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Peano's philosophical views between structuralism and logicism

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Abstract.

The paper investigates Peano's philosophical views through a detailed analysis of several mathematical practices that can be considered as markers of logicism: the link between functions and relations, the role of metatheoretical investigations, the kind of semantics, the use of definitions by abstraction, and the foundational or non-foundational value of axiomatics. Peano's view is characterized as a form of structural algebraism, which differs from both the algebra of logic tradition using mathematical symbols to express logical calculi, and from Frege's logical investigation centered on the effort to understand the functional nature of predication. The influence of Leibniz is present both in the development of a kind of semantics that is neither representational nor conceptualist (symbols mean linguistic items of ordinary language), and in the consideration of symbolic signs as mirroring what they stand for. The symbolic logic developed in Peano's *Formulario* is the root of mathematics, but not its foundation: like linguistic roots in ancient languages, logical symbols are a tool to grasp the invariant features of mathematical functions and operations.

1. Introduction¹

The main objective of this paper is to analyze Peano's philosophical views as they emerge from several mathematical practices that are usually associated with logicism: for lack of a better name, I will call this view *structural algebraism*, and will try to show some differences with respect to Frege's own approach and to a foundationalist notion of logicism.

The latter is based on some general claim about a relation of inclusion or intersection between logic and mathematics, and could come in different versions, depending on the relation that one considers. For example, the relation could subsist between: a) the symbolic notation adopted in logic and in mathematics; b) between the primitive propositions occurring in them (which may be identical or reducible one to the other or translatable one into the other); c) between the respective inference rules; d) between the axiomatic presentation of the theories; e) between the philosophical claims that ground them; or f) between their metatheoretic properties. The usual definition of logicism based on some claim about the relation between mathematics and logic has often been criticized as a product of a philosophical question that is extrinsic to mathematical and logical practice (Reck 2013b; Wilson 1992; Tappenden 1997). No matter how interesting the question about foundationalist logicism could be, this paper will follow a different path, taking logicism to be a mathematical practice that emerges from a logical investigation of definitions, from a certain notion of interpretation of the symbols of a formal language, from the distinction between relations and functions, and from the difference between primitive and derived terms or propositions in axiomatics. On the one hand, Peano was primarily a mathematician, and this explains why it is reasonable to reformulate the question about Peano's logicism as a question about which mathematical practices he preferred, and why. On the other hand, several of the mentioned mathematical practices were actually highly debated by Peano, as he discussed them with members of the

¹ I would like to thank the editors, Marco Panza and Jan Von Plato for precious insights and comments on a preliminary version of this paper.

² It has been questioned whether one could speak of a School in the absence of an academic filiation and of an explicit communication strategy (Lolli 2007), and given the difficulty in determining who should be counted as a member: Peano's students at the University of Turin, the collaborators to the *Formulario*, or the

School² involved in the joint enterprise of the *Formulario*, or with contemporaries, especially Frege. The latter is a perfect candidate in this respect, because Frege and Peano corresponded with each other and compared their respective results. Besides, they have often been associated in the literature, as if they shared a similar understanding of logic (van Heijenoort 1967).³

Of these two ways to investigate an author's stance towards logicism, the foundationalist is better suited to evaluating an explicitly foundational philosophy, the latter to evaluating a philosophical view that is embedded in a logico-mathematical practice. The former aims to compare different historical realizations of a unique philosophical and foundational notion of logicism that is preliminarily assumed to be correct, whereas the latter believes that logicism is but a schematic representation of the richness and variety of mathematical practices that accompanied the transformation of axiomatics from the end of the 19th century onwards.⁴ Following the first line of investigation, I hope not only to give further support to the thesis that “philosophical reflections can grow out of mathematical practice” (Reck 2013a, 257), but also to claim that Peano, as well as other scholars of his group, had explicit philosophical views, even if not easily translatable in the foundationalist jargon.

A third way to study Peano's relation to logicism could be an historical investigation of his relation to Leibniz and to Schröder. Although this paper cannot provide a detailed examination of this question, it will be clarified that Leibniz's influence did not urge Peano to endorse a form of logicism based on a conceptualist form of semantics.⁵

Peano's philosophical view will be described as a *structural algebraism*, i.e. a methodology that is based on: i) a metatheoretical use of definitions; ii) a particular theory of meaning (formal symbols mean words and sentences of the mathematical language); iii) a distinction between a mathematical and a pre-scientific (logical) notion of function; iv) an epistemological and dynamic understanding of axiomatics; and v) a joint investigation of logic, language and mathematics, considered both as theoretical and didactic practices.

2. Peano's logicism reconsidered

There are quite different views in the literature on Peano's philosophy of mathematics and logic: it has been claimed that Peano was a logicist (Grattan-Guinness) and an anti-logicist (Kennedy). Some scholars attribute to Peano an explicit philosophical approach (Vailati, Kennedy); others claim that he had no philosophical approach at all (Lolli), or at most an implicit and never openly declared view on philosophy (Geymonat, Grattan-

² It has been questioned whether one could speak of a School in the absence of an academic filiation and of an explicit communication strategy (Lolli 2007), and given the difficulty in determining who should be counted as a member: Peano's students at the University of Turin, the collaborators to the *Formulario*, or the more than forty people who “worked closely” with Peano (Luciano and Roero 2010). For the purpose of this paper, I will restrict the focus to Padoa, Pieri, Burali-Forti, Vacca, and Vailati.

³ Recent literature has questioned van Heijenoort's point of view, and underlined the differences between Peano and Frege's understanding of formalism. See e.g. von Plato 2021 and Bertran-San Millan 2021.

⁴ Anyway, it would be anachronistic to talk of logicism before 1929, when the term “Logizismus” began to refer to a philosophical view on the relation between mathematics and logic (Carnap 1929; Fraenkel 1928), whereas the term “Logistique” had been introduced by Couturat as a generic term for symbolic logic (Grattan-Guinness 2000, 479, 501).

⁵ Logicism is often considered as a perspective that originated in Leibniz's conceptual analysis, which aimed to determine the simple ideas from which other ideas are composed. Some scholars even go so far as suggesting that Leibniz contra Kant was at the origin of a new non-representational understanding of semantics, where terms mean concepts and only by means of concepts refer to objects. Frege's or Russell's logicism would result exactly from the combination of a Leibnizian conceptual analysis and conceptualist semantics (Coffa 1982).

Guinness).

