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# Constrained Optimal Control Problem Applied to Vaccination for COVID-19 Epidemic

Chiraz Trabelsi <sup>1 2</sup>; Rim Amami <sup>3</sup>; Walid Ben Aribi <sup>4</sup>; Hani Abidi <sup>5</sup>

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## Abstract

COVID-19 remains a major threat to the world since its emergence in December 2019, especially the lack of identification of a specific treatment, as scientific researchers continue to seek a better understanding of the epidemiological cycle and dynamics of the virus. In this work, we propose a dynamic mathematical model framework governed by a system of differential equations that integrates COVID-19 outbreaks, which is an extension of the standard SEAIR model. An optimal control problem is formulated with the aim of minimizing the number of infected individuals while considering intervention costs and the constraints of the total and maximum daily vaccine administration. We use the penalty method to approximate this constrained optimization problem and derive an optimality system that characterizes the optimal control. Finally, we carry out some numerical simulations.

**Keywords:** Covid-19 Epidemic, SEAIR, ordinary differential equations, Optimal control problem, penalty method.

**MSC Classification:** 34H05, 49J15.

## 1 Introduction

Appearing in 2019 in Wuhan, COVID-19 very quickly turned out to be a serious health problem around the world, with catastrophic consequences for the evolution of humankind. Our modern world has never faced a disease of this magnitude. All statistics data about coronavirus COVID-19 comes from the World Health Organization, Johns Hopkins CSSE, and Worldometers. Charts include the number of infected, deaths, and recovered people.

There is now no medicine or specific treatment for COVID-19, and most countries have been relying on non-pharmaceutical interventions, such as wearing face masks, washing hands taking care of personal hygiene, physical distancing, rapid-test and even more partial or total lockdown in order to curtail the spread of the disease. Therefore the study of the novel coronavirus has relatively attracted some importance in mathematical epidemiology due to its seriousness and the way its spreads worldwide.

For instance, several models have been proposed to provide insight into the effect that inoculation of a certain portion of the population will have on the dynamics of the COVID-19 pandemic.

The authors in [15] apply optimal control theory to determine optimal strategies for the implementation of non-pharmaceutical interventions to control COVID-19. The studied model was calibrated to data from the USA and focused the analysis on optimal controls from May 2020 through December 2021.

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The importance of the isolation strategy was emphasized in [3] to reduce the infection of COVID-19. The authors prove the existence and uniqueness of a global positive solution for a COVID-19 stochastic model with an isolation strategy. Some numerical simulations are presented to illustrate the theoretical results.

Many mathematical models conclude that lockdown is the best way to reduce the spread of COVID-19 effectively among all the aforementioned control strategies [12]. However, partial or total lockdown strategies are very risky for a country's economic stability, therefore some suggested mathematical models focused on analyzing the effect of COVID-19 rapid-test as an alternative to suppress the spread of COVID-19 [1, 18].

However, these measures have been not an effective protection to mitigate the pandemic globally. To bring this pandemic to an end, a large share of the world needs to be immune to the virus. The safest way to achieve this is with a vaccine. Within less than 12 months after the beginning of the COVID-19 pandemic, several research teams rose to the challenge and developed vaccines that protect from SARS-CoV-2.

Among others [5] the optimal control theory is applied to suggest the most effective mitigation strategy to minimize the number of individuals who become infected in the course of an infection while efficiently balancing vaccination and treatment applied to the models. The SIR model analysis is presented here including the proof of uniqueness and existence of the optimal control solutions.

The motivation of this study is derived from the work [4], who adopt a more modelling approach based on optimal control theory to determine the best strategy to implement until vaccine deployment. Therefore, in the present study, we incorporate the vaccination component to the model in [4], to derive an extended SEAIR model to examine the effectiveness of the COVID-19 jabs which are currently being deployed to many countries to help combat the raging pandemic situation. In the same area, the authors in [14] propose and analyze an extended SEIARD model with vaccination to examine the effectiveness of the COVID-19 jabs which are currently being deployed to many countries to help combat the raging pandemic situation. Some numerical simulations are achieved using reported data on COVID-19 infections and vaccination in Mexico.

This paper is composed of four sections. We first present in Section 2, the structure of our epidemiological model, precisely we present the equations and assumptions of the extended SEAIR model with vaccination. In Section 3 we introduce the objective function and a reformulation of the optimal control problem to minimize the incidence satisfying the constraints of the total and maximum daily vaccine administration and we derive the optimality condition. We use the penalty method to approximate this constrained optimization problem and derive an optimality system that characterizes the optimal control. In Section 4, we carry out numerical simulations using reported data on COVID-19 infections and vaccination in France, and conclude with a summary.

## 2 Model formulation

The model describes the epidemic dynamics of COVID-19 in a population after vaccine deployment and is an extension of the standard SEAIR model incorporating a temporary protection vaccine compartment to the model [4], regardless of the disease severity whether mild or severe infections. To derive the mathematical model, first, we subdivide our population into two sub-populations, the unvaccinated population and the vaccinated population (individuals who have received one, two, or three doses of the vaccine). The model classifies individuals into principle key compartments of:

- Susceptible (S).
- Latent (E): infected but asymptomatic and not infectious.
- Asymptomatic infectious (A).

- Symptomatic infectious (I).
- Immune (R) or Death (D).
- Temporary protection vaccine (V).

Now, we introduce the model 1 and describe its parameters. Figure 1 illustrates the different compartments for the extended SEAIR model with vaccination.

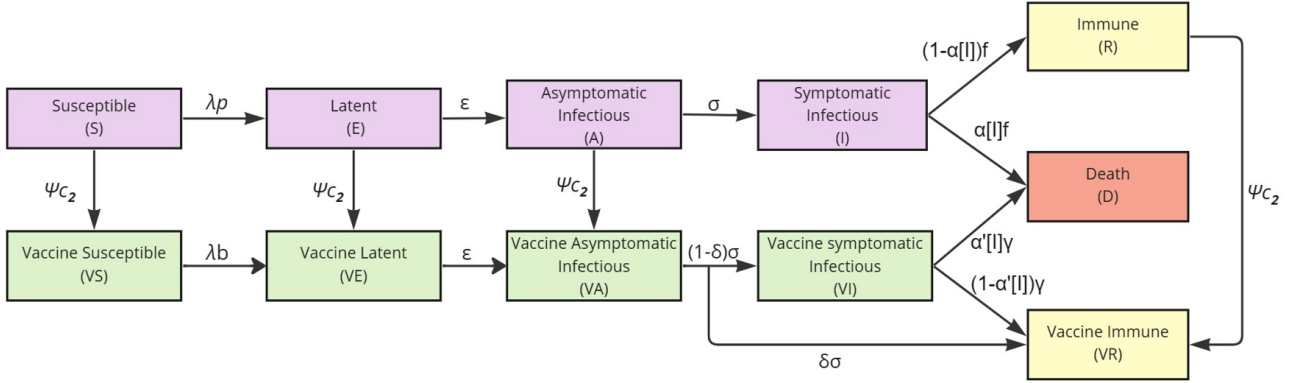


Figure 1: flow diagram for the SEAIR model with vaccination compartments

We will denote by  $N(t)$  the total population size at time  $t$  which is given by

$$\begin{aligned} N(t) &= S(t) + E(t) + A(t) + I(t) + R(t) + VS(t) + VE(t) + VA(t) + VI(t) + VR(t) \\ &= S(t) + E(t) + A(t) + I(t) + R(t) + V(t) \end{aligned}$$

We denote by  $c_1$  the control effort which represents the percentage of reduction in transmission due to public health measures at time  $t$ , and  $\beta_A, \beta_I$  are asymptomatic and symptomatic transmission rates respectively. Susceptible individuals become exposed by contact with asymptomatic infectious individuals at a rate  $\beta_A$  and by contact with symptomatic infectious individuals at a rate  $\beta_I$ . The force of infection is defined by  $\lambda = (1 - c_1(t))(\beta_A A + \beta_I I)$  [4]. A proportion  $p$  of exposed individuals ( $\lambda S$ ) move to the asymptomatic infection class at a rate  $\epsilon$ . We assume that asymptomatic infectious individuals become symptomatic infectious at rate  $\sigma$ .

