



# The minimum number of spanning trees in regular multigraphs II: the even-degree case

Jakub Pekárek, Jean-Sébastien Sereni, Zelealem B Yilma

## ► To cite this version:

Jakub Pekárek, Jean-Sébastien Sereni, Zelealem B Yilma. The minimum number of spanning trees in regular multigraphs II: the even-degree case. 2021. hal-03472283

**HAL Id: hal-03472283**

**<https://hal.science/hal-03472283>**

Preprint submitted on 9 Dec 2021

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# The minimum number of spanning trees in regular multigraphs II: the even-degree case

Jakub Pekárek\*

Jean-Sébastien Sereni†

Zealelem B. Yilma‡

## Abstract

In a companion article [The minimum number of spanning trees in regular multigraphs I: the odd-degree case, submitted for publication], we answered a question raised in earlier works by determining the minimum number of spanning trees in a connected  $d$ -regular  $n$ -vertex multigraph — and identifying all graphs achieving this smallest number — for all odd values of  $d$  and all relevant  $n$ . The general approach developed there seems relevant to study also all even values of  $d$ . However, some additional technicalities need to be dealt with, probably due to the fact the parity of  $n$  is not constrained anymore. In particular, the boundary cases (i.e. when  $d$  and  $n$  are small) show more irregularities than in the odd-degree case, where only two regimes existed regarding the relation between  $n$  and  $d$ .

Specifically, we prove that when  $n$  (and  $d$ ) are even, then every connected  $d$ -regular  $n$ -vertex multigraph  $G$  has at least  $\frac{nd}{2}(d-1)^{n/2-1}$  spanning trees, with equality if and only if  $G$  is the even cycle with edges of alternating multiplicities 1 and  $d-1$ , unless  $d=4$  and  $n \in \{6, 8, 10\}$  (in which case the bound is lower, and reached only by the padded paddle graph). If  $n$  is odd (and  $d$  is even), then every connected  $d$ -regular  $n$ -vertex multigraph  $G$  has at least  $\frac{1}{8}(d-1)^{(n-5)/2}(3d^2-4d-4)(d(n-1)-2)$  spanning trees, with equality if and only if  $G$  is the fish graph formed by a triangle and an (odd) cycle with edges of multiplicities only 1 and  $d-1$  that share a single vertex, unless  $n=3$  or the pair  $(d, n)$  belongs to a small number of sporadic values, in which case the (still unique) extremal graph is different.

## 1 Introduction

The reader is referred to the companion article [7] for more context about the problem we study. We write  $\tau(G)$  for the number of spanning trees of a multigraph  $G$ , and we define

$$\delta'_d = \liminf_{n \rightarrow \infty} (\tau(G))^{1/n}$$

where the infimum is taken over all connected  $d$ -regular  $n$ -vertex loopless multigraphs. Alon [1] sketched a neat argument proving that  $\delta'_d$  has order exactly  $\sqrt{d}$ , where the lower bound is obtained by a slight modification of his Theorem 1.1. Further, he explained that the conclusion

---

\*Computer Science Institute, Charles University, Prague, Czech Republic. Email: [pekarej@iuuk.mff.cuni.cz](mailto:pekarej@iuuk.mff.cuni.cz).

†Service public français de la recherche, Centre national de la recherche scientifique (ICube, CSTB), Strasbourg, France. Email: [sereni@kam.mff.cuni.cz](mailto:sereni@kam.mff.cuni.cz).

‡Carnegie Mellon University in Qatar. Email: [zyilma@qatar.cmu.edu](mailto:zyilma@qatar.cmu.edu). Supported by the Qatar Foundation through Carnegie Mellon Qatar's Seed Research program.

of van der Waerden’s conjecture, which had been already established by then [3, 4] (nowadays the reader can also consult Gurvits’s proof [5] for an elementary and totally different argument) implies that any  $d$ -regular  $n$ -vertex loopless multigraph actually contains  $(\Omega(\sqrt{d}))^n$  linear forests — that is, forests such that each connected component is a path.

We shall provide an exact formula for  $\delta'_d$  for all even values of  $d$ , and actually even the exact minimum value of  $\tau$  over the class of connected  $d$ -regular  $n$ -vertex loopless multigraphs for even values of  $d$  and all possible values of  $n$ . We in addition explicitly provide all graphs attaining this minimum value. (If loops are allowed, then adding loops to a simple cycle provides a  $d$ -regular  $n$ -vertex simple graphs with exactly  $n$  spanning trees for all values of  $n$  and even values of  $d$ .)

A similar result for odd values of  $d$  can be found in the companion article [7], which is motivated by and extends a recent result by Bogdanowicz [2], who had considered the number of spanning trees of cubic multigraphs.

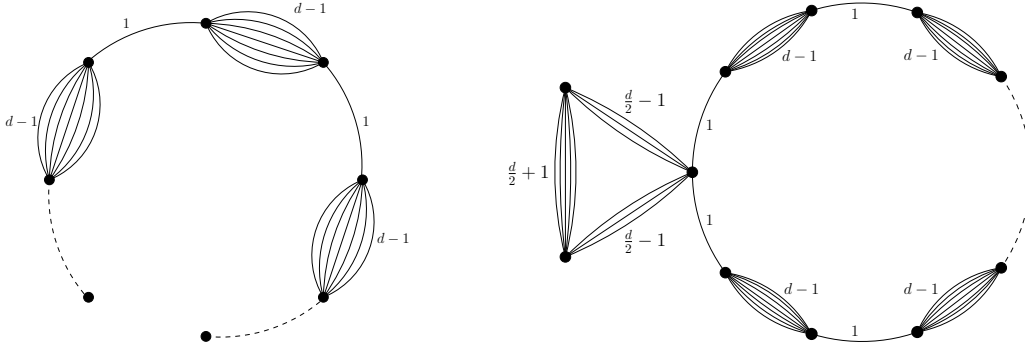


Figure 1: The graphs  $PC_{d,n}$  (left) and  $\mathcal{H}_{d,n}$  (right).

For convenience, we here recall the definition of some specific graph families, which turn out to be extremal in some cases, and we also introduce a new family.

**Definition 1.** Let  $d$  be an even integer greater than 2 and  $n$  an integer greater than 3.

- If  $n$  is even, then  $PC_{d,n}$  is the *padded cycle graph*, illustrated in Figure 1, left: it is the  $d$ -regular cycle graph on  $n$  vertices with edges of alternating multiplicities 1 and  $d - 1$ .
- A  *$d$ -regular fish graph* is any  $d$ -regular multigraph obtained from the disjoint union of two odd cycles (possibly with edges of multiplicity greater than 1) by identifying two vertices belonging to distinct cycles.
- Let  $\mathcal{H}_{d,n}$  be the  $d$ -regular fish graph on  $n$  vertices which consists of a triangle and an odd cycle  $C_{n-2}$  that share one vertex, where the multiplicities of the edges in the triangle are  $\frac{d}{2} - 1$ ,  $\frac{d}{2} - 1$  and  $\frac{d}{2} + 1$  (and hence edge multiplicities in  $C_{n-2}$  are either 1 or  $d - 1$ ), illustrated in Figure 1, right.
- For  $n \geq 5$ , let  $PP_{4,n}$  be the *degree 4 padded paddle graph* illustrated in Figure 2: we start from an  $(n - 4)$ -vertex path with end-vertices  $u_1$  and  $u_2$  and all edges of multiplicity 2, and then for each  $i \in \{1, 2\}$  we add a triangle with edge multiplicities 1, 1 and 3 and identify its vertex of degree 2 with  $u_i$  (notice that  $u_1 = u_2$  if  $n = 5$ , and then  $PP_{4,5}$  is isomorphic to the 4-regular fish graph  $\mathcal{H}_{4,5}$ ).

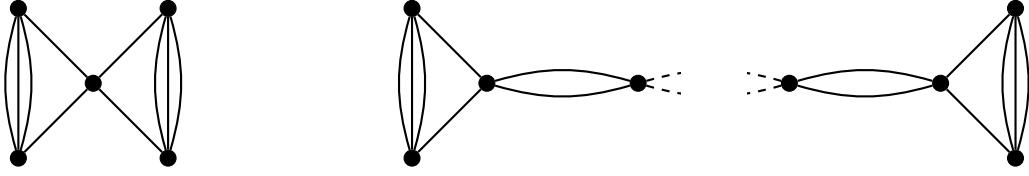


Figure 2: The 4-regular padded paddle graph  $PP_{4,n}$  for  $n = 5$  (left) and for larger values of  $n$  (right). The padded paddle graph  $PP_{4,5}$  is isomorphic to the 4-regular fish graph  $\mathcal{H}_{4,5}$ .

We gather the number of spanning trees for the graphs defined in Definition 1. The straightforward proof is omitted.

