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1 **ENRICHED NONCONFORMING MULTISCALE FINITE ELEMENT**
2 **METHOD FOR STOKES FLOWS IN HETEROGENEOUS MEDIA**
3 **BASED ON HIGH-ORDER WEIGHTING FUNCTIONS ***

4 Q. FENG[†], G. ALLAIRE[‡], AND P. OMNES[§]

5 **Abstract.** This paper addresses an enriched nonconforming Multiscale Finite Element Method
6 (MsFEM) to solve viscous incompressible flow problems in genuine heterogeneous or porous media. In
7 the work of [B. P. Muljadi, J. Narski, A. Lozinski, and P. Degond, Multiscale Modeling & Simulation
8 2015 13:4, 1146-1172] and [G. Jankowiak and A. Lozinski, arXiv:1802.04389 [math.NA], 2018], a
9 nonconforming MsFEM has been first developed for Stokes problems in such media. Based on these
10 works, we propose an innovative enriched nonconforming MsFEM where the approximation space
11 of both velocity and pressure are enriched by weighting functions which are defined by polynomials
12 of higher-degree. Numerical experiments show that this enriched nonconforming MsFEM improves
13 significantly the accuracy of the nonconforming MsFEMs. Theoretically, this method provides a
14 general framework which allows to find a good compromise between the accuracy of the method and
15 the computing costs, by varying the degrees of polynomials.

16 **Key words.** Multiscale Finite Element Method, Crouzeix-Raviart Element, Stokes Flows

17 **AMS subject classifications.** 65N30, 76D07, 76M30

18 **1. Introduction.**

19 **1.1. Multiscale problems.** Modeling of Stokes flows in heterogeneous media
20 is a topic of significant interest in many engineering practices, such as reservoir en-
21 gineering, flows through fractured porous media, flows in nuclear reactor cores, etc.
22 The challenge in numerical simulations of these problems lies in the fact that the so-
23 lution contains multiscale features such as spatial scale disparity and requires a very
24 fine mesh to resolve all the details. In some engineering circumstances, quantities
25 of interest are only related to macroscopic properties of the solution. But fine-scale
26 features cannot be omitted in physical models since they can affect significantly the
27 macroscopic behavior of the solution. However, solving these problems on a very fine
28 mesh can be prohibitively expensive or impossible with today's computing capacities.
29 As a consequence, some model reduction techniques are developed to get reliable so-
30 lutions at reasonable computational costs. These methods attempt to resolve scales
31 below the coarse mesh scale by incorporating local computations into a global problem
32 which is defined only on a coarse mesh.

33 **1.2. Multiscale methods.** A certainly not exhaustive list of Multiscale meth-
34 ods includes homogenization based methods [5, 20, 28], upscaling methods [8, 23, 32,
35 43], Multiscale Finite Element Methods (MsFEMs) [9, 19, 20, 21, 29, 30], variational
36 multiscale methods [2, 31], heterogeneous multiscale methods [17, 28], multiscale finite
37 volumes and discontinuous Galerkin methods [3, 14, 26, 34] and so on. MsFEMs were

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38 first proposed by [29, 30] and then experienced major advancements in [9, 19, 20, 21].
 39 This method has been extended to the generalized multiscale finite element method
 40 in [10, 11, 12, 18]. MsFEMs use a finite element approach and construct special ba-
 41 sis functions which incorporate the fine scale features of the problem. MsFEMs rely
 42 on two types of meshes: a global coarse mesh and a collection of local fine meshes.
 43 The basis functions are constructed by solving local problems on the fine meshes
 44 with prescribed boundary conditions. Then a coarse-scale problem, involving these
 45 basis functions, is solved on the coarse mesh. In order to improve the accuracy of
 46 MsFEMs, the oversampling method [29] was introduced to better approximate the
 47 boundary conditions in local problems. This gives a nonconforming MsFEM since the
 48 oversampling introduces discontinuities across coarse elements.

49 Another nonconforming method, namely the Crouzeix-Raviart MsFEM, was de-
 50 veloped for solving the diffusion problem [35, 36, 37], the advection-diffusion problem
 51 [16], the Stokes problem [33, 40] and the Oseen problem [39] in heterogeneous media.
 52 More recently, [13] proposes a multiscale hybrid high-order method for highly oscil-
 53 latory elliptic problems. The method in [13] can be considered as a first attempt at
 54 generalizing the Crouzeix-Raviart MsFEM to arbitrary orders of approximation. In
 55 the Crouzeix-Raviart MsFEM, basis functions are constructed associated to coarse
 56 element edges. The non-conforming nature of the Crouzeix-Raviart element [15] pro-
 57 vides great flexibility especially in case of randomly placed obstacles. The conformity
 58 between coarse elements is enforced in a weak sense, i.e., only the average of the
 59 jump of the function vanishes on the interface between coarse elements. In the case of
 60 densely placed obstacles, it is very difficult to avoid intersections between interfaces
 61 of coarse elements and obstacles. In this situation, the Crouzeix-Raviart MsFEM is
 62 very attractive since it allows the multiscale basis functions to have a natural bound-
 63 ary condition on coarse element edges, which relaxes the sensitivity of the method to
 64 complex patterns of obstacles, without using oversampling methods.

65 In [40] the Crouzeix-Raviart MsFEM was introduced for Stokes problems in het-
 66 erogeneous media. In order to improve the accuracy of the method, the authors of [33]
 67 enriched the approximation space of velocity by adding weighting functions which are
 68 defined by linear polynomials. The penalization method was applied with high dif-
 69 fusion and viscosity parameters inside the obstacles, in order to perform numerical
 70 simulations on simple Cartesian meshes, and local problems are discretized by the
 71 \mathbb{Q}_1 - \mathbb{Q}_1 finite element method [4].

72 We mention that the method proposed in [13] share some similarities with our
 73 enriched Crouzeix-Raviart MsFEM in the use of weighting functions defined by poly-
 74 nomials of higher degrees. But the method in [13] is defined only for elliptic problems
 75 in the framework of a hybrid high-order method, whereas in our work we develop an
 76 enriched method for flow problems in the framework of MsFEMs.

77 **1.3. This paper.** We propose an innovative enriched Crouzeix-Raviart MsFEM
 78 for Stokes flows in heterogeneous media with numerous solid obstacles. In this method,
 79 both approximation spaces of velocity and pressure are enriched by adding weighting
 80 functions which are defined by polynomials of higher-degrees. By varying the degrees
 81 of polynomials, the enriched Crouzeix-Raviart MsFEM allows to find a good com-
 82 promise between the accuracy of the method and the computing costs. Numerical
 83 experiments show that the enriched Crouzeix-Raviart MsFEM is significantly more
 84 accurate than the MsFEM proposed in [33, 40]. In practice, rather than using the
 85 penalization method as in [33, 40], we perform numerical simulations using body-
 86 fitted unstructured triangular meshes. Local problems are solved by the finite volume

87 element method [6] in an industrial software *TrioCFD* [41].

88 The paper is organized as follows. The problem formulation is described in [sec-](#)
 89 [tion 2](#). The enriched Crouzeix-Raviart MsFEM is presented in [section 3](#). [Section 4](#)
 90 discusses shortly the practical implementations of MsFEMs. In [section 5](#), we present
 91 some numerical experiments in 2-d (although the method is analysed in space di-
 92 mension 2 or 3) and compare the accuracy of different variants of Crouzeix-Raviart
 93 MsFEM. The conclusions are given in [section 6](#).

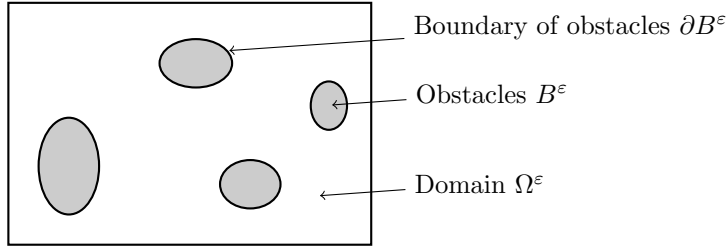


Fig. 1: Rectangular domain Ω comprising a fluid domain Ω^ϵ perforated by a set of obstacles B^ϵ

94 **2. Problem formulation.** Let $\Omega \subset \mathbb{R}^d$ be a regular bounded open set (with
 95 $d = 2$ or 3). As shown in [Figure 1](#), we divide the domain Ω into a fixed solid part
 96 B^ϵ such that $\partial B^\epsilon \cap \partial\Omega = \emptyset$ and its complementary fluid part Ω^ϵ (here ϵ denotes a
 97 small parameter equal to the ratio of the heterogeneities characteristic length with
 98 the characteristic length of the domain). The steady-state Stokes problem is to find
 99 the velocity $\mathbf{u} : \Omega^\epsilon \rightarrow \mathbb{R}^d$ and the pressure $p : \Omega^\epsilon \rightarrow \mathbb{R}$ solutions to

$$100 \quad (2.1) \quad -\mu\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega^\epsilon,$$

$$101 \quad (2.2) \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega^\epsilon,$$

103 where the boundary conditions are given by

$$104 \quad (2.3) \quad \mathbf{u} = \mathbf{0} \quad \text{on } \partial B^\epsilon \cap \partial\Omega^\epsilon,$$

$$105 \quad (2.4) \quad \mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega \cap \partial\Omega^\epsilon,$$

107 with \mathbf{f} a given force per unit volume, and μ the dynamic viscosity.

108 *The weak formulation.* We introduce the spaces $V = H_0^1(\Omega^\epsilon)^d$ for the velocity,

$$109 \quad (2.5) \quad M = L_0^2(\Omega^\epsilon) = \{p \in L^2(\Omega^\epsilon) \text{ s.t. } \int_{\Omega^\epsilon} p = 0\}$$

111 for the pressure and $X = V \times M$. For simplicity the fluid domain Ω^ϵ is assumed
 112 to be connected in order for the pressure to be uniquely defined in M . The weak
 113 formulation of the Stokes problem (2.1)–(2.2) reads: find $(\mathbf{u}, p) \in X$ such that

$$114 \quad (2.6) \quad c((\mathbf{u}, p), (\mathbf{v}, q)) = \int_{\Omega^\epsilon} \mathbf{f} \cdot \mathbf{v}, \quad \forall (\mathbf{v}, q) \in X,$$

116 where the bilinear form c is defined by

$$117 \quad c((\mathbf{u}, p), (\mathbf{v}, q)) = \int_{\Omega^\epsilon} (\mu\nabla\mathbf{u} : \nabla\mathbf{v} - p \operatorname{div} \mathbf{v} - q \operatorname{div} \mathbf{u}).$$

118

119 The existence and uniqueness of a solution to problem (2.6) is guaranteed by the
 120 Banach-Nečas-Babuška (BNB) theorem [22]. This theorem states that problem (2.6)
 121 has a unique solution provided that the bilinear form c is bounded and satisfies the
 122 so-called inf-sup condition:

$$123 \quad (2.7) \quad \exists \alpha > 0, \quad \inf_{(\mathbf{u}, p) \in X} \sup_{(\mathbf{v}, q) \in X} \frac{c((\mathbf{u}, p), (\mathbf{v}, q))}{\|\mathbf{u}, p\|_X \|\mathbf{v}, q\|_X} \geq \alpha.$$

Algorithm 3.1 main steps of a multiscale finite element method

- 1: Partition the domain Ω into a set of coarse elements (coarse mesh)
 - 2: **for** each coarse element **do**
 - 3: Partition the coarse element into a fine mesh
 - 4: Construct multiscale basis functions via local problems
 - 5: Compute rigidity matrices locally on the fine mesh
 - 6: **end for**
 - 7: Assemble global matrices and solve the coarse problem on the coarse mesh
 - 8: **for** each coarse element **do**
 - 9: Reconstruct fine-scale solutions on the fine mesh
 - 10: **end for**
-

125 **3. Crouzeix-Raviart Multiscale Finite Element Method.** Algorithm 3.1
 126 outlines the main steps of MsFEMs. We present the enriched Crouzeix-Raviart Ms-
 FEM following these steps.

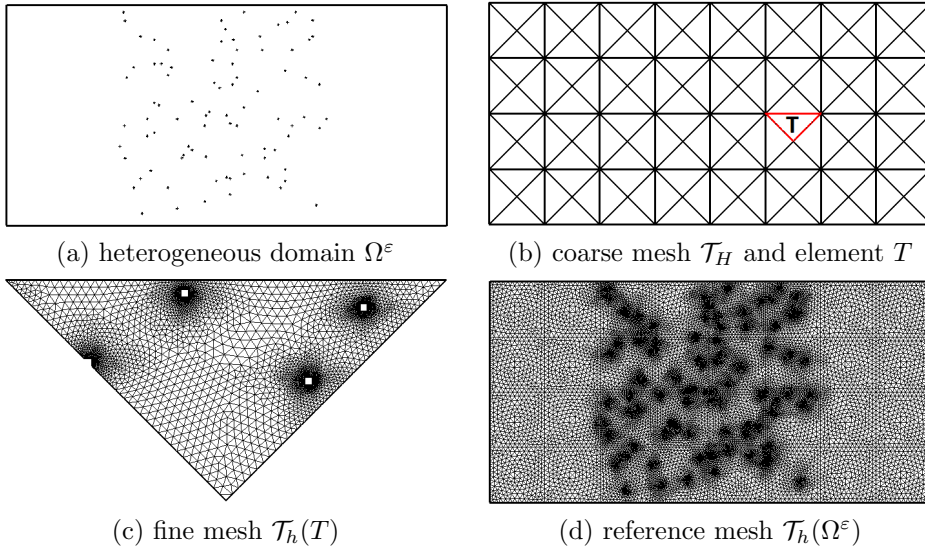


Fig. 2: Illustration of the heterogeneous domain Ω^ϵ , the coarse mesh \mathcal{T}_H , the fine mesh $\mathcal{T}_h(T)$ of a coarse element T and the reference mesh $\mathcal{T}_h(\Omega^\epsilon)$.

127

128 **3.1. Discretization of the domain.** We discretize the domain Ω into an un-
 129 structured coarse mesh \mathcal{T}_H (see Figure 2 (b)) consisting of N_H elements (triangles)

130 of width at most H . Let \mathcal{E}_H denote the set of all edges/faces of \mathcal{T}_H including those
 131 on the domain boundary $\partial\Omega$. For the sake of simplicity, we denote elements of \mathcal{E}_H as
 132 faces throughout this paper. On each element $T \in \mathcal{T}_H$, we construct a fine mesh $\mathcal{T}_h(T)$
 133 (see [Figure 2 \(c\)](#)), with cell elements of maximum width h . Typically $0 < h \leq H$ and
 134 \mathcal{T}_h is fine enough to fully resolve obstacles B^ε .

135 Note that fine meshes do not need to be matching on their common interfaces,
 136 which allows to generate them independently and in parallel. If fine meshes do match
 137 on the interface between coarse elements, their union forms a conforming reference
 138 mesh $\mathcal{T}_h(\Omega^\varepsilon)$ (see [Figure 2 \(d\)](#)) which is used to compute reference solutions, only
 139 needed for the validation of MsFEMs.

140 Let ω_E be the neighborhood of the face $E \in \mathcal{E}_H$ defined by

$$141 \quad (3.1) \quad \omega_E = \bigcup_j \{T_j \in \mathcal{T}_H \mid E \subset \partial T_j\}.$$

142
 143 When E is in the interior of the domain, ω_E is the union of two adjacent elements.
 144 When E is on the boundary of the domain, ω_E is composed of only one element.

145 **3.2. Crouzeix-Raviart functional spaces.** Now we introduce some important
 146 definitions and notations. For any integer n and any integer $1 \leq l \leq d$, we denote
 147 by \mathbb{P}_n^l the linear space spanned by l -variate polynomial functions of total degree at
 148 most n . The dimension of \mathbb{P}_n^l is

$$149 \quad N_n^l := \dim(\mathbb{P}_n^l) = \binom{n+l}{n}.$$

150 For any $T \in \mathcal{T}_H$, we denote by $\mathbb{P}_n^d(T)$ the restriction to T of polynomials in \mathbb{P}_n^d .
 151 For any $E \in \mathcal{E}_H$, we denote by $\mathbb{P}_n^{d-1}(E)$ the restriction to E of polynomials of \mathbb{P}_n^{d-1} .
 152 For the sake of simplicity, we denote $\mathbb{P}_n^d(T)$ and $\mathbb{P}_n^{d-1}(E)$ respectively by $\mathbb{P}_n(T)$ and
 153 $\mathbb{P}_n(E)$.

154 Let s be a positive integer and let $\omega_{E,i} : E \rightarrow \mathbb{R}^d$ be some vector-valued functions
 155 associated to each face $E \in \mathcal{E}_H$ for $i = 1, \dots, s$. Let r be a non-negative integer and
 156 let $\varphi_{T,k} : T \rightarrow \mathbb{R}^d$ be some vector-valued functions associated to each coarse element
 157 $T \in \mathcal{T}_H$ for $k = 1, \dots, r$. Let t be a positive integer and let $\varpi_{T,j} : T \rightarrow \mathbb{R}$ be some
 158 scalar functions associated to each coarse element $T \in \mathcal{T}_H$ for $j = 1, \dots, t$.

