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Modular-proximal gradient algorithms in variable exponent Lebesgue spaces *

Marta Lazzaretti^{†1,2}, Luca Calatroni^{‡2} and Claudio Estatico^{§1}

¹Department of Mathematics, Via Dodecaneso 35, University of Genova, Italy
²CNRS, UCA, Inria, Laboratoire I3S, Sophia-Antipolis, 06903 France

Abstract

We consider structured optimisation problems defined in terms of the sum of a smooth and convex function, and a proper, l.s.c., convex (typically non-smooth) one in reflexive variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$. Due to their intrinsic space-variant properties, such spaces can be naturally used as solution space and combined with space-variant functionals for the solution of ill-posed inverse problems. For this purpose, we propose and analyse two instances (primal and dual) of proximal gradient algorithms in $L^{p(\cdot)}(\Omega)$, where the proximal step, rather than depending on the natural (non-separable) $L^{p(\cdot)}(\Omega)$ norm, is defined in terms of its modular function, which, thanks to its separability, allows for the efficient computation of algorithmic iterates. Convergence in function values is proved for both algorithms, with convergence rates depending on problem/space smoothness. To show the effectiveness of the proposed modelling, some numerical tests highlighting the flexibility of the space $L^{p(\cdot)}(\Omega)$ are shown for exemplar deconvolution and mixed noise removal problems. Finally, a numerical verification on the convergence speed and computational costs of both algorithms in comparison with analogous ones defined in standard $L^p(\Omega)$ spaces is presented.

1 Introduction

Let $\Omega \subseteq \mathbb{R}^d$, with $d \in \mathbb{N}$, $d \geq 1$ be a Lebesgue measurable subset with positive measure and let $p(\cdot) : \Omega \rightarrow [1, +\infty]$ be a Lebesgue measurable function. The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined in terms of the function $p(\cdot)$ which, differently from conventional $L^p(\Omega)$ spaces, acts as point-wise variable exponent, inducing a specific shift-variant norm. In this work, we will study a proximal-gradient (a.k.a. forward-backward) splitting algorithm to solve composite optimization problems in the form

$$\arg \min_{x \in L^{p(\cdot)}(\Omega)} \phi(x) := f(x) + g(x) \quad (\text{P})$$

where $f : L^{p(\cdot)}(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}$ is a proper, convex and Gateaux differentiable function while $g : L^{p(\cdot)}(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}$ is lower semi-continuous, proper, convex, and possibly non-smooth. Optimization problems of this kind often arise in the context of variational regularization of inverse problems such as, e.g., the minimization of Tikhonov-like functionals, where f denotes a data-fidelity term while g stands for the regularization or penalty term, see, e.g. [6] for a review.

Variable exponent Lebesgue spaces $L^{p(\cdot)}$ are (non-Hilbertian) Banach spaces, endowed with space-variant geometrical properties which are useful in adaptive regularization techniques, see, e.g., [19, 3]. Under suitable choices of the exponent function $p(\cdot)$, having $L^{p(\cdot)}$ as solution space allows for a better recovery of sparse solutions, avoiding over-smoothing reconstruction artifacts of usual regularization models formulated in $L^2(\Omega)$, a property that was already observed in previous works (see, e.g., [15]) in the context of iterative regularization methods (with no penalty).

Forward-backward splitting algorithms are a common strategy used to solve problems in the form (P). Initially proposed in [7, 9, 18] in Hilbert spaces \mathcal{H} for functions f with \mathcal{L} -Lipschitz gradient, they are defined by the following updating rule for $k \geq 0$

$$x^{k+1} = \arg \min_{x \in \mathcal{H}} \frac{1}{2} \|x - (x^k - \tau_k \nabla f(x^k))\|_{\mathcal{H}}^2 + \tau_k g(x),$$

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[†]lazzaretti@dima.unige.it

[‡]calatroni@i3s.unice.fr

[§]estatico@dima.unige.it

where $\tau_k > 0$ is a suitably-chosen sequence of step sizes. Such schemes enable to exploit the differentiability of the smooth function f in a forward step defined in terms of the gradient of f $x^k \mapsto x^k - \tau_k \nabla f(x^k)$ which is decoupled from the backward step defined in terms of the proximal operator associated to g .

The generalization of this algorithm to a Banach space \mathcal{X} is not straightforward since the element $\nabla f(x^k) \in \mathcal{X}^*$ cannot be identified as an element of \mathcal{X} anymore due to the lack of Riesz isomorphism. As a consequence, the forward step cannot thus be performed directly on $x^k \in \mathcal{X}$. As a remedy, we need to introduce the so-called duality maps, which link primal and dual spaces and allow to perform the forward gradient step either in the dual or in the primal space. The intrinsic non-linearity of duality mappings, however, introduces new challenges in the definition of a backward step in terms of the proximal operator of g . In [4, 16, 17], forward-backward algorithms have been proposed to solve minimization problems in the form (P) and defined in a reflexive, strictly convex and smooth Banach space \mathcal{X} , by means of suitably defined notions of duality mappings and proximal operators.

In particular, in [4] Bredies introduces the following iterative procedure for smooth functions f with $(p-1)$ -Hölder continuous gradient ∇f on bounded sets with $1 < p \leq 2$ with constant $K > 0$ and step-sizes chosen as $0 < \bar{\tau} \leq \tau_k \leq \frac{p(1-\delta)}{K}$, with $0 < \delta < 1$:

$$x^{k+1} \in \arg \min_{x \in \mathcal{X}} \frac{1}{p} \|x - x^k\|_{\mathcal{X}}^p + \tau_k \langle \nabla f(x^k), x \rangle + \tau_k g(x), \quad (1)$$

where $\langle u^*, x \rangle = \langle x, u^* \rangle = u^*(x) \in \mathbb{R}$ denotes the duality pairing, for $u^* \in \mathcal{X}^*$ and $x \in \mathcal{X}$. It is interesting to notice that, for $g \equiv 0$, the updating rule (1) can be also written as

$$0 \in \partial \left(\frac{1}{p} \|\cdot - x^k\|_{\mathcal{X}}^p + \tau_k \langle \nabla f(x^k), \cdot \rangle \right) (x^{k+1}),$$

where, for a convex function $h : \mathcal{X} \rightarrow \mathbb{R}$ $\partial h(w)$ denotes the subdifferential of h evaluated at z . By linearity, this leads to the following equivalent inclusions

$$\begin{aligned} 0 &\in \mathbf{J}_{\mathcal{X}}^p(x^{k+1} - x^k) + \tau_k \nabla f(x^k) \\ x^k - x^{k+1} &\in \mathbf{J}_{\mathcal{X}^*}^{p'}(\tau_k \nabla f(x^k)) \\ x^{k+1} &\in x^k - \tau_k \mathbf{J}_{\mathcal{X}^*}^{p'}(\nabla f(x^k)), \end{aligned} \quad (2)$$

where $\mathbf{J}_{\mathcal{X}}^p : \mathcal{X} \rightarrow \mathcal{X}^*$, $\mathbf{J}_{\mathcal{X}}^p = \partial \left(\frac{1}{p} \|\cdot\|_{\mathcal{X}}^p \right)$, is the so-called p -duality map of the (primal) space \mathcal{X} and $\mathbf{J}_{\mathcal{X}^*}^{p'} : \mathcal{X}^* \rightarrow \mathcal{X}$ is its inverse, that is, the p' -duality map of the dual space \mathcal{X}^* , with p' the Hölder conjugate of p [8]. Since $\mathbf{J}_{\mathcal{X}^*}^{p'}(\nabla f(x^k)) \in \mathcal{X}$, the latter inclusion shows that (1) reduces to a gradient descent step performed in the primal space \mathcal{X} . It can thus be seen a generalisation of the so-called primal method described in [23] to solve inverse problems in the form $Ax = y \in \mathcal{Y}$, where $A : \mathcal{X} \rightarrow \mathcal{Y}$ is a linear operator between two Banach spaces $\mathcal{X} \ni x$ and \mathcal{Y} , via the minimisation of the residual functional $f : \mathcal{X} \rightarrow \mathbb{R}$ defined as $f(x) = \frac{1}{p} \|Ax - y\|_{\mathcal{Y}}^p$. Such interpretation is not obvious from (1), where forward and backward steps are defined as one single minimization problem, so that they cannot be distinguished. Moreover, it shows the tight link between proximal-gradient schemes and regularization theory in Banach spaces [24, 23], notions that relate to the fields of optimization/convex analysis and functional analysis, respectively.

In [16], the authors considered a different forward-backward splitting algorithm defined in Banach spaces, which involves a proximal step defined in terms of Bregman distance as follows

$$x^{k+1} = \arg \min_{x \in \mathcal{X}} \left\{ \frac{1}{p'} \|x^k\|_{\mathcal{X}}^p + \frac{1}{p} \|x\|_{\mathcal{X}}^p - \langle \mathbf{J}_{\mathcal{X}}^p(x^k), x \rangle + \tau_k \langle \nabla f(x^k), x \rangle + \tau_k g(x) \right\} \quad (3)$$

with f as above and the step-sizes τ_k here satisfy $0 < \bar{\tau} \leq \tau_k \leq \frac{pc}{K}$, with $c > 0$ such that

$$\langle \mathbf{J}_{\mathcal{X}}^p(x) - \mathbf{J}_{\mathcal{X}}^p(y), x - y \rangle \geq c \max \left\{ \|x - y\|_{\mathcal{X}}^p, \|\mathbf{J}_{\mathcal{X}}^p(x) - \mathbf{J}_{\mathcal{X}}^p(y)\|_{\mathcal{X}^*}^{p'} \right\}, \quad \forall x, y \in \mathcal{X}.$$

In this second case, the algorithm requires the computation of the forward step in the dual space. Similarly as before, for $g \equiv 0$ we notice that the updating rule (3) can now be written as

$$0 \in \partial \left(\frac{1}{p'} \|x^k\|_{\mathcal{X}}^p + \frac{1}{p} \|\cdot\|_{\mathcal{X}}^p - \langle \mathbf{J}_{\mathcal{X}}^p(x^k), \cdot \rangle + \tau_k \langle \nabla f(x^k), \cdot \rangle \right) (x^{k+1}),$$

so that we recover a generalisation of the so-called dual method [23]

$$\mathbf{J}_{\mathcal{X}}^p(x^{k+1}) \in \mathbf{J}_{\mathcal{X}}^p(x^k) - \tau_k \nabla f(x^k) \quad x^{k+1} = \mathbf{J}_{\mathcal{X}^*}^{p'} \left(\mathbf{J}_{\mathcal{X}}^p(x^{k+1}) \right), \quad (4)$$

introduced in [24] for $f(x) = \frac{1}{p} \|Ax - y\|_{\mathcal{Y}}^p$, where, differently from (1), the forward step is now computed in \mathcal{X}^* , since we have $\mathbf{J}_{\mathcal{X}}^p(x^k), \nabla f(x^k) \in \mathcal{X}^*$.

Algorithms (1) and (3) have been proposed for reflexive, strictly convex and smooth Banach spaces. Hence, they can a priori be applied to solve (P) in $L^{p(\cdot)}(\Omega)$ whenever $1 < p(x) < +\infty$, for a.e. $x \in \Omega$. However, the definition of $\|\cdot\|_{L^{p(\cdot)}}$ and of the associated duality mapping makes their use impracticable in real applications, since both are not (domain additively) separable, differently from what happens, for instance, in $L^p(\Omega)$ spaces. To overcome this issue, we propose here to replace the role played by the norm naturally appearing in the definition of the proximal operator by the modular functions defined by:

$$\bar{\rho}_{p(\cdot)}(x) := \int_{\Omega} \frac{1}{p(t)} |x(t)|^{p(t)} dt, \quad \rho_{p(\cdot)}(x) := \int_{\Omega} |x(t)|^{p(t)} dt,$$

for $x \in L^{p(\cdot)}(\Omega)$. Due to their additive separability, these functions allow for efficient computations of proximal points.

On these grounds, inspired by [4] we study an iterative proximal-gradient algorithm where whose iteration for $k \geq 0$ reads:

$$x^{k+1} = \arg \min_{x \in L^{p(\cdot)}(\Omega)} \bar{\rho}_{p(\cdot)}(x - x^k) + \tau_k \langle \nabla f(x^k), x \rangle + \tau_k g(x), \quad (5)$$

for a proper choice of the sequence $\tau_k > 0$. Following [16], we then propose another proximal-gradient algorithm whose update reads for $k \geq 0$:

$$x^{k+1} = \arg \min_{u \in L^{p(\cdot)}(\Omega)} \rho_{p(\cdot)}(u) - \langle \mathbf{J}_{\rho_{p(\cdot)}}(x^k), u \rangle + \tau_k \langle \nabla f(x^k), u \rangle + \tau_k g(u),$$

where $\mathbf{J}_{\rho_{p(\cdot)}}$ stands for the gradient of the modular $\rho_{p(\cdot)}$. For both algorithms, we prove convergence in function values and provide an interpretation as primal and dual algorithms, respectively. We next show how the intrinsic space-variant properties of the underlying space $L^{p(\cdot)}(\Omega)$ correspond to the use of adaptive non-standard thresholding functions which turn out to be a flexible tool in the solution of exemplar deconvolution and denoising problems, as we show in the numerical section. Finally, a numerical verification of the algorithmic convergence rates, in comparison with standard Hilbert and $L^p(\Omega)$ -type algorithms is performed.

Structure of the paper We first review in Section 2 some basic notions on variable exponent Lebesgue spaces theory. Then, in Section 3 we carefully analyze the convergence properties of algorithm (5). We provide an analogous study in Section 4 for algorithm 1. Their relationships with primal and dual methods of regularization theory in Banach spaces are addressed in the Section 5. Finally, we provide examples on the application of the proposed methods along with some numerical results in the concluding Section 6.

2 Preliminaries on variable exponent Lebesgue spaces

We introduce in this section some definitions, basic rationales and key concepts of $L^{p(\cdot)}(\Omega)$ spaces. For a thorough survey of these spaces, see [12, 11].

2.1 Modular and Luxemburg norm

Let $\mathcal{P}(\Omega) := \{p(\cdot) : \Omega \rightarrow [1, +\infty] \mid p(\cdot) \text{ is Lebesgue measurable}\}$ denote the set of all the exponents. Given $p(\cdot) \in \mathcal{P}(\Omega)$, we denote the essential infimum and essential supremum of $p(\cdot)$ by

$$p_- := \operatorname{ess\,inf}_{u \in \Omega} p(u) \quad \text{and} \quad p_+ := \operatorname{ess\,sup}_{u \in \Omega} p(u).$$

For the sake of simplicity, we assume in the following $1 < p_- \leq p_+ < +\infty$ (for the general case, see [12]). We will denote by $\mathcal{F}(\Omega)$ the set of all Lebesgue measurable functions $x : \Omega \rightarrow \mathbb{R} \cup \{+\infty\}$. In order to characterize spaces $L^{p(\cdot)}(\Omega)$, it is first necessary to introduce the key concept of modular function.

