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Ground state photon number at large distance

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Abstract

The purpose of this article is to give a result of localization in position space of the photons in the ground state for a Hamiltonian describing nuclear magnetic resonance in quantum electrodynamics. We prove that $|x|^{-5}$ is an upper bound of the photon number density as $|x|$ goes to infinity. We also obtain the asymptotic behavior of the photon number at infinity and prove in particular that it is the smallest in the direction of the expectation of the total spin in the ground state.

Keywords: Ground state, photon localization, photon number, photon density, photon asymptotic, nuclear magnetic resonance, quantum electrodynamics

1 Introduction.

This article is concerned with the ground states of a Hamiltonian describing Nuclear Magnetic Resonance (NMR) in the framework of Quantum Electrodynamics (QED), in particular with the localization in position space in \mathbb{R}^3 of the photons in these states.

NMR is the spin interaction of a finite number of static particles of \mathbb{R}^3 with a constant magnetic field and with the quantized magnetic field. This phenomenon is represented by a Hamiltonian $H(g)$ depending on a positive parameter g (the coupling constant) and acting on a Hilbert space \mathcal{H}_{tot} . This Hamiltonian, introduced by Cohen-Tannoudji, Dupont-Roc and Grynberg [8] (see also Reuse [32]), may also be seen as a simplification of the Pauli-Fierz Hamiltonian [6, 12]. It is recalled in Section 2.

The Hilbert space \mathcal{H}_{tot} is a completed tensor product $\mathcal{H}_{\text{tot}} = \mathcal{H}_{\text{ph}} \otimes \mathcal{H}_{\text{sp}}$, where \mathcal{H}_{ph} is the space of the photons and \mathcal{H}_{sp} the space of the particles with spin (see Section 2). The space \mathcal{H}_{ph} is a Fock space, in which the usual operators, as the number operator N or the annihilation operators $a(k)$, $k \in \mathbb{R}^3$, are defined.

If f belongs to the domain $D(N^{1/2} \otimes I)$ then one can localize the photons in a given state f of the Hilbert space \mathcal{H}_{tot} , in momentum space, as well as in position space. We recall in Appendix A that, for all f in $D(N^{1/2})$, one can define a function (annihilation operator) denoted $k \rightarrow a(k)f$ defined almost everywhere on \mathbb{R}^3 and taking values in $\mathcal{H}_{\text{ph}}^3$, belonging to $L^2(\mathbb{R}^3, \mathcal{H}_{\text{ph}}^3)$ and that, the expectation of photon number in this state f is given by:

$$\|N^{1/2}f\|^2 = \int_{\mathbb{R}^3} \|a(k)f\|^2 dk.$$

For our model, if $f \in D(N^{1/2} \otimes I)$ then the expectation of the photon number is:

$$\|(N^{1/2} \otimes I)f\|^2 = \int_{\mathbb{R}^3} \|(a(k) \otimes I)f\|^2 dk.$$

One can localize photons in momentum space in the following way. If E is a Borel set of \mathbb{R}^3 , one usually considers that, in the state $f \in D(N^{1/2} \otimes I)$, the expectation of the number of photons with a momentum belonging to E is given by:

$$\int_E \|(a(k) \otimes I)f\|^2 dk.$$

The localization in position space is less studied but it can be nevertheless analyzed using the Fourier transform.

For all f in $D(N^{1/2} \otimes I)$, since the function $k \rightarrow (a(k) \otimes I)f$ belongs to $L^2(\mathbb{R}^3, \mathcal{H}_{\text{tot}}^3)$, one can define its Fourier transform, which is also in $L^2(\mathbb{R}^3, \mathcal{H}_{\text{tot}}^3)$. We denote this transform by $x \rightarrow (\widehat{a}(x) \otimes I)f$. By abuse of notation, one usually writes:

$$(\widehat{a}(x) \otimes I)f = \int_{\mathbb{R}^3} e^{-ix \cdot k} (a(k) \otimes I)f dk. \quad (1.1)$$

One has:

$$(2\pi)^{-3} \int_{\mathbb{R}^3} \|(\widehat{a}(x) \otimes I)f\|^2 dx = \langle (N \otimes I)f, f \rangle.$$