Basing his argument on a passage by Peano stating that natural numbers cannot be defined from mere logical concepts (Peano 1891b, 91), Kennedy claimed that Peano had an explicit philosophical view, but only about the philosophical views he did not share (Kennedy 2002, 9). Quoting the 1895 Review of Frege's *Grundgesetze* asserting that thanks to the *Formulario*, "mathematics is now in possession of an appropriate instrument to represent all its propositions, and to analyze the various forms of reasoning" (Peano 1895, 190), Grattan-Guinness remarked that Peano looked like a logicist, or rather like an 'opportunist logician', because he introduced important logical notions and made considerable contributions to the foundations of mathematics, but lacked a general program that could induce him to pursue all the logical consequences of his own insights (Grattan-Guinness 2000, 247).⁶ Lolli claimed that the question about Peano's philosophy of mathematics is pointless, because Peano was neither interested nor competent in philosophy, as he himself declared in the review to Schröder's *Vorlesungen* (Peano 1891a, 164). That the *Formulario* was not a contribution to the foundations of logic and mathematics was already Frege's view, and actually Lolli (2007, 3) recalled one of his famous expressions as he described the *Formulario* as a project of information storage and retrieval.⁷ Geymonat attested that Peano did not want to write about philosophical issues and remained silent when asked about his own epistemological conception (Geymonat 1955).

The situation is even more complicated if one considers philosophical remarks by other collaborators of Peano, e.g. Pieri, Padoa, and Burali-Forti. Each author seems to have a particular philosophical view, which agrees on certain points with a common view of the school, but also differs quite radically on other aspects. One of the earlier interpreters of Peano's philosophy was Giovanni Vailati (1906b), who was unanimously recognized as the philosopher of the group. He claimed that Peano's ideas were similar to the views held by pragmatists, and even if (as he was a pragmatist himself) he could have had the tendency to attribute to Peano his own views, there are some effective similarities with the pragmatist epistemology.

Mario Pieri and Alessandro Padoa explicitly discussed the question of the relations between mathematics and logic from a foundational perspective. Pieri offered a restrictive interpretation of logicism, which Grattan Guinness (2000, 371) considered as the expression of the general position of the school: mathematics consisted in the application of some logical principles to special relations—like Russell's applied mathematics. Alessandro Padoa considered logic and mathematics to be distinct but seemed to consider the criteria for demarcating between the two as partly conventional or at least historically determined.

Giving up on the idea that it may be possible to offer a simple yes-no answer to the question as to whether Peano was a logicist on the basis of paying selective attention to some of his more 'philosophical' remarks, I will instead try to show that he had a coherent epistemology that can be investigated by an analysis of several mathematical practices. As Frege puts it, the question is whether Peano's *Formulario* was simply "the work that has to be done in order to write out a sentence as simply as possible in symbols" or a full effort to continue the reduction into simpler components right down to the simplest elements, developing a logical investigation on proofs, and explaining "how, from one of those formulae, or from two of them, a new one is obtained" (Frege 1897, 366, English translation

⁶ "Peano presented arithmetic in a symbolic language which contained logical techniques rather than grounded it in an ideal language which expressed such features" (Grattan-Guinness 2000, 247).

⁷ As Frege puts it in his 1897 paper *On Mr. Peano's conceptual notation and my own*, Peano's project "seems orientated towards the storage of knowledge rather than towards proof, towards brevity and international intelligibility rather than towards logical perfection". (Frege 1897, 365-366, repr. in *Kleinen Schriften*, 223; English Translation in Frege 1984, 237).

238).⁸ But what mathematical practices, if any, can be chosen as significant objects of investigation in this respect?

Apart from the practice of proving, which has already been studied in some detail,⁹ other interesting topics could also be analyzed. In Frege's logic functions are primary, and relations are defined as a special case of functions, i.e. as the mapping from the Cartesian product of two sets to the set formed by the truth values True and False, whereas in modern predicate calculi (following Russell), relations are primary (usually defined as subsets of the Cartesian product of two sets), and functions are introduced as special cases of relations that satisfy functionality (Zalta 2015, § 2.1). Another issue concerns the presence or absence of metatheoretical investigations, which van Heijenoort considered as one among many reasons to distinguish between different conceptions of logic as a language or as a calculus (van Heijenoort 1967). A comparison with Frege and Russell, who are presented by van Heijenoort as representatives of the conception of logic as language, will show the need for some clarifications and further distinctions. The main point will concern a detailed analysis of Peano's theory of meaning and of his notion of interpretation, which radically differs from Frege's notion, even if both assign a central role to the distinction between function and argument,¹⁰ as well as from the contemporary model-theoretic notion.¹¹

Drawing attention to these topics, several further philosophical issues come to the forefront: the influence of algebraic investigations, the scope and limits of formal notation, and the distinction between interpreted and non-interpreted symbols. This is in turn correlated to the understanding of the notions of equality, meaning, truth, axiom, and to the meta-theoretical role of definitions--topics that will ground my characterization of Peano's philosophical view as a structural and algebraist approach to logic, without any strong demarcation between logic, mathematics, and linguistics, and conceiving axiomatics as a dynamical investigation and comparison of scientific theories. This epistemological understanding of axiomatics and the syntactic notion of interpretation that grounds it will show some relevant differences from Frege's conception of what logic is (or should be).

3. Function, operation and relation

Peano first introduces the symbol of function and then introduces the symbol of relation. The notion of function is considered to be synonymous with that of operation and of mapping or transformation.¹²

Given a symbol h of operation, or function or mapping, which are all synonymous, one has to consider the class of individuals on which one can operate with the symbol h and the class of individuals that one obtains as a result. [...] We will thus write bfa for

⁸ On the notion of simplicity in Peano, Padoa, Frege and Russell see Bellucci et al 2018.

⁹ See von Plato 2018; 2017; 2014, but also Bertran-San Millan (2020) who claims that Peano lacked, at least in his early writings, Frege's interest for a logical calculus and for the separation of inference rules from axioms, and suggests (based on Badesa 2004) that this was because Peano used as a basis for his early mathematical logic Schröder's logic, which was not indented as a deductive system. Another reason might be that Peano's notion of interpretation, as reconstructed below (§ 4) did not favor a separate treatment of the calculus of classes and of the calculus of propositions, which should both be obtained by substitution of natural language words and sentences to the formal symbols of the logical section of the *Formulario*.

¹⁰ On Frege's distinction between function and argument in *Begriffsschrift*, see Badesa, Calixto and Bertran-San Millán 2017.

¹¹ I would like to thank Georg Schiemer for bringing this point to my attention. See also Giovannini and Schiemer 2019.

¹² I will not discuss here similarities and differences between Peano's and Dedekind's approaches. For a recent discussion, see von Plato 2019 and Kahle 2021.

