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# Rational solutions to the KdV equation depending on multi-parameters

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## Abstract

We construct multi-parametric rational solutions to the KdV equation. For this, we use solutions in terms of exponentials depending on several parameters and take a limit when one of these parameters goes to 0. Here we present degenerate rational solutions and give a result without the presence of a limit as a quotient of polynomials depending on  $3N$  parameters. We give the explicit expressions of some of these rational solutions.

**Key Words :** KdV equation, rational solutions.

**PACS numbers :**

33Q55, 37K10, 47.10A-, 47.35.Fg, 47.54.Bd

## 1 Introduction

We consider the KdV equation

$$4u_t = 6uu_x - u_{xxx}, \quad (1)$$

where as usual, the subscripts  $x$  and  $t$  denote partial derivatives and  $u$  is a function of  $x$  and  $t$ .

Korteweg and de Vries [1] introduced this equation (1) for the first time in 1895.

The KdV equation (1) is the basis of the most common tool for the (1+1)-dimensional modelling of shallow water waves. It is now known that this equation describes the propagation of waves with weak dispersion in various nonlinear media.

In 1967, Gardner et al. [2] gave a method of resolution of this equation. Zakharov and Faddeev [3] proved that this is a complete integrable equation in 1971.

Using the bilinear method, Hirota [4] constructed solutions in 1971. Among the works which have been realized, we can mention some of them. Solutions in terms of Riemann theta functions [5] were given by Its and Matveev in 1975. Lax gave in 1975 the expressions of periodic and almost periodic solutions [6]; we can also quote, Matveev in 1992 [7], Ma in 2004 [8], Kovalyov in 2005 [9].

We consider solutions to the KdV equation in this paper. From elementary exponential functions depending on several parameters, we construct rational solutions in performing a passage to the limit when one of these parameters goes to 0. We obtain rational solutions as depending on  $3N$  parameters at order  $N$ . To get a result which does not depend on a limit, we have to consider another type of functions. With this new functions, we can give explicit solutions and we get a hierarchy of rational solution of order  $N$  to the KdV equation depending on  $3N$  parameters.

## 2 Solutions to the KdV equation

### 2.1 Solutions to the KdV equation in terms of elementary exponentials

We consider the KdV equation (1)

$$4u_t = 6uu_x - u_{xxx}.$$

We use the following notations. We consider the real numbers  $\alpha_j$ ,  $\beta_j$  and  $\gamma_j$  defined by :

$$\alpha_j = \sum_{k=1}^N a_k (je)^{2k-1}, \quad \beta_j = \sum_{k=1}^N c_k (je)^{2k-1}, \quad \gamma_j = \sum_{k=1}^N d_k (je)^{2k-1}, \quad \text{for } 1 \leq j \leq N \quad (2)$$

We consider the elementary functions :

$$f_{ij}(x, t) = \alpha_j^{i-1} \exp(\alpha_j x - \alpha_j^3 t + \beta_j) - \alpha_j^{i-1} \exp(\alpha_j x - \alpha_j^3 t + \gamma_j), \quad \text{for } 1 \leq i \leq N \quad (3)$$

Then, we have the following statement :

**Theorem 2.1** *The function  $v$  defined by*

$$v(x, t) = -2\partial_x^2 \ln(\det(f_{ij})_{(i,j) \in [1,N]}) \quad (4)$$

*is a solution to the KdV equation (1) with  $e$ ,  $a_j$ ,  $c_j$  and  $d_j$ ,  $1 \leq j \leq N$  arbitrarily real parameters.*

## 2.2 Rational solutions to the KdV equation

We are going to perform a limit when the parameter  $e$  tends to 0. With this idea, we can obtain rational solutions to the KdV equation,

### 2.2.1 Rational solutions as a limit case

We get the following result :

**Theorem 2.2** *The function  $v$  defined by*

$$v(x, t) = \lim_{e \rightarrow 0} -2\partial_x^2 \ln(\det(f_{ij})_{(i,j) \in [1, N]}) \quad (5)$$

*is a rational solution to the KdV equation (1) depending on  $3N$  parameters  $a_j$ ,  $c_j$  and  $d_j$ ,  $1 \leq j \leq N$ .*

### 2.2.2 Degenerate rational solutions

We can give a more precise result avoiding the presence of a limit. For this, we consider another type of functions  $g_{ij}$  defined by :

$$g_{ij} = \frac{\partial^{2j-1} f_{ij}}{\partial e^{2j-1}} \Big|_{e=0}, \text{ for } 1 \leq i \leq N, 1 \leq j \leq N.$$

Then get the following result :

**Theorem 2.3** *The function  $v$  defined by*

$$v(x, t) = -2\partial_x^2 \ln(\det(g_{ij})_{(i,j) \in [1, N]}) \quad (6)$$

*is a rational solution to the KdV equation (1) depending on  $3N$  parameters  $a_j$ ,  $c_j$  and  $d_j$ ,  $1 \leq j \leq N$ .*

So for each positive integer  $N$ , we get rational solutions to the KdV equation depending on  $3N$  real parameters.

In the following we give some examples of rational solutions. These results are consequences of the previous result (6).