Since $\langle (N \otimes I)f, f \rangle$ is the average number of photons, one can consider that the average number of photons with a position lying in a Borel set E of \mathbb{R}^3 is given by:

$$\langle N_E f, f \rangle = (2\pi)^{-3} \int_E \|(\widehat{a}(x) \otimes I)f\|^2 dx. \quad (1.2)$$

In this way, the function $x \rightarrow \|(\widehat{a}(x) \otimes I)f\|^2$ can be seen as the density, in position space, of the number of photons in the state $f \in \mathcal{H}_{\text{tot}}$. We note that, the problem of the localization of photons is mentioned in Reuse [32] (page 296-297). See also [26] and [4]. Let us mention that the operator N_E defined in (1.2) is identical to the operator $d\Gamma(1_E(i\nabla_k))$ introduced in [11] for other issues. Our objective is to study this function when f is a state of minimal energy of the Hamiltonian given in [8]. It was shown in [6, 19, 34, 13] that the infimum of the spectrum is an eigenvalue. This result is recalled in Theorem 2.1. The eigenvectors are called ground states. Their properties are studied in, *e.g.*, [23, 5, 24, 10]. It is proved in [13] that, if the constant external field is not vanishing and if the coupling constant is sufficiently small then the subspace of ground states is one dimensional. It is well known (see [10, 21], see also [19] and [6, 13]) that every ground state U_g of the Hamiltonian $H(g)$ belongs to the domain $D(N^m \otimes I)$ for an arbitrary $m \geq 0$. It is also well known that the ground state is not in the photon vacuum. See [13] for estimates of the average number of photons in the state U_g .

Consequently, if the constant external field is not vanishing and if the coupling constant g is small enough, one can study the function $x \rightarrow \|(\widehat{a}(x) \otimes I)U_g\|^2$ when U_g is a normalized ground state, which is the purpose of our main result, namely Theorem 1.1. This function is supposed to describe the localization (in position space) of the photons in a ground state.

The main result is the following.

Theorem 1.1. *Suppose that $\mathbf{B}^{\text{ext}} \neq 0$. Let U_g be a normalized ground state of Theorem 2.1, where the coupling constant g is small enough to ensure that the space of ground states has dimension 1. Then:*

1. *The function $x \rightarrow |x|^{5/2}(\widehat{a}(x) \otimes I)U_g$ is bounded and continuous on \mathbb{R}^3 , with values in $\mathcal{H}_{\text{tot}}^3$.*

2. For all unit vectors v of \mathbb{R}^3 , one has:

$$\lim_{|x| \rightarrow \infty} |x|^{5/2} (\widehat{a}(|x|v) \otimes I) U_g = -\frac{3}{\sqrt{2}} \chi(0) (v \times \mathbf{S}^{[\text{tot}]}) U_g$$

where χ is the "ultraviolet cut-off function", used in (2.1) for the definition of the magnetic field operator, and where $\mathbf{S}^{[\text{tot}]}$ is the expectation of total spin in the ground state U_g , i.e., the vector of \mathbb{R}^3 with coordinates:

$$S_j^{[\text{tot}]} = \sum_{\lambda=1}^P \langle (I \otimes \sigma_j^{[\lambda]}) U_g, U_g \rangle$$

where $\sigma_j^{[\lambda]}$ is defined in (2.3).

In physics, one often supposes that $\chi(0) = 1$. In particular, for the local density of the photons of the ground state, there exists $C > 0$ such that:

$$\|(\widehat{a}(x) \otimes I) U_g\|^2 \leq C(1 + |x|)^{-5}.$$

Since U_g is normalized, one has, for all unit vectors $v \in \mathbb{R}^3$:

$$\lim_{|x| \rightarrow \infty} |x|^5 \|(\widehat{a}(|x|v) \otimes I) U_g\|^2 = (9/2) |\chi(0)|^2 (v \times \mathbf{S}^{[\text{tot}]})^2.$$

One sees that, for large distances, the photons are fewer in the direction of the expectation of the total spin.

Section 2 is devoted to the precise description of the model, Section 3 gives the proof of the main result (Theorem 1.1) and the appendices recall some facts concerning annihilation operators and the Pull Through Formula.

2 The Hamiltonian and its ground state.

The Hilbert space of the states of our system is a completed tensor product $\mathcal{H}_{\text{ph}} \otimes \mathcal{H}_{\text{sp}}$, where \mathcal{H}_{ph} is the Hilbert space of the free photons and \mathcal{H}_{sp} , the space of the particles with spin.