Definition 2.1. *Given $p(\cdot) \in \mathcal{P}(\Omega)$ with $p_+ < +\infty$, the function $\rho_{p(\cdot)} : \mathcal{F}(\Omega) \rightarrow [0, +\infty]$ defined by*

$$\rho_{p(\cdot)}(x) = \int_{\Omega} |x(t)|^{p(t)} dt \quad (6)$$

is called modular associated to the exponent function $p(\cdot)$. An alternative definition of modular function consists in considering $\bar{\rho}_{p(\cdot)} : \mathcal{F}(\Omega) \rightarrow [0, +\infty]$ defined by

$$\bar{\rho}_{p(\cdot)}(x) = \int_{\Omega} \frac{1}{p(t)} |x(t)|^{p(t)} dt. \quad (7)$$

We will refer to (6) and (7) as modular functions, as needed. Notice that $\rho_{p(\cdot)}(x)$ is the generalization of the p -power $\|x\|_p^p$ in $L^p(\Omega)$ with constant exponent $p \in (1, +\infty)$. Similarly, $\bar{\rho}_{p(\cdot)}(x)$ generalizes the quantity $\frac{1}{p} \|x\|_p^p$. The modular allows to characterize the variable exponent space $L^{p(\cdot)}$ and to define its norm, in the general framework of the Luxemburg norms of Orlicz spaces [12].

Definition 2.2. The space $L^{p(\cdot)}(\Omega)$ is the set of functions $x \in \mathcal{F}(\Omega)$ such that

$$\rho_{p(\cdot)}\left(\frac{x}{\lambda}\right) \leq 1,$$

for some $\lambda > 0$. For any $x \in L^{p(\cdot)}(\Omega)$, we define $\|\cdot\|_{L^{p(\cdot)}(\Omega)} : L^{p(\cdot)}(\Omega) \rightarrow \mathbb{R}$ as

$$\|x\|_{L^{p(\cdot)}} := \inf \left\{ \lambda > 0 : \rho_{p(\cdot)}\left(\frac{x}{\lambda}\right) \leq 1 \right\}. \quad (8)$$

Theorem 2.1. [12] The function $\|\cdot\|_{L^{p(\cdot)}(\Omega)}$ defined in (8) is a norm on $L^{p(\cdot)}(\Omega)$ and the space $(L^{p(\cdot)}(\Omega), \|\cdot\|_{L^{p(\cdot)}(\Omega)})$ is a Banach space.

It is interesting to point out that if $p(\cdot) \equiv p \in (1, +\infty)$, the classical notion of norm $\|x\|_p = \left(\int_{\Omega} |x(t)|^p dt\right)^{1/p}$ in $L^p(\Omega)$ can be retrieved since:

$$\rho_{p(\cdot)}\left(\frac{x}{\lambda}\right) = \rho_p\left(\frac{x}{\lambda}\right) = \frac{1}{\lambda^p} \rho_p(x) = \frac{1}{\lambda^p} \|x\|_p^p$$

so that the infimum in (8) is equal to $\|x\|_p$. We remark that the computation of the p -radical of the integral is necessary to ensure the homogeneity property $\|\alpha x\|_p = |\alpha| \|x\|_p$, for any $\alpha \in \mathbb{C}$. With a variable exponent, such computation is obviously not possible and in turn the 1-d minimization problem (8) has to be solved. However, the quantity (8) can be bounded by the p_- and p_+ radicals of the modular. At a certain extent, one can thus think of the norm as \tilde{p} -radical (which depends on x) of the modular with $p_- \leq \tilde{p} \leq p_+$, as stated by the following Proposition.

Proposition 2.1. [12, Lemma 3.2.5] Let $p(\cdot) \in \mathcal{P}(\Omega)$ with $p_+ < +\infty$.

- (i) If $\|x\|_{L^{p(\cdot)}} > 1$, then $\rho_{p(\cdot)}(x)^{1/p_+} \leq \|x\|_{L^{p(\cdot)}} \leq \rho_{p(\cdot)}(x)^{1/p_-}$.
- (ii) If $0 < \|x\|_{L^{p(\cdot)}} \leq 1$, then $\rho_{p(\cdot)}(x)^{1/p_-} \leq \|x\|_{L^{p(\cdot)}} \leq \rho_{p(\cdot)}(x)^{1/p_+}$.

An important consequence of this Proposition is that, when $p_+ < +\infty$, then $\rho_{p(\cdot)}(x) < +\infty \iff \|x\|_{L^{p(\cdot)}} < +\infty$. This equivalence is obvious in $L^p(\Omega)$ spaces, but it does not always hold true in $L^{p(\cdot)}(\Omega)$. A classical example is the following, where $p_+ = +\infty$. Consider $\Omega = [1, +\infty)$, $x(t) = 1$ and $p(t) = t$, $\forall t \in \Omega$. Then, $\rho_{p(\cdot)}(x) = \int_1^{+\infty} 1 dt = +\infty$, but for all $\lambda > 1$ there holds $\rho_{p(\cdot)}\left(\frac{x}{\lambda}\right) = \int_1^{+\infty} \frac{1}{\lambda^t} dt = \frac{1}{\lambda \log(\lambda)} < +\infty$, thus $\|x\|_{L^{p(\cdot)}} < +\infty$ (in particular, $\|x\|_{L^{p(\cdot)}} \approx 1.763$). In $L^{p(\cdot)}(\Omega)$, modular and norm are thus truly different objects. However, the following norm-modular equivalence can be proved.

Proposition 2.2. [12, Lemma 3.2.4] Let $x \in L^{p(\cdot)}(\Omega)$ and $p(\cdot) \in \mathcal{P}(\Omega)$. Then:

- (i) $\|x\|_{L^{p(\cdot)}} \leq 1$ and $\rho_{p(\cdot)}(x) \leq 1$ are equivalent.
- (ii) If $\|x\|_{L^{p(\cdot)}} \leq 1$, then $\rho_{p(\cdot)}(x) \leq \|x\|_{L^{p(\cdot)}}$.
- (iii) If $\|x\|_{L^{p(\cdot)}} > 1$, then $\rho_{p(\cdot)}(x) \geq \|x\|_{L^{p(\cdot)}}$.

Lemma 2.1. [12, Lemma 3.4.2] If $p_+ < +\infty$, the following norm-modular unit ball properties hold:

- (i) $\|x\|_{L^{p(\cdot)}} < 1$ and $\rho_{p(\cdot)}(x) < 1$ are equivalent.
- (ii) $\|x\|_{L^{p(\cdot)}} = 1$ and $\rho_{p(\cdot)}(x) = 1$ are equivalent.

If $1 < p_- \leq p_+ < +\infty$, then the following useful properties hold true.

Theorem 2.2. [12, Theorem 3.4.7, Theorem 3.4.9] Given $p(\cdot)$ such that $1 < p_- \leq p_+ < +\infty$, then $L^{p(\cdot)}(\Omega)$ is reflexive and uniformly convex.

Theorem 2.3. [14, Lemma 1] Given $p(\cdot)$ such that $1 < p_- \leq p_+ < +\infty$, then $L^{p(\cdot)}(\Omega)$ is smooth.

2.2 Dual space, duality mappings, separability

In $L^p(\Omega)$ spaces, the isometric isomorphism between the space $(L^p(\Omega))^*$ and $L^{p'}(\Omega)$, with p' being the Hölder conjugate of p , has a key role in devising regularization algorithms [23]. In $L^{p(\cdot)}(\Omega)$ spaces, such isomorphism does not hold true, as we briefly summarize in this subsection. For a comprehensive description of these arguments, see [12].

Definition 2.3. Let $G : L^{p(\cdot)}(\Omega) \rightarrow \mathbb{R}$ be a linear functional. G is bounded if

$$\sup_{u \in L^{p(\cdot)}(\Omega), \|u\|_{L^{p(\cdot)}} \leq 1} |G(u)| < +\infty.$$

The dual space of $L^{p(\cdot)}(\Omega)$ can thus be defined as the set

$$(L^{p(\cdot)}(\Omega))^* = \{G : L^{p(\cdot)}(\Omega) \rightarrow \mathbb{R} : G \text{ is linear and bounded}\},$$

which is a Banach space with the norm

$$\|G\|_{(L^{p(\cdot)}(\Omega))^*} := \sup_{u \in L^{p(\cdot)}(\Omega), \|u\|_{L^{p(\cdot)}} \leq 1} |G(u)|.$$

For $1 < p_- \leq p_+ < +\infty$, the Hölder conjugate of $p(\cdot)$ is a Lebesgue measurable function $p'(\cdot) \in \mathcal{P}(\Omega)$ such that

$$\frac{1}{p(t)} + \frac{1}{p'(t)} = 1 \quad \text{a.e. in } \Omega.$$

For any $z \in L^{p(\cdot)}(\Omega)$, it can be shown that there exists a unique $G \in (L^{p(\cdot)}(\Omega))^*$ such that

$$G(u) = \int_{\Omega} z(t)u(t)dt \quad \forall u \in L^{p(\cdot)}(\Omega).$$

Thus, we can denote unambiguously G as G_z .

Definition 2.4. [12, Definition 2.7.1] *The associate space of $L^{p(\cdot)}(\Omega)$, denoted by $\mathcal{A}(L^{p(\cdot)}(\Omega))$, is the space of functions $z \in L^{p'(\cdot)}(\Omega)$ such that*

$$\|z\|'_{p'(\cdot)} := \sup_{u \in L^{p(\cdot)}(\Omega), \|u\|_{L^{p(\cdot)}} \leq 1} \int_{\Omega} |z(t)||u(t)|dt < +\infty. \quad (9)$$

It can be proved that $\|\cdot\|'_{p'(\cdot)}$ is a norm on $\mathcal{A}(L^{p(\cdot)}(\Omega))$. Note that since

$$\begin{aligned} \|G_z\|_{(L^{p(\cdot)}(\Omega))^*} &= \sup_{u \in L^{p(\cdot)}(\Omega), \|u\|_{L^{p(\cdot)}} \leq 1} |G_z(u)| = \\ &= \sup_{u \in L^{p(\cdot)}(\Omega), \|u\|_{L^{p(\cdot)}} \leq 1} \int_{\Omega} |z(t)||u(t)|dt = \|z\|'_{p'(\cdot)}, \end{aligned} \quad (10)$$

requiring that (9) is finite is equivalent to requiring that the linear operator G_z is bounded. Thus, there exists an isometric embedding between $\mathcal{A}(L^{p(\cdot)}(\Omega))$ and $(L^{p(\cdot)}(\Omega))^*$.

Proposition 2.3. [12, Corollary 3.2.14] *For all $z \in \mathcal{A}(L^{p(\cdot)}(\Omega))$, there holds*

$$\frac{1}{2}\|z\|_{L^{p(\cdot)}} \leq \|z\|'_{p'(\cdot)} \leq 2\|z\|_{L^{p(\cdot)}},$$

and the bounds are optimal.

Remark 2.1. *From the Proposition above combined with (10), it follows that $(L^{p(\cdot)}(\Omega))^*$ and $L^{p'(\cdot)}(\Omega)$ are not isometrically isomorphic.*

A key role in the definition of iterative schemes in Banach spaces is the concept of duality mapping. A duality map allows to associate an element of a Banach space \mathcal{X} with a specific element of its dual space \mathcal{X}^* .

Definition 2.5. [8] *Let \mathcal{X} be a Banach space and $r > 1$. Then the duality map $\mathbf{J}_{\mathcal{X}}^r$ with gauge function $t \mapsto t^{r-1}$ is the operator $\mathbf{J}_{\mathcal{X}}^r : \mathcal{X} \rightarrow 2^{\mathcal{X}^*}$ such that*

$$\mathbf{J}_{\mathcal{X}}^r(x) = \{x^* \in \mathcal{X}^* \mid x^*(x) = \langle x^*, x \rangle = \|x\|_{\mathcal{X}}\|x^*\|_{\mathcal{X}^*}, \|x^*\|_{\mathcal{X}^*} = \|x\|_{\mathcal{X}}^{r-1}\} \quad \forall x \in \mathcal{X}.$$

If \mathcal{X} is smooth, then the duality map is single valued, that is $\mathbf{J}_{\mathcal{X}}^r : \mathcal{X} \rightarrow \mathcal{X}^*$. In addition, if \mathcal{X} is a Hilbert space \mathcal{H} , by virtue of Riesz Theorem, the duality map reduces to the identity operator for $r = 2$, that is $\mathbf{J}_{\mathcal{H}}^2(x) = x$, where the isometric isomorphism between \mathcal{H} and \mathcal{H}^* has been implicitly considered.

In general, the following result gives a more intuitive interpretation of the important role played by duality mappings in the definition of minimization schemes.

Theorem 2.4 ((Asplund) [8]). *Let \mathcal{X} a Banach space and $r > 1$. The duality map $\mathbf{J}_{\mathcal{X}}^r$ is the subdifferential of the convex functional $h : \mathcal{X} \rightarrow \mathbb{R}$ defined as $h(x) = \frac{1}{r}\|x\|_{\mathcal{X}}^r$, that is*

$$\mathbf{J}_{\mathcal{X}}^r = \partial h = \partial \left(\frac{1}{r} \|\cdot\|_{\mathcal{X}}^r \right).$$

Thanks to this Theorem, the duality maps are thus monotone operators.

In [13, 14] the authors proved that $\|\cdot\|_{L^{p(\cdot)}}$ is Gateaux-differentiable, for any $x \neq 0$, providing an analytical expression for its Gateau derivative. They also show in [20, 21] that it is Fréchet differentiable too, for any $x \neq 0$. From these results, it follows that $\frac{1}{r}\|\cdot\|_{L^{p(\cdot)}}^r$ for $r > 1$ is Fréchet differentiable for any $x \in \mathcal{X}$, hence its Gateaux derivative is the duality map with gauge function $t \mapsto t^{r-1}$. By simply computing the Gateaux derivative of $\frac{1}{r}\|\cdot\|_{L^{p(\cdot)}}^r$ following similar arguments of [13, 14], it is thus possible to obtain the following formula for $\mathbf{J}_{L^{p(\cdot)}}^r$.