**Lemma 2.** *Let  $d$  be an even integer greater than 2.*

- For any integer  $n \geq 2$ ,

$$\tau(PC_{d,n}) = \frac{n}{2}(d-1)^{n/2} + \frac{n}{2}(d-1)^{n/2-1} = \frac{nd}{2(d-1)}(d-1)^{n/2}.$$

- For any integer  $n \geq 5$ ,

$$\begin{aligned} \tau(\mathcal{H}_{d,n}) &= \left( \left( \frac{d}{2} - 1 \right)^2 + 2 \left( \frac{d}{2} - 1 \right) \left( \frac{d}{2} + 1 \right) \right) \left( \frac{n-1}{2}(d-1)^{(n-3)/2} + \frac{n-3}{2}(d-1)^{(n-5)/2} \right) \\ &= \frac{1}{8}(d-1)^{(n-5)/2}(3d^2 - 4d - 4)(d(n-1) - 2) \end{aligned}$$

- For any integer  $n \geq 5$ ,

$$\tau(PP_{4,n}) = 49 \cdot 2^{n-5}.$$

We state our result in several parts. We separate the even and odd cardinality into Theorems 3 and 4 — and actually prove each case separately. To ease readability, the degree-4 case is given alone in Theorem 5, being peculiar in the sense that it is the only case where the padded paddle graph can be optimal. Finally, a handful of sporadic cases excluded from Theorem 4 are presented in Proposition 7. Together these four parts give the unique minimiser graphs for each setting of  $n$  and  $d$  where  $d$  is even.

**Theorem 3.** *Let  $G$  be a connected  $d$ -regular  $n$ -vertex multigraph where  $d, n \geq 2$  are both even. Then,*

$$\tau(G) \geq \tau(PC_{d,n}) = \frac{nd}{2}(d-1)^{n/2-1}$$

*with equality if and only if  $G$  is isomorphic to  $PC_{d,n}$ , unless  $d = 4$  and  $n \in \{6, 8, 10\}$ .*

**Theorem 4.** *Let  $G$  be a connected  $d$ -regular  $n$ -vertex multigraph where  $d \geq 4$  is even and  $n \geq 5$  is odd. Then*

$$\tau(G) \geq \tau(\mathcal{H}_{d,n}) = \frac{1}{8}(d-1)^{(n-5)/2}(3d^2 - 4d - 4)(d(n-1) - 2)$$

*with equality if and only if  $G$  is isomorphic to  $\mathcal{H}_{d,n}$ , unless the pair  $(d, n)$  belongs to  $\{(4, 7), (4, 9), (4, 11), (4, 13), (6, 7), (6, 9), (6, 11), (8, 9)\}$ .*

**Theorem 5.** Let  $n \geq 4$  be an integer and let  $G$  be a connected 4-regular  $n$ -vertex multigraph. Then  $G$  is the unique graph minimising  $\tau(G)$  if and only if one of the following holds

- $G = PP_{4,n}$  and  $n \in \{5, \dots, 11\} \cup \{13\}$
- $G = PC_{4,n}$  and  $n$  is even and either 4 or at least 12
- $G = \mathcal{H}_{4,n}$  and  $n$  is odd and at least 15.

Here,  $\tau(PP_{4,n}) = 49 \cdot 2^{n-5}$ ;  $\tau(PC_{4,n}) = 2n \cdot 3^{\frac{n}{2}-1}$  and  $\tau(\mathcal{H}_{4,n}) = 7 \cdot (2n-3) \cdot 3^{\frac{n-5}{2}}$ .

**Definition 6.** For  $n$  odd and  $d$  even, let  $\mathcal{H}_{d,n}^*$  be the  $d$ -regular multigraph on  $n$  vertices obtained from the disjoint union of  $\frac{d}{2} - 1$  triangles and one odd cycle by identifying one vertex from each into a single central vertex. Note that the central vertex has exactly  $d$  distinct neighbours (hence it is incident only to edges of multiplicity 1) and all the other edge multiplicities alternate between 1 and  $d-1$  on each cycle.

**Proposition 7.** Let  $G$  be a connected  $d$ -regular  $n$ -vertex multigraph. If the pair  $(d, n)$  belongs to  $\{(6, 7), (6, 9), (6, 11), (8, 9)\}$ , then

$$\tau(G) \geq \frac{1}{2} \cdot (2d-1)^{d/2-1} (d(n-d+3) - 2)(d-1)^{(n-d-1)/2},$$

with equality if and only if  $G$  is isomorphic to  $\mathcal{H}_{d,n}^*$  (see Figure 3 for illustration).

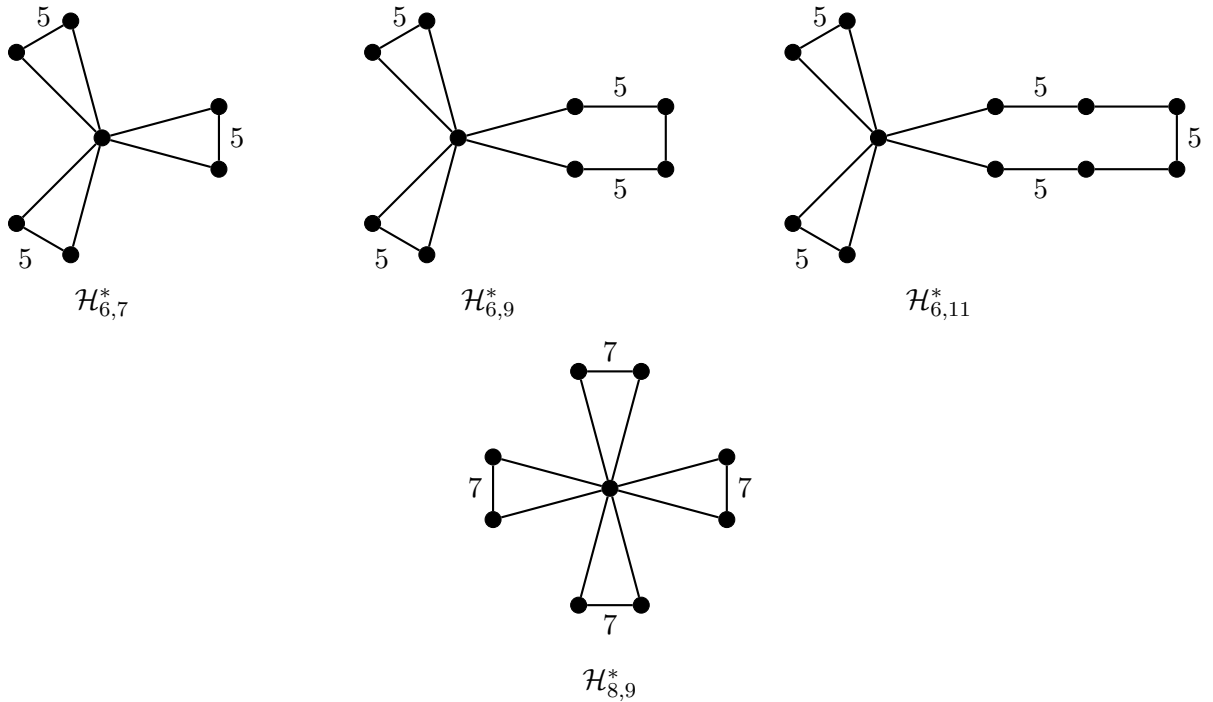


Figure 3: The unique minimiser  $\mathcal{H}_{d,n}^*$  in each of the four sporadic cases when  $d \geq 6$ . Parallel edges are represented by a single edge with the multiplicity written next to it.

The sequel is organised as follows. In Section 2 we first recall some results already mentioned in the companion article [7] and place them in our context or extend them if needed. Section 3

quickly clears out the cases where the number of vertices is at most 5, which serve as the base for an induction on the number of vertices. Sections 4 and 5 then respectively establish, when  $d \geq 6$ , Theorem 3, and Theorem 4 along with Proposition 7. The case of connected 4-regular multigraphs is kept separated and dealt with in Section 6. Contrary to the general case, when  $d = 4$  the analysis is cleaner if it is not split with respect to the parity of the number of vertices of the multigraphs considered.

## 2 Preliminaries

We here state some results established in the companion article [7]. The terminology stays the same. In particular, if  $G$  is a multigraph, then  $w_G(u, v)$  is the multiplicity of the edge  $uv$ , and by  $G - uv$  we mean the multigraph obtained from  $G$  by deleting all edges between the vertices  $u$  and  $v$ . We consider (loopless) multigraphs, and indifferently use the terms “multigraph” and “graph”. Should we want to consider a graph with no multiple edges, we will use the term *simple graph*.

**Proposition 8** ([7, Proposition 3]). *Let  $G, H_1, H_2$  be connected graphs on the same vertex set. If  $2G = H_1 + H_2$ , then  $\tau(G) \geq \tau(H_1)$  or  $\tau(G) \geq \tau(H_2)$  with at least one strict inequality unless  $G = H_1 = H_2$ .*

**Corollary 9** ([7, Corollary 4]). *Suppose  $G$  is a connected  $d$ -regular graph on  $n$  vertices that contains an even cycle with at least 4 vertices. Let  $M_1$  and  $M_2$  be the complementary perfect matchings of the cycle. If  $G$  minimises the number of spanning trees over all connected  $d$ -regular graphs on  $n$  vertices, then at least one of the graphs  $H_1 = G - M_1 + M_2$  and  $H_2 = G + M_1 - M_2$  is not connected.*

Given a subset  $X \subseteq V(G)$ , let  $\partial X$  be the number of edges with exactly one endpoint in  $X$ .

