of λ_V , we deduce that

$$\begin{aligned} \sum_i \int |\partial_{x_i} v_\epsilon| \xi_h dx - \lambda \lambda_V \sum_k \int |\partial_{x_k} v_\epsilon| \xi_h dx &\leq \lambda \sum_i \int (\Delta \xi_h)^+ |\partial_{x_i} p_\epsilon| dx \\ &+ \int \sum_i |\partial_{x_i} f_\epsilon| \xi_h dx + \lambda \int |v_\epsilon| \sum_i |\partial_{x_i}(\nabla \cdot V)| \xi_h dx, \end{aligned}$$

and then the corresponding property (4.10) follows for v_ϵ . Thanks to (3.4) and (3.5), we know that v_ϵ and $\partial_{x_i} p_\epsilon$ are bounded in $L^2(\Omega)$. This implies that, for any $\omega \subset\subset \Omega$, $\sum_i \int_\omega |\partial_{x_i} v_\epsilon| dx$ is bounded.

So, v_ϵ is bounded in $BV_{loc}(\Omega)$. Combining this with the L^1 -bound (3.4), it implies in particular, taking a subsequence if necessary, the convergence in (3.25) holds to be true also in $L^1(\Omega)$ and then $v \in BV_{loc}(\Omega)$. At last, letting $\epsilon \rightarrow \infty$ in (4.11) and, using moreover (3.26) and the lower semi-continuity of variation measures $|\partial_{x_i} v_\epsilon|$, we deduce (4.10) for the limit v , which is the solution of the problem (3.1) by Lemma 3.2. \square

Remark 5. In connection with Remark 3, taking $\xi = \omega_h$ given by (1.12) instead of ξ_h , the results of Theorem 4.6 as well as the results of all this section remains to be true. Thus we can replace the assumption (1.10) by the general one (1.12).

Proof of Theorem 4.5. Recall that under the assumptions of the theorem, the BV_{loc} estimate (4.10) is fulfilled for v_m . Since the constants C in (3.5) does not depend on m , this implies that u_m is bounded in $BV(\omega)$. Since ω is arbitrary, we deduce in particular that the convergence in (4.3) holds to be true also in $L^1(\Omega)$, $v \in BV_{loc}(\Omega)$, and (4.10) is fulfilled. \square

Proof of Theorem 1.3. Thanks to Corollary 4.2, we have

$$u_m \rightarrow u, \quad \text{in } \mathcal{C}([0, T]; L^1(\Omega)).$$

On the other hand, thanks to (3.4) and (3.5), it is clear that p_m is bounded in $L^2(0, T; H_0^1(\Omega))$. So, there exists $p \in L^2(0, T; H_0^1(\Omega))$, such that, taking a subsequence if necessary, we have

$$u_m^m \rightarrow p, \quad \text{in } L^2(0, T; H_0^1(\Omega)) - \text{weak}.$$

Then using monotonicity arguments we have $u \in \text{sign}(p)$ a.e. in Q , and letting $m \rightarrow \infty$, in the weak formulation we deduce that the couple (u, p) satisfies (1.11). Thus the results of the theorem. \square

5 Reaction case

Let us consider now the reaction-diffusion porous medium equation with linear drift

$$(5.24) \quad \begin{cases} \frac{\partial u}{\partial t} - \Delta u^m + \nabla \cdot (u V) = g(\cdot, u) & \text{in } Q \\ u = 0 & \text{on } \Sigma \\ u(0) = u_0 & \text{in } \Omega, \end{cases}$$

Thanks to Theorem 6.10 and Theorem 6.11, we assume that $g : Q \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory application ; i.e. continuous in $r \in \mathbb{R}$ and measurable in $(t, x) \in Q$, and satisfies moreover the following assumptions :

(\mathcal{G}_1) $g(\cdot, r) \in L^2(Q)$ for any $r \in \mathbb{R}$.