## 2.3 First order rational solutions

We have the following result at order  $N = 1$  :

**Proposition 2.1** *The function  $v$  defined by*

$$v(x, t) = 8 \frac{a_1^2}{(2a_1x - d_1 + c_1)^2}, \quad (7)$$

*is a solution to the KdV equation (1) with  $a_1$ ,  $c_1$ ,  $d_1$  arbitrarily real parameters.*

**Remark 2.1** *This solution independent of  $t$  does not present any interest.*

## 2.4 Second order rational solutions

**Proposition 2.2** *The function  $v$  defined by*

$$v(x, t) = -2 \frac{n(x, t)}{d(x, t)^2}, \quad (8)$$

*with*

$$\begin{aligned} n(x, t) = & -192 a_1^8 x^4 + 12 a_1^3 (32 a_1^4 d_1 - 32 a_1^4 c_1) x^3 + 12 a_1^3 (-24 a_1^3 c_1^2 + 48 a_1^3 c_1 d_1 - \\ & 24 a_1^3 d_1^2) x^2 + 12 a_1^3 (96 a_1^5 t - 48 a_1 a_2 d_1 - 8 a_1^2 c_1^3 + 24 a_1^2 d_1 c_1^2 - 48 a_1^2 c_2 + 8 a_1^2 d_1^3 - \\ & 24 a_1^2 c_1 d_1^2 + 48 a_1^2 d_2 + 48 a_1 a_2 c_1) x + 12 a_1^3 (4 c_1 a_1 d_1^3 - 24 a_1 d_1 d_2 + 24 a_2 d_1^2 + 24 a_2 c_1^2 + \\ & 48 t a_1^4 c_1 - a_1 c_1^4 - a_1 d_1^4 + 4 d_1 a_1 c_1^3 - 48 t a_1^4 d_1 - 24 a_1 c_1 c_2 - 48 c_1 a_2 d_1 + 24 c_1 d_2 a_1 + 24 d_1 c_2 a_1 - \\ & 6 c_1^2 a_1 d_1^2), \end{aligned}$$

$$\begin{aligned} d(x, t) = & 8 a_1^4 x^3 + (-12 a_1^3 d_1 + 12 a_1^3 c_1) x^2 + (6 a_1^2 c_1^2 - 12 c_1 a_1^2 d_1 + 6 a_1^2 d_1^2) x - 12 c_2 a_1 + \\ & 12 d_2 a_1 - 3 d_1 a_1 c_1^2 + a_1 c_1^3 + 3 c_1 a_1 d_1^2 + 12 a_2 c_1 - a_1 d_1^3 + 24 t a_1^4 - 12 a_2 d_1 \end{aligned}$$

*is a rational solution to the KdV equation (1), quotient of two polynomials with numerator of degree 4 in  $x$ , 1 in  $t$ , and denominator of degree 6 in  $x$ , 2 in  $t$ .*

## 2.5 Rational solutions of order three

We get the following rational solutions given by :

**Proposition 2.3** *The function  $v$  defined by*

$$v(x, t) = -2 \frac{n(x, t)}{d(x, t)^2}, \quad (9)$$

*with*

$$\begin{aligned} n(x, t) = & -24576 a_1^{16} x^{10} - 24 a_1^2 (-5120 a_1^{13} d_1 + 5120 a_1^{13} c_1) x^9 - 24 a_1^2 (11520 a_1^{12} c_1^2 - \\ & 23040 a_1^{12} c_1 d_1 + 11520 a_1^{12} d_1^2) x^8 - 24 a_1^2 (-15360 a_1^{11} d_1^3 + 46080 a_1^{11} c_1 d_1^2 + 15360 a_1^{11} c_1^3 - \\ & 46080 a_1^{11} c_1^2 d_1) x^7 - 24 a_1^2 (-53760 a_1^{10} c_1^3 d_1 + 80640 a_1^{10} c_1^2 d_1^2 + 13440 a_1^{10} d_1^4 - 53760 a_1^{10} c_1 d_1^3 + \\ & 13440 a_1^{10} c_1^4) x^6 - 24 a_1^2 (-207360 a_1^7 a_2^2 c_1 + 40320 a_1^9 c_1 d_1^4 - 80640 a_1^9 c_1^2 d_1^3 - 69120 a_1^8 a_3 d_1 + \\ & 207360 a_1^7 a_2^2 d_1 - 8064 a_1^9 d_1^5 - 69120 a_1^9 c_3 - 40320 a_1^9 c_1^4 d_1 + 80640 a_1^9 c_1^3 d_1^2 + 8064 a_1^9 c_1^5 - \\ & 207360 a_1^8 a_2 d_2 + 69120 a_1^9 d_3 + 207360 a_1^8 a_2 c_2 + 69120 a_1^8 a_3 c_1) x^5 - 24 a_1^2 (3360 a_1^8 c_1^6 + \\ & 3360 a_1^8 d_1^6 + 345600 t^2 a_1^{14} + 86400 a_1^8 c_2^2 + 86400 a_1^8 d_2^2 - 345600 t a_1^{11} c_2 + 345600 t a_1^{11} d_2 - \\ & 20160 a_1^8 c_1^5 d_1 + 50400 a_1^8 c_1^4 d_1^2 - 67200 a_1^8 c_1^3 d_1^3 + 50400 a_1^8 c_1^2 d_1^4 - 20160 a_1^8 c_1 d_1^5 - \\ & 172800 a_1^8 c_1 c_3 + 172800 a_1^8 c_1 d_3 - 172800 a_1^8 c_2 d_2 + 172800 a_1^8 c_3 d_1 - 172800 a_1^8 d_1 d_3 + 172800 a_1^7 a_3 c_1^2 + \\ & 172800 a_1^7 a_3 d_1^2 - 432000 a_1^6 a_2^2 c_1^2 - 432000 a_1^6 a_2^2 d_1^2 + 345600 t a_1^{10} a_2 c_1 - 345600 t a_1^{10} a_2 d_1 + \\ & 345600 a_1^7 a_2 c_1 c_2 - 345600 a_1^7 a_2 c_1 d_2 - 345600 a_1^7 a_2 c_2 d_1 + 345600 a_1^7 a_2 d_1 d_2 - 345600 a_1^7 a_3 c_1 d_1 + \\ & 864000 a_1^6 a_2^2 c_1 d_1) x^4 - 24 a_1^2 (960 a_1^7 c_1^7 - 960 a_1^7 d_1^7 + 691200 t^2 a_1^{13} c_1 - 691200 t^2 a_1^{13} d_1 - \\ & 6720 a_1^7 c_1^6 d_1 + 20160 a_1^7 c_1^5 d_1^2 - 33600 a_1^7 c_1^4 d_1^3 + 33600 a_1^7 c_1^3 d_1^4 - 20160 a_1^7 c_1^2 d_1^5 + \\ & 6720 a_1^7 c_1 d_1^6 + 172800 a_1^7 c_1 c_2^2 + 172800 a_1^7 c_1 d_2^2 - 172800 a_1^7 c_2^2 d_1 - 172800 a_1^7 d_1 d_2^2 - \\ & 172800 a_1^7 c_1^2 c_3 + 172800 a_1^7 c_1^2 d_3 - 172800 a_1^7 c_3 d_1^2 + 172800 a_1^7 d_1^2 d_3 + 172800 a_1^6 a_3 c_1^3 - \\ & 172800 a_1^6 a_3 d_1^3 - 345600 a_1^5 a_2^2 c_1^3 + 345600 a_1^5 a_2^2 d_1^3 - 691200 t a_1^{10} c_1 c_2 + 691200 t a_1^{10} c_1 d_2 + \\ & 691200 t a_1^{10} c_2 d_1 - 691200 t a_1^{10} d_1 d_2 + 691200 t a_1^9 a_2 c_1^2 + 691200 t a_1^9 a_2 d_1^2 + 345600 a_1^7 c_1 c_3 d_1 - \\ & 345600 a_1^7 c_1 d_1 d_3 + 172800 a_1^6 a_2 c_1^2 c_2 - 172800 a_1^6 a_2 c_1^2 d_2 + 172800 a_1^6 a_2 c_2 d_1^2 - 172800 a_1^6 a_2 d_1^2 d_2 - \end{aligned}$$

