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Rational solutions to the KdV equation depending on multi-parameters

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Abstract

We construct multi-parametric rational solutions to the KdV equation. For this, we use solutions in terms of exponentials depending on several parameters and take a limit when one of these parameters goes to 0. Here we present degenerate rational solutions and give a result without the presence of a limit as a quotient of polynomials depending on $3N$ parameters. We give the explicit expressions of some of these rational solutions.

Key Words : KdV equation, rational solutions.

PACS numbers :

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1 Introduction

We consider the KdV equation

$$4u_t = 6uu_x - u_{xxx}, \quad (1)$$

where as usual, the subscripts x and t denote partial derivatives and u is a function of x and t .

Korteweg and de Vries [1] introduced this equation (1) for the first time in 1895.

The KdV equation (1) is the basis of the most common tool for the (1+1)-dimensional modelling of shallow water waves. It is now known that this equation describes the propagation of waves with weak dispersion in various nonlinear media.

In 1967, Gardner et al. [2] gave a method of resolution of this equation. Zakharov and Faddeev [3] proved that this is a complete integrable equation in 1971.

Using the bilinear method, Hirota [4] constructed solutions in 1971. Among the works which have been realized, we can mention some of them. Solutions in terms of Riemann theta functions [5] were given by Its and Matveev in 1975. Lax gave in 1975 the expressions of periodic and almost periodic solutions [6]; we can also quote, Matveev in 1992 [7], Ma in 2004 [8], Kovalyov in 2005 [9].

We consider solutions to the KdV equation in this paper. From elementary exponential functions depending on several parameters, we construct rational solutions in performing a passage to the limit when one of these parameters goes to 0. We obtain rational solutions as depending on $3N$ parameters at order N . To get a result which does not depend on a limit, we have to consider another type of functions. With this new functions, we can give explicit solutions and we get a hierarchy of rational solution of order N to the KdV equation depending on $3N$ parameters.

2 Solutions to the KdV equation

2.1 Solutions to the KdV equation in terms of elementary exponentials

We consider the KdV equation (1)

$$4u_t = 6uu_x - u_{xxx}.$$

We use the following notations. We consider the real numbers α_j , β_j and γ_j defined by :

$$\alpha_j = \sum_{k=1}^N a_k (je)^{2k-1}, \quad \beta_j = \sum_{k=1}^N c_k (je)^{2k-1}, \quad \gamma_j = \sum_{k=1}^N d_k (je)^{2k-1}, \quad \text{for } 1 \leq j \leq N \quad (2)$$

We consider the elementary functions :

$$f_{ij}(x, t) = \alpha_j^{i-1} \exp(\alpha_j x - \alpha_j^3 t + \beta_j) - \alpha_j^{i-1} \exp(\alpha_j x - \alpha_j^3 t + \gamma_j), \quad \text{for } 1 \leq i \leq N \quad (3)$$

Then, we have the following statement :

Theorem 2.1 *The function v defined by*

$$v(x, t) = -2\partial_x^2 \ln(\det(f_{ij})_{(i,j) \in [1,N]}) \quad (4)$$

is a solution to the KdV equation (1) with e , a_j , c_j and d_j , $1 \leq j \leq N$ arbitrarily real parameters.

2.2 Rational solutions to the KdV equation

We are going to perform a limit when the parameter e tends to 0. With this idea, we can obtain rational solutions to the KdV equation,

2.2.1 Rational solutions as a limit case

We get the following result :

Theorem 2.2 *The function v defined by*

$$v(x, t) = \lim_{e \rightarrow 0} -2\partial_x^2 \ln(\det(f_{ij})_{(i,j) \in [1, N]}) \quad (5)$$

is a rational solution to the KdV equation (1) depending on $3N$ parameters a_j , c_j and d_j , $1 \leq j \leq N$.

2.2.2 Degenerate rational solutions

We can give a more precise result avoiding the presence of a limit. For this, we consider another type of functions g_{ij} defined by :

$$g_{ij} = \frac{\partial^{2j-1} f_{ij}}{\partial e^{2j-1}} \Big|_{e=0}, \text{ for } 1 \leq i \leq N, 1 \leq j \leq N.$$

Then get the following result :

Theorem 2.3 *The function v defined by*

$$v(x, t) = -2\partial_x^2 \ln(\det(g_{ij})_{(i,j) \in [1, N]}) \quad (6)$$

is a rational solution to the KdV equation (1) depending on $3N$ parameters a_j , c_j and d_j , $1 \leq j \leq N$.

So for each positive integer N , we get rational solutions to the KdV equation depending on $3N$ real parameters.

In the following we give some examples of rational solutions. These results are consequences of the previous result (6).

2.3 First order rational solutions

We have the following result at order $N = 1$:

Proposition 2.1 *The function v defined by*

$$v(x, t) = 8 \frac{a_1^2}{(2a_1x - d_1 + c_1)^2}, \quad (7)$$

is a solution to the KdV equation (1) with a_1 , c_1 , d_1 arbitrarily real parameters.

