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► To cite this version:

Qingqing Feng, Grégoire Allaire, Pascal Omnes. Enriched nonconforming multiscale finite element method for Stokes flows in heterogeneous media based on high-order weighting functions. Multiscale Modeling and Simulation: A SIAM Interdisciplinary Journal, 2022, 20 (1), 10.1137/21M141926X . hal-03475694

HAL Id: hal-03475694

<https://hal.science/hal-03475694v1>

Submitted on 11 Dec 2021

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ENRICHED NONCONFORMING MULTISCALE FINITE ELEMENT METHOD FOR STOKES FLOWS IN HETEROGENEOUS MEDIA BASED ON HIGH-ORDER WEIGHTING FUNCTIONS *

Q. FENG[†], G. ALLAIRE[‡], AND P. OMNES[§]

Abstract. This paper addresses an enriched nonconforming Multiscale Finite Element Method (MsFEM) to solve viscous incompressible flow problems in genuine heterogeneous or porous media. In the work of [B. P. Muljadi, J. Narski, A. Lozinski, and P. Degond, Multiscale Modeling & Simulation 2015 13:4, 1146-1172] and [G. Jankowiak and A. Lozinski, arXiv:1802.04389 [math.NA], 2018], a nonconforming MsFEM has been first developed for Stokes problems in such media. Based on these works, we propose an innovative enriched nonconforming MsFEM where the approximation space of both velocity and pressure are enriched by weighting functions which are defined by polynomials of higher-degree. Numerical experiments show that this enriched nonconforming MsFEM improves significantly the accuracy of the nonconforming MsFEMs. Theoretically, this method provides a general framework which allows to find a good compromise between the accuracy of the method and the computing costs, by varying the degrees of polynomials.

Key words. Multiscale Finite Element Method, Crouzeix-Raviart Element, Stokes Flows

AMS subject classifications. 65N30, 76D07, 76M30

1. Introduction.

1.1. Multiscale problems. Modeling of Stokes flows in heterogeneous media is a topic of significant interest in many engineering practices, such as reservoir engineering, flows through fractured porous media, flows in nuclear reactor cores, etc. The challenge in numerical simulations of these problems lies in the fact that the solution contains multiscale features such as spatial scale disparity and requires a very fine mesh to resolve all the details. In some engineering circumstances, quantities of interest are only related to macroscopic properties of the solution. But fine-scale features cannot be omitted in physical models since they can affect significantly the macroscopic behavior of the solution. However, solving these problems on a very fine mesh can be prohibitively expensive or impossible with today's computing capacities. As a consequence, some model reduction techniques are developed to get reliable solutions at reasonable computational costs. These methods attempt to resolve scales below the coarse mesh scale by incorporating local computations into a global problem which is defined only on a coarse mesh.

1.2. Multiscale methods. A certainly not exhaustive list of Multiscale methods includes homogenization based methods [5, 20, 28], upscaling methods [8, 23, 32, 43], Multiscale Finite Element Methods (MsFEMs) [9, 19, 20, 21, 29, 30], variational multiscale methods [2, 31], heterogeneous multiscale methods [17, 28], multiscale finite volumes and discontinuous Galerkin methods [3, 14, 26, 34] and so on. MsFEMs were

*Submitted to the editors DATE.

[†]Université Paris-Saclay, CEA, Service de Thermo-hydraulique et de Mécanique des Fluides, 91191 Gif-sur-Yvette cedex, France. Current address: Électricité de France, R&D Division, Fluid Mechanics, Energy and Environment Department (MFEE), 6 quai Watier, Chatou 78401, France. (qingqing.feng@edf.fr).

[‡]CMAF, Ecole Polytechnique, route de Saclay, 91128 Palaiseau Cedex, France (gregoire.allaire@polytechnique.fr).

[§]Université Paris-Saclay, CEA, Service de Thermo-hydraulique et de Mécanique des Fluides, 91191, Gif-sur-Yvette, France and Université Sorbonne Paris Nord, LAGA, CNRS UMR 7539, Institut Galilée, 99 Av. J.-B. Clément, 93430 Villetaneuse, France (pascal.omnes@cea.fr).

first proposed by [29, 30] and then experienced major advancements in [9, 19, 20, 21]. This method has been extended to the generalized multiscale finite element method in [10, 11, 12, 18]. MsFEMs use a finite element approach and construct special basis functions which incorporate the fine scale features of the problem. MsFEMs rely on two types of meshes: a global coarse mesh and a collection of local fine meshes. The basis functions are constructed by solving local problems on the fine meshes with prescribed boundary conditions. Then a coarse-scale problem, involving these basis functions, is solved on the coarse mesh. In order to improve the accuracy of MsFEMs, the oversampling method [29] was introduced to better approximate the boundary conditions in local problems. This gives a nonconforming MsFEM since the oversampling introduces discontinuities across coarse elements.

Another nonconforming method, namely the Crouzeix-Raviart MsFEM, was developed for solving the diffusion problem [35, 36, 37], the advection-diffusion problem [16], the Stokes problem [33, 40] and the Oseen problem [39] in heterogeneous media. More recently, [13] proposes a multiscale hybrid high-order method for highly oscillatory elliptic problems. The method in [13] can be considered as a first attempt at generalizing the Crouzeix-Raviart MsFEM to arbitrary orders of approximation. In the Crouzeix-Raviart MsFEM, basis functions are constructed associated to coarse element edges. The non-conforming nature of the Crouzeix-Raviart element [15] provides great flexibility especially in case of randomly placed obstacles. The conformity between coarse elements is enforced in a weak sense, i.e., only the average of the jump of the function vanishes on the interface between coarse elements. In the case of densely placed obstacles, it is very difficult to avoid intersections between interfaces of coarse elements and obstacles. In this situation, the Crouzeix-Raviart MsFEM is very attractive since it allows the multiscale basis functions to have a natural boundary condition on coarse element edges, which relaxes the sensitivity of the method to complex patterns of obstacles, without using oversampling methods.

In [40] the Crouzeix-Raviart MsFEM was introduced for Stokes problems in heterogeneous media. In order to improve the accuracy of the method, the authors of [33] enriched the approximation space of velocity by adding weighting functions which are defined by linear polynomials. The penalization method was applied with high diffusion and viscosity parameters inside the obstacles, in order to perform numerical simulations on simple Cartesian meshes, and local problems are discretized by the \mathbb{Q}_1 - \mathbb{Q}_1 finite element method [4].

We mention that the method proposed in [13] share some similarities with our enriched Crouzeix-Raviart MsFEM in the use of weighting functions defined by polynomials of higher degrees. But the method in [13] is defined only for elliptic problems in the framework of a hybrid high-order method, whereas in our work we develop an enriched method for flow problems in the framework of MsFEMs.

1.3. This paper. We propose an innovative enriched Crouzeix-Raviart MsFEM for Stokes flows in heterogeneous media with numerous solid obstacles. In this method, both approximation spaces of velocity and pressure are enriched by adding weighting functions which are defined by polynomials of higher-degrees. By varying the degrees of polynomials, the enriched Crouzeix-Raviart MsFEM allows to find a good compromise between the accuracy of the method and the computing costs. Numerical experiments show that the enriched Crouzeix-Raviart MsFEM is significantly more accurate than the MsFEM proposed in [33, 40]. In practice, rather than using the penalization method as in [33, 40], we perform numerical simulations using body-fitted unstructured triangular meshes. Local problems are solved by the finite volume

87 element method [6] in an industrial software *TrioCFD* [41].

88 The paper is organized as follows. The problem formulation is described in [sec-](#)
 89 [tion 2](#). The enriched Crouzeix-Raviart MsFEM is presented in [section 3](#). [Section 4](#)
 90 discusses shortly the practical implementations of MsFEMs. In [section 5](#), we present
 91 some numerical experiments in 2-d (although the method is analysed in space di-
 92 mension 2 or 3) and compare the accuracy of different variants of Crouzeix-Raviart
 93 MsFEM. The conclusions are given in [section 6](#).

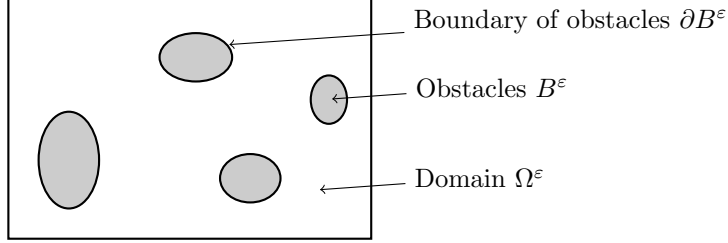


Fig. 1: Rectangular domain Ω comprising a fluid domain Ω^ε perforated by a set of obstacles B^ε

94 **2. Problem formulation.** Let $\Omega \subset \mathbb{R}^d$ be a regular bounded open set (with
 95 $d = 2$ or 3). As shown in [Figure 1](#), we divide the domain Ω into a fixed solid part
 96 B^ε such that $\partial B^\varepsilon \cap \partial\Omega = \emptyset$ and its complementary fluid part Ω^ε (here ε denotes a
 97 small parameter equal to the ratio of the heterogeneities characteristic length with
 98 the characteristic length of the domain). The steady-state Stokes problem is to find
 99 the velocity $\mathbf{u} : \Omega^\varepsilon \rightarrow \mathbb{R}^d$ and the pressure $p : \Omega^\varepsilon \rightarrow \mathbb{R}$ solutions to

$$100 \quad (2.1) \quad -\mu \Delta \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega^\varepsilon,$$

$$101 \quad (2.2) \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega^\varepsilon,$$

103 where the boundary conditions are given by

$$104 \quad (2.3) \quad \mathbf{u} = \mathbf{0} \quad \text{on } \partial B^\varepsilon \cap \partial\Omega^\varepsilon,$$

$$105 \quad (2.4) \quad \mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega \cap \partial\Omega^\varepsilon,$$

107 with \mathbf{f} a given force per unit volume, and μ the dynamic viscosity.

108 *The weak formulation.* We introduce the spaces $V = H_0^1(\Omega^\varepsilon)^d$ for the velocity,

$$109 \quad (2.5) \quad M = L_0^2(\Omega^\varepsilon) = \{p \in L^2(\Omega^\varepsilon) \text{ s.t. } \int_{\Omega^\varepsilon} p = 0\}$$

111 for the pressure and $X = V \times M$. For simplicity the fluid domain Ω^ε is assumed
 112 to be connected in order for the pressure to be uniquely defined in M . The weak
 113 formulation of the Stokes problem (2.1)–(2.2) reads: find $(\mathbf{u}, p) \in X$ such that

$$114 \quad (2.6) \quad c((\mathbf{u}, p), (\mathbf{v}, q)) = \int_{\Omega^\varepsilon} \mathbf{f} \cdot \mathbf{v}, \quad \forall (\mathbf{v}, q) \in X,$$

116 where the bilinear form c is defined by

$$117 \quad c((\mathbf{u}, p), (\mathbf{v}, q)) = \int_{\Omega^\varepsilon} (\mu \nabla \mathbf{u} : \nabla \mathbf{v} - p \operatorname{div} \mathbf{v} - q \operatorname{div} \mathbf{u}).$$

The existence and uniqueness of a solution to problem (2.6) is guaranteed by the Banach-Nečas-Babuška (BNB) theorem [22]. This theorem states that problem (2.6) has a unique solution provided that the bilinear form c is bounded and satisfies the so-called inf-sup condition:

$$(2.7) \quad \exists \alpha > 0, \quad \inf_{(\mathbf{u}, p) \in X} \sup_{(\mathbf{v}, q) \in X} \frac{c((\mathbf{u}, p), (\mathbf{v}, q))}{\|\mathbf{u}, p\|_X \|\mathbf{v}, q\|_X} \geq \alpha.$$

Algorithm 3.1 main steps of a multiscale finite element method

- 1: Partition the domain Ω into a set of coarse elements (coarse mesh)
 - 2: **for** each coarse element **do**
 - 3: Partition the coarse element into a fine mesh
 - 4: Construct multiscale basis functions via local problems
 - 5: Compute rigidity matrices locally on the fine mesh
 - 6: **end for**
 - 7: Assemble global matrices and solve the coarse problem on the coarse mesh
 - 8: **for** each coarse element **do**
 - 9: Reconstruct fine-scale solutions on the fine mesh
 - 10: **end for**
-

3. Crouzeix-Raviart Multiscale Finite Element Method. Algorithm 3.1 outlines the main steps of MsFEMs. We present the enriched Crouzeix-Raviart MsFEM following these steps.

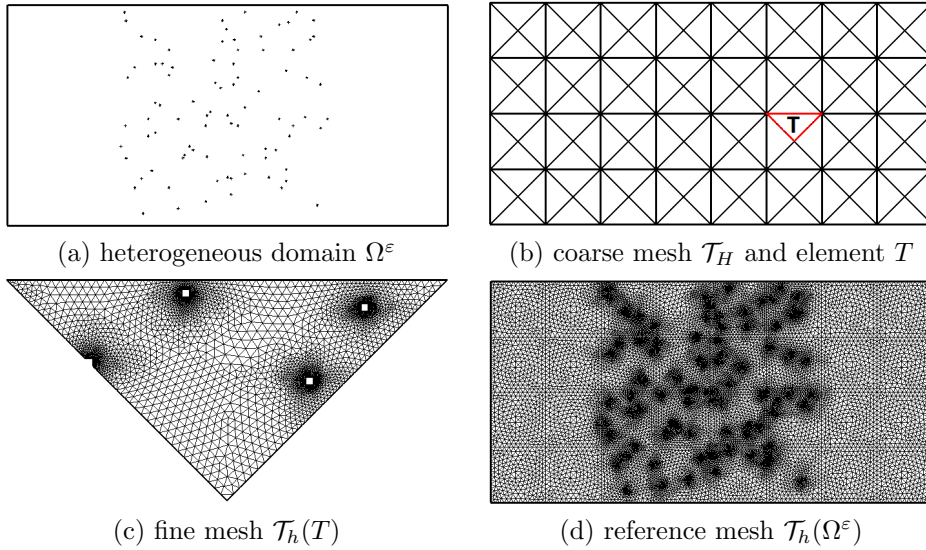


Fig. 2: Illustration of the heterogeneous domain Ω^ϵ , the coarse mesh \mathcal{T}_H , the fine mesh $\mathcal{T}_h(T)$ of a coarse element T and the reference mesh $\mathcal{T}_h(\Omega^\epsilon)$.

3.1. Discretization of the domain. We discretize the domain Ω into an unstructured coarse mesh \mathcal{T}_H (see Figure 2 (b)) consisting of N_H elements (triangles)

of width at most H . Let \mathcal{E}_H denote the set of all edges/faces of \mathcal{T}_H including those on the domain boundary $\partial\Omega$. For the sake of simplicity, we denote elements of \mathcal{E}_H as faces throughout this paper. On each element $T \in \mathcal{T}_H$, we construct a fine mesh $\mathcal{T}_h(T)$ (see Figure 2 (c)), with cell elements of maximum width h . Typically $0 < h \leq H$ and \mathcal{T}_h is fine enough to fully resolve obstacles B^ε .

Note that fine meshes do not need to be matching on their common interfaces, which allows to generate them independently and in parallel. If fine meshes do match on the interface between coarse elements, their union forms a conforming reference mesh $\mathcal{T}_h(\Omega^\varepsilon)$ (see Figure 2 (d)) which is used to compute reference solutions, only needed for the validation of MsFEMs.

Let ω_E be the neighborhood of the face $E \in \mathcal{E}_H$ defined by

$$(3.1) \quad \omega_E = \bigcup_j \{T_j \in \mathcal{T}_H \mid E \subset \partial T_j\}.$$

When E is in the interior of the domain, ω_E is the union of two adjacent elements. When E is on the boundary of the domain, ω_E is composed of only one element.

3.2. Crouzeix-Raviart functional spaces. Now we introduce some important definitions and notations. For any integer n and any integer $1 \leq l \leq d$, we denote by \mathbb{P}_n^l the linear space spanned by l -variate polynomial functions of total degree at most n . The dimension of \mathbb{P}_n^l is

$$N_n^l := \dim(\mathbb{P}_n^l) = \binom{n+l}{n}.$$

For any $T \in \mathcal{T}_H$, we denote by $\mathbb{P}_n^d(T)$ the restriction to T of polynomials in \mathbb{P}_n^d . For any $E \in \mathcal{E}_H$, we denote by $\mathbb{P}_n^{d-1}(E)$ the restriction to E of polynomials of \mathbb{P}_n^{d-1} . For the sake of simplicity, we denote $\mathbb{P}_n^d(T)$ and $\mathbb{P}_n^{d-1}(E)$ respectively by $\mathbb{P}_n(T)$ and $\mathbb{P}_n(E)$.

