Uniform bound for the volume of the solutions to Liouville type equations on the annulus

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Abstract

We consider variational problems with regular Hölderian weight or with weight and boundary singularity and, Dirichlet condition. We prove the boundedness of the volume of the solutions to these equations on the annulus.

Keywords: Regular Hölderian weight, boundary singularity, a priori estimate, annulus, Lipschitz condition.
MSC: 35J60, 35B45.

1 Introduction and Main Results

We set $\Delta = \partial_{11} + \partial_{22}$ on the annulus $\Omega = C(1, 1/2, 0)$ of $\mathbb{R}^2$ of radii 1 and $1/2$ centered at the origin.

We consider the following equation:

$$(P)\begin{cases} -\Delta u = (1 + |x - x_0|^{2\beta})Ve^u & \text{in } \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{in } \partial\Omega. \end{cases}$$

Here: $C(1)$ the unit circle and $C(1/2)$ the circle of radius $1/2$ centered at the origin.

$$\beta \in (0, 1/2),\ x_0 \in C(1/2),$$

and,

$$u \in W_0^{1,1}(\Omega),\ e^u \in L^1(\Omega) \text{ and } 0 < a \leq V \leq b.$$

This is an equation with regular Hölderian weight not Lipschitz in x_0 but have a weak derivative.

We have in [10] a nice formulation of this problem (P) in the sense of the distributions. This Problem arises in various geometrical and physical situations, see for example [1, 5, 23, 27]. The above equation was studied by many authors, with and without boundary conditions, also for Riemannian surfaces, see [1-27], where one can find some existence and compactness results. In [9] we have the following important Theorem,

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Theorem A *(Brezis-Merle [9]).For u and V two functions relative to (P) with,*

$$0 < a \leq V \leq b < +\infty$$

then it holds, for all compact set K of Ω:

$$\sup_K u \leq c,$$

with c depending on a, b, β, x_0, K and Ω.

We deduce from Theorem A and from the elliptic estimates that, u is uniformly bounded in $C^2_{loc}(\Omega)$.

In this paper we try to prove that we have on all Ω the boundedness of the volume of the solutions of (P) if we add the assumption that V is uniformly Lipschitz with particular Lispchitz number.

Here we have:

Theorem 1.1 *Assume that u is a solution of (P) relative to V with the following conditions:*

$$x_0 \in C(1/2) \subset \partial\Omega, \ \beta \in (0, 1/2),$$

and,

$$0 < a \leq V \leq b, \ ||\nabla V||_{L^\infty} \leq A = \frac{a}{2(1 + 2^{2\beta})},$$

we have,

$$\int_\Omega e^u \leq c(a, b, \beta, x_0, \Omega).$$

A consequence of this theorem is a compactness result of the solutions to this Liouville type equation, see [4].

We have the same result if we consider Liouville equation with boundary singularity on the annulus $\Omega = C(1, 1/2, 0)$ for particular Lipschitz number of the potential V:

We consider the following equation in the sense of distributions:

$$(P_\beta) \begin{cases} -\Delta u = |x - x_0|^{2\beta} V e^u & \text{in } \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{in } \partial\Omega. \end{cases}$$

Here: $C(1)$ the unit circle and $C(1/2)$ the circle of radius $1/2$ centered at the origin.

$$\beta \in (-1/2, +\infty), \ x_0 \in C(1/2),$$

and,

$$u \in W^{1,1}_0(\Omega), \ |x - x_0|^{2\beta} e^u \in L^1(\Omega) \text{ and } 0 < a \leq V \leq b.$$

This is an equation with boundary singularity and Dirichlet condition on the annulus.

Here we have:

Theorem 1.2 *Assume that u is a solution of (P_β) relative to V with the following conditions:*

$$x_0 \in C(1/2) \subset \partial\Omega,\ \beta \in (-1/2, +\infty),$$

and,

$$0 < a \leq V \leq b,\ ||\nabla V||_{L^\infty} \leq A = \frac{(\beta+1)a}{2},$$

we have,

$$\int_\Omega |x - x_0|^{2\beta} e^u \leq c(a, b, \beta, x_0, \Omega).$$

For the previous theorem, the condition $\beta > -1/2$ is in relation with the regularity of the solutions to apply the Pohozaev identity.