Infective individuals leave the compartment at the rate  $(1 - \alpha[I])$  with the fraction  $f$  recovering from disease, whereas the rest dying of infection, we assume the disease-induced mortality rate  $\alpha$  to be a step function as follows:

$$\alpha[I] = \begin{cases} \alpha_{min} & \text{if } I < I^* \\ \alpha_{max} & \text{if } I \geq I^* \end{cases}$$

and  $I^*$  is the total number of infected hosts in the healthcare system, or simply the healthcare capacity and  $\alpha$  bound disease-induced mortality rate.

Since an antibody test is not required or recommended before getting the COVID-19 vaccination, we assume the time-dependent control function  $c_2(t)$  measures the rate at which Susceptible, Latent, Asymptomatic infectious and recovered individuals are vaccinated with vaccine efficacy  $\psi$ . As the vaccine does not completely remove the infection, we also assume that a proportion  $b$ ,  $b \ll p$ , of exposed vaccine individuals ( $\lambda VS$ ) move to asymptomatic infection at the same rate  $\epsilon$ .

A proportion  $\delta$  of vaccine asymptomatic infectious individuals become recovered at the rate  $\sigma$ , while the remainder  $(1 - \delta)$  move to vaccine Symptomatic infection class at the same rate  $\sigma$  and recovered at rate  $\delta(1 - \alpha'[I])$  with  $\alpha' \ll \alpha$ , denotes the disease-induced mortality for vaccinated individuals, whereas the rest dying of infection.

For the whole population level, we assume also that natural mortality increases because of hospital saturation. We capture this using the following step function for the mortality rate  $\mu$ ,

$$\mu[I] = \begin{cases} 0 & \text{if } I < I^* \\ \mu & \text{if } I \geq I^* \end{cases}$$

with  $\mu$  the natural mortality rate with hospital saturation.

The principal parameters used throughout this paper and their interpretation are as follows:

- $\mu$  : natural mortality rate with hospital saturation.
- $\alpha, \alpha'$  : bounds disease induced mortality rate with  $\alpha' \ll \alpha$ .
- $c_1 \in [0, 1]$  control effort: the percentage of reduction in transmission due to public health measure at time  $t$ .
- $c_2 \in [0, 1]$  control effort. The time-dependent control function  $c_2(t)$  measures the rate at which susceptible individuals are vaccinated.
- force infection:  $\lambda = (1 - c_1(t))(\beta_A A + \beta_I I)$  where  $\begin{cases} \beta_I = & \text{symptomatic transmission rate} \\ \beta_A = & \text{asymptomatic transmission rate} \end{cases}$
- $\psi$  : susceptible, latent, asymptomatic, and recovered individuals are vaccinated with vaccine efficacy  $\psi$ .
- $p$  : proportion of infections unvaccinated individuals.
- $b$  : proportion of infections vaccinated individuals (It can be considered too as the loss of vaccine protection).
- $\epsilon$  : waiting rate to viral shedding.
- $\sigma$  : waiting rate to symptom onset.
- $f$  : recovery rate from infections unvaccinated individuals.
- $\gamma$  : recovery rate from infections vaccinated individuals.
- $\delta$  : proportion of recovered among vaccine asymptomatic infectious individuals.

Hence, our model 1 is described by the following system of ODEs:

$$\begin{aligned} (1) \quad S'(t) &= -\lambda p S(t) - \mu[I] S(t) - \psi c_2(t) S(t) \\ (2) \quad E'(t) &= \lambda p S(t) - (\epsilon + \mu[I]) E(t) - \psi c_2(t) E(t) \\ (3) \quad A'(t) &= \epsilon E(t) - (\sigma + \mu[I]) A(t) - \psi c_2(t) A(t) \\ (4) \quad I'(t) &= \sigma A(t) - (f + \mu[I]) I(t) \\ (5) \quad R'(t) &= (1 - \alpha[I]) f I(t) - \mu[I] R(t) - \psi c_2(t) R(t) \\ (6) \quad VS'(t) &= \psi c_2(t) S(t) - (\lambda b + \mu[I]) VS(t) \\ (7) \quad VE'(t) &= \psi c_2(t) E(t) + \lambda b VS(t) - (\epsilon + \mu[I]) VE(t) \\ (8) \quad VA'(t) &= \psi c_2(t) A(t) + \epsilon VE(t) - (\sigma + \mu[I]) VA(t) \\ (9) \quad VI'(t) &= (1 - \delta) \sigma VA(t) - (\gamma + \mu[I]) VI(t) \\ (10) \quad VR'(t) &= \psi c_2(t) R(t) + \delta \sigma VA(t) + (1 - \alpha'[I]) \gamma VI(t) - \mu[I] VR(t) \\ (11) \quad D'(t) &= \alpha[I] f I(t) + \alpha'[I] \gamma VI(t) + \mu[I] N \end{aligned}$$

with the following initial conditions

$$\begin{aligned}
(12) \quad & S(0) = S_0 = N_0 - (I_0 + E_0 + A_0 + D_0) \geq 0 \\
& E(0) = E_0 \geq 0, \quad A(0) = A_0 \geq 0, \quad I(0) = I_0 \geq 0, \quad R(0) = R_0 \geq 0 \\
& VS(0) \geq 0, \quad VE(0) \geq 0, \quad VA(0) \geq 0, \quad VI(0) \geq 0, \quad VR(0) \geq 0 \\
& D(0) = D_0 \geq 0.
\end{aligned}$$

**Remark 2.1** We should mention that the previous equations (6), (7), (8), (9), and (10) can be replaced by the following global equation:

$$(13) \quad V'(t) = \psi c_2(t)(S(t) + E(t) + A(t) + R(t)) - \mu[I]V(t).$$

In summary, using the global equation (13) the previous nonlinear system of ODEs describing the COVID-19 dynamics under the initial conditions (12) can be written as follows :

$$(14) \quad \begin{cases} S'(t) = & -\lambda p S(t) - \mu[I]S(t) - \psi c_2(t)S(t) \\ E'(t) = & \lambda p S(t) - (\epsilon + \mu[I])E(t) - \psi c_2(t)E(t) \\ A'(t) = & \epsilon E(t) - (\sigma + \mu[I])A(t) - \psi c_2(t)A(t) \\ I'(t) = & \sigma A(t) - (f + \mu[I])I(t) \\ R'(t) = & (1 - \alpha[I])fI(t) - \mu[I]R(t) - \psi c_2(t)R(t) \\ V'(t) = & \psi c_2(t)(S(t) + E(t) + A(t) + R(t)) - \mu[I]V(t) \\ D'(t) = & \alpha[I]fI(t) + \alpha'[I]\gamma VI(t) + \mu[I]N. \end{cases}$$

with the initial conditions :

$$\begin{aligned}
(15) \quad & S(0) = S_0 = N_0 - (I_0 + E_0 + A_0 + D_0) \geq 0 \\
& E(0) = E_0, \quad A(0) = A_0, \quad I(0) = I_0, \quad R(0) = R_0, \quad V(0) = V_0 \geq 0 \\
& D_0 = D_0 \geq 0.
\end{aligned}$$

### 3 Optimal control problem

In the present section, the optimal control theory is applied to suggest the most effective mitigation strategy to minimize the number of the infectious people, while also minimizing the effort of vaccinating the population and the effort of the public health measures during a fixed time period.

We recall that the control variable  $c_1(t)$  is used for the reduction in transmission due to public health measures at time  $t$  and to analyze vaccine efficacy enhancement, time-dependent control  $c_2(t)$  is introduced. The purpose of introducing these time-dependent controls is to analyze the effect of its variations with time on the dynamic of COVID-19.

#### 3.1 Objective function

Let us first define the objective function and then derive the necessary optimality condition. Our goal is to minimize the number of people who become infected, and thus the number of people who die due to the COVID-19 infection at a minimal efforts. Thus, we seek to minimize the following objective functional :

$$(16) \quad J(c) = \int_0^T I(t) + B_1 c_1^2(t) + B_2 c_2^2(t) dt,$$

where the control effort pair  $c = (c_1, c_2)$ . We define the set of admissible controls to be

$$U = \{(c_1, c_2) \text{ are Lebesgue measurable functions: } (c_1(t), c_2(t)) \in [0, 1]^2, \forall t \in [0, T]\}.$$

$B_1, B_2$  are constants that can be chosen to balance the relative costs of the public health restriction and vaccination. Quadratic terms  $c_1^2$  and  $c_2^2$  are introduced to account for nonlinear

costs potentially arising at high intervention levels and since implementation of any public health intervention and the expense of vaccination does not have a linear cost, see, for example [9].