Proposition 2.4. *For each $x, u \in L^{p(\cdot)}(\Omega)$ with $1 < p_- \leq p_+ < +\infty$, and for any $r \in (1, +\infty)$, the duality mapping $\mathbf{J}_{L^{p(\cdot)}}^r : L^{p(\cdot)}(\Omega) \rightarrow (L^{p(\cdot)}(\Omega))^*$ is a linear operator with expression:*

$$\langle \mathbf{J}_{L^{p(\cdot)}}^r(x), u \rangle = \frac{1}{\int_{\Omega} \frac{p(t)|x(t)|^{p(t)}}{\|x\|_{L^{p(\cdot)}}^{p(t)}} dt} \int_{\Omega} \frac{1}{\|x\|_{L^{p(\cdot)}}^{p(t)-r}} p(t) \text{sign}(x(t)) |x(t)|^{p(t)-1} u(t) dt. \quad (11)$$

It is easy to check that, if $p(\cdot) \equiv p$ is constant, with $1 < p < +\infty$, then $\mathbf{J}_{L^{p(\cdot)}}^r$ coincides with \mathbf{J}_p^r , the duality map of classical (i.e. constant exponent) Lebesgue spaces:

$$\langle \mathbf{J}_p^r(x), u \rangle = \|x\|_p^{r-p} \int_{\Omega} \text{sign}(x(t)) |x(t)|^{p-1} u(t) dt.$$

Proposition 2.5. *For any $x, u \in L^{p(\cdot)}(\Omega)$, with $p_+ < +\infty$, the functions $\rho_{p(\cdot)}(\cdot)$ and $\bar{\rho}_{p(\cdot)}(\cdot)$ are Gateaux differentiable, with derivatives*

$$\langle \mathbf{J}_{\rho_{p(\cdot)}}(x), u \rangle = \int_{\Omega} p(t) \text{sign}(x(t)) |x(t)|^{p(t)-1} u(t) dt$$

$$\langle \mathbf{J}_{\bar{\rho}_{p(\cdot)}}(x), u \rangle = \int_{\Omega} \text{sign}(x(t)) |x(t)|^{p(t)-1} u(t) dt.$$

Proof. Let $x, u \in L^{p(\cdot)}(\Omega)$. The Gateaux derivative of $\rho_{p(\cdot)}$ at x along direction u is given by

$$\lim_{t \rightarrow 0} \frac{\rho_{p(\cdot)}(x + tu) - \rho_{p(\cdot)}(x)}{t} = \left[\frac{\partial}{\partial t} \rho_{p(\cdot)}(x + tu) \right]_{t=0}.$$

First of all, note that

$$\rho_{p(\cdot)}(x + tu) = \int_{\Omega} |x(s) + tu(s)|^{p(s)} ds < +\infty.$$

Since $x, u \in L^{p(\cdot)}(\Omega)$, then $x + tu \in L^{p(\cdot)}(\Omega)$ and consequently $\|x + tu\|_{L^{p(\cdot)}} < +\infty$ by definition of $L^{p(\cdot)}(\Omega)$ space. Since $p_+ < +\infty$ and by Proposition 2.1, $\|x + tu\|_{L^{p(\cdot)}} < +\infty$ implies $\rho_{p(\cdot)}(x + tu) < +\infty$. We can thus compute the partial derivative of $\rho_{p(\cdot)}(x + tu)$ with respect to t by “differentiating under the integral sign”. To do so, we first need to verify the regularity of the integrand function. Let $f : \Omega \times (-1, 1) \rightarrow \mathbb{R}$ be defined by

$$f(s, t) := |x(s) + tu(s)|^{p(s)}, \quad s \in \Omega, \quad t \in (-1, 1).$$

By direct computations, we obtain

$$\frac{\partial f}{\partial t}(s, t) = p(s) |x(s) + tu(s)|^{p(s)-1} \text{sign}(x(s) + tu(s)) u(s)$$

and, since $|t| < 1$,

$$\begin{aligned} \left| \frac{\partial f}{\partial t}(s, t) \right| &\leq p_+ |x(s) + tu(s)|^{p(s)-1} |u(s)| \\ &\leq p_+ |x(s) + tu(s)|^{p(s)-1} (|u(s)| + |x(s)|) \\ &\leq p_+ (|u(s)| + |x(s)|)^{p(s)} =: g(s), \end{aligned}$$

with $g(s)$ integrable. We conclude thanks to the dominated convergence theorem. \square

Remark 2.2. *We stress that although $\mathbf{J}_{\rho_{p(\cdot)}}$ and $\mathbf{J}_{\bar{\rho}_{p(\cdot)}}$ are not duality mappings, we adopt nonetheless the same notation for consistency.*

It is interesting to observe the following property of the modular function and its gradient, which will be used in the following convergence analysis.

Lemma 2.2. *For any $x \in L^{p(\cdot)}(\Omega)$,*

$$\langle \mathbf{J}_{\bar{\rho}_{p(\cdot)}}(x), x \rangle = \rho_{p(\cdot)}(x). \quad (12)$$

Proof. By direct computation:

$$\langle \mathbf{J}_{\bar{\rho}_{p(\cdot)}}(x), x \rangle = \int_{\Omega} \text{sign}(x(t)) |x(t)|^{p(t)-1} x(t) dt = \rho_{p(\cdot)}(x). \quad \square$$

Note that this is the analog of a general property of duality mappings in Banach spaces. Indeed, by Definition 2.5, if \mathcal{X} is a smooth Banach space, then for any $r > 1$ and $x \in \mathcal{X}$ there holds

$$\langle \mathbf{J}_{\mathcal{X}}^r(x), x \rangle = \|x\|_{\mathcal{X}} \|x^*\|_{\mathcal{X}^*} = \|x\|_{\mathcal{X}} \|x\|_{\mathcal{X}}^{r-1} = \|x\|_{\mathcal{X}}^r.$$

As it will be better explained in Section 5, it is handy having functionals and operators defined in Banach spaces that are *separable*, that is, their global computation can be decomposed into the sum of low-dimensional functionals. To make this property more precise, we consider the following definition of domain additively separability.

Definition 2.6. Let \mathcal{X} a functional Banach space on Ω . An operator $S : \mathcal{X} \rightarrow \mathcal{X}^*$ or a functional $S : \mathcal{X} \rightarrow \mathbb{R}$ is said to be domain additively separable, if, for any finite family of Lebesgue measurable subsets $(\Omega_i)_{i=1}^n$ of Ω such that $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$, and $\Omega = \bigcup_{i=1}^n \Omega_i$, there holds that

$$S(x) = \sum_{i=1}^n S(\chi_i x)$$

for any $x \in \mathcal{X}$, where $\chi_i \in \mathcal{X}$ is the characteristic function of Ω_i , that is $\chi_i(t) = 1$ for $t \in \Omega_i$, and $\chi_i(t) = 0$ for $t \notin \Omega_i$.

In the following, domain additively separability will be often simply referred to as separability. It is quite evident that, in conventional $L^p(\Omega)$ spaces, the norm functional $\|\cdot\|_p^p$, as well as the operator $\mathbf{J}_p^p(\cdot)$, are domain additively separable, since, for any suitable family of subsets $(\Omega_i)_{i=1}^n$, there holds

$$\|x\|_p^p = \sum_{i=1}^n \|\chi_i x\|_p^p \quad \text{and} \quad \langle \mathbf{J}_p^p(x), u \rangle = \sum_{i=1}^n \langle \mathbf{J}_p^p(\chi_i x), u \rangle = \langle \sum_{i=1}^n \mathbf{J}_p^p(\chi_i x), u \rangle.$$

On the contrary, norms and duality maps in variable exponent spaces are not separable.

Lemma 2.3. *The norm and the duality mapping in $L^{p(\cdot)}(\Omega)$ are not domain additively separable in the sense of Definition 2.6.*

Proof. It is quite evident that the Luxemburg norm (8) requires the solution of a 1D minimization problem on the entire domain Ω , which, in general, cannot be divided into the solutions onto single sets of the partition, that is, $\|x\|_{L^{p(\cdot)}} \neq \sum_{i=1}^n \|\chi_i x\|_{L^{p(\cdot)}(\Omega)}$. As the duality mapping is concerned, the two norms in the denominators of $\mathbf{J}_{L^{p(\cdot)}}^r$ (11) show that its computation cannot be decomposed into the computation of n integrals involving only the restriction of the function x onto single sets of the partition, or, in other words, $\mathbf{J}_{L^{p(\cdot)}(\Omega)}^r(x) \neq \sum_{i=1}^n \mathbf{J}_{L^{p(\cdot)}(\Omega)}^r(\chi_i x)$. \square

The modular functions introduced in Definition 6 as well as their gradients turn out instead to satisfy the separability property.

Lemma 2.4. *The modular functions in $L^{p(\cdot)}(\Omega)$ and their gradients are domain additively separable, in the sense of Definition 2.6.*

Proof. We consider the modular function $\bar{\rho}_{p(\cdot)}(x)$ defined in (7) (for (6), the proof is similar). By direct computation, by the linearity property of the integral w.r.t. the integration domain, we have

$$\begin{aligned} \bar{\rho}_{p(\cdot)}(x) &= \int_{\Omega} \frac{1}{p(t)} |x(t)|^{p(t)} dt = \sum_{i=1}^n \int_{\Omega_i} \frac{1}{p(t)} |x(t)|^{p(t)} dt \\ &= \sum_{i=1}^n \int_{\Omega} \frac{1}{p(t)} |\chi_i(t) x(t)|^{p(t)} dt = \sum_{i=1}^n \bar{\rho}_{p(\cdot)}(\chi_i x). \end{aligned}$$

Similarly, for $\mathbf{J}_{\bar{\rho}_{p(\cdot)}}$, we can write

$$\begin{aligned} \langle \mathbf{J}_{\bar{\rho}_{p(\cdot)}}(x), u \rangle &= \int_{\Omega} \text{sign}(x(t)) |x(t)|^{p(t)-1} u(t) dt = \sum_{i=1}^n \int_{\Omega_i} \text{sign}(x(t)) |x(t)|^{p(t)-1} u(t) dt \\ &= \sum_{i=1}^n \int_{\Omega} \text{sign}(x(t)) |\chi_i(t) x(t)|^{p(t)-1} u(t) dt = \sum_{i=1}^n \langle \mathbf{J}_{\bar{\rho}_{p(\cdot)}}(\chi_i x), u \rangle \\ &= \langle \sum_{i=1}^n \mathbf{J}_{\bar{\rho}_{p(\cdot)}}(\chi_i x), u \rangle, \end{aligned}$$

which concludes the proof. \square

3 A modular-proximal gradient algorithm

In this section, we propose and analyze an iterative procedure to solve the minimization problem (P). We set $\bar{\phi} := \inf_{x \in L^{p(\cdot)}(\Omega)} \phi(x)$, and define $\text{Sol}(P)$ as $\text{Sol}(P) := \{x \in L^{p(\cdot)}(\Omega) : \phi(x) = \bar{\phi}\} \neq \emptyset$. We consider the following two assumptions.

Assumption 3.1. *The exponent function $p(\cdot)$ is such that $1 < p_- \leq p_+ \leq 2$.*

Assumption 3.2. $\nabla f : L^{p(\cdot)}(\Omega) \rightarrow L^{p(\cdot)}(\Omega)^*$ is $(p-1)$ Hölder-continuous with exponent $p_+ \leq p \leq 2$ and constant $K > 0$, i.e.:

$$\|\nabla f(u) - \nabla f(v)\|_{(L^{p(\cdot)})^*} \leq K \|u - v\|_{L^{p(\cdot)}}^{p-1} \quad \forall u, v \in L^{p(\cdot)}(\Omega).$$

Algorithm 1 Modular-proximal gradient algorithm in $L^{p(\cdot)}(\Omega)$ spaces

Parameters: $\rho \in (0, 1)$, $\{\tau_k\}_k$ s.t.

$$0 < \bar{\tau} \leq \tau_k \leq \frac{p(1-\delta)}{K} \quad \text{with } 0 < \delta < 1. \quad (13)$$

Initialization: Start with $x^0 \in L^{p(\cdot)}(\Omega)$.

FOR $k = 0, 1, \dots$ REPEAT

FOR $i = 0, 1, \dots$ REPEAT

1. Set $\tau_k = \rho^i \tau_k$.
2. Compute the next iterate as:

$$x^{k+1} = \arg \min_{x \in L^{p(\cdot)}(\Omega)} \bar{\rho}_{p(\cdot)}(x - x^k) + \tau_k \langle \nabla f(x^k), x \rangle + \tau_k g(x). \quad (14)$$

UNTIL $\rho_{p(\cdot)}(x^k - x^{k+1}) < 1$

UNTIL convergence

The first modular-proximal gradient algorithm we propose is reported in Algorithm 1, as pseudocode. The inner loop is needed to select at each iteration $k \geq 0$ a sufficiently small step-size τ_k such that $\rho_{p(\cdot)}(x^k - x^{k+1}) < 1$, which is required in the following convergence analysis as we will see in the proofs of Proposition 3.3 and Lemma 3.2. It should be thought as a backtracking-like procedure affecting more the first algorithmic iterations where the quantity $\rho_{p(\cdot)}(x^k - x^{k+1})$ is likely to be large.

We start our analysis discussing the well-definition of the step (14).

Proposition 3.1. For each $x \in L^{p(\cdot)}(\Omega)$, $v^* \in L^{p(\cdot)}(\Omega)^*$ and $\tau > 0$, the problem

$$\arg \min_{u \in L^{p(\cdot)}(\Omega)} \bar{\rho}_{p(\cdot)}(u - x) + \tau \langle v^*, u \rangle + \tau g(u) \quad (15)$$

has a unique solution.

Proof. Let $\tau > 0$. Note that when $\|u - x\|_{L^{p(\cdot)}} > 1$, by Proposition 2.1(i) there holds:

$$\bar{\rho}_{p(\cdot)}(u - x) \geq \frac{1}{p_+} \rho_{p(\cdot)}(u - x) \geq \frac{1}{p_+} \|u - x\|_{L^{p(\cdot)}}^{p_-}.$$

Let now $\bar{x} \in \text{Sol}(P)$. The optimality condition reads: $0 \in \nabla f(\bar{x}) + \partial g(\bar{x})$ or, equivalently, $\bar{\omega} := -\nabla f(\bar{x}) \in \partial g(\bar{x})$. By definition of subdifferential, there holds $g(u) \geq g(\bar{x}) + \langle \bar{\omega}, u - \bar{x} \rangle = g(\bar{x}) + \langle \bar{\omega}, u \rangle - \langle \bar{\omega}, \bar{x} \rangle$ for all $u \in L^{p(\cdot)}(\Omega)$. By combining such inequality with the Cauchy-Schwartz inequality, we get:

$$\begin{aligned} & \bar{\rho}_{p(\cdot)}(u - x) + \tau \langle v^*, u \rangle + \tau g(u) \\ & \geq \frac{1}{p_+} \|u - x\|_{L^{p(\cdot)}}^{p_-} + \tau \langle v^* + \bar{\omega}, u \rangle + \tau g(\bar{x}) - \tau \langle \bar{\omega}, \bar{x} \rangle \\ & \geq \|u\|_{L^{p(\cdot)}} \left[\frac{1}{p_+} \frac{\|u - x\|_{L^{p(\cdot)}}^{p_-}}{\|u\|_{L^{p(\cdot)}}} + \tau \frac{\langle v^* + \bar{\omega}, u \rangle}{\|u\|_{L^{p(\cdot)}}} + \frac{\tau g(\bar{x}) - \tau \langle \bar{\omega}, \bar{x} \rangle}{\|u\|_{L^{p(\cdot)}}} \right] \\ & \geq \|u\|_{L^{p(\cdot)}} \left[\frac{1}{p_+} \frac{\|u - x\|_{L^{p(\cdot)}}^{p_-}}{\|u\|_{L^{p(\cdot)}}} - \tau \|v^* + \bar{\omega}\|_{(L^{p(\cdot)})^*} + \tau \frac{g(\bar{x}) - \langle \bar{\omega}, \bar{x} \rangle}{\|u\|_{L^{p(\cdot)}}} \right] \geq L \|u\|_{L^{p(\cdot)}} \end{aligned}$$

for some $L > 0$ and all $u \in L^{p(\cdot)}(\Omega)$ such that $\|u\|_{L^{p(\cdot)}}$ is large enough. Note, in particular, that $\frac{1}{p_+} \frac{\|u - x\|_{L^{p(\cdot)}}^{p_-}}{\|u\|_{L^{p(\cdot)}}} \rightarrow +\infty$ as $\|u\|_{L^{p(\cdot)}} \rightarrow +\infty$ since $p_- > 1$ and x is fixed. Hence, the functional in (15) is coercive. Moreover, it is convex, proper and lower semi-continuous in $L^{p(\cdot)}(\Omega)$, which, by Theorem 2.2 is reflexive, thus at least one solution exists. Moreover, since $1 < p_- \leq p_+ \leq 2$, the functional is strictly convex, hence the solution is unique. \square

Clearly, the convergence analysis of Algorithm 1 is related to the study of fixed points of iterations (15). More precisely, we have the following result.