159 *Hypothesis 3.1.* For $n = 0$, for any $T \in \mathcal{T}_H$ and for any $E \in \mathcal{E}_H$, we choose

$$160 \quad (3.2) \quad \begin{cases} s = d : \omega_{E,1} = \mathbf{e}_1, \dots, \omega_{E,s} = \mathbf{e}_d, \\ r = 0, \\ t = 1 : \varpi_{T,1} = 1, \end{cases}$$

161 where $(\mathbf{e}_1, \dots, \mathbf{e}_d)$ is the canonical basis of \mathbb{R}^d . For $n \geq 1$, for any $E \in \mathcal{E}_H$ and for
 162 any $T \in \mathcal{T}_H$, we assume that

- 163 1. $s = d N_n^{d-1}$ and $(\omega_{E,i})_{1 \leq i \leq s}$ is a set of basis functions of $(\mathbb{P}_n(E))^d$.
- 164 2. $r = d N_{n-1}^d$ and $(\varphi_{T,k})_{1 \leq k \leq r}$ is a set of basis functions of $(\mathbb{P}_{n-1}(T))^d$.
- 165 3. $t = N_n^d$ and $(\varpi_{T,j})_{1 \leq j \leq t}$ is a set of basis functions of $\mathbb{P}_n(T)$.

166 *Remark 3.2.* For $n = 0$ the choice [\(3.2\)](#) of weighting functions implies that the
 167 enriched Crouzeix-Raviart MsFEM becomes exactly the Crouzeix-Raviart MsFEM
 168 proposed in [\[40\]](#). This also corresponds to the so-called CR2 method investigated
 169 in [\[33\]](#). Furthermore, in a domain without obstacles, i.e. $B^\varepsilon = \emptyset$, the Crouzeix-Raviart
 170 MsFEM space with $n = 0$ becomes the classical Crouzeix-Raviart finite element space.

173 *Remark 3.3.* For $n = 1$ and for the space dimension $d = 2$, our choice of weighting
 174 functions is, for any $T \in \mathcal{T}_H$ and for any $E \in \mathcal{E}(T)$,

$$175 \quad (3.3) \quad \begin{cases} \boldsymbol{\omega}_{E,1} = \mathbf{e}_1, \boldsymbol{\omega}_{E,2} = \mathbf{e}_2, \boldsymbol{\omega}_{E,3} = \mathbf{n}_E \psi_E, \boldsymbol{\omega}_{E,4} = \boldsymbol{\tau}_E \phi_E, \\ \varpi_{T,1} = 1, \varpi_{T,2} = x, \varpi_{T,3} = y, \\ 176 \quad \varphi_{T,1} = \mathbf{e}_1, \varphi_{T,2} = \mathbf{e}_2, \end{cases}$$

177 where ψ_E and ϕ_E are non-vanishing functions in $\mathbb{P}_1(E)$ with vanishing mean-values on
 178 $E \cap \Omega^\varepsilon$ and $(\mathbf{e}_1, \mathbf{e}_2)$ is the canonical basis of \mathbb{R}^2 . In this paper, we have implemented
 179 and tested only the enriched MsFEM with $n = 1$ which we denote by CR4_high.

180 *Remark 3.4.* Our choice of weighting functions for $n \geq 1$ is driven by the following
 181 fact: once the polynomial degree for the weighting functions $(\boldsymbol{\omega}_{E,i})$ has been chosen, it
 182 is seen in (3.13)–(3.15) that the pressure decomposition into resolved and unresolved
 183 contributions motivates the fact that the boundary trace of the resolved pressure
 184 belongs (at most) to the span of $(\boldsymbol{\omega}_{E,i})$, and that the space of unresolved velocities
 185 is orthogonal to (at least) the gradients of the resolved pressures. So our choice,
 186 although not the only possible one, corresponds to these two limit cases: resolved
 187 pressures will have the same polynomial (maximal) degree as the set of $(\boldsymbol{\omega}_{E,i})$, and
 188 unresolved velocities will be (minimally) orthogonal to polynomials one order lower
 189 than the resolved pressures.

190 To construct the approximation spaces of enriched Crouzeix-Raviart MsFEM, we
 191 define V_H^{ext} differently from that of [33, 40]:

$$192 \quad V_H^{ext} = \left\{ \begin{array}{l} \mathbf{u} \in (L^2(\Omega^\varepsilon))^d \text{ s. t. } \mathbf{u}|_T \in (H^1(T \cap \Omega^\varepsilon))^d \text{ for any } T \in \mathcal{T}_H, \\ 193 \quad \mathbf{u} = \mathbf{0} \text{ on } \partial B^\varepsilon, \int_{E \cap \Omega^\varepsilon} [[\mathbf{u}]] \cdot \boldsymbol{\omega}_{E,j} = 0 \text{ for all } E \in \mathcal{E}_H, j = 1, \dots, s \end{array} \right\},$$

194 where $[[\mathbf{u}]]$ denotes the jump of \mathbf{u} across an internal face and $[[\mathbf{u}]] = \mathbf{u}$ on $\partial\Omega$. The
 195 space V_H^{ext} enhances the natural velocity space $(H_0^1(\Omega^\varepsilon))^d$ so that we have at our
 196 disposal discontinuous vector fields across faces of the coarse mesh \mathcal{T}_H . Therefore V_H^{ext}
 197 is not included in $(H_0^1(\Omega^\varepsilon))^d$, the MsFEM is nonconforming and the homogeneous
 198 boundary condition is only weakly enforced.

The extended velocity-pressure space is defined as

$$X_H^{ext} = V_H^{ext} \times M$$

199 with M defined by (2.5). We want to decompose X_H^{ext} into a direct sum of a finite di-
 200 mensional subspace X_H containing coarse scales and an infinite dimensional subspace
 201 X_H^0 containing unsolved fine scales, i.e., $X_H^{ext} = X_H \oplus X_H^0$. This will be achieved in
 202 Theorem 3.15.

203 Taking into account Hypothesis 3.1, we first define the infinite dimensional space
 204 X_H^0 as follows.

DEFINITION 3.5. *The velocity-pressure space X_H^0 is defined as a subspace of X_H^{ext}
 by*

$$205 \quad X_H^0 = V_H^0 \times M_H^0, \quad \text{with}$$

$$206 \quad (3.4) \quad V_H^0 = \left\{ \begin{array}{l} \mathbf{u} \in V_H^{ext} \text{ s. t. } \int_{E \cap \Omega^\varepsilon} \mathbf{u} \cdot \boldsymbol{\omega}_{E,j} = 0, \int_{T \cap \Omega^\varepsilon} \mathbf{u} \cdot \boldsymbol{\varphi}_{T,k} = 0, \\ \forall T \in \mathcal{T}_H, \forall E \in \mathcal{E}_H, j = 1, \dots, s, k = 1, \dots, r. \end{array} \right\},$$

$$207 \quad (3.5) \quad M_H^0 = \left\{ p \in M \text{ s. t. } \int_{T \cap \Omega^\varepsilon} p \varpi_{T,j} = 0, \forall T \in \mathcal{T}_H, j = 1, \dots, t \right\}.$$

208

209 It is important to stress that enriching only the set of edge weights $\omega_{E,j}$ seems
 210 insufficient: indeed, in that case, a given function \mathbf{u} vanishing on the edges of any
 211 T would belong to the unresolved fine scales whatever the number of edge weights,
 212 but imposing the condition $\int_{T \cap \Omega^\varepsilon} \mathbf{u} \cdot \boldsymbol{\varphi}_{T,k} = 0$ for an increasing number of triangle
 213 weights will reduce the norm of its component in the unresolved subspace.

214 Besides, for any $T \in \mathcal{T}_H$, we define $M_H^0(T)$ by

$$215 \quad (3.6) \quad M_H^0(T) = \left\{ p \in L^2(T \cap \Omega^\varepsilon) \text{ s. t. } \int_{T \cap \Omega^\varepsilon} p \varpi_{T,i} = 0, i = 1, \dots, t \right\}.$$

217 DEFINITION 3.6. *The velocity-pressure space X_H is defined as a subspace of X_H^{ext} ,
 218 being the "orthogonal" complement of X_H^0 with respect to the bilinear form $c_H(\cdot, \cdot)$ as
 219 follows:*

$$220 \quad (\mathbf{u}_H, p_H) \in X_H \iff (\mathbf{u}_H, p_H) \in X_H^{ext} \text{ such that}$$

$$221 \quad (3.7) \quad c_H((\mathbf{u}_H, p_H), (\mathbf{v}, q)) = 0, \forall (\mathbf{v}, q) \in X_H^0,$$

223 where $c_H(\cdot, \cdot)$ is defined by

$$224 \quad c_H((\mathbf{u}_H, p_H), (\mathbf{v}, q)) = \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} (\mu \nabla \mathbf{u}_H : \nabla \mathbf{v} - p_H \operatorname{div} \mathbf{v} - q \operatorname{div} \mathbf{u}_H).$$

226 Remark 3.7. The word "orthogonal" is written between quotes since the bilinear
 227 form $c_H(\cdot, \cdot)$ is not a scalar product (not positive definite).

228 DEFINITION 3.8. *Define functional spaces M_H and V_H by*

$$229 \quad (3.8) \quad M_H = \{q \in M \text{ s. t. } q|_T \in \mathbb{P}_n(T), \forall T \in \mathcal{T}_H\},$$

$$230 \quad (3.9) \quad V_H = \left\{ \begin{array}{l} \mathbf{v} \in V_H^{ext} : \forall T \in \mathcal{T}_H, \exists \zeta^T \in M_H^0(T) \text{ such that} \\ -\mu \Delta \mathbf{v} + \nabla \zeta^T \in \operatorname{span} \{\boldsymbol{\varphi}_{T,1}, \dots, \boldsymbol{\varphi}_{T,r}\} \text{ in } T \cap \Omega^\varepsilon \\ \operatorname{div} \mathbf{v} \in \operatorname{span} \{\varpi_{T,1}, \dots, \varpi_{T,t}\} \text{ in } T \cap \Omega^\varepsilon \\ \mu \nabla \mathbf{v} \mathbf{n} - \zeta^T \mathbf{n} \in \operatorname{span} \{\boldsymbol{\omega}_{E,1}, \dots, \boldsymbol{\omega}_{E,s}\} \text{ on } E \cap \Omega^\varepsilon \forall E \in \mathcal{E}(T) \end{array} \right\},$$

232 where $\mathcal{E}(T)$ is the set of faces composing ∂T .

233 Remark 3.9. Clearly, M_H , defined by (3.8), is orthogonal to M_H^0 , defined by (3.5),
 234 in the set M .

235 Remark 3.10. In Sections 3.3 and 3.5 it will be proved that V_H is finite-dimension-
 236 al and spanned by velocity fields associated to each element $T \in \mathcal{T}_H$ and weight
 237 $\boldsymbol{\varphi}_{T,k}$ with $1 \leq k \leq r$ on the one hand and to each edge $E \in \mathcal{E}_H$ and weight $\boldsymbol{\omega}_{E,i}$ with
 238 $1 \leq i \leq s$ on the other hand.

239 LEMMA 3.11. *In definition (3.9), the pressure ζ^T is uniquely defined for a given
 240 velocity $\mathbf{v} \in V_H$. Therefore, gluing together the pressures ζ^T on all triangles $T \in \mathcal{T}_H$
 241 yields a single function $\pi_H(\mathbf{v}) \in M_H^0$ such that $\pi_H(\mathbf{v}) = \zeta^T$ on any triangle $T \in \mathcal{T}_H$.
 242 The operator $\pi_H : V_H \rightarrow M_H^0$ is linear and continuous.*

243 *Proof.* For a given $\mathbf{v} \in V_H$, we rewrite definition (3.9)

$$244 \quad -\mu \Delta \mathbf{v} + \nabla \zeta^T = \sum_{k=1}^r \lambda_{T,k} \boldsymbol{\varphi}_{T,k} \text{ in } T \cap \Omega^\varepsilon,$$

$$245 \quad \operatorname{div} \mathbf{v} \in \operatorname{span} \{\varpi_{T,1}, \dots, \varpi_{T,t}\} \text{ in } T \cap \Omega^\varepsilon,$$

$$246 \quad \mathbf{v} = 0 \text{ on } \partial B^\varepsilon \cap T,$$

$$247 \quad \mu \nabla \mathbf{v} \mathbf{n} - \zeta^T \mathbf{n} \in \operatorname{span} \{\boldsymbol{\omega}_{E,1}, \dots, \boldsymbol{\omega}_{E,s}\} \text{ on } E \cap \Omega^\varepsilon, \forall E \in \mathcal{E}(T).$$

249 for some vector $\lambda_T = (\lambda_{T,1}, \dots, \lambda_{T,r})$. Let us show that the pressure $\zeta^T \in M_H^0(T)$
 250 and the vector λ_T are uniquely defined. Assume there exist $(\zeta_1^T, \lambda_T^1) \in M_H^0(T) \times \mathbb{R}^r$
 251 and $(\zeta_2^T, \lambda_T^2) \in M_H^0(T) \times \mathbb{R}^r$ satisfying the system of equations above for the same
 252 velocity \mathbf{v} . In other words,

$$253 \quad \nabla \zeta_1^T = \sum_{k=1}^r \lambda_{T,k}^1 \varphi_{T,k} + \mu \Delta \mathbf{v},$$

$$254 \quad \nabla \zeta_2^T = \sum_{k=1}^r \lambda_{T,k}^2 \varphi_{T,k} + \mu \Delta \mathbf{v}.$$

255
 256 Subtracting these equations, we obtain

$$257 \quad \nabla (\zeta_1^T - \zeta_2^T) = \sum_{k=1}^r (\lambda_{T,k}^1 - \lambda_{T,k}^2) \varphi_{T,k}.$$

258
 259 The fact $\varphi_{T,k} \in (\mathbb{P}_{n-1}(T))^d$ implies that $(\zeta_1^T - \zeta_2^T) \in \mathbb{P}_n(T)$. At the same time,
 260 $(\zeta_1^T - \zeta_2^T) \in M_H^0(T)$ defined by (3.6) which is orthogonal to $\mathbb{P}_n(T)$. Thus we deduce
 261 that $\zeta_1^T = \zeta_2^T$ and then $\lambda_T^1 = \lambda_T^2$. Therefore we conclude that there exists an operator
 262 $\pi_H : V_H \rightarrow M_H^0$ such that $\zeta^T = [\pi_H(\mathbf{v})]_T$. Moreover, π_H is obviously linear and,
 263 since V_H is finite-dimensional, π_H is continuous. \square

264 **THEOREM 3.12.** *Using the notations of Definition 3.8, let us introduce the space*

$$265 \quad (3.10) \quad \underline{X}_H = \text{span} \{(\mathbf{u}_H, \pi_H(\mathbf{u}_H) + \bar{p}_H), \mathbf{u}_H \in V_H, \bar{p}_H \in M_H\}.$$

267 *Then it coincides with the subspace X_H defined in (3.7), namely*

$$268 \quad (3.11) \quad X_H = \underline{X}_H.$$

270

271 *Proof.* We first prove the inclusion $X_H \subset \underline{X}_H$. Let $(\mathbf{u}_H, p_H) \in X_H$ in the sense
 272 of definition (3.7), i.e. $\forall (\mathbf{v}, q) \in X_H^0$:

$$(3.12) \quad c_H((\mathbf{u}_H, p_H), (\mathbf{v}, q)) = \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} (\mu \nabla \mathbf{u}_H : \nabla \mathbf{v} - p_H \text{div } \mathbf{v} - q \text{div } \mathbf{u}_H) = 0.$$

275 Let $\bar{p}_H \in M_H$ be the orthogonal projection of p_H on M_H , and let $p'_H = (p_H - \bar{p}_H)$.
 276 The fact that M_H and M_H^0 are orthogonal implies that

$$277 \quad p_H = \bar{p}_H + p'_H \text{ with } \bar{p}_H \in M_H \text{ and } p'_H \in M_H^0.$$

279 By virtue of this decomposition, the term concerning p_H in (3.12) can be decomposed
 280 as

$$281 \quad (3.13) \quad \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} p_H \text{div } \mathbf{v} = \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} \bar{p}_H \text{div } \mathbf{v} + \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} p'_H \text{div } \mathbf{v}.$$

283 Now we compute the first term in the right hand side of (3.13). Integrating by parts
 284 on each triangle T yields:

$$285 \quad (3.14) \quad \int_{T \cap \Omega^\varepsilon} \bar{p}_H \text{div } \mathbf{v} = \int_{\partial(T \cap \Omega^\varepsilon)} \mathbf{v} \cdot \mathbf{n} \bar{p}_H - \int_{T \cap \Omega^\varepsilon} \mathbf{v} \cdot \nabla \bar{p}_H, \forall \mathbf{v} \in V_H^0.$$