“symbol of function that being written before an a produces a b ” or “operation that transforms the a in b ”, or “correspondence between the a and the b ” or “ b function defined in a ” or “ b function of the a ”. (Peano 1894, § 23, 27–28)

Peano considers four properties of this functional symbol. The first says, that if h maps a set A into a set B , then an element x of A is mapped by h into an element hx of B . The second property is functionality: if $x=y$, then $hx=hy$, i.e. the feature that is nowadays used as a defining condition for functions. Two further properties are mentioned: let’s call them co-domain extendibility (if B is contained in a set C , then h maps A into C , i.e. it can be defined on a superset of its original co-domain), and domain contractibility (if C is contained in A , then hx maps C into B , i.e. can be defined on a subset of its original domain) (Peano 1894, §23, 27 ff.). The insistence on the latter two properties already reveals Peano’s interest in a dynamical use of functional symbols, and particularly in questions concerning generally valid ways to modify the definition of a function by transformation of its domain and co-domain.

Relations are defined by means of functions (§ 3): if xay expresses a relation between x and y , then it can be considered as a function of y that produces the x that are in relation with y .¹³ The first examples that Peano mentions are the algebraic relations of equality, greater than, and less than.¹⁴

Padoa and Pieri considered the notion of function as an ordinary logical notion, and at some point this became the dominant view in the Peano School (Pieri 1906, 204–6). Yet, Peano long hesitated to consider the notion of function as a logical concept.¹⁵ Is this a sign of anti-logicism? As a mathematician, Peano distinguished two quite distinct practices concerning the use of function symbols: sometimes one considers a general notion of function, without specifying the domain of variability of the variable, and sometimes one operates on functions whose domain is fixed.

In mathematics there is no single definition for example of “multiplication”, nor does there exist in the *Formulario* an equality of the form $\times =$ (expression composed by other signs). But there exists a definition of multiplication between two natural numbers, then between two relative numbers, then between two rational numbers, and so on. In the *Formulario* one can easily find more than 30 definitions of $x \times y$, with different hypotheses. Thus to the sign of function it is not connected a domain onto which the function is determined, which is also called variability domain of the function. As a matter of fact, it is not possible to talk about equality of two functions, because two functions may produce identical results in one domain and different results in another domain. But two arbitrary functions u and v always have a common domain that can be expressed by $x \ni (ux = vx)$. We cannot talk about the number of functions that satisfy a

¹³ For example, the relation $y \leq x$, with $x, y \in N$ can be expressed as a function f from N to $P(N)$ such that to each $x \in N$ is associated a unique $f(x) = \{y: y \leq x\}$.

¹⁴ Russell and Dedekind always took the order relation as a primitive. Peano considers also the possibility of defining it by means of an operation, i.e. he envisages both an additive and an order approach (it depends on the theory to be axiomatized). Rodolfo Bettazzi, who was not, strictly speaking, a scholar of Peano, but had many contacts with him and his school, preferred the additive approach in his axiomatization of quantities (Bettazzi 1890). In the *additive approach*, the primitives are an equality or inequality relation and an operation of addition; the order relation is defined by means of the former. In the *order approach*, the primitives are an order relation and an operation (of successor, division, passage to the limit...). The additive approach had its origins in the Greek theory of proportions, and was interestingly applied to physical measurement by Helmholtz: to measure a class of physical magnitudes one should identify a physical operation that can be executed on such quantities, and then investigate its algebraic properties (commutativity, associativity, etc.), so as to choose the appropriate system of numbers to measure them (Cantù 2018).

¹⁵ Vailati explained that properties of functions and correspondences have been included in the list of logical formulas, “because, *from a certain point of view*, they belong to logic rather than mathematics” (my emphasis) (Vailati 1893).

given condition: no function is invertible, and so on. When mathematicians talk about equality, number, inverse of a function, the term “function” denotes the system $(u; a)$, where u is the function considered in § 1 and a is the variability domain. We call it “definite function”. (Peano 1906, §4, 80, my translation).

The latter is the usual notion of function that occurs in mathematical practices: Peano calls it ‘definite function’, but what is the former? Is it the logical notion of a propositional function? Or rather a pre-formal notion of function that logic tries to capture?

A comparison with Hermann Grassmann’s *General Theory of Forms* might suggest a different reading: the general notion of function is an abstraction from the notions of functions occurring in different mathematical domains: it retains their common properties and remains undetermined with respect to further mathematical features. Grassmann distinguishes between formal and real operations. Formal operations of addition and multiplication are operations characterized by a structure of commutative group (in the case of addition) and of a ring (in the case of addition and multiplication taken together). Formal operations are denoted by the usual arithmetical symbols $+$ and \cdot , but are not defined with respect to a specific mathematical domain. These general features of operations are studied in the *General Theory of Forms* (Cantù 2020).

Then there are real operations of addition and multiplication that occur when the formal notion is applied to a specific domain (e.g. natural numbers, vectors). Real operations may have different properties depending on the domain they are applied to: e.g. an operation of multiplication between natural numbers is commutative, whereas the multiplication between vectors is not. According to Grassmann, the *General Theory of Forms* is the study of some fundamental relations and operations that occur in all branches of mathematics, but is not itself part of mathematics, because it contains formal operations that are underdetermined and that could receive full determination only when they are applied to a specific domain and become real operations in mathematical specific theories.¹⁶ Is the *General Theory of Forms* a logical theory? As with proportion theory in Euclid’s *Elements*, it does not imply the creation of a new genus of objects (Cantù 2008, §3), but merely assembles a list of propositions that “relate to all branches of mathematics in the same way” (Grassmann 1844, 33; 1995, 33). Peano seems to consider logic as a list of all propositions that are later used and applied in the *Formulario* and that can be put together in an introductory volume to mathematical theories.

Mathematical logic is the only instrument that could express and treat propositions of ordinary mathematics. It is not an end in itself (Peano 1913).

Both the calculus of the classes and the calculus of propositions appear as applications of the general abstract algebra to the domain of logic (as usually treated in the Boolean tradition), as the following table suggests:

Symbols	\cup	\cap	\supset	Λ
Reading for classes	union	intersection	is contained in	nothing
Reading for propositions	disjunction	conjunction	implies	absurd

Table 1. Peano’s logical symbols and interpretation

Some remarks from Gödel’s *Philosophical Notebooks* confirm that he considered Peano’s general notion of function as an alternative to Frege’s and Russell’s view, because “Peano

¹⁶ A different but similar way to distinguish between functions as expressions of algebraic symbols and functions acquiring mathematical meaning can be found in Lagrange: “algebraic analysis is pure and general, because the arguments are taken as being previously indeterminate” (Panza 2015).

seems to assume that all functions have a meaning (Sinn) for any argument” (Crocco et al. 2017, 51, my translation). Gödel interpreted this peculiar notion of function as the most general operation that can be applied to any two things, having usual mathematical and logical relations as special cases.