Proposition 3.2. The solutions of (P) coincide with the fixed points of the iteration defined by Algorithm 1.

Proof. Suppose that for some $k \geq 0$ $x^k \in \text{Sol}(P)$. Then, by optimality, $-\nabla f(x^k) \in \partial g(x^k)$. Note that u solves (14) if and only if the following inclusion holds for all $u \in L^{p(\cdot)}(\Omega)$

$$-\tau_k \nabla f(x^k) \in \mathbf{J}_{\bar{\rho}_{p(\cdot)}}(u - x^k) + \tau_k \partial g(u),$$

where $\mathbf{J}_{\bar{\rho}_{p(\cdot)}}$ is the derivative of $\bar{\rho}_{p(\cdot)}$, according to Proposition 2.5. Clearly, $u = x^k$ is a solution of (14), hence $x^{k+1} = x^k$, so x^k is a fixed point of the iteration process.

Conversely, suppose now $x^k = x^{k+1} \in L^{p(\cdot)}(\Omega)$, then

$$-\tau_k \nabla f(x^k) \in \mathbf{J}_{\bar{\rho}_{p(\cdot)}}(x^{k+1} - x^k) + \tau_k \partial g(x^{k+1}) = \tau_k \partial g(x^{k+1}) = \tau_k \partial g(x^k),$$

meaning that x^k is optimal, that is, $x^k \in \text{Sol}(\text{P})$. \square

3.1 Convergence analysis

We provide here a detailed convergence analysis of Algorithm 1 in order to provide an insight on its convergence speed in function values. Our analysis is inspired by the one conducted [4], although it relies on properties related to the modular function $\bar{\rho}_{p(\cdot)}$ rather than to the norm $\|\cdot\|_{L^{p(\cdot)}}$. As such, it relies on different arguments, as we will make precise in the following.

Lemma 3.1. *For each $u \in L^{p(\cdot)}(\Omega)$, the following inequality holds true:*

$$\left\langle \frac{\mathbf{J}_{\bar{\rho}_{p(\cdot)}}(x^k - x^{k+1})}{\tau_k}, u - x^{k+1} \right\rangle \leq g(u) - g(x^{k+1}) + \langle \nabla f(x^k), u - x^{k+1} \rangle \quad (16)$$

Moreover, the following inequality holds true

$$\rho_{p(\cdot)}(x^k - x^{k+1}) \leq \tau_k D(x^k) \quad (17)$$

where by $D(x^k)$ we denote the quantity:

$$D(x^k) := g(x^k) - g(x^{k+1}) + \langle \nabla f(x^k), x^k - x^{k+1} \rangle. \quad (18)$$

Proof. Note that x^{k+1} solves (14) if and only if

$$0 \in \mathbf{J}_{\bar{\rho}_{p(\cdot)}}(x^{k+1} - x^k) + \tau_k \nabla f(x^k) + \tau_k \partial g(x^{k+1}) \iff \frac{\mathbf{J}_{\bar{\rho}_{p(\cdot)}}(x^k - x^{k+1})}{\tau_k} - \nabla f(x^k) \in \partial g(x^{k+1}).$$

By definition of subdifferential, we thus have that, $\forall u \in L^{p(\cdot)}(\Omega)$,

$$\left\langle \frac{\mathbf{J}_{\bar{\rho}_{p(\cdot)}}(x^k - x^{k+1})}{\tau_k} - \nabla f(x^k), u - x^{k+1} \right\rangle \leq g(u) - g(x^{k+1}),$$

which, by rearranging, coincides with (16). Choosing now $u = x^k$ above and recalling (18), we get

$$\left\langle \frac{\mathbf{J}_{\bar{\rho}_{p(\cdot)}}(x^k - x^{k+1})}{\tau_k}, x^k - x^{k+1} \right\rangle \leq D(x^k).$$

Applying now (12) entails

$$\left\langle \frac{\mathbf{J}_{\bar{\rho}_{p(\cdot)}}(x^k - x^{k+1})}{\tau_k}, x^k - x^{k+1} \right\rangle = \frac{\rho_{p(\cdot)}(x^k - x^{k+1})}{\tau_k},$$

by which (17) follows directly. \square

The following proposition shows that the iteration scheme (14) leads to a descent of the functional ϕ of our minimization problem (P). This will be crucial for the following convergence analysis.

Proposition 3.3. *For every $k \geq 0$, if $\rho_{p(\cdot)}(x^k - x^{k+1}) < 1$, then the iteration defined by Algorithm 1 satisfies*

$$\phi(x^{k+1}) \leq \phi(x^k) - \left(1 - \frac{K\tau_k}{p}\right) D(x^k). \quad (19)$$

Proof. By (18), we have

$$\phi(x^k) - \phi(x^{k+1}) = D(x^k) + f(x^k) - f(x^{k+1}) - \langle \nabla f(x^k), x^k - x^{k+1} \rangle. \quad (20)$$

For the last three terms, we notice that we can write

$$\begin{aligned} & f(x^k) - f(x^{k+1}) - \langle \nabla f(x^k), x^k - x^{k+1} \rangle \\ &= \int_0^1 \langle \nabla f(x^k + t(x^{k+1} - x^k)) - \nabla f(x^k), x^k - x^{k+1} \rangle dt. \end{aligned} \quad (21)$$

By applying now the $(p-1)$ -Hölder continuity of ∇f (Assumption 3.2), we can provide an estimate of the absolute value of the right-hand side of (21) in terms of $D(x^k)$, since

$$\begin{aligned} & \left| \int_0^1 \langle \nabla f(x^k + t(x^{k+1} - x^k)) - \nabla f(x^k), x^k - x^{k+1} \rangle dt \right| \\ & \leq \int_0^1 \|\nabla f(x^k + t(x^{k+1} - x^k)) - \nabla f(x^k)\|_{(L^{p(\cdot)})^*} \|x^k - x^{k+1}\|_{L^{p(\cdot)}} dt \\ & \leq \int_0^1 K \|x^k - x^{k+1}\|_{L^{p(\cdot)}}^p t^{p-1} dt \leq \frac{K}{p} \|x^k - x^{k+1}\|_{L^{p(\cdot)}}^p \end{aligned}$$

Since $\rho_{p(\cdot)}(x^k - x^{k+1}) < 1$ by construction, by Lemma 2.1(i) there holds $\|x^k - x^{k+1}\|_{L^{p(\cdot)}} < 1$ and $\|x^k - x^{k+1}\|_{L^{p(\cdot)}} < \rho_{p(\cdot)}(x^k - x^{k+1})^{1/p_+} \leq \rho_{p(\cdot)}(x^k - x^{k+1})^{1/p}$ by Proposition 2.1(ii). Hence, by (17) we obtain

$$\begin{aligned} & \left| \int_0^1 \langle \nabla f(x^k + t(x^{k+1} - x^k)) - \nabla f(x^k), x^k - x^{k+1} \rangle dt \right| \\ & \leq \frac{K}{p} \|x^k - x^{k+1}\|_{L^{p(\cdot)}}^p \leq \frac{K}{p} \rho_{p(\cdot)}(x^k - x^{k+1}) \leq \frac{K\tau_k}{p} D(x^k), \end{aligned}$$

which concludes the proof by combining this with (20) and (21). \square

For each $k \geq 1$, let us now define for simplicity the k -th residual $r_k := \phi(x^k) - \bar{\phi}$. Note that $r_k \geq 0$ by definition. We can thus rewrite (19) as

$$r_k - r_{k+1} \geq \left(1 - \frac{K\tau_k}{p}\right) D(x^k). \quad (22)$$

Thanks to the bounds on the step-sizes τ_k , there holds $r_k - r_{k+1} \geq 0$, hence the descent of the functional $\phi = f + g$ is guaranteed.

Consider now the conjugate exponent $p'(\cdot) \in \mathcal{P}(\Omega)$. Since $1 < p(\cdot) \leq 2$ and $\frac{1}{p(t)} + \frac{1}{p'(t)} = 1$ a.e., there holds $2 \leq p'(\cdot) < +\infty$ and

$$(p')_- := \operatorname{ess\,inf}_{t \in \Omega} p'(t) = (p_+)' \quad (p')_+ := \operatorname{ess\,sup}_{t \in \Omega} p'(t) = (p_-)'$$

where by $(p_+)'$ we denote the conjugate of p_+ and by $(p_-)'$ the conjugate of p_- . The following Lemma shows that by assuming the boundedness of the sequence $(x_k)_k$, an estimate of the right hand side of (22) depending on $(p_-)'$ can be found.

Lemma 3.2. *Suppose that $(x^k)_k$ is bounded. Then, $r_k - r_{k+1} \geq c_0 r_k^{(p_-)'} with $c_0 > 0$.$*

Proof. Since $(x^k)_k$ is bounded, there exists $C_1 > 0$ such that for all $k \geq 1$ $\|x^k - \bar{x}\|_{L^{p(\cdot)}} \leq C_1$ for some $\bar{x} \in \operatorname{Sol}(P)$. The convexity of f as well as (16) with $u = \bar{x}$ gives

$$\begin{aligned} r_k &= f(x^k) + g(x^k) - f(\bar{x}) - g(\bar{x}) \\ &\leq g(x^k) - g(\bar{x}) + \langle \nabla f(x^k), x^k - \bar{x} \rangle \\ &= \langle \nabla f(x^k), x^k - x^{k+1} \rangle + g(x^k) - g(\bar{x}) + \langle \nabla f(x^k), x^{k+1} - \bar{x} \rangle \\ &= D(x^k) + g(x^{k+1}) - g(\bar{x}) + \langle \nabla f(x^k), x^{k+1} - \bar{x} \rangle \\ &\leq D(x^k) + \left\langle \frac{\mathbf{J}_{\bar{\rho}_{p(\cdot)}}(x^k - x^{k+1})}{\tau_k}, x^{k+1} - \bar{x} \right\rangle \\ &\leq D(x^k) + \tau_k^{-1} \|\mathbf{J}_{\bar{\rho}_{p(\cdot)}}(x^k - x^{k+1})\|_{(L^{p(\cdot)})^*} \|x^{k+1} - \bar{x}\|_{L^{p(\cdot)}} \\ &\leq D(x^k) + \tau_k^{-1} \|\mathbf{J}_{\bar{\rho}_{p(\cdot)}}(x^k - x^{k+1})\|_{(L^{p(\cdot)})^*} C_1. \end{aligned} \quad (23)$$

Recalling now that $\langle \mathbf{J}_{\bar{\rho}_{p(\cdot)}}(u), v \rangle = \int_{\Omega} \operatorname{sign}(u(t)) |u(t)|^{p(t)-1} v(t) dt$ for any $u, v \in L^{p(\cdot)}(\Omega)$, by(10) and Proposition 2.3 we have

$$\|\mathbf{J}_{\bar{\rho}_{p(\cdot)}}(u)\|_{(L^{p(\cdot)})^*} = \|\operatorname{sign}(u) |u|^{p(\cdot)-1}\|_{L^{p'(\cdot)}} \leq 2 \|\operatorname{sign}(u) |u|^{p(\cdot)-1}\|_{L^{p'(\cdot)}}.$$

Observe now that the following equality holds true:

$$\begin{aligned} \rho_{p'(\cdot)}(\operatorname{sign}(u) |u|^{p(\cdot)-1}) &= \int_{\Omega} \left(\operatorname{sign}(u(t)) |u(t)|^{p(t)-1} \right)^{p'(t)} dt \\ &= \int_{\Omega} \left(\operatorname{sign}(u(t)) |u(t)| \right)^{p(t)} dt = \int_{\Omega} |u(t)|^{p(t)} dt = \rho_{p(\cdot)}(u). \end{aligned}$$

Since by construction, there holds $\rho_{p(\cdot)}(x^k - x^{k+1}) < 1$, we deduce

$$\rho_{p'(\cdot)}\left(\operatorname{sign}(x^k - x^{k+1}) |x^k - x^{k+1}|^{p(\cdot)-1}\right) < 1,$$

which by Lemma 2.1(i) entails

$$\|\operatorname{sign}(x^k - x^{k+1}) |x^k - x^{k+1}|^{p(\cdot)-1}\|_{L^{p'(\cdot)}} < 1.$$

Furthermore note that for all $u \in L^{p(\cdot)}(\Omega)$, Proposition 2.1(ii) entails

$$\|\operatorname{sign}(u) |u|^{p(\cdot)-1}\|_{L^{p'(\cdot)}} \leq \left(\rho_{p'(\cdot)}(\operatorname{sign}(u) |u|^{p(\cdot)-1})\right)^{1/(p')_+} = \left(\rho_{p'(\cdot)}(\operatorname{sign}(u) |u|^{p(\cdot)-1})\right)^{1/(p_-)'},$$

which can now be evaluated in $u = x^k - x^{k+1}$ and combined with the previous inequalities to get from (23)

$$\begin{aligned} r_k &\leq D(x^k) + \tau_k^{-1} \|\mathbf{J}_{\bar{\rho}_{p(\cdot)}}(x^k - x^{k+1})\|_{(L^{p(\cdot)})^*} C_1 \\ &\leq D(x^k) + \tau_k^{-1} C_1 2 \|\text{sign}(x^k - x^{k+1}) |x^k - x^{k+1}|^{p(\cdot)-1}\|_{L^{p'(\cdot)}} \\ &\leq D(x^k) + \tau_k^{-1} C_1 2 \left(\rho_{p'(\cdot)}(\text{sign}(x^k - x^{k+1}) |x^k - x^{k+1}|^{p(\cdot)-1}) \right)^{1/(p_-)'} \\ &= D(x^k) + \tau_k^{-1} C_1 2 \left(\rho_{p(\cdot)}(x^k - x^{k+1}) \right)^{1/(p_-)'}, \end{aligned}$$

so that, by (17), we have

$$r_k \leq D(x^k) + 2C_1 \tau_k^{-1} (\tau_k D(x^k))^{1/(p_-)'}$$

The step-size constraints (13) together with (19) entail $r_k - r_{k+1} \geq \delta D(x^k)$ and $\tau_k \geq \bar{\tau}$. Plugging these quantities into the last inequality above, we obtain

$$r_k \leq \frac{r_k - r_{k+1}}{\delta} + 2C_1 \bar{\tau}^{-\frac{(p_-)'-1}{(p_-)'}} \left(\frac{r_k - r_{k+1}}{\delta} \right)^{1/(p_-)'}$$

Note that since r_k is a non-negative decreasing sequence, then $r_k - r_{k+1}$ is bounded, so that we can write

$$\delta r_k \leq (r_k - r_{k+1})^{1/(p_-)' } \left(R^{-\frac{(p_-)'-1}{(p_-)'}} + 2C_1 \bar{\tau}^{-\frac{(p_-)'-1}{(p_-)'}} \delta^{\frac{(p_-)'-1}{(p_-)'}} \right),$$

for a sufficiently large $R > 0$, which finally gives

$$r_k - r_{k+1} \geq \frac{\delta^{(p_-)'}}{\left(R^{-\frac{(p_-)'-1}{(p_-)'}} + 2C_1 \bar{\tau}^{-\frac{(p_-)'-1}{(p_-)'}} \delta^{\frac{(p_-)'-1}{(p_-)'}} \right)^{(p_-)'}} r_k^{(p_-)'}$$

□

Thanks to the previous lemma and following analogous arguments as in [4, Proposition 4], the following convergence rate in function values can be found.