$$\begin{aligned}
& 518400 a_1^6 a_3 c_1^2 d_1 + 518400 a_1^6 a_3 c_1 d_1^2 + 1036800 a_1^5 a_2^2 c_1^2 d_1 - 1036800 a_1^5 a_2^2 c_1 d_1^2 - 345600 a_1^7 c_1 c_2 d_2 + \\
& 345600 a_1^7 c_2 d_1 d_2 - 345600 a_1^6 a_2 c_1 c_2 d_1 + 345600 a_1^6 a_2 c_1 d_1 d_2 - 1382400 t a_1^9 a_2 c_1 d_1 x^3 - 24 a_1^2 (180 a_1^6 c_1^8 + \\
& 180 a_1^6 d_1^8 + 518400 t^2 a_1^{12} c_1^2 + 518400 t^2 a_1^{12} d_1^2 - 1440 a_1^6 c_1^7 d_1 + 5040 a_1^6 c_1^6 d_1^2 - 10080 a_1^6 c_1^5 d_1^3 + \\
& 12600 a_1^6 c_1^4 d_1^4 - 10080 a_1^6 c_1^3 d_1^5 + 5040 a_1^6 c_1^2 d_1^6 - 1440 a_1^6 c_1 d_1^7 - 86400 a_1^6 c_1^3 c_3 + 86400 a_1^6 c_1^3 d_3 + \\
& 86400 a_1^6 c_3 d_1^3 - 86400 a_1^6 d_1^3 d_3 + 86400 a_1^5 a_3 c_1^4 + 86400 a_1^5 a_3 d_1^4 - 129600 a_1^4 a_2^2 c_1^4 - \\
& 129600 a_1^4 a_2^2 d_1^4 + 129600 a_1^6 c_1^2 c_2^2 + 129600 a_1^6 c_1^2 d_2^2 + 129600 a_1^6 c_2^2 d_1^2 + 129600 a_1^6 d_1^2 d_2^2 - \\
& 259200 a_1^6 c_1 c_2^2 d_1 - 259200 a_1^6 c_1 d_1 d_2^2 - 259200 a_1^6 c_2 d_1^2 d_2 - 1036800 t^2 a_1^{12} c_1 d_1 - 518400 t a_1^9 c_1^2 c_2 + \\
& 518400 t a_1^9 c_1^2 d_2 - 518400 t a_1^9 c_2 d_1^2 + 518400 t a_1^9 d_1^2 d_2 + 518400 t a_1^8 a_2 c_1^3 - 518400 t a_1^8 a_2 d_1^3 + \\
& 259200 a_1^6 c_1^2 c_3 d_1 - 259200 a_1^6 c_1^2 d_1 d_3 - 259200 a_1^6 c_1 c_3 d_1^2 + 259200 a_1^6 c_1 d_1^2 d_3 - 345600 a_1^5 a_3 c_1^3 d_1 + \\
& 518400 a_1^5 a_3 c_1^2 d_1^2 - 345600 a_1^5 a_3 c_1 d_1^3 + 518400 a_1^4 a_2^2 c_1^3 d_1 - 777600 a_1^4 a_2^2 c_1^2 d_1^2 + 518400 a_1^4 a_2^2 c_1 d_1^3 - \\
& 259200 a_1^6 c_1^2 c_2 d_2 + 518400 a_1^6 c_1 c_2 d_1 d_2 + 1036800 t a_1^9 c_1 c_2 d_1 - 1036800 t a_1^9 c_1 d_1 d_2 - 1555200 t a_1^8 a_2 c_1^2 d_1 + \\
& 1555200 t a_1^8 a_2 c_1 d_1^2 x^2 - 24 a_1^2 (-86400 a_1^5 c_2^3 + 86400 a_1^5 d_2^3 + 691200 t^3 a_1^{14} + 20 a_1^5 c_1^9 - \\
& 20 a_1^5 d_1^9 + 518400 t a_1^8 c_1^2 c_2 d_1 - 518400 t a_1^8 c_1^2 d_1 d_2 - 518400 t a_1^8 c_1 c_2 d_1^2 + 518400 t a_1^8 c_1 d_1^2 d_2 - \\
& 691200 t a_1^7 a_2 c_1^3 d_1 + 1036800 t a_1^7 a_2 c_1^2 d_1^2 - 691200 t a_1^7 a_2 c_1 d_1^3 - 1036800 t a_1^7 a_2 c_1 c_2 + 1036800 t a_1^7 a_2 c_1 d_2 + \\
& 1036800 t a_1^7 a_2 c_2 d_1 - 1036800 t a_1^7 a_2 d_1 d_2 + 172800 t^2 a_1^{11} c_1^3 - 172800 t^2 a_1^{11} d_1^3 + 259200 a_1^5 c_2^2 d_2 - \\