I. What is the most general operation that can be applied to two arbitrary things a , b and that contains as special cases “ \mathcal{E} ” and “application” —and that corresponds to the operation of moving side by side [Nebeneinanderschiebens]? That there is such an operation follows from the fact that each concept has an extension in almost all directions and that the application is the extension of \in . This relation [is] maybe definable by: 1) the simplest kind of combination, 2) that which owes its existence immediately to the existence of this pair, 3) that which is perceived (that towards which one turns its gaze) when one directs its vision to a and b , 4) the relation that subsists between a and b (or the connection that subsists between them), e.g. the vector $a - b$. In particular it is to be expected that for two things emerges a pair, for two propositions a product, for something that needs completion what is obtained by the completion (as for example by an operation), for classes and numbers maybe the sum, for concepts the product. Notably, depending on the type, ab would denote $a \in b$, $a - b$, $b \in a$, $a | b$ (for relations and functions) (Crocco et al. 2017, 53-54, my translation).

A relevant difference with respect to Frege’s and Russell’s theories (especially in the typed versions) is that there is not, in principle, a distinction between a domain of entities that occur only as arguments of a function, and entities that have a functional nature, as is the case in the Fregean analysis of predication, where there is a clear distinction between predicates conceived as unsaturated functions and subjects that saturate them, or in Russell’s tradition, where at each type level there are objects that can occur only as arguments of a function. The lack of this asymmetry, which is presumably related to Peano’s algebraic concern for a representation of logical truths by means of equalities and for geometric duality, could correspond to what Gödel had in mind as he wrote:

II. If one explains the concept of function in Peano’s way on the basis of this operation of combination, then each thing is both a function and a transformation, because each thing can be combined with other things (the concept application, \in , etc., in this operation of combination is “limited” to certain classes of things). Yet, on the other hand there are functions in a strict sense, i.e. objects whose essence consists in the possibility of being combined, whereas for the others the fact of being combined is so to say something ‘external’ (Crocco et al. 2017, 54, my translation).

Peano seems to distinguish the pre-formal non-mathematical notion of indefinite function from the formal logical notions of union and intersection of classes, or the notions of conjunction and disjunction of propositions. To further clarify the relations between them, and to investigate whether the pre-formal notion should be considered as logical, we will have to analyze in some more detail Peano’s linguistic notion of meaning (see below § 4). Yet, the distance from Frege’s approach is already evident. Frege requires all functions to be defined, and requires, in particular, the existence of a value in output for any value taken as argument. As remarked by Gödel, Peano used the symbol of (indefinite) function for the maximum of a function, because he introduced it without ascertaining the arguments on which the function could be defined, and whether it could have a value for each of these arguments (Cantù 2016). Peano, unlike Frege, wanted to preserve as much as possible the ordinary way in which mathematicians use functional symbols¹⁷, introducing them before having fully determined their specific properties in a given domain. One could say that Peano’s functional symbols,

¹⁷ For a further analysis of Peano’s notational practices and symbolism see Schlimm 2021.

like the symbols occurring in Grassmann's *General Theory of Forms*, are endowed with a sort of minimal, incomplete meaning described by the four properties mentioned above (that of being a mapping, of being functional, of being extendible with respect to the codomain and being contractible with respect to the domain). This algebraic interpretation of Peano's approach, which explains why he had no problems in accepting conditional and case by case definitions, is supported by another passage of Gödel's *Max Phil*:

In the case of an incomplete definition of 'function' (Peano), there is also the possibility that it might be defined for certain arguments, undefined for certain arguments, defined by cases for certain arguments (Crocco et al. 2017, 55-56, my translation).

The distance from Frege's perspective is even greater, if one considers that Peano did not share Frege's association between the concept/object distinction and the function/argument distinction.¹⁸ But this dissimilarity is explained by a radically different understanding of meaning: instead of concepts and objects, formal symbols acquire their full meaning when they are interpreted ('read as') ordinary mathematical language terms and sentences of a specific mathematical theory.

4. Meaning and interpretation

Criticizing Peano in *Begründung meiner strengeren Grundsätze des Definierens* (1897/98), Frege explicitly mentions that he is using the term 'meanings' [Bedeutungen] in the sense of Peano's 'significations' (1969, English Translation in Frege 1983, 153, fn.). Yet, what 'significations' should mean is not fully clear, neither in Frege's commentary nor in Peano's own writings. The term 'to mean' that Peano writes in Italian, French and Latino sine flexione respectively [*significare, signifïer, significat*] is first used in *I principii di geometria* (Peano 1889b) and in the *Arithmetices Principia* (Peano 1889a), even if other expressions like 'to indicate' [*indica*] or 'can be read as' [*si legge come*] are used as if they were synonymous. Apart from some inconsistencies, two main uses can be distinguished.

Firstly, Peano uses the term when a symbol stands for a longer symbolic expression, as for example when 'ab.cd significat (ab)(cd)' (Peano 1889a, 45). Here he introduces a typographical abbreviation, suggesting that a symbol means another symbol when the latter can be substituted *salva veritate* by the former in the appropriate context. Secondly, Peano uses the term when a symbol means different things, depending on the context in which it is interpreted (or applied). For example, in *I principii di geometria* and in the *Arithmetices Principia* he mentions that the symbol \wedge means 'nothing' [*nulla, nihil*] in the context of classes and 'absurd' [*assurdo, absurdum*] in the context of propositions. Another example is offered by the symbol \supset , which means 'it is deduced' [*si deduce, deducitur*] in the context of propositions and 'is contained' [*è contenuto, continetur*] in the context of classes. A formal mathematical symbol (non-interpreted and to which no fully determinate concept could correspond) means another symbol (a word of the natural language) that is always used in a specific mathematical or logical context, and is thus endowed with the meaning it usually has in that context. Recalling the distinction introduced in § 3, this relation of meaning is the one that allows the passage from formal to real operations, both in the case of logical symbols (as here) and in the case of mathematical symbols, where the same symbol could be read as the multiplication between natural numbers N or between integer numbers n .

In other words, a symbol of this formal general language means a word of the language of a specific mathematical theory: meaning is again a relation that concerns only the

¹⁸ Yet some scholars have recently pointed out that Frege did not make this association either in his early writings, but only from the *Grundgesetze* onwards (Badesa and Bertran-San Millan, 2017).

linguistic level. Symbols can be interpreted as varying on linguistic terms. Paolo Mancosu's remarks on Padoa's notion of interpretation support the idea that Peano had a linguistic notion of interpretation.

In contemporary model theory we think of an interpretation as specifying a domain of individuals with relations on them satisfying the propositions of the system, by means of an appropriate function sending individual constants to objects and relation symbols to subsets of the domain. It is important to remark that in Padoa's notion of interpretation something else is going on. An interpretation of a generic system is given by a concrete set of propositions with meaning. In this sense the abstract theory captures all of the individual theories, just as the expression $x + y = y + x$ captures all the particular expressions of the form $2+3 = 3+2$, $5+7 = 7+5$, etc. (Mancosu, Zach, and Badesa 2009, 7).