Remark 2.1 *This solution independent of t does not present any interest.*

2.4 Second order rational solutions

Proposition 2.2 *The function v defined by*

$$v(x, t) = -2 \frac{n(x, t)}{d(x, t)^2}, \quad (8)$$

with

$$\begin{aligned} n(x, t) = & -192 a_1^8 x^4 + 12 a_1^3 (32 a_1^4 d_1 - 32 a_1^4 c_1) x^3 + 12 a_1^3 (-24 a_1^3 c_1^2 + 48 a_1^3 c_1 d_1 - \\ & 24 a_1^3 d_1^2) x^2 + 12 a_1^3 (96 a_1^5 t - 48 a_1 a_2 d_1 - 8 a_1^2 c_1^3 + 24 a_1^2 d_1 c_1^2 - 48 a_1^2 c_2 + 8 a_1^2 d_1^3 - \\ & 24 a_1^2 c_1 d_1^2 + 48 a_1^2 d_2 + 48 a_1 a_2 c_1) x + 12 a_1^3 (4 c_1 a_1 d_1^3 - 24 a_1 d_1 d_2 + 24 a_2 d_1^2 + 24 a_2 c_1^2 + \\ & 48 t a_1^4 c_1 - a_1 c_1^4 - a_1 d_1^4 + 4 d_1 a_1 c_1^3 - 48 t a_1^4 d_1 - 24 a_1 c_1 c_2 - 48 c_1 a_2 d_1 + 24 c_1 d_2 a_1 + 24 d_1 c_2 a_1 - \\ & 6 c_1^2 a_1 d_1^2), \end{aligned}$$

$$\begin{aligned} d(x, t) = & 8 a_1^4 x^3 + (-12 a_1^3 d_1 + 12 a_1^3 c_1) x^2 + (6 a_1^2 c_1^2 - 12 c_1 a_1^2 d_1 + 6 a_1^2 d_1^2) x - 12 c_2 a_1 + \\ & 12 d_2 a_1 - 3 d_1 a_1 c_1^2 + a_1 c_1^3 + 3 c_1 a_1 d_1^2 + 12 a_2 c_1 - a_1 d_1^3 + 24 t a_1^4 - 12 a_2 d_1 \end{aligned}$$

is a rational solution to the KdV equation (1), quotient of two polynomials with numerator of degree 4 in x , 1 in t , and denominator of degree 6 in x , 2 in t .

2.5 Rational solutions of order three

We get the following rational solutions given by :

Proposition 2.3 *The function v defined by*

$$v(x, t) = -2 \frac{n(x, t)}{d(x, t)^2}, \quad (9)$$

with

$$\begin{aligned} n(x, t) = & -24576 a_1^{16} x^{10} - 24 a_1^2 (-5120 a_1^{13} d_1 + 5120 a_1^{13} c_1) x^9 - 24 a_1^2 (11520 a_1^{12} c_1^2 - \\ & 23040 a_1^{12} c_1 d_1 + 11520 a_1^{12} d_1^2) x^8 - 24 a_1^2 (-15360 a_1^{11} d_1^3 + 46080 a_1^{11} c_1 d_1^2 + 15360 a_1^{11} c_1^3 - \\ & 46080 a_1^{11} c_1^2 d_1) x^7 - 24 a_1^2 (-53760 a_1^{10} c_1^3 d_1 + 80640 a_1^{10} c_1^2 d_1^2 + 13440 a_1^{10} d_1^4 - 53760 a_1^{10} c_1 d_1^3 + \\ & 13440 a_1^{10} c_1^4) x^6 - 24 a_1^2 (-207360 a_1^7 a_2^2 c_1 + 40320 a_1^9 c_1 d_1^4 - 80640 a_1^9 c_1^2 d_1^3 - 69120 a_1^8 a_3 d_1 + \\ & 207360 a_1^7 a_2^2 d_1 - 8064 a_1^9 d_1^5 - 69120 a_1^9 c_3 - 40320 a_1^9 c_1^4 d_1 + 80640 a_1^9 c_1^3 d_1^2 + 8064 a_1^9 c_1^5 - \\ & 207360 a_1^8 a_2 d_2 + 69120 a_1^9 d_3 + 207360 a_1^8 a_2 c_2 + 69120 a_1^8 a_3 c_1) x^5 - 24 a_1^2 (3360 a_1^8 c_1^6 + \\ & 3360 a_1^8 d_1^6 + 345600 t^2 a_1^{14} + 86400 a_1^8 c_2^2 + 86400 a_1^8 d_2^2 - 345600 t a_1^{11} c_2 + 345600 t a_1^{11} d_2 - \\ & 20160 a_1^8 c_1^5 d_1 + 50400 a_1^8 c_1^4 d_1^2 - 67200 a_1^8 c_1^3 d_1^3 + 50400 a_1^8 c_1^2 d_1^4 - 20160 a_1^8 c_1 d_1^5 - \\ & 172800 a_1^8 c_1 c_3 + 172800 a_1^8 c_1 d_3 - 172800 a_1^8 c_2 d_2 + 172800 a_1^8 c_3 d_1 - 172800 a_1^8 d_1 d_3 + 172800 a_1^7 a_3 c_1^2 + \\ & 172800 a_1^7 a_3 d_1^2 - 432000 a_1^6 a_2^2 c_1^2 - 432000 a_1^6 a_2^2 d_1^2 + 345600 t a_1^{10} a_2 c_1 - 345600 t a_1^{10} a_2 d_1 + \\ & 345600 a_1^7 a_2 c_1 c_2 - 345600 a_1^7 a_2 c_1 d_2 - 345600 a_1^7 a_2 c_2 d_1 + 345600 a_1^7 a_2 d_1 d_2 - 345600 a_1^7 a_3 c_1 d_1 + \\ & 864000 a_1^6 a_2^2 c_1 d_1) x^4 - 24 a_1^2 (960 a_1^7 c_1^7 - 960 a_1^7 d_1^7 + 691200 t^2 a_1^{13} c_1 - 691200 t^2 a_1^{13} d_1 - \\ & 6720 a_1^7 c_1^6 d_1 + 20160 a_1^7 c_1^5 d_1^2 - 33600 a_1^7 c_1^4 d_1^3 + 33600 a_1^7 c_1^3 d_1^4 - 20160 a_1^7 c_1^2 d_1^5 + \\ & 6720 a_1^7 c_1 d_1^6 + 172800 a_1^7 c_1 c_2^2 + 172800 a_1^7 c_1 d_2^2 - 172800 a_1^7 c_2^2 d_1 - 172800 a_1^7 d_1 d_2^2 - \\ & 172800 a_1^7 c_1^2 c_3 + 172800 a_1^7 c_1^2 d_3 - 172800 a_1^7 c_3 d_1^2 + 172800 a_1^7 d_1^2 d_3 + 172800 a_1^6 a_3 c_1^3 - \\ & 172800 a_1^6 a_3 d_1^3 - 345600 a_1^5 a_2^2 c_1^3 + 345600 a_1^5 a_2^2 d_1^3 - 691200 t a_1^{10} c_1 c_2 + 691200 t a_1^{10} c_1 d_2 + \\ & 691200 t a_1^{10} c_2 d_1 - 691200 t a_1^{10} d_1 d_2 + 691200 t a_1^9 a_2 c_1^2 + 691200 t a_1^9 a_2 d_1^2 + 345600 a_1^7 c_1 c_3 d_1 - \\ & 345600 a_1^7 c_1 d_1 d_3 + 172800 a_1^6 a_2 c_1^2 c_2 - 172800 a_1^6 a_2 c_1^2 d_2 + 172800 a_1^6 a_2 c_2 d_1^2 - 172800 a_1^6 a_2 d_1^2 d_2 - \end{aligned}$$