& 259200 a_1^5 c_2 d_2^2 + 86400 a_1^2 a_2^3 c_1^3 - 86400 a_1^2 a_2^3 d_1^3 - 1036800 t^2 a_1^{11} c_2 + 1036800 t^2 a_1^{11} d_2 - \\
& 180 a_1^5 c_1^8 d_1 + 720 a_1^5 c_1^7 d_1^2 - 1680 a_1^5 c_1^6 d_1^3 + 2520 a_1^5 c_1^5 d_1^4 - 2520 a_1^5 c_1^4 d_1^5 + 1680 a_1^5 c_1^3 d_1^6 - \\
& 720 a_1^5 c_1^2 d_1^7 + 180 a_1^5 c_1 d_1^8 + 518400 t a_1^8 c_2^2 + 518400 t a_1^8 d_2^2 - 21600 a_1^5 c_1^4 c_3 + 21600 a_1^5 c_1^4 d_3 - \\
& 21600 a_1^5 c_3 d_1^4 + 21600 a_1^5 d_1^4 d_3 + 21600 a_1^4 a_3 c_1^5 - 21600 a_1^4 a_3 d_1^5 - 21600 a_1^3 a_2^2 c_1^5 + \\
& 21600 a_1^3 a_2^2 d_1^5 + 43200 a_1^5 c_1^3 c_2^2 + 43200 a_1^5 c_1^3 d_2^2 - 43200 a_1^5 c_2^2 d_1^3 - 43200 a_1^5 d_1^3 d_2^2 - \\
& 86400 a_1^5 c_1^3 c_2 d_2 - 129600 a_1^5 c_1^2 c_2^2 d_1 - 129600 a_1^5 c_1^2 d_1 d_2^2 + 129600 a_1^5 c_1 c_2^2 d_1^2 + 129600 a_1^5 c_1 d_1^2 d_2^2 + \\
& 86400 a_1^5 c_2 d_1^3 d_2 - 518400 t^2 a_1^{11} c_1^2 d_1 + 518400 t^2 a_1^{11} c_1 d_1^2 - 172800 t a_1^8 c_1^3 c_2 + 172800 t a_1^8 c_1^3 d_2 + \\
& 172800 t a_1^8 c_2 d_1^3 - 172800 t a_1^8 d_1^3 d_2 + 172800 t a_1^7 a_2 c_1^4 + 172800 t a_1^7 a_2 d_1^4 - 1036800 t a_1^8 c_2 d_2 + \\
& 518400 t a_1^6 a_2^2 c_1^2 + 518400 t a_1^6 a_2^2 d_1^2 + 86400 a_1^5 c_1^3 c_3 d_1 - 86400 a_1^5 c_1^3 d_1 d_3 - 129600 a_1^5 c_1^2 c_3 d_1^2 + \\
& 129600 a_1^5 c_1^2 d_1^2 d_3 + 86400 a_1^5 c_1 c_3 d_1^3 - 86400 a_1^5 c_1 d_1^3 d_3 - 21600 a_1^4 a_2 c_1^4 c_2 + 21600 a_1^4 a_2 c_1^4 d_2 - \\
& 21600 a_1^4 a_2 c_2 d_1^4 + 21600 a_1^4 a_2 d_1^4 d_2 - 108000 a_1^4 a_3 c_1^4 d_1 + 216000 a_1^4 a_3 c_1^3 d_1^2 - 216000 a_1^4 a_3 c_1^2 d_1^3 + \\
& 108000 a_1^4 a_3 c_1 d_1^4 + 108000 a_1^3 a_2^2 c_1^4 d_1 - 216000 a_1^3 a_2^2 c_1^3 d_1^2 + 216000 a_1^3 a_2^2 c_1^2 d_1^3 - 108000 a_1^3 a_2^2 c_1 d_1^4 + \\
& 259200 a_1^4 a_2 c_1 c_2^2 + 259200 a_1^4 a_2 c_1 d_2^2 - 259200 a_1^4 a_2 c_2^2 d_1 - 259200 a_1^4 a_2 d_1 d_2^2 - 259200 a_1^3 a_2^2 c_1^2 c_2 + \\
& 259200 a_1^3 a_2^2 c_1^2 d_2 - 259200 a_1^3 a_2^2 c_2 d_1^2 + 259200 a_1^3 a_2^2 d_1^2 d_2 - 259200 a_1^2 a_2^3 c_1^2 d_1 + 259200 a_1^2 a_2^3 c_1 d_1^2 + \\
& 1036800 t^2 a_1^{10} a_2 c_1 - 1036800 t^2 a_1^{10} a_2 d_1 + 259200 a_1^5 c_1^2 c_2 d_1 d_2 - 259200 a_1^5 c_1 c_2 d_1^2 d_2 + \\
& 86400 a_1^4 a_2 c_1^3 c_2 d_1 - 86400 a_1^4 a_2 c_1^3 d_1 d_2 - 129600 a_1^4 a_2 c_1^2 c_2 d_1^2 + 129600 a_1^4 a_2 c_1^2 d_1^2 d_2 + \\