(\mathcal{G}_2) There exists $0 \leq \theta$, such that

$$\frac{\partial g}{\partial r}(t, x, \cdot) \leq \theta, \quad \text{in } \mathcal{D}'(\mathbb{R}), \text{ for a.e. } (t, x) \in Q.$$

(\mathcal{G}_3) There exists $\omega_1, \omega_2 \in W^{1, \infty}(0, T)$ such that $u_0 \leq \omega_2(0)$ (resp. $\omega_1(0) \leq u_0$), for any $t \in (0, T)$,

$$\dot{\omega}_2(t) + \omega_2(t) \nabla \cdot V \geq g(\cdot, \omega_2(t)) \quad \text{a.e. in } \Omega$$

(rep. $\dot{\omega}_1(t) + \omega_1(t) \nabla \cdot V \leq g(\cdot, \omega_1(t))$, a.e. in Ω).

Remark 6. *One sees in particular that (G2) implies that, for any $a, b \in \mathbb{R}$, we have*

$$\text{sign}_0^+(b-a)(g(t,x,b) - g(t,x,a)) \leq \theta(b-a)^+, \quad \text{for a.e. } (t,x) \in Q.$$

and then, for any $r \in [-M, M]$, we have

$$(5.25) \quad -g^-(\cdot, M) - \theta(M-r) \leq g(\cdot, r) \leq g^+(\cdot, -M) + \theta(M+r), \quad \text{a.e. in } Q, .$$

Theorem 5.7. *Assume $V \in W^{1,2}(\Omega)$, $\nabla \cdot V \in L^\infty(\Omega)$ and V satisfies the outpointing condition (1.1). Under the assumption (\mathcal{G}_1) , (\mathcal{G}_2) and (\mathcal{G}_3) , for any $u_0 \in L^2(\Omega)$, the problem (5.24) has a unique weak solution u_m in the sense of Definition 1.1 with $f = g(\cdot, u)$. Moreover, we have*

1. *u is the unique mild solution of the Cauchy problem (3.31) with $f(\cdot) = g(\cdot, u(\cdot))$ a.e. in Q .*
2. *for any $0 \leq t < T$, $\omega_1(t) \leq u(t) \leq \omega_2(t)$ a.e. in Ω .*

Corollary 5.3. *Under the assumptions of Theorem 5.7, assume moreover that $0 \leq u_0$ a.e. in Ω , and*

$$(\mathcal{G}_4) \quad 0 \leq g(\cdot, 0) \text{ a.e. in } Q.$$

then the solution of (5.24) satisfies

$$0 \leq u(t) \leq \omega_2(t), \quad \text{a.e. in } \Omega, \text{ for any } t \in (0, T).$$

Proof of Theorem 5.7. Let $F : [0, T] \times L^1(\Omega) \rightarrow L^1(\Omega)$ be given by

$$F(t, z(\cdot)) = g(t, \cdot, (z(\cdot) \vee (-M)) \wedge M) \quad \text{a.e. in } \Omega, \text{ for any } (t, z) \in [0, T] \times L^1(\Omega),$$

where $M := \max(\|\omega_1\|_\infty, \|\omega_2\|_\infty)$. Thanks to Remark 6, one sees that F satisfies all the assumptions of Theorem 6.10. Then, thanks to Theorem 6.9, we consider $u \in \mathcal{C}([0, T], L^1(\Omega))$ the mild solution of the evolution problem

$$\begin{cases} u_t + \mathcal{A}_m u \ni F(\cdot, u) & \text{in } (0, T) \\ u(0) = u_0. \end{cases}$$