Let s be a positive integer and let $\omega_{E,i} : E \rightarrow \mathbb{R}^d$ be some vector-valued functions associated to each face $E \in \mathcal{E}_H$ for $i = 1, \dots, s$. Let r be a non-negative integer and let $\varphi_{T,k} : T \rightarrow \mathbb{R}^d$ be some vector-valued functions associated to each coarse element $T \in \mathcal{T}_H$ for $k = 1, \dots, r$. Let t be a positive integer and let $\varpi_{T,j} : T \rightarrow \mathbb{R}$ be some scalar functions associated to each coarse element $T \in \mathcal{T}_H$ for $j = 1, \dots, t$.

Hypothesis 3.1. For $n = 0$, for any $T \in \mathcal{T}_H$ and for any $E \in \mathcal{E}_H$, we choose

$$(3.2) \quad \begin{cases} s = d : \omega_{E,1} = \mathbf{e}_1, \dots, \omega_{E,s} = \mathbf{e}_d, \\ r = 0, \\ t = 1 : \varpi_{T,1} = 1, \end{cases}$$

where $(\mathbf{e}_1, \dots, \mathbf{e}_d)$ is the canonical basis of \mathbb{R}^d . For $n \geq 1$, for any $E \in \mathcal{E}_H$ and for any $T \in \mathcal{T}_H$, we assume that

1. $s = d N_n^{d-1}$ and $(\omega_{E,i})_{1 \leq i \leq s}$ is a set of basis functions of $(\mathbb{P}_n(E))^d$.
2. $r = d N_{n-1}^d$ and $(\varphi_{T,k})_{1 \leq k \leq r}$ is a set of basis functions of $(\mathbb{P}_{n-1}(T))^d$.
3. $t = N_n^d$ and $(\varpi_{T,j})_{1 \leq j \leq t}$ is a set of basis functions of $\mathbb{P}_n(T)$.

Remark 3.2. For $n = 0$ the choice (3.2) of weighting functions implies that the enriched Crouzeix-Raviart MsFEM becomes exactly the Crouzeix-Raviart MsFEM proposed in [40]. This also corresponds to the so-called CR2 method investigated in [33]. Furthermore, in a domain without obstacles, i.e. $B^\varepsilon = \emptyset$, the Crouzeix-Raviart MsFEM space with $n = 0$ becomes the classical Crouzeix-Raviart finite element space.

Remark 3.3. For $n = 1$ and for the space dimension $d = 2$, our choice of weighting functions is, for any $T \in \mathcal{T}_H$ and for any $E \in \mathcal{E}(T)$,

$$(3.3) \quad \begin{cases} \boldsymbol{\omega}_{E,1} = \mathbf{e}_1, \boldsymbol{\omega}_{E,2} = \mathbf{e}_2, \boldsymbol{\omega}_{E,3} = \mathbf{n}_E \psi_E, \boldsymbol{\omega}_{E,4} = \boldsymbol{\tau}_E \phi_E, \\ \varpi_{T,1} = 1, \varpi_{T,2} = x, \varpi_{T,3} = y, \\ \boldsymbol{\varphi}_{T,1} = \mathbf{e}_1, \boldsymbol{\varphi}_{T,2} = \mathbf{e}_2, \end{cases}$$

where ψ_E and ϕ_E are non-vanishing functions in $\mathbb{P}_1(E)$ with vanishing mean-values on $E \cap \Omega^\varepsilon$ and $(\mathbf{e}_1, \mathbf{e}_2)$ is the canonical basis of \mathbb{R}^2 . In this paper, we have implemented and tested only the enriched MsFEM with $n = 1$ which we denote by CR4_high.

Remark 3.4. Our choice of weighting functions for $n \geq 1$ is driven by the following fact: once the polynomial degree for the weighting functions $(\boldsymbol{\omega}_{E,i})$ has been chosen, it is seen in (3.13)–(3.15) that the pressure decomposition into resolved and unresolved contributions motivates the fact that the boundary trace of the resolved pressure belongs (at most) to the span of $(\boldsymbol{\omega}_{E,i})$, and that the space of unresolved velocities is orthogonal to (at least) the gradients of the resolved pressures. So our choice, although not the only possible one, corresponds to these two limit cases: resolved pressures will have the same polynomial (maximal) degree as the set of $(\boldsymbol{\omega}_{E,i})$, and unresolved velocities will be (minimally) orthogonal to polynomials one order lower than the resolved pressures.

To construct the approximation spaces of enriched Crouzeix-Raviart MsFEM, we define V_H^{ext} differently from that of [33, 40]:

$$V_H^{ext} = \left\{ \begin{array}{l} \mathbf{u} \in (L^2(\Omega^\varepsilon))^d \text{ s. t. } \mathbf{u}|_T \in (H^1(T \cap \Omega^\varepsilon))^d \text{ for any } T \in \mathcal{T}_H, \\ \mathbf{u} = \mathbf{0} \text{ on } \partial B^\varepsilon, \int_{E \cap \Omega^\varepsilon} [[\mathbf{u}]] \cdot \boldsymbol{\omega}_{E,j} = 0 \text{ for all } E \in \mathcal{E}_H, j = 1, \dots, s \end{array} \right\},$$

where $[[\mathbf{u}]]$ denotes the jump of \mathbf{u} across an internal face and $[[\mathbf{u}]] = \mathbf{u}$ on $\partial\Omega$. The space V_H^{ext} enhances the natural velocity space $(H_0^1(\Omega^\varepsilon))^d$ so that we have at our disposal discontinuous vector fields across faces of the coarse mesh \mathcal{T}_H . Therefore V_H^{ext} is not included in $(H_0^1(\Omega^\varepsilon))^d$, the MsFEM is nonconforming and the homogeneous boundary condition is only weakly enforced.

The extended velocity-pressure space is defined as

$$X_H^{ext} = V_H^{ext} \times M$$

with M defined by (2.5). We want to decompose X_H^{ext} into a direct sum of a finite dimensional subspace X_H containing coarse scales and an infinite dimensional subspace X_H^0 containing unsolved fine scales, i.e., $X_H^{ext} = X_H \oplus X_H^0$. This will be achieved in Theorem 3.15.

Taking into account Hypothesis 3.1, we first define the infinite dimensional space X_H^0 as follows.

DEFINITION 3.5. *The velocity-pressure space X_H^0 is defined as a subspace of X_H^{ext} by*

$$X_H^0 = V_H^0 \times M_H^0, \quad \text{with}$$

$$(3.4) \quad V_H^0 = \left\{ \begin{array}{l} \mathbf{u} \in V_H^{ext} \text{ s. t. } \int_{E \cap \Omega^\varepsilon} \mathbf{u} \cdot \boldsymbol{\omega}_{E,j} = 0, \int_{T \cap \Omega^\varepsilon} \mathbf{u} \cdot \boldsymbol{\varphi}_{T,k} = 0, \\ \forall T \in \mathcal{T}_H, \forall E \in \mathcal{E}_H, j = 1, \dots, s, k = 1, \dots, r. \end{array} \right\},$$

$$(3.5) \quad M_H^0 = \left\{ p \in M \text{ s. t. } \int_{T \cap \Omega^\varepsilon} p \varpi_{T,j} = 0, \forall T \in \mathcal{T}_H, j = 1, \dots, t \right\}.$$

It is important to stress that enriching only the set of edge weights $\omega_{E,j}$ seems insufficient: indeed, in that case, a given function \mathbf{u} vanishing on the edges of any T would belong to the unresolved fine scales whatever the number of edge weights, but imposing the condition $\int_{T \cap \Omega^\varepsilon} \mathbf{u} \cdot \boldsymbol{\varphi}_{T,k} = 0$ for an increasing number of triangle weights will reduce the norm of its component in the unresolved subspace.

Besides, for any $T \in \mathcal{T}_H$, we define $M_H^0(T)$ by

$$(3.6) \quad M_H^0(T) = \left\{ p \in L^2(T \cap \Omega^\varepsilon) \text{ s. t. } \int_{T \cap \Omega^\varepsilon} p \varpi_{T,i} = 0, i = 1, \dots, t \right\}.$$

DEFINITION 3.6. The velocity-pressure space X_H is defined as a subspace of X_H^{ext} , being the "orthogonal" complement of M_H^0 with respect to the bilinear form $c_H(\cdot, \cdot)$ as follows:

$$(3.7) \quad (\mathbf{u}_H, p_H) \in X_H \iff (\mathbf{u}_H, p_H) \in X_H^{ext} \text{ such that } c_H((\mathbf{u}_H, p_H), (\mathbf{v}, q)) = 0, \forall (\mathbf{v}, q) \in M_H^0,$$

where $c_H(\cdot, \cdot)$ is defined by

$$c_H((\mathbf{u}_H, p_H), (\mathbf{v}, q)) = \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} (\mu \nabla \mathbf{u}_H : \nabla \mathbf{v} - p_H \operatorname{div} \mathbf{v} - q \operatorname{div} \mathbf{u}_H).$$

Remark 3.7. The word "orthogonal" is written between quotes since the bilinear form $c_H(\cdot, \cdot)$ is not a scalar product (not positive definite).

DEFINITION 3.8. Define functional spaces M_H and V_H by

$$(3.8) \quad M_H = \{q \in M \text{ s. t. } q|_T \in \mathbb{P}_n(T), \forall T \in \mathcal{T}_H\},$$

$$(3.9) \quad V_H = \left\{ \begin{array}{l} \mathbf{v} \in V_H^{ext} : \forall T \in \mathcal{T}_H, \exists \zeta^T \in M_H^0(T) \text{ such that} \\ -\mu \Delta \mathbf{v} + \nabla \zeta^T \in \operatorname{span} \{\boldsymbol{\varphi}_{T,1}, \dots, \boldsymbol{\varphi}_{T,r}\} \text{ in } T \cap \Omega^\varepsilon \\ \operatorname{div} \mathbf{v} \in \operatorname{span} \{\varpi_{T,1}, \dots, \varpi_{T,t}\} \text{ in } T \cap \Omega^\varepsilon \\ \mu \nabla \mathbf{v} \mathbf{n} - \zeta^T \mathbf{n} \in \operatorname{span} \{\boldsymbol{\omega}_{E,1}, \dots, \boldsymbol{\omega}_{E,s}\} \text{ on } E \cap \Omega^\varepsilon \forall E \in \mathcal{E}(T) \end{array} \right\},$$

where $\mathcal{E}(T)$ is the set of faces composing ∂T .

Remark 3.9. Clearly, M_H , defined by (3.8), is orthogonal to M_H^0 , defined by (3.5), in the set M .

Remark 3.10. In Sections 3.3 and 3.5 it will be proved that V_H is finite-dimensional and spanned by velocity fields associated to each element $T \in \mathcal{T}_H$ and weight $\boldsymbol{\varphi}_{T,k}$ with $1 \leq k \leq r$ on the one hand and to each edge $E \in \mathcal{E}_H$ and weight $\boldsymbol{\omega}_{E,i}$ with $1 \leq i \leq s$ on the other hand.

LEMMA 3.11. In definition (3.9), the pressure ζ^T is uniquely defined for a given velocity $\mathbf{v} \in V_H$. Therefore, gluing together the pressures ζ^T on all triangles $T \in \mathcal{T}_H$ yields a single function $\pi_H(\mathbf{v}) \in M_H^0$ such that $\pi_H(\mathbf{v}) = \zeta^T$ on any triangle $T \in \mathcal{T}_H$. The operator $\pi_H : V_H \rightarrow M_H^0$ is linear and continuous.

Proof. For a given $\mathbf{v} \in V_H$, we rewrite definition (3.9)

$$\begin{aligned} -\mu \Delta \mathbf{v} + \nabla \zeta^T &= \sum_{k=1}^r \lambda_{T,k} \boldsymbol{\varphi}_{T,k} \text{ in } T \cap \Omega^\varepsilon, \\ \operatorname{div} \mathbf{v} &\in \operatorname{span} \{\varpi_{T,1}, \dots, \varpi_{T,t}\} \text{ in } T \cap \Omega^\varepsilon, \\ \mathbf{v} &= 0 \text{ on } \partial B^\varepsilon \cap T, \\ \mu \nabla \mathbf{v} \mathbf{n} - \zeta^T \mathbf{n} &\in \operatorname{span} \{\boldsymbol{\omega}_{E,1}, \dots, \boldsymbol{\omega}_{E,s}\} \text{ on } E \cap \Omega^\varepsilon, \forall E \in \mathcal{E}(T). \end{aligned}$$

for some vector $\lambda_T = (\lambda_{T,1}, \dots, \lambda_{T,r})$. Let us show that the pressure $\zeta^T \in M_H^0(T)$ and the vector λ_T are uniquely defined. Assume there exist $(\zeta_1^T, \lambda_T^1) \in M_H^0(T) \times \mathbb{R}^r$ and $(\zeta_2^T, \lambda_T^2) \in M_H^0(T) \times \mathbb{R}^r$ satisfying the system of equations above for the same velocity \mathbf{v} . In other words,

$$\nabla \zeta_1^T = \sum_{k=1}^r \lambda_{T,k}^1 \boldsymbol{\varphi}_{T,k} + \mu \Delta \mathbf{v},$$

$$\nabla \zeta_2^T = \sum_{k=1}^r \lambda_{T,k}^2 \boldsymbol{\varphi}_{T,k} + \mu \Delta \mathbf{v}.$$

Subtracting these equations, we obtain

$$\nabla (\zeta_1^T - \zeta_2^T) = \sum_{k=1}^r (\lambda_{T,k}^1 - \lambda_{T,k}^2) \boldsymbol{\varphi}_{T,k}.$$

The fact $\boldsymbol{\varphi}_{T,k} \in (\mathbb{P}_{n-1}(T))^d$ implies that $(\zeta_1^T - \zeta_2^T) \in \mathbb{P}_n(T)$. At the same time, $(\zeta_1^T - \zeta_2^T) \in M_H^0(T)$ defined by (3.6) which is orthogonal to $\mathbb{P}_n(T)$. Thus we deduce that $\zeta_1^T = \zeta_2^T$ and then $\lambda_T^1 = \lambda_T^2$. Therefore we conclude that there exists an operator $\pi_H : V_H \rightarrow M_H^0$ such that $\zeta^T = [\pi_H(\mathbf{v})]_T$. Moreover, π_H is obviously linear and, since V_H is finite-dimensional, π_H is continuous. \square

THEOREM 3.12. *Using the notations of Definition 3.8, let us introduce the space*

$$(3.10) \quad \underline{X}_H = \text{span} \{(\mathbf{u}_H, \pi_H(\mathbf{u}_H) + \bar{p}_H), \mathbf{u}_H \in V_H, \bar{p}_H \in M_H\}.$$

Then it coincides with the subspace X_H defined in (3.7), namely

$$(3.11) \quad X_H = \underline{X}_H.$$

Proof. We first prove the inclusion $X_H \subset \underline{X}_H$. Let $(\mathbf{u}_H, p_H) \in X_H$ in the sense of definition (3.7), i.e. $\forall (\mathbf{v}, q) \in X_H^0$:

$$(3.12) \quad c_H((\mathbf{u}_H, p_H), (\mathbf{v}, q)) = \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} (\mu \nabla \mathbf{u}_H : \nabla \mathbf{v} - p_H \text{div } \mathbf{v} - q \text{div } \mathbf{u}_H) = 0.$$

Let $\bar{p}_H \in M_H$ be the orthogonal projection of p_H on M_H , and let $p'_H = (p_H - \bar{p}_H)$. The fact that M_H and M_H^0 are orthogonal implies that

$$p_H = \bar{p}_H + p'_H \text{ with } \bar{p}_H \in M_H \text{ and } p'_H \in M_H^0.$$

By virtue of this decomposition, the term concerning p_H in (3.12) can be decomposed as

$$(3.13) \quad \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} p_H \text{div } \mathbf{v} = \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} \bar{p}_H \text{div } \mathbf{v} + \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} p'_H \text{div } \mathbf{v}.$$

Now we compute the first term in the right hand side of (3.13). Integrating by parts on each triangle T yields:

$$(3.14) \quad \int_{T \cap \Omega^\varepsilon} \bar{p}_H \text{div } \mathbf{v} = \int_{\partial(T \cap \Omega^\varepsilon)} \mathbf{v} \cdot \mathbf{n} \bar{p}_H - \int_{T \cap \Omega^\varepsilon} \mathbf{v} \cdot \nabla \bar{p}_H, \quad \forall \mathbf{v} \in V_H^0.$$