& 86400 a_1^4 a_2 c_1 c_2 d_1^3 - 86400 a_1^4 a_2 c_1 d_1^3 d_2 - 518400 a_1^4 a_2 c_1 c_2 d_2 + 518400 a_1^4 a_2 c_2 d_1 d_2 + 518400 a_1^3 a_2^2 c_1 c_2 d_1 - \\
& 518400 a_1^3 a_2^2 c_1 d_1 d_2 - 1036800 t a_1^6 a_2^2 c_1 d_1 x - 24 a_1^2 (518400 a_1^3 a_2 c_1 c_2 d_1 d_2 + 1036800 t a_1^6 a_2 c_1 c_2 d_1 - \\
& 1036800 t a_1^6 a_2 c_1 d_1 d_2 + 518400 a_1^2 a_2 a_3 c_1 c_2 - 518400 a_1^2 a_2 a_3 c_1 d_2 - 518400 a_1^2 a_2 a_3 c_2 d_1 + \\
& 518400 a_1^2 a_2 a_3 d_1 d_2 + 1036800 a_1 a_2^2 a_3 c_1 d_1 + 43200 a_1^4 c_1^3 c_2 d_1 d_2 - 64800 a_1^4 c_1^2 c_2 d_1^2 d_2 + \\
& 43200 a_1^4 c_1 c_2 d_1^3 d_2 - 259200 a_1^3 a_2 c_1^2 c_2 d_2 - 259200 a_1^3 a_2 c_1 c_2^2 d_1 - 259200 a_1^3 a_2 c_1 d_1 d_2^2 - \\
& 259200 a_1^3 a_2 c_2 d_1^2 d_2 - 86400 t a_1^7 c_1^3 d_1 d_2 - 129600 t a_1^7 c_1^2 c_2 d_1^2 + 129600 t a_1^7 c_1^2 d_1^2 d_2 + 86400 t a_1^7 c_1 c_2 d_1^3 - \\
& 86400 t a_1^7 c_1 d_1^3 d_2 - 108000 t a_1^6 a_2 c_1^4 d_1 + 216000 t a_1^6 a_2 c_1^3 d_1^2 - 216000 t a_1^6 a_2 c_1^2 d_1^3 + 108000 t a_1^6 a_2 c_1 d_1^4 - \\
& 777600 t a_1^5 a_2^2 c_1^2 d_1 + 777600 t a_1^5 a_2^2 c_1 d_1^2 + 86400 t a_1^7 c_1^3 c_2 d_1 - 518400 t a_1^7 c_1 c_2 d_2 + 518400 t a_1^7 c_2 d_1 d_2 - \\
& 518400 t a_1^6 a_2 c_1^2 c_2 + 518400 t a_1^6 a_2 c_1^2 d_2 - 518400 t a_1^6 a_2 c_2 d_1^2 + 518400 t a_1^6 a_2 d_1^2 d_2 + 21600 a_1^3 a_2 c_1^4 c_2 d_1 - \\
& 21600 a_1^3 a_2 c_1^4 d_1 d_2 - 43200 a_1^3 a_2 c_1^3 c_2 d_1^2 + 43200 a_1^3 a_2 c_1^3 d_1^2 d_2 + 43200 a_1^3 a_2 c_1^2 c_2 d_1^3 - \\
& 43200 a_1^3 a_2 c_1^2 d_1^3 d_2 - 21600 a_1^3 a_2 c_1 c_2 d_1^4 + 21600 a_1^3 a_2 c_1 d_1^4 d_2 + 388800 a_1^2 a_2^2 c_1^2 c_2 d_1 - \\
& 388800 a_1^2 a_2^2 c_1^2 d_1 d_2 - 388800 a_1^2 a_2^2 c_1 c_2 d_1^2 + 388800 a_1^2 a_2^2 c_1 d_1^2 d_2 - 1036800 t^2 a_1^9 a_2 c_1 d_1 + \\
& 518400 a_1^3 a_2 c_3 d_2 - 518400 a_1^3 a_2 d_2 d_3 - 172800 a_1^3 a_3 c_1 c_3 + 172800 a_1^3 a_3 c_1 d_3 + 172800 a_1^3 a_3 c_3 d_1 - \\
& 172800 a_1^3 a_3 d_1 d_3 + 518400 a_1^2 a_2^2 c_1 c_3 - 518400 a_1^2 a_2^2 c_1 d_3 - 1555200 a_1^2 a_2^2 c_2 d_2 - 518400 a_1^2 a_2^2 c_3 d_1 + \\
& 518400 a_1^2 a_2^2 d_1 d_3 - 172800 a_1^2 a_2^3 c_1 d_1 - 1555200 a_1 a_2^3 c_1 c_2 + 1555200 a_1 a_2^3 c_1 d_2 + 1555200 a_1 a_2^3 c_2 d_1 -
\end{aligned}$$

