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Giuseppe Peano and his School: Logic, Epistemology and Didactics

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Peano's axioms for arithmetic, published in 1889, are ubiquitously cited in the writings on modern axiomatics, and his *Formulario* is often quoted as the precursor of Russell's *Principia Mathematica*. Yet, a comprehensive historical and philosophical evaluation of the contributions of the Peano School to mathematics, logic, and the foundation of mathematics is still to be achieved. In line with an increased interest of philosophy of mathematics in the investigation of mathematical practices, this thematic issue adds some contributions to a possible reconstruction of the philosophical views of the Peano School emerging from logical, mathematical, linguistic, and educational practices, as well as from the interactions with contemporary scholars in Italy and abroad (Cantor, Dedekind, Frege, Russell, Hilbert, Bernays, Wilson, Amaldi, Enriques, Veronese, Vivanti and Bettazzi).

1. The Peano School

It is debatable whether the group can be classified as a “scientific school”, and an exhaustive list of all the members be uniquely determined. The category of research mathematical school, explored in its distinctions and national features by David Rowe (2002), has recently been opposed to the category of a mathematical tradition. According to José Ferreiros (1999, pp. xxii-xxiii) a research mathematical school is “a group led normally by only one mathematician, localized within a single institutional setting, and counting on a significant supply of advanced students”, whereas a mathematical tradition “implies that one can find a common research orientation in different actors that do not share a common institutional site, but are linked by traceable influences on each other”. To settle the question whether the Peano group should be considered as a research school or as a mathematical tradition, one has to deconstruct several clichés occurring in the literature and clarify the nature of Peano's leadership, the circulation of knowledge within the Peano School, and the role of other collective enterprises beside the *Formulario* (e.g., the *Rivista di Matematica*, the journal *Schola et Vita*, the *Dizionario*, as well as other contemporary articles and teaching materials). Original contributions have been recently achieved through the exploitation of new archival sources: the discovery of new collaborators not previously mentioned, the distinction of different levels of decision-making in Peano's redaction of the *Formulario*, as well as new insights on the original contribution of each member to shared knowledge in the group (Luciano 2017).

2. The philosophical interest

The school is generally considered as a phenomenon that appeared suddenly in all its splendor at the Paris congress of 1900 and then extinguished with the rapidity of a firework that leaves a vivid but indefinite memory. Given the long-lasting impression made on Russell and other participants in the 1900 Paris Conferences in Mathematics, Philosophy and Psychology by Peano's, Burali-Forti's, Padoa's, Pieri's, and Vailati's contributions, the literature has often searched for reasons to explain a general loss of philosophical interest in the Peano school in the first half of the 20th century. General explanations abound in the literature: the non-academic nature of the group; the multiform topics of interest going from mathematical analysis to geometry, from linguistics to universal languages, from philosophical pragmatism to logicism (Roero 2010); the scarce attention given to the transformation of mathematics and

to the development of set theory after 1910; a general belief that Peano was not really interested in the theory of inferential reasoning, or in the metalogical and metamathematical investigation of the properties of axiomatic theories (van Hejenhoort 1967).

Other philosophical explanations have also been suggested: Peano's utilitarian approach to logic (Grattan Guinness 2000); the lack of a shared and explicit epistemological framework concerning relevant logical and methodological issues such as functions (Luciano 2017, Cantù 2021), logical identities (Cantù 2007), definitions by abstraction (Mancosu 2018), and questions of purity (Arana and Mancosu 2012); a subdivision of labor that made Giovanni Vailati in Italy (Arrighi et al. 2009) and Louis Couturat in France (Luciano and Roero 2005) the chief philosophical spokesmen of the group; the belief that Peano's presentation of arithmetical axioms had less interesting philosophical implications with respect to logicism and structuralism than that of Dedekind (Ferreiros 2005); the interest of Peano's collaborators in pedagogical and political issues (Giacardi 2006, Luciano 2012).

The topic is reconsidered in a new light in this special issue, as the authors discuss the relationship between Dedekind's and Peano's axioms (Kahle, this volume), the absence of the universal quantifier among the primitive symbols of Peano's *Formulario* and its relation to the use of free variables (von Plato, this volume), the peculiarities of Peano's symbolic notation (Schlimm, this volume), the lack of recognition of Pieri's pedagogical remarks in Italy (Marchisotto and Millan Gasca, this volume), the early association in the USA with Russell's point of view (Lolli, this volume), the interaction between Peano's auxiliary international language project and the internationalization movement at the beginning of the century (Aray, this volume), the limits of Peano's proof of the impossibility of infinitesimals (Freguglia, this volume).