286

287 Since $\bar{p}_H \in M_H$, definition (3.8) implies that for any $T \in \mathcal{T}_H$, $\bar{p}_H|_T \in \mathbb{P}_n(T)$. Thus,
 288 for any $E \in \mathcal{E}(T)$, $\bar{p}_H|_E \mathbf{n} \in (\mathbb{P}_n(E))^d = \text{span}\{\boldsymbol{\omega}_{E,1}, \dots, \boldsymbol{\omega}_{E,s}\}$, according to **Hypothesis 3.1**.
 289 Then, since $\mathbf{v} \in V_H^0$, definition (3.4) implies that

$$290 \quad \int_{\partial(T \cap \Omega^\varepsilon)} \mathbf{v} \cdot \mathbf{n} \bar{p}_H = 0.$$

292 Then (3.14) reduces to

$$293 \quad \int_{T \cap \Omega^\varepsilon} \bar{p}_H \operatorname{div} \mathbf{v} = - \int_{T \cap \Omega^\varepsilon} \mathbf{v} \cdot \nabla \bar{p}_H.$$

295 As $\bar{p}_H|_T \in \mathbb{P}_n(T)$, it is obvious that $\nabla \bar{p}_H = \mathbf{0}$ for $n = 0$ and that $\nabla \bar{p}_H \in (\mathbb{P}_{n-1}(T))^d =$
 296 $\text{span}\{\boldsymbol{\varphi}_{T,1}, \dots, \boldsymbol{\varphi}_{T,r}\}$ on $T \in \mathcal{T}_H$ for $n \geq 1$, according to **Hypothesis 3.1**. Then, since
 297 $\mathbf{v} \in V_H^0$, definition (3.4) implies that

$$298 \quad \int_{T \cap \Omega^\varepsilon} \bar{p}_H \operatorname{div} \mathbf{v} = - \int_{T \cap \Omega^\varepsilon} \mathbf{v} \cdot \nabla \bar{p}_H = 0.$$

300 As a result, (3.13) reduces to

$$301 \quad \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} p_H \operatorname{div} \mathbf{v} = \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} p'_H \operatorname{div} \mathbf{v}$$

303 and (3.12) reduces to: $\forall (\mathbf{v}, q) \in X_H^0$,

(3.15)

$$304 \quad c_H((\mathbf{u}_H, p_H), (\mathbf{v}, q)) = \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} (\mu \nabla \mathbf{u}_H : \nabla \mathbf{v} - p'_H \operatorname{div} \mathbf{v} - q \operatorname{div} \mathbf{u}_H) = 0.$$

306 In a second step, choosing the test function $\mathbf{v} = \mathbf{0}$ and, for a given element
 307 $T \in \mathcal{T}_H$, the test function $q \in M_H^0$ with q vanishing outside T , (3.15) becomes

$$308 \quad \int_{T \cap \Omega^\varepsilon} q \operatorname{div} \mathbf{u}_H = 0.$$

310 By definition (3.5) of M_H^0 , this implies that

$$311 \quad (\operatorname{div} \mathbf{u}_H)|_T \in \text{span}\{\boldsymbol{\varpi}_{T,1}, \dots, \boldsymbol{\varpi}_{T,t}\} \text{ i.e. } (\operatorname{div} \mathbf{u}_H)|_T \in \mathbb{P}_n(T) \text{ in } T \cap \Omega^\varepsilon,$$

313 thanks to **Hypothesis 3.1**.

314 In a third step, we observe that for any face $E \in \mathcal{E}(T)$, there exist some (not
 315 unique) functions $\mathbf{v}_{E,i} \in (H^1(T \cap \Omega^\varepsilon))^d$, $i = 1, \dots, s$ such that

$$316 \quad (3.17) \quad \begin{cases} \int_{F \cap \Omega^\varepsilon} \mathbf{v}_{E,i} \cdot \boldsymbol{\omega}_{F,j} = \delta_{E,F} \delta_{i,j}, \quad \forall F \in \mathcal{E}(T), \quad \forall j = 1, \dots, s, \\ \int_{T \cap \Omega^\varepsilon} \mathbf{v}_{E,i} \cdot \boldsymbol{\varphi}_{T,l} = 0, \quad \forall l = 1, \dots, r, \\ \mathbf{v}_{E,i} = \mathbf{0}, \quad \text{on } \partial B^\varepsilon \cap T. \end{cases}$$

318 We also observe that there exist (not unique) functions $\mathbf{v}_{T,k} \in (H^1(T \cap \Omega^\varepsilon))^d$, $k =$
 319 $1, \dots, r$ such that

$$320 \quad (3.18) \quad \begin{cases} \int_{F \cap \Omega^\varepsilon} \mathbf{v}_{T,k} \cdot \boldsymbol{\omega}_{F,j} = 0, \quad \forall F \in \mathcal{E}(T), \quad \forall j = 1, \dots, s, \\ \int_{T \cap \Omega^\varepsilon} \mathbf{v}_{T,k} \cdot \boldsymbol{\varphi}_{T,l} = \delta_{k,l}, \quad \forall l = 1, \dots, r, \\ \mathbf{v}_{T,k} = \mathbf{0}, \quad \text{on } \partial B^\varepsilon \cap T. \end{cases}$$

321
322

323 *Remark 3.13.* One may construct such functions as follows. Let θ be a given non-
 324 vanishing positive function in $H_0^1(T \cap \Omega^\varepsilon)$. Then, functions $\mathbf{v}_{T,k}$ are sought under
 325 the form $(\sum_{1 \leq k' \leq r} z_{k,k'} \boldsymbol{\varphi}_{T,k'}) \theta$. They evidently satisfy the first and last line of (3.18)
 326 and, for any k , coefficients $z_{k,k'}$ are chosen such that $\sum_{1 \leq k' \leq r} a_{k',l} z_{k,k'} = \delta_{k,l}$ for all
 327 $l = 1, \dots, r$, with $a_{k',l} = \int_{T \cap \Omega^\varepsilon} \theta \boldsymbol{\varphi}_{T,k'} \cdot \boldsymbol{\varphi}_{T,l}$. This is a square linear system which
 328 has a unique solution since $(\mathbf{v}, \mathbf{w}) \mapsto \int_{T \cap \Omega^\varepsilon} \theta \mathbf{v} \cdot \mathbf{w}$ is a scalar product on $(\mathbb{P}_{n-1}(T))^d$
 329 and $(\boldsymbol{\varphi}_{T,k})_{1 \leq k \leq r}$ a basis of that space. Next, for a given face $E \in \mathcal{E}(T)$, considering
 330 a function $\theta_E \in H_0^1(E \cap \Omega^\varepsilon)$ a similar process leads to functions $\tilde{\mathbf{v}}_{E,i}$ defined on
 331 $E \cap \Omega^\varepsilon$, vanishing on $\partial(E \cap \Omega^\varepsilon)$ and such that $\int_{E \cap \Omega^\varepsilon} \tilde{\mathbf{v}}_{E,i} \cdot \boldsymbol{\omega}_{E,j} = \delta_{i,j}$. Extending
 332 these functions by 0 on the other faces $F \neq E$ of T and on $\partial B^\varepsilon \cap T$, they belong to
 333 $H^{1/2}(\partial(T \cap \Omega^\varepsilon))$ and can be lifted to functions of $H^1(T \cap \Omega^\varepsilon)$ still denoted by $\tilde{\mathbf{v}}_{E,i}$
 334 and which verify by construction the first and third lines of (3.17). We conclude by
 335 setting $\mathbf{v}_{E,i} = \tilde{\mathbf{v}}_{E,i} - \sum_{1 \leq k \leq r} a_k \mathbf{v}_{T,k}$ with $a_k = \int_{T \cap \Omega^\varepsilon} \tilde{\mathbf{v}}_{E,i} \cdot \boldsymbol{\varphi}_{T,k}$.

336 We denote by $V(T)$ the set of functions in $(H^1(T \cap \Omega^\varepsilon))^d$ vanishing on $\partial B^\varepsilon \cap T$:

$$337 \quad (3.19) \quad V(T) = \left\{ \mathbf{v} \in (H^1(T \cap \Omega^\varepsilon))^d \text{ such that } \mathbf{v} = \mathbf{0} \text{ on } \partial B^\varepsilon \cap T \right\}.$$

338 We now check that the space $V(T)$ can be decomposed as

$$340 \quad V(T) = V_{f_0}(T) \oplus \text{span} \{ \mathbf{v}_{E,i}, \mathbf{v}_{T,k}, \forall E \in \mathcal{E}(T), i = 1, \dots, s, k = 1, \dots, r \}$$

341 where

$$342 \quad (3.20) \quad V_{f_0}(T) = \left\{ \begin{array}{l} \mathbf{v} \in (H^1(T \cap \Omega^\varepsilon))^d : \int_{F \cap \Omega^\varepsilon} \mathbf{v} \cdot \boldsymbol{\omega}_{F,j} = 0, \int_{T \cap \Omega^\varepsilon} \mathbf{v} \cdot \boldsymbol{\varphi}_{T,l} = 0, \\ \mathbf{v} = \mathbf{0} \text{ on } \partial B^\varepsilon \cap T, \forall F \in \mathcal{E}(T), j = 1, \dots, s, l = 1, \dots, r. \end{array} \right\}.$$

343 Indeed, the fact that these two subspaces are in direct sum is obvious. Next, we
 344 decompose any $\mathbf{v} \in V(T)$ into

$$345 \quad (3.21) \quad \mathbf{v} = \tilde{\mathbf{v}} + \sum_{E \in \mathcal{E}_H} \sum_{i=1}^s \beta_{E,i} \mathbf{v}_{E,i} + \sum_{k=1}^r \beta_{T,k} \mathbf{v}_{T,k}.$$

346 Choosing

$$347 \quad (3.22) \quad \beta_{E,i} = \int_{E \cap \Omega^\varepsilon} \mathbf{v} \cdot \boldsymbol{\omega}_{E,i} \text{ and } \beta_{T,k} = \int_{T \cap \Omega^\varepsilon} \mathbf{v} \cdot \boldsymbol{\varphi}_{T,k},$$

348 one may easily check that $\tilde{\mathbf{v}} \in V_{f_0}(T)$.

349 Now, for a given T and a given $\mathbf{v} \in V(T)$, let $\underline{\tilde{\mathbf{v}}}$ be the function which is equal to $\tilde{\mathbf{v}}$
 350 on $T \cap \Omega^\varepsilon$ as constructed in (3.21), and equal to $\mathbf{0}$ elsewhere. Hence, since $\tilde{\mathbf{v}} \in V_{f_0}(T)$,
 351 it holds that $\underline{\tilde{\mathbf{v}}} \in V_H^0$. Taking $q = 0$ and $\mathbf{v} = \underline{\tilde{\mathbf{v}}}$ in (3.15) we obtain

$$352 \quad (3.23) \quad c_H((\mathbf{u}_H, p_H), (\underline{\tilde{\mathbf{v}}}, 0)) = \int_{T \cap \Omega^\varepsilon} (\mu \nabla \mathbf{u}_H : \nabla \underline{\tilde{\mathbf{v}}} - p'_H \text{div } \underline{\tilde{\mathbf{v}}}) = 0.$$

353 Substituting $\underline{\tilde{\mathbf{v}}}$ defined in (3.21) into (3.23), we obtain

$$354 \quad (3.24) \quad \int_{T \cap \Omega^\varepsilon} (\mu \nabla \mathbf{u}_H : \nabla \mathbf{v} - p'_H \text{div } \mathbf{v}) = \sum_{k=1}^r \beta_{T,k} \int_{T \cap \Omega^\varepsilon} (\mu \nabla \mathbf{u}_H : \nabla \mathbf{v}_{T,k} - p'_H \text{div } \mathbf{v}_{T,k})$$

$$355 \quad + \sum_{E \in \mathcal{E}(T)} \sum_{i=1}^s \beta_{E,i} \int_{T \cap \Omega^\varepsilon} (\mu \nabla \mathbf{u}_H : \nabla \mathbf{v}_{E,i} - p'_H \text{div } \mathbf{v}_{E,i}).$$

356

361 Denoting for any $E \in \mathcal{E}(T)$, $i = 1, \dots, s$ and $k = 1, \dots, r$

$$362 \quad \lambda_{E,i} = \int_{T \cap \Omega^\varepsilon} \mu \nabla \mathbf{u}_H : \nabla \mathbf{v}_{E,i} - p'_H \operatorname{div} \mathbf{v}_{E,i},$$

$$363 \quad \lambda_{T,k} = \int_{T \cap \Omega^\varepsilon} \mu \nabla \mathbf{u}_H : \nabla \mathbf{v}_{T,k} - p'_H \operatorname{div} \mathbf{v}_{T,k},$$

364

365 and using the definition of $\beta_{E,i}$ and $\beta_{T,k}$ given by (3.22), equation (3.24) can be
366 written as

$$367 \quad \int_{T \cap \Omega^\varepsilon} (\mu \nabla \mathbf{u}_H : \nabla \mathbf{v} - p'_H \operatorname{div} \mathbf{v}) = \sum_{E \in \mathcal{E}(T)} \sum_{i=1}^s \lambda_{E,i} \int_{E \cap \Omega^\varepsilon} \mathbf{v} \cdot \boldsymbol{\omega}_{E,i}$$

$$368 \quad (3.25) \quad + \sum_{k=1}^r \lambda_{T,k} \int_{T \cap \Omega^\varepsilon} \mathbf{v} \cdot \boldsymbol{\varphi}_{T,k},$$

369

370 which holds for any $\mathbf{v} \in V(T)$ defined in (3.19). In (3.25), considering an arbitrary
371 \mathbf{v} that vanishes on all edges $E \in \mathcal{E}(T)$ and integrating by parts in the left-hand side,
372 we obtain the following strong form:

$$373 \quad (3.26) \quad -\mu \Delta \mathbf{u}_H + \nabla p'_H = \sum_{k=1}^r \lambda_{T,k} \boldsymbol{\varphi}_{T,k} \text{ in } T \cap \Omega^\varepsilon.$$

374

375 Then by writing the variational formulation of (3.26) for $\mathbf{v} \in V(T)$ and comparing
376 with (3.25), we deduce that

$$377 \quad (3.27) \quad \mu \nabla \mathbf{u}_H \mathbf{n} - p'_H \mathbf{n} \in \operatorname{span} \{ \boldsymbol{\omega}_{E,1}, \dots, \boldsymbol{\omega}_{E,s} \} \text{ on } E \cap \Omega^\varepsilon \quad \forall E \in \mathcal{E}(T).$$

379 Finally, combining equations (3.16), (3.26) and (3.27), we obtain

$$380 \quad -\mu \Delta \mathbf{u}_H + \nabla p'_H = \sum_{k=1}^r \lambda_{T,k} \boldsymbol{\varphi}_{T,k} \text{ in } T \cap \Omega^\varepsilon,$$

$$381 \quad \operatorname{div} \mathbf{u}_H \in \operatorname{span} \{ \boldsymbol{\varpi}_{T,1}, \dots, \boldsymbol{\varpi}_{T,t} \} \text{ in } T \cap \Omega^\varepsilon,$$

$$382 \quad \mathbf{u}_H = 0 \text{ on } \partial B^\varepsilon \cap T,$$

$$383 \quad \mu \nabla \mathbf{u}_H \mathbf{n} - p'_H \mathbf{n} \in \operatorname{span} \{ \boldsymbol{\omega}_{E,1}, \dots, \boldsymbol{\omega}_{E,s} \} \text{ on } E \cap \Omega^\varepsilon, \quad \forall E \in \mathcal{E}(T).$$

384

385 Using Lemma 3.11, the pressure $p'_H = \pi_H(\mathbf{u}_H)$ and the vector $\lambda_T = (\lambda_{T,1}, \dots, \lambda_{T,r})$
386 are both uniquely determined by \mathbf{u}_H . We recall that the pressure p_H is defined by
387 $p_H = \bar{p}_H + p'_H$. Thus we proved that (\mathbf{u}_H, p_H) defined in (3.7) belongs to the space
388 defined by (3.10), i.e. $X_H \subset \underline{X}_H$.