The first term in the objective function ( $I(t)$ ) corresponds to the total number of infected individuals by the COVID-19 epidemic. The second term ( $B_1c_1^2(t) + B_2c_2^2(t)$ ) represents the total cost associated with the implementation of the control measures. It's a quadratic expression to find a known solution (for more details see [11]).

The main goal of our observations is to search for optimal control variables  $c_i^*$ , for  $i = 1, 2$  associated with public health restrictions and efficacy of vaccination respectively, mainly to find a function  $c^* = (c_1^*, c_2^*)$  such that

$$J(c^*) = J(c_1^*, c_2^*) = \min_{\mathcal{U}} J(c).$$

To find the optimal control pair  $c^*(t)$ , that minimizes  $J(c)$ , we follow standard results from optimal control theory applied to systems of ordinary differential equations.

In general, when the world has faced its most dangerous pandemic, the vaccination coverage (proportion of vaccinated people in a population at a given time) and the maximum daily vaccine administration are limited. Therefore, we assume there are practical limitations in our optimal control problem.

### Realistic restrictions

We define positive constants  $c_{max}$  as the maximum daily vaccination and  $c_{total}$  as the vaccine coverage, and our optimal control problem integrates these realistic constraints using state variable inequality constraints, or stated mathematically

$$(17) \quad \begin{cases} c_2(t)(S(t) + E(t) + A(t) + R(t)) & \leq c_{max} \\ \int_0^T c_2(t)(S(t) + E(t) + A(t) + R(t))dt & \leq c_{total} \end{cases}$$

The problem we are now facing is to minimize the number of infected individuals using a limited total vaccination. This can be stated

$$(18) \quad \text{minimize } J(c) \text{ subject to (14) and (17).}$$

Pontryagin's Maximum Principle cannot be used to deal with this problem as stated due to the constraint on vaccination coverage, though we can use a simple trick to convert this problem to a more familiar form. We introduce a new state variable, we denote  $z(t)$  and set

$$z(t) = \int_0^t c_2(s)(S(s) + E(s) + A(s) + R(s))ds.$$

Then, it follows

$$(19) \quad \begin{cases} z'(t) & = c_2(t)(S(t) + E(t) + A(t) + R(t)) \\ z(0) & = 0 \\ z(T) & \leq c_{total} \end{cases}$$

Thus, the constrained minimization problem (18) is transformed into:

$$\text{minimize } J(c) \text{ subject to (14)}$$

and to

$$(20) \quad \left\{ \begin{array}{l} z'(t) = c_2(t)(S(t) + E(t) + A(t) + R(t)) \\ z(0) = 0 \\ z(T) \leq c_{total} \\ 0 \leq c_1(t) \leq 1, \quad \text{and} \quad 0 \leq c_2(t) \leq 1 \\ c_2(t)(S(t) + E(t) + A(t) + R(t)) \leq c_{max} \end{array} \right.$$

are constraints that are required to be satisfied.

Using the monotonically increasing property of  $z(t)$ , we get

$$z(T) \leq c_{total} \quad \text{is equivalent to} \quad z(t) \leq c_{total}, \quad \text{for all } 0 \leq t \leq T.$$

Finally, our transformed constrained minimization problem takes the following form

$$(21) \quad \text{minimize } J(c) = \int_0^T I(t)dt + \int_0^T (B_1 c_1^2(t) + B_2 c_2^2(t))dt,$$

subject to

$$(22) \quad \begin{aligned} S'(t) &= -\lambda p S(t) - \mu[I]S(t) - \psi c_2(t)S(t) \\ E'(t) &= \lambda p S(t) - (\epsilon + \mu[I])E(t) - \psi c_2(t)E(t) \\ A'(t) &= \epsilon E(t) - (\sigma + \mu[I])A(t) - \psi c_2(t)A(t) \\ I'(t) &= \sigma A(t) - (f + \mu[I])I(t) \\ R'(t) &= (1 - \alpha[I])f I(t) - \mu[I]R(t) - \psi c_2(t)R(t) \\ V'(t) &= \psi c_2(t)(S(t) + E(t) + A(t) + R(t)) - \mu[I]V(t) \\ D'(t) &= \alpha[I]f I(t) + \alpha'[I]\gamma V I(t) + \mu[I]N \\ z'(t) &= c_2(t)(S(t) + E(t) + A(t) + R(t)), \end{aligned}$$

$$\text{and } S(0) = S_0 = N_0 - (I_0 + E_0 + A_0 + D_0), \quad E_0, A_0, R_0, V_0, D_0 \geq 0, \\ z(0) = 0, \quad z(T) \leq c_{total},$$

$$0 \leq c_1(t) \leq 1, \quad 0 \leq c_2(t) \leq 1, \quad c_2(t)(S(t) + E(t) + A(t) + R(t)) \leq c_{max}.$$

Our main goal is to approximate the solution of this constrained optimization problem.

### 3.2 Existence of an optimal control pair

Now, we focus on establishing a criterion for the existence of optimal solutions to the constrained minimization problem (21) subject to (22). We begin by examining the conditions of the Filippov-Cesari existence theorem [2]. Here we first establish the following theorem on the existence of optimal control.

**Theorem 3.1** *There exists an optimal control pair  $c_1^*(t)$ ,  $c_2^*(t)$ , such that the objective functional  $J(c)$  subject to (22) is minimized over  $U$ .*

For this purpose, let us recall the Filippov-Cesari existence theorem [2].

**Theorem 3.2** *Let  $x(t) = (x_1(t), \dots, x_n(t)) \in \mathbb{R}^n$  be a state vector and  $u(t) = (u_1(t), \dots, u_r(t)) \in \mathbb{R}^r$  be a control vector associated to the following optimal control problem*

$$(23) \quad \min_{x,u} \int_{t_0}^{t_1} F(t, x(t), u(t))dt,$$

with

$$(24) \quad \dot{x} = g_1(t, x(t), u(t)), \quad x(t_0) = x_0,$$



with the terminal conditions

$$(25) \quad \begin{aligned} & x_i(t_1) \geq x_i^1, \quad i = 1, \dots, m-1 \\ & x_i(t_1) \text{ free}, \quad i = m, \dots, n, \quad \text{it is meant that the value of } x_i(t_1) \text{ is unrestricted,} \end{aligned}$$

and for  $u(t) \in U$ , with  $U$  is a fixed set in  $\mathbb{R}^r$  we have the following constraints

$$(26) \quad g_2(t, x(t), u(t)) \geq 0,$$

where the function  $F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^r \rightarrow \mathbb{R}$ ,  $g_1 : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^r \rightarrow \mathbb{R}^n$  and  $g_2 : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^r \rightarrow \mathbb{R}^r$  are continuously differentiable in all variables. Assume that there exists an admissible pair  $(x(t), u(t))$ , namely if  $u(t)$  is any piecewise continuous control and  $x(t)$  is a continuously differentiable function such that (24) and (26) are satisfied, and

1.  $U$  is closed.
2.  $N(t, x) = \{\tilde{y} \equiv (y, y_{n+1}) : y = g_1(t, x, u), y_{n+1} \geq F(t, x, u), g_2(t, x, u) \geq 0, u \in U\}$  is convex for all  $(t, x) \in [t_0, t_1] \times \mathbb{R}^n$ .
3. There exists a number  $\theta_1 > 0$  such that  $\|x(t)\| < \theta_1$  for all admissible pairs  $(x(t), u(t))$ , and all  $t \in [t_0, t_1]$ .
4. There exists an open ball  $B(0, \theta_2) \subset \mathbb{R}^r$  which contains the set  $\Omega(t, x) = \{u \in U : g_2(t, x, u) \geq 0\}$  for all  $x \in B(0, \theta_1)$ .