Since $\bar{p}_H \in M_H$, definition (3.8) implies that for any $T \in \mathcal{T}_H$, $\bar{p}_H|_T \in \mathbb{P}_n(T)$. Thus, for any $E \in \mathcal{E}(T)$, $\bar{p}_H|_E \mathbf{n} \in (\mathbb{P}_n(E))^d = \text{span}\{\boldsymbol{\omega}_{E,1}, \dots, \boldsymbol{\omega}_{E,s}\}$, according to Hypothesis 3.1. Then, since $\mathbf{v} \in V_H^0$, definition (3.4) implies that

$$\int_{\partial(T \cap \Omega^\varepsilon)} \mathbf{v} \cdot \mathbf{n} \bar{p}_H = 0.$$

Then (3.14) reduces to

$$\int_{T \cap \Omega^\varepsilon} \bar{p}_H \operatorname{div} \mathbf{v} = - \int_{T \cap \Omega^\varepsilon} \mathbf{v} \cdot \nabla \bar{p}_H.$$

As $\bar{p}_H|_T \in \mathbb{P}_n(T)$, it is obvious that $\nabla \bar{p}_H = \mathbf{0}$ for $n = 0$ and that $\nabla \bar{p}_H \in (\mathbb{P}_{n-1}(T))^d = \text{span}\{\boldsymbol{\varphi}_{T,1}, \dots, \boldsymbol{\varphi}_{T,r}\}$ on $T \in \mathcal{T}_H$ for $n \geq 1$, according to Hypothesis 3.1. Then, since $\mathbf{v} \in V_H^0$, definition (3.4) implies that

$$\int_{T \cap \Omega^\varepsilon} \bar{p}_H \operatorname{div} \mathbf{v} = - \int_{T \cap \Omega^\varepsilon} \mathbf{v} \cdot \nabla \bar{p}_H = 0.$$

As a result, (3.13) reduces to

$$\sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} p_H \operatorname{div} \mathbf{v} = \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} p'_H \operatorname{div} \mathbf{v}$$

and (3.12) reduces to: $\forall (\mathbf{v}, q) \in X_H^0$,

(3.15)

$$c_H((\mathbf{u}_H, p_H), (\mathbf{v}, q)) = \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} (\mu \nabla \mathbf{u}_H : \nabla \mathbf{v} - p'_H \operatorname{div} \mathbf{v} - q \operatorname{div} \mathbf{u}_H) = 0.$$

In a second step, choosing the test function $\mathbf{v} = \mathbf{0}$ and, for a given element $T \in \mathcal{T}_H$, the test function $q \in M_H^0$ with q vanishing outside T , (3.15) becomes

$$\int_{T \cap \Omega^\varepsilon} q \operatorname{div} \mathbf{u}_H = 0.$$

By definition (3.5) of M_H^0 , this implies that

$$(\operatorname{div} \mathbf{u}_H)|_T \in \text{span}\{\boldsymbol{\varpi}_{T,1}, \dots, \boldsymbol{\varpi}_{T,t}\} \text{ i.e. } (\operatorname{div} \mathbf{u}_H)|_T \in \mathbb{P}_n(T) \text{ in } T \cap \Omega^\varepsilon,$$

thanks to Hypothesis 3.1.

In a third step, we observe that for any face $E \in \mathcal{E}(T)$, there exist some (not unique) functions $\mathbf{v}_{E,i} \in (H^1(T \cap \Omega^\varepsilon))^d$, $i = 1, \dots, s$ such that

$$\begin{cases} \int_{F \cap \Omega^\varepsilon} \mathbf{v}_{E,i} \cdot \boldsymbol{\omega}_{F,j} = \delta_{E,F} \delta_{i,j}, \quad \forall F \in \mathcal{E}(T), \quad \forall j = 1, \dots, s, \\ \int_{T \cap \Omega^\varepsilon} \mathbf{v}_{E,i} \cdot \boldsymbol{\varphi}_{T,l} = 0, \quad \forall l = 1, \dots, r, \\ \mathbf{v}_{E,i} = \mathbf{0}, \text{ on } \partial B^\varepsilon \cap T. \end{cases}$$

We also observe that there exist (not unique) functions $\mathbf{v}_{T,k} \in (H^1(T \cap \Omega^\varepsilon))^d$, $k = 1, \dots, r$ such that

$$\begin{cases} \int_{F \cap \Omega^\varepsilon} \mathbf{v}_{T,k} \cdot \boldsymbol{\omega}_{F,j} = 0, \quad \forall F \in \mathcal{E}(T), \quad \forall j = 1, \dots, s, \\ \int_{T \cap \Omega^\varepsilon} \mathbf{v}_{T,k} \cdot \boldsymbol{\varphi}_{T,l} = \delta_{k,l}, \quad \forall l = 1, \dots, r, \\ \mathbf{v}_{T,k} = \mathbf{0}, \text{ on } \partial B^\varepsilon \cap T. \end{cases}$$

Remark 3.13. One may construct such functions as follows. Let θ be a given non-vanishing positive function in $H_0^1(T \cap \Omega^\varepsilon)$. Then, functions $\mathbf{v}_{T,k}$ are sought under the form $(\sum_{1 \leq k' \leq r} z_{k,k'} \boldsymbol{\varphi}_{T,k'}) \theta$. They evidently satisfy the first and last line of (3.18) and, for any k , coefficients $z_{k,k'}$ are chosen such that $\sum_{1 \leq k' \leq r} a_{k',l} z_{k,k'} = \delta_{k,l}$ for all $l = 1, \dots, r$, with $a_{k',l} = \int_{T \cap \Omega^\varepsilon} \theta \boldsymbol{\varphi}_{T,k'} \cdot \boldsymbol{\varphi}_{T,l}$. This is a square linear system which has a unique solution since $(\mathbf{v}, \mathbf{w}) \mapsto \int_{T \cap \Omega^\varepsilon} \theta \mathbf{v} \cdot \mathbf{w}$ is a scalar product on $(\mathbb{P}_{n-1}(T))^d$ and $(\boldsymbol{\varphi}_{T,k})_{1 \leq k \leq r}$ a basis of that space. Next, for a given face $E \in \mathcal{E}(T)$, considering a function $\theta_E \in H_0^1(E \cap \Omega^\varepsilon)$ a similar process leads to functions $\tilde{\mathbf{v}}_{E,i}$ defined on $E \cap \Omega^\varepsilon$, vanishing on $\partial(E \cap \Omega^\varepsilon)$ and such that $\int_{E \cap \Omega^\varepsilon} \tilde{\mathbf{v}}_{E,i} \cdot \boldsymbol{\omega}_{E,j} = \delta_{i,j}$. Extending these functions by 0 on the other faces $F \neq E$ of T and on $\partial B^\varepsilon \cap T$, they belong to $H^{1/2}(\partial(T \cap \Omega^\varepsilon))$ and can be lifted to functions of $H^1(T \cap \Omega^\varepsilon)$ still denoted by $\tilde{\mathbf{v}}_{E,i}$ and which verify by construction the first and third lines of (3.17). We conclude by setting $\mathbf{v}_{E,i} = \tilde{\mathbf{v}}_{E,i} - \sum_{1 \leq k \leq r} a_k \mathbf{v}_{T,k}$ with $a_k = \int_{T \cap \Omega^\varepsilon} \tilde{\mathbf{v}}_{E,i} \cdot \boldsymbol{\varphi}_{T,k}$.

We denote by $V(T)$ the set of functions in $(H^1(T \cap \Omega^\varepsilon))^d$ vanishing on $\partial B^\varepsilon \cap T$:

$$(3.19) \quad V(T) = \left\{ \mathbf{v} \in (H^1(T \cap \Omega^\varepsilon))^d \text{ such that } \mathbf{v} = \mathbf{0} \text{ on } \partial B^\varepsilon \cap T \right\}.$$

We now check that the space $V(T)$ can be decomposed as

$$V(T) = V_{f0}(T) \oplus \text{span} \{ \mathbf{v}_{E,i}, \mathbf{v}_{T,k}, \forall E \in \mathcal{E}(T), i = 1, \dots, s, k = 1, \dots, r \}$$

where

$$(3.20) \quad V_{f0}(T) = \left\{ \begin{array}{l} \mathbf{v} \in (H^1(T \cap \Omega^\varepsilon))^d : \int_{F \cap \Omega^\varepsilon} \mathbf{v} \cdot \boldsymbol{\omega}_{F,j} = 0, \int_{T \cap \Omega^\varepsilon} \mathbf{v} \cdot \boldsymbol{\varphi}_{T,l} = 0, \\ \mathbf{v} = \mathbf{0} \text{ on } \partial B^\varepsilon \cap T, \forall F \in \mathcal{E}(T), j = 1, \dots, s, l = 1, \dots, r. \end{array} \right\}.$$

Indeed, the fact that these two subspaces are in direct sum is obvious. Next, we decompose any $\mathbf{v} \in V(T)$ into

$$(3.21) \quad \mathbf{v} = \tilde{\mathbf{v}} + \sum_{E \in \mathcal{E}_H} \sum_{i=1}^s \beta_{E,i} \mathbf{v}_{E,i} + \sum_{k=1}^r \beta_{T,k} \mathbf{v}_{T,k}.$$

Choosing

$$(3.22) \quad \beta_{E,i} = \int_{E \cap \Omega^\varepsilon} \mathbf{v} \cdot \boldsymbol{\omega}_{E,i} \text{ and } \beta_{T,k} = \int_{T \cap \Omega^\varepsilon} \mathbf{v} \cdot \boldsymbol{\varphi}_{T,k},$$

one may easily check that $\tilde{\mathbf{v}} \in V_{f0}(T)$.

Now, for a given T and a given $\mathbf{v} \in V(T)$, let $\underline{\mathbf{v}}$ be the function which is equal to $\tilde{\mathbf{v}}$ on $T \cap \Omega^\varepsilon$ as constructed in (3.21), and equal to $\mathbf{0}$ elsewhere. Hence, since $\tilde{\mathbf{v}} \in V_{f0}(T)$, it holds that $\underline{\mathbf{v}} \in V_H^0$. Taking $q = 0$ and $\mathbf{v} = \underline{\mathbf{v}}$ in (3.15) we obtain

$$(3.23) \quad c_H((\mathbf{u}_H, p_H), (\underline{\mathbf{v}}, 0)) = \int_{T \cap \Omega^\varepsilon} (\mu \nabla \mathbf{u}_H : \nabla \tilde{\mathbf{v}} - p'_H \text{div } \tilde{\mathbf{v}}) = 0.$$

Substituting $\tilde{\mathbf{v}}$ defined in (3.21) into (3.23), we obtain

$$(3.24) \quad \begin{aligned} \int_{T \cap \Omega^\varepsilon} (\mu \nabla \mathbf{u}_H : \nabla \mathbf{v} - p'_H \text{div } \mathbf{v}) &= \sum_{k=1}^r \beta_{T,k} \int_{T \cap \Omega^\varepsilon} (\mu \nabla \mathbf{u}_H : \nabla \mathbf{v}_{T,k} - p'_H \text{div } \mathbf{v}_{T,k}) \\ &+ \sum_{E \in \mathcal{E}(T)} \sum_{i=1}^s \beta_{E,i} \int_{T \cap \Omega^\varepsilon} (\mu \nabla \mathbf{u}_H : \nabla \mathbf{v}_{E,i} - p'_H \text{div } \mathbf{v}_{E,i}). \end{aligned}$$

Denoting for any $E \in \mathcal{E}(T)$, $i = 1, \dots, s$ and $k = 1, \dots, r$

$$\begin{aligned}\lambda_{E,i} &= \int_{T \cap \Omega^\varepsilon} \mu \nabla \mathbf{u}_H : \nabla \mathbf{v}_{E,i} - p'_H \operatorname{div} \mathbf{v}_{E,i}, \\ \lambda_{T,k} &= \int_{T \cap \Omega^\varepsilon} \mu \nabla \mathbf{u}_H : \nabla \mathbf{v}_{T,k} - p'_H \operatorname{div} \mathbf{v}_{T,k},\end{aligned}$$

and using the definition of $\beta_{E,i}$ and $\beta_{T,k}$ given by (3.22), equation (3.24) can be written as

$$\begin{aligned}\int_{T \cap \Omega^\varepsilon} (\mu \nabla \mathbf{u}_H : \nabla \mathbf{v} - p'_H \operatorname{div} \mathbf{v}) &= \sum_{E \in \mathcal{E}(T)} \sum_{i=1}^s \lambda_{E,i} \int_{E \cap \Omega^\varepsilon} \mathbf{v} \cdot \boldsymbol{\omega}_{E,i} \\ &+ \sum_{k=1}^r \lambda_{T,k} \int_{T \cap \Omega^\varepsilon} \mathbf{v} \cdot \boldsymbol{\varphi}_{T,k},\end{aligned}$$

which holds for any $\mathbf{v} \in V(T)$ defined in (3.19). In (3.25), considering an arbitrary \mathbf{v} that vanishes on all edges $E \in \mathcal{E}(T)$ and integrating by parts in the left-hand side, we obtain the following strong form:

$$-\mu \Delta \mathbf{u}_H + \nabla p'_H = \sum_{k=1}^r \lambda_{T,k} \boldsymbol{\varphi}_{T,k} \text{ in } T \cap \Omega^\varepsilon.$$

Then by writing the variational formulation of (3.26) for $\mathbf{v} \in V(T)$ and comparing with (3.25), we deduce that

$$\mu \nabla \mathbf{u}_H \mathbf{n} - p'_H \mathbf{n} \in \operatorname{span} \{ \boldsymbol{\omega}_{E,1}, \dots, \boldsymbol{\omega}_{E,s} \} \text{ on } E \cap \Omega^\varepsilon \quad \forall E \in \mathcal{E}(T).$$

Finally, combining equations (3.16), (3.26) and (3.27), we obtain

$$\begin{aligned}-\mu \Delta \mathbf{u}_H + \nabla p'_H &= \sum_{k=1}^r \lambda_{T,k} \boldsymbol{\varphi}_{T,k} \text{ in } T \cap \Omega^\varepsilon, \\ \operatorname{div} \mathbf{u}_H &\in \operatorname{span} \{ \boldsymbol{\varpi}_{T,1}, \dots, \boldsymbol{\varpi}_{T,t} \} \text{ in } T \cap \Omega^\varepsilon, \\ \mathbf{u}_H &= 0 \text{ on } \partial B^\varepsilon \cap T, \\ \mu \nabla \mathbf{u}_H \mathbf{n} - p'_H \mathbf{n} &\in \operatorname{span} \{ \boldsymbol{\omega}_{E,1}, \dots, \boldsymbol{\omega}_{E,s} \} \text{ on } E \cap \Omega^\varepsilon, \quad \forall E \in \mathcal{E}(T).\end{aligned}$$

Using Lemma 3.11, the pressure $p'_H = \pi_H(\mathbf{u}_H)$ and the vector $\lambda_T = (\lambda_{T,1}, \dots, \lambda_{T,r})$ are both uniquely determined by \mathbf{u}_H . We recall that the pressure p_H is defined by $p_H = \bar{p}_H + p'_H$. Thus we proved that (\mathbf{u}_H, p_H) defined in (3.7) belongs to the space defined by (3.10), i.e. $X_H \subset \underline{X}_H$.