3. Logic and epistemology

Some of the usual explanations lose their effectiveness, not only because of some new specific results, but also because of the interdisciplinary and practical turn suggested by the intertwining of logic and epistemology, whereby the latter is taken to mean, as in neo-Latin languages, both the analysis of scientific knowledge and the critique of scientific theories. This perspective constituted the red thread of an international project (PICS INTEREPISTEME 2018-2020) co-funded by the French National Research Center and the Vienna Circle Institute and co-directed by Paola Cantù and Georg Schiemer in collaboration with Erika Luciano at the University of Turin. The objective was to compare three distinct collaborative and interdisciplinary epistemologies developed by the members of the Peano School, the editorial board of the *Revue de Métaphysique et de Morale*, and the Vienna Circle. The project showed various points of connection between collaborative and interdisciplinary approaches and educational and political aims, such as the vulgarization of scientific knowledge, and the criticism of disciplinary and national boundaries. But it focused on the origins and development of non-mainstream philosophical views that cannot be reduced to logicism or structuralism, and investigated the underestimated influence of Leibniz's philosophy (Luciano 2006, Cantù 2014), 19th century positivism, empiricism, and neo-criticism on these standard views in philosophy of mathematics (Cantù and Schiemer, forthcoming).

The partial receiving of the specificity of the school's research program was due to the misunderstanding of the deep relation between education, linguistics, and axiomatics, but also to a simplistic association of Peano's ideas with Russell's philosophy. This tendency did not only emerge in van Hejenhoort's remarks on the lack of inference rules and metatheoretical investigations, or in the quick tendency to classify Peano as a logicist but was already evident in the early reception of Peano in the USA. Gabriele Lolli shows how the works of the Peano's school were discussed by Edwing B. Wilson already in 1904 in a review of two writings of

Bertrand Russell, so that the two conceptions were eventually combined in “the Peano-Russell point of view”.

The ability to discriminate subtle differences between the positions of Russell, Frege and Peano characterized a fine reader of Peano’s writings: Kurt Gödel. If the philosophical notebooks (*Max Phil*) reveal a deep understanding of differences on the notions of function and definite description (Crocco et al. 2017, Cantù 2016a), the summary of the *Formulario* to be found in one of his *Excerptenhefte* shows the analysis of the rules of inferences used in deductive chains. The accurate summary of Peano’s *Arithmetices Principia* written in the Gabelsberger shorthand on a loose paper at the time when he was preparing the article on Russell’s logic (early 1943) and edited by Jan von Plato for this special issue, attests that Gödel read not only the *Formulario*, but also the *Arithmetices Principia*, focusing his comments on the formal character of proofs.

4. *The philosophy implicit in the school’s practices*

The attention to mathematical practices showed that Peano not only had a strong impact on the writings by Frege, Russell, Carnap and Gödel, but developed a proper philosophical view that emerges from the logical investigation of definitions, the logical interpretation of the symbols of a formal language, the distinction between relations and functions, and the difference between primitive and derived terms or propositions in an axiomatic system. Peano’s philosophical view, distinct both from logicism and from structuralism, emerges as a result of a joint investigation of logic, language and mathematics, considered both as theoretical and didactic practices. The interest in definitions and the analysis of language had significant effects on Peano’s semantics, which differs from what is usually described as a conceptualist (or as a three-level: words / concepts / objects) semantics, because symbols refer to concepts only through the mediation of language. Like entries in a dictionary that get their meaning only when they are inserted in a given linguistic context, the symbols’ meaning can be determined only through a preliminary substitution with linguistic sentences, and in each substitution the symbols refer to the concepts expressed by the corresponding words in ordinary mathematical language (Cantù 2021).

This volume constitutes a further decisive step in the direction of reconstructing Peano’s philosophical views from a detailed analysis of logical, mathematical, pedagogical but also linguistic practices. The essays gathered here focus on Giuseppe Peano’s, Alessandro Padoa’s and Mario Pieri’s works, but the same method can be fruitfully applied to other members of the school, such as Giovanni Vailati (Cantù and de Zan 2009), Cesare Burali-Forti and Alessandro Padoa. Besides, the contributions of Peano and other members of the school are evaluated by comparison with contemporaries (Richard Dedekind, Gottlob Frege, Bertrand Russell, David Hilbert and Paul Bernays), resulting in a historically accurate analysis of some subtle but fundamental differences between their respective projects, which aimed aim to present, analyze or ground mathematics as a rigorous, deductive science.