Then there exists an optimal pair  $(x^*(t), u^*(t))$  to the problem (24), (25), (26) with  $u^*(t)$  measurable.

### Proof of Theorem 3.1.

The previous nontrivial requirements from Filippov-Cesari's theorem are verified below in order to establish the existence of a solution to the optimal control problem.

We therefore introduce the state variable

$x(t) = (S(t), E(t), A(t), I(t), R(t), V(t), D(t), z(t))^T$ , and control vector  $u(t) = (c_1(t), c_2(t)) \in \mathcal{U}$  and  $F(t, x(t), u(t)) = x_4(t) + B_1 c_1^2(t) + B_2 c_2^2(t)$ , the dynamics of the system is of the form,

$$(27) \quad g_1(t, x(t), u(t)) = \begin{bmatrix} -\lambda p S(t) - \mu [I] S(t) - \psi c_2(t) S(t) \\ \lambda p S(t) - (\epsilon + \mu [I]) E(t) - \psi c_2(t) E(t) \\ \epsilon E(t) - (\sigma + \mu [I]) A(t) - \psi c_2(t) A(t) \\ \sigma A(t) - (f + \mu [I]) I(t) \\ (1 - \alpha [I]) f I(t) - \mu [I] R(t) - \psi c_2(t) R(t) \\ \psi c_2(t) (S(t) + E(t) + A(t) + R(t)) - \mu [I] V(t) \\ \alpha [I] f I(t) + \alpha' [I] \gamma V I(t) + \mu [I] N \\ c_2(t) (S(t) + E(t) + A(t) + R(t)) \end{bmatrix}$$

and  $g_2(t, x(t), u(t)) = c_{max} - c_2(t)(x_1(t) + x_2(t) + x_3(t) + x_5(t))$ .

For ease of notation we omit the dependence on time of the variables. For the finite time interval, the variables  $S$ ,  $E$ ,  $A$ ,  $I$ ,  $R$  and  $V$  remain nonnegative if the initial values are nonnegative, (decrease only proportional to their present sizes, respectively). The variables  $z$  and  $D$  are also nonnegative since the changes in these variables are nonnegative.

To derive the upper bounds for the solutions, we use the approach explained in [5] and [8] to show that the total population size  $N$  is bounded above  $N(0)$ . Since none of  $S$ ,  $E$ ,  $A$ ,  $I$ ,  $R$  and  $V$  can be negative the upper bound of  $N$  is also an upper bound for  $S$ ,  $E$ ,  $A$ ,  $I$ ,  $R$  and  $V$ . The boundedness of the auxiliary state variable  $z$  follows from the boundedness of the control  $c_2$ ,  $S$ ,  $E$ ,  $A$ ,  $R$  and  $V$ . Note that  $F$ ,  $g_1$ , and  $g_2$  are of class  $C^1$ , and  $g_1$  is bounded. Thus, there exists a solution for the system (22) which guarantees an admissible pair  $(x(t), u(t))$ .

The control set  $\mathcal{U} = [0, 1]^2$  is closed then the condition 1 is trivial, and as is also a compact set then condition 4 holds.

Now, we verify the condition 2. We first note that the control set  $\mathcal{U}$  is closed and convex, the integrand of the objective functional  $F(t, x, \cdot)$  and the constraint control function  $g_2(t, x, \cdot)$  are convex on  $\mathcal{U}$ . The function  $g_1(t, x, \cdot)$  in the system (27) can be written as linear function of the control variable with coefficients depending on the state variables and time therefore

$$g_1(t, x, u) = k(t, x) + l(t, x)u,$$

thus  $g_1$  is also a convex function on  $\mathcal{U}$ . Let  $\tilde{y}^1, \tilde{y}^2 \in N$ , for  $\omega \in [0, 1]$ , we can prove easily that the affine combination  $\omega\tilde{y}^1 + (1 - \omega)\tilde{y}^2$  belongs to  $N$  for all  $(t, x) \in [t_0, t_1] \times \mathbb{R}^8$  due to the convexity of the functions  $F, g_1$  and  $g_2$ , which shows that  $N$  is convex set.

Finally, condition 3 follows from the boundedness of solutions to the system (22) for a finite time interval. Filippov-Cesari's conditions theorem are verified. ■

### 3.3 Penalty Method

Several methods have been proposed for handling constrained optimal control problems. The most common approach is to convert them into unconstrained optimization problems, often via the use of a penalty method. Namely, the unconstrained problems are formed by adding a term, called penalty function, to the objective function that consists of a penalty parameter multiplied by a measure of violation of the constraints.

There are different ways to construct the penalty term, a widely used one is the quadratic penalty. An appropriate quadratic penalty function associated with the minimization problem (20) is:

$$J_p(c) = \int_0^T \left[ I(t) + B_1 c_1^2(t) + B_2 c_2^2(t) + \mu_1 \max(0, h_1(t))^2 + \mu_2 \max(0, h_2(t))^2 \right] dt$$

where  $B_1, B_2$  are constants,  $\mu_1, \mu_2$  are penalty coefficients that change the relative severity of the constraints violation and  $h_1, h_2$  are inequality constraints given by

$$h_1(t) = c_2(t)(S(t) + E(t) + A(t) + R(t)) - c_{max} \quad \text{and} \quad h_2(t) = z(t) - c_{total}.$$

Thus, the penalty function  $J_p(c)$  can be written as follows

$$\begin{aligned} J_p(c) = & \int_0^T \left[ I(t) + B_1 c_1^2(t) + B_2 c_2^2(t) \right. \\ & + \mu_1 (c_2(t)(S(t) + E(t) + A(t) + R(t)) - c_{max})^2 H_1(c_2(t)(S(t) + E(t) + A(t) + R(t)) - c_{max}) \\ & \left. + \mu_2 (z(t) - c_{total})^2 H_2(z(t) - c_{total}) \right] dt \end{aligned}$$

where  $H_1$  and  $H_2$  denote the Heaviside step functions, given by :

$$H_1(c_2(t)(S(t) + E(t) + A(t) + R(t)) - c_{max}) = \begin{cases} 0 & \text{if } c_2(t)(S(t) + E(t) + A(t) + R(t)) \leq c_{max} \\ 1 & \text{if } c_2(t)(S(t) + E(t) + A(t) + R(t)) > c_{max} \end{cases}$$

$$H_2(z(t) - c_{total}) = \begin{cases} 0 & \text{if } z(t) \leq c_{total} \\ 1 & \text{if } z(t) > c_{total} \end{cases}$$

Then, to minimize the solution of the constrained optimization problem (21), we should minimize  $J_p(c)$ , mainly to find the control  $c^*$  so as to

$$(28) \quad \text{minimise } J_p(c) \text{ subject to the nonlinear system of ODEs } \quad (22)$$

$$z'(t) = c_2(t)(S(t) + E(t) + A(t) + R(t)), \quad z(0) = 0, \quad 0 \leq c_1(t), c_2(t) \leq 1.$$

Pontryagin's Maximum principle is used to derive the optimality system which provides the necessary conditions of the optimal solutions of (28).