389 Reciprocally, we now prove that $(\mathbf{u}_H, p_H) \in \underline{X}_H$ defined by (3.10), with $p_H =$
390 $\bar{p}_H + \pi_H(\mathbf{u}_H)$, also satisfies relation (3.7). The first step is to remark that (\mathbf{u}_H, p_H)
391 is indeed in X_H^{ext} ; the fact that p_H belongs to M is obvious, and the fact that \mathbf{u}_H
392 belongs to V_H^{ext} comes from the definition of V_H itself. As far as the "orthogonality"
393 relation in (3.7) is concerned, since $\mathbf{u}_H \in V_H$ then Lemma 3.11 shows that there exist

394 a unique $\zeta^T = \pi_H(\mathbf{u}_H) \in M_H^0(T)$ and a unique $(\lambda_{T,1}, \dots, \lambda_{T,r}) \in \mathbb{R}^r$ such that

$$395 \quad (3.28) \quad -\mu \Delta \mathbf{u}_H + \nabla \zeta^T = \sum_{k=1}^r \lambda_{T,k} \boldsymbol{\varphi}_{T,k} \text{ in } T \cap \Omega^\varepsilon,$$

$$396 \quad (3.29) \quad \operatorname{div} \mathbf{u}_H \in \operatorname{span} \{\boldsymbol{\varpi}_{T,1}, \dots, \boldsymbol{\varpi}_{T,t}\} \text{ in } T \cap \Omega^\varepsilon,$$

$$397 \quad (3.30) \quad \mathbf{u}_H = \mathbf{0} \text{ on } \partial B^\varepsilon \cap T,$$

$$398 \quad (3.31) \quad \mu \nabla \mathbf{u}_H \mathbf{n} - \zeta^T \mathbf{n} \in \operatorname{span} \{\boldsymbol{\omega}_{E,1}, \dots, \boldsymbol{\omega}_{E,s}\} \text{ on } E \cap \Omega^\varepsilon, \forall E \in \mathcal{E}(T).$$

400 Equation (3.31) shows that for any $E \in \mathcal{E}(T)$, there exist $\lambda_{E,1}, \dots, \lambda_{E,s} \in \mathbb{R}$ such
401 that

$$402 \quad \mu \nabla \mathbf{u}_H \mathbf{n} - \zeta^T \mathbf{n} = \sum_{i=1}^s \lambda_{E,i} \boldsymbol{\omega}_{E,i} \text{ on } E \cap \Omega^\varepsilon, \forall E \in \mathcal{E}(T).$$

404 Taking the scalar product of (3.28) with any $\mathbf{v} \in V_H^0$, integrating by parts and making
405 use of the equations above, we obtain

$$406 \quad \int_{T \cap \Omega^\varepsilon} \mu \nabla \mathbf{u}_H : \nabla \mathbf{v} - \int_{T \cap \Omega^\varepsilon} \zeta^T \operatorname{div} \mathbf{v} = \sum_{k=1}^r \lambda_{T,k} \int_{T \cap \Omega^\varepsilon} \boldsymbol{\varphi}_{T,k} \cdot \mathbf{v}$$

$$407 \quad (3.32) \quad + \sum_{E \in \mathcal{E}(T)} \sum_{i=1}^s \lambda_{E,i} \int_{E \cap \Omega^\varepsilon} \boldsymbol{\omega}_{E,i} \cdot \mathbf{v}.$$

409 Then, definition (3.4) of V_H^0 implies that

$$410 \quad \int_{T \cap \Omega^\varepsilon} \boldsymbol{\varphi}_{T,k} \cdot \mathbf{v} = 0 \text{ and } \int_{E \cap \Omega^\varepsilon} \boldsymbol{\omega}_{E,i} \cdot \mathbf{v} = 0.$$

412 Thus (3.32) reduces to

$$413 \quad (3.33) \quad \int_{T \cap \Omega^\varepsilon} \mu \nabla \mathbf{u}_H : \nabla \mathbf{v} - \int_{T \cap \Omega^\varepsilon} \pi_H(\mathbf{u}_H)|_T \operatorname{div} \mathbf{v} = 0.$$

415 Besides, for $\bar{p}_H \in M_H$, integration by parts yields

$$416 \quad \int_{T \cap \Omega^\varepsilon} \bar{p}_H|_T \operatorname{div} \mathbf{v} = \int_{\partial(T \cap \Omega^\varepsilon)} \mathbf{v} \cdot \mathbf{n} \bar{p}_H|_T - \int_{T \cap \Omega^\varepsilon} \mathbf{v} \cdot \nabla \bar{p}_H|_T, \forall \mathbf{v} \in V_H^0.$$

418 Definition (3.8) of M_H implies that $\bar{p}_H|_T \in \mathbb{P}_n(T)$, so that $\nabla \bar{p}_H|_T \in (\mathbb{P}_{n-1}(T))^d$
419 for $n \geq 1$ and $\nabla \bar{p}_H|_T = \mathbf{0}$ for $n = 0$. We also have that the restriction of $\mathbf{n} \bar{p}_H|_T$
420 to $E \in \mathcal{E}(T)$ belongs to $(\mathbb{P}_n(E))^d$. Making use of Hypothesis 3.1, definition of V_H^0
421 implies that

$$422 \quad (3.34) \quad \begin{cases} \int_{\partial(T \cap \Omega^\varepsilon)} \mathbf{v} \cdot \mathbf{n} \bar{p}_H|_T = 0 \\ \int_{T \cap \Omega^\varepsilon} \mathbf{v} \cdot \operatorname{grad} \bar{p}_H|_T = 0 \end{cases} \implies \int_{T \cap \Omega^\varepsilon} \bar{p}_H|_T \operatorname{div} \mathbf{v} = 0.$$

424 Moreover, for any $\mathbf{u}_H \in V_H$, $\operatorname{div} \mathbf{u}_H \in \operatorname{span} \{\boldsymbol{\varpi}_{T,1}, \dots, \boldsymbol{\varpi}_{T,t}\}$. The fact that
425 $M_H^0(T)$ is orthogonal to $\operatorname{span} \{\boldsymbol{\varpi}_{T,1}, \dots, \boldsymbol{\varpi}_{T,t}\}$ implies that for any $q \in M_H^0(T)$

$$426 \quad (3.35) \quad \int_{T \cap \Omega^\varepsilon} q \operatorname{div} \mathbf{u}_H = 0.$$

428 As a result, summing (3.33), (3.34) and (3.35), we obtain that $\forall (\mathbf{v}, q) \in X_H^0$, and
 429 for all $T \in \mathcal{T}_H$

$$430 \int_{T \cap \Omega^\varepsilon} \mu \nabla \mathbf{u}_H : \nabla \mathbf{v} - \int_{T \cap \Omega^\varepsilon} (\pi_H(\mathbf{u}_H)|_T + \bar{p}_H|_T) \operatorname{div} \mathbf{v} - \int_{T \cap \Omega^\varepsilon} q \operatorname{div} \mathbf{u}_H = 0,$$

432 Now by definition $p_H|_T = \pi_H(\mathbf{u}_H)|_T + \bar{p}_H|_T$ on each element $T \in \mathcal{T}_H$. Summing
 433 the equation above on all elements $T \in \mathcal{T}_H$, we obtain that for any $(\mathbf{v}, q) \in X_H^0$,

$$434 \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} \mu \nabla \mathbf{u}_H : \nabla \mathbf{v} - \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} p_H \operatorname{div} \mathbf{v} - \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} q \operatorname{div} \mathbf{u}_H = 0.$$

436 which exactly means that $(\mathbf{u}_H, p_H) \in X_H$, and therefore $\underline{X}_H \subset X_H$. Consequently,
 437 equality (3.11) is proved. \square

438 **3.3. Local problems defined by Stokes equations.** Now we construct some
 439 functions associated to elements or faces of the coarse mesh, we shall prove in sec-
 440 tion 3.5 that they form a basis of the resolved space V_H . From their definitions below,
 441 they evidently belong to V_H^{ext} .

442 *The strong form.* We first construct functions associated to faces of the coarse
 443 mesh. For any $E \in \mathcal{E}_H$, for $i = 1, \dots, s$, find the function $\Phi_{E,i} : \Omega^\varepsilon \rightarrow \mathbb{R}^d$, the
 444 pressure $\pi_{E,i} : \Omega^\varepsilon \rightarrow \mathbb{R}$ such that $\Phi_{E,i}$ and $\pi_{E,i}$ vanish outside the coarse element(s)
 445 $T_k \subset \omega_E$ (defined by (3.1)) for $k \in \{1, 2\}$ (only one coarse element if $E \in \partial\Omega$) and
 446 solve on T_k :

$$447 \quad (3.36) \quad \begin{cases} -\mu \Delta \Phi_{E,i} + \nabla \pi_{E,i} \in \operatorname{span} \{\varphi_{T_k,1}, \dots, \varphi_{T_k,r}\} \text{ in } T_k \cap \Omega^\varepsilon, \\ \operatorname{div} \Phi_{E,i} \in \operatorname{span} \{\varpi_{T_k,1}, \dots, \varpi_{T_k,t}\} \text{ in } T_k \cap \Omega^\varepsilon, \\ \mu \nabla \Phi_{E,i} \mathbf{n} - \pi_{E,i} \mathbf{n} \in \operatorname{span} \{\omega_{F,1}, \dots, \omega_{F,s}\} \text{ on } F \cap \Omega^\varepsilon, \forall F \in \mathcal{E}(T_k), \\ \Phi_{E,i} = \mathbf{0} \text{ on } \partial B^\varepsilon \cap T_k, \\ \int_{F \cap \Omega^\varepsilon} \Phi_{E,i} \cdot \omega_{F,j} = \begin{cases} \delta_{ij}, & F = E \\ 0, & F \neq E \end{cases} \quad \forall F \in \mathcal{E}(T_k), \forall j = 1, \dots, s, \\ \int_{T_k \cap \Omega^\varepsilon} \Phi_{E,i} \cdot \varphi_{T_k,l} = 0 \quad \forall l = 1, \dots, r, \\ \int_{T_k \cap \Omega^\varepsilon} \pi_{E,i} \cdot \varpi_{T_k,m} = 0 \quad \forall m = 1, \dots, t. \end{cases}$$

449 Then, we construct functions associated to elements of the coarse mesh. For each
 450 $T \in \mathcal{T}_H$, for $k = 1, \dots, r$, the support of the function $\Psi_{T,k}$ is reduced to $T \cap \Omega^\varepsilon$. We
 451 find $\Psi_{T,k} : \Omega^\varepsilon \rightarrow \mathbb{R}^d$ and $\pi_{T,k} : \Omega^\varepsilon \rightarrow \mathbb{R}$ by solving on T :

$$452 \quad (3.37) \quad \begin{cases} -\mu \Delta \Psi_{T,k} + \nabla \pi_{T,k} \in \operatorname{span} \{\varphi_{T,1}, \dots, \varphi_{T,r}\} \text{ in } T \cap \Omega^\varepsilon, \\ \operatorname{div} \Psi_{T,k} \in \operatorname{span} \{\varpi_{T,1}, \dots, \varpi_{T,t}\} \text{ in } T \cap \Omega^\varepsilon, \\ \mu \nabla \Psi_{T,k} \mathbf{n} - \pi_{T,k} \mathbf{n} \in \operatorname{span} \{\omega_{F,1}, \dots, \omega_{F,s}\} \text{ on } F \cap \Omega^\varepsilon, \forall F \in \mathcal{E}(T), \\ \Psi_{T,k} = 0 \text{ on } \partial B^\varepsilon \cap T, \\ \int_{F \cap \Omega^\varepsilon} \Psi_{T,k} \cdot \omega_{F,j} = 0 \quad \forall F \in \mathcal{E}(T), \forall j = 1, \dots, s, \\ \int_{T \cap \Omega^\varepsilon} \Psi_{T,k} \cdot \varphi_{T,l} = \delta_{kl} \quad \forall l = 1, \dots, r, \\ \int_{T \cap \Omega^\varepsilon} \pi_{T,k} \cdot \varpi_{T,m} = 0 \quad \forall m = 1, \dots, t. \end{cases}$$

454 *The weak form.* The weak form of (3.36) is as follows. Let $n_s = s \times \operatorname{Card}(\mathcal{E}(T_k))$.
 455 For any $E \in \mathcal{E}_H$, for $i = 1, \dots, s$, on the coarse element $T_k \subset \omega_E$ for $k \in \{1, 2\}$ (only
 456 one coarse element if $E \in \partial\Omega$), find $\Phi_{E,i} \in (H^1(T_k \cap \Omega^\varepsilon))^d$ such that $\Phi_{E,i} = \mathbf{0}$ on

457 $\partial B^\varepsilon \cap T_k$, $\pi_{E,i} \in L^2(T_k \cap \Omega^\varepsilon)$, $(\lambda_{F,j})_{F \in \mathcal{E}(T_k), j=1, \dots, s} \in \mathbb{R}^{n_s}$ and $(\lambda_{T_k,l})_{l=1, \dots, r} \in \mathbb{R}^r$
 458 by solving:

(3.38)

$$\begin{cases}
 \int_{T_k \cap \Omega^\varepsilon} \mu \nabla \Phi_{E,i} : \nabla \mathbf{v} - \int_{T_k \cap \Omega^\varepsilon} \pi_{E,i} \operatorname{div} \mathbf{v} + \sum_{F \in \mathcal{E}(T_k)} \sum_{j=1}^s \lambda_{F,j} \int_{F \cap \Omega^\varepsilon} \mathbf{v} \cdot \boldsymbol{\omega}_{F,j} = \\
 \sum_{l=1}^r \lambda_{T_k,l} \int_{T_k \cap \Omega^\varepsilon} \boldsymbol{\varphi}_{T_k,l} \cdot \mathbf{v}, \forall \mathbf{v} \in (H^1(T_k \cap \Omega^\varepsilon))^d \text{ such that } \mathbf{v} = \mathbf{0} \text{ on } \partial B^\varepsilon \cap T_k, \\
 \int_{T_k \cap \Omega^\varepsilon} q \operatorname{div} \Phi_{E,i} = 0, \forall q \in M_H^0(T_k), \\
 \sum_{F \in \mathcal{E}(T_k)} \sum_{j=1}^s \mu_{F,j} \int_{F \cap \Omega^\varepsilon} \Phi_{E,i} \cdot \boldsymbol{\omega}_{F,j} = \mu_{E,i}, \forall (\mu_{F,j})_{F \in \mathcal{E}(T_k), j=1, \dots, s} \in \mathbb{R}^{n_s}, \\
 \sum_{l=1}^r \mu_l \int_{T_k \cap \Omega^\varepsilon} \Phi_{E,i} \cdot \boldsymbol{\varphi}_{T_k,l} = 0, \forall (\mu_l)_{l=1, \dots, r} \in \mathbb{R}^r, \\
 \int_{T_k \cap \Omega^\varepsilon} \pi_{E,i} \cdot \boldsymbol{\omega}_{T_k,m} = 0, \forall m = 1, \dots, t.
 \end{cases}$$

461 The weak form of (3.37) is as follows. Recall that $n_s = s \times \operatorname{Card}(\mathcal{E}(T))$. For
 462 $k = 1, \dots, r$, find $\Psi_{T,k} \in (H^1(T \cap \Omega^\varepsilon))^d$ such that $\Psi_{T,k} = \mathbf{0}$ on $\partial B^\varepsilon \cap T$, $\pi_{T,k} \in$
 463 $L^2(T \cap \Omega^\varepsilon)$, $(\lambda_{F,j})_{F \in \mathcal{E}(T), j=1, \dots, s} \in \mathbb{R}^{n_s}$ and $(\lambda_{T,l})_{l=1, \dots, r} \in \mathbb{R}^r$ by solving

(3.39)

$$\begin{cases}
 \int_{T \cap \Omega^\varepsilon} \mu \nabla \Psi_{T,k} : \nabla \mathbf{v} - \int_{T \cap \Omega^\varepsilon} \pi_{T,k} \operatorname{div} \mathbf{v} + \sum_{F \in \mathcal{E}(T)} \sum_{j=1}^s \lambda_{F,j} \int_{F \cap \Omega^\varepsilon} \mathbf{v} \cdot \boldsymbol{\omega}_{F,j} = \\
 \sum_{l=1}^r \lambda_{T,l} \int_{T \cap \Omega^\varepsilon} \boldsymbol{\varphi}_{T,l} \cdot \mathbf{v}, \forall \mathbf{v} \in (H^1(T \cap \Omega^\varepsilon))^d \text{ such that } \mathbf{v} = \mathbf{0} \text{ on } \partial B^\varepsilon \cap T, \\
 \int_{T \cap \Omega^\varepsilon} q \operatorname{div} \Psi_{T,k} = 0, \forall q \in M_H^0(T), \\
 \sum_{F \in \mathcal{E}(T)} \sum_{j=1}^s \mu_{F,j} \int_{F \cap \Omega^\varepsilon} \Psi_{T,k} \cdot \boldsymbol{\omega}_{F,j} = 0, \forall (\mu_{F,j})_{F \in \mathcal{E}(T), j=1, \dots, s} \in \mathbb{R}^{n_s}, \\
 \sum_{l=1}^r \mu_{T,l} \int_{T \cap \Omega^\varepsilon} \Psi_{T,k} \cdot \boldsymbol{\varphi}_{T,l} = \mu_{T,k}, \forall (\mu_{T,l})_{l=1, \dots, r} \in \mathbb{R}^r, \\
 \int_{T \cap \Omega^\varepsilon} \pi_{T,k} \cdot \boldsymbol{\omega}_{T,m} = 0, \forall m = 1, \dots, t.
 \end{cases}$$

466 **3.4. Well-posedness of local problems.** We prove that the local problems
 467 (3.36) and (3.37) are well-posed. Choosing a coarse element $T \in \mathcal{T}_H$, let n_T be the
 468 number of edges composing ∂T and $n_s = n_T \times s$ be the dimension of the vector
 469 containing Lagrange multipliers $(\lambda_{F,j})$ for all $F \in \mathcal{E}(T)$ and $j = 1, \dots, s$. Moreover,
 470 r is the dimension of the Lagrange multiplier vector $(\lambda_{T,l})$ for $l = 1, \dots, r$. We
 471 introduce the velocity space

$$V_T = \left\{ \mathbf{v} \in (H^1(T \cap \Omega^\varepsilon))^d, \mathbf{v} = \mathbf{0} \text{ on } \partial B^\varepsilon \cap T \right\},$$

474 as well as $M_H^0(T)$, the pressure space defined by (3.6). The variational formulation of
 475 local problems (3.38) and (3.39) are of the form :