**Lemma 10** ([7, Lemma 10]). *Suppose  $G$  is a connected  $d$ -regular  $n$ -vertex multigraph minimising the number of spanning trees. If  $\partial_G\{u, v, w\} \leq d - 2$ , then, without loss of generality,  $N_G(v) \cup N_G(w) \subset \{u, v, w\}$ .*

Lemma 10 hints at the structural relevance of what we call pendant triangles. Let  $G$  be a connected multigraph with at least 4 vertices, and suppose that  $\{u, v, w\}$  is a set of three vertices inducing a triangle  $T$  in  $G$ . We say that  $T$  is a *pendant triangle* if at most, and hence exactly, one vertex in  $T$  has a neighbour not in  $T$ . This vertex is then an *articulation point* of  $G$ , and the other two vertices in  $T$  are the *terminal vertices* of  $T$ .

The next lemma will be useful to perform a graph operation central to our argument. It directly follows from the even parity of  $d$ .

**Lemma 11.** *If  $d$  is an even integer and  $G$  is a connected  $d$ -regular graph, then, for every vertex  $x$ , the graph  $G - x$  has at most  $d/2$  components. Moreover, if  $xy$  is an edge of multiplicity  $w_G(x, y)$ , then  $G - x - y$  has at most  $d - w_G(x, y)$  components.*

*Proof.* If  $C$  is a component of  $G - x$ , or of  $G - x - y$ , then because  $d$  is even there must be an even (and positive, as  $G$  is connected) number of edges in  $G$  between  $C$  and  $\{x\}$ , or between  $C$  and  $\{x, y\}$ , respectively. The statements follow.  $\square$

As in the companion article [7], a central component of our argument is a ‘lifting’ operation similar to one used by Ok and Thomassen [6]. Let  $x, y_1$  and  $y_2$  be three distinct vertices in a graph  $H$ , and suppose that  $f_i$  is an edge in  $H$  between  $x$  and  $y_i$ , for  $i \in \{1, 2\}$ . *Lifting*  $f_1$  and  $f_2$  means deleting the two edges  $f_1$  and  $f_2$  and adding an edge between  $y_1$  and  $y_2$ . If  $x$  is a vertex of degree  $2m$  in  $H$ , a *complete lift* of  $x$  is the process of first performing a sequence of  $m$  lifts of pairs of edges incident with  $x$  and then deleting the vertex  $x$  (which is, by then, isolated), thereby producing a multigraph  $H_x$ . Observe that it is possible to perform a complete lift of  $x$  if there does not exist a vertex  $y$  such that  $w_H(x, y) \geq m + 1$ . It is possible to produce a connected multigraph  $H_x$  via a complete lift of  $x$  if, in addition,  $H$  is connected and  $H - x$  has at most  $m + 1$  components. Together with Lemma 11, this implies that in a connected graph where every vertex has even degree, it is possible to obtain a connected graph by performing a complete lift of an arbitrary vertex as long as the vertex shares at most half its edges with any particular neighbour. The following proposition shows that we can sequentially lift two such vertices in an optimal graph.

**Proposition 12.** *Suppose  $G$  is a connected  $d$ -regular multigraph minimising the number of spanning trees and let  $u$  and  $v$  be two vertices in  $G$  not incident to edges of multiplicity more than  $d/2$ . Then it is possible to sequentially completely lift  $u$  and  $v$  to produce a connected  $d$ -regular graph.*

*Proof.* First, suppose that there is no vertex  $x$  such that  $\{u, v, x\}$  forms a triangle in  $G$ . Let  $G_u$  be any connected multigraph obtained by completely lifting  $u$  in  $G$ . Then for every neighbour  $w$  of  $v$  in  $G$  different from  $u$ , the multiplicities of the edge  $vw$  in  $G$  and in  $G_u$  are the same. In addition, for any new neighbour  $z$  that  $v$  might gain when lifting  $u$ , the multiplicity of the edge  $vz$  in  $G_u$  cannot exceed the multiplicity of the edge  $vu$  in  $G$ , and hence is at most  $\frac{d}{2}$ . It follows that  $v$  can be completely lifted in  $G_u$ , as required.

Now suppose that there is a vertex  $x$  such that  $\{u, v, x\}$  forms a triangle  $T$  in  $G$ . Since  $G$  does not contain a diamond by Corollary 9, the vertices  $u$  and  $v$  have no common neighbour other than  $x$ . Recall also that by Lemma 10, if  $\{u, v, x\}$  induces more than  $d$  edges, then the triangle  $T$  must be pendant. As a pendant triangle contains at most one vertex not incident to an edge of multiplicity greater than  $\frac{d}{2}$ , we deduce that  $w_G(u, v) + w_G(u, x) + w_G(v, x) \leq d$ . Since  $d = w_G(u, v) + w_G(u, x) + \sum_{y \neq v, x} w_G(u, y)$ , we infer that  $w_G(v, x) \leq \sum_{y \neq v, x} w_G(u, y)$ . Consequently, when lifting  $u$ , we may choose to pair at least  $\min\{w_G(u, v), w_G(v, x)\}$  edges between  $u$  and  $v$  with edges incident to  $u$  but not to  $x$ . Then, after the complete lift of  $u$ , the multiplicity of the edge  $vx$  in the obtained graph  $G_u$  is at most  $\max\{w(u, v), w(v, x)\} \leq d/2$ , and therefore we can completely lift  $v$  in  $G_u$ .  $\square$

**Theorem 13** (Ok and Thomassen [6]). *Let  $H$  be a graph with a vertex  $x$  of degree  $2m$ . Let  $H_x$*

be a graph obtained from  $H$  by a complete lift of  $x$ . Then

$$\tau(H) \geq c_m \tau(H_x),$$

where

$$c_m = \min_{d_1, d_2, \dots, d_k} \min_X \frac{\prod_{i=1}^k d_i}{\tau(X)},$$

where the minimum is taken over all sequences of positive integers  $d_1, \dots, d_k$  with varying length  $k$  such that  $\sum_{i=1}^k d_i = 2m$ , and over all connected  $k$ -vertex graphs  $X$  with degree sequence  $d'_1, d'_2, \dots, d'_k$  such that  $d'_i \leq d_i$  for each  $i \in \{1, \dots, k\}$ .

Ok and Thomassen [6] have determined the values of  $c_m$  for  $m \in \{1, \dots, 4\}$ . We have calculated a few more values using exhaustive computer search. The values, and the graphs attaining them, are given in Table 1. (For  $m = 5$ , the diamond graph, the graph obtained from  $K_4$  by removing one edge, also attains the value  $c_5 = 9/2$ .)










m	1	2	3	4	5	6	7	8	9
$c_m$	1	2	$\frac{8}{3}$	$\frac{18}{5}$	$\frac{9}{2}$	$\frac{81}{16}$	6	$\frac{48}{7}$	$\frac{375}{49}$
$\frac{2(m+1)}{3}$	$\frac{4}{3}$	2	$\frac{8}{3}$	$\frac{10}{3}$	$\frac{4}{3}$	$\frac{14}{3}$	$\frac{16}{3}$	6	$\frac{20}{3}$
extremal graphs									

Table 1: Values of the Ok-Thomassen  $c_m$  term, the lower bound from Proposition 14, and graphs attaining the  $c_m$  values.

The next statement is established in the companion article.

**Proposition 14** ([7, Proposition 9]). *For all positive integers  $m$  we have  $c_m > 2m/e$ , where  $e$  is the base of the natural logarithm. In addition,  $c_m \geq 2(m+1)/3$  for all  $m \geq 2$  with strict inequality if  $m \geq 4$ .*

We are now ready to perform a full analysis of the even-degree case.

### 3 Base Cases

**Lemma 15.** *For all even  $d \geq 4$ , the unique connected  $n$ -vertex  $d$ -regular multigraph minimising the number of spanning trees is*

1. for  $n = 2$ : an edge of multiplicity  $d$ ;
2. for  $n = 3$ : a triangle with all edges of multiplicity  $d/2$ ;
3. for  $n = 4$ : the padded cycle graph  $PC_{d,4}$ ;
4. for  $n = 5$ : the fish graph  $\mathcal{H}_{d,5}$ .



*Proof.* Let  $G$  be a connected  $n$ -vertex  $d$ -regular multigraph that minimises the number of spanning trees. For  $n = 2$  and  $n = 3$ , the graphs given in the statement are, in fact, the unique connected  $d$ -regular  $n$ -vertex multigraphs. It is possible to look at these graphs as degenerate forms of the padded cycle and of  $\mathcal{H}_{d,n}$ , respectively.