Thanks to (5.25), it is clear that $F(\cdot, u) \in L^2(Q)$, so that, using Proposition 3.3, we can deduce that u is a weak solution of (1.2). The uniqueness follows from the equivalence between weak solution and mild solution as well as the uniqueness result of Theorem 6.10. To end up the proof, it is enough to show that $\omega_1(t) \leq u(t) \leq \omega_2(t)$ a.e. in Ω , for any $0 \leq t < T$. Indeed, in particular this implies that $F(t, u(t)) = g(t, \cdot, u(t))$. To this aim, we use Theorem 1.1 with the the fact that ω_2 is a weak solution of (1.2) with $f = \dot{\omega}_2 + \omega_2 \nabla \cdot V$, to see that

$$\begin{aligned} \frac{d}{dt} \int (u - \omega_2)^+ dx &\leq \int_{[u \geq \omega_2]} (g(\cdot, u) - \dot{\omega}_2 - \omega_2 \nabla \cdot V) dx \\ &\leq \int_{[u \geq \omega_2]} (g(\cdot, u) - g(\cdot, \omega_2)) dx \\ &\leq \theta \int (u(t) - \omega_2)^+ dx. \end{aligned}$$

Applying Gronwall and using the fact that $u(0) \leq \omega_2(0)$, we obtain $u(t) \leq \omega_2$ a.e. in Q . The proof of $u \geq \omega_1$ in Q follows in the same way by proving that

$$\frac{d}{dt} \int (\omega_1 - u)^+ \leq \theta \int (\omega_1 - u)^+.$$

Thus the results of the theorem. □

Now, for the limit of the solution of (5.24), we have the following result.

Theorem 5.8. *Assume $V \in W^{1,2}(\Omega)$, $\nabla \cdot V \in L^\infty(\Omega)$ and V satisfies the outpointing condition (1.10). Let g_m be a sequence of Carathéodory applications satisfying (\mathcal{G}_1) , (\mathcal{G}_2) and (\mathcal{G}_3) with θ independent of m . For any $u_{0m} \in L^2(\Omega)$ being a sequence of initial data let u_m be the sequence of corresponding solution of (5.24). If*

$$(5.29) \quad g_m(\cdot, r) \rightarrow g(\cdot, r), \quad \text{in } L^1(Q), \quad \text{for any } r \in \mathbb{R},$$

and

$$u_{0m} \rightarrow u_0, \quad \text{in } L^1(\Omega), \quad \text{and } |u_0| \leq 1 \text{ a.e. in } \Omega,$$

then, we have

1. $u_m \rightarrow u$ in $\mathcal{C}([0, T], L^1(\Omega))$
2. $u_m^m \rightarrow p$ in $L^2(0, T; H_0^1(\Omega))$ -weak
3. (u, p) is the solution of the Hele-Shaw problem

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t} - \Delta p + \nabla \cdot (u V) = g(\cdot, u) \\ u \in \text{sign}(p) \end{array} \right\} \quad \text{in } Q$$

$$\left\{ \begin{array}{l} u = 0 \end{array} \right\} \quad \text{on } \Sigma$$

$$\left\{ \begin{array}{l} u(0) = u_0 \end{array} \right\} \quad \text{in } \Omega,$$

in the sense that (u, p) is the solution of (1.9) with $f(\cdot) = g(\cdot, u(\cdot))$ a.e. in Q satisfying $u(0) = u_0$.

Proof. To begin with we prove compactness of u_m in $\mathcal{C}([0, T]; L^1(\Omega))$. We know that u_m is the mild solution of the sequence of Cauchy problems

$$\left\{ \begin{array}{l} u_t + \mathcal{A}_m u \ni F_m(\cdot, u) \quad \text{in } (0, T) \\ u(0) = u_{0m}, \end{array} \right.$$

where, for a.e. $t \in (0, T)$, $F_m(t, z) = g_m(t, \cdot, z(\cdot)) \vee (-M) \wedge M$, a.e. in Ω , for any $z \in L^1(\Omega)$, and

$$M := \max(\|\omega_1\|_\infty, \|\omega_2\|_\infty).$$

Thanks to (5.25), one sees that F_m satisfies all the assumptions of Theorem 6.11. This implies, by Theorem 6.11, that

$$(5.30) \quad u_m \rightarrow u, \quad \text{in } \mathcal{C}([0, T]; L^1(\Omega)), \text{ as } m \rightarrow \infty.$$