476 Find $(\mathbf{u}, p, (\lambda_{F,j}), (\lambda_{T,l})) \in V_T \times M_H^0(T) \times \mathbb{R}^{n_s} \times \mathbb{R}^r$ such that

$$(3.40) \quad \widehat{a}_T(\mathbf{u}, \mathbf{v}) + b_T(\mathbf{v}, p) + c_E(\mathbf{v}, (\lambda_{F,j}^T)) + d_T(\mathbf{v}, (\lambda_{T,l})) = 0, \forall \mathbf{v} \in V_T$$

$$(3.41) \quad b_T(\mathbf{u}, q) = 0, \forall q \in M_H^0(T)$$

$$(3.42) \quad c_E(\mathbf{u}, (\mu_{F,j})) = \ell_1((\mu_{F,j})), \forall (\mu_{F,j}) \in \mathbb{R}^{n_s},$$

$$(3.43) \quad d_T(\mathbf{u}, (\mu_{T,l})) = \ell_2((\mu_{T,l})), \forall (\mu_{T,l}) \in \mathbb{R}^r.$$

482 where the bilinear forms are defined by

$$\begin{aligned}
483 \quad \widehat{a}_T(\mathbf{u}, \mathbf{v}) &= \int_{T \cap \Omega^\varepsilon} \mu \nabla \mathbf{u} : \nabla \mathbf{v}, \\
484 \quad b_T(\mathbf{v}, p) &= - \int_{T \cap \Omega^\varepsilon} p \operatorname{div} \mathbf{v}, \\
485 \quad c_E(\mathbf{v}, (\lambda_{F,j})) &= \sum_{F \in \mathcal{E}(T)} \sum_{j=1}^s \lambda_{F,j} \int_{F \cap \Omega^\varepsilon} \mathbf{v} \cdot \boldsymbol{\omega}_{F,j}, \\
486 \quad d_T(\mathbf{v}, (\mu_{T,l})) &= \sum_{l=1}^r \mu_{T,l} \int_{T \cap \Omega^\varepsilon} \mathbf{v} \cdot \boldsymbol{\varphi}_{T,l}, \\
487
\end{aligned}$$

and where the linear forms ℓ_1 and ℓ_2 are defined by

$$\begin{aligned}
&\ell_1((\mu_{F,j})) = \mu_{E,i} \quad , \quad \ell_2((\mu_{T,l})) = 0 \quad \text{for (3.38) and} \\
&\ell_1((\mu_{F,j})) = 0 \quad , \quad \ell_2((\mu_{T,l})) = \mu_{T,k} \quad \text{for (3.39).}
\end{aligned}$$

488 System (3.40)–(3.43) is called a twofold saddle point problem in the literature and
489 can be viewed as a single saddle point problem defined on $V_T \times (M_H^0(T) \times \mathbb{R}^{n_s} \times \mathbb{R}^r)$.
490 By introducing the following bilinear form

$$\begin{aligned}
491 \quad \forall (\mathbf{v}, (p, (\lambda_{F,j}), (\lambda_{T,l}))) &\in V_T \times (M_H^0(T) \times \mathbb{R}^{n_s} \times \mathbb{R}^r) \\
492 \quad \tilde{b}_T(\mathbf{v}, (p, (\lambda_{F,j}), (\lambda_{T,l}))) &= b_T(\mathbf{v}, p) + c_E(\mathbf{v}, (\lambda_{F,j})) + d_T(\mathbf{v}, (\lambda_{T,l})),
\end{aligned}$$

494 system (3.40)–(3.43) can be reformulated as

$$\begin{aligned}
495 \quad (3.44) \quad \widehat{a}_T(\mathbf{u}, \mathbf{v}) + \tilde{b}_T(\mathbf{v}, (p, (\lambda_{F,j}), (\lambda_{T,l}))) &= 0, \quad \forall \mathbf{v} \in V_T, \\
496 \quad (3.45) \quad \tilde{b}_T(\mathbf{u}, (q, (\mu_{F,j}), (\mu_{T,l}))) &= \ell_1((\mu_{F,j})) + \ell_2((\mu_{T,l})), \\
497 \quad \forall (q, (\mu_{F,j}), (\mu_{T,l})) &\in M_H^0(T) \times \mathbb{R}^{n_s} \times \mathbb{R}^r.
\end{aligned}$$

499 Now we can apply the inf-sup theory to prove the well-posedness of (3.44)–(3.45).
500 It is easy to check that the bilinear forms \widehat{a}_T and \tilde{b}_T are both continuous over their
501 spaces of definition. Additionally, the linear forms ℓ_1 and ℓ_2 are also continuous.
502 Moreover, the bilinear form \widehat{a}_T is coercive over the velocity kernel of the bilinear form
503 \tilde{b}_T , since this kernel, for all $n \geq 0$, contains vector functions which, at least, have
504 mean values that vanish on the edges (or faces) of T , and this implies a Poincaré-like
505 inequality.

506 Finally, it remains only to prove that the bilinear form $\tilde{b}_T(\cdot, \cdot)$ satisfies the fol-
507 lowing inf-sup condition: there exists $\tilde{\alpha} > 0$ such that $\forall (p, (\lambda_{F,j}), (\lambda_{T,l})) \in M_H^0(T) \times$
508 $\mathbb{R}^{n_s} \times \mathbb{R}^r$

$$509 \quad (3.46) \quad \sup_{\mathbf{v} \in V_T} \frac{\tilde{b}_T(\mathbf{v}, (p, (\lambda_{F,j}), (\lambda_{T,l})))}{\left(\|p\|_{L^2(T \cap \Omega^\varepsilon)} + \|(\lambda_{F,j})\| + \|(\lambda_{T,l})\| \right) \|\mathbf{v}\|_{H^1(T \cap \Omega^\varepsilon)}} \geq \tilde{\alpha}.$$

511 To check the inf-sup condition for $\tilde{b}_T(\cdot, \cdot)$, for any $p \in M_H^0(T)$, $(\lambda_{F,j}) \in \mathbb{R}^{n_s}$ and
512 $(\lambda_{T,l}) \in \mathbb{R}^r$, we would like to find a velocity field $\mathbf{v} \in V_T$ and a polynomial $q \in \mathbb{P}_n(T)$
513 such that

$$514 \quad (3.47) \quad - \operatorname{div} \mathbf{v} = p + q,$$

$$515 \quad (3.48) \quad \int_{F \cap \Omega^\varepsilon} \mathbf{v} \cdot \boldsymbol{\omega}_{F,j} = \lambda_{F,j}, \quad \forall F \in \mathcal{E}(T), j = 1, \dots, s,$$

$$516 \quad (3.49) \quad \int_{T \cap \Omega^\varepsilon} \mathbf{v} \cdot \boldsymbol{\varphi}_{T,l} = \lambda_{T,l}, \quad \forall l = 1, \dots, r.$$

517 Indeed, in that case we would have (since $(p, q)_{T \cap \Omega^\varepsilon} = 0$)

$$518 \quad (3.50) \quad \tilde{b}_T(\mathbf{v}, (p, (\lambda_{F,j}), (\lambda_{T,l}))) = \|p\|_{L^2(T \cap \Omega^\varepsilon)}^2 + \|(\lambda_{F,j})\|^2 + \|(\lambda_{T,l})\|^2.$$

519 Note that it is necessary to include a polynomial $q \in \mathbb{P}_n(T)$ in (3.47); if not, for any
520 $q' \in \mathbb{P}_n(T)$, we would have

$$521 \quad (3.51) \quad (\operatorname{div} \mathbf{v}, q') = 0 = \sum_F \int_{F \cap \Omega^\varepsilon} \mathbf{v} \cdot (q' \mathbf{n}) - \int_{T \cap \Omega^\varepsilon} \mathbf{v} \cdot \nabla q'.$$

522 Since $(q' \mathbf{n})|_F \in \operatorname{span}\{\boldsymbol{\omega}_{F,j}, j = 1, \dots, s\}$ and $\nabla q' \in \operatorname{span}\{\boldsymbol{\varphi}_{T,l}, l = 1, \dots, r\}$, then
523 (3.51) with (3.48) and (3.49) would imply that the sets $(\lambda_{F,j})_{F \in \mathcal{E}(T), j=1, \dots, s}$ and
524 $(\lambda_{T,l})_{l=1, \dots, r}$ could not be arbitrary.

525 Using functions (3.17) and (3.18), we may look for \mathbf{v} under the following form

$$526 \quad (3.52) \quad \mathbf{v} = \sum_{F \in \mathcal{E}(T)} \sum_{j=1}^s \lambda_{F,j} \mathbf{v}_{F,j} + \sum_{l=1}^r \lambda_{T,l} \mathbf{v}_{T,l} + \mathbf{w}$$

527 with $\mathbf{w} \in V_T$ verifying

$$528 \quad (3.53) \quad -\operatorname{div} \mathbf{w} = \tilde{p},$$

$$529 \quad (3.54) \quad \int_{F \cap \Omega^\varepsilon} \mathbf{w} \cdot \boldsymbol{\omega}_{F,j} = 0, \forall F \in \mathcal{E}(T), j = 1, \dots, s,$$

$$530 \quad (3.55) \quad \int_{T \cap \Omega^\varepsilon} \mathbf{w} \cdot \boldsymbol{\varphi}_{T,l} = 0, \forall l = 1, \dots, r.$$

where

$$\tilde{p} = p - (a - \Pi_n a),$$

and Π_n is the $L^2(T \cap \Omega^\varepsilon)$ orthogonal projection on $\mathbb{P}_n(T)$ and a is defined by

$$a = - \sum_{F \in \mathcal{E}(T)} \sum_{j=1}^s \lambda_{F,j} \operatorname{div} \mathbf{v}_{F,j} - \sum_{l=1}^r \lambda_{T,l} \operatorname{div} \mathbf{v}_{T,l}.$$

531 Moreover, in (3.47), we chose $q = \Pi_n a$. Note that \tilde{p} is orthogonal to \mathbb{P}_n since this is
532 the case for both p and $(a - \Pi_n a)$.

533 The standard inf-sup condition (with constant $\beta > 0$) is equivalent to the fact
534 that for any $\tilde{q} \in L_0^2(T \cap \Omega^\varepsilon)$, there exists $\mathbf{w}_0 \in (H_0^1(T \cap \Omega^\varepsilon))^d \subset V_T$ such that
535 $\operatorname{div} \mathbf{w}_0 = \tilde{q}$ and $\|\mathbf{w}_0\|_{H^1(T \cap \Omega^\varepsilon)} \leq \beta \|\tilde{q}\|_{L^2(T \cap \Omega^\varepsilon)}$. We pick such a \mathbf{w}_0 for the particular
536 case $\tilde{q} = \tilde{p}$ since \tilde{p} is orthogonal to \mathbb{P}_n and thus belongs to $L_0^2(T \cap \Omega^\varepsilon)$. Since \tilde{p}
537 depends continuously on p and a and since a depends continuously on the sets of $\lambda_{F,j}$
538 and $\lambda_{T,l}$ (recall that $\mathbf{v}_{F,j}$ and $\mathbf{v}_{T,l}$ are given functions), then \mathbf{w}_0 depends continuously
539 on p and on the sets of $\lambda_{F,j}$ and $\lambda_{T,l}$. Moreover, such a \mathbf{w}_0 automatically verifies
540 boundary conditions (3.54) since it vanishes on the boundary. The sequel of the proof
541 is performed for the case $d = 3$; the case $d = 2$ can be performed in a similar way, the
542 only difference being that there are two curl operators when $d = 2$, one that applies
543 on vector fields and returns a scalar field, and the other that acts conversely. Let \mathcal{G}
544 be a given open ball included in $T \cap \Omega^\varepsilon$, we shall look for \mathbf{w} under the following form:

$$545 \quad (3.56) \quad \mathbf{w} = \mathbf{w}_0 + \nabla \times \mathbf{z},$$

546 in which \mathbf{z} will be a smooth function compactly supported in \mathcal{G} (and thus $\nabla \times \mathbf{z}$ will
547 be in $(H_0^1(T \cap \Omega^\varepsilon))^d$) such that

$$548 \quad (3.57) \quad \int_{T \cap \Omega^\varepsilon} \nabla \times \mathbf{z} \cdot \boldsymbol{\varphi}_{T,l} = \int_{\mathcal{G}} \mathbf{z} \cdot \nabla \times \boldsymbol{\varphi}_{T,l} = - \int_{T \cap \Omega^\varepsilon} \mathbf{w}_0 \cdot \boldsymbol{\varphi}_{T,l}, \forall l = 1, \dots, r.$$

549 Let us now prove that it is possible to construct such a function \mathbf{z} . We start by
550 proving that the equalities in (3.57) will be verified for all $l = 1, \dots, r$ if they are
551 verified on a subset of indices. For this, let us consider the curl operator

$$552 \quad \nabla \times : (\mathbb{P}_{n-1})^d \mapsto (\mathbb{P}_{n-2})^d$$

553 and denote by $\mathcal{R}(\nabla \times)$ its range and $r_c := \dim \mathcal{R}(\nabla \times)$ the dimension of this range.
554 Like for any linear operator in finite dimensions, it holds that:

$$555 \quad (3.58) \quad (\mathbb{P}_{n-1})^d = \text{Ker}(\nabla \times) \oplus \text{Span}(\boldsymbol{\varphi}_{T,l_k}),$$

556 where $(\boldsymbol{\varphi}_{T,l_k})_{1 \leq k \leq r_c}$ is a subset of $(\boldsymbol{\varphi}_{T,l})_{1 \leq l \leq r}$ such that $(\nabla \times \boldsymbol{\varphi}_{T,l_k})_{1 \leq k \leq r_c}$ is a basis
557 of $\mathcal{R}(\nabla \times)$. We now prove that if the equalities in (3.57) are verified for all $l = l_k$,
558 with $1 \leq k \leq r_c$, then they will be verified for all $1 \leq l \leq r$. Indeed, taking into
559 account (3.58) and since it can easily be proved that $\text{Ker}(\nabla \times) = \nabla(\mathbb{P}_n)$, then if $l \neq l_k$
560 it holds that there exists $r \in \mathbb{P}_n$ and a set of real numbers (α_k) such that

$$561 \quad \boldsymbol{\varphi}_{T,l} = \nabla r + \sum_{1 \leq k \leq r_c} \alpha_k \boldsymbol{\varphi}_{T,l_k}.$$

562 According to this decomposition, we have on the one hand

$$563 \quad \int_{\mathcal{G}} \mathbf{z} \cdot \nabla \times \boldsymbol{\varphi}_{T,l} = \sum_{1 \leq k \leq r_c} \alpha_k \int_{\mathcal{G}} \mathbf{z} \cdot \nabla \times \boldsymbol{\varphi}_{T,l_k} = - \sum_{1 \leq k \leq r_c} \alpha_k \int_{T \cap \Omega^\varepsilon} \mathbf{w}_0 \cdot \boldsymbol{\varphi}_{T,l_k}$$

564 since (3.57) is assumed to hold true for all l_k , with $1 \leq k \leq r_c$. On the other hand,
565 we have

$$566 \quad - \int_{T \cap \Omega^\varepsilon} \mathbf{w}_0 \cdot \boldsymbol{\varphi}_{T,l} = - \int_{T \cap \Omega^\varepsilon} \mathbf{w}_0 \cdot \nabla r - \sum_{1 \leq k \leq r_c} \alpha_k \int_{T \cap \Omega^\varepsilon} \mathbf{w}_0 \cdot \boldsymbol{\varphi}_{T,l_k}$$

567 so that (3.57) will hold for the index l if $\int_{T \cap \Omega^\varepsilon} \mathbf{w}_0 \cdot \nabla r$ vanishes for any $r \in \mathbb{P}_n$. This
568 is indeed the case since

$$569 \quad \int_{T \cap \Omega^\varepsilon} \mathbf{w}_0 \cdot \nabla r = - \int_{T \cap \Omega^\varepsilon} \text{div } \mathbf{w}_0 r = - \int_{T \cap \Omega^\varepsilon} \tilde{p} r = 0$$

570 because \tilde{p} is orthogonal to \mathbb{P}_n and $r \in \mathbb{P}_n$.