Introduce a piecewise differentiable vector-valued functions  $\lambda(t) = (\lambda_1(t), \lambda_2(t), \dots, \lambda_8(t))$  where each  $\lambda_i$  is the adjoint variable corresponding to  $x_i$ , (with 8 states we will need 8 adjoints). Setting  $\dot{x}(t) = dx/dt$ , we introduce the augmented Hamiltonian for the constraints control as follows:

$$\begin{aligned} H(t, x, u, \lambda) &= \langle \lambda(t), \dot{x}(t) \rangle + I(t) + B_1 c_1^2(t) + B_2 c_2^2(t) \\ &+ \mu_1(z'(t) - c_{max})^2 H_1(z'(t) - c_{max}) + \mu_2(z(t) - c_{total})^2 H_2(z(t) - c_{total}) \\ &= \lambda_1(t) [-p(1 - c_1(t))(\beta_A A + \beta_I I)S(t) - \mu[I]S(t) - \psi c_2(t)S(t)] \\ &+ \lambda_2(t) [p(1 - c_1(t))(\beta_A A + \beta_I I)S(t) - (\epsilon - \mu[I])E(t) - \psi c_2(t)E(t)] \\ &+ \lambda_3(t) [\epsilon E(t) - (\sigma + \mu[I])A(t) - \psi c_2(t)A(t)] + \lambda_4(t) [\sigma A(t) - (f + \mu[I])I(t)] \\ &+ \lambda_5(t) [(1 - \alpha[I])fI(t) - \mu[I]R(t) - \psi c_2(t)R(t)] + \lambda_6(t) [\psi z'(t) - \mu[I]V(t)] \\ &+ \lambda_7(t)z'(t) + \lambda_8(t) [\alpha[I]fI(t) + \alpha'[I]\gamma V(t) + \mu[I]N] \\ &+ I(t) + B_1 c_1^2(t) + B_2 c_2^2(t) + \mu_1(z'(t) - c_{max})^2 H_1(z'(t) - c_{max}) \\ &+ \mu_2(z(t) - c_{total})^2 H_2(z(t) - c_{total}) - \sum_{i=1}^2 \omega_{i1}(t)c_i(t) - \sum_{i=1}^2 \omega_{i2}(t)(1 - c_i(t)), \end{aligned}$$

where  $z'(t) = c_2(t)(S(t) + E(t) + A(t) + R(t))$  and  $\omega_{ij}(t) \geq 0$  are the penalty multipliers satisfying

$$\omega_{i1}(t)c_i(t) = \omega_{i2}(t)(1 - c_i(t)) = 0 \quad \text{at} \quad c_i(t) = c_i^*(t) \quad \text{for} \quad i = 1, 2$$

with  $c_i^*(t) = (c_1^*(t), c_2^*(t))$  is the optimal control pair should be found.

On differentiating the augmented Lagrangian  $H$  with respect to state variables and setting the result to zero, we get the following adjoint system:

$$\begin{aligned} \lambda_1'(t) = -\frac{\partial H}{\partial S} &= \lambda_1(t)(-p(1 - c_1(t))(\beta_A A + \beta_I I) + \mu[I]\lambda_1(t) + \psi c_2(t)\lambda_1(t) \\ &- p(1 - c_1(t))(\beta_A A + \beta_I I)\lambda_2(t) - \psi c_2(t)\lambda_6(t) - c_2(t)\lambda_7(t) - \mu[I]\lambda_8(t) \\ &- 2c_2(t)\mu_1(z'(t) - c_{max})H_1(z'(t) - c_{max}) \\ &= (\lambda_1(t) - \lambda_2(t))(p(1 - c_1(t))(\beta_A A + \beta_I I)) + (\lambda_1(t) - \lambda_8(t))\mu[I] + (\lambda_1(t) - \lambda_6(t))\psi c_2(t) \\ &- c_2(t)\lambda_7(t) - 2c_2(t)\mu_1(z'(t) - c_{max})H_1(z'(t) - c_{max}). \end{aligned}$$

$$\begin{aligned} \lambda_2'(t) = -\frac{\partial H}{\partial E} &= (\epsilon + \mu[I])\lambda_2(t) + \psi c_2(t)\lambda_2(t) - \epsilon\lambda_3(t) - \psi c_2(t)\lambda_6(t) - c_2(t)\lambda_7(t) - \mu[I]\lambda_8(t) \\ &- 2c_2(t)\mu_1(z'(t) - c_{max})H_1(z'(t) - c_{max}) \\ &= \epsilon(\lambda_2(t) - \lambda_3(t)) + (\lambda_2(t) - \lambda_8(t))\mu[I] + (\lambda_2(t) - \lambda_6(t))\psi c_2(t) \\ &- c_2(t)\lambda_7(t) - 2c_2(t)\mu_1(z'(t) - c_{max})H_1(z'(t) - c_{max}). \end{aligned}$$

$$\begin{aligned} \lambda_3'(t) = -\frac{\partial H}{\partial A} &= \beta_A \lambda_1(t)(p(1 - c_1(t))S(t) - \beta_A \lambda_2(t)(p(1 - c_1(t))S(t) + (\sigma + \mu[I])\lambda_3(t) + \psi c_2(t)\lambda_3(t) \\ &- \sigma \lambda_4(t) - \psi c_2(t)\lambda_6(t) - c_2(t)\lambda_7(t) - \mu[I]\lambda_8(t) - 2c_2(t)\mu_1(z'(t) - c_{max})H_1(z'(t) - c_{max})) \\ &= \beta_A(p(1 - c_1(t))S(t)(\lambda_1(t) - \lambda_2(t)) + \sigma(\lambda_3(t) - \lambda_4(t)) + \mu[I](\lambda_3(t) - \lambda_8(t)) \\ &+ \psi c_2(t)(\lambda_3(t) - \lambda_6(t)) - c_2(t)\lambda_7(t) - 2c_2(t)\mu_1(z'(t) - c_{max})H_1(z'(t) - c_{max}). \end{aligned}$$

$$\begin{aligned} \lambda_4'(t) = -\frac{\partial H}{\partial I} &= \beta_I(p(1 - c_1(t))S(t)(\lambda_1(t) - \lambda_2(t)) + (f + \mu[I])\lambda_4(t) - (1 - \alpha[I])f\lambda_5(t) \\ &- \psi c_2(t)\lambda_6(t) - c_2(t)\lambda_7(t) - \alpha[I]f\lambda_8(t) - \mu[I]\lambda_8(t) - 1 \\ &= \beta_I(p(1 - c_1(t))S(t)(\lambda_1(t) - \lambda_2(t)) + f(\lambda_4(t) - \lambda_5(t)) + \mu[I](\lambda_4(t) - \lambda_8(t)) \\ &+ \alpha[I]f(\lambda_5(t) - \lambda_8(t)) - \psi c_2(t)\lambda_6(t) - c_2(t)\lambda_7(t) - 1. \end{aligned}$$

$$\begin{aligned}
\lambda'_5(t) &= -\frac{\partial H}{\partial R} = \mu[I]\lambda_5(t) + \psi c_2(t)\lambda_5(t) - \psi c_2(t)\lambda_6(t) - c_2(t)\lambda_7(t) - \mu[I]\lambda_8(t) \\
&\quad - c_2(t)\mu_1(z'(t) - c_{max})H_1(z'(t) - c_{max}) \\
&= \mu[I](\lambda_5(t) - \lambda_8(t)) + \psi c_2(t)(\lambda_5(t) - \lambda_6(t)) - c_2(t)\lambda_7(t) \\
&\quad - c_2(t)\mu_1(z'(t) - c_{max})H_1(z'(t) - c_{max}).
\end{aligned}$$

$$\lambda'_6(t) = -\frac{\partial H}{\partial V} = \mu[I]\lambda_6(t) - \alpha'[I]\gamma\lambda_8(t) - \mu[I]\lambda_8(t).$$

$$\lambda'_7(t) = -\frac{\partial H}{\partial z} = -2\mu_2(z(t) - c_{total})H_2(z(t) - c_{total}).$$

$$\lambda'_8(t) = -\frac{\partial H}{\partial D} = 0.$$

with the transversality conditions  $\lambda_i(T) = 0$ , for  $i = \{1, \dots, 8\}$ . Now, we differentiate the augmented Lagrangian  $H$  with respect to  $c = (c_1, c_2)$ :

$$\frac{\partial H}{\partial c_1} = \lambda_1(t)p(\beta_A A + \beta_I I)S(t) - \lambda_2(t)p(\beta_A A + \beta_I I)S(t) + 2B_1 c_1(t) - \omega_{11}(t) + \omega_{12}(t).$$

$$\begin{aligned}
\frac{\partial H}{\partial c_2} &= -\psi S(t)\lambda_1(t) - \psi E(t)\lambda_2(t) - \psi A(t)\lambda_3(t) - \psi R(t)\lambda_5(t) + \psi(S(t) + E(t) + A(t) + R(t))\lambda_6(t) \\
&\quad + (S(t) + E(t) + A(t) + R(t))\lambda_7(t) + 2B_2 c_2(t) \\
&\quad + 2\mu_1(S(t) + E(t) + A(t) + R(t))(z'(t) - c_{max})H_1(z'(t) - c_{max}) - \omega_{21}(t) + \omega_{22}(t).
\end{aligned}$$

Recall that  $z'(t) = c_2(t)(S(t) + E(t) + A(t) + R(t))$ .