Photons. The one photon configuration Hilbert space \mathcal{H} is the set of mappings $f \in L^2(\mathbb{R}^3, \mathbb{R}^3)$ satisfying $k \cdot f(k) = 0$ almost everywhere in $k \in \mathbb{R}^3$ (see [29]) where $|f|^2 = \int_{\mathbb{R}^3} |f(k)|^2 dk$. Then, $\mathcal{H}_{\mathbf{C}}$ stands for its standard complexification and one denotes by $\langle f, g \rangle$ the scalar product of two elements f and g of $\mathcal{H}_{\mathbf{C}}$, where the mapping $g \rightarrow \langle f, g \rangle$ is antilinear. The Hilbert space \mathcal{H}_{ph} of photon quantum states is the symmetrized Fock space $\mathcal{F}_s(\mathcal{H}_{\mathbf{C}})$ over $\mathcal{H}_{\mathbf{C}}$ the complexified space of \mathcal{H} . We follow [31] for Fock spaces considerations and notations, in particular, for the usual operators in these spaces: the Segal field $\Phi_S(V)$ associated with an element V in \mathcal{H}^2 , the $\Gamma(T)$ and $d\Gamma(T)$ operators associated with some operator T acting in \mathcal{H}^2 . Note that, throughout this paper, the space \mathcal{H}^2 is sometimes identified to the complexified space $\mathcal{H}_{\mathbf{C}}$ but this identification is not everywhere systematically effectuated in order to avoid possible confusions.

Let M_ω be the operator with domain $D(M_\omega) \subset \mathcal{H}$ such that $M_\omega q(k) = |k|q(k)$ almost everywhere in $k \in \mathbb{R}^3$. We denote in the same way the analogous operators defined on \mathcal{H}^2 or on the complexified space

$\mathcal{H}_{\mathbf{C}}$. In the Fock space framework, the photon free energy Hamiltonian operator H_{ph} is usually defined as $H_{\text{ph}} = d\Gamma(M_\omega)$.

The photon number operator denoted by N is $N = d\Gamma(I)$.

The three components of the magnetic field at each point x in \mathbb{R}^3 are defined using the elements B_{jx} belonging to \mathcal{H}^2 and written as follows, when one identifies \mathcal{H}^2 with the complexified space $\mathcal{H}_{\mathbf{C}}$:

$$B_{jx}(k) = \frac{i\chi(|k|)|k|^{\frac{1}{2}}}{(2\pi)^{\frac{3}{2}}} e^{-i(k \cdot x)} \frac{k \times e_j}{|k|}, \quad k \in \mathbb{R}^3 \setminus \{0\} \quad (2.1)$$

where the function χ (ultraviolet cutoff) belongs to $\mathcal{S}(\mathbb{R})$.

One then defines the magnetic fields components operators at each point x of \mathbb{R}^3 by:

$$B_m(x) = \Phi_{\mathcal{S}}(B_{mx}),$$

for $m = 1, 2, 3$.

Spins. The configuration space of the system of P spins is then the space $\mathcal{H}_{\text{sp}} = (\mathbb{C}^2)^{\otimes P}$. The fermion property for the spin- $\frac{1}{2}$ fixed particles is omitted here. In the space \mathcal{H}_{sp} , we use the operators related to the spins of the different particles. Let σ_j ($1 \leq j \leq 3$) be the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.2)$$

For all $\lambda \leq P$ and all $m \leq 3$, we denote by $\sigma_m^{[\lambda]}$ the operator in \mathcal{H}_{sp} defined by:

$$\sigma_m^{[\lambda]} = I \otimes \cdots \otimes I \otimes \sigma_m \otimes I \otimes \cdots \otimes I, \quad (2.3)$$

where σ_m is located at the λ^{th} position.