571 Setting $\boldsymbol{\psi}_k := \nabla \times \boldsymbol{\varphi}_{T,l_k}$, the conclusion of the first part of this proof is that it
572 is sufficient to construct a function \mathbf{z} , regular enough and compactly supported in \mathcal{G}
573 such that (3.57) is satisfied for all indices l_k , namely

$$574 \quad (3.59) \quad \int_{\mathcal{G}} \mathbf{z} \cdot \boldsymbol{\psi}_k = - \int_{T \cap \Omega^\varepsilon} \mathbf{w}_0 \cdot \boldsymbol{\varphi}_{T,l_k}, \forall k = 1, \dots, r_c.$$

575 Choosing a positive non-vanishing smooth scalar function θ with compact support
576 in \mathcal{G} , we look for the function \mathbf{z} under the form

$$577 \quad \mathbf{z} = \left(\sum_{1 \leq i \leq r_c} z_i \boldsymbol{\psi}_i \right) \theta,$$

578 with a set of real numbers (z_i) with $i = 1, \dots, r_c$. Then, (3.59) reduces to finding a
579 vector (z_i) such that

$$580 \quad (3.60) \quad \sum_{1 \leq i \leq r_c} \left(\int_{\mathcal{G}} \theta \psi_i \cdot \psi_k \right) z_i = - \int_{T \cap \Omega^\varepsilon} \mathbf{w}_0 \cdot \boldsymbol{\varphi}_{T,l_k}, \forall k = 1, \dots, r_c.$$

581 This is a square linear system of size r_c which has a unique solution since

$$582 \quad (\mathbf{p}, \mathbf{q}) \mapsto \int_{\mathcal{G}} \theta \mathbf{p} \cdot \mathbf{q}$$

583 is a scalar product on $\mathcal{R}(\nabla \times)$ and (ψ_k) a basis of that space. Moreover, since this is
584 a finite dimensional linear system, the dependence of the coefficients z_i , and thus of
585 the function $\nabla \times \mathbf{z}$ with respect to the right-hand side in (3.60), and thus with respect
586 to \mathbf{w}_0 , is continuous.

587 As a conclusion, we have constructed a function \mathbf{w} under the form (3.56) which
588 verifies (3.53)–(3.54)–(3.55). We have shown that, by construction, both \mathbf{w}_0 and $\nabla \times \mathbf{z}$
589 depend linearly and continuously on p , $\lambda_{F,j}$ and $\lambda_{T,l}$. Therefore, this is also the case
590 for \mathbf{v} defined from \mathbf{w} by (3.52). Thus from (3.50), we conclude that (3.46) holds with
591 a constant $\tilde{\alpha} > 0$. This continuity constant may of course depend on ε , in a way which
592 is out of the scope of this article.

593 3.5. Basis of the space V_H .

594 **THEOREM 3.14.** *The functions $\Phi_{E,i}$ for $E \in \mathcal{E}_H$ and $i = 1, \dots, s$ defined by their*
595 *restrictions (3.36) over each element T_k whose boundary includes E and $\Psi_{T,k}$ for*
596 *$T \in \mathcal{T}_H$ and $k = 1, \dots, r$ defined by (3.37) form a basis of V_H defined by (3.9). In*
597 *other words,*

$$598 \quad (3.61) \quad V_H = \text{span} \{ \Phi_{E,i}, \Psi_{T,k}, E \in \mathcal{E}_H, T \in \mathcal{T}_H, i = 1, \dots, s, k = 1, \dots, r \}$$

600 and $\{ \Phi_{E,i}, E \in \mathcal{E}_H, i = 1, \dots, s \} \cup \{ \Psi_{T,k}, T \in \mathcal{T}_H, k = 1, \dots, r \}$ forms a linearly
601 independent family.

Proof. First of all, it is easy to verify that

$$\{ \Phi_{E,i}, E \in \mathcal{E}_H, i = 1, \dots, s \} \cup \{ \Psi_{T,k}, T \in \mathcal{T}_H, k = 1, \dots, r \}$$

602 forms a linearly independent family.

603 Then, let us prove (3.61). It is obvious that functions $\Phi_{E,i}$, for $E \in \mathcal{E}_H$ and
604 $i = 1, \dots, s$ defined by (3.36) and $\Psi_{T,k}$ for $T \in \mathcal{T}_H$ and $k = 1, \dots, r$ defined by (3.37)
605 all belong to V_H . Consequently, we have

$$606 \quad \text{span} \{ \Phi_{E,i}, \Psi_{T,k}, E \in \mathcal{E}_H, T \in \mathcal{T}_H, i = 1, \dots, s, k = 1, \dots, r \} \subset V_H$$

608 Reciprocally, let $\mathbf{u} \in V_H$; definition of V_H implies that on each $T \in \mathcal{T}_H$, there
609 exist a unique $\zeta^T \in M_H^0(T)$ and a unique vector $(\lambda_{T,1}, \dots, \lambda_{T,r}) \in \mathbb{R}^r$ that satisfy
610 (3.28)–(3.31). We introduce \mathbf{v} and σ as follows:

$$611 \quad \mathbf{v} = \mathbf{u} - \sum_{T \in \mathcal{T}_H} \sum_{k=1}^r \left(\int_{T \cap \Omega^\varepsilon} \mathbf{u} \cdot \boldsymbol{\varphi}_{T,k} \right) \Psi_{T,k} - \sum_{E \in \mathcal{E}_H} \sum_{i=1}^s \left(\int_{E \cap \Omega^\varepsilon} \mathbf{u} \cdot \boldsymbol{\omega}_{E,i} \right) \Phi_{E,i},$$

$$612 \quad \sigma = \sum_{T \in \mathcal{T}_H} \zeta^T - \sum_{T \in \mathcal{T}_H} \sum_{k=1}^r \left(\int_{T \cap \Omega^\varepsilon} \mathbf{u} \cdot \boldsymbol{\varphi}_{T,k} \right) \pi_{T,k} - \sum_{E \in \mathcal{E}_H} \sum_{i=1}^s \left(\int_{E \cap \Omega^\varepsilon} \mathbf{u} \cdot \boldsymbol{\omega}_{E,i} \right) \pi_{E,i}.$$

613

614 For any $T \in \mathcal{T}_H$, it is easy to check that the restrictions of \mathbf{v} and σ to T verify:

615 (3.62) $-\mu\Delta\mathbf{v} + \nabla\sigma \in \text{span}\{\boldsymbol{\varphi}_{T,1}, \dots, \boldsymbol{\varphi}_{T,r}\}$ in $T \cap \Omega^\varepsilon$

616 (3.63) $\text{div } \mathbf{v} \in \text{span}\{\boldsymbol{\varpi}_{T,1}, \dots, \boldsymbol{\varpi}_{T,t}\}$ in $T \cap \Omega^\varepsilon$

617 (3.64) $\mu\nabla\mathbf{v}\mathbf{n} - \sigma\mathbf{n} \in \text{span}\{\boldsymbol{\omega}_{F,1}, \dots, \boldsymbol{\omega}_{F,s}\}$ on $F \cap \Omega^\varepsilon$, $\forall F \in \mathcal{E}(T)$
 618 $\mathbf{v} = \mathbf{0}$ on $\partial B^\varepsilon \cap T$

619 (3.65) $\int_{E \cap \Omega^\varepsilon} \mathbf{v} \cdot \boldsymbol{\omega}_{E,i} = 0 \forall E \in \mathcal{E}(T), \forall i = 1, \dots, s$

620 (3.66) $\int_{T \cap \Omega^\varepsilon} \mathbf{v} \cdot \boldsymbol{\varphi}_{T,l} = 0 \forall l = 1, \dots, r$

621 (3.67) $\int_{T \cap \Omega^\varepsilon} \sigma \cdot \boldsymbol{\varpi}_{T,j} = 0 \forall j = 1, \dots, t$
 622

623 Equation (3.62) implies that there exists $(\lambda_{T,1}, \dots, \lambda_{T,r}) \in \mathbb{R}^r$ such that

624 (3.68) $-\mu\Delta\mathbf{v} + \nabla\sigma = \sum_{l=1}^r \lambda_{T,l} \boldsymbol{\varphi}_{T,l}$
 625

626 Choosing \mathbf{v} as the test function in the variational formulation of (3.68) implies
 627 that

628 (3.69) $\int_{T \cap \Omega^\varepsilon} \mu|\nabla\mathbf{v}|^2 - \int_{T \cap \Omega^\varepsilon} \sigma \text{div } \mathbf{v} = \sum_{l=1}^r \lambda_{T,l} \int_{T \cap \Omega^\varepsilon} \boldsymbol{\varphi}_{T,l} \cdot \mathbf{v} + \int_{\partial(T \cap \Omega^\varepsilon)} (\mu\nabla\mathbf{v}\mathbf{n} - \sigma\mathbf{n}) \cdot \mathbf{v}$
 629

630 It is straightforward to deduce from (3.63) and (3.67) that

631 $\int_{T \cap \Omega^\varepsilon} \sigma \text{div } \mathbf{v} = 0$
 632

633 Then equation (3.66) implies that

634 $\sum_{l=1}^r \lambda_{T,l} \int_{T \cap \Omega^\varepsilon} \boldsymbol{\varphi}_{T,l} \cdot \mathbf{v} = 0$
 635

636 Combining equations (3.64) and (3.65), we can deduce that

637 $\int_{\partial(T \cap \Omega^\varepsilon)} (\mu\nabla\mathbf{v}\mathbf{n} - \sigma\mathbf{n}) \cdot \mathbf{v} = 0$
 638

639 Finally, equation (3.69) reduces to

640 $\int_{T \cap \Omega^\varepsilon} |\nabla\mathbf{v}|^2 = 0$,
 641

642 which implies that \mathbf{v} is constant on $T \cap \Omega^\varepsilon$. With (3.66), we deduce that $\mathbf{v} = \mathbf{0}$.

643 Thus we have proved that

644 $V_H \subset \text{span}\{\boldsymbol{\Phi}_{E,i}, \boldsymbol{\Psi}_{T,k}, E \in \mathcal{E}_H, T \in \mathcal{T}_H, i = 1, \dots, s, k = 1, \dots, r\}$.

645 Consequently, combing the results above, we have proved that

646 $V_H = \text{span}\{\boldsymbol{\Phi}_{E,i}, \boldsymbol{\Psi}_{T,k}, E \in \mathcal{E}_H, T \in \mathcal{T}_H, i = 1, \dots, s, k = 1, \dots, r\}$. \square
 647

649 We conclude that any function V_H can be represented as

$$650 \quad \forall \mathbf{u}_H \in V_H, \quad \mathbf{u}_H = \sum_{E \in \mathcal{E}_H} \sum_{i=1}^s u_{E,i} \Phi_{E,i} + \sum_{T \in \mathcal{T}_H} \sum_{k=1}^r u_{T,k} \Psi_{T,k}.$$

652 An explicit formulation of the linear operator π_H (see [Lemma 3.11](#)) is

$$653 \quad (3.70) \quad \forall \mathbf{u}_H \in V_H, \quad \pi_H(\mathbf{u}_H) = \sum_{E \in \mathcal{E}_H} \sum_{i=1}^s u_{E,i} \pi_{E,i} + \sum_{T \in \mathcal{T}_H} \sum_{k=1}^r u_{T,k} \pi_{T,k}.$$

655 **3.6. Decomposition of the space X_H^{ext} .** [Subsection 3.2](#) results in the following
656 decomposition.

657 **THEOREM 3.15.** *The space X_H^{ext} can be decomposed as:*

$$658 \quad X_H^{ext} = X_H \oplus X_H^0,$$

660 where X_H^0 is defined in [Definition 3.5](#) and X_H is defined in [Definition 3.6](#).

661 *Proof.* This theorem is proved in two steps:

$$662 \quad (i) \ X_H^{ext} = X_H + X_H^0, \quad (ii) \ X_H \cap X_H^0 = \{0\}.$$

664 We start by proving step (i). First, by definition X_H and X_H^0 are both subsets of
665 X_H^{ext} . Next, we prove that $X_H^{ext} \subset X_H + X_H^0$, i.e. for any $(\mathbf{u}, p) \in X_H^{ext}$, there exist
666 $(\mathbf{u}_H, p_H) \in X_H$, $\mathbf{u}^0 \in V_H^0$ and $p^0 \in M_H^0$ such that

$$667 \quad (3.71) \quad \mathbf{u} = \mathbf{u}_H + \mathbf{u}^0, \quad p = p_H + p^0.$$

669 In order to prove this, we consider a given $(\mathbf{u}, p) \in X_H^{ext}$ and we first pick any
670 triangle $T \in \mathcal{T}_H$; then we consider the velocity space $V_{f_0}(T)$ defined by [\(3.20\)](#), we
671 recall the definition of $M_H^0(T)$ by [\(3.6\)](#) and we consider the following problem: Find
672 $(\mathbf{u}^0, p^0) \in V_{f_0}(T) \times M_H^0(T)$ such that

$$673 \quad (3.72) \quad \int_{T \cap \Omega^\varepsilon} (\mu \nabla \mathbf{u}^0 : \nabla \mathbf{v} - p^0 \operatorname{div} \mathbf{v}) = \int_{T \cap \Omega^\varepsilon} (\mu \nabla \mathbf{u} : \nabla \mathbf{v} - p \operatorname{div} \mathbf{v}), \quad \forall \mathbf{v} \in V_{f_0}(T)$$

$$674 \quad (3.73) \quad \int_{T \cap \Omega^\varepsilon} q \operatorname{div} \mathbf{u}^0 = \int_{T \cap \Omega^\varepsilon} q \operatorname{div} \mathbf{u}, \quad \forall q \in M_H^0(T).$$

676 This problem has a unique solution because:

- 677 (a) $(\mathbf{u}, \mathbf{v}) \mapsto \int_{T \cap \Omega^\varepsilon} (\mu \nabla \mathbf{u} : \nabla \mathbf{v})$ is coercive on $V_{f_0}(T)$,
- 678 (b) the existence of a function $\mathbf{w} \in V_T$ satisfying [\(3.53\)](#)–[\(3.54\)](#)–[\(3.55\)](#) with a linear
679 and continuous dependence with respect to an arbitrary $\tilde{p} \in M_H^0(T)$ proves the inf-sup
680 condition for the divergence operator over the couple of spaces $V_{f_0}(T) \times M_H^0(T)$.

681 By gluing together (\mathbf{u}^0, p^0) on each element $T \in \mathcal{T}_H$, we obtain that $(\mathbf{u}^0, p^0) \in$
682 $V_H^0 \times M_H^0$. Consequently, we have proved that for any $(\mathbf{u}, p) \in X_H^{ext}$, there exists
683 $(\mathbf{u}^0, p^0) \in X_H^0$ such that

$$684 \quad c_H((\mathbf{u}^0, p^0), (\mathbf{v}, q)) = c_H((\mathbf{u}, p), (\mathbf{v}, q)), \quad \forall (\mathbf{v}, q) \in V_H^0 \times M_H^0.$$

686 Thus, defining \mathbf{u}_H and p_H by

$$687 \quad \mathbf{u}_H = \mathbf{u} - \mathbf{u}^0, \quad p_H = p - p^0,$$

689 we get that (\mathbf{u}_H, p_H) satisfies

$$690 \quad c_H((\mathbf{u}_H, p_H), (\mathbf{v}, q)) = 0, \quad \forall (\mathbf{v}, q) \in V_H^0 \times M_H^0$$

692 and we deduce by [Definition 3.6](#) that $(\mathbf{u}_H, p_H) \in X_H$. Finally, we conclude that
693 $X_H^{ext} = X_H + X_H^0$.