Reciprocally, we now prove that $(\mathbf{u}_H, p_H) \in \underline{X}_H$ defined by (3.10), with $p_H = \bar{p}_H + \pi_H(\mathbf{u}_H)$, also satisfies relation (3.7). The first step is to remark that (\mathbf{u}_H, p_H) is indeed in X_H^{ext} ; the fact that p_H belongs to M is obvious, and the fact that \mathbf{u}_H belongs to V_H^{ext} comes from the definition of V_H itself. As far as the "orthogonality" relation in (3.7) is concerned, since $\mathbf{u}_H \in V_H$ then Lemma 3.11 shows that there exist

394 a unique $\zeta^T = \pi_H(\mathbf{u}_H) \in M_H^0(T)$ and a unique $(\lambda_{T,1}, \dots, \lambda_{T,r}) \in \mathbb{R}^r$ such that

$$395 \quad (3.28) \quad -\mu \Delta \mathbf{u}_H + \nabla \zeta^T = \sum_{k=1}^r \lambda_{T,k} \boldsymbol{\varphi}_{T,k} \text{ in } T \cap \Omega^\varepsilon,$$

$$396 \quad (3.29) \quad \operatorname{div} \mathbf{u}_H \in \operatorname{span} \{\boldsymbol{\varpi}_{T,1}, \dots, \boldsymbol{\varpi}_{T,t}\} \text{ in } T \cap \Omega^\varepsilon,$$

$$397 \quad (3.30) \quad \mathbf{u}_H = \mathbf{0} \text{ on } \partial B^\varepsilon \cap T,$$

$$398 \quad (3.31) \quad \mu \nabla \mathbf{u}_H \mathbf{n} - \zeta^T \mathbf{n} \in \operatorname{span} \{\boldsymbol{\omega}_{E,1}, \dots, \boldsymbol{\omega}_{E,s}\} \text{ on } E \cap \Omega^\varepsilon, \forall E \in \mathcal{E}(T).$$

400 Equation (3.31) shows that for any $E \in \mathcal{E}(T)$, there exist $\lambda_{E,1}, \dots, \lambda_{E,s} \in \mathbb{R}$ such
401 that

$$402 \quad \mu \nabla \mathbf{u}_H \mathbf{n} - \zeta^T \mathbf{n} = \sum_{i=1}^s \lambda_{E,i} \boldsymbol{\omega}_{E,i} \text{ on } E \cap \Omega^\varepsilon, \forall E \in \mathcal{E}(T).$$

404 Taking the scalar product of (3.28) with any $\mathbf{v} \in V_H^0$, integrating by parts and making
405 use of the equations above, we obtain

$$406 \quad \int_{T \cap \Omega^\varepsilon} \mu \nabla \mathbf{u}_H : \nabla \mathbf{v} - \int_{T \cap \Omega^\varepsilon} \zeta^T \operatorname{div} \mathbf{v} = \sum_{k=1}^r \lambda_{T,k} \int_{T \cap \Omega^\varepsilon} \boldsymbol{\varphi}_{T,k} \cdot \mathbf{v} \\ 407 \quad (3.32) \quad + \sum_{E \in \mathcal{E}(T)} \sum_{i=1}^s \lambda_{E,i} \int_{E \cap \Omega^\varepsilon} \boldsymbol{\omega}_{E,i} \cdot \mathbf{v}.$$

409 Then, definition (3.4) of V_H^0 implies that

$$410 \quad \int_{T \cap \Omega^\varepsilon} \boldsymbol{\varphi}_{T,k} \cdot \mathbf{v} = 0 \text{ and } \int_{E \cap \Omega^\varepsilon} \boldsymbol{\omega}_{E,i} \cdot \mathbf{v} = 0.$$

412 Thus (3.32) reduces to

$$413 \quad (3.33) \quad \int_{T \cap \Omega^\varepsilon} \mu \nabla \mathbf{u}_H : \nabla \mathbf{v} - \int_{T \cap \Omega^\varepsilon} \pi_H(\mathbf{u}_H)|_T \operatorname{div} \mathbf{v} = 0.$$

415 Besides, for $\bar{p}_H \in M_H$, integration by parts yields

$$416 \quad \int_{T \cap \Omega^\varepsilon} \bar{p}_H|_T \operatorname{div} \mathbf{v} = \int_{\partial(T \cap \Omega^\varepsilon)} \mathbf{v} \cdot \mathbf{n} \bar{p}_H|_T - \int_{T \cap \Omega^\varepsilon} \mathbf{v} \cdot \nabla \bar{p}_H|_T, \forall \mathbf{v} \in V_H^0.$$

418 Definition (3.8) of M_H implies that $\bar{p}_H|_T \in \mathbb{P}_n(T)$, so that $\nabla \bar{p}_H|_T \in (\mathbb{P}_{n-1}(T))^d$
419 for $n \geq 1$ and $\nabla \bar{p}_H|_T = \mathbf{0}$ for $n = 0$. We also have that the restriction of $\mathbf{n} \bar{p}_H|_T$
420 to $E \in \mathcal{E}(T)$ belongs to $(\mathbb{P}_n(E))^d$. Making use of Hypothesis 3.1, definition of V_H^0
421 implies that

$$422 \quad (3.34) \quad \begin{cases} \int_{\partial(T \cap \Omega^\varepsilon)} \mathbf{v} \cdot \mathbf{n} \bar{p}_H|_T = 0 \\ \int_{T \cap \Omega^\varepsilon} \mathbf{v} \cdot \operatorname{grad} \bar{p}_H|_T = 0 \end{cases} \implies \int_{T \cap \Omega^\varepsilon} \bar{p}_H|_T \operatorname{div} \mathbf{v} = 0.$$

424 Moreover, for any $\mathbf{u}_H \in V_H$, $\operatorname{div} \mathbf{u}_H \in \operatorname{span} \{\boldsymbol{\varpi}_{T,1}, \dots, \boldsymbol{\varpi}_{T,t}\}$. The fact that
425 $M_H^0(T)$ is orthogonal to $\operatorname{span} \{\boldsymbol{\varpi}_{T,1}, \dots, \boldsymbol{\varpi}_{T,t}\}$ implies that for any $q \in M_H^0(T)$

$$426 \quad (3.35) \quad \int_{T \cap \Omega^\varepsilon} q \operatorname{div} \mathbf{u}_H = 0.$$

As a result, summing (3.33), (3.34) and (3.35), we obtain that $\forall (\mathbf{v}, q) \in X_H^0$, and for all $T \in \mathcal{T}_H$

$$\int_{T \cap \Omega^\varepsilon} \mu \nabla \mathbf{u}_H : \nabla \mathbf{v} - \int_{T \cap \Omega^\varepsilon} (\pi_H(\mathbf{u}_H)|_T + \bar{p}_H|_T) \operatorname{div} \mathbf{v} - \int_{T \cap \Omega^\varepsilon} q \operatorname{div} \mathbf{u}_H = 0,$$

Now by definition $p_H|_T = \pi_H(\mathbf{u}_H)|_T + \bar{p}_H|_T$ on each element $T \in \mathcal{T}_H$. Summing the equation above on all elements $T \in \mathcal{T}_H$, we obtain that for any $(\mathbf{v}, q) \in X_H^0$,

$$\sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} \mu \nabla \mathbf{u}_H : \nabla \mathbf{v} - \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} p_H \operatorname{div} \mathbf{v} - \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} q \operatorname{div} \mathbf{u}_H = 0.$$

which exactly means that $(\mathbf{u}_H, p_H) \in X_H$, and therefore $\underline{X}_H \subset X_H$. Consequently, equality (3.11) is proved. \square

3.3. Local problems defined by Stokes equations. Now we construct some functions associated to elements or faces of the coarse mesh, we shall prove in section 3.5 that they form a basis of the resolved space V_H . From their definitions below, they evidently belong to V_H^{ext} .

The strong form. We first construct functions associated to faces of the coarse mesh. For any $E \in \mathcal{E}_H$, for $i = 1, \dots, s$, find the function $\Phi_{E,i} : \Omega^\varepsilon \rightarrow \mathbb{R}^d$, the pressure $\pi_{E,i} : \Omega^\varepsilon \rightarrow \mathbb{R}$ such that $\Phi_{E,i}$ and $\pi_{E,i}$ vanish outside the coarse element(s) $T_k \subset \omega_E$ (defined by (3.1)) for $k \in \{1, 2\}$ (only one coarse element if $E \in \partial\Omega$) and solve on T_k :

$$(3.36) \quad \begin{cases} -\mu \Delta \Phi_{E,i} + \nabla \pi_{E,i} \in \operatorname{span} \{\varphi_{T_k,1}, \dots, \varphi_{T_k,r}\} \text{ in } T_k \cap \Omega^\varepsilon, \\ \operatorname{div} \Phi_{E,i} \in \operatorname{span} \{\varpi_{T_k,1}, \dots, \varpi_{T_k,t}\} \text{ in } T_k \cap \Omega^\varepsilon, \\ \mu \nabla \Phi_{E,i} \mathbf{n} - \pi_{E,i} \mathbf{n} \in \operatorname{span} \{\omega_{F,1}, \dots, \omega_{F,s}\} \text{ on } F \cap \Omega^\varepsilon, \forall F \in \mathcal{E}(T_k), \\ \Phi_{E,i} = \mathbf{0} \text{ on } \partial B^\varepsilon \cap T_k, \\ \int_{F \cap \Omega^\varepsilon} \Phi_{E,i} \cdot \omega_{F,j} = \begin{cases} \delta_{ij}, & F = E \\ 0, & F \neq E \end{cases} \quad \forall F \in \mathcal{E}(T_k), \forall j = 1, \dots, s, \\ \int_{T_k \cap \Omega^\varepsilon} \Phi_{E,i} \cdot \varphi_{T_k,l} = 0 \quad \forall l = 1, \dots, r, \\ \int_{T_k \cap \Omega^\varepsilon} \pi_{E,i} \cdot \varpi_{T_k,m} = 0 \quad \forall m = 1, \dots, t. \end{cases}$$

Then, we construct functions associated to elements of the coarse mesh. For each $T \in \mathcal{T}_H$, for $k = 1, \dots, r$, the support of the function $\Psi_{T,k}$ is reduced to $T \cap \Omega^\varepsilon$. We find $\Psi_{T,k} : \Omega^\varepsilon \rightarrow \mathbb{R}^d$ and $\pi_{T,k} : \Omega^\varepsilon \rightarrow \mathbb{R}$ by solving on T :

$$(3.37) \quad \begin{cases} -\mu \Delta \Psi_{T,k} + \nabla \pi_{T,k} \in \operatorname{span} \{\varphi_{T,1}, \dots, \varphi_{T,r}\} \text{ in } T \cap \Omega^\varepsilon, \\ \operatorname{div} \Psi_{T,k} \in \operatorname{span} \{\varpi_{T,1}, \dots, \varpi_{T,t}\} \text{ in } T \cap \Omega^\varepsilon, \\ \mu \nabla \Psi_{T,k} \mathbf{n} - \pi_{T,k} \mathbf{n} \in \operatorname{span} \{\omega_{F,1}, \dots, \omega_{F,s}\} \text{ on } F \cap \Omega^\varepsilon, \forall F \in \mathcal{E}(T), \\ \Psi_{T,k} = \mathbf{0} \text{ on } \partial B^\varepsilon \cap T, \\ \int_{F \cap \Omega^\varepsilon} \Psi_{T,k} \cdot \omega_{F,j} = 0 \quad \forall F \in \mathcal{E}(T), \forall j = 1, \dots, s, \\ \int_{T \cap \Omega^\varepsilon} \Psi_{T,k} \cdot \varphi_{T,l} = \delta_{kl} \quad \forall l = 1, \dots, r, \\ \int_{T \cap \Omega^\varepsilon} \pi_{T,k} \cdot \varpi_{T,m} = 0 \quad \forall m = 1, \dots, t. \end{cases}$$

The weak form. The weak form of (3.36) is as follows. Let $n_s = s \times \operatorname{Card}(\mathcal{E}(T_k))$. For any $E \in \mathcal{E}_H$, for $i = 1, \dots, s$, on the coarse element $T_k \subset \omega_E$ for $k \in \{1, 2\}$ (only one coarse element if $E \in \partial\Omega$), find $\Phi_{E,i} \in (H^1(T_k \cap \Omega^\varepsilon))^d$ such that $\Phi_{E,i} = \mathbf{0}$ on

457 $\partial B^\varepsilon \cap T_k$, $\pi_{E,i} \in L^2(T_k \cap \Omega^\varepsilon)$, $(\lambda_{F,j})_{F \in \mathcal{E}(T_k), j=1, \dots, s} \in \mathbb{R}^{n_s}$ and $(\lambda_{T_k,l})_{l=1, \dots, r} \in \mathbb{R}^r$
 458 by solving:

(3.38)

$$\begin{cases}
 \int_{T_k \cap \Omega^\varepsilon} \mu \nabla \Phi_{E,i} : \nabla \mathbf{v} - \int_{T_k \cap \Omega^\varepsilon} \pi_{E,i} \operatorname{div} \mathbf{v} + \sum_{F \in \mathcal{E}(T_k)} \sum_{j=1}^s \lambda_{F,j} \int_{F \cap \Omega^\varepsilon} \mathbf{v} \cdot \boldsymbol{\omega}_{F,j} = \\
 \sum_{l=1}^r \lambda_{T_k,l} \int_{T_k \cap \Omega^\varepsilon} \boldsymbol{\varphi}_{T_k,l} \cdot \mathbf{v}, \quad \forall \mathbf{v} \in (H^1(T_k \cap \Omega^\varepsilon))^d \text{ such that } \mathbf{v} = \mathbf{0} \text{ on } \partial B^\varepsilon \cap T_k, \\
 \int_{T_k \cap \Omega^\varepsilon} q \operatorname{div} \Phi_{E,i} = 0, \quad \forall q \in M_H^0(T_k), \\
 \sum_{F \in \mathcal{E}(T_k)} \sum_{j=1}^s \mu_{F,j} \int_{F \cap \Omega^\varepsilon} \Phi_{E,i} \cdot \boldsymbol{\omega}_{F,j} = \mu_{E,i}, \quad \forall (\mu_{F,j})_{F \in \mathcal{E}(T_k), j=1, \dots, s} \in \mathbb{R}^{n_s}, \\
 \sum_{l=1}^r \mu_l \int_{T_k \cap \Omega^\varepsilon} \Phi_{E,i} \cdot \boldsymbol{\varphi}_{T_k,l} = 0, \quad \forall (\mu_l)_{l=1, \dots, r} \in \mathbb{R}^r, \\
 \int_{T_k \cap \Omega^\varepsilon} \pi_{E,i} \cdot \boldsymbol{\omega}_{T_k,m} = 0, \quad \forall m = 1, \dots, t.
 \end{cases}$$

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460

461 The weak form of (3.37) is as follows. Recall that $n_s = s \times \operatorname{Card}(\mathcal{E}(T))$. For
 462 $k = 1, \dots, r$, find $\Psi_{T,k} \in (H^1(T \cap \Omega^\varepsilon))^d$ such that $\Psi_{T,k} = \mathbf{0}$ on $\partial B^\varepsilon \cap T$, $\pi_{T,k} \in$
 463 $L^2(T \cap \Omega^\varepsilon)$, $(\lambda_{F,j})_{F \in \mathcal{E}(T), j=1, \dots, s} \in \mathbb{R}^{n_s}$ and $(\lambda_{T,l})_{l=1, \dots, r} \in \mathbb{R}^r$ by solving

(3.39)

$$\begin{cases}
 \int_{T \cap \Omega^\varepsilon} \mu \nabla \Psi_{T,k} : \nabla \mathbf{v} - \int_{T \cap \Omega^\varepsilon} \pi_{T,k} \operatorname{div} \mathbf{v} + \sum_{F \in \mathcal{E}(T)} \sum_{j=1}^s \lambda_{F,j} \int_{F \cap \Omega^\varepsilon} \mathbf{v} \cdot \boldsymbol{\omega}_{F,j} = \\
 \sum_{l=1}^r \lambda_{T,l} \int_{T \cap \Omega^\varepsilon} \boldsymbol{\varphi}_{T,l} \cdot \mathbf{v}, \quad \forall \mathbf{v} \in (H^1(T \cap \Omega^\varepsilon))^d \text{ such that } \mathbf{v} = \mathbf{0} \text{ on } \partial B^\varepsilon \cap T, \\
 \int_{T \cap \Omega^\varepsilon} q \operatorname{div} \Psi_{T,k} = 0, \quad \forall q \in M_H^0(T), \\
 \sum_{F \in \mathcal{E}(T)} \sum_{j=1}^s \mu_{F,j} \int_{F \cap \Omega^\varepsilon} \Psi_{T,k} \cdot \boldsymbol{\omega}_{F,j} = 0, \quad \forall (\mu_{F,j})_{F \in \mathcal{E}(T), j=1, \dots, s} \in \mathbb{R}^{n_s}, \\
 \sum_{l=1}^r \mu_{T,l} \int_{T \cap \Omega^\varepsilon} \Psi_{T,k} \cdot \boldsymbol{\varphi}_{T,l} = \mu_{T,k}, \quad \forall (\mu_{T,l})_{l=1, \dots, r} \in \mathbb{R}^r, \\
 \int_{T \cap \Omega^\varepsilon} \pi_{T,k} \cdot \boldsymbol{\omega}_{T,m} = 0, \quad \forall m = 1, \dots, t.
 \end{cases}$$