The Hamiltonian. This Hamiltonian is often used for modeling NMR in quantum field theory (see [30, 27] and Section 4.11 of [32]). It is a selfadjoint extension of the following operator, initially defined in a dense subspace of $\mathcal{H}_{\text{ph}} \otimes \mathcal{H}_{\text{sp}}$:

$$H(g) = H_0 + gH_{\text{int}}, \quad (2.4)$$

where g is a positive constant and:

$$H_0 = H_{\text{ph}} \otimes I + \sum_{\lambda=1}^P \sum_{m=1}^3 B_m^{\text{ext}} \otimes \sigma_m^{[\lambda]} \quad (2.5)$$

where $H_{\text{ph}} = d\Gamma(M_\omega)$ is the photon free energy operator, acting in a domain $D(H_{\text{ph}}) \subset \mathcal{H}_{\text{ph}}$ and $\mathbf{B}^{\text{ext}} = (B_1^{\text{ext}}, B_2^{\text{ext}}, B_3^{\text{ext}}) \neq 0$ is the constant magnetic field. Moreover:

$$H_{\text{int}} = \sum_{\lambda=1}^P \sum_{m=1}^3 B_m(x_\lambda) \otimes \sigma_m^{[\lambda]} \quad (2.6)$$

and the x_λ ($1 \leq \lambda \leq P$) are the points of \mathbb{R}^3 where the static particles are located.

If an element U of H^2 lies in the domain $D(M_\omega^{-1/2})$ then the Segal field $\Phi_S(U)$ is bounded from $D(H_{\text{ph}})$ into \mathcal{H}_{ph} , see point ii) of Proposition 3.4 in [1] or see [9]. This is therefore the case for the operators $B_j(x)$ according to the assumptions on the ultraviolet cutoff function χ in (2.1). Thus, according to the Kato-Rellich Theorem, $H(g)$ has a selfadjoint extension with the same domain as the free operator $H_{\text{ph}} \otimes I$ domain.

The ground state. Let us now recall the results of [19], [34] and [13] about the ground state of this Hamiltonian, or more precisely of Hamiltonians very close to this one.

Theorem 2.1. *The operator $H(g)$ defined by (2.4), (2.5) and (2.6), admits a selfadjoint extension with the same domain as the free operator H_0 . There exists a unit element U_g of $D(H_0)$, such that $H(g)U_g = E(g)U_g$, where $E(g)$ is the infimum of the spectrum of $H(g)$. This element U_g is in the domain of $N^m \otimes I$, for all $m \geq 0$. If g is small enough and if $\mathbf{B}^{\text{ext}} \neq 0$, this element U_g is unique up to a phase factor.*

This theorem follows from [13] (Theorem 1 page 447), and, for the uniqueness, from [34] and [22].

3 Asymptotic behavior at infinity of the photon number.

Theorem 1.1 is proved in this Section. We first note that, according to equality (B.1), the mapping $k \rightarrow (a(k) \otimes I)U_g$ taking values in $\mathcal{H}_{\text{tot}}^3$, belongs to $L^1(\mathbb{R}^3)$, and consequently, its Fourier transform denoted by $x \rightarrow \hat{a}(x)U_g$, also taking values in $\mathcal{H}_{\text{tot}}^3$, is continuous on \mathbb{R}^3 .

The following result is useful in order to prove the other points in Theorem 1.1.

Proposition 3.1. *Let $(H, D(H))$ be a self-adjoint operator in a Hilbert space \mathcal{H} and suppose that the spectrum of the operator H is the half line $[E, \infty)$ (with $E \in \mathbb{R}$). Then,*

$$F(z, H, E) = z(H - E + z)^{-1}$$

is a holomorphic mapping on the half plane $\{z \in \mathbf{C}, \text{Re}z > 0\}$, continuous on the closure of this half plane without the origin, and with a norm smaller than or equal to one. Let (z_n) be a sequence of complex numbers satisfying $\text{Re}z_n \geq 0$ and converging to zero. Then, for any $f \in \mathcal{H}$:

$$\lim_{n \rightarrow \infty} F(z_n, H, E)f = Pf$$

where P is the orthogonal projection on $\ker(H - E)$.