The example should not induce one to believe that this *sui generis* notion of interpretation holds only for individual constants; it holds also for relation symbols. An important difference with respect to the model-theoretic interpretation should be spelled out in some more detail. The model-theoretic notion requires two steps: the individuation of the right domain of objects and the choice of the appropriate function. Different interpretations could then arise from the choice of different functions on the same domain of objects. Besides, the interpretation maps symbols into individual objects and sets of individual objects.

In Padoa's as well as in Peano's notion of interpretation, there is just a one-step move, because the symbols are interpreted by substitution with a set of propositions from ordinary mathematical language, and these propositions have a fixed domain and a unique function that maps the proper names into individual objects and the predicates into sets of such objects. Padoa's remarks about the difference between general and specialized theories seem to confirm this idea:

The system of undefined symbols can then be regarded as the abstraction obtained from all these interpretations, and the generic theory can then be regarded as the abstraction obtained from the specialized theories that result when in the generic theory the system of undefined symbols is successively replaced by each of the interpretations of this theory. Thus, by means of just one argument that proves a proposition of the generic theory we prove implicitly a proposition in each of the specialized theories (Padoa 1901, 121).

This move from generic to specialized theories, from uninterpreted formal symbols to a concrete set of meaningful propositions of the ordinary mathematical language, from the symbols for formal operation to those for real operations requires the passage from a list of formal sentences that summarize some common traits of different specific theories to specific theories themselves. And this holds for Peano in the case of mathematical and logical theories. This explains Peano's development of piecemeal definitions of numeric systems, as well as the distance from Frege's need to have a unique definition of each symbol. What guides the admissible substitutions and avoids confusion are for Peano the actual mathematical practices, embedded in their ordinary language formulation. Even if the distinction between function and argument is clearly identified, the meanings of symbols are ordinary mathematics words and propositions, and not concepts or objects.

What then can be said about Peano's logicism? Is it a kind of heretic logicism that has nothing to share with the idea of reducing arithmetic to logic? There are at least two senses in which mathematics still has to be 'preceded' by logic in Peano's work, even if logic emerges as the result of a bottom-up process of successive abstraction from specialized theories (as their invariant). Contrary to what is usually said, this view was not first developed by Padoa, but had already been developed by Peano in his *Calcolo geometrico*,¹⁹ and commonly shared

19 Yet in the *Calcolo Geometrico*, Peano seemed to consider logic as part of mathematics, probably

by the members of the school already at the beginning of the 1880s.²⁰

First, logical signs occur in mathematical language, and they are usually given one or other of the two previously mentioned interpretations (see e.g. Table 1 in § 3): it is therefore useful to let them precede the treatment of mathematical theories in a treatise, so that repetitions could be avoided and proofs economized. Just as in proportion theory, we can prove theorems that hold both in geometry and in arithmetic: “by means of just one argument that proves a proposition of the generic theory we prove implicitly a proposition in each of the specialized theories” (Padoa 1901, 121).

Secondly, logic precedes mathematics in the sense that it is used to actually do mathematics, because: a) logical principles of inferences are used in proving theorems;²¹ b) certain general properties of uninterpreted symbols (for example the codomain expandability or the domain contractility of the symbol of function) are used in the transformation of functions into new functions or used to define them case by case; and c) because the kind of abstraction that one has at the level of formal, uninterpreted, or rather underdetermined symbols derives from a wider, metamathematical understanding of logic as an algebraic investigation of structural similarities between different operations and functions.

Before analyzing Peano’s metatheory (§ 5), I would just like to add some brief remarks on abstraction, a question that is quite relevant for logicism, and especially for neo-logicism.²² It has been claimed that the Peano School first recognized definitions by abstraction as a specific form of abstraction and that Frege was the first to use them “as abstraction principles in the service of a philosophical program such as logicism” (Mancosu 2016, p. 110). Besides, it is well known that Russell’s definition of number as the class of equinumerous classes did not raise Peano’s enthusiasm, even if it was readily accepted by Padoa and Pieri. Should this mean that Peano was in some sense an anti-logicist? The question is complex.

First, some of the reasons why Peano disliked definitions by abstraction do not depend on philosophical prejudices, but on some defects of the corresponding mathematical practice. Any adequate definition should satisfy a criterion of homogeneity (Peano’s condition is more liberal than conservativeness, as adopted in the traditional account of definitions), but if for example one defines rational numbers as the class of equinumerous classes, then one has to introduce a non-homogeneous definition of the quotient between numbers.

Second, it is questionable whether for Peano the value of the abstraction function was really the equivalence class itself rather than one of its members taken as a representative of the whole class (Mancosu 2018). Even if Peano said that a definition by abstraction introduces a new entity, it is disputable whether this entity should be considered as the class

because he adopted the algebra of logic perspective and language, from which he later departed: “Deductive logic, which forms part of the science of mathematics, has not previously advanced very far, although it was a subject of study by Leibniz, Hamilton, Cayley, Boole, H. and R. Grassmann, Schroder, etc. The few questions treated in this introduction already constitute an organic whole, which may serve in much research. Many of the notations introduced are adopted in the geometric calculus.” (Peano 1889, English Translation in Peano 2013, x).

²⁰ Vailati’s 1892 paper *On the fundamental principles of straight line geometry* clearly expressed the geometrical origin of this notion of interpretation. The point is to deduce the principles of geometry from a minimal set of conventions as postulates, which do not refer to any privileged interpretation of the considered signs of relations and operations, but only to combinational properties of relations and operations themselves: e.g. the relation of two points a, b on a straight line, or the relation of succession between two instants, or the relation of ‘greater than’ between two quantities (Vailati 1892, 71–72).

²¹ von Plato (2018; 2017; 2014) has recently shown how rich Peano’s *Formulario* is from this point of view.

²² For reasons of space, I cannot develop here a detailed discussion of Peano’s genuine or non genuine abstraction, and its relations to his theory of definitions. For the analysis of Peano’s definitions by abstraction see Mancosu 2016 and Mancosu 2018.

of equivalence of the abstraction function. On the one hand, the required relation could be weaker than an equivalence relation (Burali-Forti 1894).²³ On the other hand, Padoa and Vailati often mentioned that whenever a new word or a new expression is introduced in mathematics, the latter need not necessarily mean a new concept or object: for example in a case by case definition of a function, the value of the abstraction function could be the class of equivalence for certain cases, and a term for other cases.