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465

466 **3.4. Well-posedness of local problems.** We prove that the local problems
 467 (3.36) and (3.37) are well-posed. Choosing a coarse element $T \in \mathcal{T}_H$, let n_T be the
 468 number of edges composing ∂T and $n_s = n_T \times s$ be the dimension of the vector
 469 containing Lagrange multipliers $(\lambda_{F,j})$ for all $F \in \mathcal{E}(T)$ and $j = 1, \dots, s$. Moreover,
 470 r is the dimension of the Lagrange multiplier vector $(\lambda_{T,l})$ for $l = 1, \dots, r$. We
 471 introduce the velocity space

$$V_T = \left\{ \mathbf{v} \in (H^1(T \cap \Omega^\varepsilon))^d, \mathbf{v} = \mathbf{0} \text{ on } \partial B^\varepsilon \cap T \right\},$$

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473

474 as well as $M_H^0(T)$, the pressure space defined by (3.6). The variational formulation of
 475 local problems (3.38) and (3.39) are of the form :

476 Find $(\mathbf{u}, p, (\lambda_{F,j}), (\lambda_{T,l})) \in V_T \times M_H^0(T) \times \mathbb{R}^{n_s} \times \mathbb{R}^r$ such that

$$(3.40) \quad \hat{a}_T(\mathbf{u}, \mathbf{v}) + b_T(\mathbf{v}, p) + c_E(\mathbf{v}, (\lambda_{F,j}^T)) + d_T(\mathbf{v}, (\lambda_{T,l})) = 0, \quad \forall \mathbf{v} \in V_T$$

$$(3.41) \quad b_T(\mathbf{u}, q) = 0, \quad \forall q \in M_H^0(T)$$

$$(3.42) \quad c_E(\mathbf{u}, (\mu_{F,j})) = \ell_1((\mu_{F,j})), \quad \forall (\mu_{F,j}) \in \mathbb{R}^{n_s},$$

$$(3.43) \quad d_T(\mathbf{u}, (\mu_{T,l})) = \ell_2((\mu_{T,l})), \quad \forall (\mu_{T,l}) \in \mathbb{R}^r.$$

480

where the bilinear forms are defined by

$$\begin{aligned}\widehat{a}_T(\mathbf{u}, \mathbf{v}) &= \int_{T \cap \Omega^\varepsilon} \mu \nabla \mathbf{u} : \nabla \mathbf{v}, \\ b_T(\mathbf{v}, p) &= - \int_{T \cap \Omega^\varepsilon} p \operatorname{div} \mathbf{v}, \\ c_E(\mathbf{v}, (\lambda_{F,j})) &= \sum_{F \in \mathcal{E}(T)} \sum_{j=1}^s \lambda_{F,j} \int_{F \cap \Omega^\varepsilon} \mathbf{v} \cdot \boldsymbol{\omega}_{F,j}, \\ d_T(\mathbf{v}, (\mu_{T,l})) &= \sum_{l=1}^r \mu_{T,l} \int_{T \cap \Omega^\varepsilon} \mathbf{v} \cdot \boldsymbol{\varphi}_{T,l},\end{aligned}$$

and where the linear forms ℓ_1 and ℓ_2 are defined by

$$\begin{aligned}\ell_1((\mu_{F,j})) &= \mu_{E,i} \quad , \quad \ell_2((\mu_{T,l})) = 0 \quad \text{for (3.38) and} \\ \ell_1((\mu_{F,j})) &= 0 \quad , \quad \ell_2((\mu_{T,l})) = \mu_{T,k} \quad \text{for (3.39)}.\end{aligned}$$

System (3.40)–(3.43) is called a twofold saddle point problem in the literature and can be viewed as a single saddle point problem defined on $V_T \times (M_H^0(T) \times \mathbb{R}^{n_s} \times \mathbb{R}^r)$. By introducing the following bilinear form

$$\begin{aligned}\forall (\mathbf{v}, (p, (\lambda_{F,j}), (\lambda_{T,l}))) &\in V_T \times (M_H^0(T) \times \mathbb{R}^{n_s} \times \mathbb{R}^r) \\ \tilde{b}_T(\mathbf{v}, (p, (\lambda_{F,j}), (\lambda_{T,l}))) &= b_T(\mathbf{v}, p) + c_E(\mathbf{v}, (\lambda_{F,j})) + d_T(\mathbf{v}, (\lambda_{T,l})),\end{aligned}$$

system (3.40)–(3.43) can be reformulated as

$$\begin{aligned}(3.44) \quad &\widehat{a}_T(\mathbf{u}, \mathbf{v}) + \tilde{b}_T(\mathbf{v}, (p, (\lambda_{F,j}), (\lambda_{T,l}))) = 0, \quad \forall \mathbf{v} \in V_T, \\ (3.45) \quad &\tilde{b}_T(\mathbf{u}, (q, (\mu_{F,j}), (\mu_{T,l}))) = \ell_1((\mu_{F,j})) + \ell_2((\mu_{T,l})), \\ &\forall (q, (\mu_{F,j}), (\mu_{T,l})) \in M_H^0(T) \times \mathbb{R}^{n_s} \times \mathbb{R}^r.\end{aligned}$$

Now we can apply the inf-sup theory to prove the well-posedness of (3.44)–(3.45). It is easy to check that the bilinear forms \widehat{a}_T and \tilde{b}_T are both continuous over their spaces of definition. Additionally, the linear forms ℓ_1 and ℓ_2 are also continuous. Moreover, the bilinear form \widehat{a}_T is coercive over the velocity kernel of the bilinear form \tilde{b}_T , since this kernel, for all $n \geq 0$, contains vector functions which, at least, have mean values that vanish on the edges (or faces) of T , and this implies a Poincaré-like inequality.

Finally, it remains only to prove that the bilinear form $\tilde{b}_T(.,.)$ satisfies the following inf-sup condition: there exists $\tilde{\alpha} > 0$ such that $\forall (p, (\lambda_{F,j}), (\lambda_{T,l})) \in M_H^0(T) \times \mathbb{R}^{n_s} \times \mathbb{R}^r$

$$(3.46) \quad \sup_{\mathbf{v} \in V_T} \frac{\tilde{b}_T(\mathbf{v}, (p, (\lambda_{F,j}), (\lambda_{T,l})))}{\left(\|p\|_{L^2(T \cap \Omega^\varepsilon)} + \|(\lambda_{F,j})\| + \|(\lambda_{T,l})\| \right) \|\mathbf{v}\|_{H^1(T \cap \Omega^\varepsilon)}} \geq \tilde{\alpha}.$$

To check the inf-sup condition for $\tilde{b}_T(.,.)$, for any $p \in M_H^0(T)$, $(\lambda_{F,j}) \in \mathbb{R}^{n_s}$ and $(\lambda_{T,l}) \in \mathbb{R}^r$, we would like to find a velocity field $\mathbf{v} \in V_T$ and a polynomial $q \in \mathbb{P}_n(T)$ such that

$$(3.47) \quad -\operatorname{div} \mathbf{v} = p + q,$$

$$(3.48) \quad \int_{F \cap \Omega^\varepsilon} \mathbf{v} \cdot \boldsymbol{\omega}_{F,j} = \lambda_{F,j}, \quad \forall F \in \mathcal{E}(T), j = 1, \dots, s,$$

$$(3.49) \quad \int_{T \cap \Omega^\varepsilon} \mathbf{v} \cdot \boldsymbol{\varphi}_{T,l} = \lambda_{T,l}, \quad \forall l = 1, \dots, r.$$

Indeed, in that case we would have (since $(p, q)_{T \cap \Omega^\varepsilon} = 0$)

$$(3.50) \quad \tilde{b}_T(\mathbf{v}, (p, (\lambda_{F,j}), (\lambda_{T,l}))) = \|p\|_{L^2(T \cap \Omega^\varepsilon)}^2 + \|(\lambda_{F,j})\|^2 + \|(\lambda_{T,l})\|^2.$$

Note that it is necessary to include a polynomial $q \in \mathbb{P}_n(T)$ in (3.47); if not, for any $q' \in \mathbb{P}_n(T)$, we would have

$$(3.51) \quad (\operatorname{div} \mathbf{v}, q') = 0 = \sum_F \int_{F \cap \Omega^\varepsilon} \mathbf{v} \cdot (q' \mathbf{n}) - \int_{T \cap \Omega^\varepsilon} \mathbf{v} \cdot \nabla q'.$$

Since $(q' \mathbf{n})|_F \in \operatorname{span}\{\boldsymbol{\omega}_{F,j}, j = 1, \dots, s\}$ and $\nabla q' \in \operatorname{span}\{\boldsymbol{\varphi}_{T,l}, l = 1, \dots, r\}$, then (3.51) with (3.48) and (3.49) would imply that the sets $(\lambda_{F,j})_{F \in \mathcal{E}(T), j=1, \dots, s}$ and $(\lambda_{T,l})_{l=1, \dots, r}$ could not be arbitrary.

Using functions (3.17) and (3.18), we may look for \mathbf{v} under the following form

$$(3.52) \quad \mathbf{v} = \sum_{F \in \mathcal{E}(T)} \sum_{j=1}^s \lambda_{F,j} \mathbf{v}_{F,j} + \sum_{l=1}^r \lambda_{T,l} \mathbf{v}_{T,l} + \mathbf{w}$$

with $\mathbf{w} \in V_T$ verifying

$$(3.53) \quad -\operatorname{div} \mathbf{w} = \tilde{p},$$

$$(3.54) \quad \int_{F \cap \Omega^\varepsilon} \mathbf{w} \cdot \boldsymbol{\omega}_{F,j} = 0, \forall F \in \mathcal{E}(T), j = 1, \dots, s,$$

$$(3.55) \quad \int_{T \cap \Omega^\varepsilon} \mathbf{w} \cdot \boldsymbol{\varphi}_{T,l} = 0, \forall l = 1, \dots, r.$$

where

$$\tilde{p} = p - (a - \Pi_n a),$$

and Π_n is the $L^2(T \cap \Omega^\varepsilon)$ orthogonal projection on $\mathbb{P}_n(T)$ and a is defined by

$$a = - \sum_{F \in \mathcal{E}(T)} \sum_{j=1}^s \lambda_{F,j} \operatorname{div} \mathbf{v}_{F,j} - \sum_{l=1}^r \lambda_{T,l} \operatorname{div} \mathbf{v}_{T,l}.$$

Moreover, in (3.47), we chose $q = \Pi_n a$. Note that \tilde{p} is orthogonal to \mathbb{P}_n since this is the case for both p and $(a - \Pi_n a)$.

The standard inf-sup condition (with constant $\beta > 0$) is equivalent to the fact that for any $\tilde{q} \in L_0^2(T \cap \Omega^\varepsilon)$, there exists $\mathbf{w}_0 \in (H_0^1(T \cap \Omega^\varepsilon))^d \subset V_T$ such that $\operatorname{div} \mathbf{w}_0 = \tilde{q}$ and $\|\mathbf{w}_0\|_{H^1(T \cap \Omega^\varepsilon)} \leq \beta \|\tilde{q}\|_{L^2(T \cap \Omega^\varepsilon)}$. We pick such a \mathbf{w}_0 for the particular case $\tilde{q} = \tilde{p}$ since \tilde{p} is orthogonal to \mathbb{P}_n and thus belongs to $L_0^2(T \cap \Omega^\varepsilon)$. Since \tilde{p} depends continuously on p and a and since a depends continuously on the sets of $\lambda_{F,j}$ and $\lambda_{T,l}$ (recall that $\mathbf{v}_{F,j}$ and $\mathbf{v}_{T,l}$ are given functions), then \mathbf{w}_0 depends continuously on p and on the sets of $\lambda_{F,j}$ and $\lambda_{T,l}$. Moreover, such a \mathbf{w}_0 automatically verifies boundary conditions (3.54) since it vanishes on the boundary. The sequel of the proof is performed for the case $d = 3$; the case $d = 2$ can be performed in a similar way, the only difference being that there are two curl operators when $d = 2$, one that applies on vector fields and returns a scalar field, and the other that acts conversely. Let \mathcal{G} be a given open ball included in $T \cap \Omega^\varepsilon$, we shall look for \mathbf{w} under the following form:

$$(3.56) \quad \mathbf{w} = \mathbf{w}_0 + \nabla \times \mathbf{z},$$

in which \mathbf{z} will be a smooth function compactly supported in \mathcal{G} (and thus $\nabla \times \mathbf{z}$ will be in $(H_0^1(T \cap \Omega^\varepsilon))^d$) such that

$$(3.57) \quad \int_{T \cap \Omega^\varepsilon} \nabla \times \mathbf{z} \cdot \boldsymbol{\varphi}_{T,l} = \int_{\mathcal{G}} \mathbf{z} \cdot \nabla \times \boldsymbol{\varphi}_{T,l} = - \int_{T \cap \Omega^\varepsilon} \mathbf{w}_0 \cdot \boldsymbol{\varphi}_{T,l}, \forall l = 1, \dots, r.$$

Let us now prove that it is possible to construct such a function \mathbf{z} . We start by proving that the equalities in (3.57) will be verified for all $l = 1, \dots, r$ if they are verified on a subset of indices. For this, let us consider the curl operator

$$\nabla \times : (\mathbb{P}_{n-1})^d \mapsto (\mathbb{P}_{n-2})^d$$

and denote by $\mathcal{R}(\nabla \times)$ its range and $r_c := \dim \mathcal{R}(\nabla \times)$ the dimension of this range. Like for any linear operator in finite dimensions, it holds that:

$$(3.58) \quad (\mathbb{P}_{n-1})^d = \text{Ker}(\nabla \times) \oplus \text{Span}(\boldsymbol{\varphi}_{T,l_k}),$$

where $(\boldsymbol{\varphi}_{T,l_k})_{1 \leq k \leq r_c}$ is a subset of $(\boldsymbol{\varphi}_{T,l})_{1 \leq l \leq r}$ such that $(\nabla \times \boldsymbol{\varphi}_{T,l_k})_{1 \leq k \leq r_c}$ is a basis of $\mathcal{R}(\nabla \times)$. We now prove that if the equalities in (3.57) are verified for all $l = l_k$, with $1 \leq k \leq r_c$, then they will be verified for all $1 \leq l \leq r$. Indeed, taking into account (3.58) and since it can easily be proved that $\text{Ker}(\nabla \times) = \nabla(\mathbb{P}_n)$, then if $l \neq l_k$ it holds that there exists $r \in \mathbb{P}_n$ and a set of real numbers (α_k) such that

$$\boldsymbol{\varphi}_{T,l} = \nabla r + \sum_{1 \leq k \leq r_c} \alpha_k \boldsymbol{\varphi}_{T,l_k}.$$

According to this decomposition, we have on the one hand

$$\int_{\mathcal{G}} \mathbf{z} \cdot \nabla \times \boldsymbol{\varphi}_{T,l} = \sum_{1 \leq k \leq r_c} \alpha_k \int_{\mathcal{G}} \mathbf{z} \cdot \nabla \times \boldsymbol{\varphi}_{T,l_k} = - \sum_{1 \leq k \leq r_c} \alpha_k \int_{T \cap \Omega^\varepsilon} \mathbf{w}_0 \cdot \boldsymbol{\varphi}_{T,l_k}$$

since (3.57) is assumed to hold true for all l_k , with $1 \leq k \leq r_c$. On the other hand, we have

$$- \int_{T \cap \Omega^\varepsilon} \mathbf{w}_0 \cdot \boldsymbol{\varphi}_{T,l} = - \int_{T \cap \Omega^\varepsilon} \mathbf{w}_0 \cdot \nabla r - \sum_{1 \leq k \leq r_c} \alpha_k \int_{T \cap \Omega^\varepsilon} \mathbf{w}_0 \cdot \boldsymbol{\varphi}_{T,l_k}$$

so that (3.57) will hold for the index l if $\int_{T \cap \Omega^\varepsilon} \mathbf{w}_0 \cdot \nabla r$ vanishes for any $r \in \mathbb{P}_n$. This is indeed the case since

$$\int_{T \cap \Omega^\varepsilon} \mathbf{w}_0 \cdot \nabla r = - \int_{T \cap \Omega^\varepsilon} \text{div } \mathbf{w}_0 r = - \int_{T \cap \Omega^\varepsilon} \tilde{p} r = 0$$

because \tilde{p} is orthogonal to \mathbb{P}_n and $r \in \mathbb{P}_n$.