Proof. Set φ the function defined on the spectrum of H by $\varphi(E) = 1$ and $\varphi(x) = 0$ if $x > E$. For any x belonging to the spectrum of H , set:

$$\varphi_n(x) = \frac{z_n}{z_n + x - E}.$$

If $\text{Re}z_n \geq 0$ then $|\varphi_n(x)| \leq 1$ for all x in the spectrum of H . In addition, $\varphi_n(x)$ tends to $\varphi(x)$ for each x in the spectrum of H . According to a standard result (for example, see [7] Lemma 3, Chapter 5, Section 4), $\varphi_n(H)f$ tends to $\varphi(H)f$ for each f . Observe that $\varphi_n(H)f = F(z_n, H, E)f$. It is also well known (see

[7], Theorem 3, Chapter 6, Section 1) that $\varphi(H)$ is the orthogonal projection on $\ker(H - E)$. The proof of the Proposition then follows. □

The mapping $(\widehat{a}(x) \otimes I)U_g$ is given by (1.1) and $(a(k) \otimes I)U_g$ by (B.1) and (2.1) where U_g is the ground state. That is:

$$(\widehat{a}(x) \otimes I)U_g = -\frac{ig}{4\pi^{\frac{3}{2}}} \sum_{\lambda=1}^P \sum_{m=1}^3 \int_{\mathbb{R}^3} e^{-ix \cdot k} \chi(|k|) |k|^{\frac{1}{2}} \frac{k \times e_m}{|k|} (H - E + |k|)^{-1} f_m^{[\lambda]} dk$$

with $f_m^{[\lambda]} = (I \otimes \sigma_m^{[\lambda]})U_g$. In the sequel, $\chi(|k|)$ is approximated by $\chi(0)e^{-|k|}$. Therefore, the following function is under consideration:

$$b(x)U_g = -\frac{ig}{4\pi^{\frac{3}{2}}} \sum_{\lambda=1}^P \sum_{m=1}^3 \int_{\mathbb{R}^3} e^{-ix \cdot k} \chi(0) e^{-|k|} |k|^{\frac{1}{2}} \frac{k \times e_m}{|k|} (H - E + |k|)^{-1} f_m^{[\lambda]} dk. \quad (3.1)$$

Lemma 3.2. *There exists $C > 0$ such that:*

$$\|(\widehat{a}(x) \otimes I)U_g - b(x)U_g\| \leq \frac{Cg}{|x|^3}.$$

Proof of the Lemma. The standard measure on the unit sphere S^2 is denoted by μ . For any $v \in S^2$, $m \leq 3$ and $\lambda > 0$, one has:

$$\int_{S^2} e^{-i\lambda v \cdot \omega} (\omega \times e_m) d\mu(\omega) = 4i\pi (v \times e_m) \left[\frac{\cos \lambda}{\lambda} - \frac{\sin \lambda}{\lambda^2} \right]. \quad (3.2)$$

One then deduces the existence of $C > 0$ satisfying:

$$\|(\widehat{a}(x) \otimes I)U - b(x)U\| \leq C \sum_{m=1}^3 |I_m(x)|$$

with, if $x = |x|v$, $|v| = 1$:

$$I_m(x) = \int_0^\infty \left[\frac{\cos |x|\rho}{|x|\rho} - \frac{\sin |x|\rho}{(|x|\rho)^2} \right] \Phi_m(\rho) d\rho$$

where:

$$\Phi_m(\rho) = \rho^{5/2} (\chi(\rho) - \chi(0)e^{-\rho}) (H - E + \rho)^{-1} f_m.$$

One checks that:

$$\int_0^\infty \frac{\sin(|x|\rho)}{|x|^2 \rho^2} (\Phi_m(\rho)) d\rho = \frac{1}{|x|^3} \int_0^\infty \cos(|x|\rho) \frac{d}{d\rho} \left[\frac{\Phi_m(\rho)}{\rho^2} \right] d\rho$$

and:

$$\int_0^\infty \frac{\cos(|x|\rho)}{|x|\rho} (\Phi_m(\rho)) d\rho = \frac{1}{|x|^3} \int_0^\infty \cos(|x|\rho) \frac{d^2}{d\rho^2} \left[\frac{\Phi_m(\rho)}{\rho} \right] d\rho.$$

These integrations by parts holds true since as $\rho \rightarrow 0$:

$$\left| \frac{d}{d\rho} \left[\frac{\Phi_m(\rho)}{\rho^2} \right] \right| + \left| \frac{d^2}{d\rho^2} \left[\frac{\Phi_m(\rho)}{\rho} \right] \right| \leq \frac{C}{\rho^{1/2}}.$$

Moreover, the left hand side is rapidly decreasing as $\rho \rightarrow \infty$. The proof is thus completed. \square