If one agrees with Ignacio Angelelli that there were genuine abstraction in Peano, and no principles of abstraction that “dispense with abstraction and allow us to do well without worrying about abstraction” (Angelelli 2004, 18), then the value of an abstraction function could not have generally coincided with the equivalence class (even if this is certainly the case in Padoa 1907).

5. Metatheory and axiomatics

Whether the Peano school contributed to the development of metatheory is a controversial issue in the literature. Marco Borga claimed that there was no metalogic in the Peano School, because not even the notion of logical law was deeply investigated, and there was no real development of any modern meta-theoretical inquiries (Borga 1985). Casari on the contrary claimed that Peano contributed both to metalogic and to the metatheoretical investigation of arithmetic: “it is he who, making the concept of numerical system depend on the concepts of formal language and of the satisfaction of formal conditions, created, albeit quite unaware, the premises for the possible variation of the very concept of numerical system; this, in fact, is transformed when the language adopted and the accepted relationship of satisfaction, vary.” (Casari 2011, 145).

If Peano definitely had a metalogical interest in the investigations of independence (von Plato 2019, Bertran-San-Millan 2021), one reason that could explain why the issue was controversial is the fact that when one investigates metatheory in Peano, one should not look for a metalinguistic study of general properties of formal deductive systems, but, as in the case of Frege, one should check whether there is some “theoretical scrutiny of a particular theory using notions of ‘language’, ‘reference’, ‘interpretation, etc.’” (Tappenden 1997, 213). A further reason that could explain the divergence among scholars is the fact that Peano, Padoa and other members of the school disagreed not only on the notion and use of definitions by abstraction (see § 4 above) but also on other topics that are quite relevant for a metatheoretical inquiry. The following issues, though only sketched out briefly, will shed some light on the difference between the meta-logical approach developed by Padoa and Peano’s meta-mathematical perspective.

Concerning definitions, there were strong dissimilarities about axioms as implicit definitions. Peano claimed that they determine rather than define what is common to an infinity of systems that satisfy the axioms (Peano 1891b, 93–94; 1898, 1–2); Pieri remarked that they define the primitive ideas denoted by the symbols occurring in the axioms (e.g. number, unity, addition in arithmetical axioms) (Pieri 1901, 378, 387); Vailati said that geometric axioms define a class of objects that are indirectly defined by means of other definitions (e.g. Hilbert’s notions of points, straight lines, planes) or a single primitive relation (e.g. Hilbert’s betweenness) (Vailati 2010, 151); Vacca claimed that they define the meaning of primitive terms (e.g. the symbols 0, N_0 , + in the arithmetical axioms) (Vacca 1896-99, 186); Burali-Forti asserted that they define an entity in itself, i.e. belonging to a class whose individuals verify the primitive propositions (e.g. number in the arithmetical axioms) (Burali-

²³ For a modern revival of this idea, see Joinet 2017, who claims that definitions by abstraction could be introduced in the presence of relations that are much weaker than equivalence relations.

Forti 1894); Padoa specified that what determines the meaning directly of the primitive ideas and indirectly of the derivative ideas are not the axioms themselves, but the interpretations of the symbols occurring in them that verify the axioms (e.g. the interpretations given to the symbols N and $+$ in the arithmetical axioms) (Padoa 1900a; Padoa 1900b).

Concerning equality and logical identity, Peano strongly disagreed with Frege, and refuted the idea that logic should have a unique domain with a fixed equality on it, but Padoa seemed rather sympathetic to this view, and, following Hilbert, suggested reflexivity and substitutivity as a definition of identity (Cantù 2010; Cantù 2007).

Peano was mainly interested in inference rules that were actually used in mathematics, whereas Padoa was generally interested in syllogistic reasoning (Padoa 1912). Even when there was some explicit metatheoretical investigation of the properties of axiomatic systems, different members were especially interested in different features: Pieri in consistency; Peano and Padoa in independence, whereas Burali-Forti, at least in the last edition of his *Logic*, was very dismissive.

Even if the various members of the school shared similar views on the classification of definitions (by operators, by abstraction, by induction, and so on), they diverged on the kind of definitions that should be preferred in a mathematical treatise: Burali-Forti was initially quite critical of abstract definitions, and claimed that only nominal definitions express the concept, whereas the other kinds of definitions merely express an intuition; later on, he accepted definitions by abstraction, convinced by Russell's claim that one could always transform them into nominal definitions; Peano admitted that each definition should be evaluated with respect to the specific aims of the theory in which it occurs; Padoa defended definitions by abstraction, and investigated the nature of equivalence relations that ground them.

Finally, slightly different formulations of the notions of formal language and logical consequence occur in the works of different authors (especially Peano, Padoa and Pieri).

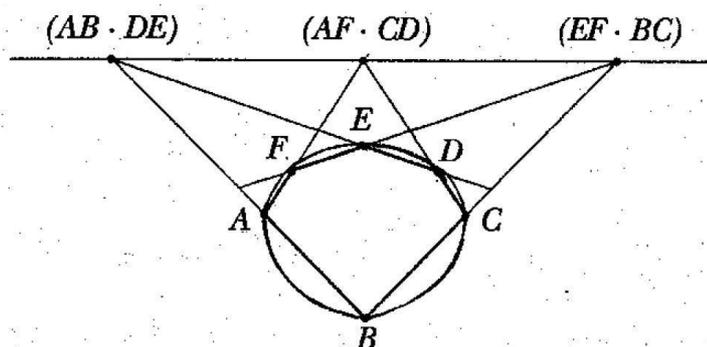
We have already mentioned Peano's tendency to view logic as a tool to prove the propositions of ordinary mathematics (Peano 1913, 48), but also as an investigation of deduction, even if, according to Frege, he did not go far enough in this direction (Frege 1984, 237). Two examples of Peano's use of definitions in logical and geometric writings respectively explain why the role of logic in mathematics was not only that of deducing theorems from the axioms of a specific mathematical theory, but also that of directing the axiomatic presentation of mathematics in the *Formulario*. My claim will be that Peano, like 19th century mathematicians,²⁴ understood logic in a broad sense as an epistemological enterprise, interested both in the investigation of the logical components of mathematics (an interest that Peano shared with Frege), but also in the architecture of mathematical axiomatic theories (an interest that survived in successive encyclopedic works, as in Bourbaki's *Elements of Mathematics*) (Segre 1955: 35).

The first example shows that Peano was deeply interested in what he called "possible" definitions, i.e. alternative definitions of the same symbol leading to a different architecture of the axiomatic presentation of the logic of classes. Λ is a symbol that can be defined in two different ways in the same logical theory (of classes): as the class that contains the objects that are common to any class, or as the only element of a set obtained by intersection of a property and its contradictory, i.e. as the result of the application of an operation and its inverse.