Setting $\boldsymbol{\psi}_k := \nabla \times \boldsymbol{\varphi}_{T,l_k}$, the conclusion of the first part of this proof is that it is sufficient to construct a function \mathbf{z} , regular enough and compactly supported in \mathcal{G} such that (3.57) is satisfied for all indices l_k , namely

$$(3.59) \quad \int_{\mathcal{G}} \mathbf{z} \cdot \boldsymbol{\psi}_k = - \int_{T \cap \Omega^\varepsilon} \mathbf{w}_0 \cdot \boldsymbol{\varphi}_{T,l_k}, \forall k = 1, \dots, r_c.$$

Choosing a positive non-vanishing smooth scalar function θ with compact support in \mathcal{G} , we look for the function \mathbf{z} under the form

$$\mathbf{z} = \left(\sum_{1 \leq i \leq r_c} z_i \boldsymbol{\psi}_i \right) \theta,$$

with a set of real numbers (z_i) with $i = 1, \dots, r_c$. Then, (3.59) reduces to finding a vector (z_i) such that

$$(3.60) \quad \sum_{1 \leq i \leq r_c} \left(\int_{\mathcal{G}} \theta \psi_i \cdot \psi_k \right) z_i = - \int_{T \cap \Omega^\varepsilon} \mathbf{w}_0 \cdot \boldsymbol{\varphi}_{T,l_k}, \forall k = 1, \dots, r_c.$$

This is a square linear system of size r_c which has a unique solution since

$$(p, q) \mapsto \int_{\mathcal{G}} \theta p \cdot q$$

is a scalar product on $\mathcal{R}(\nabla \times)$ and (ψ_k) a basis of that space. Moreover, since this is a finite dimensional linear system, the dependence of the coefficients z_i , and thus of the function $\nabla \times \mathbf{z}$ with respect to the right-hand side in (3.60), and thus with respect to \mathbf{w}_0 , is continuous.

As a conclusion, we have constructed a function \mathbf{w} under the form (3.56) which verifies (3.53)–(3.54)–(3.55). We have shown that, by construction, both \mathbf{w}_0 and $\nabla \times \mathbf{z}$ depend linearly and continuously on p , $\lambda_{F,j}$ and $\lambda_{T,l}$. Therefore, this is also the case for \mathbf{v} defined from \mathbf{w} by (3.52). Thus from (3.50), we conclude that (3.46) holds with a constant $\tilde{\alpha} > 0$. This continuity constant may of course depend on ε , in a way which is out of the scope of this article.

3.5. Basis of the space V_H .

THEOREM 3.14. *The functions $\Phi_{E,i}$ for $E \in \mathcal{E}_H$ and $i = 1, \dots, s$ defined by their restrictions (3.36) over each element T_k whose boundary includes E and $\Psi_{T,k}$ for $T \in \mathcal{T}_H$ and $k = 1, \dots, r$ defined by (3.37) form a basis of V_H defined by (3.9). In other words,*

$$(3.61) \quad V_H = \text{span} \{ \Phi_{E,i}, \Psi_{T,k}, E \in \mathcal{E}_H, T \in \mathcal{T}_H, i = 1, \dots, s, k = 1, \dots, r \}$$

and $\{ \Phi_{E,i}, E \in \mathcal{E}_H, i = 1, \dots, s \} \cup \{ \Psi_{T,k}, T \in \mathcal{T}_H, k = 1, \dots, r \}$ forms a linearly independent family.

Proof. First of all, it is easy to verify that

$$\{ \Phi_{E,i}, E \in \mathcal{E}_H, i = 1, \dots, s \} \cup \{ \Psi_{T,k}, T \in \mathcal{T}_H, k = 1, \dots, r \}$$

forms a linearly independent family.

Then, let us prove (3.61). It is obvious that functions $\Phi_{E,i}$, for $E \in \mathcal{E}_H$ and $i = 1, \dots, s$ defined by (3.36) and $\Psi_{T,k}$ for $T \in \mathcal{T}_H$ and $k = 1, \dots, r$ defined by (3.37) all belong to V_H . Consequently, we have

$$\text{span} \{ \Phi_{E,i}, \Psi_{T,k}, E \in \mathcal{E}_H, T \in \mathcal{T}_H, i = 1, \dots, s, k = 1, \dots, r \} \subset V_H$$

Reciprocally, let $\mathbf{u} \in V_H$; definition of V_H implies that on each $T \in \mathcal{T}_H$, there exist a unique $\zeta^T \in M_H^0(T)$ and a unique vector $(\lambda_{T,1}, \dots, \lambda_{T,r}) \in \mathbb{R}^r$ that satisfy (3.28)–(3.31). We introduce \mathbf{v} and σ as follows:

$$\begin{aligned} \mathbf{v} &= \mathbf{u} - \sum_{T \in \mathcal{T}_H} \sum_{k=1}^r \left(\int_{T \cap \Omega^\varepsilon} \mathbf{u} \cdot \boldsymbol{\varphi}_{T,k} \right) \Psi_{T,k} - \sum_{E \in \mathcal{E}_H} \sum_{i=1}^s \left(\int_{E \cap \Omega^\varepsilon} \mathbf{u} \cdot \boldsymbol{\omega}_{E,i} \right) \Phi_{E,i}, \\ \sigma &= \sum_{T \in \mathcal{T}_H} \zeta^T - \sum_{T \in \mathcal{T}_H} \sum_{k=1}^r \left(\int_{T \cap \Omega^\varepsilon} \mathbf{u} \cdot \boldsymbol{\varphi}_{T,k} \right) \pi_{T,k} - \sum_{E \in \mathcal{E}_H} \sum_{i=1}^s \left(\int_{E \cap \Omega^\varepsilon} \mathbf{u} \cdot \boldsymbol{\omega}_{E,i} \right) \pi_{E,i}. \end{aligned}$$

For any $T \in \mathcal{T}_H$, it is easy to check that the restrictions of \mathbf{v} and σ to T verify:

$$(3.62) \quad -\mu\Delta\mathbf{v} + \nabla\sigma \in \text{span}\{\boldsymbol{\varphi}_{T,1}, \dots, \boldsymbol{\varphi}_{T,r}\} \text{ in } T \cap \Omega^\varepsilon$$

$$(3.63) \quad \text{div } \mathbf{v} \in \text{span}\{\varpi_{T,1}, \dots, \varpi_{T,t}\} \text{ in } T \cap \Omega^\varepsilon$$

$$(3.64) \quad \begin{aligned} \mu\nabla\mathbf{v}\mathbf{n} - \sigma\mathbf{n} &\in \text{span}\{\boldsymbol{\omega}_{F,1}, \dots, \boldsymbol{\omega}_{F,s}\} \text{ on } F \cap \Omega^\varepsilon, \forall F \in \mathcal{E}(T) \\ \mathbf{v} &= \mathbf{0} \text{ on } \partial B^\varepsilon \cap T \end{aligned}$$

$$(3.65) \quad \int_{E \cap \Omega^\varepsilon} \mathbf{v} \cdot \boldsymbol{\omega}_{E,i} = 0 \quad \forall E \in \mathcal{E}(T), \quad \forall i = 1, \dots, s$$

$$(3.66) \quad \int_{T \cap \Omega^\varepsilon} \mathbf{v} \cdot \boldsymbol{\varphi}_{T,l} = 0 \quad \forall l = 1, \dots, r$$

$$(3.67) \quad \int_{T \cap \Omega^\varepsilon} \sigma \cdot \varpi_{T,j} = 0 \quad \forall j = 1, \dots, t$$

Equation (3.62) implies that there exists $(\lambda_{T,1}, \dots, \lambda_{T,r}) \in \mathbb{R}^r$ such that

$$(3.68) \quad -\mu\Delta\mathbf{v} + \nabla\sigma = \sum_{l=1}^r \lambda_{T,l} \boldsymbol{\varphi}_{T,l}.$$

Choosing \mathbf{v} as the test function in the variational formulation of (3.68) implies that

$$(3.69) \quad \int_{T \cap \Omega^\varepsilon} \mu |\nabla \mathbf{v}|^2 - \int_{T \cap \Omega^\varepsilon} \sigma \text{div } \mathbf{v} = \sum_{l=1}^r \lambda_{T,l} \int_{T \cap \Omega^\varepsilon} \boldsymbol{\varphi}_{T,l} \cdot \mathbf{v} + \int_{\partial(T \cap \Omega^\varepsilon)} (\mu \nabla \mathbf{v} \mathbf{n} - \sigma \mathbf{n}) \cdot \mathbf{v}.$$

It is straightforward to deduce from (3.63) and (3.67) that

$$\int_{T \cap \Omega^\varepsilon} \sigma \text{div } \mathbf{v} = 0.$$

Then equation (3.66) implies that

$$\sum_{l=1}^r \lambda_{T,l} \int_{T \cap \Omega^\varepsilon} \boldsymbol{\varphi}_{T,l} \cdot \mathbf{v} = 0.$$

Combining equations (3.64) and (3.65), we can deduce that

$$\int_{\partial(T \cap \Omega^\varepsilon)} (\mu \nabla \mathbf{v} \mathbf{n} - \sigma \mathbf{n}) \cdot \mathbf{v} = 0.$$

Finally, equation (3.69) reduces to

$$\int_{T \cap \Omega^\varepsilon} |\nabla \mathbf{v}|^2 = 0,$$

which implies that \mathbf{v} is constant on $T \cap \Omega^\varepsilon$. With (3.66), we deduce that $\mathbf{v} = \mathbf{0}$.

Thus we have proved that

$$V_H \subset \text{span}\{\boldsymbol{\Phi}_{E,i}, \boldsymbol{\Psi}_{T,k}, E \in \mathcal{E}_H, T \in \mathcal{T}_H, i = 1, \dots, s, k = 1, \dots, r\}.$$

Consequently, combining the results above, we have proved that

$$V_H = \text{span}\{\boldsymbol{\Phi}_{E,i}, \boldsymbol{\Psi}_{T,k}, E \in \mathcal{E}_H, T \in \mathcal{T}_H, i = 1, \dots, s, k = 1, \dots, r\}. \quad \square$$

We conclude that any function V_H can be represented as

$$\forall \mathbf{u}_H \in V_H, \mathbf{u}_H = \sum_{E \in \mathcal{E}_H} \sum_{i=1}^s u_{E,i} \Phi_{E,i} + \sum_{T \in \mathcal{T}_H} \sum_{k=1}^r u_{T,k} \Psi_{T,k}.$$

An explicit formulation of the linear operator π_H (see Lemma 3.11) is

$$(3.70) \quad \forall \mathbf{u}_H \in V_H, \pi_H(\mathbf{u}_H) = \sum_{E \in \mathcal{E}_H} \sum_{i=1}^s u_{E,i} \pi_{E,i} + \sum_{T \in \mathcal{T}_H} \sum_{k=1}^r u_{T,k} \pi_{T,k}.$$

3.6. Decomposition of the space X_H^{ext} . Subsection 3.2 results in the following decomposition.

THEOREM 3.15. *The space X_H^{ext} can be decomposed as:*

$$X_H^{ext} = X_H \oplus X_H^0,$$

where X_H^0 is defined in Definition 3.5 and X_H is defined in Definition 3.6.

Proof. This theorem is proved in two steps:

$$(i) X_H^{ext} = X_H + X_H^0, \quad (ii) X_H \cap X_H^0 = \{0\}.$$

We start by proving step (i). First, by definition X_H and X_H^0 are both subsets of X_H^{ext} . Next, we prove that $X_H^{ext} \subset X_H + X_H^0$, i.e. for any $(\mathbf{u}, p) \in X_H^{ext}$, there exist $(\mathbf{u}_H, p_H) \in X_H, \mathbf{u}^0 \in V_H^0$ and $p^0 \in M_H^0$ such that

$$(3.71) \quad \mathbf{u} = \mathbf{u}_H + \mathbf{u}^0, \quad p = p_H + p^0.$$

In order to prove this, we consider a given $(\mathbf{u}, p) \in X_H^{ext}$ and we first pick any triangle $T \in \mathcal{T}_H$; then we consider the velocity space $V_{f0}(T)$ defined by (3.20), we recall the definition of $M_H^0(T)$ by (3.6) and we consider the following problem: Find $(\mathbf{u}^0, p^0) \in V_{f0}(T) \times M_H^0(T)$ such that

$$(3.72) \quad \int_{T \cap \Omega^\varepsilon} (\mu \nabla \mathbf{u}^0 : \nabla \mathbf{v} - p^0 \operatorname{div} \mathbf{v}) = \int_{T \cap \Omega^\varepsilon} (\mu \nabla \mathbf{u} : \nabla \mathbf{v} - p \operatorname{div} \mathbf{v}), \quad \forall \mathbf{v} \in V_{f0}(T)$$

$$(3.73) \quad \int_{T \cap \Omega^\varepsilon} q \operatorname{div} \mathbf{u}^0 = \int_{T \cap \Omega^\varepsilon} q \operatorname{div} \mathbf{u}, \quad \forall q \in M_H^0(T).$$

This problem has a unique solution because:

(a) $(\mathbf{u}, \mathbf{v}) \mapsto \int_{T \cap \Omega^\varepsilon} (\mu \nabla \mathbf{u} : \nabla \mathbf{v})$ is coercive on $V_{f0}(T)$,

(b) the existence of a function $\mathbf{w} \in V_T$ satisfying (3.53)–(3.54)–(3.55) with a linear and continuous dependence with respect to an arbitrary $\tilde{p} \in M_H^0(T)$ proves the inf-sup condition for the divergence operator over the couple of spaces $V_{f0}(T) \times M_H^0(T)$.

By gluing together (\mathbf{u}^0, p^0) on each element $T \in \mathcal{T}_H$, we obtain that $(\mathbf{u}^0, p^0) \in V_H^0 \times M_H^0$. Consequently, we have proved that for any $(\mathbf{u}, p) \in X_H^{ext}$, there exists $(\mathbf{u}^0, p^0) \in X_H^0$ such that

$$c_H((\mathbf{u}^0, p^0), (\mathbf{v}, q)) = c_H((\mathbf{u}, p), (\mathbf{v}, q)), \quad \forall (\mathbf{v}, q) \in V_H^0 \times M_H^0.$$

Thus, defining \mathbf{u}_H and p_H by

$$\mathbf{u}_H = \mathbf{u} - \mathbf{u}^0, \quad p_H = p - p^0,$$

we get that (\mathbf{u}_H, p_H) satisfies

$$c_H((\mathbf{u}_H, p_H), (\mathbf{v}, q)) = 0, \forall (\mathbf{v}, q) \in V_H^0 \times M_H^0$$

and we deduce by [Definition 3.6](#) that $(\mathbf{u}_H, p_H) \in X_H$. Finally, we conclude that $X_H^{ext} = X_H + X_H^0$.

Now it remains to prove step (ii), namely that $X_H \cap X_H^0 = \{0\}$. Let $(\mathbf{u}, p) \in X_H \cap X_H^0$, then this implies that for each $T \in \mathcal{T}_H$, $p|_T$ belongs at the same time to $\mathbb{P}_n(T)$ according to (3.8), and to its orthogonal according to definition (3.6) and [Hypothesis 3.1](#). This implies that $p = 0$. Next, Definition 3.6, implies that $c_H((\mathbf{u}, p), (\mathbf{u}, p)) = 0$, which reduces to $\int_{T \cap \Omega^\varepsilon} \mu \nabla \mathbf{u} : \nabla \mathbf{u} = 0$, leading to $\mathbf{u} = \mathbf{0}$ which finishes this part of the proof. \square

3.7. Coarse-scale problem. The coarse-scale formulation of the Stokes problem (2.1)–(2.2) reads: find $(\mathbf{u}_H, p_H) \in X_H$ such that

$$c_H((\mathbf{u}_H, p_H), (\mathbf{v}, q)) = (\mathbf{f}, \mathbf{v}), \forall (\mathbf{v}, q) \in X_H.$$

[Theorem 3.12](#) implies that p_H can be decomposed as $p_H = \pi_H(\mathbf{u}_H) + \bar{p}_H$ with $\pi_H(\mathbf{u}_H) \in M_H^0$ and $\bar{p}_H \in M_H$. It is easy to verify that $(\pi_H(\mathbf{u}_H), \operatorname{div} \mathbf{v}) = 0$ for all $\mathbf{u}_H, \mathbf{v} \in V_H$. The problem above can thus be reformulated as: find $\mathbf{u}_H \in V_H$ and $\bar{p}_H \in M_H$ such that

$$(3.74) \quad a_H(\mathbf{u}_H, \mathbf{v}) + b_H(\mathbf{v}, \bar{p}_H) = F_H(\mathbf{v}), \forall \mathbf{v} \in V_H,$$

$$(3.75) \quad b_H(\mathbf{u}_H, q) = 0, \forall q \in M_H,$$

where

$$a_H(\mathbf{u}_H, \mathbf{v}) = \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} \mu \nabla \mathbf{u}_H : \nabla \mathbf{v},$$

$$b_H(\mathbf{v}, \bar{p}_H) = - \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} \bar{p}_H \operatorname{div} \mathbf{v},$$

$$F_H(\mathbf{v}) = \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} \mathbf{f} \cdot \mathbf{v}.$$

THEOREM 3.16. *The space V_H and the space M_H have the following relation*

$$\operatorname{div}_H V_H = M_H,$$

where div_H is the broken divergence operator defined triangle by triangle:

$$(\operatorname{div}_H \mathbf{v})|_T = \operatorname{div} \mathbf{v}|_T.$$

Proof.