$$\begin{aligned}
& 518400 a_1^6 a_3 c_1^2 d_1 + 518400 a_1^6 a_3 c_1 d_1^2 + 1036800 a_1^5 a_2^2 c_1^2 d_1 - 1036800 a_1^5 a_2^2 c_1 d_1^2 - 345600 a_1^7 c_1 c_2 d_2 + \\
& 345600 a_1^7 c_2 d_1 d_2 - 345600 a_1^6 a_2 c_1 c_2 d_1 + 345600 a_1^6 a_2 c_1 d_1 d_2 - 1382400 t a_1^9 a_2 c_1 d_1 x^3 - 24 a_1^2 (180 a_1^6 c_1^8 + \\
& 180 a_1^6 d_1^8 + 518400 t^2 a_1^{12} c_1^2 + 518400 t^2 a_1^{12} d_1^2 - 1440 a_1^6 c_1^7 d_1 + 5040 a_1^6 c_1^6 d_1^2 - 10080 a_1^6 c_1^5 d_1^3 + \\
& 12600 a_1^6 c_1^4 d_1^4 - 10080 a_1^6 c_1^3 d_1^5 + 5040 a_1^6 c_1^2 d_1^6 - 1440 a_1^6 c_1 d_1^7 - 86400 a_1^6 c_1^3 c_3 + 86400 a_1^6 c_1^3 d_3 + \\
& 86400 a_1^6 c_3 d_1^3 - 86400 a_1^6 d_1^3 d_3 + 86400 a_1^5 a_3 c_1^4 + 86400 a_1^5 a_3 d_1^4 - 129600 a_1^4 a_2^2 c_1^4 - \\
& 129600 a_1^4 a_2^2 d_1^4 + 129600 a_1^6 c_1^2 c_2^2 + 129600 a_1^6 c_1^2 d_2^2 + 129600 a_1^6 c_2^2 d_1^2 + 129600 a_1^6 d_1^2 d_2^2 - \\
& 259200 a_1^6 c_1 c_2^2 d_1 - 259200 a_1^6 c_1 d_1 d_2^2 - 259200 a_1^6 c_2 d_1^2 d_2 - 1036800 t^2 a_1^{12} c_1 d_1 - 518400 t a_1^9 c_1^2 c_2 + \\
& 518400 t a_1^9 c_1^2 d_2 - 518400 t a_1^9 c_2 d_1^2 + 518400 t a_1^9 d_1^2 d_2 + 518400 t a_1^8 a_2 c_1^3 - 518400 t a_1^8 a_2 d_1^3 + \\
& 259200 a_1^6 c_1^2 c_3 d_1 - 259200 a_1^6 c_1^2 d_1 d_3 - 259200 a_1^6 c_1 c_3 d_1^2 + 259200 a_1^6 c_1 d_1^2 d_3 - 345600 a_1^5 a_3 c_1^3 d_1 + \\
& 518400 a_1^5 a_3 c_1^2 d_1^2 - 345600 a_1^5 a_3 c_1 d_1^3 + 518400 a_1^4 a_2^2 c_1^3 d_1 - 777600 a_1^4 a_2^2 c_1^2 d_1^2 + 518400 a_1^4 a_2^2 c_1 d_1^3 - \\
& 259200 a_1^6 c_1^2 c_2 d_2 + 518400 a_1^6 c_1 c_2 d_1 d_2 + 1036800 t a_1^9 c_1 c_2 d_1 - 1036800 t a_1^9 c_1 d_1 d_2 - 1555200 t a_1^8 a_2 c_1^2 d_1 + \\
& 1555200 t a_1^8 a_2 c_1 d_1^2 x^2 - 24 a_1^2 (-86400 a_1^5 c_2^3 + 86400 a_1^5 d_2^3 + 691200 t^3 a_1^{14} + 20 a_1^5 c_1^9 - \\
& 20 a_1^5 d_1^9 + 518400 t a_1^8 c_1^2 c_2 d_1 - 518400 t a_1^8 c_1^2 d_1 d_2 - 518400 t a_1^8 c_1 c_2 d_1^2 + 518400 t a_1^8 c_1 d_1^2 d_2 - \\
& 691200 t a_1^7 a_2 c_1^3 d_1 + 1036800 t a_1^7 a_2 c_1^2 d_1^2 - 691200 t a_1^7 a_2 c_1 d_1^3 - 1036800 t a_1^7 a_2 c_1 c_2 + 1036800 t a_1^7 a_2 c_1 d_2 + \\
& 1036800 t a_1^7 a_2 c_2 d_1 - 1036800 t a_1^7 a_2 d_1 d_2 + 172800 t^2 a_1^{11} c_1^3 - 172800 t^2 a_1^{11} d_1^3 + 259200 a_1^5 c_2^2 d_2 - \\