Proof of Theorem 1.1. It is sufficient to consider $b(x)U_g$. For each $v \in S^2$, one sees using (3.1) and (3.2):

$$b(|x|v)U_g = \chi(0)\pi^{-1/2} \sum_{\lambda=1}^P \sum_{m=1}^3 v \times e_m I_{m,\text{aux}}^{[\lambda]}(|x|)$$

with:

$$I_{m,\text{aux}}^{[\lambda]}(|x|) = \int_0^\infty \left[\frac{\cos|x|\rho}{|x|\rho} - \frac{\sin|x|\rho}{(|x|\rho)^2} \right] \rho^{5/2} e^{-\rho(H-E+\rho)^{-1}} f_m^{[\lambda]} d\rho.$$

One has:

$$I_{m,\text{aux}}^{[\lambda]}(|x|) = \frac{2}{|x|^{5/2}} \int_0^\infty (t^2 \cos(t^2) - \sin(t^2)) e^{-t^2/|x|} F\left(\frac{t^2}{|x|}, H, E\right) f_m^{[\lambda]} dt \quad (3.3)$$

where $F(s, H, E) = s(H - E + s)^{-1}$.

We can write $I_{m,\text{aux}}^{[\lambda]} = I_{m,\text{aux}}^{\lambda+} + I_{m,\text{aux}}^{\lambda-}$ setting:

$$I_{m,\text{aux}}^{\lambda\varepsilon} = \frac{1}{|x|^{5/2}} \int_0^\infty e^{\varepsilon it^2} (t^2 + \varepsilon i) e^{-t^2/|x|} F\left(\frac{t^2}{|x|}, H, E\right) f_m^{[\lambda]} dt \quad (3.4)$$

with $\varepsilon = \pm$. If $|x| \geq 1$, a change of contour of integration shows that:

$$I_{m,\text{aux}}^{\lambda\varepsilon} = \frac{1}{|x|^{5/2}} (\varepsilon i e^{\varepsilon i\pi/4}) \int_0^\infty (r^2 + 1) e^{-r^2} e^{-\varepsilon i r^2/|x|} F\left(\frac{\varepsilon i r^2}{|x|}, H, E\right) f_m^{[\lambda]} dr. \quad (3.5)$$

It is therefore deduced that:

$$|x|^{5/2} I_{m,\text{aux}}^{[\lambda]}(|x|) \leq 2|f_m^{[\lambda]}| \int_0^\infty (r^2 + 1) e^{-r^2} dr. \quad (3.6)$$

Consequently, the function $x \rightarrow |x|^{5/2}|b(x)|$ is bounded on $\{|x| \geq 1\}$ proving point i) of the Theorem when using Lemma 3.2.

Using Lebesgue's dominated convergence Theorem, point ii) comes from:

$$\lim_{|x| \rightarrow \infty} |x|^{5/2} I_{m,\text{aux}}^{\lambda\varepsilon}(|x|) = (\varepsilon i e^{\varepsilon i\pi/4}) \lim_{z \rightarrow 0, \text{Re} z \geq 0} F(z, H, E) f_m^{[\lambda]} \int_0^\infty (r^2 + 1) e^{-r^2} dr.$$

The existence of the above limit comes from Proposition 3.1 which also shows:

$$\lim_{z \rightarrow 0, \text{Re} z \geq 0} F(z, H, E) f_m^{[\lambda]} = P f_m = \langle f_m^{[\lambda]}, U_g \rangle U_g$$

where P is the projection on the eigenspace of the infimum of the spectrum of $H(g)$. Therefore:

$$\begin{aligned} \lim_{|x| \rightarrow \infty} |x|^{5/2} b(|x|v)U_g &= -\sqrt{2/\pi} \chi(0) \sum_{\lambda=1}^P \sum_{m=1}^3 v \times e_m \langle (I \otimes \sigma_m^{[\lambda]}) U_g, U_g \rangle U_g \int_0^\infty (r^2 + 1) e^{-r^2} dr \\ &= -\sqrt{2/\pi} \chi(0) (v \times \mathbf{S}^{[\text{tot}]}) U_g \int_0^\infty (r^2 + 1) e^{-r^2} dr \\ &= -\frac{3}{\sqrt{2}} \chi(0) (v \times \mathbf{S})^{[\text{tot}]} U_g \end{aligned}$$

proving point ii) with the help of Lemma 3.2.