We prove first that $\operatorname{div}_H V_H \subset M_H$. For any $\mathbf{v} \in V_H$, for any $T \in \mathcal{T}_H$, it holds that $\operatorname{div} \mathbf{v}|_T \in \mathbb{P}_n(T)$ thanks to (3.9) and [Hypothesis 3.1](#). It thus remains to prove that $\operatorname{div}_H \mathbf{v} \in M$, which reduces to prove that

$$\int_{\Omega^\varepsilon} \operatorname{div}_H \mathbf{v} = 0.$$

This is true. Indeed, since $V_H \subset V_H^{ext}$, definition of V_H^{ext} implies that $\int_{E \cap \Omega^\varepsilon} [[\mathbf{v}]] \cdot \mathbf{n} = 0$, and we thus deduce that

$$\int_{\Omega^\varepsilon} \operatorname{div}_H \mathbf{v} = \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} \operatorname{div} \mathbf{v} = \sum_{T \in \mathcal{T}_H} \int_{\partial(T \cap \Omega^\varepsilon)} \mathbf{v} \cdot \mathbf{n} = \sum_{E \in \mathcal{E}_H} \int_{E \cap \Omega^\varepsilon} [[\mathbf{v} \cdot \mathbf{n}]] = 0.$$

Hence we have proved that $\operatorname{div}_H V_H \subset M_H$.

Reciprocally, we now prove that $M_H \subset \operatorname{div}_H V_H$. For any $q \in M_H$, definition of M_H shows that $q \in L_0^2(\Omega)$. Thus, there exists $\mathbf{v} \in (H_0^1(\Omega^\varepsilon))^d$ such that $\operatorname{div} \mathbf{v} = q$. Since $(H_0^1(\Omega^\varepsilon))^d \subset V_H^{ext}$, [Theorem 3.15](#) implies that \mathbf{v} can be decomposed as

$$(3.76) \quad \mathbf{v} = \mathbf{v}_H + \mathbf{v}_H^0 \text{ with } \mathbf{v}_H \in V_H, \mathbf{v}_H^0 \in V_H^0.$$

For any element $T \in \mathcal{T}_H$, let $\varpi_T \in \mathbb{P}_n(T)$. Integration by parts gives:

$$\sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} \varpi_T \operatorname{div} \mathbf{v}_H = \sum_{T \in \mathcal{T}_H} \int_{\partial(T \cap \Omega^\varepsilon)} \varpi_T \mathbf{v}_H \cdot \mathbf{n} - \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} \mathbf{v}_H \cdot \nabla \varpi_T$$

Substituting \mathbf{v}_H defined in (3.76) into this equation, we obtain

$$\begin{aligned} \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} \varpi_T \operatorname{div} \mathbf{v}_H &= \sum_{T \in \mathcal{T}_H} \int_{\partial(T \cap \Omega^\varepsilon)} \varpi_T (\mathbf{v} - \mathbf{v}_H^0) \cdot \mathbf{n} \\ &\quad - \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} (\mathbf{v} - \mathbf{v}_H^0) \cdot \nabla \varpi_T \end{aligned}$$

The fact that $\varpi_T \in \mathbb{P}_n(T)$ implies that $\mathbf{n} \varpi_T \in (\mathbb{P}_n(T))^d$ and $\nabla \varpi_T \in \mathbb{P}_{n-1}(T)$. Then, definition of V_H^0 by (3.4), [Hypothesis 3.1](#) implies that

$$\int_{\partial(T \cap \Omega^\varepsilon)} \mathbf{v}_H^0 \cdot \mathbf{n} \varpi_T = 0, \quad \int_{T \cap \Omega^\varepsilon} \mathbf{v}_H^0 \cdot \nabla \varpi_T = 0.$$

Finally (3.77) reduces to

$$\begin{aligned} \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} \varpi_T \operatorname{div} \mathbf{v}_H &= \sum_{T \in \mathcal{T}_H} \int_{\partial(T \cap \Omega^\varepsilon)} \varpi_T \mathbf{v} \cdot \mathbf{n} - \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} \mathbf{v} \cdot \nabla \varpi_T \\ &= \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} \varpi_T \operatorname{div} \mathbf{v} \\ &= \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} q \varpi_T. \end{aligned}$$

Since $\mathbf{v}_H \in V_H$, we know by (3.9) that, on T , $\operatorname{div} \mathbf{v}_H \in \mathbb{P}_n(T)$. Moreover, $q \in \mathbb{P}_n(T)$ and ϖ_T is arbitrary in $\mathbb{P}_n(T)$ in the above equality. Thus, we deduce that $q|_T = (\operatorname{div} \mathbf{v}_H)|_T$ and so $q = \operatorname{div}_H \mathbf{v}_H$. Thus $M_H \subset \operatorname{div}_H V_H$. Consequently, combining the results above, we conclude that $\operatorname{div}_H V_H = M_H$. \square

Making use of [Theorem 3.16](#), it is trivial to deduce from (3.75) that $\operatorname{div} \mathbf{u}_H = 0$ in $T \cap \Omega^\varepsilon$ for $T \in \mathcal{T}_H$. We can thus eliminate the pressure unknown from (3.74)–(3.75) by introducing a subspace of V_H :

$$Z_H = \{\mathbf{v} \in V_H \text{ such that } \operatorname{div}_H \mathbf{v} = 0, \forall T \in \mathcal{T}_H\}.$$

Therefore (3.74)–(3.75) is equivalent to: find $\mathbf{u}_H \in Z_H$ such that

$$\sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} \mu \nabla \mathbf{u}_H : \nabla \mathbf{v} = \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} \mathbf{f} \cdot \mathbf{v}, \quad \forall \mathbf{v} \in Z_H$$

The existence and uniqueness of a solution \mathbf{u}_H to this problem is guaranteed by the coercivity of the bilinear form over V_H . Then the existence and uniqueness of pressure \bar{p}_H follows from the fact that $\operatorname{div}_H V_H = M_H$. As a result, we have proved that (3.74)–(3.75) has one and only one solution $(\mathbf{u}_H, \bar{p}_H) \in V_H \times M_H$.

To solve the coarse-scale problem on the coarse mesh, we approximate the non-homogeneous Dirichlet boundary condition $\mathbf{u}_H = \mathbf{0}$ defined by (2.4) in a weak form:

$$\int_{E \cap \partial \Omega} \mathbf{u}_H \cdot \boldsymbol{\omega}_{E,j} = 0, \quad \forall E \in \mathcal{E}_H \cap \partial \Omega, \quad j = 1, \dots, s.$$

3.8. Reconstruction of fine-scale solutions. After obtaining coarse solutions $\mathbf{u}_H = (\mathbf{u}_{E,1}, \dots, \mathbf{u}_{E,s})_{E \in \mathcal{E}_H}$ and $\bar{p}_H = (\bar{p}_H|_T)_{T \in \mathcal{T}_H}$, we reconstruct on any coarse element $T \in \mathcal{T}_H$ fine-scale solutions by:

$$\begin{aligned} \mathbf{u}_H|_T &= \sum_{E \in \mathcal{E}(T)} \sum_{i=1}^s u_{E,i} \boldsymbol{\Phi}_{E,i} + \sum_{k=1}^r u_{T,k} \boldsymbol{\Psi}_{T,k}, \\ p_H|_T &= \sum_{E \in \mathcal{E}(T)} \sum_{i=1}^s u_{E,i} \pi_{E,i} + \sum_{k=1}^r u_{T,k} \pi_{T,k} + \bar{p}_H|_T. \end{aligned}$$

4. Implementation issues and comparisons to previous methods. We implement MsFEMs in a CFD software called *TrioCFD* [41] developed by the CEA. In *TrioCFD*, incompressible Navier-Stokes equations are discretized by the finite volume element method [6, 7], which is a certain combination of the finite element method [22, 27] and the finite volume method [24, 42]. The main idea is to approximate fluxes on the boundary of control volumes by replacing unknowns by a finite element approximation. The finite volume element method consists of two meshes: a primal mesh and a dual mesh. Equations are discretized by volume integrals on the dual mesh and unknowns are discretized into a finite element space on the primal mesh.

In our work, we implement the CR4_high MsFEM where weighting functions are defined by (3.3). Local problems (3.36)–(3.37) are solved by the finite volume element method where the velocity and pressure are discretized by the \mathbb{P}_1 nonconforming/ \mathbb{P}_1 element. The coarse-scale problem is implemented using (3.74)–(3.75), with a standard orthogonal basis of the fully discontinuous $\mathbb{P}_n(T)$ space for the pressure and the basis of V_H that was constructed in subsection 3.5 for the velocity. The global matrix can be computed in parallel over multiple processors and remains sparse since the supports of these basis functions are local.

We want to compare the accuracy of CR4_high to that of MsFEMs proposed originally in [33, 40]. Thus we implement also the CR2 and CR3 MsFEMs defined in [33]. The CR2 method is the original method of [40] and corresponds to the choice $n = 0$ of the present article. The CR3 method is obtained by enriching the edge weighting functions:

$$\boldsymbol{\omega}_{E,1} = \mathbf{e}_1, \quad \boldsymbol{\omega}_{E,2} = \mathbf{e}_2, \quad \boldsymbol{\omega}_{E,3} = \mathbf{n}_E \psi_E,$$

where ψ_E is a non-vanishing function in $\mathbb{P}_1(E)$ with vanishing mean-value on $E \cap \Omega^\varepsilon$ and $(\mathbf{e}_1, \mathbf{e}_2)$ is the canonical basis of \mathbb{R}^2 . We also investigate the CR4 method which

enriches the CR3 method to reach the full $[\mathbb{P}_1(E)]^2$ weighting space on the edges:

$$\omega_{E,1} = \mathbf{e}_1, \omega_{E,2} = \mathbf{e}_2, \omega_{E,3} = \mathbf{n}_E \psi_E, \omega_{E,4} = \boldsymbol{\tau}_E \phi_E,$$

where $\mathbf{n}_E \psi_E, \boldsymbol{\tau}_E \phi_E$ are the same as those in (3.3). In all these methods, the pressure weighting function is limited to $\varpi_{T,1} = 1$, and we recall that $\boldsymbol{\varphi}_{T,k}$ for $k = 1, \dots, r$ are only defined in the enriched MsFEM and do not exist in CR2, CR3 and CR4. The idea in comparing CR4 and CR4_high is to show that enriching only the edge weighting functions $\omega_{E,i}$ is not enough to significantly enhance the accuracy of the scheme. We also need to add constraints in the triangle in order to make the non-resolved space V_H^0 (3.4) smaller and smaller.

Unknowns in local problems of CR2, CR3 and CR4 are discretized by the \mathbb{P}_1 nonconforming/ \mathbb{P}_0 element. Reference solutions are computed by the same finite volume element method. Instead of using the penalization technique and Cartesian meshes as in [33, 40], we impose the no-slip condition directly on obstacles and numerical simulations are performed on body-fitted unstructured meshes.

Enriching the basic CR2 method by additional coarse basis functions rather than using finer meshes is an alternative to reduce the computational error, just like p enrichment is an alternative to h refinement in standard finite elements. Both approaches imply growing costs which have to be compared to guide a choice between both approaches. In order to detail this (for $d = 2$), let us consider a coarse mesh made up of n_T triangles with n_E edges. Then, Table 1 displays the number of unknowns that will be necessary for the different methods; for the sake of comparison, the asymptotic relationship $n_E \approx \frac{3}{2}n_T$ will be used. We conclude from this table that

Table 1: Number of unknowns in the different methods

Field	Method			
	CR2	CR3	CR4	CR4_high
Pressure	n_T	n_T	n_T	$3n_T$
Velocity	$2n_E$	$3n_E$	$4n_E$	$4n_E + 2n_T$
Total	$n_T + 2n_E$	$n_T + 3n_E$	$n_T + 4n_E$	$5n_T + 4n_E$
Asympt. total	$4n_T$	$5.5n_T$	$7n_T$	$11n_T$

using CR4_high is worth the additional complexity as soon as it is at least $\frac{11}{4}$ times more accurate than CR2, or twice as accurate as CR3, on a given mesh.

5. Numerical simulations. We consider Stokes flows in a two-dimensional open channel $\Omega = [0 \leq x \leq 2, 0 \leq y \leq 1]$, where the heterogeneity is represented by arbitrary placements of solid obstacles. Three-dimensional numerical simulations were also carried out with CR2 and CR3 and can be found in [25]. As shown in Figure 3, case A contains 26 obstacles of width $\varepsilon = 0.015$ and case B contains 100 obstacles of width $\varepsilon = 0.003$. Case B corresponds to a denser distribution of obstacles and is similar somehow to a homogenization setting, while case A with fewer and bigger obstacles is not so close to an asymptotic homogenized regime. We assign $\mu = 1$ and $\mathbf{f} = \mathbf{0}$. The parabolic inflow boundary condition $\mathbf{u} = y(1 - y)\mathbf{e}_1$ is imposed at the inlet. The Neumann boundary condition $\mu \nabla \mathbf{u} \cdot \mathbf{n} - p\mathbf{n} = 0$ is imposed at the outlet and the no-slip condition is applied on other boundaries. In numerical convergence studies, we fix the element size h of fine meshes and vary only the element size H of coarse meshes. We ensure that $h \ll \varepsilon < H$. The element size h is small enough to

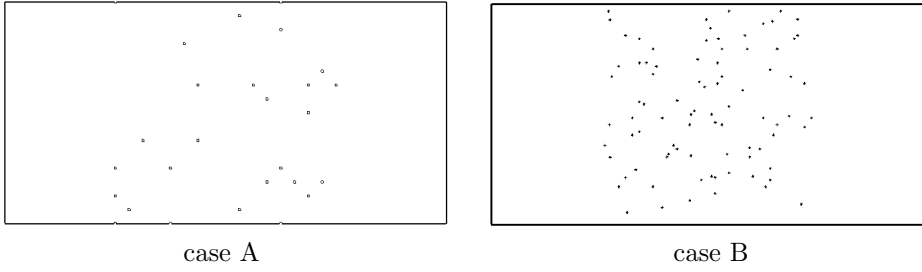


Fig. 3: Two computational domains with square obstacles

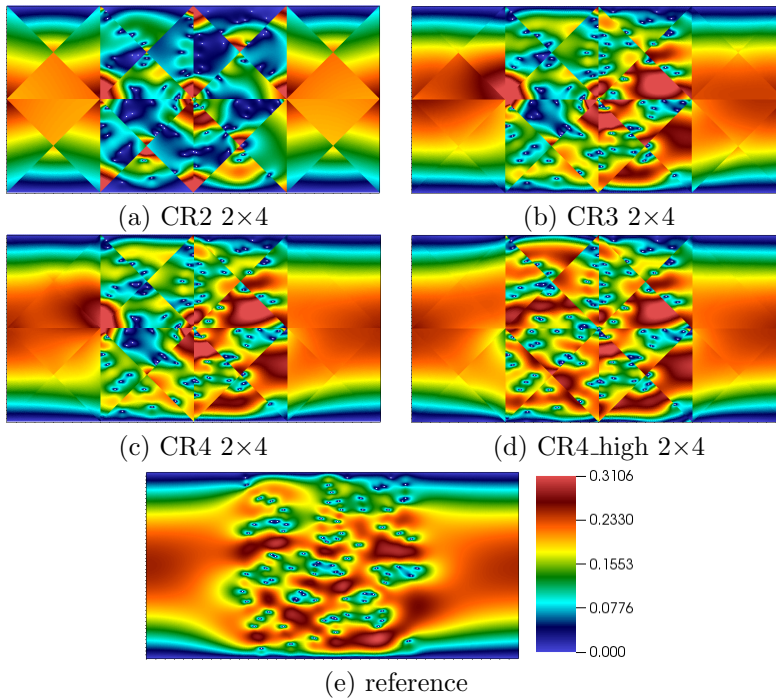


Fig. 4: $|u|$ contours of Stokes flows in case B computed on the coarse mesh 2×4

well capture small obstacles in the media. We are interested only in the case in which $H > \varepsilon$, since the opposite case $H < \varepsilon$ is covered by classical finite element methods and MsFEMs are not needed.