Proposition 3.4. *If $(x^k)_k$ is bounded, then the following convergence rate in function values can be found for the iterates of Algorithm 1*

$$r_k \leq \eta \frac{1}{k^{p_- - 1}}. \quad (24)$$

Such convergence rate is related to the smoothness of the space $L^{p(\cdot)}$ considered and, in particular, to the infimum exponent value p_- appearing in the analytical proof of Lemma 3.2. It can thus be read as the worst-case convergence speed. It is highly expected that such convergence result can be improved and that, practically, faster convergence could be achieved, as our numerical tests will show.

We now provide a result relative to the convergence of the sequence $(x_k)_k$ itself.

Proposition 3.5. *If $(x^k)_k$ is bounded, then the sequence $(x^k)_k$ has at least one accumulation point. All accumulation points belong to $\text{Sol}(P)$. If $\text{Sol}(P) = \{\bar{x}\}$, then $(x^k)_k$ converges weakly to \bar{x} .*

Proof. See [4], Proposition 5 and [16], Proposition 3.4 (iii). □

We conclude this section recalling the definition of totally convex and r -convex functions. Under this further hypothesis, it is possible to show that the sequence of iterates defined by Algorithm 1 converges strongly to a solution of the minimization problem (P) and moreover that (P) has a unique solution.

Definition 3.1. *Let $h : L^{p(\cdot)}(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper, convex and lower semi-continuous. The functional h is said to be totally convex in $\hat{u} \in L^{p(\cdot)}(\Omega)$ if for all $\omega \in \partial h(\hat{u})$ and for each $(u^n)_n$ such that*

$$h(u^n) - h(\hat{u}) - \langle \omega, u^n - \hat{u} \rangle \rightarrow 0,$$

there holds

$$\|u^n - \hat{u}\|_{L^{p(\cdot)}} \rightarrow 0,$$

for $n \rightarrow +\infty$. We say that h is totally convex if it is totally convex in \hat{u} for all $\hat{u} \in L^{p(\cdot)}(\Omega)$.

Similarly, h is convex of power-type r (or, r -convex) in $\hat{u} \in L^{p(\cdot)}(\Omega)$ with $r \geq 2$ if for all $M > 0$ and $\omega \in \partial h(\hat{u})$ there exists $\beta > 0$ such that for all $\|u - \hat{u}\|_{L^{p(\cdot)}} \leq M$

$$h(u) - h(\hat{u}) - \langle \omega, u - \hat{u} \rangle \geq \beta \|u - \hat{u}\|_{L^{p(\cdot)}}^r.$$

We say that h is convex of power-type r if it is convex of power-type r in \hat{u} for all $\hat{u} \in L^{p(\cdot)}(\Omega)$.

Proposition 3.6. *If f or g are totally convex or r -convex with $r \geq 2$, then the solution of problem (P) is unique. Denoting it by $\bar{x} \in L^{p(\cdot)}(\Omega)$, we further have that under the same conditions the sequence $(x^k)_k$ defined by (14) converges strongly to \bar{x} .*

Proof. First, note that r -convexity implies total-convexity. Then, focus on the case that f is totally convex. Suppose now by contradiction that there exists $\tilde{x} \in \text{Sol}(\text{P})$ with $\tilde{x} \neq \bar{x}$. Thus $\|\tilde{x} - \bar{x}\|_{L^{p(\cdot)}} > 0$. By defining:

$$R(x) := f(x) - f(\bar{x}) - \langle \nabla f(\bar{x}), x - \bar{x} \rangle, \quad (25)$$

we have that $R(x) \geq 0$ for all $x \in L^{p(\cdot)}(\Omega)$. Moreover, by the optimality of \bar{x} and the subgradient inequality, there holds $\phi(x) - \phi(\bar{x}) \geq R(x)$ for all $x \in L^{p(\cdot)}(\Omega)$. Choosing $x = \tilde{x}$, we thus get

$$0 = \phi(\tilde{x}) - \phi(\bar{x}) \geq R(\tilde{x}) \geq 0,$$

whence $R(\tilde{x}) = 0$. Taking now $u^n = \tilde{x}$ for all $n \geq 1$, we find a contradiction as the total convexity property is violated, whence we deduce $\tilde{x} = \bar{x}$.

To complete the proof, let us consider (25) once again. By optimality of \bar{x} and thanks to the subgradient inequality, there holds $r_k \geq R(x^k)$, whence, by Proposition 3.4, we deduce that $R(x^k) \leq \eta_{\frac{1}{k^p-1}}$. Letting now $k \rightarrow +\infty$ we thus infer that $R(x^k) \rightarrow 0$ and, by the total convexity of f , that $\|x^k - \bar{x}\|_{L^{p(\cdot)}} \rightarrow 0$ which completes the proof.

In the case that g is totally convex, the proof is analogous by defining $R(x) := g(x) - g(\bar{x}) + \langle \nabla f(\bar{x}), x - \bar{x} \rangle$. \square

4 A Bregmanized modular-proximal gradient algorithm

In this section, we introduce a different modular-proximal gradient algorithm solving (P) where the proximal step is defined in terms of a modular Bregman-like distance. Our study is here inspired to the analysis carried out in [16] where an analogous algorithm is studied for a general Banach space \mathcal{X} . Similarly as above, we start this section by stating the required assumptions.

Assumption 4.1. $\nabla f : L^{p(\cdot)}(\Omega) \rightarrow L^{p(\cdot)}(\Omega)^*$ is $(p-1)$ Hölder-continuous with $1 < p \leq 2$ with constant K , i.e.:

$$\|\nabla f(u) - \nabla f(v)\|_{(L^{p(\cdot)})^*} \leq K \|u - v\|_{L^{p(\cdot)}}^{p-1} \quad \forall u, v \in L^{p(\cdot)}(\Omega).$$

Assumption 4.2. *There exists $c > 0$ such that for all $u, v \in L^{p(\cdot)}(\Omega)$*

$$\langle \mathbf{J}_{\rho_{p(\cdot)}}(u) - \mathbf{J}_{\rho_{p(\cdot)}}(v), u - v \rangle \geq c \max \left\{ \|u - v\|_{L^{p(\cdot)}}^p, \|\mathbf{J}_{\rho_{p(\cdot)}}(u) - \mathbf{J}_{\rho_{p(\cdot)}}(v)\|_{(L^{p(\cdot)})^*}^{p'} \right\}$$

where p' is the Hölder-conjugate of p , i.e. $\frac{1}{p} + \frac{1}{p'} = 1$.

The pseudocode of the proposed algorithm is reported in Algorithm 2.

Algorithm 2 Bregmanized modular-proximal gradient algorithm in $L^{p(\cdot)}(\Omega)$ spaces

Parameters: $\{\tau_k\}_k$ s.t.

$$0 < \bar{\tau} \leq \tau_k \leq \frac{pc(1-\delta)}{K} \quad \text{with } 0 < \delta < 1.$$

Initialization: Start with $x^0 \in L^{p(\cdot)}(\Omega)$.

FOR $k = 0, 1, \dots$ REPEAT

 Compute the next iterate as:

$$x^{k+1} = \arg \min_{u \in L^{p(\cdot)}(\Omega)} \rho_{p(\cdot)}(u) - \langle \mathbf{J}_{\rho_{p(\cdot)}}(x^k), u \rangle + \tau_k \langle \nabla f(x^k), u \rangle + \tau_k g(u) \quad (26)$$

UNTIL convergence

Assumption 4.2 links the geometrical properties of the space $L^{p(\cdot)}(\Omega)$ with the Hölder smoothness properties of f . It has to be interpreted as a sufficient compatibility condition between the ambient space $L^{p(\cdot)}(\Omega)$ and the function f for achieving the desired convergence result. We will comment more on the practical verifiability of this condition in the following sections.

The optimization problem (2) characterizing Algorithm 2 can be linked to suitably defined Bregman distances, see, e.g., [5, 2] for more details. We recall in the following their definition.

Definition 4.1. *Given $h : L^{p(\cdot)}(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}$ smooth, convex, proper, l.s.c., the Bregman distance between $x, y \in L^{p(\cdot)}(\Omega)$ with respect to h is defined by*

$$\mathcal{D}_h(x, y) := h(x) - h(y) - \langle \nabla h(y), x - y \rangle, \quad \forall x, y \in L^{p(\cdot)}(\Omega).$$

We now observe that the updating rule in Algorithm 2 can be equivalently written in terms of the Bregman distance associated to $\rho_{p(\cdot)}$ as follows:

$$x^{k+1} \in \arg \min_{u \in L^{p(\cdot)}(\Omega)} \{ \mathcal{D}_{\rho_{p(\cdot)}}(u, x^k) + \tau_k \langle \nabla f(x^k), u \rangle + \tau_k g(u) \},$$

since the terms which are constant with respect to u can trivially be neglected.

The following proposition is the analog of Proposition 3.1. It shows that the minimization problem (26) has a unique solution at each iteration, thus x^{k+1} is well defined.

Proposition 4.1. *The problem*

$$\arg \min_{u \in L^{p(\cdot)}(\Omega)} \{ \rho_{p(\cdot)}(u) - \langle \mathbf{J}_{\rho_{p(\cdot)}}(x), u \rangle + \tau \langle v^*, u \rangle + \tau g(u) \}$$

has a unique solution for each $x \in L^{p(\cdot)}(\Omega)$, $v^* \in L^{p(\cdot)}(\Omega)^*$ and $\tau > 0$.

Proof. Let $\tau > 0$ and $\bar{x} \in \text{Sol}(P)$. By optimality, $0 \in \nabla f(\bar{x}) + \partial g(\bar{x})$, or, equivalently, $\bar{\omega} := -\nabla f(\bar{x}) \in \partial g(\bar{x})$. As g is convex, then $g(u) \geq g(\bar{x}) + \langle \bar{\omega}, u \rangle - \langle \bar{\omega}, \bar{x} \rangle$ for all $u \in L^{p(\cdot)}(\Omega)$, which entails

$$\begin{aligned} & \rho_{p(\cdot)}(u) - \langle \mathbf{J}_{\rho_{p(\cdot)}}(x), u \rangle + \tau \langle v^*, u \rangle + \tau g(u) \\ & \geq \rho_{p(\cdot)}(u) + \langle \tau v^* - \mathbf{J}_{\rho_{p(\cdot)}}(x) + \tau \bar{\omega}, u \rangle + \tau g(\bar{x}) - \tau \langle \bar{\omega}, \bar{x} \rangle \\ & = \|u\|_{L^{p(\cdot)}} \left[\frac{\rho_{p(\cdot)}(u)}{\|u\|_{L^{p(\cdot)}}} + \frac{\langle \omega, u \rangle}{\|u\|_{L^{p(\cdot)}}} + \frac{\alpha}{\|u\|_{L^{p(\cdot)}}} \right], \end{aligned}$$

where we set for simplicity $\omega := \tau v^* - \mathbf{J}_{\rho_{p(\cdot)}}(x) + \tau \bar{\omega} \in L^{p(\cdot)}(\Omega)^*$ and $\alpha := \tau g(\bar{x}) - \tau \langle \bar{\omega}, \bar{x} \rangle \in \mathbb{R}$. By Cauchy-Schwartz inequality we have $|\langle \omega, u \rangle| \leq \|w\|_{(L^{p(\cdot)})^*} \|u\|_{L^{p(\cdot)}}$ and by Proposition 2.1(i), we deduce

$$\begin{aligned} \|u\|_{L^{p(\cdot)}} \left[\frac{\rho_{p(\cdot)}(u)}{\|u\|_{L^{p(\cdot)}}} + \frac{\langle \omega, u \rangle}{\|u\|_{L^{p(\cdot)}}} + \frac{\alpha}{\|u\|_{L^{p(\cdot)}}} \right] & \geq \|u\|_{L^{p(\cdot)}} \left[\frac{\rho_{p(\cdot)}(u)}{\|u\|_{L^{p(\cdot)}}} - \|w\|_{(L^{p(\cdot)})^*} + \frac{\alpha}{\|u\|_{L^{p(\cdot)}}} \right] \\ & \geq \|u\|_{L^{p(\cdot)}} \left[\|u\|_{L^{p(\cdot)}}^{p_- - 1} - \|w\|_{(L^{p(\cdot)})^*} + \frac{\alpha}{\|u\|_{L^{p(\cdot)}}} \right] \geq L \|u\|_{L^{p(\cdot)}} \end{aligned}$$

for some $L > 0$ and for all u whose norm is large enough. This proves that the functional is coercive. Moreover, it is strictly convex since $p_- > 1$, proper and lower semi-continuous in a reflexive Banach space, it has a unique solution. \square

In order to interpret (26) as a fixed point iterative scheme, it is useful to introduce the following notion which shows analogies with the standard one of Moreau envelope (see, e.g., [1, Chapter 12]).

Definition 4.2. *Given $h : L^{p(\cdot)}(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}$ smooth, convex, proper, l.s.c., we define the Moreau-like envelope $e_h : L^{p(\cdot)}(\Omega)^* \rightarrow \mathbb{R}$ and the modular-proximal mapping $\pi_h : L^{p(\cdot)}(\Omega)^* \rightarrow L^{p(\cdot)}(\Omega)$ as follows*

$$\begin{aligned} e_h(x^*) & := \inf_{u \in L^{p(\cdot)}(\Omega)} h(u) + \Delta(x^*, u), \quad x^* \in L^{p(\cdot)}(\Omega)^* \\ \pi_h(x^*) & := \arg \min_{u \in L^{p(\cdot)}(\Omega)} h(u) + \Delta(x^*, u), \quad x^* \in L^{p(\cdot)}(\Omega)^* \end{aligned} \quad (27)$$

where $\Delta(\cdot, \cdot)$ denotes the Bregman-like distance associated to $\rho_{p(\cdot)}$, that is $\Delta(x^*, u) := \rho_{p(\cdot)}(u) - \langle x^*, u \rangle$.