& 259200 a_1^5 c_2 d_2^2 + 86400 a_1^2 a_2^3 c_1^3 - 86400 a_1^2 a_2^3 d_1^3 - 1036800 t^2 a_1^{11} c_2 + 1036800 t^2 a_1^{11} d_2 - \\
& 180 a_1^5 c_1^8 d_1 + 720 a_1^5 c_1^7 d_1^2 - 1680 a_1^5 c_1^6 d_1^3 + 2520 a_1^5 c_1^5 d_1^4 - 2520 a_1^5 c_1^4 d_1^5 + 1680 a_1^5 c_1^3 d_1^6 - \\
& 720 a_1^5 c_1^2 d_1^7 + 180 a_1^5 c_1 d_1^8 + 518400 t a_1^8 c_2^2 + 518400 t a_1^8 d_2^2 - 21600 a_1^5 c_1^4 c_3 + 21600 a_1^5 c_1^4 d_3 - \\
& 21600 a_1^5 c_3 d_1^4 + 21600 a_1^5 d_1^4 d_3 + 21600 a_1^4 a_3 c_1^5 - 21600 a_1^4 a_3 d_1^5 - 21600 a_1^3 a_2^2 c_1^5 + \\
& 21600 a_1^3 a_2^2 d_1^5 + 43200 a_1^5 c_1^3 c_2^2 + 43200 a_1^5 c_1^3 d_2^2 - 43200 a_1^5 c_2^2 d_1^3 - 43200 a_1^5 d_1^3 d_2^2 - \\
& 86400 a_1^5 c_1^3 c_2 d_2 - 129600 a_1^5 c_1^2 c_2^2 d_1 - 129600 a_1^5 c_1^2 d_1 d_2^2 + 129600 a_1^5 c_1 c_2^2 d_1^2 + 129600 a_1^5 c_1 d_1^2 d_2^2 + \\
& 86400 a_1^5 c_2 d_1^3 d_2 - 518400 t^2 a_1^{11} c_1^2 d_1 + 518400 t^2 a_1^{11} c_1 d_1^2 - 172800 t a_1^8 c_1^3 c_2 + 172800 t a_1^8 c_1^3 d_2 + \\
& 172800 t a_1^8 c_2 d_1^3 - 172800 t a_1^8 d_1^3 d_2 + 172800 t a_1^7 a_2 c_1^4 + 172800 t a_1^7 a_2 d_1^4 - 1036800 t a_1^8 c_2 d_2 + \\
& 518400 t a_1^6 a_2^2 c_1^2 + 518400 t a_1^6 a_2^2 d_1^2 + 86400 a_1^5 c_1^3 c_3 d_1 - 86400 a_1^5 c_1^3 d_1 d_3 - 129600 a_1^5 c_1^2 c_3 d_1^2 + \\
& 129600 a_1^5 c_1^2 d_1^2 d_3 + 86400 a_1^5 c_1 c_3 d_1^3 - 86400 a_1^5 c_1 d_1^3 d_3 - 21600 a_1^4 a_2 c_1^4 c_2 + 21600 a_1^4 a_2 c_1^4 d_2 - \\
& 21600 a_1^4 a_2 c_2 d_1^4 + 21600 a_1^4 a_2 d_1^4 d_2 - 108000 a_1^4 a_3 c_1^4 d_1 + 216000 a_1^4 a_3 c_1^3 d_1^2 - 216000 a_1^4 a_3 c_1^2 d_1^3 + \\
& 108000 a_1^4 a_3 c_1 d_1^4 + 108000 a_1^3 a_2^2 c_1^4 d_1 - 216000 a_1^3 a_2^2 c_1^3 d_1^2 + 216000 a_1^3 a_2^2 c_1^2 d_1^3 - 108000 a_1^3 a_2^2 c_1 d_1^4 + \\
& 259200 a_1^4 a_2 c_1 c_2^2 + 259200 a_1^4 a_2 c_1 d_2^2 - 259200 a_1^4 a_2 c_2^2 d_1 - 259200 a_1^4 a_2 d_1 d_2^2 - 259200 a_1^3 a_2^2 c_1^2 c_2 + \\
& 259200 a_1^3 a_2^2 c_1^2 d_2 - 259200 a_1^3 a_2^2 c_2 d_1^2 + 259200 a_1^3 a_2^2 d_1^2 d_2 - 259200 a_1^2 a_2^3 c_1^2 d_1 + 259200 a_1^2 a_2^3 c_1 d_1^2 + \\
& 1036800 t^2 a_1^{10} a_2 c_1 - 1036800 t^2 a_1^{10} a_2 d_1 + 259200 a_1^5 c_1^2 c_2 d_1 d_2 - 259200 a_1^5 c_1 c_2 d_1^2 d_2 + \\
& 86400 a_1^4 a_2 c_1^3 c_2 d_1 - 86400 a_1^4 a_2 c_1^3 d_1 d_2 - 129600 a_1^4 a_2 c_1^2 c_2 d_1^2 + 129600 a_1^4 a_2 c_1^2 d_1^2 d_2 + \\