A consequence of the previous theorem is a compactness result for the solutions to a Liouville type equation with boundary singularity on the annulus, see [3].

2 Proof of the Theorems:

Proof of the theorems:

By corollary 1 of the paper of Brezis-Merle, we have: $e^{ku} \in L^1(\Omega)$ for all $k > 2$ and the elliptic estimates and the Sobolev embedding imply that: $u \in W^{2,k}(\Omega) \cap C^{1,\epsilon}(\bar{\Omega}), \epsilon > 0$. By the maximum principle $u \geq 0$.

Step 1: By using the first eigenvalue and the first eigenfunction, with Dirichlet boundary condition, the volume is locally uniformly bounded, and thus the solutions are locally uniformly bounded by Brezis-Merle result. The solutions $u > 0$ are locally uniformly bounded in $C^{1,\epsilon}(\Omega)$ for ϵ small.

Step 2: Let's consider $C_1 = C(1, 3/4, 0)$ and $C_2 = C(3/4, 1/2, 0)$ the two annulus wich are the neighborhood of the two components of the boundary.

We multiply the equation by $(x-x_0)\cdot\nabla u$ on C_1 and C_2 and use the Pohozaev identity. We use the fact that u is uniformly bounded around the circle $C(3/4)$. We obtain:

1) We have on C_1:

$$\int_{C_1} (\Delta u)[(x - x_0) \cdot \nabla u]dx = \int_{C_1} -[(1 + |x - x_0|^{2\beta})V(x - x_0) \cdot \nabla(e^u)]dx,$$

Thus,

$$\int_{\partial C_1} [(x - x_0) \cdot \nabla u](\nabla u \cdot \nu) - \frac{1}{2}[(x - x_0) \cdot \nu]|\nabla u|^2 =$$

$$= \int_{C_1} (2 + 2(\beta+1)|x - x_0|^{2\beta})Ve^u dx + \int_{C_1} (1 + |x - x_0|^{2\beta})(x - x_0) \cdot \nabla V e^u dx +$$

$$- \int_{\partial C_1} (1 + |x - x_0|^{2\beta})[(x - x_0) \cdot \nu]Ve^u d\sigma$$

We can write, ($u = 0$ on $C(1)$):

$$\int_{C(1)} \frac{1}{2}[(x - x_0) \cdot \nu](\partial_\nu u)^2 d\sigma + O(1) =$$

$$= \int_{C_1} (2+2(\beta+1)|x-x_0|^{2\beta}) V e^u dx + \int_{C_1} (1+|x-x_0|^{2\beta})(x-x_0) \cdot \nabla V e^u dx + O(1) =$$

$$= C \int_{C(1)} \partial_\nu u d\sigma + O(1),$$

with $C > 0$ not depends on u.

Be cause $\nu = x, ||x|| = 1, ||x_0|| = 1/2$ and then by Cauchy-Schwarz, $(x - x_0) \cdot x = ||x||^2 - x_0 \cdot x \geq 1/2$, we obtain:

$$\int_{C(1)} (\partial_\nu u)^2 d\sigma \leq 4C \int_{C(1)} \partial_\nu u d\sigma + O(1),$$

and, by the Cauchy-Schwarz inequality applied to the integral of the right hand side we obtain:

$$\int_{C(1)} (\partial_\nu u)^2 d\sigma = O(1), \text{ and } \int_{C(1)} (\partial_\nu u) d\sigma = O(1).$$

2) We have on C_2, we use again, the uniform boundedness of u in C^1 norm around $C(3/4)$:

$$\int_{C_2} (\Delta u)[(x - x_0) \cdot \nabla u] dx = \int_{C_2} -[(1 + |x - x_0|^{2\beta}) V (x - x_0) \cdot \nabla (e^u)] dx,$$