For  $n = 4$  and  $n = 5$ , consider first the underlying simple graph  $H$  of  $G$ . As  $G$  is connected and regular, every vertex must have degree at least 2 in  $H$ . In addition, we know, by Corollary 9 of Proposition 8, that a graph containing a cycle of length 4 and a path (edge-disjoint from the cycle) connecting opposite vertices of the cycle cannot be optimal. When  $n = 4$ , this leaves  $C_4$  as the only option for  $H$ . Applying Corollary 9 again, we see that  $G$  must be the padded cycle graph  $PC_{d,4}$ .

For  $n = 5$ , we may also disallow, by Lemma 10, those underlying simple graphs containing a triangle with only one vertex of degree 2. Consequently, we infer that  $H$  is isomorphic to either  $C_5$  or the butterfly graph (obtained from the disjoint union of two triangles by identifying two vertices belonging to distinct triangles). If  $H$  is isomorphic to  $C_5$ , then in  $G$  all edges have equal multiplicity  $d/2$ , implying that  $\tau(G) = 5 \cdot (d/2)^4 = \frac{5}{16} \cdot d^4$ . If  $H$  is isomorphic to the butterfly graph, then Proposition 8 implies that the two triangles have to be as lopsided as possible. It follows that  $G$  is isomorphic to  $\mathcal{H}_{d,5}$ , and hence  $\tau(G) = \frac{1}{4} \cdot (3d^2 - 4d - 4)(2d - 1)$ , which is smaller than  $\frac{5}{16} \cdot d^4$ . Therefore  $G$  must be  $\mathcal{H}_{d,5}$ , as announced.  $\square$

## 4 Proof of Theorem 3

We prove Theorem 3 by induction on the number  $n$  of vertices. Throughout this section, the integer  $n$  is assumed to be even.

A key element of our inductive proof is the observation that, for  $n \geq 6$ ,

$$\frac{\tau(PC_{d,n})}{\tau(PC_{d,n-2})} = \frac{n(d-1)}{n-2} \leq \frac{3}{2}(d-1).$$

Then, whenever  $PC_{d,n-2}$  minimises the number of spanning trees among connected  $d$ -regular multigraphs on  $n-2$  vertices, we can argue that a connected  $d$ -regular multigraph  $G$  on  $n$  vertices cannot be optimal if  $\tau(G) > \frac{3}{2}(d-1)\tau(G')$  for some connected  $d$ -regular multigraph  $G'$  on  $n-2$  vertices.

Fix  $n \geq 6$  and suppose that  $\tau(G') \geq \tau(PC_{d,n-2})$  for every connected  $d$ -regular multigraph  $G'$  on  $n-2$  vertices. Let  $G$  be a connected  $d$ -regular  $n$ -vertex multigraph minimising the number of spanning trees.

**Lemma 16.** *Let  $u \sim_G v$ . If  $d/2 + 1 \leq w_G(u, v) \leq d - 2$ , then there exists a vertex  $z$  such that*

$$w_G(u, v) + w_G(u, z) = d = w_G(v, u) + w_G(v, z).$$

*In other words,  $u$  and  $v$  are terminal vertices of a (same) pendant triangle.*

*Proof.* First note that the hypothesis of the lemma cannot hold for  $d = 4$ . So suppose that  $d \geq 6$  and set  $m = w_G(u, v)$ . Suppose there is no vertex  $z$  such that  $w(u, z) + w(v, z) \geq d - m + 1$ . Let  $G'$  be a connected  $d$ -regular multigraph on  $n-2$  vertices obtained from  $G$  by deleting

all  $m$  edges between  $u$  and  $v$  and identifying the two vertices  $u$  and  $v$  into a new vertex  $x$ . Notice that  $\tau(G) \geq m \cdot \tau(G')$ . Now, since  $x$  has degree  $2(d - m)$  in  $G'$  and  $w_{G'}(x, z) = w_G(u, z) + w_G(v, z) \leq d - m$  for every neighbour  $z$  of  $x$  in  $V(G')$ , we can completely lift the vertex  $x$  in  $G'$ , which produces a connected  $d$ -regular multigraph  $G'_x$  on  $n - 2$  vertices. It follows from Theorem 13 and Proposition 14 that

$$\begin{aligned} \tau(G) &\geq m \cdot c_{d-m} \cdot \tau(G'_x) \geq \frac{2m(d-m+1)}{3} \cdot \tau(G'_x) \\ &\geq 2(d-2) \cdot \tau(G'_x) \geq 2(d-2) \cdot \tau(PC_{d,n-2}) \\ &> \frac{3}{2}(d-1) \cdot \tau(PC_{d,n-2}) \geq \tau(PC_{d,n}), \end{aligned}$$

where we used that  $d \geq 6$  for the third line and  $m \mapsto m(d-m+1)$  is decreasing over  $[d/2+1, d-2]$  for the second line. Therefore,  $G$  fails to be optimal unless there is a vertex  $z$  such that  $w_G(u, z) + w_G(v, z) \geq d - m + 1$ . Consequently, the set  $\{u, v, z\}$  induces at least  $d+1$  edges in  $G$ , and hence Lemma 10 yields that  $\{u, v, z\}$  induces a pendant triangle in  $G$ . As  $w_G(u, v) \geq d/2 + 1$ , the vertex  $z$  must be the articulation point. This concludes the proof.  $\square$

It follows from Lemma 16 that every vertex of  $G$  that is not a terminal vertex of a pendant triangle either is incident to an edge of multiplicity  $d-1$ , or is incident only to edges of multiplicity at most  $d/2$ . We call a vertex *exceptional* if all its incident edges have multiplicity at most  $d/2$ .

**Lemma 17.** *If  $d \geq 6$ , then  $G$  has at most one exceptional vertex.*

*Proof.* Suppose, for the sake of contradiction, that  $u$  and  $v$  are distinct exceptional vertices. By Proposition 12, it is possible to produce a connected  $d$ -regular  $(n-2)$ -vertex multigraph  $G'$  by sequentially lifting  $u$  and  $v$ . Theorem 13 ensures that  $\tau(G) \geq (c_{d/2})^2 \cdot \tau(G')$ , and hence  $\tau(G) > \frac{d^2}{e^2} \cdot \tau(G')$  by Proposition 14.

If  $d \geq 10$  then  $\frac{d^2}{e^2} > \frac{3(d-1)}{2}$ , and it thus follows that

$$\tau(G) > \frac{3}{2}(d-1)\tau(G') \geq \frac{3}{2}(d-1)\tau(PC_{d,n-2}) \geq \tau(PC_{d,n}).$$

If  $d = 8$ , then we use the exact value of  $c_4$  to deduce the following:

$$\tau(G) \geq (c_4)^2 \tau(G') = \frac{18^2}{5^2} \tau(G') > \frac{3 \cdot 7}{2} \tau(PC_{8,n-2}) \geq \tau(PC_{8,n}).$$

For  $d = 6$ , as  $c_3 = 8/3$  we have  $c_3^2 = 64/9 < 15/2 = 3(d-1)/2$ . However,  $64/9 > 4(d-1)/3 \geq \tau(PC_{6,n})/\tau(PC_{6,n-2})$  for  $n \geq 8$  and our assertion will hold if we show that  $PC_{6,6}$  is optimal for  $n = 6$ . Indeed, let  $G_u$  be a connected 6-regular multigraph obtained from  $G$  by completely lifting  $u$ . As  $\tau(\mathcal{H}_{6,5}) = 220 \leq \tau(G_u)$  by Lemma 15, it follows that

$$\tau(G) \geq c_3 \cdot \tau(G_u) \geq \frac{8}{3} \cdot 220 > 450 = \tau(PC_{6,6}),$$

which conclude the proof.  $\square$

As mentioned earlier, it follows from Lemma 16 that every non-exceptional vertex of  $G$  either has an edge of multiplicity  $d-1$  or belongs to a pendant triangle. If all vertices are of the

former type, then  $G$  must be the padded cycle  $PC_{d,n}$ . Otherwise, since  $G$  is connected, edges of alternating multiplicities 1 and  $d - 1$  must form odd cycles. Consequently, if  $d \geq 6$ , then Lemma 17 implies that  $G$  must be a collection of odd cycles (and pendant triangles) all sharing one vertex. However, this requires  $G$  to have an odd number of vertices, a contradiction. This establishes the statement of Theorem 3 whenever  $d \geq 6$ .

## 5 Proof of Theorem 4 and Proposition 7

In this section, the integer  $n$  is assumed to be odd. We will simultaneously prove Theorem 4 and Proposition 7 by first identifying a family of graphs which includes the stated extremal structures for the two statements. Next, we show that, within this family, the graphs  $\mathcal{H}_{d,n}$  and  $\mathcal{H}_{d,n}^*$  are the minimisers of  $\tau$  for the stated values of  $d$  and  $n$ . We then establish the desired result by proving that, for  $d \geq 6$ , any minimiser of  $\tau$  over the whole class of connected  $d$ -regular multigraphs must belong to the above-identified family.

Let  $\mathcal{F}_{d,n}$  be the class of all connected  $d$ -regular  $n$ -vertex multigraphs that consist of pendant triangles and padded odd cycles (i.e., containing only edges of multiplicity 1 or  $d - 1$ ) all sharing the same vertex. This vertex is called the *centre* of the graph. Since the centre is a cut-vertex, the number of spanning trees of such a graph is the product of the number of spanning trees of the odd cycles and triangles composing it. Let  $F_{d,n}$  be a minimiser of  $\tau$  over the class  $\mathcal{F}_{d,n}$ .