& 86400 a_1^4 a_2 c_1 c_2 d_1^3 - 86400 a_1^4 a_2 c_1 d_1^3 d_2 - 518400 a_1^4 a_2 c_1 c_2 d_2 + 518400 a_1^4 a_2 c_2 d_1 d_2 + 518400 a_1^3 a_2^2 c_1 c_2 d_1 - \\
& 518400 a_1^3 a_2^2 c_1 d_1 d_2 - 1036800 t a_1^6 a_2^2 c_1 d_1 x - 24 a_1^2 (518400 a_1^3 a_2 c_1 c_2 d_1 d_2 + 1036800 t a_1^6 a_2 c_1 c_2 d_1 - \\
& 1036800 t a_1^6 a_2 c_1 d_1 d_2 + 518400 a_1^2 a_2 a_3 c_1 c_2 - 518400 a_1^2 a_2 a_3 c_1 d_2 - 518400 a_1^2 a_2 a_3 c_2 d_1 + \\
& 518400 a_1^2 a_2 a_3 d_1 d_2 + 1036800 a_1 a_2^2 a_3 c_1 d_1 + 43200 a_1^4 c_1^3 c_2 d_1 d_2 - 64800 a_1^4 c_1^2 c_2 d_1^2 d_2 + \\
& 43200 a_1^4 c_1 c_2 d_1^3 d_2 - 259200 a_1^3 a_2 c_1^2 c_2 d_2 - 259200 a_1^3 a_2 c_1 c_2^2 d_1 - 259200 a_1^3 a_2 c_1 d_1 d_2^2 - \\
& 259200 a_1^3 a_2 c_2 d_1^2 d_2 - 86400 t a_1^7 c_1^3 d_1 d_2 - 129600 t a_1^7 c_1^2 c_2 d_1^2 + 129600 t a_1^7 c_1^2 d_1^2 d_2 + 86400 t a_1^7 c_1 c_2 d_1^3 - \\
& 86400 t a_1^7 c_1 d_1^3 d_2 - 108000 t a_1^6 a_2 c_1^4 d_1 + 216000 t a_1^6 a_2 c_1^3 d_1^2 - 216000 t a_1^6 a_2 c_1^2 d_1^3 + 108000 t a_1^6 a_2 c_1 d_1^4 - \\
& 777600 t a_1^5 a_2^2 c_1^2 d_1 + 777600 t a_1^5 a_2^2 c_1 d_1^2 + 86400 t a_1^7 c_1^3 c_2 d_1 - 518400 t a_1^7 c_1 c_2 d_2 + 518400 t a_1^7 c_2 d_1 d_2 - \\
& 518400 t a_1^6 a_2 c_1^2 c_2 + 518400 t a_1^6 a_2 c_1^2 d_2 - 518400 t a_1^6 a_2 c_2 d_1^2 + 518400 t a_1^6 a_2 d_1^2 d_2 + 21600 a_1^3 a_2 c_1^4 c_2 d_1 - \\
& 21600 a_1^3 a_2 c_1^4 d_1 d_2 - 43200 a_1^3 a_2 c_1^3 c_2 d_1^2 + 43200 a_1^3 a_2 c_1^3 d_1^2 d_2 + 43200 a_1^3 a_2 c_1^2 c_2 d_1^3 - \\
& 43200 a_1^3 a_2 c_1^2 d_1^3 d_2 - 21600 a_1^3 a_2 c_1 c_2 d_1^4 + 21600 a_1^3 a_2 c_1 d_1^4 d_2 + 388800 a_1^2 a_2^2 c_1^2 c_2 d_1 - \\
& 388800 a_1^2 a_2^2 c_1^2 d_1 d_2 - 388800 a_1^2 a_2^2 c_1 c_2 d_1^2 + 388800 a_1^2 a_2^2 c_1 d_1^2 d_2 - 1036800 t^2 a_1^9 a_2 c_1 d_1 + \\
& 518400 a_1^3 a_2 c_3 d_2 - 518400 a_1^3 a_2 d_2 d_3 - 172800 a_1^3 a_3 c_1 c_3 + 172800 a_1^3 a_3 c_1 d_3 + 172800 a_1^3 a_3 c_3 d_1 - \\
& 172800 a_1^3 a_3 d_1 d_3 + 518400 a_1^2 a_2^2 c_1 c_3 - 518400 a_1^2 a_2^2 c_1 d_3 - 1555200 a_1^2 a_2^2 c_2 d_2 - 518400 a_1^2 a_2^2 c_3 d_1 + \\
& 518400 a_1^2 a_2^2 d_1 d_3 - 172800 a_1^2 a_2^3 c_1 d_1 - 1555200 a_1 a_2^3 c_1 c_2 + 1555200 a_1 a_2^3 c_1 d_2 + 1555200 a_1 a_2^3 c_2 d_1 -
\end{aligned}$$