694 Now it remains to prove step (ii), namely that $X_H \cap X_H^0 = \{0\}$. Let $(\mathbf{u}, p) \in X_H \cap$
695 X_H^0 , then this implies that for each $T \in \mathcal{T}_H$, $p|_T$ belongs at the same time to $\mathbb{P}_n(T)$ ac-
696 cording to [\(3.8\)](#), and to its orthogonal according to definition [\(3.6\)](#) and [Hypothesis 3.1](#).
697 This implies that $p = 0$. Next, [Definition 3.6](#), implies that $c_H((\mathbf{u}, p), (\mathbf{u}, p)) = 0$,
698 which reduces to $\int_{T \cap \Omega^\varepsilon} \mu \nabla \mathbf{u} : \nabla \mathbf{u} = 0$, leading to $\mathbf{u} = \mathbf{0}$ which finishes this part of
699 the proof. \square

700 **3.7. Coarse-scale problem.** The coarse-scale formulation of the Stokes prob-
701 lem [\(2.1\)–\(2.2\)](#) reads: find $(\mathbf{u}_H, p_H) \in X_H$ such that

$$702 \quad c_H((\mathbf{u}_H, p_H), (\mathbf{v}, q)) = (\mathbf{f}, \mathbf{v}), \quad \forall (\mathbf{v}, q) \in X_H.$$

704 [Theorem 3.12](#) implies that p_H can be decomposed as $p_H = \pi_H(\mathbf{u}_H) + \bar{p}_H$ with
705 $\pi_H(\mathbf{u}_H) \in M_H^0$ and $\bar{p}_H \in M_H$. It is easy to verify that $(\pi_H(\mathbf{u}_H), \operatorname{div} \mathbf{v}) = 0$ for all
706 $\mathbf{u}_H, \mathbf{v} \in V_H$. The problem above can thus be reformulated as: find $\mathbf{u}_H \in V_H$ and
707 $\bar{p}_H \in M_H$ such that

$$708 \quad (3.74) \quad a_H(\mathbf{u}_H, \mathbf{v}) + b_H(\mathbf{v}, \bar{p}_H) = F_H(\mathbf{v}), \quad \forall \mathbf{v} \in V_H,$$

$$709 \quad (3.75) \quad b_H(\mathbf{u}_H, q) = 0, \quad \forall q \in M_H,$$

711 where

$$712 \quad a_H(\mathbf{u}_H, \mathbf{v}) = \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} \mu \nabla \mathbf{u}_H : \nabla \mathbf{v},$$

$$713 \quad b_H(\mathbf{v}, \bar{p}_H) = - \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} \bar{p}_H \operatorname{div} \mathbf{v},$$

$$714 \quad F_H(\mathbf{v}) = \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} \mathbf{f} \cdot \mathbf{v}.$$

716 **THEOREM 3.16.** *The space V_H and the space M_H have the following relation*

$$717 \quad \operatorname{div}_H V_H = M_H,$$

where div_H is the broken divergence operator defined triangle by triangle:

$$(\operatorname{div}_H \mathbf{v})|_T = \operatorname{div} \mathbf{v}|_T.$$

719

Proof. We prove first that $\operatorname{div}_H V_H \subset M_H$. For any $\mathbf{v} \in V_H$, for any $T \in \mathcal{T}_H$, it holds that $\operatorname{div} \mathbf{v}|_T \in \mathbb{P}_n(T)$ thanks to [\(3.9\)](#) and [Hypothesis 3.1](#). It thus remains to prove that $\operatorname{div}_H \mathbf{v} \in M$, which reduces to prove that

$$\int_{\Omega^\varepsilon} \operatorname{div}_H \mathbf{v} = 0.$$

720 This is true. Indeed, since $V_H \subset V_H^{ext}$, definition of V_H^{ext} implies that $\int_{E \cap \Omega^\varepsilon} [[\mathbf{v}]] \cdot \mathbf{n} = 0$,
721 and we thus deduce that

$$722 \int_{\Omega^\varepsilon} \operatorname{div}_H \mathbf{v} = \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} \operatorname{div} \mathbf{v} = \sum_{T \in \mathcal{T}_H} \int_{\partial(T \cap \Omega^\varepsilon)} \mathbf{v} \cdot \mathbf{n} = \sum_{E \in \mathcal{E}_H} \int_{E \cap \Omega^\varepsilon} [[\mathbf{v} \cdot \mathbf{n}]] = 0.$$

724 Hence we have proved that $\operatorname{div}_H V_H \subset M_H$.

725 Reciprocally, we now prove that $M_H \subset \operatorname{div}_H V_H$. For any $q \in M_H$, definition of
726 M_H shows that $q \in L_0^2(\Omega)$. Thus, there exists $\mathbf{v} \in (H_0^1(\Omega^\varepsilon))^d$ such that $\operatorname{div} \mathbf{v} = q$.
727 Since $(H_0^1(\Omega^\varepsilon))^d \subset V_H^{ext}$, [Theorem 3.15](#) implies that \mathbf{v} can be decomposed as

$$728 \quad (3.76) \quad \mathbf{v} = \mathbf{v}_H + \mathbf{v}_H^0 \text{ with } \mathbf{v}_H \in V_H, \mathbf{v}_H^0 \in V_H^0.$$

730 For any element $T \in \mathcal{T}_H$, let $\varpi_T \in \mathbb{P}_n(T)$. Integration by parts gives:

$$731 \quad \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} \varpi_T \operatorname{div} \mathbf{v}_H = \sum_{T \in \mathcal{T}_H} \int_{\partial(T \cap \Omega^\varepsilon)} \varpi_T \mathbf{v}_H \cdot \mathbf{n} - \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} \mathbf{v}_H \cdot \nabla \varpi_T$$

733 Substituting \mathbf{v}_H defined in [\(3.76\)](#) into this equation, we obtain

$$734 \quad \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} \varpi_T \operatorname{div} \mathbf{v}_H = \sum_{T \in \mathcal{T}_H} \int_{\partial(T \cap \Omega^\varepsilon)} \varpi_T (\mathbf{v} - \mathbf{v}_H^0) \cdot \mathbf{n} \\ 735 \quad (3.77) \quad - \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} (\mathbf{v} - \mathbf{v}_H^0) \cdot \nabla \varpi_T$$

737 The fact that $\varpi_T \in \mathbb{P}_n(T)$ implies that $\mathbf{n} \varpi_T \in (\mathbb{P}_n(T))^d$ and $\nabla \varpi_T \in \mathbb{P}_{n-1}(T)$.
738 Then, definition of V_H^0 by [\(3.4\)](#), [Hypothesis 3.1](#) implies that

$$739 \quad \int_{\partial(T \cap \Omega^\varepsilon)} \mathbf{v}_H^0 \cdot \mathbf{n} \varpi_T = 0, \quad \int_{T \cap \Omega^\varepsilon} \mathbf{v}_H^0 \cdot \nabla \varpi_T = 0.$$

741 Finally [\(3.77\)](#) reduces to

$$742 \quad \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} \varpi_T \operatorname{div} \mathbf{v}_H = \sum_{T \in \mathcal{T}_H} \int_{\partial(T \cap \Omega^\varepsilon)} \varpi_T \mathbf{v} \cdot \mathbf{n} - \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} \mathbf{v} \cdot \nabla \varpi_T \\ 743 \quad = \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} \varpi_T \operatorname{div} \mathbf{v} \\ 744 \quad = \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} q \varpi_T.$$

746 Since $\mathbf{v}_H \in V_H$, we know by [\(3.9\)](#) that, on T , $\operatorname{div} \mathbf{v}_H \in \mathbb{P}_n(T)$. Moreover,
747 $q \in \mathbb{P}_n(T)$ and ϖ_T is arbitrary in $\mathbb{P}_n(T)$ in the above equality. Thus, we deduce
748 that $q|_T = (\operatorname{div} \mathbf{v}_H)|_T$ and so $q = \operatorname{div}_H \mathbf{v}_H$. Thus $M_H \subset \operatorname{div}_H V_H$. Consequently,
749 combining the results above, we conclude that $\operatorname{div}_H V_H = M_H$. \square

750 Making use of [Theorem 3.16](#), it is trivial to deduce from [\(3.75\)](#) that $\operatorname{div} \mathbf{u}_H = 0$
751 in $T \cap \Omega^\varepsilon$ for $T \in \mathcal{T}_H$. We can thus eliminate the pressure unknown from [\(3.74\)](#)–[\(3.75\)](#)
752 by introducing a subspace of V_H :

$$753 \quad Z_H = \{\mathbf{v} \in V_H \text{ such that } \operatorname{div}_H \mathbf{v} = 0, \forall T \in \mathcal{T}_H\}.$$

755 Therefore (3.74)–(3.75) is equivalent to: find $\mathbf{u}_H \in Z_H$ such that

$$756 \quad \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} \mu \nabla \mathbf{u}_H : \nabla \mathbf{v} = \sum_{T \in \mathcal{T}_H} \int_{T \cap \Omega^\varepsilon} \mathbf{f} \cdot \mathbf{v}, \quad \forall \mathbf{v} \in Z_H$$

758 The existence and uniqueness of a solution \mathbf{u}_H to this problem is guaranteed
759 by the coercivity of the bilinear form over V_H . Then the existence and uniqueness of
760 pressure \bar{p}_H follows from the fact that $\operatorname{div}_H V_H = M_H$. As a result, we have proved
761 that (3.74)–(3.75) has one and only one solution $(\mathbf{u}_H, \bar{p}_H) \in V_H \times M_H$.

762 To solve the coarse-scale problem on the coarse mesh, we approximate the non-
763 homogeneous Dirichlet boundary condition $\mathbf{u}_H = \mathbf{0}$ defined by (2.4) in a weak form:

$$764 \quad \int_{E \cap \partial \Omega} \mathbf{u}_H \cdot \boldsymbol{\omega}_{E,j} = 0, \quad \forall E \in \mathcal{E}_H \cap \partial \Omega, \quad j = 1, \dots, s.$$

766 **3.8. Reconstruction of fine-scale solutions.** After obtaining coarse solutions
767 $\mathbf{u}_H = (\mathbf{u}_{E,1}, \dots, \mathbf{u}_{E,s})_{E \in \mathcal{E}_H}$ and $\bar{p}_H = (\bar{p}_H|_T)_{T \in \mathcal{T}_H}$, we reconstruct on any coarse
768 element $T \in \mathcal{T}_H$ fine-scale solutions by:

$$769 \quad \mathbf{u}_H|_T = \sum_{E \in \mathcal{E}(T)} \sum_{i=1}^s u_{E,i} \boldsymbol{\Phi}_{E,i} + \sum_{k=1}^r u_{T,k} \boldsymbol{\Psi}_{T,k},$$

$$770 \quad p_H|_T = \sum_{E \in \mathcal{E}(T)} \sum_{i=1}^s u_{E,i} \pi_{E,i} + \sum_{k=1}^r u_{T,k} \pi_{T,k} + \bar{p}_H|_T.$$

772 **4. Implementation issues and comparisons to previous methods.** We
773 implement MsFEMs in a CFD software called *TrioCFD* [41] developed by the CEA. In
774 *TrioCFD*, incompressible Navier-Stokes equations are discretized by the finite volume
775 element method [6, 7], which is a certain combination of the finite element method
776 [22, 27] and the finite volume method [24, 42]. The main idea is to approximate
777 fluxes on the boundary of control volumes by replacing unknowns by a finite element
778 approximation. The finite volume element method consists of two meshes: a primal
779 mesh and a dual mesh. Equations are discretized by volume integrals on the dual
780 mesh and unknowns are discretized into a finite element space on the primal mesh.

781 In our work, we implement the CR4_high MsFEM where weighting functions are
782 defined by (3.3). Local problems (3.36)–(3.37) are solved by the finite volume element
783 method where the velocity and pressure are discretized by the \mathbb{P}_1 nonconforming/ \mathbb{P}_1
784 element. The coarse-scale problem is implemented using (3.74)–(3.75), with a stan-
785 dard orthogonal basis of the fully discontinuous $\mathbb{P}_n(T)$ space for the pressure and the
786 basis of V_H that was constructed in subsection 3.5 for the velocity. The global matrix
787 can be computed in parallel over multiple processors and remains sparse since the
788 supports of these basis functions are local.

789 We want to compare the accuracy of CR4_high to that of MsFEMs proposed
790 originally in [33, 40]. Thus we implement also the CR2 and CR3 MsFEMs defined
791 in [33]. The CR2 method is the original method of [40] and corresponds to the choice
792 $n = 0$ of the present article. The CR3 method is obtained by enriching the edge
793 weighting functions:

$$794 \quad \boldsymbol{\omega}_{E,1} = \mathbf{e}_1, \quad \boldsymbol{\omega}_{E,2} = \mathbf{e}_2, \quad \boldsymbol{\omega}_{E,3} = \mathbf{n}_E \psi_E,$$

796 where ψ_E is a non-vanishing function in $\mathbb{P}_1(E)$ with vanishing mean-value on $E \cap \Omega^\varepsilon$
797 and $(\mathbf{e}_1, \mathbf{e}_2)$ is the canonical basis of \mathbb{R}^2 . We also investigate the CR4 method which

798 enriches the CR3 method to reach the full $[\mathbb{P}_1(E)]^2$ weighting space on the edges:

799
$$\boldsymbol{\omega}_{E,1} = \mathbf{e}_1, \boldsymbol{\omega}_{E,2} = \mathbf{e}_2, \boldsymbol{\omega}_{E,3} = \mathbf{n}_E \psi_E, \boldsymbol{\omega}_{E,3} = \boldsymbol{\tau}_E \phi_E,$$

801 where $\mathbf{n}_E \psi_E, \boldsymbol{\tau}_E \phi_E$ are the same as those in (3.3). In all these methods, the pressure
 802 weighting function is limited to $\varpi_{T,1} = 1$, and we recall that $\boldsymbol{\varphi}_{T,k}$ for $k = 1, \dots, r$ are
 803 only defined in the enriched MsFEM and do not exist in CR2, CR3 and CR4. The idea
 804 in comparing CR4 and CR4_high is to show that enriching only the edge weighting
 805 functions $\boldsymbol{\omega}_{E,i}$ is not enough to significantly enhance the accuracy of the scheme. We
 806 also need to add constraints in the triangle in order to make the non-resolved space
 807 V_H^0 (3.4) smaller and smaller.

808 Unknowns in local problems of CR2, CR3 and CR4 are discretized by the \mathbb{P}_1
 809 nonconforming/ \mathbb{P}_0 element. Reference solutions are computed by the same finite
 810 volume element method. Instead of using the penalization technique and Cartesian
 811 meshes as in [33, 40], we impose the no-slip condition directly on obstacles and numerical
 812 simulations are performed on body-fitted unstructured meshes.

813 Enriching the basic CR2 method by additional coarse basis functions rather than
 814 using finer meshes is an alternative to reduce the computational error, just like p
 815 enrichment is an alternative to h refinement in standard finite elements. Both ap-
 816 proaches imply growing costs which have to be compared to guide a choice between
 817 both approaches. In order to detail this (for $d = 2$), let us consider a coarse mesh
 818 made up of n_T triangles with n_E edges. Then, Table 1 displays the number of un-
 819 knowns that will be necessary for the different methods; for the sake of comparison,
 the asymptotic relationship $n_E \approx \frac{3}{2}n_T$ will be used. We conclude from this table that

Table 1: Number of unknowns in the different methods

Field	Method			
	CR2	CR3	CR4	CR4_high
Pressure	n_T	n_T	n_T	$3n_T$
Velocity	$2n_E$	$3n_E$	$4n_E$	$4n_E + 2n_T$
Total	$n_T + 2n_E$	$n_T + 3n_E$	$n_T + 4n_E$	$5n_T + 4n_E$
Asympt. total	$4n_T$	$5.5n_T$	$7n_T$	$11n_T$

820 using CR4_high is worth the additional complexity as soon as it is at least $\frac{11}{4}$ times
 821 more accurate than CR2, or twice as accurate as CR3, on a given mesh.
 822

823 **5. Numerical simulations.** We consider Stokes flows in a two-dimensional
 824 open channel $\Omega = [0 \leq x \leq 2, 0 \leq y \leq 1]$, where the heterogeneity is represented by
 825 arbitrary placements of solid obstacles. Three-dimensional numerical simulations were
 826 also carried out with CR2 and CR3 and can be found in [25]. As shown in Figure 3,
 827 case A contains 26 obstacles of width $\varepsilon = 0.015$ and case B contains 100 obstacles
 828 of width $\varepsilon = 0.003$. Case B corresponds to a denser distribution of obstacles and
 829 is similar somehow to a homogenization setting, while case A with fewer and bigger
 830 obstacles is not so close to an asymptotic homogenized regime. We assign $\mu = 1$ and
 831 $\mathbf{f} = \mathbf{0}$. The parabolic inflow boundary condition $\mathbf{u} = y(1 - y)\mathbf{e}_1$ is imposed at the
 832 inlet. The Neumann boundary condition $\mu \nabla \mathbf{u} \cdot \mathbf{n} - p\mathbf{n} = 0$ is imposed at the outlet
 833 and the no-slip condition is applied on other boundaries. In numerical convergence
 834 studies, we fix the element size h of fine meshes and vary only the element size H of
 835 coarse meshes. We ensure that $h \ll \varepsilon < H$. The element size h is small enough to

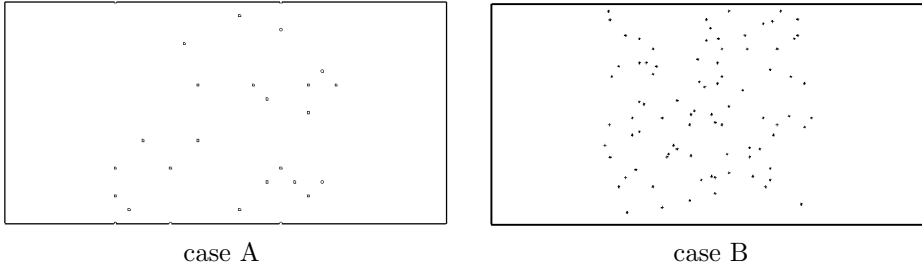


Fig. 3: Two computational domains with square obstacles

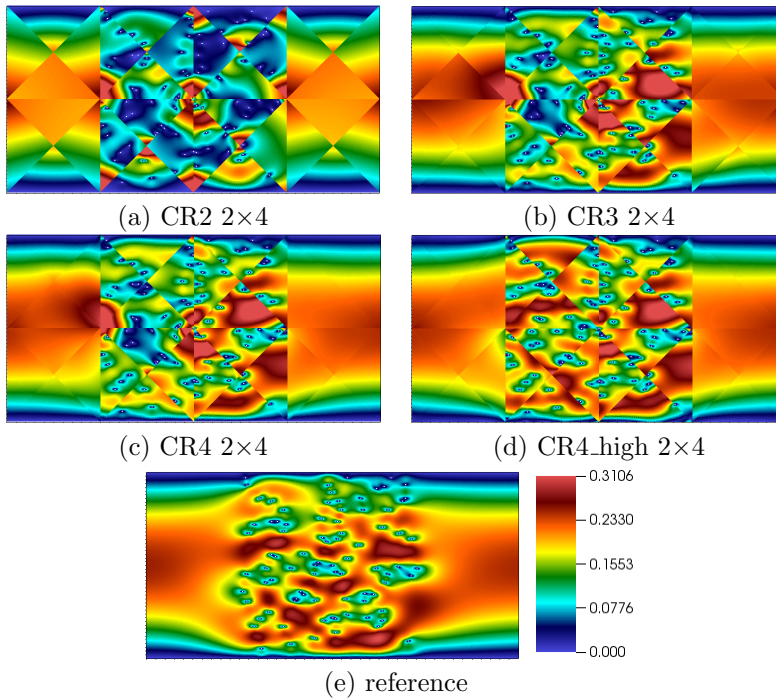


Fig. 4: $|\mathbf{u}|$ contours of Stokes flows in case B computed on the coarse mesh 2×4

836 well capture small obstacles in the media. We are interested only in the case in which
837 $H > \varepsilon$, since the opposite case $H < \varepsilon$ is covered by classical finite element methods
838 and MsFEMs are not needed.