To sum up, we find the optimal control by solving the state system with initial conditions and adjoint equations, we obtain  $c_1^*, c_2^*$  defined as:

$$\begin{aligned}
c_1^* &= \frac{-\lambda_1(t)p(\beta_A A + \beta_I I)S(t) + \lambda_2(t)p(\beta_A A + \beta_I I)S(t) + \omega_{11}(t) - \omega_{12}(t)}{2B_1} \\
c_2^* &= \frac{\psi(S(t)\lambda_1(t) + E(t)\lambda_2(t) + A(t)\lambda_3(t) + R(t)\lambda_5(t))}{2B_2 + 2\mu_1(S(t) + E(t) + A(t) + R(t))^2 H_1} \\
&\quad + \frac{-(S(t) + E(t) + A(t) + R(t))(\psi\lambda_6(t) - \lambda_7(t) + 2\mu_1 c_{max})H_1}{2B_2 + 2\mu_1(S(t) + E(t) + A(t) + R(t))^2 H_1} + \omega_{21}(t) - \omega_{22}(t).
\end{aligned}$$

To find an explicit expression for the optimal control without the penalty multipliers  $\omega_{ij}$ , where  $i, j = 1, 2$ , we start the discussion for the first component  $c_1^*$  with  $c_1 \in [0, 1)$  to avoid the case where the force infection  $\lambda = 0$  (case where  $c_1 = 1$ ). For that, we consider the following cases:

**Case 1:** On the set  $\{t | 0 < c_1^*(t) < 1\}$ , we have  $\omega_{11}(t) = \omega_{12}(t) = 0$ . Hence the optimal control  $c_1^*$  becomes

$$c_1^* = \frac{-\lambda_1(t)p(\beta_A A + \beta_I I)S(t) + \lambda_2(t)p(\beta_A A + \beta_I I)S(t)}{2B_1}.$$

**Case 2 :** On the set  $\{t | c_1^*(t) = 0\}$ , we have  $\omega_{12}(t) = 0$  and  $\omega_{11}(t) \geq 0$ . Hence

$$c_1^* = \frac{-\lambda_1(t)p(\beta_A A + \beta_I I)S(t) + \lambda_2(t)p(\beta_A A + \beta_I I)S(t) + \omega_{11}(t)}{2B_1},$$

which implies that

$$c_1^* = \frac{-\lambda_1(t)p(\beta_A A + \beta_I I)S(t) + \lambda_2(t)p(\beta_A A + \beta_I I)S(t)}{2B_1} \leq 0.$$

Combining these two cases, the optimal control  $c_1^*$  is characterized as

$$(29) \quad c_1^* = \min \left[ 1, \max \left\{ 0, \frac{(\lambda_2(t) - \lambda_1(t))p(\beta_A A + \beta_I I)S(t)}{2B_1} \right\} \right].$$

We proceed with the same reasoning to find  $c_2^*$ , where  $c_2 \in [0, 1]$  and we add the following third case:

**Case 3:** On the set  $\{t|c_2^*(t) = 1\}$ , we have  $\omega_{21}(t) = 0$  and  $\omega_{22}(t) \geq 0$ . Hence

$$\begin{aligned} 1 = c_2^* &= \frac{\psi \left( S(t)\lambda_1(t) + E(t)\lambda_2(t) + A(t)\lambda_3(t) + R(t)\lambda_5(t) \right)}{2B_2 + 2\mu_1(S(t) + E(t) + A(t) + R(t))^2 H_1} \\ &+ \frac{-(S(t) + E(t) + A(t) + R(t)) \left( \psi \lambda_6(t) - \lambda_7(t) + 2\mu_1 c_{max} H_1 \right) - \omega_{22}(t)}{2B_2 + 2\mu_1(S(t) + E(t) + A(t) + R(t))^2 H_1} \end{aligned}$$

which implies that

$$\begin{aligned} c_2^* &= \frac{\psi \left( S(t)\lambda_1(t) + E(t)\lambda_2(t) + A(t)\lambda_3(t) + R(t)\lambda_5(t) \right)}{2B_2 + 2\mu_1(S(t) + E(t) + A(t) + R(t))^2 H_1} \\ &+ \frac{-(S(t) + E(t) + A(t) + R(t)) \left( \psi \lambda_6(t) - \lambda_7(t) + 2\mu_1 c_{max} H_1 \right)}{2B_2 + 2\mu_1(S(t) + E(t) + A(t) + R(t))^2 H_1} \geq 1. \end{aligned}$$

To conclude, we get

$$(30) \quad c_2^* = \min \left[ 1, \max \left\{ 0, \frac{\psi \left( S(t)\lambda_1(t) + E(t)\lambda_2(t) + A(t)\lambda_3(t) + R(t)\lambda_5(t) \right)}{2B_2 + 2\mu_1(S(t) + E(t) + A(t) + R(t))^2 H_1} \right. \right. \\ \left. \left. + \frac{-(S(t) + E(t) + A(t) + R(t)) \left( \psi \lambda_6(t) - \lambda_7(t) + 2\mu_1 c_{max} H_1 \right)}{2B_2 + 2\mu_1(S(t) + E(t) + A(t) + R(t))^2 H_1} \right\} \right].$$

## 4 Numerical simulations and discussion

In this section, we perform numerical simulations for a previous model to estimate the evolution of the COVID-19 outbreak in France. The simulations focus on solving a constrained optimal control problem applied to vaccination.

The values of the model parameters are obtained from available local data and previous studies.

The simulations consider the period before the vaccination program in France, as well as the period after vaccination. Figures are presented to illustrate the simulation results, including the number of infected individuals and the number of deaths. The impact of control measures and vaccination on the spread of the infection and the reduction of deaths is analyzed.

Overall, the simulations provide insights into the effectiveness of vaccination and control measures in managing the COVID-19 outbreak in France.

## 4.1 Model state variables and parameters

Table 1 provides a summary of the model's state variables and parameters. The state variables include the densities of susceptible, vaccine susceptible, latent, vaccine latent, asymptomatic infectious, vaccine asymptomatic infectious, symptomatic infectious, vaccine symptomatic infectious, recovered individuals, vaccine recovered individuals and total deaths. The parameters include the basic reproduction number, proportion of infections, proportion of infections vaccinated, proportion of recovery among vaccinated, transmission rates, reduced transmission factor of infections, control effort, immigration rate, waiting rates to viral shedding and symptom onset, recovery rates from infections, natural mortality rate, disease-induced mortality rates, health-care capacity, cost weight, and initial conditions. The best-fit values of these parameters were obtained from [4] and the available local data.

Variables	Description	
$S, VS$	Density of susceptible and susceptible vaccinated individuals	
$E, VE$	Density of latent and latent vaccinated individuals	
$A, VA$	Density of asymptomatic and asymptomatic vaccinated infectious individuals	
$I, VI$	Density of symptomatic and symptomatic vaccinated infectious individuals	
$R, VR$	Density of recovered and recovered vaccinated individuals	
D	Total deaths	
Parameters	Description (unit)	Value(range) [ref.]
$p$	Proportion of infections	0.9(0.85 – 0.95)[9]
$\beta_I$	Symptomatic transmission rate	Calculated <sup>a</sup>
$\beta_A$	Asymptomatic transmission rate	Calculated <sup>a</sup>
$b$	Proportion of infections vaccinated individuals	0.2 [assumed]
$c_1$	Control public health effort	(0, 1)
$c_2$	Control effort due to the vaccination	(0,1)
$\varepsilon$	Waiting rate to viral shedding ( day <sup>-1</sup> )	1/4.2(0.21 – 0.27)[16]
$\sigma$	Waiting rate to symptom onset ( day <sup>-1</sup> )	1(0.9 – 1.1)[7]
$\gamma$	Recovery rate from infections ( day <sup>-1</sup> )	1/17(0.025 – 0.1)[27]
$\delta$	proportion of recovered among vaccine asymptomatic	0.8 [assumed]
$\mu$	Natural mortality rate with hospital saturation ( day <sup>-1</sup> )	10 <sup>-5</sup> [assumed]
$\alpha_{\min}$	Lower bound disease-induced mortality rate ( day <sup>-1</sup> )	Calculated <sup>a</sup>
$\alpha_{\max}$	Higher bound induced mortality rate (day <sup>-1</sup> )	2 $\alpha_{\min}$ [assumed]
$I^*$	Healthcare capacity	12000 [assumed]
$B_1, B_2$	Cost weight	800(0, $\infty$ ) [assumed]
Initial conditions		
$N_0$	Total population	67 × 10 <sup>6</sup>
$I_0$	Size of infected population	0.01 × $I^*$ ( variable )
$S_0$	Size of susceptible population	$N_0 - I_0$

The parameters  $\alpha_{\min}, \beta_I, \beta_A$  are calculated by Djidjou-Demasse et al., see [4].