$$\begin{aligned}
& 1555200 a_1 a_2^3 d_1 d_2 - 518400 a_1 a_2^2 a_3 c_1^2 - 518400 a_1 a_2^2 a_3 d_1^2 + 259200 t a_1^5 a_2^2 c_1^3 - 259200 t a_1^5 a_2^2 d_1^3 - \\
& 10800 a_1^4 c_1^4 c_2 d_2 - 21600 a_1^4 c_1^3 c_2^2 d_1 - 21600 a_1^4 c_1^3 d_1 d_2^2 + 32400 a_1^4 c_1^2 c_2^2 d_1^2 + 32400 a_1^4 c_1^2 d_1^2 d_2^2 - \\
& 21600 a_1^4 c_1 c_2^2 d_1^3 - 21600 a_1^4 c_1 d_1^3 d_2^2 - 10800 a_1^4 c_2 d_1^4 d_2 + 129600 a_1^3 a_2 c_1^2 c_2^2 + 129600 a_1^3 a_2 c_1^2 d_2^2 + \\
& 129600 a_1^3 a_2 c_2^2 d_1^2 + 129600 a_1^3 a_2 d_1^2 d_2^2 - 518400 a_1^3 a_2 c_2 c_3 + 518400 a_1^3 a_2 c_2 d_3 - 21600 t a_1^7 c_1^4 c_2 + \\
& 21600 t a_1^7 c_1^4 d_2 - 21600 t a_1^7 c_2 d_1^4 + 21600 t a_1^7 d_1^4 d_2 + 21600 t a_1^6 a_2 c_1^5 - 21600 t a_1^6 a_2 d_1^5 - \\
& 172800 a_1 a_2^3 c_1^3 d_1 + 259200 a_1 a_2^3 c_1^2 d_1^2 - 172800 a_1 a_2^3 c_1 d_1^3 - 86400 t^2 a_1^{10} c_1^3 d_1 + 129600 t^2 a_1^{10} c_1^2 d_1^2 - \\
& 86400 t^2 a_1^{10} c_1 d_1^3 + 21600 a_1^2 a_2^2 c_1^3 d_1^3 - 16200 a_1^2 a_2^2 c_1^2 d_1^4 + 6480 a_1^2 a_2^2 c_1 d_1^5 + 129600 a_1^4 c_1 c_2^2 d_2 - \\
& 129600 a_1^4 c_1 c_2 d_2^2 - 129600 a_1^4 c_2^2 d_1 d_2 + 129600 a_1^4 c_2 d_1 d_2^2 - 518400 t^2 a_1^{10} c_1 c_2 + 518400 t^2 a_1^{10} c_1 d_2 + \\
& 518400 t^2 a_1^{10} c_2 d_1 - 518400 t^2 a_1^{10} d_1 d_2 + 518400 t^2 a_1^9 a_2 c_1^2 + 518400 t^2 a_1^9 a_2 d_1^2 - 129600 a_1^2 a_2^2 c_1^3 c_2 + \\
& 129600 a_1^2 a_2^2 c_1^3 d_2 + 129600 a_1^2 a_2^2 c_2 d_1^3 - 129600 a_1^2 a_2^2 d_1^3 d_2 + 259200 t a_1^7 c_1 c_2^2 + 259200 t a_1^7 c_1 d_2^2 - \\
& 259200 t a_1^7 c_2^2 d_1 - 259200 t a_1^7 d_1 d_2^2 + 10800 a_1^4 c_1^4 c_3 d_1 - 10800 a_1^4 c_1^4 d_1 d_3 - 21600 a_1^4 c_1^3 c_3 d_1^2 + \\
& 21600 a_1^4 c_1^3 d_1^2 d_3 + 21600 a_1^4 c_1^2 c_3 d_1^3 - 21600 a_1^4 c_1^2 d_1^3 d_3 - 10800 a_1^4 c_1 c_3 d_1^4 + 10800 a_1^4 c_1 d_1^4 d_3 - \\
& 4320 a_1^3 a_2 c_1^5 c_2 + 4320 a_1^3 a_2 c_1^5 d_2 + 4320 a_1^3 a_2 c_2 d_1^5 - 4320 a_1^3 a_2 d_1^5 d_2 - 12960 a_1^3 a_3 c_1^5 d_1 + \\
& 32400 a_1^3 a_3 c_1^4 d_1^2 - 43200 a_1^3 a_3 c_1^3 d_1^3 + 32400 a_1^3 a_3 c_1^2 d_1^4 - 12960 a_1^3 a_3 c_1 d_1^5 + 6480 a_1^2 a_2^2 c_1^5 d_1 - \\
& 16200 a_1^2 a_2^2 c_1^4 d_1^2 + 5400 a_1^4 c_1^4 c_2^2 + 5400 a_1^4 c_1^4 d_2^2 + 5400 a_1^4 c_2^2 d_1^4 + 5400 a_1^4 d_1^4 d_2^2 - \\
& 172800 a_1^4 c_3 d_3 + 777600 a_1^2 a_2^2 c_2^2 + 777600 a_1^2 a_2^2 d_2^2 + 86400 a_1^2 a_3^2 c_1^2 + 86400 a_1^2 a_3^2 d_1^2 - \\
& 1555200 a_2^4 c_1 d_1 + 45 a_1^4 c_1^8 d_1^2 - 120 a_1^4 c_1^7 d_1^3 + 210 a_1^4 c_1^6 d_1^4 - 252 a_1^4 c_1^5 d_1^5 + 210 a_1^4 c_1^4 d_1^6 - \\
& 120 a_1^4 c_1^3 d_1^7 + 45 a_1^4 c_1^2 d_1^8 - 10 a_1^4 c_1 d_1^9 - 2160 a_1^4 c_1^5 c_3 + 2160 a_1^4 c_1^5 d_3 + 2160 a_1^4 c_3 d_1^5 - \\
& 2160 a_1^4 d_1^5 d_3 + 2160 a_1^3 a_3 c_1^6 + 2160 a_1^3 a_3 d_1^6 - 1080 a_1^2 a_2^2 c_1^6 - 1080 a_1^2 a_2^2 d_1^6 - 43200 a_1^4 c_1 c_2^3 + \\
& 43200 a_1^4 c_1 d_2^3 + 43200 a_1^4 c_2^3 d_1 - 43200 a_1^4 d_1 d_2^3 + 345600 t^3 a_1^{13} c_1 - 345600 t^3 a_1^{13} d_1 - \\
& 10 a_1^4 c_1^9 d_1 + 43200 a_1 a_2^3 c_1^4 + 43200 a_1 a_2^3 d_1^4 + 21600 t^2 a_1^{10} c_1^4 + 21600 t^2 a_1^{10} d_1^4 + a_1^4 c_1^{10} + \\
& a_1^4 d_1^{10} + 86400 a_1^4 c_3^2 + 86400 a_1^4 d_3^2 + 777600 a_2^4 c_1^2 + 777600 a_2^4 d_1^2),
\end{aligned}$$