Thus the compactness of u_m . Remember that u_m is a weak solution of

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u^m + \nabla \cdot (u V) = f_m & \text{in } Q := (0, T) \times \Omega \\ u = 0 & \text{on } \Sigma := (0, T) \times \partial\Omega \\ u(0) = u_{0m} & \text{in } \Omega. \end{cases}$$

with $f_m := g(\cdot, u_m)$. Using again (5.25), (5.29) and (5.30), we have

$$f_m \rightarrow g(\cdot, u) \quad \text{in } L^1(Q), \quad \text{as } m \rightarrow \infty.$$

So, by Corollary 4.2 and Theorem 6.11, we deduce that u is a solution of (1.4) and p is given by the limit of u_m^m in $L^2(0, T; H_0^1(\Omega))$ -weak. At last the uniqueness follows from the results of the recent paper [27]. □

6 Appendix

6.1 Reminder on evolution problem governed by accretive operator

Our aim here is to remind the reader on some basic tools on L^1 -nonlinear semi-group theory. We are interested in PDE which can be written in the following form

$$(6.1) \quad \begin{cases} \frac{du}{dt} + Bu \ni f & \text{in } (0, T) \\ u(0) = u_0, \end{cases}$$

where B is a possibly multivalued operator defined on $L^1(\Omega)$ by its graph

$$B = \{(x, y) \in L^1(\Omega) \times L^1(\Omega) : y \in Bx\},$$

$f \in L^1(0, T; L^1(\Omega))$ and $u_0 \in L^1(\Omega)$. An operator B is said to be accretif if and only if the operator $J_\lambda := (I + \lambda B)^{-1}$ defines a contraction in $L^1(\Omega)$, for any $\lambda > 0$; i.e. if for $i = 1, 2$, $(f_i - u_i) \in \lambda B u_i$, then $\|u_1 - u_2\|_1 \leq \|f_1 - f_2\|_1$.

To study the evolution problem (6.1), the main ingredient is to use the operator J_λ , through the Euler-Implicit time discretization scheme. For an arbitrary $0 < \varepsilon \leq \varepsilon_0$, and $n \in \mathbb{N}^*$ be such that $(n+1)\varepsilon = T$, we consider the sequence of $(u_i, p_i)_{i=0, \dots, n}$ given by :

$$u_i + \varepsilon B u_i \ni \varepsilon f_i + u_{i-1}, \quad \text{for } i = 1, \dots, n,$$

where, for each $i = 0, \dots, n-1$, f_i is given by

$$f_i = \frac{1}{\varepsilon} \int_{i\varepsilon}^{(i+1)\varepsilon} f(s) ds, \quad \text{a.e. in } \Omega.$$

Then, for a given ε -time discretization $0 = t_0 < t_1 < t_2 < \dots < t_n < t_{n+1} = T$, satisfying $t_{i+1} - t_i = \varepsilon$, we define the ε -approximate solution

$$u_\varepsilon := \sum_{i=0}^{n-1} u_i \chi_{[t_i, t_{i+1})}.$$

Moreover, we denote by \tilde{u}_ε its linear interpolate given by

$$\tilde{u}_\varepsilon(t) = \sum_{i=0}^{n-1} \frac{(t - t_i)u_{i+1} - (t - t_{i+1})u_i}{t_{i+1} - t_i} \chi_{[t_i, t_{i+1})}(t), \quad \text{for any } t \in [0, T).$$

In particular, one sees that u_ε , \tilde{u}_ε and f_ε satisfies the following ε -approximate dynamic

$$\frac{d\tilde{u}_\varepsilon}{dt} + B u_\varepsilon \ni f_\varepsilon, \quad \text{in } (0, T).$$

The main goal afterwards is to let $\varepsilon \rightarrow 0$, to cover the "natural" solution of the Cauchy problem (6.1). The following theorem known as Crandall-Liggett theorem (at least in the case where $f \equiv 0$, cf. [19]) pictures the limit of u_ε and \tilde{u}_ε .