Table 1: Model state variables and parameters

## 4.2 SEAIR model before vaccination

First, we consider the model (1) before the vaccine deployment, during the period from December 2020 through October 2021 in France. We assume that the vaccinated sub-populations,  $VS, VE, VA, VI$  and  $VR$  are equal to zero. We use Runge Kutta scheme to solve the set of differential equations.

For simplicity and straightforward analysis of causality, we take initial conditions

$$\begin{aligned}
 I_0 &= 6458 \\
 E_0 &= 2 * 6458 \\
 A_0 &= 1.5 * 6458 \\
 S_0 &= N_0 - (I_0 + E_0 + A_0 + D_0) \\
 R_0 &= VS_0 = VE_0 = VA_0 = VI_0 = VP_0 = VR_0 = 0 \\
 D_0 &= 34
 \end{aligned}$$

On the following Figure 2, each color corresponds to one of compartment of the model (Susceptible, Latent, Asymptomatic, Infected, Recovered and Death).

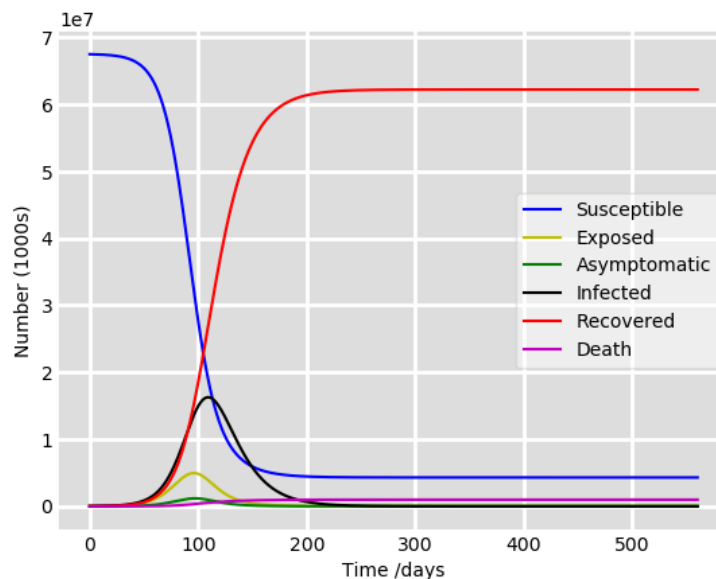


Figure 2: The model simulation before the vaccination program in France

We can see that in the absence of any control the number of infected individuals grows exponentially fast at the beginning of the epidemic (curve in black color) and the epidemic reaches its peak, equilibrium point as the intersection of the curves for those still susceptible and those immunized following recovery, 110 days after the beginning.

This is a natural behavior of an epidemic. However, waiting for a large part of the population to become infected in order to mitigate the epidemic is certainly not the best strategy, especially when the disease presents a high mortality due to the severity of the infection or to the saturation of the healthcare system.

In the following section, we will simulate the solutions of the extended SEAIR model (1) to estimate the impact of the vaccination program in France.

### 4.3 Vaccination effect in France

We will now simulate the solution to the model (1) to assess the impact of the vaccination program to combat the infectious diseases caused by COVID-19. The first dose of COVID-19 vaccine in France was received for the first time on December 27, 2020. Since that, the number of doses has multiplied rapidly to protect the population as quickly as possible.

Vaccines used	Efficiency rate	Administered doses
Pfizer	95%	81 490 504
AstraZeneca/Oxford	70%	7 838 931
Moderna	50.38%	12 357 533
Johnson and Johnson	66%	1 074 197

Table 2: Distribution of vaccine in France

The table 2 provides an overview of the distribution of vaccines in France, highlighting their efficacy rates.

In the numerical simulations, we compare the optimal intervention strategies under different settings of a proportion of infections, waiting rate to viral shedding, symptom onset, and the infection rate among vaccines. In France, these results are given for the period from February 15, 2021 to April 30, 2021 when the number of cases go past 50000 infected cases (Figure 3). Vaccination is highly concentrated.

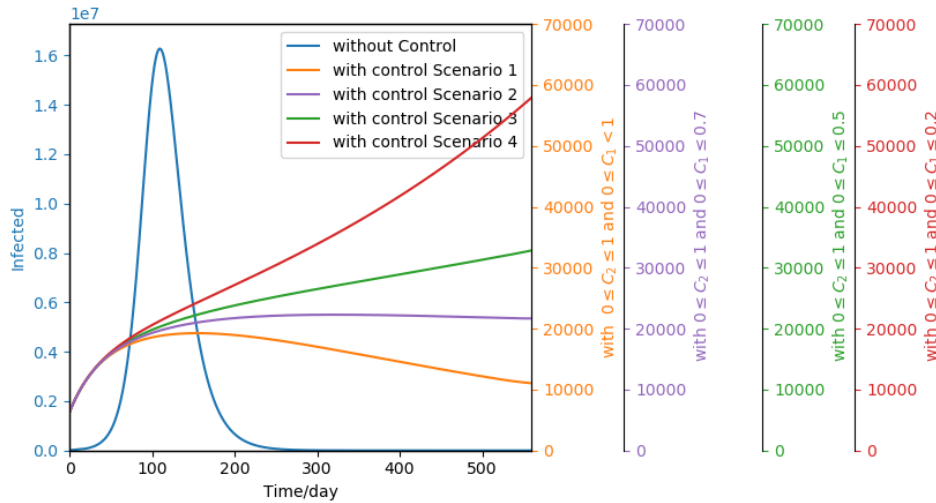


Figure 3: Comparaison infected people in France before and after vaccination

From Figure 3, we can observe that the number of infected individuals is influenced by the control parameter of the measures taken to control the spread of diseases  $c_1$ . In the absence of control measures, the number of infected individuals exceeds 16 million (1.6e7). Under the optimal vaccination, the spread of infection is significantly reduced in Scenario 1 when  $c_1$  and  $c_2$  range between 0 and 0.95, resulting in a lower rate of new infections.

Furthermore, they show a clear correlation between an increase in the maximum vaccination threshold and a significant reduction in the peak of critical infections. This underlines the importance of achieving high vaccination coverage rates to mitigate the spread of infectious diseases and protect vulnerable populations



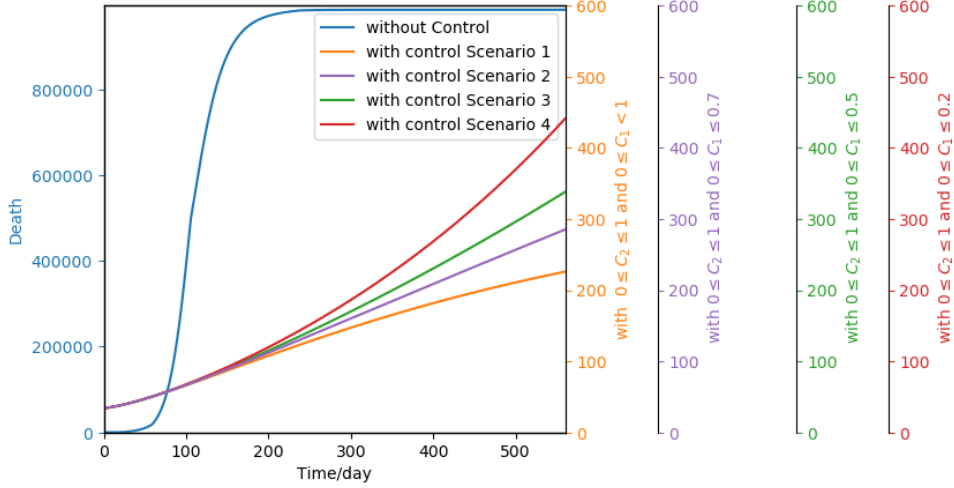


Figure 4: Comparaision between death people in France before and after vaccination

In this optimal case, the improved control measures not only impact the number of infected individuals but also have an influence on the number of deaths, illustrated in Figure 4. By implementing the optimal control measures in scenario 1, the number of deaths is expected to be reduced to 205 compared to the other scenarios.