$$\begin{aligned}
d(x, t) = & -64 a_1^8 x^6 + (192 a_1^7 d_1 - 192 a_1^7 c_1) x^5 + (-240 a_1^6 d_1^2 - 240 a_1^6 c_1^2 + 480 a_1^6 c_1 d_1) x^4 + \\
& (-160 a_1^5 c_1^3 + 480 a_1^5 c_2 - 480 a_1^5 d_2 + 160 a_1^5 d_1^3 + 480 a_1^5 c_1^2 d_1 - 960 t a_1^8 + 480 a_1^4 a_2 d_1 - \\
& 480 a_1^4 a_2 c_1 - 480 a_1^5 c_1 d_1^2) x^3 + (-1440 t a_1^7 c_1 - 60 a_1^4 d_1^4 - 720 a_1^4 c_1 d_2 - 60 a_1^4 c_1^4 + 1440 a_1^3 a_2 c_1 d_1 + \\
& 720 a_1^4 d_1 d_2 - 720 a_1^4 c_2 d_1 - 720 a_1^3 a_2 d_1^2 + 720 a_1^4 c_1 c_2 + 240 a_1^4 c_1 d_1^3 + 1440 t a_1^7 d_1 + 240 a_1^4 c_1^3 d_1 - \\
& 360 a_1^4 c_1^2 d_1^2 - 720 a_1^3 a_2 c_1^2) x^2 + (4320 a_1 a_2^2 d_1 - 1440 a_1^3 c_3 + 1440 a_1^3 d_3 - 720 a_1^6 d_1^2 t - \\
& 720 a_1^6 c_1^2 t - 4320 a_1 a_2^2 c_1 + 12 a_1^3 d_1^5 - 12 a_1^3 c_1^5 + 360 a_2 a_1^2 d_1^3 - 4320 a_2 a_1^2 d_2 + 4320 a_2 a_1^2 c_2 - \\
& 360 a_2 a_1^2 c_1^3 - 360 a_1^3 d_1^2 d_2 + 360 a_1^3 d_1^2 c_2 - 120 a_1^3 d_1^2 c_1^3 + 120 a_1^3 c_1^2 d_1^3 - 360 a_1^3 c_1^2 d_2 + \\
& 360 a_1^3 c_1^2 c_2 + 1440 a_3 a_1^2 c_1 - 1440 a_3 a_1^2 d_1 - 60 a_1^3 d_1^4 c_1 + 60 a_1^3 c_1^4 d_1 + 1080 a_1^2 c_1^2 a_2 d_1 + \\
& 720 a_1^3 d_1 d_2 c_1 - 720 a_1^3 c_1 c_2 d_1 + 1440 t a_1^6 c_1 d_1 - 1080 a_1^2 d_1^2 a_2 c_1) x + 60 a_1^2 d_1^3 d_2 + 720 a_1 a_3 d_1^2 + \\
& 120 t a_1^5 d_1^3 + 2880 t^2 a_1^8 + 720 a_1 a_2 d_1 d_2 + 2880 t a_1^4 a_2 c_1 - 60 a_1 a_2 d_1^4 - 2880 t a_1^4 d_1 a_2 + \\
& 720 a_1 a_3 c_1^2 - d_1^6 a_1^2 + 720 a_1 a_2 c_1 c_2 - c_1^6 a_1^2 - 15 a_1^2 c_1^2 d_1^4 - 15 a_1^2 c_1^4 d_1^2 + 720 d_3 a_1^2 c_1 + \\
& 720 c_3 a_1^2 d_1 + 6 d_1^5 a_1^2 c_1 + 6 c_1^5 a_1^2 d_1 - 60 c_2 a_1^2 d_1^3 - 1440 c_2 a_1^2 d_2 + 20 d_1^3 a_1^2 c_1^3 - 60 d_2 a_1^2 c_1^3 - \\
& 2880 t a_1^5 c_2 - 720 a_1^2 c_1 c_3 + 60 a_1^2 c_1^3 c_2 - 120 t a_1^5 c_1^3 - 60 a_1 a_2 c_1^4 + 2880 t a_1^5 d_2 - 720 a_1^2 d_1 d_3 - \\
& 1440 a_2^2 d_1^2 + 720 a_1^2 c_2^2 + 720 a_1^2 d_2^2 + 2880 d_1 a_2^2 c_1 + 180 d_1 a_1^2 c_1^2 d_2 - 1440 d_1 a_3 a_1 c_1 - \\
& 360 a_1 c_1^2 a_2 d_1^2 + 180 a_1^2 c_1 c_2 d_1^2 - 180 d_1^2 d_2 a_1^2 c_1 - 180 c_1^2 c_2 a_1^2 d_1 - 360 d_1^2 t a_1^5 c_1 + 360 c_1^2 t a_1^5 d_1 - \\
& 720 c_2 a_1 a_2 d_1 + 240 d_1^3 a_1 a_2 c_1 - 720 d_2 a_1 a_2 c_1 + 240 c_1^3 a_1 a_2 d_1 - 1440 a_2^2 c_1^2
\end{aligned}$$