Note that the minimum in (27) is uniquely attained as $\rho_{p(\cdot)}$ is a strictly convex function. Moreover, the unique point $\pi_h(x^*)$ satisfies

$$0 \in \partial h(\pi_h(x^*)) + \mathbf{J}_{\rho_{p(\cdot)}}(\pi_h(x^*)) - x^* \iff x^* \in \partial h(\pi_h(x^*)) + \mathbf{J}_{\rho_{p(\cdot)}}(\pi_h(x^*)). \quad (28)$$

Proposition 4.2. *Let $\gamma > 0$. Then, $\bar{x} \in \text{Sol}(P)$ if and only if $\bar{x} = \pi_{\gamma g}(\mathbf{J}_{\rho_{p(\cdot)}}(\bar{x}) - \gamma \nabla f(\bar{x}))$.*

Proof. Since \bar{x} solves (P), we have

$$0 \in \nabla f(\bar{x}) + \partial g(\bar{x}) \iff 0 \in \gamma \nabla f(\bar{x}) - \mathbf{J}_{\rho_{p(\cdot)}}(\bar{x}) + \mathbf{J}_{\rho_{p(\cdot)}}(\bar{x}) + \gamma \partial g(\bar{x}),$$

that is

$$\mathbf{J}_{\rho_{p(\cdot)}}(\bar{x}) - \gamma \nabla f(\bar{x}) \in \mathbf{J}_{\rho_{p(\cdot)}}(\bar{x}) + \gamma \partial g(\bar{x}).$$

By (28), we thus deduce:

$$\bar{x} = \pi_{\gamma g}(\mathbf{J}_{\rho_{p(\cdot)}}(\bar{x}) - \gamma \nabla f(\bar{x})),$$

as required. \square

Remark 4.1. The iteration (26) can be equivalently formulated as

$$\begin{aligned} x^{k+1} &= \mathbf{J}_{\rho_{p(\cdot)}}(x^k) - \tau_k \nabla f(x^k) \in \mathbf{J}_{\rho_{p(\cdot)}}(x^{k+1}) + \tau_k \partial g(x^{k+1}) \iff \\ x^{k+1} &= \pi_{\tau_k g}(\mathbf{J}_{\rho_{p(\cdot)}}(x^k) - \tau_k \nabla f(x^k)). \end{aligned}$$

This shows that (26) can be read as a fixed-point iteration scheme.

Similarly as in Proposition (3.4), it is possible to prove the convergence in function values for the iterates of Algorithm 2 and achieve a convergence rate. Here, we omit the proof, as it follows verbatim the one in [16] which is itself inspired by [4, Proposition 4].

Proposition 4.3. *If $(x^k)_k$ is bounded, then the following convergence rate in function values can be found for the iterates of Algorithm 2*

$$r_k \leq \eta \frac{1}{k^{p-1}}. \quad (29)$$

It is interesting to compare the rate (29) with the analogous one in (24) obtained for Algorithm 1. The dependence on the Hölder exponent p in (29) links the speed of convergence to the smoothness of the smooth function f rather than to the one of the underlying $L^{p(\cdot)}(\Omega)$ space, which appears in (24). For reasonably smooth problems we thus expect Algorithm 2 to show better performance than Algorithm 1.

5 Sparse reconstruction models: thresholding functions and primal-dual interpretation

In this section, we consider an exemplar sparse reconstruction model used in a variety of signal/image inverse problems and discuss the application of the algorithms presented in this paper for the computation of its numerical solution. Given a Lebesgue measurable map $p(\cdot) : \Omega \rightarrow (1, 2]$, we consider a bounded linear operator $A : L^{p(\cdot)}(\Omega) \rightarrow L^{p_+}(\Omega)$ and an observation $y \in L^{p_+}(\Omega)$, $p_+ \leq 2$. For $\lambda > 0$, we aim to minimize the Tikhonov-like functional

$$\arg \min_{x \in L^{p(\cdot)}(\Omega)} \frac{1}{p_+} \|Ax - y\|_{p_+}^{p_+} + \lambda \|x\|_1$$

in the Banach space $L^{p(\cdot)}(\Omega)$, where $f(x) = \frac{1}{p_+} \|Ax - y\|_{p_+}^{p_+}$ is proper, convex and smooth, while $g(x) = \lambda \|x\|_1$ is proper, l.s.c, convex and non-smooth.

The gradient of f can be computed as $\nabla f(x) = A^* \mathbf{J}_{p_+}^{p_+}(Ax - y) \in (L^{p(\cdot)}(\Omega))^*$. In agreement to Assumption 3.2 for Algorithm 1 and to Assumption 4.1 for Algorithm 2, we now need to study the Hölder continuity of ∇f . We claim that ∇f is $(p_+ - 1)$ -Hölder continuous. To show that, we recall the following useful definitions and properties.

Definition 5.1. [23] *A Banach space \mathcal{X} is called smooth of power-type r (or r -smooth) with $r \in (1, 2]$, if there exists a constant $C > 0$ such that for all $u, v \in \mathcal{X}$ it holds that*

$$\frac{\|v\|_{\mathcal{X}}^r}{r} - \frac{\|u\|_{\mathcal{X}}^r}{r} - \langle \mathbf{J}_{\mathcal{X}}^r(u), v - u \rangle \leq C \|v - u\|_{\mathcal{X}}^r,$$

where $\mathbf{J}_{\mathcal{X}}^r(\cdot)$ denotes the duality mapping between \mathcal{X} and \mathcal{X}^* .

Proposition 5.1. [4] *If a Banach space \mathcal{X} is smooth of power-type r , then $\frac{\|\cdot\|_{\mathcal{X}}^r}{r}$ is continuously differentiable with derivative $\mathbf{J}_{\mathcal{X}}^r$ which is $(r - 1)$ -Hölder continuous.*

Furthermore, for $s \geq r$, the functional $\frac{\|\cdot\|_{\mathcal{X}}^s}{s}$ is continuously differentiable. Its derivative is given by $\mathbf{J}_{\mathcal{X}}^s$ which is still $(r - 1)$ -Hölder continuous on each bounded subset of \mathcal{X} .

Lemma 5.1. [23] *Constant exponent Lebesgue spaces $L^r(\Omega)$ are $\min\{2, r\}$ -smooth.*

The space $L^{p_+}(\Omega)$ is thus p_+ -smooth and $\mathbf{J}_{p_+}^{p_+}$ is $(p_+ - 1)$ -Hölder continuous, i.e.

$$\exists K_1 > 0 \quad \text{s.t.} \quad \forall y_1, y_2 \in L^{p_+}(\Omega) \quad \|\mathbf{J}_{p_+}^{p_+}(y_1) - \mathbf{J}_{p_+}^{p_+}(y_2)\|_{(p_+)^*} \leq K_1 \|y_1 - y_2\|_{p_+}^{p_+-1}.$$

Using this combined with the linearity of A and the sub-multiplicativity of the norm, we can thus write

$$\begin{aligned} \|\nabla f(u) - \nabla f(v)\|_{(L^{p(\cdot)})^*} &= \left\| A^* \left[\mathbf{J}_{p_+}^{p_+}(Au - y) - \mathbf{J}_{p_+}^{p_+}(Av - y) \right] \right\|_{(L^{p(\cdot)})^*} \\ &\leq \|A^*\|_{(L^{p(\cdot)})^*} \|\mathbf{J}_{p_+}^{p_+}(Au - y) - \mathbf{J}_{p_+}^{p_+}(Av - y)\|_{(L^{p(\cdot)})^*} \leq \|A^*\|_{(L^{p(\cdot)})^*} K_1 \|A(u - v)\|_{p_+}^{p_+-1} \\ &\leq K_1 \|A^*\|_{(L^{p(\cdot)})^*} \|A\|_{p_+}^{p_+-1} \|u - v\|_{L^{p(\cdot)}}^{p_+-1} \leq K \|u - v\|_{L^{p(\cdot)}}^{p_+-1} \quad \forall u, v \in L^{p(\cdot)}(\Omega), \end{aligned}$$

showing that ∇f is $(p_+ - 1)$ -Hölder continuous, as required.

We can thus focus now on the computation of the solutions of (14) in the discrete setting, where the domain Ω is discretized into the disjoint sum of n non-empty subsets, i.e. $\Omega = \uplus_{i=1}^n \Omega_i$. By considering a single real value on each subset Ω_i , with a slight

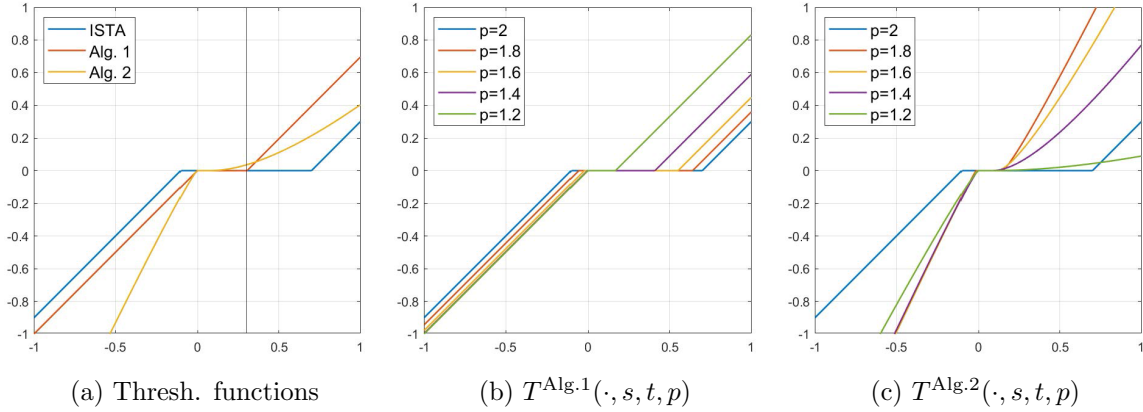


Figure 1: (a) 1-dimensional thresholding functions $T^{\text{Alg.1}}(\cdot, s, t, p)$, $T^{\text{Alg.2}}(\cdot, s, t, p)$ and $T^{\text{ISTA}}(\cdot, s, t)$ with $p = 1.3$, $s = 0.3$, $t = 0.4$. (b) $T^{\text{Alg.1}}(\cdot, s, t, p)$ with $s = 0.3$, $t = 0.4$ and $p \in \{1.2, 1.4, 1.6, 1.8, 2\}$. (c) $T^{\text{Alg.2}}(\cdot, s, t, p)$ with $s = 0.3$, $t = 0.4$ and $p \in \{1.2, 1.4, 1.6, 1.8, 2\}$.

abuse of notation we simply denote by $\ell^{p(\cdot)}(\mathbb{R}^n)$, the n -th dimensional subspace of the sequence space $\ell^{p(\cdot)}$ generated by the first n elements e_1, e_2, \dots, e_n of the canonical basis. To allow effective numerical resolution, we heavily exploit the separability property of the operators involved in the sense of Definition 2.6. By setting $\sigma^k := A^* \mathbf{J}_{p_+}^{p_+}(Ax^k - y) \in \mathbb{R}^n$, the iteration (14) of Algorithm 1 reads:

$$\begin{aligned} x^{k+1} &= \arg \min_{u \in \ell^{p(\cdot)}(\mathbb{R}^n)} \{ \bar{\rho}_{p(\cdot)}(u - x^k) + \tau_k \langle \sigma^k, u \rangle + \tau_k \lambda \|u\|_1 \} \\ &= \arg \min_{u \in \ell^{p(\cdot)}(\mathbb{R}^n)} \sum_{i=1}^n \left\{ \frac{1}{p_i} |u_i - x_i^k|^{p_i} + \tau_k \sigma_i^k u_i + \tau_k \lambda |u_i| \right\}. \end{aligned} \quad (30)$$

Note that, thanks to the additive separability property, at each k -th iteration, with $k \geq 1$, the n -dimensional minimization problem in (30) corresponds to the sum of n 1-dimensional ones. Hence, each component can be treated independently, so one can consider the independent minimization of the 1D functions:

$$\Psi_{x,s,t,p}^{\text{Alg.1}}(u) := \frac{1}{p} |u - x|^p + su + t|u|,$$

with $x = x_i^k$, $s = \tau_k \sigma_i^k$, $t = \tau_k \lambda$ and $p = p_i$. The minimisers of $\Psi_{x,s,t,p}^{\text{Alg.1}}$ can be computed by optimality as $(\partial \Psi_{x,s,t,p}^{\text{Alg.1}})^{-1}(0)$ and can be expressed in a compact form in terms of the thresholding function (see [4] for more details):

$$T^{\text{Alg.1}}(x, s, t, p) = \begin{cases} x - \text{sign}(s+t)|s+t|^{\frac{1}{p-1}} & \text{if } x \geq \text{sign}(s+t)|s+t|^{\frac{1}{p-1}} \\ x - \text{sign}(s-t)|s-t|^{\frac{1}{p-1}} & \text{if } x \leq \text{sign}(s-t)|s-t|^{\frac{1}{p-1}} \\ 0 & \text{otherwise,} \end{cases} \quad (31)$$

As discussed in the previous sections, a direct application of the proximal-gradient algorithm in Banach spaces studied in [4] to the $\ell^{p(\cdot)}(\mathbb{R}^n)$ scenario would have rendered more challenging due to the non separability of the norm $\|\cdot\|_{\ell^{p(\cdot)}}$ and of the corresponding r -duality map, given by

$$\mathbf{J}_{\ell^{p(\cdot)}}^r(x) = \frac{1}{\sum_{j=1}^n \frac{p_j |x_j|^{p_j}}{\|x\|_{\ell^{p(\cdot)}}}} \left(\frac{1}{\|x\|_{\ell^{p(\cdot)}}^{p_i-1}} p_i |x_i|^{p_i-1} \text{sign}(x_i) \right)_{i=1, \dots, n}.$$

We can similarly focus on Algorithm 2, for which, we remark, Assumption 4.1 holds with $\mathbf{p} = p_+$. Assumption 4.2 is hard to verify in practice, although numerical tests show that it is not that challenging to find a step-size for which convergence is guaranteed. Proceeding similarly as above, we exploit again the separability of the modular appearing in the computation of the k -th iteration (26) of Algorithm 2, which leads to the computation of the minimizers of the following 1D function

$$\Psi_{x,s,t,p}^{\text{Alg.2}}(u) = \frac{1}{p} |u|^p - |x|^{p-1} \text{sign}(x)u + su + t|u|$$

where $x = (x^k)_i$, $s = \tau_k \sigma_i^k$, $t = \tau_k \lambda$, $p = p_i$. Such minimisers are given by $(\partial \Psi_{x,s,t,p}^{\text{Alg.2}})^{-1}(0)$ and correspond to the following thresholding function:

$$T^{\text{Alg.2}}(x, s, t, p) = \begin{cases} -(s-t - |x|^{p-1} \text{sign}(x))^{\frac{1}{p-1}} & \text{if } s-t - |x|^{p-1} \text{sign}(x) > 0 \\ (|x|^{p-1} \text{sign}(x) - s-t)^{\frac{1}{p-1}} & \text{if } |x|^{p-1} \text{sign}(x) - s-t > 0 \\ 0 & \text{otherwise.} \end{cases} \quad (32)$$

Figure 1a shows a comparison between the thresholding functions (31) and (32) with the classical soft-thresholding function

$$T^{\text{ISTA}}(x, s, t) = \begin{cases} x - s + t & \text{if } x - s + t < 0 \\ x - s - t & \text{if } x - s - t > 0 \\ 0 & \text{otherwise,} \end{cases} \quad (33)$$

which corresponds to the minimization of the 1D function $\Psi_{x,s,t}^{\text{ISTA}}(u) = \frac{1}{2}(u-x+s)^2 + t|u|$ appearing in the solution of the $\ell_2 - \ell_1$ LASSO optimization problem in the Hilbert space ℓ_2 by means of the standard ISTA algorithm [10]. It is interesting to point out that differently from the ISTA thresholding function (33), both thresholding functions (31) and (32) are no longer symmetrical with respect to the vertical line $x = s$. Moreover, $T^{\text{Alg.1}}(\cdot, s, t, p)$ remains linear in x similarly as $T^{\text{ISTA}}(\cdot, s, t)$, as shown in Figure 1b for some exemplar values of p , while the thresholding function $T^{\text{Alg.2}}$ is, in general, non-linear, as shown in Figure 1. Note that for $p = 2$, both $T^{\text{Alg.1}}(\cdot, s, t, 2)$ and $T^{\text{Alg.2}}(\cdot, s, t, 2)$ coincide with the standard soft-thresholding $T^{\text{ISTA}}(\cdot, s, t)$ operator.