$$\begin{aligned}
& 1555200 a_1 a_2^3 d_1 d_2 - 518400 a_1 a_2^2 a_3 c_1^2 - 518400 a_1 a_2^2 a_3 d_1^2 + 259200 t a_1^5 a_2^2 c_1^3 - 259200 t a_1^5 a_2^2 d_1^3 - \\
& 10800 a_1^4 c_1^4 c_2 d_2 - 21600 a_1^4 c_1^3 c_2^2 d_1 - 21600 a_1^4 c_1^3 d_1 d_2^2 + 32400 a_1^4 c_1^2 c_2^2 d_1^2 + 32400 a_1^4 c_1^2 d_1^2 d_2^2 - \\
& 21600 a_1^4 c_1 c_2^2 d_1^3 - 21600 a_1^4 c_1 d_1^3 d_2^2 - 10800 a_1^4 c_2 d_1^4 d_2 + 129600 a_1^3 a_2 c_1^2 c_2^2 + 129600 a_1^3 a_2 c_1^2 d_2^2 + \\
& 129600 a_1^3 a_2 c_2^2 d_1^2 + 129600 a_1^3 a_2 d_1^2 d_2^2 - 518400 a_1^3 a_2 c_2 c_3 + 518400 a_1^3 a_2 c_2 d_3 - 21600 t a_1^7 c_1^4 c_2 + \\
& 21600 t a_1^7 c_1^4 d_2 - 21600 t a_1^7 c_2 d_1^4 + 21600 t a_1^7 d_1^4 d_2 + 21600 t a_1^6 a_2 c_1^5 - 21600 t a_1^6 a_2 d_1^5 - \\
& 172800 a_1 a_2^3 c_1^3 d_1 + 259200 a_1 a_2^3 c_1^2 d_1^2 - 172800 a_1 a_2^3 c_1 d_1^3 - 86400 t^2 a_1^{10} c_1^3 d_1 + 129600 t^2 a_1^{10} c_1^2 d_1^2 - \\
& 86400 t^2 a_1^{10} c_1 d_1^3 + 21600 a_1^2 a_2^2 c_1^3 d_1^3 - 16200 a_1^2 a_2^2 c_1^2 d_1^4 + 6480 a_1^2 a_2^2 c_1 d_1^5 + 129600 a_1^4 c_1 c_2^2 d_2 - \\
& 129600 a_1^4 c_1 c_2 d_2^2 - 129600 a_1^4 c_2^2 d_1 d_2 + 129600 a_1^4 c_2 d_1 d_2^2 - 518400 t^2 a_1^{10} c_1 c_2 + 518400 t^2 a_1^{10} c_1 d_2 + \\
& 518400 t^2 a_1^{10} c_2 d_1 - 518400 t^2 a_1^{10} d_1 d_2 + 518400 t^2 a_1^9 a_2 c_1^2 + 518400 t^2 a_1^9 a_2 d_1^2 - 129600 a_1^2 a_2^2 c_1^3 c_2 + \\
& 129600 a_1^2 a_2^2 c_1^3 d_2 + 129600 a_1^2 a_2^2 c_2 d_1^3 - 129600 a_1^2 a_2^2 d_1^3 d_2 + 259200 t a_1^7 c_1 c_2^2 + 259200 t a_1^7 c_1 d_2^2 - \\
& 259200 t a_1^7 c_2^2 d_1 - 259200 t a_1^7 d_1 d_2^2 + 10800 a_1^4 c_1^4 c_3 d_1 - 10800 a_1^4 c_1^4 d_1 d_3 - 21600 a_1^4 c_1^3 c_3 d_1^2 + \\
& 21600 a_1^4 c_1^3 d_1^2 d_3 + 21600 a_1^4 c_1^2 c_3 d_1^3 - 21600 a_1^4 c_1^2 d_1^3 d_3 - 10800 a_1^4 c_1 c_3 d_1^4 + 10800 a_1^4 c_1 d_1^4 d_3 - \\
& 4320 a_1^3 a_2 c_1^5 c_2 + 4320 a_1^3 a_2 c_1^5 d_2 + 4320 a_1^3 a_2 c_2 d_1^5 - 4320 a_1^3 a_2 d_1^5 d_2 - 12960 a_1^3 a_3 c_1^5 d_1 + \\
& 32400 a_1^3 a_3 c_1^4 d_1^2 - 43200 a_1^3 a_3 c_1^3 d_1^3 + 32400 a_1^3 a_3 c_1^2 d_1^4 - 12960 a_1^3 a_3 c_1 d_1^5 + 6480 a_1^2 a_2^2 c_1^5 d_1 - \\
& 16200 a_1^2 a_2^2 c_1^4 d_1^2 + 5400 a_1^4 c_1^4 c_2^2 + 5400 a_1^4 c_1^4 d_2^2 + 5400 a_1^4 c_2^2 d_1^4 + 5400 a_1^4 d_1^4 d_2^2 - \\
& 172800 a_1^4 c_3 d_3 + 777600 a_1^2 a_2^2 c_2^2 + 777600 a_1^2 a_2^2 d_2^2 + 86400 a_1^2 a_3^2 c_1^2 + 86400 a_1^2 a_3^2 d_1^2 - \\
& 1555200 a_2^4 c_1 d_1 + 45 a_1^4 c_1^8 d_1^2 - 120 a_1^4 c_1^7 d_1^3 + 210 a_1^4 c_1^6 d_1^4 - 252 a_1^4 c_1^5 d_1^5 + 210 a_1^4 c_1^4 d_1^6 - \\
& 120 a_1^4 c_1^3 d_1^7 + 45 a_1^4 c_1^2 d_1^8 - 10 a_1^4 c_1 d_1^9 - 2160 a_1^4 c_1^5 c_3 + 2160 a_1^4 c_1^5 d_3 + 2160 a_1^4 c_3 d_1^5 - \\
& 2160 a_1^4 d_1^5 d_3 + 2160 a_1^3 a_3 c_1^6 + 2160 a_1^3 a_3 d_1^6 - 1080 a_1^2 a_2^2 c_1^6 - 1080 a_1^2 a_2^2 d_1^6 - 43200 a_1^4 c_1 c_2^3 + \\
& 43200 a_1^4 c_1 d_2^3 + 43200 a_1^4 c_2^3 d_1 - 43200 a_1^4 d_1 d_2^3 + 345600 t^3 a_1^{13} c_1 - 345600 t^3 a_1^{13} d_1 - \\
& 10 a_1^4 c_1^9 d_1 + 43200 a_1 a_2^3 c_1^4 + 43200 a_1 a_2^3 d_1^4 + 21600 t^2 a_1^{10} c_1^4 + 21600 t^2 a_1^{10} d_1^4 + a_1^4 c_1^{10} + \\
& a_1^4 d_1^{10} + 86400 a_1^4 c_3^2 + 86400 a_1^4 d_3^2 + 777600 a_2^4 c_1^2 + 777600 a_2^4 d_1^2),
\end{aligned}$$