□

Appendices

A Standard facts on annihilation operators.

The following results are classical but adapted to the Hilbert space \mathcal{H} in Section 2. We denote by $\mathcal{F}_s^{\text{reg}}(\mathcal{H}_{\mathbf{C}})$ the subspace of $\mathcal{F}_s(\mathcal{H}_{\mathbf{C}})$ constituted by the finite linear combinations of the symmetrized products of $g_1 \otimes \cdots \otimes g_m$ where the $g_j \in \mathcal{S}(\mathbb{R}^3, \mathbb{R}^3)$ satisfy $k \cdot g_j(k) = 0$ for all $k \in \mathbb{R}^3$.

For all $k \in \mathbb{R}^3$ and $f \in \mathcal{F}_s^{\text{reg}}(\mathcal{H}_{\mathbf{C}})$, $a(k)f$ is classically defined as follows. For any (non symmetrized) product $g = g_1 \otimes \cdots \otimes g_m$, with the g_j satisfying the above conditions, one set:

$$a(k)g = \sqrt{m}g_1(k)g_2 \otimes \cdots \otimes g_m.$$

This definition is next adapted to the symmetrized space. Since the function g_1 takes values in \mathbb{R}^3 , one sees that $k \rightarrow a(k)f$ takes values in $(\mathcal{F}_s(\mathcal{H}_{\mathbf{C}}))^3$ and belongs to $\mathcal{S}(\mathbb{R}^3)$. One also notes:

$$\langle Nf, f \rangle = \int_{\mathbb{R}^3} \|a(k)f\|^2 dk.$$

The next two propositions are useful for the purpose of an extension by density in $D(N^{1/2})$.

Proposition A.1. $\mathcal{S}(\mathbb{R}^3, \mathbb{R}^3) \cap \mathcal{H}$ is dense in \mathcal{H} .

Proof. Set $f \in \mathcal{H}$ and ρ a C^∞ smooth cut-off function defined on \mathbb{R}^+ , vanishing on $[0, 1]$ and equal to one on $[2, +\infty)$. One has:

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} |f(k) - f(k)\rho(n|k|^2)|^2 dk = 0$$

using Lebesgue Theorem. The above norm $||$ is the one in \mathbb{R}^3 . Thus, the set \mathcal{H}_0 of functions in \mathcal{H} vanishing on a ball centered at the origin is dense in \mathcal{H} . Take $f \in \mathcal{H}_0$ vanishing on a ball centered at 0 with radius $\eta > 0$. There exist sequences $f_i^{(n)}$ of $\mathcal{S}(\mathbb{R}^3, \mathbb{R})$ converging to f_i , for $i = 1, 2, 3$ (n is the coordinate index for these sequences). It is naturally not clear that:

$$\sum_1^3 k_i f_i^{(n)}(k) = 0$$

and therefore that $f^{(n)}$ (whose components are the $f_i^{(n)}$) belongs to \mathcal{H} . We then set, again with a cut-off function:

$$g^{(n)} = (f^{(n)} - \frac{k}{|k|^2} f^{(n)} \cdot k) \rho(\frac{4}{\eta^2} |k|^2). \quad (\text{A.1})$$

One checks that $g^{(n)} \in \mathcal{H} \cap \mathcal{S}(\mathbb{R}^3, \mathbb{R}^3)$ and tends to f in \mathcal{H} , i.e., with the $L^2(\mathbb{R}^3, \mathbb{R}^3)$ norm.

□

Proposition A.2. $\mathcal{F}_s^{\text{reg}}(\mathcal{H}_{\mathbf{C}})$ is dense in $D(N^{1/2})$.

Proof. One chooses a basis (u_i) of \mathcal{H} in $\mathcal{H} \cap \mathcal{S}(\mathbb{R}^3, \mathbb{R}^3)$ and sets:

$$\zeta_\alpha = \sqrt{\frac{|\alpha|!}{\alpha!}} S_n \left(\otimes_i u_i^{\otimes \alpha_i} \right), \quad n = |\alpha|. \quad (\text{A.2})$$

The set of ζ_α (with $|\alpha| = n$) is an orthonormal basis of $\otimes_s^n \mathcal{H}$ (see Janson [25]). Thus, the set of finite linear combinations of ζ_α (with arbitrary $|\alpha|$) is dense in the set of finite number particles $\mathcal{F}^{\text{fin}}(\mathcal{H})$ which is dense in $D(N^{1/2})$. The proof of the Proposition is then complete since the set of finite linear combinations of ζ_α is included in $\mathcal{F}_s^{\text{reg}}(\mathcal{H}_{\mathbf{C}})$.