5.1 Interpretation as primal and dual algorithms

In this section, we analyze the relationship between the thresholding functions (31) and (32) of algorithms 1 and 2, respectively, and the primal and dual iterative methods described, e.g., in [23], similarly to the analogous considerations sketched in the Introduction 1 for the algorithms in [4] and [16] in the case $g \equiv 0$. In order to extend our considerations to the non-smooth case, we need to provide an analytical expressions of duality mappings in $L^p(\Omega)$ spaces, which, for better readability, will be done in the discrete setting, that is in $\ell^{p(\cdot)}(\mathbb{R}^n)$.

As a first remark, notice that given a fixed p and r such that $1 < p, r < +\infty$, the r -duality mapping of $\ell^p(\mathbb{R}^n)$ can be defined in terms of component-wise maps $\mathbf{j}_p : \mathbb{R} \rightarrow \mathbb{R}$ defined by $\mathbf{j}_p(t) := |t|^{p-1} \text{sign}(t)$, as follows

$$\mathbf{J}_{\ell^p}^r(x) = \|x\|_p^{r-p} (\mathbf{j}_p(x_i))_{i=1,\dots,n}.$$

Thanks to the isometric isomorphism between $(\ell^p)^*$ and $\ell^{p'}$, the inverse of such duality mapping is $(\mathbf{J}_{\ell^p}^r)^{-1} = \mathbf{J}_{(\ell^p)^*}^{r'} = \mathbf{J}_{\ell^{p'}}^{r'}$, which can be explicitly written as:

$$\begin{aligned} \mathbf{J}_{\ell^{p'}}^{r'}(x) &= \|x\|_{p'}^{r'-p'} (\mathbf{j}_{p'}(x_i))_{i=1,\dots,n} = \|x\|_{p'}^{r'-p'} (|x_i|^{p'-1} \text{sign}(x_i))_{i=1,\dots,n} \\ &= \|x\|_{p'}^{r'-p'} (|x_i|^{\frac{1}{p-1}} \text{sign}(x_i))_{i=1,\dots,n}. \end{aligned}$$

Comparing now the thresholding function $T^{\text{Alg.1}}(\cdot, s, t, p)$ in (31) with the above formulas, we observe that we can rewrite it in terms of the component-wise maps $\mathbf{j}_{p'}$ associated to the duality map $\mathbf{J}_{\ell^{p'}}^{r'}$ as:

$$T^{\text{Alg.1}}(x, s, t, p) = \begin{cases} x - \mathbf{j}_{p'}(s+t) & \text{if } x \geq \text{sign}(s+t)|s+t|^{\frac{1}{p-1}} \\ x - \mathbf{j}_{p'}(s-t) & \text{if } x \leq \text{sign}(s-t)|s-t|^{\frac{1}{p-1}} \\ 0 & \text{otherwise,} \end{cases} \quad (34)$$

This new formulation is helpful to better show that as the forward gradient step is computed in the primal space: the point-wise map $\mathbf{j}_{p'}$ is required to associate the element $s+t$ of the dual space to an element in the primal space, thus making the computation well-defined. In agreement with what has been discussed in the Introduction, see (2), we can thus interpret this proximal-gradient iteration as a *primal* algorithmic iteration.

As far as the thresholding function $T^{\text{Alg.2}}(\cdot, s, t, p)$ in (32) is concerned, we notice that we can proceed similarly and rewrite it as:

$$\begin{aligned} T^{\text{Alg.2}}(x, s, t, p) &= \begin{cases} -(s-t - \mathbf{j}_p(x))^{\frac{1}{p-1}} & \text{if } s-t - |x|^{p-1} \text{sign}(x) > 0 \\ (\mathbf{j}_p(x) - s-t)^{\frac{1}{p-1}} & \text{if } |x|^{p-1} \text{sign}(x) - s-t > 0 \\ 0 & \text{otherwise.} \end{cases} \\ &= \begin{cases} -\mathbf{j}_{p'}(s-t - \mathbf{j}_p(x)) & \text{if } s-t - |x|^{p-1} \text{sign}(x) > 0 \\ \mathbf{j}_{p'}(\mathbf{j}_p(x) - s-t) & \text{if } |x|^{p-1} \text{sign}(x) - s-t > 0 \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (35)$$

We can observe that both point-wise maps \mathbf{j}_p and $\mathbf{j}_{p'}$ associated to the duality maps \mathbf{J}_p^r and $\mathbf{J}_{p'}^{r'}$, respectively, play a role here as both a primal-to-dual map is required to define the forward step in the dual space and a dual-to-primal one is needed to compute the backward step, analogously as we have seen before. In agreement to what has been observed in the Introduction, see (4), we thus interpret such iteration as a *dual* proximal-gradient step.

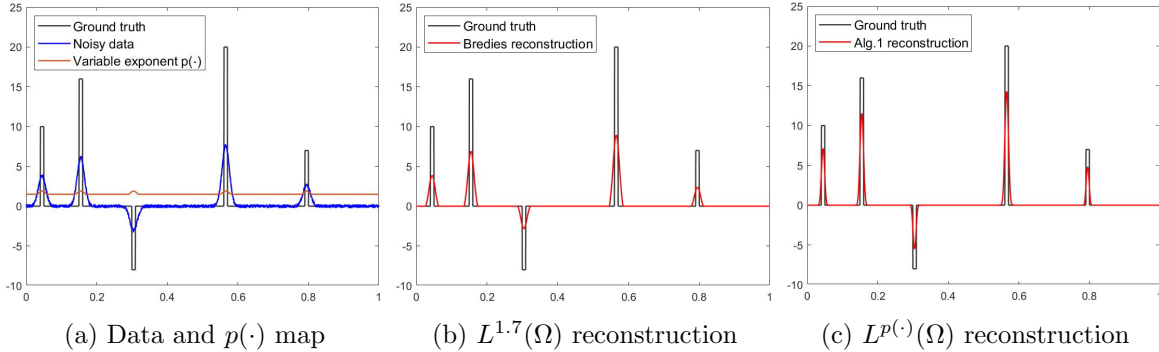


Figure 2: Parameters: $\tau_k \equiv 0.5$; $\lambda = 10^{-2}$; Stopping criterion based on the normalized relative change between x^k and x^{k+1} : $\|x^k - x^{k+1}\|_2 / \|x^k\|_2 < 10^{-4}$.

Such considerations can now be similarly applied to our modular-based variable exponent case, where the role of the primal-to-dual map \mathbf{J}_p^r is replaced by the map $\mathbf{J}_{\rho_{p(\cdot)}}$, whose expression in $\ell^{p(\cdot)}(\mathbb{R}^n)$ reads

$$\mathbf{J}_{\rho_{p(\cdot)}}(x) = (\text{sign}(x_i)|x_i|^{p_i-1})_{i=1,\dots,n} = (\mathbf{j}_{p_i}(x_i))_{i=1,\dots,n},$$

and depends on the point-wise mappings $\mathbf{j}_{p_i}(\cdot)$ defined above. Algorithm 1 should thus be interpreted as a *primal* proximal-gradient algorithm with point-wise thresholding function (34), while Algorithm 2 should be understood as a *dual* proximal-gradient one with point-wise thresholding function (35).

6 Numerical tests

In this section, we provide some numerical tests showing how the proposed model adapts to deal with a variety of signal and image deconvolution and denoising problems. We include further tests providing a numerical verification of the computational convergence properties of Algorithms 1 and 2.

6.1 Spike reconstruction

As a first example, we consider a 1D signal reconstruction problem where we seek for sparse spikes defined on $\Omega = [0, 1]$ to be reconstructed from blurred measurements corrupted with Gaussian noise, see Figure 2a. In order to favor sparse reconstructions and reduce possible over-smoothing, the formulation of a reconstruction model in a Banach space \mathcal{X} is considered, see, e.g., [4, 23]. Denoting by $A : \mathcal{X} \rightarrow L^2(\Omega)$ the blurring operator and by $y \in \mathcal{X}$ the measured data, we thus aim solve

$$\arg \min_{x \in \mathcal{X}} \frac{1}{2} \|Ax - y\|_2^2 + \lambda \|x\|_1, \quad \lambda > 0 \quad (36)$$

where, in particular, we set $\mathcal{X} = L^{p(\cdot)}(\Omega)$ with $p_- = 1.6$ and $p_+ = 2$, as shown in orange in Figure 2a. The idea is to choose a higher value $p(\cdot)$ where it is more likely to have a (spike) signal, while lower values are preferred elsewhere, so that, for these points, sparsity is enforced at a stronger extent. Note that the choice of the exponent map $p(\cdot)$ acts in fact as a prior model on the signal, together with the penalty term. To incorporate such prior knowledge, one can look directly at the shape of the data y , or, for instance, to the structure of a standard $\ell_2 - \ell_1$ reconstruction computed after a small number of iterations, in order to have a variable exponent $p(\cdot)$ consistent with an approximated (possibly over-smoothed) solution of the problem, see [15] for more details on the choice of $p(\cdot)$. Having chosen the exponent map $p(\cdot)$, we can thus solve (36) on $\mathcal{X} = L^{p(\cdot)}(\Omega)$ by means, e.g., of Algorithm 1.

In order to provide a comparison with existing models, we further consider problem (36) on $\mathcal{X} = L^p(\Omega)$ for $p = 1.7$ and solve it by means of the Algorithm in [4]. We observe that using $L^{p(\cdot)}(\Omega)$ modeling improves the quality of the reconstruction with respect to a fixed $L^p(\Omega)$ modeling.

6.2 Deconvolution of heterogeneous signals

We now consider a signal deconvolution problem of a one-dimensional heterogeneous signal $x \in L^{p(\cdot)}(\Omega)$ with $\Omega = [0, 1]$ composed of spikes and a smooth part, see Figure 3a. Given noisy and blurred data $y \in L^{p(\cdot)}(\Omega)$, we consider the following reconstruction criterion

$$\arg \min_{x \in L^{p(\cdot)}(\Omega)} \rho_{p(\cdot)}(Ax - y) + \lambda \|x\|_1 \quad (37)$$

where $A : L^{p(\cdot)}(\Omega) \rightarrow L^{p(\cdot)}(\Omega)$ denotes the blurring operator and $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \rightarrow \mathbb{R}_+$ is the modular function defined in (6). As exponent map, we choose $p_- = 1.5 \leq p(\cdot) \leq 2 = p_+$ (see Figure 3a) and solve (37) using Algorithm 2. For comparison, we further consider the standard Hilbertian case where $\mathcal{X} = L^2(\Omega)$ and (37) is defined for $p(\cdot) \equiv 2$ and the $L^p(\Omega)$ case with $p(\cdot) \equiv 1.5$. Note that upon these two choices, the data fidelity term in (37) takes the form of a standard L^2 -squared data term and of a $L^{1.5}$ one, respectively. We solve these models with the ISTA algorithm [10] and the algorithm studied in [16] for problems in Banach spaces. Comparing Figure 3b and 3c, we clearly see that spikes are better reconstructed whenever $p = 1.5$ due to the locally enhanced sparsifying property of $\rho_{p(\cdot)}$, whilst smooth regions are better restored with reduced ringing effect artifacts whenever $p = 2$. To take the best of both worlds, the use of a variable exponent $p(\cdot)$ seems to be a natural choice, as shown by the reconstruction in Figure 3d. Working with a variable exponent allows us to deal with the different nature of the signal in a more flexible way.

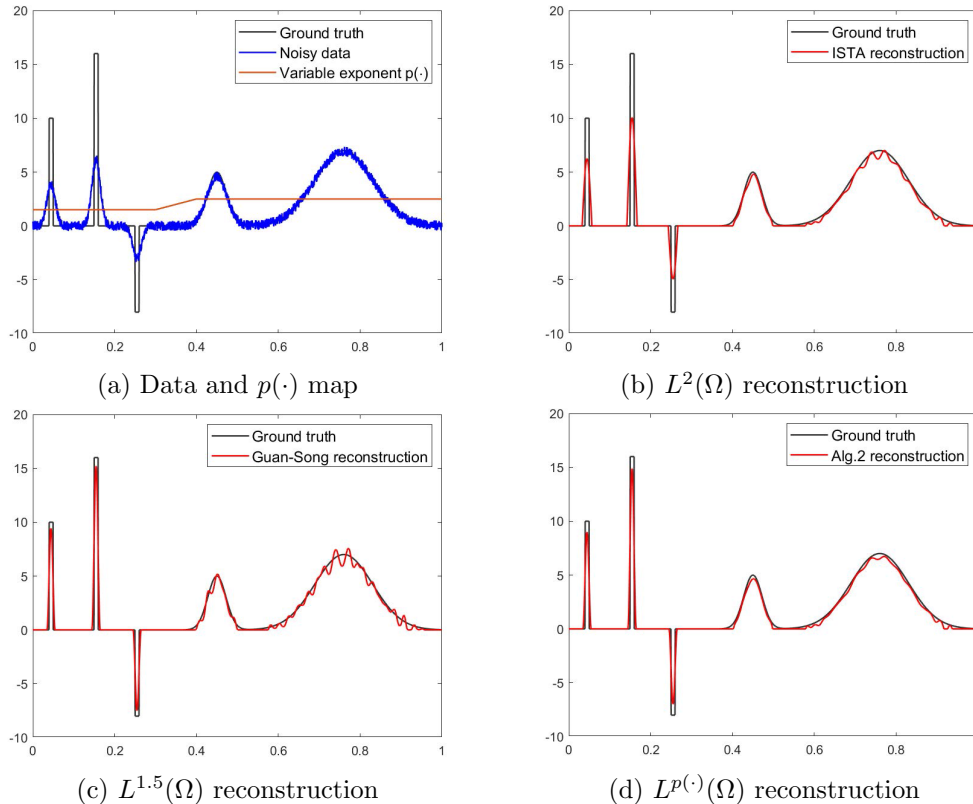


Figure 3: Parameters: $\tau_k \equiv 0.5$; $\lambda = 5 * 10^{-3}$. Stopping criterion based on the normalized relative change between x^k and x^{k+1} : $\|x^k - x^{k+1}\|_2 / \|x^k\|_2 < 4 * 10^{-6}$.