### 5.1 Optimising within $\mathcal{F}_{d,n}$ .

We start by analyzing edge multiplicities within the pendant triangles.

**Lemma 18.** *All but at most one pendant triangle in  $F_{d,n}$  contain edges of multiplicities 1 and  $d - 1$ .*

*Proof.* Suppose, on the contrary, that  $T_1 = \{u, v, w\}$  and  $T_2 = \{u, x, y\}$  are two pendant triangles in  $F_{d,n}$  sharing the vertex  $u$  and containing no edges of multiplicity 1. Consider an even walk  $W = uvwuxyu$  that starts with  $u$  then traverses both triangles (arbitrarily choosing any of the (at least two) edges between consecutive vertices) and let  $M_1 = \{uv, wu, xy\}$  and  $M_2 = \{vw, ux, yu\}$  be the complementary sets of alternating edges in  $W$ . We note that both  $H_1 = F_{d,n} - M_1 + M_2$  and  $H_2 = F_{d,n} + M_1 - M_2$  are connected, since every edge induced by  $T_1 \cup T_2$  has multiplicity at least 2. Furthermore,  $H_1, H_2 \in \mathcal{F}_{d,n}$  and  $2F_{d,n} = H_1 + H_2$ . Proposition 8 then applies and contradicts the optimality of  $F_{d,n}$ .  $\square$

**Lemma 19.** *There is at most one cycle in  $F_{d,n}$  that is not a triangle.*

*Proof.* Suppose, on the contrary, that  $C_1$  and  $C_2$  are cycles in  $F_{d,n}$  of lengths  $2k + 1$  and  $2\ell + 1$ , respectively, where  $2 \leq k \leq \ell$ . Recall that each edge in  $C_1 \cup C_2$  has multiplicity either 1 or  $d - 1$  so

$$\tau(C_1 \cup C_2) = (d - 1)^{k-1}(d(k + 1) - 1)(d - 1)^{\ell-1}(d(\ell + 1) - 1).$$

Consider the graph  $F'$  where  $C_1$  and  $C_2$  are replaced by  $C'_1$  and  $C'_2$  where  $C'_1$  is a triangle and  $C'_2$  has length  $2(k + \ell) - 1$ . Then  $F' \in \mathcal{F}_{d,n}$  and

$$\tau(C'_1 \cup C'_2) = (2d - 1)(d - 1)^{k+\ell-2}(d(k + \ell) - 1).$$

As  $2 < k + 1 \leq \ell + 1 < k + \ell$ , it follows by convexity that  $\tau(C'_1 \cup C'_2) < \tau(C_1 \cup C_2)$ , contradicting the optimality of  $F_{d,n}$  in  $\mathcal{F}_{d,n}$ .  $\square$

For every integer  $t$  such that  $1 \leq t \leq \min\{d/2 - 1, (n - 3)/2\}$ , let us define the  $d$ -regular multigraph  $H_{d,n}^t \in \mathcal{F}_{d,n}$  as one having  $t - 1$  triangles with edge multiplicities 1 and  $d - 1$  one odd cycle of length  $n - 2t$  and one triangle with edge multiplicities at the centre  $v$  being  $(d - 2t)/2$ . Lemmas 18 and 19 show that  $F_{d,n}$  must be isomorphic to  $H_{d,n}^t$  for some integer  $t$ . In fact, we can make the following stronger statement.

**Lemma 20.** *We have  $F_{d,n} \in \{H_{d,n}^1, H_{d,n}^{t^*}\}$  where  $t^* = \min\{d/2 - 1, (n - 3)/2\}$ .*

*Proof.* Observe that

$$\tau(H_{d,n}^t) = \frac{1}{8}(2d - 1)^{t-1}(d - 2t)(3d + 2t)(d(n - 2t + 1) - 2)(d - 1)^{(n-2t-3)/2}.$$

If  $2 \leq t \leq t^* - 1$ , we compare the number of spanning trees of  $H_{d,n}^t$  with the number of spanning trees of  $H_{d,n}^{t-1}$  and  $H_{d,n}^{t+1}$  to obtain

$$\begin{aligned} \frac{\tau(H_{d,n}^{t+1})\tau(H_{d,n}^{t-1})}{\tau(H_{d,n}^t)^2} &= \frac{(d - 2t - 2)(d - 2t + 2)}{(d - 2t)^2} \times \frac{(3d + 2t + 2)(3d + 2t - 2)}{(3d + 2t)^2} \\ &\quad \times \frac{(d(n - 2t - 1) - 2)(d(n - 2t + 3) - 2)}{(d(n - 2t + 1) - 2)^2} \\ &< 1, \end{aligned}$$

since each of the three fractions is less than 1 by convexity. Consequently,  $\tau(H_{d,n}^t)$  is minimised either at  $t = 1$  or at  $t = t^* = \min\{d/2 - 1, (n - 3)/2\}$ .  $\square$

All that remains is to compare these two configurations. Note that for  $d = 4$ , the two configurations are identical and therefore optimal within  $\mathcal{F}_{d,n}$ . Observe also that if  $n = 5$ , then necessarily  $t^* = 1$ , and hence  $H_{d,5}^1 = H_{d,5}^{t^*}$ . We now consider the case where  $n \geq 7$ .

**Lemma 21.** *For all even  $d \geq 6$  and odd  $n \geq 7$  we have*

- $\tau(H_{d,n}^1) < \tau(H_{d,n}^{t^*})$  if  $(d, n) \notin \{(6, 7), (6, 9), (6, 11), (8, 9)\}$ .
- $\tau(H_{d,n}^{t^*}) < \tau(H_{d,n}^1)$  if  $(d, n) \in \{(6, 7), (6, 9), (6, 11), (8, 9)\}$ .

*Proof.* First note that

$$\tau(H_{d,n}^1) = \tau(\mathcal{H}_{d,n}) = \frac{1}{8}(d - 2)(3d + 2)(d(n - 1) - 2)(d - 1)^{(n-5)/2}$$

and

$$\tau(H_{d,n}^{t^*}) = \begin{cases} \frac{1}{2}(2d - 1)^{d/2-1}(d(n - d + 3) - 2)(d - 1)^{(n-d-1)/2} & \text{if } n \geq d + 1 \text{ and} \\ \frac{1}{4}(2d - 1)^{(n-3)/2}(d - n + 3)(3d + n - 3) & \text{if } n \leq d + 1. \end{cases}$$

Consider the function

$$f_d: n \mapsto \frac{\tau(H_{d,n}^1)}{\tau(H_{d,n}^{t*})} = \begin{cases} \frac{(d-2)(3d+2)(d-1)^{d/2-2}}{4(2d-1)^{d/2-1}} \cdot \frac{(d(n-1)-2)}{(d(n-d+3)-2)} & \text{if } n \geq d+1 \text{ and} \\ \left(\frac{d-1}{2d-1}\right)^{(n-5)/2} \cdot \frac{1}{2} \cdot \frac{d-2}{2d-1} \cdot \frac{(3d+2)(d(n-1)-2)}{(d-n+3)(3d+n-3)} & \text{if } n \leq d+1. \end{cases}$$

Let us first look at the cases  $d = 6$  and  $d = 8$ . We can simplify the above expressions to obtain

$$f_6(n) = \frac{100(3n-4)}{121(3n-10)} \quad \text{for } n \geq 7; \quad f_8(7) = \frac{299}{300} \quad \text{and} \quad f_8(n) = \frac{637(4n-5)}{1125(4n-21)} \quad \text{for } n \geq 9.$$

It follows that  $f_6(n) \geq 1$  if and only if  $n \in \{7, 9, 11\}$  and  $f_8(n) \geq 1$  if and only if  $n = 9$ .

We now study the function  $x \mapsto f_d(x)$  for  $x \in [5, \infty)$ , for any fixed even value of  $d \geq 10$ . The typical shape for  $f_d(x)$  is given in Figure 4, where  $f_d(5) = 1$  and the curve has a local minimum at  $x = d - 1$  and a local maximum at  $x = d + 1$ . Indeed, we will prove that  $f_d(x)$  is decreasing on the two intervals  $[5, d - 1]$  and  $[d + 1, \infty)$ . Then, establishing that  $f_d(d + 1) < 1$  completes the proof of the lemma.

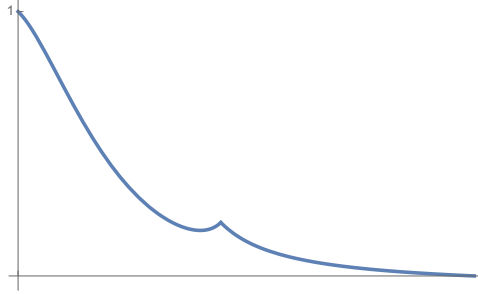


Figure 4: Typical curve for the ratio of  $\tau(H_1)$  and  $\tau(H_{t*})$ .