Theorem 6.9. *Let B be an accretive operator in $L^1(\Omega)$ and $u_0 \in \overline{D(B)}$. If for each $\varepsilon > 0$, the ε -approximate solution u_ε is well defined, then there exists a unique $u \in \mathcal{C}([0, T], L^1(\Omega))$ such that $u(0) = u_0$,*

$$u_\varepsilon \rightarrow u \quad \text{and} \quad \tilde{u}_\varepsilon \rightarrow u \quad \text{in } \mathcal{C}([0, T], L^1(\Omega)), \quad \text{as } \varepsilon \rightarrow 0.$$

The function u is called the mild solution of the evolution problem (6.1). Moreover, if u_1 and u_2 are two mild solutions associated with f_1 and f_2 , then there exists $\kappa \in L^\infty(\Omega)$, such that $\kappa \in \text{sign}(u_1 - u_2)$ a.e. in Q , and

$$\frac{d}{dt} \|u_1 - u_2\|_1 \leq \int_{[u_1=u_2]} |f_1 - f_2| dx + \int_{[u_1 \neq u_2]} (f_1 - f_2) \text{sign}_0(u_1 - u_2), \quad \text{in } \mathcal{D}'(0, T).$$

One sees that this theorem figures out in a natural way a solution to the Cauchy problem (6.1) to settle existence and uniqueness questions for the associate PDE. However, in general we do not know in which sense the limit u satisfies the concluding PDE ; this is connected to the regularity of u as well

as to the compactness of $\frac{d\tilde{u}_\varepsilon}{dt}$. We refer interested readers to [5] and [6] for more developments and examples in this direction. One can see also the book [14] in the case of Hilbert space, for which the concept of m-accretive operator is reappointed by maximal monotone graph notion.

One sees that besides the accretivity (monotonicity in the case of Hilbert space) the well posedness for the "generic" associate stationary problem

$$u + \lambda Bu \ni g, \quad \text{for a given } g$$

is first need. Thereby, a sufficient condition for the results of Theorem 6.9 is given by the so called range condition

$$\overline{\mathcal{R}(I + \lambda B)} = L^1(\Omega), \quad \text{for small } \lambda > 0.$$

Indeed, in this case Euler-Implicit time discretization scheme is well pose for any $i = 0, \dots, n - 1$, and the ε -approximate solution is well defined (for small $\varepsilon > 0$). Then the convergence to unique mild solution u follows by accretivity (monotonicity in the case of Hilbert space).

In particular, Theorem 6.9 enables to associate to each accretif operator B satisfying the range condition a nonlinear semi-group of contraction in $L^1(\Omega)$. It is given by Crandall-Ligget exponential formula

$$e^{-tB}u_0 = L^1 - \lim \left(I + \frac{t}{n}B \right)^{-n} u_0, \quad \text{for any } u_0 \in \overline{\mathcal{D}(B)}.$$

In other words the mild solution of (6.1) with $f \equiv 0$ is given by $e^{-tB}u_0$.

The attendance of a reaction in nonlinear PDE hints to study evolution problem of the type

$$(6.2) \quad \begin{cases} \frac{du}{dt} + Bu \ni F(\cdot, u) & \text{in } (0, T) \\ u(0) = u_0, \end{cases}$$

where $F : (0, T) \times L^1(\Omega) \rightarrow L^1(\Omega)$, is assumed to be Carathéodory, i.e. $F(t, z)$ is measurable in $t \in (0, T)$ and continuous in $z \in L^1(\Omega)$. To solve the evolution problem (6.2) in the framework of ε -approximate/mild solution, we say that $u \in \mathcal{C}([0, T]; L^1(\Omega))$ is a mild solution of (6.2) if and only if u is a mild solution of (6.1) with $f(t) = F(t, u(t))$ for a.e. $t \in (0, T)$. Existence and uniqueness are more or less well known in the case where $F(t, r) = f(t) + F_0(r)$, with $f(t) \in L^1(\Omega)$, for a.e. $t \in [0, T]$, and F_0 a Lipschitz continuous function in \mathbb{R} . The following theorems set up general assumptions on F to ensure existence and uniqueness of mild solution for (6.2), as well as continuous dependence with respect to u_0 and F . We refer the readers to [10] for the detailed of proofs in abstract Banach spaces.