6.3 1D and 2D mixed noise removal

We now focus on a mixed noise removal problem for blurred signals and images affected by Gaussian and impulsive (Salt & Pepper) noise, in different and disjoint parts of their spatial domain Ω , with $\Omega = [0, 1]$ and Ω being a compact of \mathbb{R}^2 , respectively. We exploit here the flexibility of $L^{p(\cdot)}(\Omega)$ spaces by treating effectively the different noise nature at the same time.

For the 1D example, the measured blurred and noisy signal is showed in Figure 4a: Gaussian noise is artificially added on the left part of the domain while impulsive noise to the right one. To retrieve the sparse underlying signal, we consider problem (37). Choosing $p(\cdot) \equiv 2$ everywhere (thus naturally setting $\mathcal{X} = L^2(\Omega)$) in the domain forces the fidelity term to reduce to $\|Ax - y\|_2^2$ which is well-known to be the most appropriate choice in the presence of Gaussian noise. For such model, the standard ISTA algorithm [10] can be used. We can observe in Figure 4b that upon this choice the reconstruction on the left-hand side is good, while it presents several artifacts on the right-hand side, due to the poor adaptivity of the model to the different noise nature there. To do so, a data term more adapted to describe the sparse nature of impulsive noise should be considered. Ideally, choosing ℓ_1 fidelity would be the most appropriate choice, see [22]. In our modeling, however, exponents $p = 1$ cannot be chosen as they would correspond to non-smooth data terms and non-reflexive Banach spaces. However, exponents $p \approx 1$ can still be considered, as they fit the modeling considered. Choosing in particular an $L^p(\Omega)$ framework with $p \equiv 1.4$ and solving the resulting model with the Algorithm detailed in [16] favors the removal of impulsive noise as it can be seen in Figure 4c (right-hand side). However, such choice concentrates intensities around the spikes on the left-hand

side, creating artifacts (in particular, changing the intensity range). Choosing a variable exponent $p_- = 1.4 \leq p(\cdot) \leq p_+ = 2$ and solving the resulting problem by, e.g., Algorithm 2 endows the modeling with enough flexibility to provide a good reconstruction of the signal over the whole image domain, as it can be shown in Figure 4d.

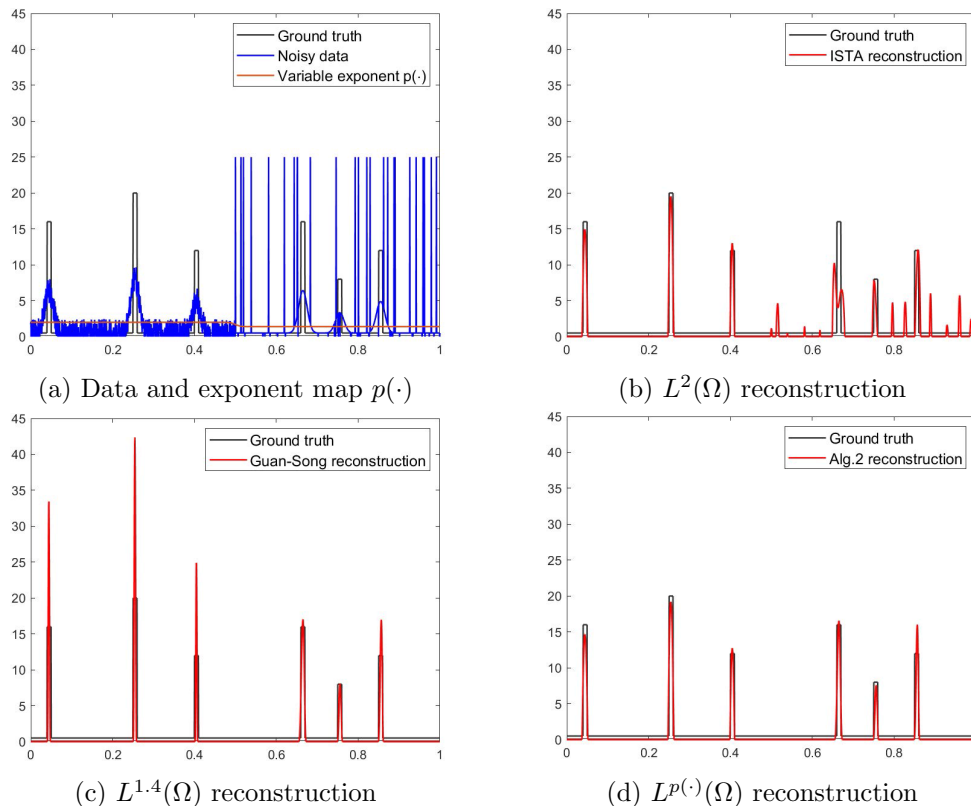


Figure 4: Parameters: $\tau_k \equiv 0.1$, $\lambda = 2 * 10^{-2}$. Stopping criterion based on the normalized relative change between x^k and x^{k+1} : $\|x^k - x^{k+1}\|_2 / \|x^k\|_2 < 4 * 10^{-6}$.

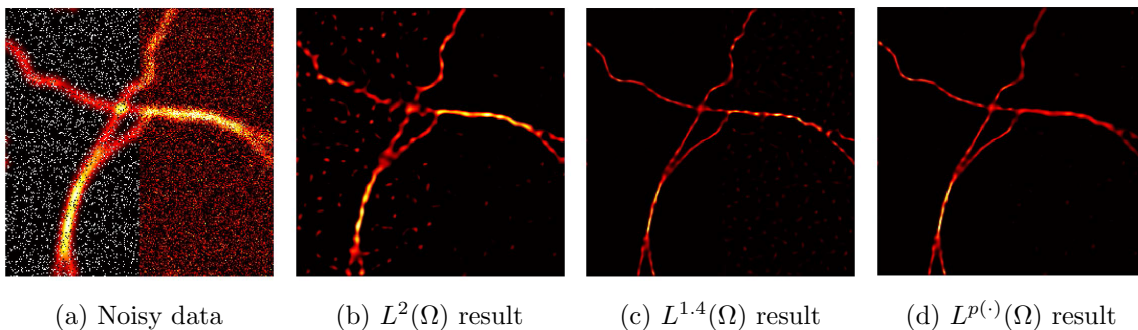


Figure 5: Parameters: $\tau_k \equiv 0.1$, $\lambda = 0.1$. Stopping criterion based on the normalized relative change between x^k and x^{k+1} : $\|x^k - x^{k+1}\|_2 / \|x^k\|_2 < 10^{-4}$.

As a similar test, we considered a mixed denoising problem for the blurred and noisy image in Figure 5a. Similarly as in the previous test, we artificially added white Gaussian noise of fixed variance on the right-hand side and impulsive noise on the left-hand side. We solved again (37) for $p(\cdot) \equiv 2$ by the ISTA algorithm, for $p(\cdot) \equiv p = 1.4$ by the algorithm in [16] and for a variable exponent $p_- = 1.4 \leq p(\cdot) \leq p_+ = 2$ such that $p_i = 2$ on the Gaussian noisy half and $p_i = 1.4$ on the impulsive noisy one by Algorithm 2. Analogous observations as the ones discussed for the 1D case can still be made and as one can clearly see, reconstruction artifacts are significantly reduced in the case of a variable exponent modelling.

The flexibility of the model given by the choice of the map $p(\cdot)$ allows to reconstruct signals with different spatial properties on the whole domain.

6.4 A numerical study on convergence rates

The previous examples show that the use of a variable exponent can help in improving reconstruction quality. It is thus natural asking which algorithm between Algorithm 1 and Algorithm 2 one should use in practice. As remarked already in [4] and as it can be observed from the convergence rate (24), Algorithm 1 is expected to be very slow in practice, in particular, slower than a gradient-type algorithm whose well-know convergence speed is of the order $O(1/k)$.

In this section, we compare the speed of convergence of different algorithms when used as numerical solvers for the deblurring problem (36) in the different Hilbert and Banach scenarios discussed in Section 6.1. In particular, we compare the speed of convergence for the ISTA algorithm used to solve (36) in $\mathcal{X} = L^2(\Omega)$, with the algorithms proposed by Bredies in [4] and Guan and Song in [16] for solving the same problem on $\mathcal{X} = L^p(\Omega)$ with $p = 1.7$ and with the two Algorithms 1 and 2 proposed in this work for $\mathcal{X} = L^{p(\cdot)}(\Omega)$. As exponent map $p(\cdot)$, we stick with the choice shown in orange Figure 2a.

Given $x^* \in \mathcal{X}$, solution of (36) in the different spaces \mathcal{X} , we recall the convergence rates (24) and (29) for Algorithm 1 and 2, respectively. In principle, to compare the speed of convergence in a precise way, a pre-computation of x^* by means of benchmark algorithms should be done for all the different scenarios discussed above. However, since up to our knowledge there is no existing algorithm for solving (36) in $\mathcal{X} = L^{p(\cdot)}(\Omega)$, instead of computing x^* we computed as a reference the value of \tilde{x} , solution of (36) with $\mathcal{X} = L^2(\Omega)$ by running ISTA or $2 * 10^4$ iterations. Note that by simple manipulations we have that for Algorithm 1

$$\phi(x^k) - \phi(\tilde{x}) = \phi(x^k) \pm \phi(x^*) - \phi(\tilde{x}) \leq \eta_1 \left(\frac{1}{k}\right)^{p-1} + c$$

and, similarly, for Algorithm 2 $\phi(x^k) - \phi(\tilde{x}) \leq \eta_2 \left(\frac{1}{k}\right)^{p-1} + c$ with $p = 2$ in (36), so that rates can still be compared up to an additive constant. We use \tilde{x} also for the computation of the convergence rates for Bredies (3) and Guan and Song (4) algorithms having $\phi(x^k) - \phi(\tilde{x}) \leq \eta_{3,4} \left(\frac{1}{k}\right)^{p-1} + c'$. Finally, recall that for ISTA the convergence rate in function values is $\phi(x^k) - \phi(\tilde{x}) \leq \eta_5 \frac{1}{k}$. As stopping criterion, for all the tested algorithms we used the normalized relative rates with respect to \tilde{x} up to a tolerance parameter $\epsilon = 10^{-4}$: $|\phi(x^k) - \phi(\tilde{x})|/\phi(\tilde{x}) < \epsilon$.

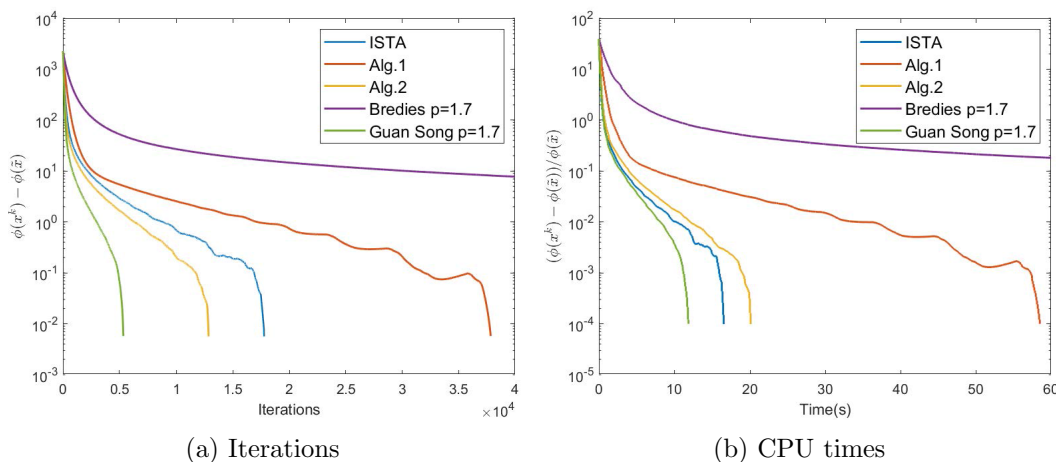


Figure 6: (a) Relative rates along the first $4 * 10^4$ iterations. (b) Normalized relative rates along the first 60 seconds of CPU time.

	ISTA	Bredies [4]	Guan-Song [16]	Alg. 1	Alg. 2
# iterations	17768	$5 * 10^5$	5352	37850	12849
CPU time	16.5s	1025.7s	11.9s	58.5s	20.0s

Table 1: Algorithmic comparison: iterations required and CPU time till convergence.

The results reported in Figure 6 and Table 1 show several interesting numerical convergence properties. First of all, we note that although Bredies' (violet line) and Guan-Song's (green line) algorithms are supposed to have the same convergence rate $O(1/k^{0.7})$ in theory for the specific problem at hand, they clearly have a very different behavior. While the first needs more than $5 * 10^5$ iterations and more than 1000 seconds of CPU time to reach convergence, the second converges with a much faster speed. The same behavior is observed also for the modular Algorithms 1 and 2 too, with the first (red line) being very slow and the second (yellow line) much faster. With respect to standard ISTA, we observe that the proposed Bregmanised algorithm and the Guan-Song one are faster in terms of number of iterations and comparable in terms of computational time. From these numerical verifications, we thus considered Algorithm 2 instead of Algorithm 1 in Section 6.2 and 6.3. The acceleration of these algorithms is indeed an interesting matter a future research.

7 Conclusions

In this paper, we presented two possible ways of generalising proximal gradient algorithms to solve structured minimisation problems in $L^{p(\cdot)}(\Omega)$. Due to the difficulties introduced by working in a Banach space setting (i.e. lack of Riesz isometric isomorphism, need of duality mappings) and the further lack of separability of the norm in $L^{p(\cdot)}(\Omega)$, standard strategies cannot be applied here, thus requiring the use of alternative iterative schemes based on the separable modular function. Two algorithms are presented, a primal one (Algorithm 1) inspired to the one studied in [4] for problems in $L^p(\Omega)$, and a dual one (Algorithm 2), analogous to the one studied in [16]. We provided a detailed convergence analysis for Algorithm 1 showing the descent of the functional and convergence in function values with rate $\mathcal{O}\left(\frac{1}{k}\right)^{p_- - 1}$, with p_- being the essential infimum of the exponent map $p(\cdot)$. For Algorithm 2 a modular Bregman-like distance is used in a dual update fashion and analogous convergence rates are showed in function values. Several numerical results are reported to compare the use of the adaptive $L^{p(\cdot)}(\Omega)$ space with Hilbert and $L^p(\Omega)$ ones on some exemplar signal deconvolution and signal/image mixed noise removal problems. Finally, a numerical verification on the speed of convergence and on the computational efficiency of the proposed algorithms is given, showing that *primal* algorithms have slower convergence in comparison with *dual* ones.

Future work shall address the crucial question on how the map $p(\cdot)$ shall be chosen in order to adapt to local signal/image prior contents. Moreover, it would be interesting to find a strategy to incorporate extrapolation/acceleration in order to improve convergence speed which, in practice, is particularly slow for primal algorithms.

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