Let us start with the case where  $x \geq d + 1$ . Observe that

$$\frac{f_d(x+2)}{f_d(x)} = \frac{(d(x+1)-2)(d(x-d+3)-2)}{(d(x-1)-2)(d(x-d+5)-2)}.$$

In the numerator as well as in the denominator, the two terms of the product sum to  $2dx - d^2 + 4d - 4$ . As  $x - d + 3 < x - d + 5 \leq x - 1 < x + 1$ , convexity implies that the quotient is less than 1 and, therefore, that  $f_d(x+2) < f_d(x)$ .

Next assume that  $5 \leq x \leq d-3$ . Then

$$\begin{aligned}
\frac{f_d(x+2)}{f_d(x)} &= \frac{d-1}{2d-1} \cdot \frac{(d-x+3)(3d+x-3)}{(d-x+1)(3d+x-1)} \cdot \frac{dx+d-2}{dx-d-2} \\
&< \frac{d-1}{2d-1} \cdot 1 \cdot \frac{(d-x+3)}{(d-x+1)} \cdot \frac{dx+d-2}{dx-d-2} \\
&= \frac{d-1}{2d-1} \cdot \left( 1 + \frac{2(d^2+2d-2)}{d(d-x+1)(x-1-2/d)} \right) \\
&\leq \frac{d-1}{2d-1} \cdot \left( 1 + \frac{2(d^2+2d-2)}{(d-4)(4d-2)} \right) \tag{1} \\
&= \frac{d-1}{2d-1} \cdot \frac{3d^2-7d+2}{2d^2-9d+4} \\
&= \frac{3d^3-10d^2+9d-2}{4d^3-20d^2+17d-4} \\
&= 1 - \frac{d^3-10d^2+8d-2}{4d^3-20d^2+17d-4} \\
&< 1.
\end{aligned}$$

The last inequality holds because  $d \geq 10$ , and inequality (1) uses the fact that for every fixed  $d \geq 10$ , the function  $x \mapsto -d(x-d-1)(x-1-2/d)$  is minimised over  $[5, d-3]$  when  $x = 5$ .

It remains to deal with the special case where  $x = d+1$ . Then we can check directly that  $f_{10}(11) = \frac{2286144}{2476099} < 1$ . Moreover, for  $d \geq 12$  we have

$$\begin{aligned}
f_d(d+1) &= \frac{(d-2)(3d+2)(d^2-2)(d-1)^{(d-4)/2}}{8(2d-1)^{d/2}} \\
&< \frac{(d-1)(3d+2)(d+1)(d-1)(d-1)^{d/2-2}}{8(2d-1)^{d/2}} \\
&= \frac{(3d+2)(d+1)(d-1)^{d/2}}{8(2d-1)^{d/2}} \\
&< (3d+2)(d+1)2^{-3-d/2} \\
&< 1.
\end{aligned}$$

This concludes the proof. □

We gather here our observations about the ratios  $\tau(F_{d,n})/\tau(F_{d,n-2})$ .

**Observation 22.** *For all even  $d \geq 6$  and odd  $n \geq 7$  we have*

$$\frac{\tau(F_{d,n})}{\tau(F_{d,n-2})} \leq \frac{3d-1}{2d-1} \cdot (d-1).$$

For  $d = 6$ , the above inequality holds with equality only for  $n = 9$ . Otherwise, we have the stronger bound

$$\frac{\tau(F_{6,n})}{\tau(F_{6,n-2})} \leq \frac{23 \times 5}{17} < 7 \quad \text{for odd } n \neq 9.$$

*Proof.* The statement follows by direct computation, recalling that  $F_{d,n} = H_{d,n}^{t*}$  if  $(d,n) \in \{(6,7), (6,9), (6,11), (8,9)\}$  and  $F_{d,n} = H_{d,n}^1$  otherwise. One needs to note that for every fixed  $d$ , the function  $x \mapsto \frac{(d(x-1)-2)}{(d(x-3)-2)}$  is decreasing with  $x$ . □

## 5.2 A minimiser must belong to $\mathcal{F}_{d,n}$ .

**Lemma 23.** *Suppose  $G$  is a minimiser of  $\tau$  over the class of connected  $d$ -regular  $n$ -vertex multigraphs, where  $d \geq 6$ . Then  $G \in \mathcal{F}_{d,n}$ .*

*Proof.* Note first that if  $n = 5$ , then the lemma holds. We now assume that  $n \geq 7$  and that the lemma holds for graphs on  $n - 2$  vertices. We begin by showing that edges of multiplicity  $m$  where  $3 \leq m \leq d - 2$  must be part of pendant triangles. Suppose that  $3 \leq m = w_G(u, v) \leq d - 2$  and the vertices  $u$  and  $v$  do not belong to a same pendant triangle. This implies there is no vertex  $z$  such that  $\{u, v, z\}$  induce a triangle with more than  $d$  edges. Let  $G'$  be a connected graph obtained by deleting all edges between  $u$  and  $v$ , identifying the vertices  $u$  and  $v$  into a new vertex  $x$  and completely lifting the vertex  $x$ . Then  $G'$  is a connected  $d$ -regular  $(n - 2)$ -vertex multigraph and

$$\tau(G) \geq m \cdot c_{d-m} \cdot \tau(G') \geq \frac{2m(d-m+1)}{3} \tau(G') \geq 2(d-2) \tau(G') \geq 2(d-2) \tau(F_{d,n-2}).$$

Since  $2(d-2) > \frac{3d-1}{2d-1}(d-1)$ , we then deduce from Observation 22 that  $\tau(G) > \tau(F_{d,n})$ , a contradiction.

We deduce from the above statement that any vertex that is incident to an edge of multiplicity between  $d/2$  and  $d - 2$  must be a terminal vertex in a pendant triangle — recalling that  $d \geq 6$ . Now consider vertices that are not terminal vertices in a pendant triangle. It follows that such vertices are either exceptional or incident to an edge of multiplicity  $d - 1$ . We next argue that an optimal graph cannot have more than one exceptional vertex. Before proving this, let us immediately show how this property allows us to conclude the proof. Suppose that the optimal graph must have at most one exceptional vertex and all remaining vertices must either be terminal vertices in a pendant triangle or incident to an edge of multiplicity  $d - 1$ . Then a path with edges of alternating multiplicities 1 and  $d - 1$  must eventually close and form an odd cycle. Therefore, the optimal graph must be a member of  $\mathcal{F}_{d,n}$ .

It thus remains to prove the announced property. Suppose, for the sake of contradiction, that  $u$  and  $v$  are both exceptional vertices. By Proposition 12, we can obtain a connected  $d$ -regular  $(n - 2)$ -vertex multigraph  $G'$  by sequentially lifting the two vertices  $u$  and  $v$ . Then,

$$\tau(G) \geq (c_{d/2})^2 \cdot \tau(G') \geq (c_{d/2})^2 \cdot \tau(F_{d,n-2}).$$

Therefore,  $G$  fails to be optimal if  $(c_{d/2})^2 > \tau(F_{d,n})/\tau(F_{d,n-2})$ . For  $d \geq 12$ , we observe, using Proposition 14, that

$$(c_{d/2})^2 > \frac{d^2}{e^2} > \frac{3d-1}{2d-1}(d-1) = \max_{\substack{n \geq 7 \\ n \text{ odd}}} \left\{ \frac{\tau(F_{d,n})}{\tau(F_{d,n-2})} \right\}.$$

For  $d \in \{8, 10\}$ , we use the actual values of  $c_5$  and  $c_4$  and obtain

$$c_5^2 = 81/4 > \frac{9 \cdot 29}{19} = \max_{\substack{n \geq 7 \\ n \text{ odd}}} \left\{ \frac{\tau(F_{10,n})}{\tau(F_{10,n-2})} \right\} \quad \text{and} \quad c_4^2 = \frac{324}{25} > \frac{7 \cdot 23}{15} = \max_{\substack{n \geq 7 \\ n \text{ odd}}} \left\{ \frac{\tau(F_{8,n})}{\tau(F_{8,n-2})} \right\}.$$

If  $d = 6$ , then

$$c_3^2 = \frac{64}{9} > 7 > \frac{23 \cdot 5}{17} \geq \frac{\tau(F_{6,n})}{\tau(F_{6,n-2})} \quad \text{for odd } n \neq 9.$$

These contradictions establish that  $G$  has at most one exceptional vertex unless  $d = 6$  and  $n = 9$ .

We now deal with the special case where  $d = 6$  and  $n = 9$ , for which we have to use a different, more structural, approach to prove that  $G$  cannot have more than one exceptional vertex. One fact we need is that  $H_{6,7}^2$  is the unique connected 6-regular 7-vertex with the fewest number of spanning trees. Indeed, the last two paragraphs do imply that  $F_{6,7} \in \mathcal{F}_{6,7}$  and then Lemma 21 yields the statement.