To call back these results, we assume moreover that F satisfies the following assumptions :

(F₁) There exists $k \in L^1_{loc}(0, T)$ such that

$$\int (F(t, z) - F(t, \hat{z})) \text{sign}_0(z - \hat{z}) dx \leq k(t) \|z - \hat{z}\|_1, \quad \text{a.e. } t \in (0, T),$$

for every $z, \hat{z} \in \overline{\mathcal{D}(B)}$.

(F₂) There exists $c \in L^1_{loc}(0, T)$ such that

$$\|F(t, z)\|_1 \leq c(t), \quad \text{a.e. } t \in (0, T)$$

for every $z \in \overline{D(B)}$.

In particular, one sees that under these assumptions, $F(\cdot, u) \in L^1_{loc}(0, T; L^1(\Omega))$ for any $u \in \mathcal{C}([0, T]; L^1(\Omega))$.

Theorem 6.10. (cf. [10]) *If B be an accretive operator in $L^1(\Omega)$ such that J_λ well defined in a dense subset of $L^1(\Omega)$, then, for any $u_0 \in \overline{D(B)}$ there exists a unique mild solution u of (6.2); i.e. u is the unique function in $\mathcal{C}([0, T]; X)$, s.t. u is the unique mild solution of*

$$\begin{cases} \frac{du}{dt} + Bu \ni f & \text{in } (0, T) \\ u(0) = u_0, \end{cases}$$

with $f(t) = F(t, u(t))$ a.e. $t \in (0, T)$.

Another important results concerns the continuous dependence of the solution with respect to the operator B as well to the data f_n and u_{0n} is given in the following theorem. The proof may be found in [10].

Theorem 6.11. (cf. [10]) *For $m = 1, 2, \dots$, let B_m be an accretive operators in $L^1(\Omega)$ satisfying the range condition and $F_m : (0, T) \times \overline{D(B_m)} \rightarrow L^1(\Omega)$ a Carathéodory applications satisfying (F₁) and (F₂) with k and c independent of m . For each $m = 1, 2, \dots$ we consider $u_{0m} \in \overline{D(B_m)}$ and u_m the mild solution of the evolution problem*

$$\begin{cases} \frac{du}{dt} + B_m u \ni f_m & \text{in } (0, T) \\ u(0) = u_{0m}, \end{cases}$$

with $f_m = F_m(\cdot, u)$. If, there exists an accretive operators B in $L^1(\Omega)$ and $F : (0, T) \times \overline{D(B)} \rightarrow L^1(\Omega)$ such that

- a) $(I + \lambda B_m)^{-1} \rightarrow (I + \lambda B)^{-1}$ in $L^1(\Omega)$, for any $0 < \lambda < \lambda_0$
- b) $F_m(t, z_m) \rightarrow F(t, z)$ in $L^1(\Omega)$, for a.e. $t \in (0, T)$, and any $z_m \in \overline{D(B_m)}$ such that $\lim_{m \rightarrow \infty} z_m = z \in \overline{D(B)}$.
- c) there exists $u_0 \in \overline{D(B)}$, such that $u_{0m} \rightarrow u_0$,

then

$$u_m \rightarrow u, \quad \text{in } \mathcal{C}([0, T], L^1(\Omega)),$$

and u is the unique mild solution of

$$\begin{cases} \frac{du}{dt} + Bu \ni F(\cdot, u) & \text{in } (0, T) \\ u(0) = u_0. \end{cases}$$

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