For case B, Figure 4, Figure 5 and Figure 6 show the velocity contours computed respectively by CR2, CR3, CR4 and CR4_high on various coarse meshes as well as the reference solution. Figure 7, Figure 8 and Figure 9 show the pressure computed by CR2, CR3, CR4 and CR4_high on various coarse meshes and the reference pressure. The reference solution is computed on a mesh containing 2,000,000 elements with an average element size $h = 0.00025$. Globally, we observe that the velocity and pressure computed by MsFEMs converge to the reference solutions. It is shown that CR3 is much more accurate than CR2 whereas CR4 is only slightly better than CR3. It is

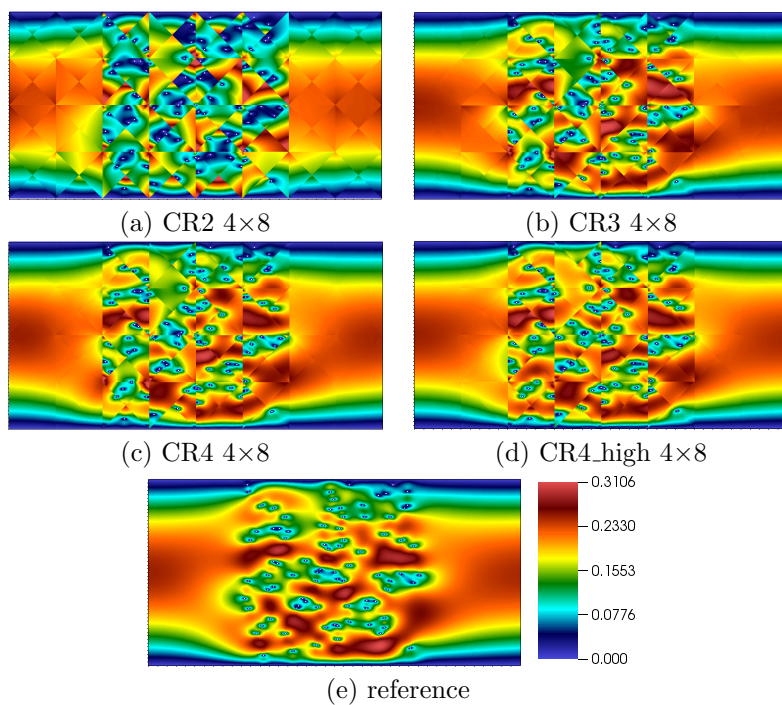


Fig. 5: $|u|$ contours of Stokes flows in case B computed on the coarse mesh 4×8

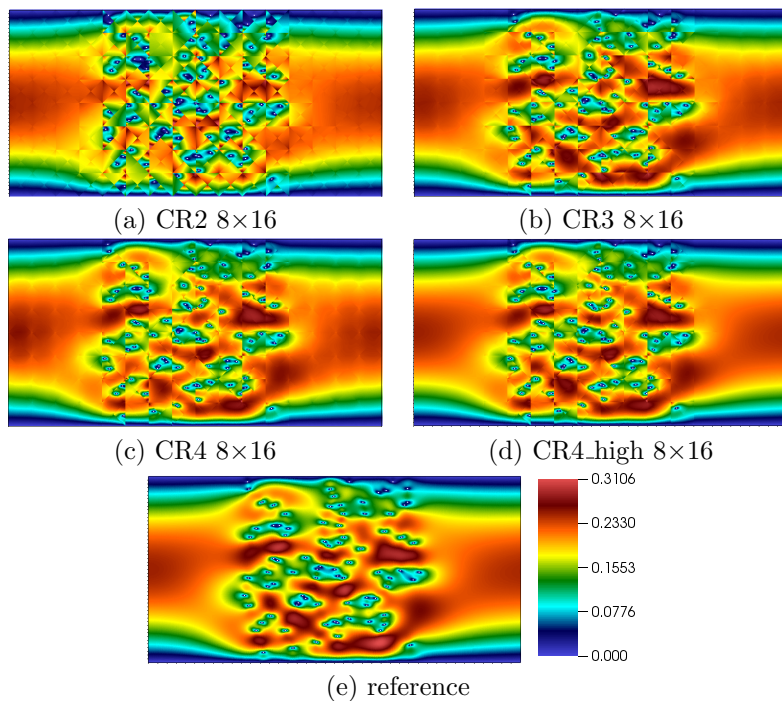


Fig. 6: $|u|$ contours of Stokes flows in case B computed on the coarse mesh 8×16

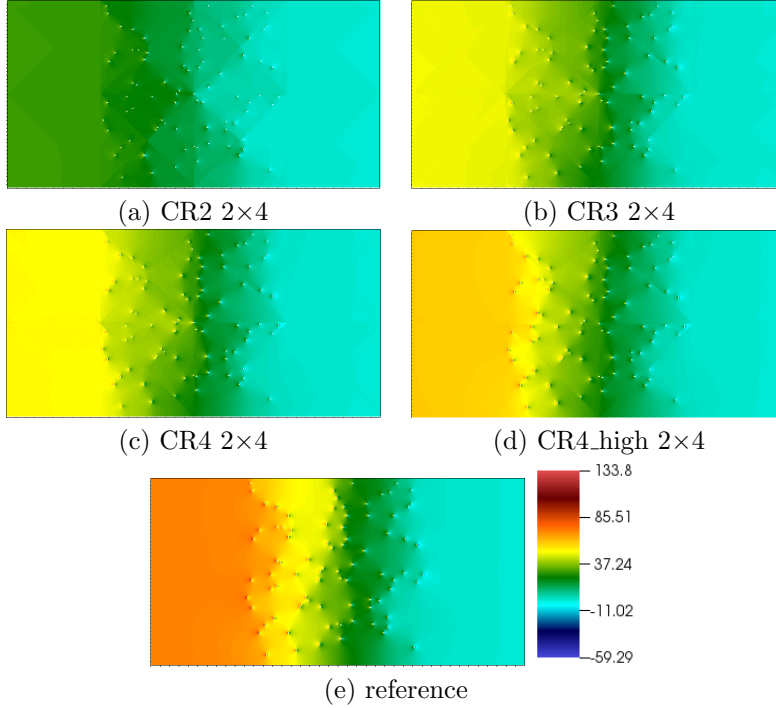


Fig. 7: Pressure of Stokes flows in case B computed on the coarse mesh 2×4

847 evident that both the velocity and pressure computed by CR4_high are significantly
 848 more accurate than those given by other MsFEMs. As shown in Figure 4 and Figure 7,
 849 CR4_high already captures most of the important flow features on the 2×4 coarse
 850 mesh and the solution is in good agreement with the reference solution. Table 2
 851 and Table 3 present the error study of different MsFEMs on various coarse meshes,
 852 relatively to the reference fine-scale solution (in L^2 norm). Figure 10 shows the
 853 numerical convergence of velocity computed by different MsFEMs in both case A and
 854 B. We observe a converging trend of the velocity and the pressure in both case A and
 855 case B. It is observed that these methods have the same rate of convergence, which
 856 is about 1 in case A and a bit lower in case B. The enriched MsFEM does not show a
 857 larger rate of convergence than lower-order methods. We only mention that a related
 858 high-order method proposed in [13] proves an error in the energy norm that behaves
 859 like $\sqrt{\varepsilon} + H^{n+1} + \sqrt{\frac{\varepsilon}{H}}$ for smooth solutions of periodic elliptic problems with highly
 860 oscillating coefficients. Ref. [1] additionally considers the effect of the fine mesh on the
 861 error. These theoretical results and the numerical results presented in these references
 862 show that it is not easy to isolate the sole effect of refining H ; this issue needs further
 863 investigation. In the present work, the errors of MsFEMs are relatively small even on
 864 coarse meshes with a rather large (H/ε) ratio and errors of CR4_high are much smaller
 865 than those of other MsFEMs. For a fair comparison, one should refer to Table 1 which
 866 takes into account the additional complexity of CR4_high with respect to the other
 867 methods. We conclude that it is always advantageous to implement CR4_high rather
 868 than CR2, and that the comparison with CR3 depends on the test case, on the mesh
 869 and on the considered variable (pressure or velocity).

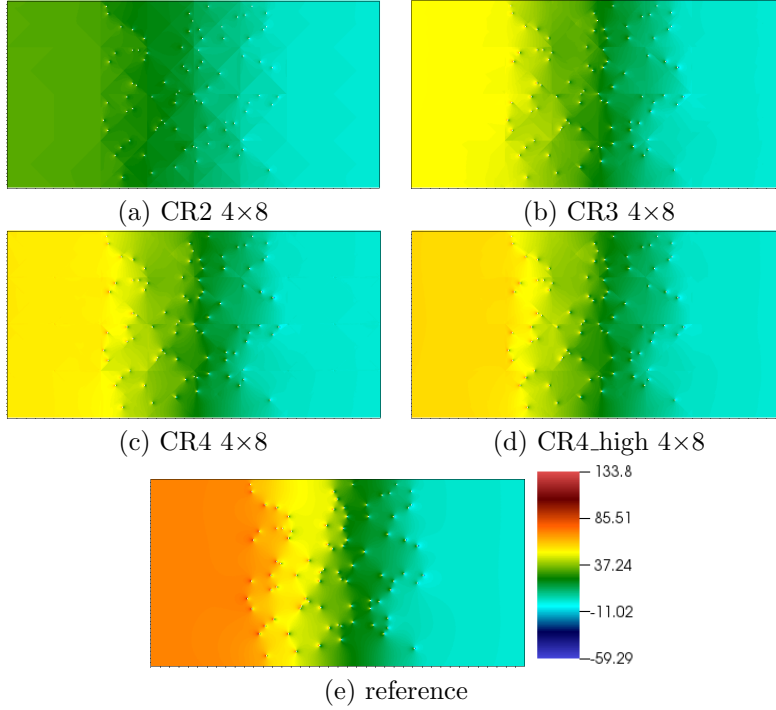
Fig. 8: Pressure of Stokes flows in case B computed on the coarse mesh 4×8

Table 2: Error analysis of Stokes flows in case A

Config.	L^2 Rel. U				L^2 Rel. P			
	CR2	CR3	CR4	CR4_high	CR2	CR3	CR4	CR4_high
2×4	0.334	0.210	0.190	0.091	0.475	0.212	0.146	0.096
4×8	0.174	0.072	0.065	0.039	0.321	0.144	0.083	0.055
8×16	0.061	0.026	0.022	0.015	0.173	0.075	0.044	0.034
16×32	0.030	0.014	0.011	0.006	0.116	0.056	0.025	0.020

From Table 2 and Table 3, Case B seems to be more difficult in that the errors with respect to the fine scale reference solution are larger in that case. This may be due to the larger number of obstacles which generate richer flow patterns.

6. Conclusions. An innovative enriched Crouzeix-Raviart MsFEM has been proposed for solving Stokes flows in highly heterogeneous media with solid obstacles. This method has been validated in numerical experiments by comparing to Crouzeix-Raviart MsFEMs originally proposed in [33, 40]. Local problems are implemented in *TrioCFD* using the finite volume element method where physical unknowns are discretized by the \mathbb{P}_1 nonconforming/ \mathbb{P}_0 or \mathbb{P}_1 nonconforming/ \mathbb{P}_1 element. Instead of using the penalization method, the no-slip condition is imposed on obstacles and numerical simulations are performed on body-fitted unstructured meshes.

Numerical experiments show a good converging trend of solutions computed by MsFEMs to reference solutions. In particular, it has been shown that the enriched

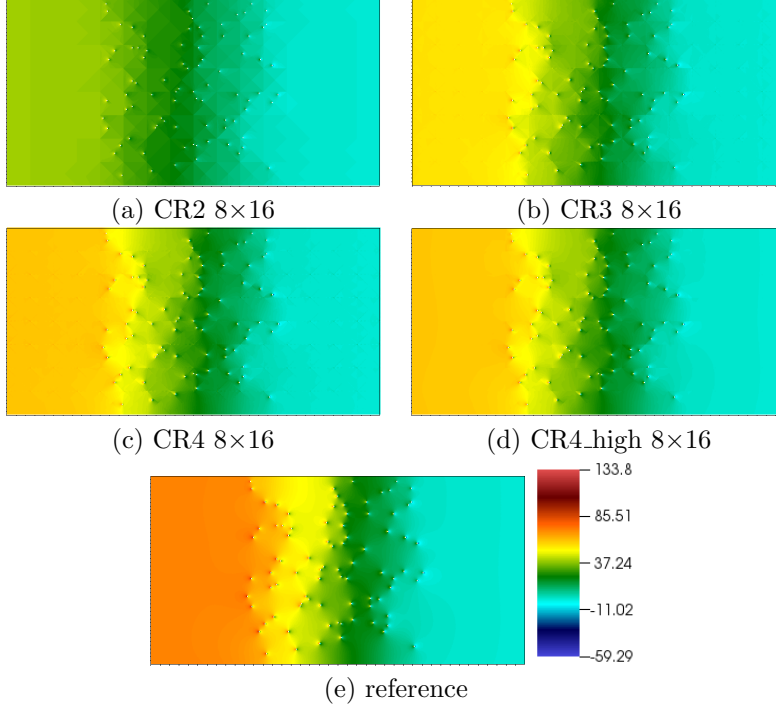


Fig. 9: Pressure of Stokes flows in case B computed on the coarse mesh 8×16

Table 3: Error analysis of Stokes flows in case B

Config.	L^2 Rel. U				L^2 Rel. P			
	CR2	CR3	CR4	CR4_high	CR2	CR3	CR4	CR4_high
2×4	0.478	0.301	0.274	0.159	0.595	0.315	0.281	0.227
4×8	0.329	0.185	0.158	0.114	0.556	0.289	0.249	0.189
8×16	0.189	0.096	0.084	0.061	0.449	0.232	0.164	0.130
16×32	0.116	0.055	0.045	0.030	0.307	0.184	0.118	0.102

Crouzeix-Raviart MsFEM with $n = 1$ is significantly more accurate than Crouzeix-Raviart MsFEMs proposed in [33, 40]. A clear perspective is to perform an error analysis in order to obtain the relation between the error and the order of the approximation n . Besides, it would be interesting to implement the enriched MsFEM with $n = 2$, $n = 3$ and so on, and then compare the enriched method with other Crouzeix-Raviart MsFEMs. In addition, when there is no obstacle and if the elements of the mesh (in 2D) are triangles, then the method presented here shares a large number of features with the non-conforming finite elements presented in [38]: pressures are fully discontinuous P_n functions, velocities contain the non-conforming P_{n+1} space and the degrees of freedom are the same, as can be seen from [38, page 298]; further investigation is needed to check whether these two methods are the same.

The enriched Crouzeix-Raviart MsFEM proposed in this paper can easily be extended to Oseen flows (see [25] for more details) which is a step further toward the

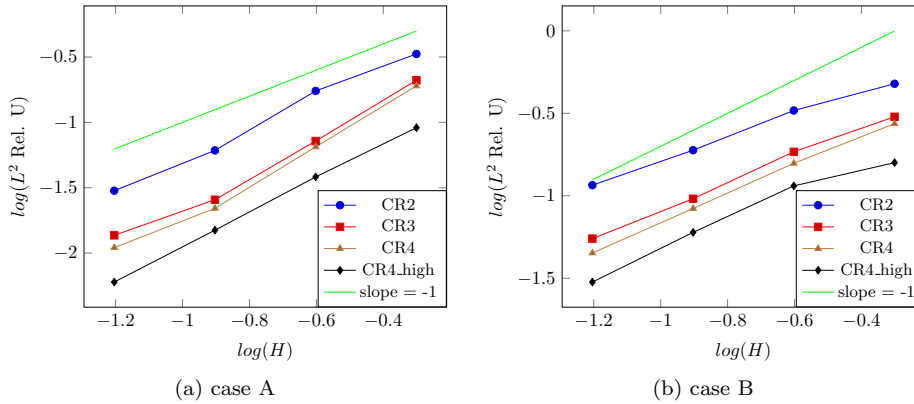


Fig. 10: Numerical convergence of MsFEMs in case B

nonlinear Navier-Stokes flows, which is the ultimate goal of this study. It would be interesting to test whether the enriched MsFEM could significantly improve the accuracy of the method proposed in [39] for Oseen flows in heterogeneous media.

Acknowledgments. This work was granted access to the HPC resources of CINES under the allocation R0051010339 made by GENCI.

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