We already know that all edges of multiplicity 3 or 4 are part of pendant triangles — actually, edges of multiplicity  $d/2$  can never be part of a pendant triangle in a  $d$ -regular multigraph on more than 3 vertices, so that  $G$  has no edges of multiplicity 3. Now suppose that  $u$  and  $v$  do not belong to a same pendant triangle, and yet are linked by exactly 2 edges in  $G$ . If  $G - uv$  is disconnected, then let  $G_1$  and  $G_2$  be the two components of  $G - uv$  where  $G_1$  has fewer vertices than  $G_2$ . Without loss of generality, assume that  $u \in G_1$ . Note that  $G_1$  has at most 4 vertices and each of its vertices, except possibly  $u$ , has at least 2 neighbours in  $G_1$ .

Suppose first that  $G_1$  has 4 vertices. Then  $u$  has a unique neighbour  $u'$  in  $G_1$  for otherwise  $G_1$ , and therefore  $G$ , would contain a diamond as a subgraph, which is impossible. Since  $u$  and  $v$  are linked by 2 edges, it follows that  $u$  and  $u'$  are linked by 4 edges, and hence  $u, u'$  is part of a pendant triangle, which is a contradiction since  $G_1$  has 4 vertices. We deduce that  $G_1$  has 3 vertices, and hence is a triangle with edge multiplicities 2, 2, 4. In particular,  $\tau(G_1) = 20$ .

In  $G_2$ , the vertex  $v$  has degree 4 (and no edge of multiplicity more than 2). Let  $G'$  be a connected 6-regular 5-vertex multigraph obtained by completely lifting  $v$  in  $G_2$ . Now,

$$\tau(G) = 2\tau(G_1)\tau(G_2) \geq 40 \cdot c_2 \cdot \tau(G') \geq 80 \cdot \tau(F_{6,5}) = 80 \cdot 220 > 10285 = \tau(F_{6,9}),$$

which is a contradiction.

Consequently,  $G - uv$  is connected. In particular, every spanning tree of  $G - uv$  is a spanning tree of  $G$  that does not use any of the two edges between  $u$  and  $v$ . Let  $G''$  be the connected 8-vertex multigraph obtained by deleting all edges between  $u$  and  $v$  and next identifying  $u$  and  $v$  into a single vertex  $x$ . Note that every spanning tree of  $G''$  corresponds in a natural way to two spanning trees of  $G$  both containing an edge between  $u$  and  $v$ , and differing only on this edge. We infer that

$$\tau(G) \geq \tau(G - uv) + 2 \cdot \tau(G''). \quad (2)$$

We now perform some complete lifts in  $G - uv$  and in  $G''$  to obtain back connected 6-regular multigraphs. (The complete lifts we are going to make are possible since  $u$  and  $v$  are exceptional and do not belong to a same pendant triangle.)

Let  $G'$  be a connected 6-regular 7-vertex multigraph obtained from  $G - uv$  by subsequently lifting  $u$  and  $v$ . We have

$$\tau(G') \geq c_2^2 \cdot \tau(G - uv) \geq c_2^2 \cdot \tau(F_{6,7}) = 4 \cdot \tau(H_{6,7}^2). \quad (3)$$

Let  $G''_x$  be a connected 6-regular 7-vertex multigraph obtained from  $G''$  by completely lifting the new vertex  $x$ , so

$$\tau(G'') \geq c_4 \cdot \tau(G''_x) \geq \frac{18}{5} \cdot \tau(H_{6,7}^2). \quad (4)$$



Combining (2), (3) and (4), we obtain

$$\tau(G) \geq \left(4 + \frac{36}{5}\right) \tau(F_{6,7}) = \frac{56}{5} \cdot \tau(F_{6,7}) > \tau(F_{6,9}),$$

where the last inequality follows from Observation 22. This contradiction finishes to establish that edges of multiplicity 2 also belong to pendant triangles.

In what follows, most assertions implicitly rely on the fact that every edge of multiplicity greater than 1 and less than 5 is contained in a pendant triangle. Suppose now that  $G$  contains an exceptional vertex  $u$  that is not a cut-vertex. Then  $u$  must have 6 distinct neighbours, all in the same 2-connected block. Otherwise,  $u$  would be incident to an edge of multiplicity 2 (recalling that  $G$  has no edge of multiplicity 3), which has to be contained in a pendant triangle. Consequently,  $u$  would have to be a terminal vertex of this pendant triangle, contradicting that  $u$  is exceptional. Similarly, because  $u$  is not a cut-vertex, the subgraph  $H$  induced in  $G$  by the neighbours of  $u$  is simple. Moreover,  $H$  has maximum degree 1, for otherwise a vertex of degree at least 2 in  $H$  along with two of its neighbours and  $u$  would induce a subgraph of  $G$  containing a diamond, a contradiction to the optimality of  $G$ . But then  $u$  and all its neighbours induce at most 9 edges, and the two remaining vertices are incident to 12 edges for a total of at most 21 edges in  $G$ , a contradiction to the 6-regularity of  $G$ .

As a result, every exceptional vertex of  $G$  is a cut-vertex. We now establish some properties of the 2-connected blocks of  $G$ . First, we note that all edges inside a 2-connected block must have multiplicity 1 or 5. This in particular implies that every 2-connected block of  $G$  contains at least 3 vertices, for if a 2-connected block is composed of only two vertices, then the 1 or 5 edges joining them would form an odd-cut in  $G$ , which is impossible in a regular multigraph of even degree.

Second, because  $G$  has 9 vertices, no 2-connected block of  $G$  can contain more than 3 cut-vertices, since to each cut-vertex  $x$  of a block  $B$ , we can associate a distinct component of  $G - x$  not intersecting  $B$ , and such a component contains at least 2 vertices by degree regularity of  $G$ . Moreover, if a 2-connected block  $B$  of  $G$  contains exactly 3 cut-vertices  $\{u, v, w\}$ , then necessarily  $G$  consists of the simple triangle  $\{u, v, w\}$ , with pendant triangles at all three vertices. It follows that  $\tau(G) = 3 \cdot 20^3 > 10285 = \tau(F_{6,9})$ , a contradiction. We thus proved that each 2-connected block of  $G$  contains at most 2 cut-vertices.

It now follows that a 2-connected block with 2 cut-vertices must contain at least 4 (and at most 5) vertices, for if  $B$  is a 2-connected block with 3 vertices containing exactly two cut-vertices of  $G$ , then they must induce a triangle that is not pendant and yet contain edges of multiplicities greater than 1 and less than 5, a contradiction.

As a consequence, if  $G$  contains at least 2 exceptional vertices, which in particular must be cut-vertices, then we can find two of them,  $u$  and  $v$ , that belong to the same 2-connected block  $B$ . So  $B$  has 4 or 5 vertices, and  $u$  and  $v$  are the only cut-vertices of  $G$  in  $B$ . Since  $B$  contains no pendant triangle, all vertices of  $B$  except  $u$  and  $v$  (which are exceptional) are incident to an edge of multiplicity 5 in  $B$ . This is a contradiction, as the only possibility is that  $B$  be a path  $u, u', v', v$  with edge multiplicities 1, 5, 1, which is not 2-connected. It follows that  $G$  contains at most one exceptional vertex, as needed. This concludes the proof.  $\square$

## 6 The case $d = 4$

The degree-4 case is exceptional in the sense that it is the only one (among regular multigraphs of even degree) for which the padded paddle graph can be optimal, specifically if the number of vertices is either at most 11 or precisely 13. It makes the analysis more tedious if we split it according to the parity of the number of vertices, as we did when  $d$  is at least 6. This is why we present this case separately: wrapping the whole argument in a single recurrence avoids some systematic checking of the possibilities for two consecutive lifts yielding a specific graph.

We shall proceed by induction on the number of vertices and, to this end, we first recall that for  $n \in \{4, 5\}$ , which corresponds to our base cases, Lemma 15 ensures the following.

- Every 4-regular connected multigraph  $G$  on 4 vertices has at least 24 spanning trees, with equality if and only if  $G$  is isomorphic to  $PC_{4,4}$ .
- Every 4-regular connected multigraph  $G$  on 5 vertices has at least 49 spanning trees, with equality if and only if  $G$  is isomorphic to  $PP_{4,5} = \mathcal{H}_{4,5}$ .

*Proof of Theorem 5.* We proceed by induction on the number  $n$  of vertices. The statement follows from Lemma 15 if  $n \in \{4, 5\}$ . Fix an integer  $n \geq 6$  and assume that the conclusion holds for graphs with fewer than  $n$  vertices. In particular, for each  $n' \in \{4, \dots, n-1\}$ , there exists a unique  $d$ -regular  $n'$ -vertex multigraph with the fewest number of spanning trees, which we name  $G_{4,n'}^*$ . Let  $G$  be a connected 4-regular  $n$ -vertex multigraph with the fewest spanning trees.

We split the analysis regarding whether or not  $G$  has an edge of multiplicity 2, and then regarding the range  $n$  is in. In each case, we either obtain a contradiction or identify a unique possibility for  $G$ .