Three examples will be briefly mentioned in this introduction: axiomatics, linguistic symbolization and rigor. Different terms are often used to characterize the school’s foundational enterprise: symbolization, formalization, axiomatization, reduction. A deep investigation of the Peano school’s practices might help disentangle some of the differences between these fundamental notions, and shed new light on different ways to conceive generality, ideography, metatheoretical inquiries, and the role of notation, intuition, and rigor.

5. *Axiomatics*

General philosophical and historical reconstructions of the development of logic in the early 20th century have accustomed us to think of Peano as one of the fathers of modern

axiomatics because of his contribution to the formulation of the axioms of arithmetic, which still bear his name. Yet, a detailed analysis of the connections between logical, linguistic, mathematical and pedagogic writings of the Peano School might help reevaluate his contributions to logic and philosophy of mathematics and discover a specific approach to axiomatics. An axiomatization is a particular kind of presentation of a theory, where the logical and the mathematical content is specified by the respective axioms. Reinhard Kahle's contribution traces a history of the formulation of the properties of real numbers, considered as *Sätze* by Dedekind and explicitly formulated as axioms by Peano, acquiring a non-logical nature in Hilbert's works and a first-order formulation in Bernay's contributions. This is a historically fruitful example of how the investigation of different uses and presentations of the same mathematical properties of numbers can reveal very different conceptions of the axiomatization of arithmetic.

Yet, axiomatics cannot be reduced to the investigation of the axiomatic formulation of single theories. It is a back-and-forth between syntactic, semantic and pragmatic-linguistic level and their goals: 1) to make explicit the implicit assumptions of a theory (e.g. by stating all the hypotheses necessary to prove a given theorem); 2) to investigate the tacit assumptions of a theory, considering what happens if they are not implicitly assumed (e.g. by testing the possibility of creating non-standard models of a theory); 3) to define the scope and goals of a research program or discipline (Woodger 1959). The linguistic analysis is a pillar of Peano's approach, and cannot be dissociated from epistemological goals, such as the search for the good order and the minimal number of concepts, and the questioning of the relation between mathematical practices and a rigorous mathematical language. The symbolization of logic, far from being exclusively aimed at the construction of axiomatic systems or the investigation of deductive inferences, rests on questions very similar to those that have developed in the social sciences: the need to distinguish the simple from the complex, the first for us from the first for itself, a canonical form from deviant forms, the definition of a term from the formation of a concept, the pragmatic consequences of a hypothesis from its theoretical role (Cantù 2020).

Axiomatics heavily relies on complex practices of symbolization and formalization, practices that have a social and interactive nature, and that should be studied in their different components: phases of redaction (exam of the pertaining bibliography, construction of a hypothetic-deductive order of the collected results, codification in symbols), division of the tasks among group members, circulation of knowledge within the group in a hierarchical or peer context, and the construction of domains of shared knowledge, that need not be mirrored in the final version of publications (Luciano 2017).

6. *Symbolization and language*

Symbolization is a process that associates symbols to words, but symbols can play the role of schematic letters, as in Hilbert's formalization, i.e. as terms having a merely formal sense that allows for a variety of interpretations, or have a substantive role, as terms whose meanings have to be conveyed by elucidation (Klev 2011). Bertran San-Millan's contribution explains how Frege used the symbols of arithmetic as canonical names, i.e. as symbols with a specific and fixed meaning, so that the mathematical letters always have a specific domain, determined by the intended application. Peano shared a similar substantive understanding of mathematical symbols in his early writings but moved towards a view of undefined symbols as uninterpreted non-logical constants devoid of meaning, as soon as he investigated, together with Padoa, metatheoretical questions concerning the independence of the axioms.

The comparative investigation of logic, linguistics and notational practices offer further insights on the particular version of ideography that is developed in the *Formulario*: symbols

mean ideas, but they are first introduced as names for terms of an interpreted mathematical language having those ideas as meaning, and then also considered as schematic variables that might receive different interpretations by substitution of different linguistic terms. The relation between mathematical symbols, words of mathematical language, mathematical concepts and mathematical objects is a complex history that a comparative investigation of Peano's contributions to logics, mathematics, linguistics, and symbolic notation might help disentangle.

The symbolization of mathematics is often discussed in the light of a reduction of mathematics to logic or as a translation that preserves the relevant mathematical meaning, but it cannot be fully understood without a detailed investigation of the design principles, the didactical and practical constraints that accompany the search for technical symbols in a new notation. The formalization is a way to distinguish the logical form from the non-logical content but can also be conceived as a method of conceptual analysis that identifies the relevant logical and mathematical ideas. As Dirk Schlimm shows in his contribution, this analysis might be used to determine the primitive terms and propositions, but also to check the adequacy of the analysis itself, thereby evaluating whether definitions are correct, and proofs are rigorous.