The second example shows the epistemological significance of shifting from one to

²⁴ See for example Bolzano, who considered logic as a theory of science, i.e. "the collection of all rules which we must follow, if we want to do a competent piece of work, when we divide the total domain of truths into individual sciences, and present them in their respective treatises". (*Wissenschaftslehre*, I, §1, 9; Bolzano 1973, 38).

another interpretation of the same string of symbols in the geometrical calculus.



The equation $(AB \cdot DE)(BC \cdot EF)(CD \cdot AF) = 0$ expresses a relation between six elements of first species $ABCDEF$; it is of second degree in each of these elements, and is satisfied if two of the elements coincide. Suppose five of the elements are given. This is an equation of second degree in the sixth, representing a locus of second order that contains the five elements given. The preceding relation is nothing but the relation between six points of a conic expressed by the hexagram of Pascal. [...] (Peano 1889)

The above-mentioned equation can be interpreted (in Peano's sense) as saying something about the six points A, B, C, D, E, F , i.e. that they are inscribed in a conic (they occur in a quadratic equation), or as saying something about the three points of intersection of the segments joining the conic points in two by two (AB, DE) , (BC, EF) and (CD, AF) , i.e. that they are collinear (the equation says that the product is zero).²⁵ Shifting from one interpretation to the other is a way to grasp the theorem of Pappus-Pascal, which affirms that if six arbitrary points are chosen on a conic and joined by line segments to form a hexagon, then the three pairs of opposite sides of the hexagon (extended if necessary) meet at three points which lie on a straight line.

The two examples explain both the origin of Peano's notion of meaning (different 'readings' of the same symbol by substitution of sentences of the ordinary mathematical language correspond to different interpretations) and the epistemological fruitfulness of the move from one interpretation to another in the dynamical, axiomatic construction of a theory, or in the epistemological understanding of a result expressed by means of mathematical equations. A mathematical theory (in some informal sense) is not fully contained in or exhausted by its axiomatic presentation, which should work as a tool to grasp similarities and differences with respect to other theories or other parts of the same theory. In this sense I claim that Peano's interest in logic was truly meta-mathematical, because he investigated different possible definitions of the same symbol, and considered different possible interpretations of formal symbols through substitution of ordinary language sentences.

6. Peano and Leibniz

It is well known that Peano often mentions Leibniz in the *Formulario* and in his

²⁵ The different interpretation is related to the fact that in the second case the primitives are the segments AB, DE, BC, EF, CD, AF , which allows us to apply the rule that the product of two segments is the point of intersection of their lines. When one is saying that the product of three points is zero if they are aligned, one is actually saying that the product of the intersection of the segments AB and DE , BC and EF and CD and AF is zero. So we can read the equation $(AB \cdot DE)(BC \cdot EF)(CD \cdot AF) = 0$ as expressing a property of the points A, B, C, D, E, F or a property of the points AB, DE, BC, EF, CD, AF .

writings on the universal language. Many references have been detected and commented on in the literature: scholars oscillate between considering Peano's readings of Leibniz as guided by a true interest for his logico-linguistic enterprise (Roero 2011, 89) or by the search for an authoritative forerunner of his own work (Luciano 2012, 41). This was not uncommon at the time, and Peano was strongly influenced by Grassmann's geometric calculus, which was developed independently from Leibniz's *analysis situs*.²⁶ But Peano did not only relate his geometrical calculus to Leibniz's *analysis situs*: he also explicitly described Leibniz's project of a *Speciosa Generalis* as a sort of universal Language or Writing System, where the symbols guide reasoning; he made appeal to Leibniz's *calculemus* as a means to put useless controversies to an end; and---using Schröder's words---claimed he had solved the Leibnizian problem in the *Arithmetices Principia*.²⁷ Not only does Peano mention Leibniz very often in the *Formulario*, but as he became more and more interested in the history of logic and mathematics, he suggested that Vacca, and afterwards Couturat, conduct research on Leibniz's unpublished manuscripts at the Hannover Library (Roero 2011). But is Leibniz's undeniable influence on Peano a proof of his logicism?

First, I claim that Leibniz's heritage, combined with the influence of the algebra of logic tradition, generates a linguistic understanding of semantics, which is neither representational nor conceptualist, but rather is based on the idea that the terms of a symbolic formal language have several possible interpretations, each one determined by substitution of words and propositions of the natural language for the symbols of the formal language.

Second, Leibniz was interested not only in conceptual analyses as ways to determine the relation between simple and complex ideas, but also in logical calculi as ways to determine the relation between simple and complex terms that are used to express those ideas. It is exactly this understanding of a logical calculus that can be found in Peano, who aimed to express propositions as equations, and deduction between propositions as a resolution method for systems of equations (Peano 1889a, Lolli 2011). Peano, like Leibniz and Grassmann, oscillates between the aim of building as many characteristics as there are domains of investigation, and the aim to build a characteristic of all characteristics: "what holds for the characteristic language holds for the characteristic calculus too: there is a tension between specific calculi, like the geometric calculus, and the idea of a general calculus ratiocinator that should operate on real characters" (Cantù 2014).²⁸ Here, I further claim that what holds for the characteristic and the calculus also holds for semantics: a conceptualist type of semantics at the level of specific theories is combined with what I called a linguistic type of semantics at the formal level (see § 3).

There are two main differences with respect to what is usually described as a conceptualist (or as a three-level: words / concepts / objects) semantics. First, a term is primitive with respect to a given set of symbols: being a primitive is a relative and not an absolute property of symbols (there is no need for a primitive term to express some fundamental idea at the level of the *ordo essendi*). Second, Peano's symbols do not express exactly the same concept in all contexts: the symbol of equality is used to express a relation of equivalence between individuals in one section and a mutual implication between propositions in another section. This is usually taken to be a mistake or a limit in Peano's

²⁶ For a further discussion of this point see De Risi 2007, and Cantù 2003, 319-320.

²⁷ Incidentally, Schröder disagreed on that point, claiming that the members of the Peano school were still using 'sailing boats' by the time 'steamboats' had been invented (Grattan-Guinness).

²⁸ Whereas in Leibniz the tension involved the right order of concepts (should it be first established as a metaphysical *ordo essendi* in the general characteristic?), Peano believes that a specific calculus can be developed without preliminarily establishing a general characteristic: the foundational and didactical advantage of each part of the *Formulario* is already evident before a complete dictionary of all uses of linguistic terms can be achieved [Peano 1896, 4].

move toward a conceptualist form of semantics. On the contrary, we have taken it to be a sign of the fact that Peano developed a linguistic approach to semantics. Symbols do not mean concepts. Like entries in a dictionary that get their meaning only when they are inserted in a given linguistic context, so the meaning of symbols can be determined only through a preliminary substitution with linguistic sentences, and in each substitution they express the concepts expressed by the corresponding words in ordinary mathematical language.