□

We can now extend the definition of $a(k)$ to $D(N^{1/2})$.

Proposition A.3. Set $f \in D(N^{1/2})$ and a sequence (f_n) in $\mathcal{F}_s^{\text{reg}}(\mathcal{H}_{\mathbf{C}})$ converging to f in $D(N^{1/2})$. Then, the $k \rightarrow a(k)f_n$ has a limit in $L^2(\mathbb{R}^3, (\mathcal{F}_s(\mathcal{H}_{\mathbf{C}}))^3)$. This limit is denoted (definition) by $a(k)f$.

Indeed, according to the previous points, one has if $m < n$:

$$\int_{\mathbb{R}^3} \|a(k)(f_m - f_n)\|^2 dk \leq (\|N^{1/2}f_m\| + \|N^{1/2}f_n\|) \|N^{1/2}(f_m - f_n)\|$$

and one also has ($\varphi \in D(N^{1/2})$):

$$\|N^{1/2}\varphi\|^2 = \int_{\mathbb{R}^3} \|a(k)\varphi\|^2 dk. \quad (\text{A.3})$$

B Pull Through Formula

Let U_g be a normalized ground state given by Theorem 2.1. It is recalled in Theorem 2.1 that U_g belongs to the domain $N^m \otimes I$ for all integers m . According to Proposition A.3, the function $k \rightarrow (a(k) \otimes I)U_g$ is well defined as an element of $L^2(\mathbb{R}^3, \mathcal{H}_{\text{tot}}^3)$. In particular, this function is defined almost everywhere. We give in the next result an explicit expression of this function, recalling the Pull through formula and its proof (see [13] and see also [33, 16, 12, 19, 17, 6]). The uniqueness of the ground state is not useful.

Theorem B.1. Let U_g be a (normalized) ground state of Theorem 2.1. Then, for almost every $k \in \mathbb{R}^3 \setminus \{0\}$:

$$(a(k) \otimes I)U_g = -(g/\sqrt{2}) \sum_{\lambda=1}^P \sum_{m=1}^3 B_{m,x_\lambda}(k) (H(g) - E(g) + |k|)^{-1} (I \otimes \sigma_m^{[\lambda]}) U_g \quad (\text{B.1})$$

where $B_{m,x_\lambda}(k)$ is defined in (2.1). Consequently:

$$\|(a(k) \otimes I)U_g\| \leq (g/\sqrt{2}) \sum_{\lambda=1}^P \sum_{m=1}^3 \frac{|B_{m,x_\lambda}(k)|}{|k|} \|(I \otimes \sigma_m^{[\lambda]}) U_g\|. \quad (\text{B.2})$$

Proof. In view of Lemma 2.5 in [9]:

$$[d\Gamma(M_\omega), a(k)] = -|k|a(k). \quad (\text{B.3})$$

Thus, from (2.4) and (2.5), for almost every k in \mathbb{R}^3 :

$$(H(g) - E(g) + |k|)(a(k) \otimes I)U_g = (a(k) \otimes I)(H(g) - E(g))U_g + g[H_{\text{int}}, (a(k) \otimes I)]U_g.$$

One has:

$$[H_{\text{int}}, (a(k) \otimes I)] = \sum_{\lambda=1}^P \sum_{m=1}^3 [B_m(x_\lambda), a(k)] \otimes \sigma_m^{[\lambda]}.$$

Besides:

$$\sqrt{2}[B_m(x_\lambda), a(k)] = [a^*(B_{m,x_\lambda}), a(k)] = -B_{m,x_\lambda}(k). \quad (\text{B.4})$$

Consequently:

$$[H_{\text{int}}, (a(k) \otimes I)] = -(1/\sqrt{2}) \sum_{\lambda=1}^P \sum_{m=1}^3 B_{m,x_\lambda}(k)(I \otimes \sigma_m^{[\lambda]}).$$

For all $k \neq 0$, the operator $(H(g) - E(g) + |k|)$ is invertible. One then deduces equality (B.1). \square

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