The distinction between symbolization and formalization is often difficult to trace, but the investigation of definitions and of metatheoretical issues of independence between axioms and an attentive investigation of the interactions with linguistics might be of help. There are several aspects of Peano's approach to the *Interlingua* that relate it to mathematical logic: in both cases a language in use (mathematical language and Latin) is taken as a starting point for the development of a universally understandable language (logical symbolism, Interlingua); secondly the two enterprises are based on collaborative networks; both are grounded in the heritage of Leibniz' *characteristica universalis*; finally they are combined in the last edition of the *Formulario*, written in Latin and symbolic language (Cantù 2016b). In her contribution, Basak Aray highlights another similarity: the connection between algebra and grammar developed in the *Formulario*, and suggests that the symbolization developed in Peano's mathematical practice guided the design of his proposal for an international auxiliary language: the *Latino sine flexione*.

This algebraic understanding of grammar better explains how logic and language are both presented in an equational form, and generality is expressed using free variables instead of assuming a universal quantifier as a primitive logical term. Jan von Plato takes this to be the reason why Peano's axiomatic systems, like Schröder's algebraic logic, lacked some principles of reasoning with the quantifiers, even if they contained other rules of inferences.

7. *Mathematical education and rigor*

The interrelation between mathematical education and conceptual analysis offers further hints to understand the main traits of the Peano school's epistemology, the importance of rigor in scientific knowledge and education as well as the interpretation of axiomatics as a metatheoretical investigation based on a variety of alternative conceptual analyses leading to different axiomatic presentations and definitions of the mathematical concepts. The attention to the pedagogic component in the Peano School showed that to the sociological singularity of this research team (the only non academic-based group in the international panorama) corresponds a unique educational project to rigor that was deeply intertwined with the mathematical, philosophical, logical and linguistic views of the group, and that had non-negligible effects on the evolution of mathematical teaching in Italy, and beyond. Rigor is not an accessory or external element that can be imposed on mathematical teaching, but a result of the development of rational mathematics and the evolution of all sciences towards the structure

of axiomatic-deductive systems. Rigor is not primarily a foundational problem, but vice versa the foundational enterprise is intertwined with didactic concerns (Luciano 2020).

Rigor is as a distinctive feature of the Peano School's style but also an essential topic in the Italian debate on mathematical pedagogic theory and teaching practice at the turn of the century. Peano's crusade in defense of rigor is not only a distinctive mark of his axiomatics, but a feature of the school's linguistic, mathematical and educational research programs. It was neither a negation of the importance of experimental methods in the early stages of mathematical education (Luciano 2020), nor a simplistic negation of mathematical intuition, which was banished from the proofs of a theory but remained central in the choice of the axioms (Rizza 2009). It was rather a didactical objective developed through exchanges with school teachers and their associations, the publication of new textbooks, and the participation to educational Governmental Committees (Giacardi 2006).

Elena Marchisotto and Ana Millan Gasca illustrate in their contribution Pieri's belief that an integration of sensible and rational intuition can deeply renew the teaching of geometry but also deploy a profound heuristic value. The analysis of Pieri's axiomatization of geometry exemplifies the almost symbiotic relation between axiomatics and pedagogy that is typical of the Peano school, as well as the partial and complex receiving of this idea in the works by Italian contemporary mathematicians, such as Enriques and Amaldi.

Yet, the very same idea of rigor gave rise to famous debates, whose philosophical objectives were sometimes obscured by putting forth educational motivations or formal demonstrations. The famous debate with Segre on rigor and intuition was not only a manifesto of Peano's style, but also an implicit criticism of Veronese's hyperspaces and the expression of the rivalry with the geometrical Italian school (Luciano 2020).

Similarly, Freguglia claims that the famous proof of the impossibility of infinitesimals was not only developed to complete and rectify an untenable proof by Cantor—undergoing the similar mistake of presupposing an axiom that is equivalent to the Archimedean axiom and therefore incompatible with the existence of infinitesimals. It was also an implicit criticism of Veronese's theory of a geometrical non-Archimedean continuum and the occasion for hosting a scientific discussion of the topic in the newly founded *Rivista di Matematica*.

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¹ The published and unpublished writings of Peano are collected in Roero 2008. For a rich literature on Peano and other members of the school see in particular Luciano 2017. A complete list of Padoa's writings can be found in Cantù 2007. The publications of Vailati and a rich literature on his life and works are listed in Cantù and De Zan 2009. On Pieri see in particular the references quoted in Marchisotto and Smith 2007.

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