$$\begin{aligned}
d(x, t) = & -64 a_1^8 x^6 + (192 a_1^7 d_1 - 192 a_1^7 c_1) x^5 + (-240 a_1^6 d_1^2 - 240 a_1^6 c_1^2 + 480 a_1^6 c_1 d_1) x^4 + \\
& (-160 a_1^5 c_1^3 + 480 a_1^5 c_2 - 480 a_1^5 d_2 + 160 a_1^5 d_1^3 + 480 a_1^5 c_1^2 d_1 - 960 t a_1^8 + 480 a_1^4 a_2 d_1 - \\
& 480 a_1^4 a_2 c_1 - 480 a_1^5 c_1 d_1^2) x^3 + (-1440 t a_1^7 c_1 - 60 a_1^4 d_1^4 - 720 a_1^4 c_1 d_2 - 60 a_1^4 c_1^4 + 1440 a_1^3 a_2 c_1 d_1 + \\
& 720 a_1^4 d_1 d_2 - 720 a_1^4 c_2 d_1 - 720 a_1^3 a_2 d_1^2 + 720 a_1^4 c_1 c_2 + 240 a_1^4 c_1 d_1^3 + 1440 t a_1^7 d_1 + 240 a_1^4 c_1^3 d_1 - \\
& 360 a_1^4 c_1^2 d_1^2 - 720 a_1^3 a_2 c_1^2) x^2 + (4320 a_1 a_2^2 d_1 - 1440 a_1^3 c_3 + 1440 a_1^3 d_3 - 720 a_1^6 d_1^2 t - \\
& 720 a_1^6 c_1^2 t - 4320 a_1 a_2^2 c_1 + 12 a_1^3 d_1^5 - 12 a_1^3 c_1^5 + 360 a_2 a_1^2 d_1^3 - 4320 a_2 a_1^2 d_2 + 4320 a_2 a_1^2 c_2 - \\
& 360 a_2 a_1^2 c_1^3 - 360 a_1^3 d_1^2 d_2 + 360 a_1^3 d_1^2 c_2 - 120 a_1^3 d_1^2 c_1^3 + 120 a_1^3 c_1^2 d_1^3 - 360 a_1^3 c_1^2 d_2 + \\
& 360 a_1^3 c_1^2 c_2 + 1440 a_3 a_1^2 c_1 - 1440 a_3 a_1^2 d_1 - 60 a_1^3 d_1^4 c_1 + 60 a_1^3 c_1^4 d_1 + 1080 a_1^2 c_1^2 a_2 d_1 + \\
& 720 a_1^3 d_1 d_2 c_1 - 720 a_1^3 c_1 c_2 d_1 + 1440 t a_1^6 c_1 d_1 - 1080 a_1^2 d_1^2 a_2 c_1) x + 60 a_1^2 d_1^3 d_2 + 720 a_1 a_3 d_1^2 + \\
& 120 t a_1^5 d_1^3 + 2880 t^2 a_1^8 + 720 a_1 a_2 d_1 d_2 + 2880 t a_1^4 a_2 c_1 - 60 a_1 a_2 d_1^4 - 2880 t a_1^4 d_1 a_2 + \\
& 720 a_1 a_3 c_1^2 - d_1^6 a_1^2 + 720 a_1 a_2 c_1 c_2 - c_1^6 a_1^2 - 15 a_1^2 c_1^2 d_1^4 - 15 a_1^2 c_1^4 d_1^2 + 720 d_3 a_1^2 c_1 + \\
& 720 c_3 a_1^2 d_1 + 6 d_1^5 a_1^2 c_1 + 6 c_1^5 a_1^2 d_1 - 60 c_2 a_1^2 d_1^3 - 1440 c_2 a_1^2 d_2 + 20 d_1^3 a_1^2 c_1^3 - 60 d_2 a_1^2 c_1^3 - \\
& 2880 t a_1^5 c_2 - 720 a_1^2 c_1 c_3 + 60 a_1^2 c_1^3 c_2 - 120 t a_1^5 c_1^3 - 60 a_1 a_2 c_1^4 + 2880 t a_1^5 d_2 - 720 a_1^2 d_1 d_3 - \\
& 1440 a_2^2 d_1^2 + 720 a_1^2 c_2^2 + 720 a_1^2 d_2^2 + 2880 d_1 a_2^2 c_1 + 180 d_1 a_1^2 c_1^2 d_2 - 1440 d_1 a_3 a_1 c_1 - \\
& 360 a_1 c_1^2 a_2 d_1^2 + 180 a_1^2 c_1 c_2 d_1^2 - 180 d_1^2 d_2 a_1^2 c_1 - 180 c_1^2 c_2 a_1^2 d_1 - 360 d_1^2 t a_1^5 c_1 + 360 c_1^2 t a_1^5 d_1 - \\
& 720 c_2 a_1 a_2 d_1 + 240 d_1^3 a_1 a_2 c_1 - 720 d_2 a_1 a_2 c_1 + 240 c_1^3 a_1 a_2 d_1 - 1440 a_2^2 c_1^2
\end{aligned}$$