839 For case B, Figure 4, Figure 5 and Figure 6 show the velocity contours computed
840 respectively by CR2, CR3, CR4 and CR4_high on various coarse meshes as well as the
841 reference solution. Figure 7, Figure 8 and Figure 9 show the pressure computed by
842 CR2, CR3, CR4 and CR4_high on various coarse meshes and the reference pressure.
843 The reference solution is computed on a mesh containing 2,000,000 elements with an
844 average element size $h = 0.00025$. Globally, we observe that the velocity and pressure
845 computed by MsFEMs converge to the reference solutions. It is shown that CR3 is
846 much more accurate than CR2 whereas CR4 is only slightly better than CR3. It is

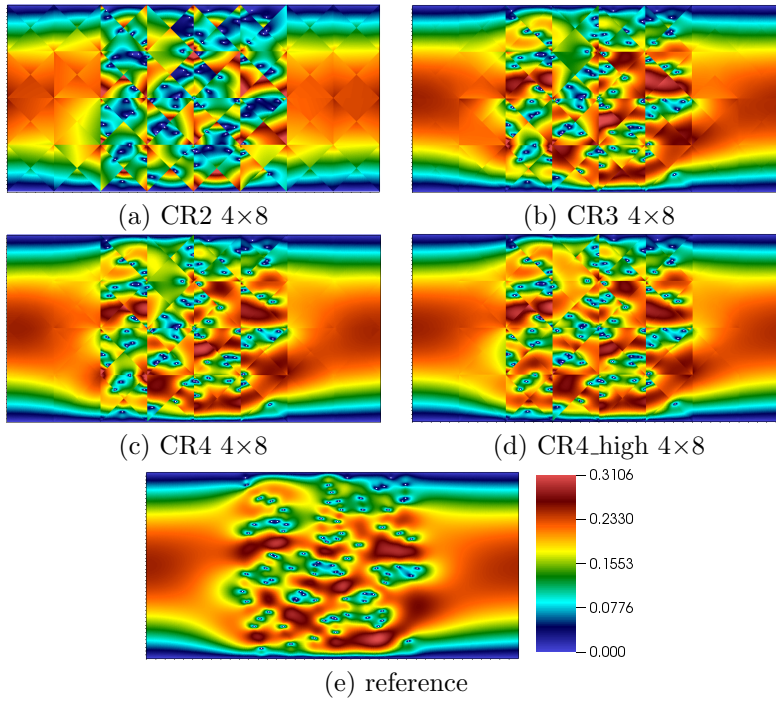


Fig. 5: $|\mathbf{u}|$ contours of Stokes flows in case B computed on the coarse mesh 4×8

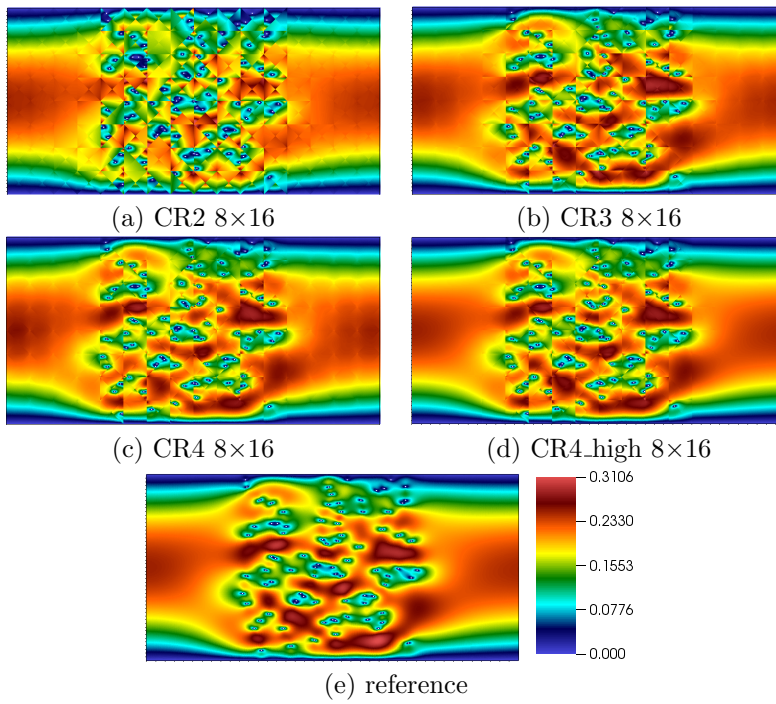


Fig. 6: $|\mathbf{u}|$ contours of Stokes flows in case B computed on the coarse mesh 8×16

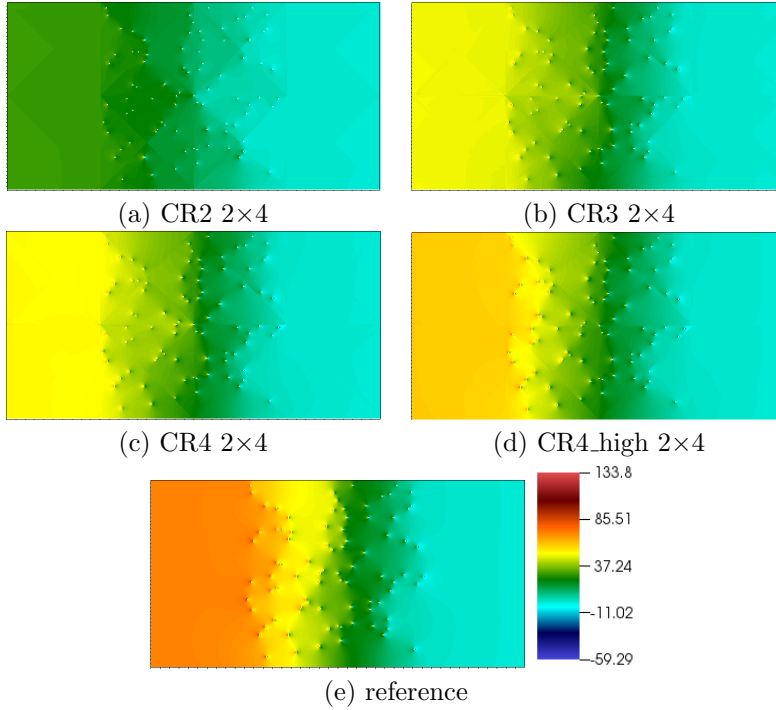


Fig. 7: Pressure of Stokes flows in case B computed on the coarse mesh 2×4

847 evident that both the velocity and pressure computed by CR4_high are significantly
 848 more accurate than those given by other MsFEMs. As shown in Figure 4 and Figure 7,
 849 CR4_high already captures most of the important flow features on the 2×4 coarse
 850 mesh and the solution is in good agreement with the reference solution. Table 2
 851 and Table 3 present the error study of different MsFEMs on various coarse meshes,
 852 relatively to the reference fine-scale solution (in L^2 norm). Figure 10 shows the
 853 numerical convergence of velocity computed by different MsFEMs in both case A and
 854 B. We observe a converging trend of the velocity and the pressure in both case A and
 855 case B. It is observed that these methods have the same rate of convergence, which
 856 is about 1 in case A and a bit lower in case B. The enriched MsFEM does not show a
 857 larger rate of convergence than lower-order methods. We only mention that a related
 858 high-order method proposed in [13] proves an error in the energy norm that behaves
 859 like $\sqrt{\varepsilon} + H^{n+1} + \sqrt{\frac{\varepsilon}{H}}$ for smooth solutions of periodic elliptic problems with highly
 860 oscillating coefficients. Ref. [1] additionally considers the effect of the fine mesh on the
 861 error. These theoretical results and the numerical results presented in these references
 862 show that it is not easy to isolate the sole effect of refining H ; this issue needs further
 863 investigation. In the present work, the errors of MsFEMs are relatively small even on
 864 coarse meshes with a rather large (H/ε) ratio and errors of CR4_high are much smaller
 865 than those of other MsFEMs. For a fair comparison, one should refer to Table 1 which
 866 takes into account the additional complexity of CR4_high with respect to the other
 867 methods. We conclude that it is always advantageous to implement CR4_high rather
 868 than CR2, and that the comparison with CR3 depends on the test case, on the mesh
 869 and on the considered variable (pressure or velocity).

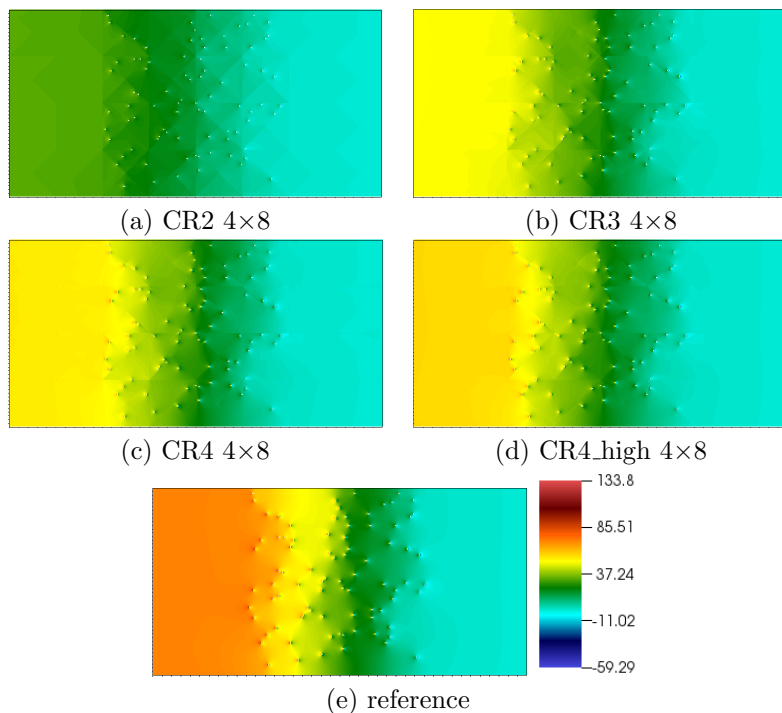
Fig. 8: Pressure of Stokes flows in case B computed on the coarse mesh 4×8

Table 2: Error analysis of Stokes flows in case A

Config.	L^2 Rel. U				L^2 Rel. P			
	CR2	CR3	CR4	CR4_high	CR2	CR3	CR4	CR4_high
2×4	0.334	0.210	0.190	0.091	0.475	0.212	0.146	0.096
4×8	0.174	0.072	0.065	0.039	0.321	0.144	0.083	0.055
8×16	0.061	0.026	0.022	0.015	0.173	0.075	0.044	0.034
16×32	0.030	0.014	0.011	0.006	0.116	0.056	0.025	0.020

870 From Table 2 and Table 3, Case B seems to be more difficult in that the errors
 871 with respect to the fine scale reference solution are larger in that case. This may be
 872 due to the larger number of obstacles which generate richer flow patterns.

873 **6. Conclusions.** An innovative enriched Crouzeix-Raviart MsFEM has been
 874 proposed for solving Stokes flows in highly heterogeneous media with solid obstacles.
 875 This method has been validated in numerical experiments by comparing to Crouzeix-
 876 Raviart MsFEMs originally proposed in [33, 40]. Local problems are implemented
 877 in *TrioCFD* using the finite volume element method where physical unknowns are
 878 discretized by the \mathbb{P}_1 nonconforming/ \mathbb{P}_0 or \mathbb{P}_1 nonconforming/ \mathbb{P}_1 element. Instead
 879 of using the penalization method, the no-slip condition is imposed on obstacles and
 880 numerical simulations are performed on body-fitted unstructured meshes.

881 Numerical experiments show a good converging trend of solutions computed by
 882 MsFEMs to reference solutions. In particular, it has been shown that the enriched

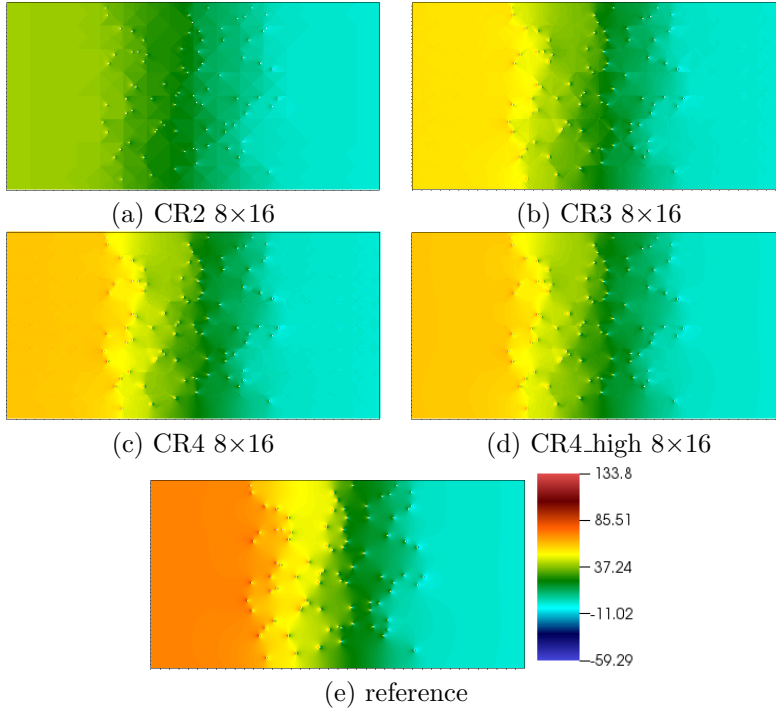


Fig. 9: Pressure of Stokes flows in case B computed on the coarse mesh 8×16

Table 3: Error analysis of Stokes flows in case B

Config.	L^2 Rel. U				L^2 Rel. P			
	CR2	CR3	CR4	CR4_high	CR2	CR3	CR4	CR4_high
2×4	0.478	0.301	0.274	0.159	0.595	0.315	0.281	0.227
4×8	0.329	0.185	0.158	0.114	0.556	0.289	0.249	0.189
8×16	0.189	0.096	0.084	0.061	0.449	0.232	0.164	0.130
16×32	0.116	0.055	0.045	0.030	0.307	0.184	0.118	0.102

883 Crouzeix-Raviart MsFEM with $n = 1$ is significantly more accurate than Crouzeix-
884 Raviart MsFEMs proposed in [33, 40]. A clear perspective is to perform an error
885 analysis in order to obtain the relation between the error and the order of the approx-
886 imation n . Besides, it would be interesting to implement the enriched MsFEM with
887 $n = 2$, $n = 3$ and so on, and then compare the enriched method with other Crouzeix-
888 Raviart MsFEMs. In addition, when there is no obstacle and if the elements of the
889 mesh (in 2D) are triangles, then the method presented here shares a large number
890 of features with the non-conforming finite elements presented in [38]: pressures are
891 fully discontinuous P_n functions, velocities contain the non-conforming P_{n+1} space
892 and the degrees of freedom are the same, as can be seen from [38, page 298]; further
893 investigation is needed to check whether these two methods are the same.

894 The enriched Crouzeix-Raviart MsFEM proposed in this paper can easily be ex-
895 tended to Oseen flows (see [25] for more details) which is a step further toward the

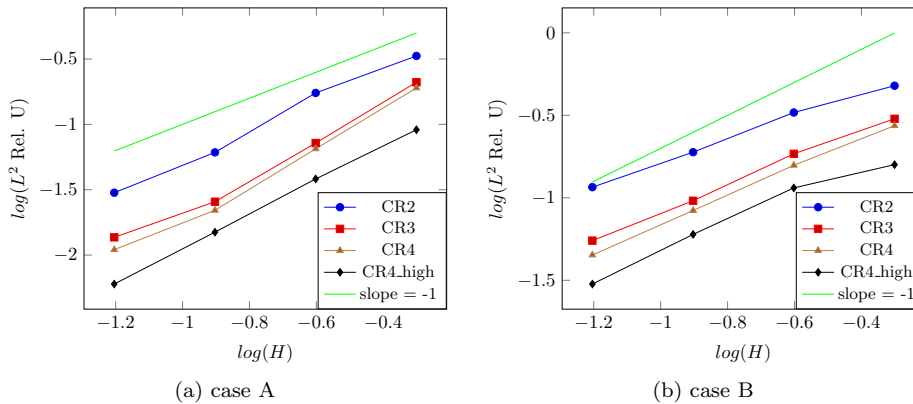


Fig. 10: Numerical convergence of MsFEMs in case B

896 nonlinear Navier-Stokes flows, which is the ultimate goal of this study. It would
 897 be interesting to test whether the enriched MsFEM could significantly improve the
 898 accuracy of the method proposed in [39] for Oseen flows in heterogeneous media.

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