Thus,

$$\int_{\partial C_2} [(x - x_0) \cdot \nabla u](\nabla u \cdot \nu) - \frac{1}{2}[(x - x_0) \cdot \nu]|\nabla u|^2 =$$

$$= \int_{C_2} (2 + 2(\beta + 1)|x - x_0|^{2\beta}) V e^u dx + \int_{C_2} (1 + |x - x_0|^{2\beta})(x - x_0) \cdot \nabla V e^u dx +$$

$$- \int_{\partial C_2} (1 + |x - x_0|^{2\beta})[(x - x_0) \cdot \nu] V e^u d\sigma$$

We can write,($u = 0$ on $C(1/2)$):

$$\int_{C(1/2)} \frac{1}{2}[(x - x_0) \cdot \nu](\partial_\nu u)^2 d\sigma + O(1) =$$

$$= \int_{C_2} (2+2(\beta+1)|x-x_0|^{2\beta}) V e^u dx + \int_{C_2} (1+|x-x_0|^{2\beta})(x-x_0) \cdot \nabla V e^u dx + O(1),$$

But here, $\nu = -2x$ and $(x - x_0) \cdot \nu = -(x - x_0) \cdot x \leq 0$ and thus:

$$\int_{C_2}(2+2(\beta+1)|x-x_0|^{2\beta})Ve^u dx+\int_{C_2}(1+|x-x_0|^{2\beta})(x-x_0)\cdot\nabla Ve^u dx+$$

$$+\int_{C(1/2)}\frac{1}{2}[-(x-x_0)\cdot\nu](\partial_\nu u)^2 d\sigma = O(1),$$

If we choose:

$$\frac{|(x-x_0)\cdot\nabla V|}{V}\leq\frac{1}{2}\inf_{x\in\bar{\Omega}}\frac{(2+2(\beta+1)|x-x_0|^{2\beta})}{1+|x-x_0|^{2\beta}}, \tag{1}$$

We obtain:

$$\int_{C_2}(2+2(\beta+1)|x-x_0|^{2\beta})Ve^u dx+\int_{C_2}(1+|x-x_0|^{2\beta})(x-x_0)\cdot\nabla Ve^u dx\geq$$

$$\geq\frac{1}{2}\int_{C_2}(2+2(\beta+1)|x-x_0|^{2\beta})Ve^u dx\geq 0$$

thus,

$$\int_{C_2}(2+2(\beta+1)|x-x_0|^{2\beta})Ve^u dx = O(1),$$

(This condition is true for the second theorem for $x\in\Omega$, we have: $\frac{|(x-x_0)\cdot\nabla V|}{V}\leq\frac{1}{2}\inf_{x\in\Omega}\frac{2(\beta+1)|x-x_0|^{2\beta}}{|x-x_0|^{2\beta}}=\beta+1$, this is true if $A\leq\frac{a(\beta+1)}{2}$.)
and thus,

$$\int_{C(1/2)}\partial_\nu u d\sigma = O(1),$$

Thus, if: $2\frac{diam(\Omega)||\nabla V||_\infty}{a}\leq\frac{2}{1+[diam(\Omega)]^{2\beta}}\Rightarrow$ if $2A\leq\frac{2a}{2(1+[diam(\Omega)]^{2\beta})}$, we obtain a uniform bound for $\int_{C(1/2)}\partial_\nu u d\sigma$. But $diam(\Omega)=diameter(\Omega)=2$, because it is the annulus.

If we use 1) and 2), we obtain: if $A\leq\frac{a}{2(1+2^{2\beta})}$ for the first theorem, then:

$$\int_{C(1)}(\partial_\nu u)d\sigma = O(1),\text{ and }\int_{C(1/2)}\partial_\nu u d\sigma = O(1),$$

and, thus:

$$\int_\Omega(1+|x-x_0|^{2\beta})Ve^u dx=\int_{\partial\Omega}\partial_\nu u d\sigma = O(1).$$

We have the same conclusion for the second theorem if $A=\frac{a(\beta+1)}{2}$.

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