is a rational solution to the KdV equation (1), quotient of two polynomials with the numerator of order 10 in  $x$ , 3 in  $t$ , the denominator of degree 12 in  $x$ , 4 in  $t$ .

## 2.6 Another orders

For greater orders, the solutions become very complex and we cannot give them here. We only give solutions with parameters  $a_j$  equal to 1 and all other parameters equal to 0.

### Order 4

**Proposition 2.4** *The function  $v$  defined by*

$$v(x, t) = -2 \frac{n(x, t)}{d(x, t)^2}, \quad (10)$$

$$n(x, t) = -10x^{18} - 360tx^{15} - 14175t^2x^{12} + 330750t^3x^9 + 5953500t^4x^6 - 22325625t^6,$$

$$d(x, t) = x^{10} + 45tx^7 + 4725t^3x$$

*is a rational solution to the KdV equation (1), quotient of two polynomials with numerator of degree 18 in  $x$ , 6 in  $t$ , and denominator of degree 20 in  $x$ , 6 in  $t$ .*

### Order 5

**Proposition 2.5** *The function  $v$  defined by*

$$v(x, t) = -2 \frac{n(x, t)}{d(x, t)^2}, \quad (11)$$

$$n(x, t) = -15x^{28} - 1890tx^{25} - 113400t^2x^{22} - 84837375t^4x^{16} - 7501410000t^5x^{13} - 64699661250t^6x^{10} - 168781725000t^7x^7 - 4430520281250t^8x^4 + 8861040562500t^9x,$$

$$d(x, t) = -x^{15} - 105tx^{12} - 1575t^2x^9 - 33075t^3x^6 + 992250t^4x^3 + 1488375t^5$$

*is a rational solution to the KdV equation (1), quotient of two polynomials with numerator of degree 28 in  $x$ , 9 in  $t$ , and denominator of degree 30 in  $x$ , 10 in  $t$ .*

### Order 6

**Proposition 2.6** *The function  $v$  defined by*

$$v(x, t) = -2 \frac{n(x, t)}{d(x, t)^2}, \quad (12)$$

$$n(x, t) = -21x^{40} - 6300tx^{37} - 793800t^2x^{34} - 39690000t^3x^{31} - 1909585125t^4x^{28} - 43320642750t^5x^{25} + 8174661547500t^6x^{22} + 376889591925000t^7x^{19} + 5541251715759375t^8x^{16} + 60042410851500000t^9x^{13} - 1339696292124093750t^{10}x^{10} + 2026431366238125000t^{11}x^7 - 88656372272917968750t^{12}x^4 - 106387646727501562500t^{13}x,$$

$$d(x, t) = -x^{21} - 210tx^{18} - 10395t^2x^{15} - 264600t^3x^{12} + 5457375t^4x^9 - 343814625t^5x^6 - 3438146250t^6x^3 + 5157219375t^7$$

*is a rational solution to the KdV equation (1), quotient of two polynomials with numerator of degree 40 in  $x$ , 13 in  $t$ , and denominator of degree 42 in  $x$ , 14 in  $t$ .*



### 3 Conclusion

We have given three types of representations of solutions to the KdV equation. First, solutions in terms of elementary exponential functions have been constructed. In particular, performing a passage to the limit when one parameter goes to 0 we get rational solutions to the KdV equation. We give another representation in terms of determinants without the presence of a limit. So we obtain an infinite hierarchy of multi-parametric families of rational solutions to the KdV equation as a quotient of a polynomials depending on  $3N$  real parameters.

We can formulate some remarks about the structure of these solutions. For the  $N$ -order solution, the numerator of the solution is a polynomial of degree  $N(N - 1) - 2$  in  $x$  and the denominator a polynomial of degree  $N(N + 1)$  in  $x$ . The structure relative to  $t$  seems more complicated. It would be relevant to study these polynomials in more details.

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