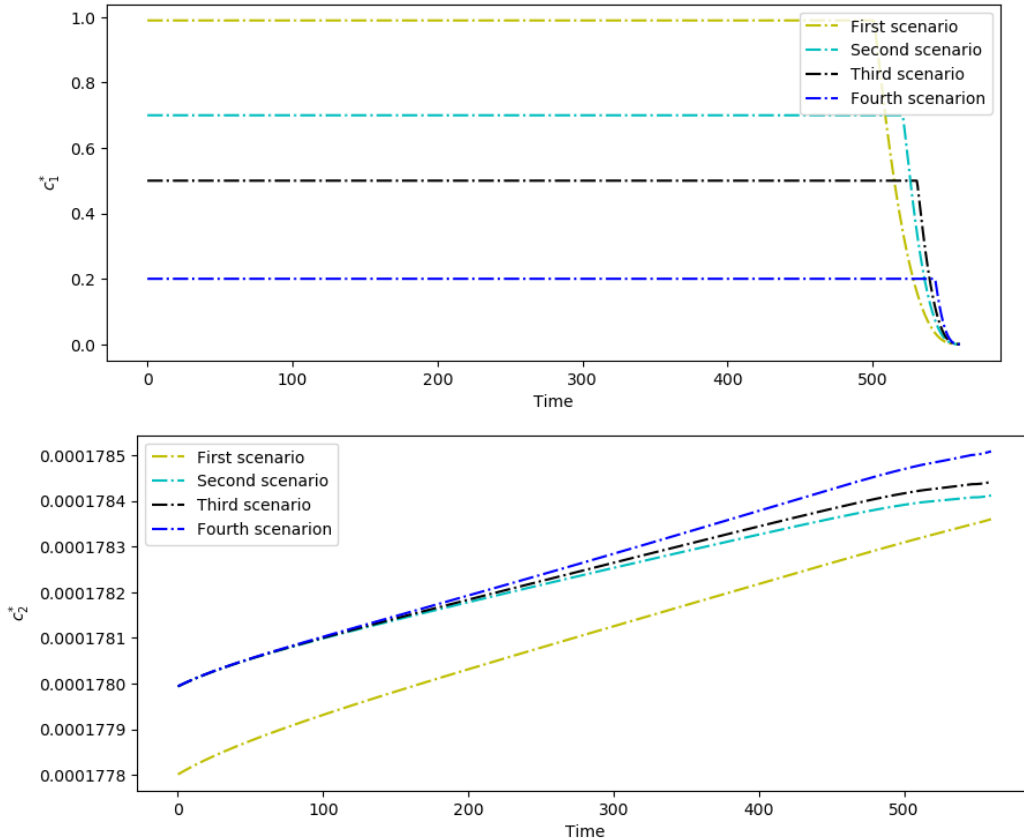


Figure 5: Progression of the optimal Controls  $c_1^*$  and  $c_2^*$  over time.

Figure 5 shows the progression of optimal controls  $c_1^*$  and  $c_2^*$  over time. For all four scenarios, we observe that control  $c_1^*$ , defined by the percentage reduction in transmission due to a public health measure or sanitary control at time  $t$ , always takes the maximum that is defined in equation (29) this shows the necessity of sanitary restraint to the extinction of the epidemic.

Whereas for the  $c_2^*$  control, which measures the rate at which susceptible individuals are vaccinated, we observe that the curves for the different scenarios are characterized by low and almost constant growth throughout the vaccination period (between  $1.778e - 04$  and  $1.785e - 04$ ),

indicating that the optimal vaccination strategy is to vaccinate all individuals simultaneously when total coverage is possible.

The following figures 6 describe the evolution of the  $\lambda_i, i \in \{1, \dots, 8\}$  weights of the Lagrangian to control the relative importance of constraints compared to the objective function  $J_p$  during the optimization problem-solving.

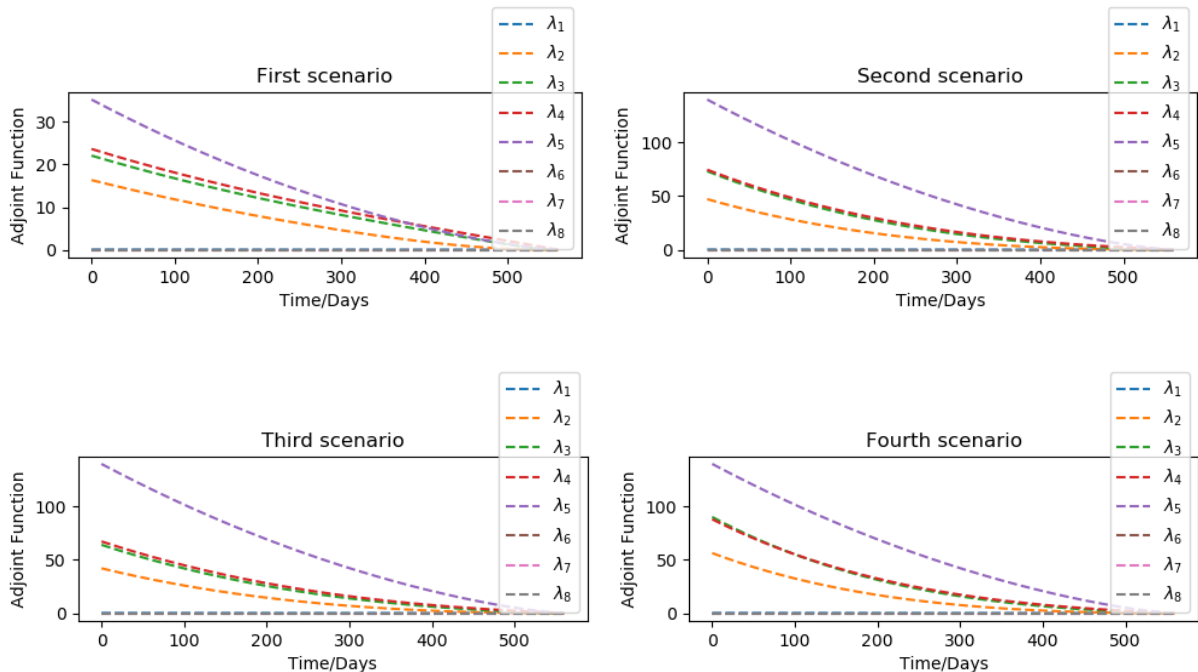


Figure 6: Adjoint Functions

## 5 Conclusion

In this work, a SEAIR mathematical model for the spread of infectious diseases such as COVID-19 is developed. Then, an optimal control problem aiming at minimizing the number of infected people with minimal effort is developed and studied.

The main goal of our studies is to search for optimal control variables  $c^* = (c_1^*, c_2^*)$  associated respectively with public health restrictions and vaccination effectiveness. We formulated an optimal control problem and used the conditions of the Filippov-Cesari existence theorem to characterize optimal control. Unconstrained problems are formed by adding to the objective function a penalty function.

Then, we solved numerically the model for the COVID-19 pandemic in France using parameters that have been developed in the literature. The optimal strategy was verified under different scenarios: health restrictions limited to 20, 50, 75, and 100 percent.

In fact, the results suggest starting vaccination at the same time, whatever the current constraints.

Our results show that higher maximum  $c_1^*$  restriction rates reduce the number of severely infected people in all countries, but do not lead to a slight reduction in infection. We therefore recommend the mandatory introduction of parallel vaccination for all populations, as the only way to reduce mortality.

In conclusion, this study provides valuable information on the optimal vaccination control strategy and public health measures. The results may help in making decisions aimed at reducing the number of people infected and mortality caused by infectious diseases.

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