is a rational solution to the KdV equation (1), quotient of two polynomials with the numerator of order 10 in x , 3 in t , the denominator of degree 12 in x , 4 in t .

2.6 Another orders

For greater orders, the solutions become very complex and we cannot give them here. We only give solutions with parameters a_j equal to 1 and all other parameters equal to 0.

Order 4

Proposition 2.4 *The function v defined by*

$$v(x, t) = -2 \frac{n(x, t)}{d(x, t)^2}, \quad (10)$$

$$n(x, t) = -10x^{18} - 360tx^{15} - 14175t^2x^{12} + 330750t^3x^9 + 5953500t^4x^6 - 22325625t^6,$$

$$d(x, t) = x^{10} + 45tx^7 + 4725t^3x$$

is a rational solution to the KdV equation (1), quotient of two polynomials with numerator of degree 18 in x , 6 in t , and denominator of degree 20 in x , 6 in t .

Order 5

Proposition 2.5 *The function v defined by*

$$v(x, t) = -2 \frac{n(x, t)}{d(x, t)^2}, \quad (11)$$

$$n(x, t) = -15x^{28} - 1890tx^{25} - 113400t^2x^{22} - 84837375t^4x^{16} - 7501410000t^5x^{13} - 64699661250t^6x^{10} - 168781725000t^7x^7 - 4430520281250t^8x^4 + 8861040562500t^9x,$$

$$d(x, t) = -x^{15} - 105tx^{12} - 1575t^2x^9 - 33075t^3x^6 + 992250t^4x^3 + 1488375t^5$$

is a rational solution to the KdV equation (1), quotient of two polynomials with numerator of degree 28 in x , 9 in t , and denominator of degree 30 in x , 10 in t .

Order 6

Proposition 2.6 *The function v defined by*

$$v(x, t) = -2 \frac{n(x, t)}{d(x, t)^2}, \quad (12)$$

$$n(x, t) = -21x^{40} - 6300tx^{37} - 793800t^2x^{34} - 39690000t^3x^{31} - 1909585125t^4x^{28} - 43320642750t^5x^{25} + 8174661547500t^6x^{22} + 376889591925000t^7x^{19} + 5541251715759375t^8x^{16} + 60042410851500000t^9x^{13} - 1339696292124093750t^{10}x^{10} + 2026431366238125000t^{11}x^7 - 88656372272917968750t^{12}x^4 - 106387646727501562500t^{13}x,$$

$$d(x, t) = -x^{21} - 210tx^{18} - 10395t^2x^{15} - 264600t^3x^{12} + 5457375t^4x^9 - 343814625t^5x^6 - 3438146250t^6x^3 + 5157219375t^7$$

is a rational solution to the KdV equation (1), quotient of two polynomials with numerator of degree 40 in x , 13 in t , and denominator of degree 42 in x , 14 in t .

3 Conclusion

We have given three types of representations of solutions to the KdV equation. First, solutions in terms of elementary exponential functions have been constructed. In particular, performing a passage to the limit when one parameter goes to 0 we get rational solutions to the KdV equation. We give another representation in terms of determinants without the presence of a limit. So we obtain an infinite hierarchy of multi-parametric families of rational solutions to the KdV equation as a quotient of a polynomials depending on $3N$ real parameters.

We can formulate some remarks about the structure of these solutions. For the N -order solution, the numerator of the solution is a polynomial of degree $N(N - 1) - 2$ in x and the denominator a polynomial of degree $N(N + 1)$ in x . The structure relative to t seems more complicated. It would be relevant to study these polynomials in more details.

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