Peano adopts a linguistic semantics for the symbols of the formal language, as he declares that symbols ‘mean’ words of the ordinary mathematical language that have a fixed meaning, which corresponds to their use in the mathematical practice. Even if Peano shared this tendency with the algebra of logic tradition, he was further influenced by Leibniz’s characteristic with respect to the idea that the signs should ‘naturally’ mirror what they stand for. Yet this should not be interpreted as being grounded in a representational or conceptualist type of semantics: symbols do not stand for concepts nor for things, but for natural language words: for example ε (being a member of) stands for the Greek word ‘ $\varepsilon\sigma\tau\iota$ ’, and ι (definite description) stands for the word ‘illo’, and the symbol for the inverse operation should be an inverted iota, mirroring the inversion of the function.²⁹ It is in this sense that Leibniz’s idea of a characteristic containing ‘real’ characters is not completely abandoned in Peano’s perspective. This emerges with even more force in Peano’s investigations into universal languages. Like the root of a word in ancient languages (e.g Arabic or Sanskrit) already expresses a part of the meaning of the words that are obtained by adding prefixes and suffixes to it, so formal symbols already express a part of the meaning of the natural language words that can be substituted for them in specific mathematical theories. At least in this sense Peano can be considered as an heir of Leibniz’s logicism: the idea that signs ‘naturally’ mirror what they stand for is compatible with Peano’s linguistic semantics.

7. Conclusion

Peano’s approach could thus be characterized as a form of structural algebraism, influenced by Leibniz (§6), based on the distinction between a logical and a mathematical notion of function (§3), on a peculiar notion of interpretation and meaning (symbols mean linguistic entities), (§4) on a genuine use of abstraction (§4), and on a strong metatheoretical interest for alternative definitions that is related to his epistemological and dynamic understanding of axiomatics (§5).

Why I call this approach *algebraism* should be clear from what was said in § 3 about Grassmann’s influence,³⁰ and more could be said about its relation to Schröder’s algebra of logic. Its structural features were illustrated in § 5, where I suggested that Peano’s main interest in axiomatics was to identify the logical as the common ‘structuring element’ of different theories. But the primacy of the notion of function over that of relation and the influence of geometric duality on the development of the notion of interpretation also suggests a structural approach, intended not as a philosophical theory based on the idea that all there is to mathematical objects is their structure, but as a methodological structuralism that considers that objects could carry many structures, and that their relations should be investigated.

This structural algebraism cannot be understood without taking into account that

²⁹ Quine considers Peano’s exploitation of inverses to reflect “his sensitivity to the inner logic of natural language” (Quine 1987).

³⁰ See for example Grattan-Guinness’ (2000, 126) distinction between the algebraic logic of Schröder and the mathematical logic of Frege, but note also that he includes Peano in the tradition of mathematical rather than algebraic logic.

Peano's enterprise was an investigation that concerned at the same time logic, mathematics, and linguistics, as well as theoretical and didactic practices.³¹ But can it then be considered as a variety of logicism? If logicism is considered to be related in its various forms to a mathematical practice that is centered on the analysis of the relation between logical and mathematical notions, on the study of the conditions and limits of a formal treatment of mathematical objects based on a genuine abstraction, on the effort to highlight the logical inferences drawn between primitive and derived propositions of an axiomatic system, then structural algebraism could still be a variety of logicism, especially if one also takes into account Leibniz's interest in looking for a general characteristic and for a 'natural' correspondence between symbols and what they stand for. If logicism is taken to be a reductionist enterprise aiming at introducing a unique symbolic calculus that can be interpreted on a unique domain of objects, and at giving a unique and definitive presentation of arithmetic by means of logical concepts, then algebraist structuralism is definitely not a variety of logicism.

To conclude, Peano's approach to logic does not coincide with a mathematical treatment of logic, nor with a general and systematic "investigation of the logical components of mathematics" (Zermelo 1908, 1). In other terms—using a distinction by Jourdain (1914, viii) that was later popularized by van Heijenoort (van Heijenoort 1967)—Peano did not develop a *calculus ratiocinator* nor a *lingua characteristica*, or at least he had an objective that was broader than both. van Heijenoort's characterization neglects the epistemological and metamathematical aspect of Peano's enterprise that was highlighted through the analysis of what he says about definitions. The choice between different kinds of definitions cannot be done in purely abstract terms, without taking into account specific mathematical practices and specific theoretical systems, without considering syntactic and semantic virtues that a mathematical definition should satisfy, as well as the consequences of a given definition on the order and virtues of successive definitions in the presentation of a theory. These epistemological interests cannot be separated from Peano's didactic motivation.

Peano's 'mathematical logic' differs both from the algebra of logic tradition that used mathematical symbols to express logical calculi, and from logical investigations centered on the effort to understand the functional nature of predication. Logic is rather, in a Grassmannian sense, the invariant that emerges by algebraic syntactic substitutions, the root of mathematics intended in a linguistic rather than reductionist sense, or, in Bourbaki's words, the structuring element of mathematics. The content of the logical inquiry includes the analysis of mathematical notions and theories as well as the mathematical practices in which they are formed and the procedures that characterize them. As Vailati himself remarked, a distinctive feature of the *Formulario* was a joint interest in history, linguistics, mathematics and logic that can be evinced from the historical annotations, from the application of linguistic categories to the analysis of function (pre-functions and post-functions are introduced by analogy with prefixes and suffixes in natural language), from the effort to study the dynamics and the evolution of theories, as if they were living organisms, or, we could add, as if they were living languages.³²

³¹ For an analysis of the strict relation between Peano's logics and linguistics, as well as of the deep interaction between Peano's philosophical views and didactics see the contributions published in Roero 2010 and Lolli 2001. As an example, I will just quote Peano's remarks on definitions: "the best definition is the one that the teacher prefers" (Peano 1915, 112).

³² "In the *Formulario* of Peano the importance given to historical data has steadily increased, especially under the inspiration of one of the principal collaborators, Vacca (among other things an enthusiastic investigator of the development of mathematics in the Far East); and the importance attributed to articles of this kind now constitutes one of the most noteworthy among the distinctive characteristics of the method of treatment of the various branches of mathematics that the said *Formulario* presents. Theories are therein expounded, not as in the

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ordinary treatment, under their “static” aspect—as one could express it—their aspect of repose; but under that of movement and development—not in the conventional attitudes of stuffed animals, with glass eyes; but as organisms, which live, eat, struggle, reproduce: or at least like figures in a cinematograph, with some naturalness of progression and development. To this tendency to recognize the identity of theories, beyond or under differences of expression, symbolism, language, representative conventions and the rest, is to be attributed also the constant interest of the mathematical logicians in linguistic questions” (Vailati 1906a)

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