Suppose first that  $G$  has an edge of multiplicity 2 between two vertices  $u$  and  $v$ . Let  $G'$  be obtained from  $G$  by first deleting these two edges and then identifying  $u$  and  $v$  into a single vertex  $x$ . It follows that  $G'$  is a connected 4-regular  $(n-1)$ -vertex multigraph and  $\tau(G) \geq 2 \cdot \tau(G')$ , with equality only if  $G - uv$  is disconnected. We now consider three cases regarding the range of  $n$ , which reveals whether  $G_{4,n-1}^*$  is  $PP_{4,n-1}$ , or  $PC_{4,n-1}$  or  $\mathcal{H}_{4,n-1}$ .

If  $n \in \{6, \dots, 12\} \cup \{14\}$ , then the induction hypothesis implies that  $G_{4,n-1}^* = PP_{4,n-1}$ . As  $\tau(PP_{4,n}) = 2 \cdot \tau(PP_{4,n-1})$  we deduce that, necessarily,  $\tau(G) = \tau(PP_{4,n})$  and  $G'$  is  $PP_{4,n-1}$ . Since  $G - uv$  is disconnected, we deduce that  $G$  must be isomorphic to  $PP_{4,n}$ .

If  $n$  is even and at least 16, then  $G_{4,n-1}^* = \mathcal{H}_{4,n-1}$ . For any such value of  $n$ , however,  $\tau(PC_{4,n}) < 2 \cdot \tau(\mathcal{H}_{4,n-1}) \leq \tau(G)$ , a contradiction.

Last, if  $n$  is odd and at least 13, then by induction  $G_{4,n-1}^* = PC_{4,n-1}$ . However, this information is not useful because it turns out that no connected  $d$ -regular  $n$ -vertex multigraph has as few as  $2 \cdot \tau(PC_{4,n-1})$  spanning trees. We need to perform a little structural analysis. We want to show that  $G$  contains two vertices that can be subsequently completely lifted, thus yielding a connected  $d$ -regular  $(n-2)$ -vertex multigraph  $G''$  such that  $\tau(G'') \geq c_2^2 \cdot \tau(G_{4,n-2}^*)$ . To this end, observe that both  $u$  and  $v$  are exceptional vertices in  $G$ , then they cannot be subsequently completely lifted if and only if they have a common neighbour  $w$  such that (up to swapping  $u$  and  $v$ ) there are exactly 2 edges between  $u$  and  $w$  and 1 edge between  $v$  and  $w$ . Let  $G_u$  be obtained

from  $G$  by completely lifting  $u$  (which, in this situation, amounts to deleting  $u$  and adding 2 edges between  $v$  and  $w$ ). If no vertex of  $G_u$  can be completely lifted, then all vertices in  $G_u$  are incident to an edge of multiplicity 3, which implies that  $G_u$  is  $PC_{4,n-1}$ . Consequently,  $G$  is isomorphic to the graph depicted in Figure 5, which has  $(16n - 12) \cdot 3^{(n-5)/2}$  spanning trees. This is more than  $\tau(F_{4,n})$ , which is a contradiction. As a result, it is indeed possible to consecutively completely lift two vertices in  $G$ , thereby obtaining a connected  $d$ -regular  $(n - 2)$ -vertex multigraph  $G''$  with  $\tau(G'') \geq 4 \cdot \tau(G_{4,n-2}^*)$ . This implies that  $n = 13$ , as if  $n \geq 15$  then  $4 \cdot \tau(G_{4,n-2}^*) > \tau(H_{4,n})$ . Indeed, if  $n \geq 17$  then  $G_{4,n-2}^* = \mathcal{H}_{4,n-2}$  and  $4 \cdot \tau(H_{4,n-2}) > \tau(H_{4,n})$  if  $n \geq 11$ , while if  $n = 15$  then  $G_{4,n-2}^* = PP_{4,13}$  and  $4 \cdot \tau(PP_{4,13}) > \tau(H_{4,15})$ . Consequently,  $G_{4,n-2}^* = PP_{4,11}$ . We thus proved that not only is it possible to subsequently completely lift two vertices in  $G$ , but also that every such sequence of two consecutive complete lifts results in the graph  $PP_{4,11}$ . This is possible if and only if  $G$  is isomorphic to  $PP_{4,13}$ .

We thus established that if  $G$  has an edge of multiplicity 2, then necessarily  $G = PP_{4,n}$  and  $n \in \{6, \dots, 14\}$ .

Suppose now that  $G$  has no edge of multiplicity 2. Then either  $G$  has an exceptional vertex  $u$  with 4 different neighbours, or  $G$  is  $PC_{4,n}$ . Suppose the former and let  $G_u$  be a connected 4-regular  $(n-1)$ -vertex multigraph obtained by completely lifting  $u$  in  $G$ . Then  $\tau(G) \geq c_2 \cdot \tau(G_u) = 2 \cdot \tau(G_u)$ . We let  $e_1$  and  $e_2$  be the two edges of  $G_u$  arising from the complete lift of  $u$ , that is, the two edges of  $G_u$  that do not belong to  $G$ . Observe that  $e_1$  and  $e_2$  must be disjoint since  $u$  has 4 different neighbours in  $G$ . As before, we split the analysis with respect to the value of  $n$ .

If  $n \in \{6, \dots, 12\} \cup \{14\}$ , then the induction hypothesis implies that  $G_{n-1}^* = PP_{4,n-1}$ , and therefore  $\tau(G) \geq 2 \cdot \tau(PP_{4,n-1})$ , with equality if and only if  $G_u$  is  $PP_{4,n-1}$ . Since  $\tau(PP_{4,n}) = 2 \cdot \tau(PP_{4,n-1})$ , we deduce that  $G_u$  must be  $PP_{4,n-1}$ . Because  $G$  itself has no edge of multiplicity 2, the construction of  $G_u$  implies that  $n - 1 \in \{5, 6\}$ . Moreover, if  $n = 5$  then by symmetry there are only two choices for  $e_1$  and  $e_2$ , both of which yield an edge of multiplicity 2 in  $G$ , a contradiction. It follows that  $G_u$  cannot be  $PP_{4,5}$ , and hence  $n - 1 = 6$ . Then, up to symmetry, there is only one choice for  $e_1$  and  $e_2$  (without loss of generality,  $e_1$  must be an edge between the two exceptional vertices and  $e_2$  is disjoint from  $e_1$ ), and reversing the operation to recover  $G$  must again create an edge of multiplicity 2, a contradiction.

If  $n$  is even and at least 16, then  $G_{4,n-1}^* = \mathcal{H}_{4,n-1}$ , and hence  $\tau(G) \geq 2 \cdot \tau(\mathcal{H}_{4,n-1})$ . However,  $\tau(PC_{4,2k}) < 2 \cdot \tau(\mathcal{H}_{4,2k-1})$  when  $k \geq 4$ , which contradicts the optimality of  $G$ .

It remains to deal with the case where  $n$  is odd and at least 13, in which case  $G_{4,n-1}^* = PC_{4,n-1}$  by induction. If  $u$  is the unique exceptional vertex of  $G$ , then  $G_u$  has no exceptional vertex, because completely lifting a vertex cannot create an exceptional vertex (the multiplicities of the edges incident to the other vertices can only increase). Consequently, every vertex of  $G_u$  is incident to an edge of multiplicity 3, implying that  $G_u$  is isomorphic to  $PC_{4,n-1}$ . Now, the edges  $e_1$  and  $e_2$  of  $G_u$  must be two disjoint edges of multiplicity 1, as  $G$  has no edge of multiplicity 2. It follows that  $G$  belongs to  $\mathcal{F}_{4,n}$ , as defined in Section 5.1. Therefore, Lemma 20 implies that  $G$  is isomorphic to  $\mathcal{H}_{4,n}$ .

If  $G$  has at least one other exceptional vertex  $v$ , then because  $G$  has no edge of multiplicity 2 we know that  $v$  is still exceptional in  $G_u$ . As a result, we can obtain a 4-regular  $(n - 2)$ -vertex

multigraph  $G_{uv}$  by completely lifting  $v$  in  $G_u$ . Furthermore, we note that  $G_{uv}$  cannot be isomorphic to  $PP_{4,n-2}$ . Indeed, note that  $PP_{4,n-2}$  contains  $n - 7 \geq 6$  edges of multiplicity 2. However, as each of  $u$  and  $v$  has 4 different neighbours in  $G$ , and  $G$  itself has no edge of multiplicity 2, the number of edges of multiplicity 2 in  $G_{uv}$  is at most 4. This implies that  $n \geq 17$ , and it follows that

$$\tau(G) \geq (c_2)^2 \cdot \tau(G_{uv}) = 4 \cdot \tau(G_{uv}) \geq 4 \cdot \tau(G_{4,n-2}^*) = 4 \cdot \tau(\mathcal{H}_{4,n-2}).$$

This provides a contradiction, because as reported earlier  $\tau(\mathcal{H}_{4,n}) < 4 \cdot \tau(\mathcal{H}_{4,n-2})$  when  $n \geq 11$ .

We thus proved that if  $G$  contains no edge of multiplicity 2, then either  $n$  is even and  $G = PC_{4,n}$ , or  $n$  is odd and at least 13 and then  $G = \mathcal{H}_{4,n}$ .

Summing-up both situations (i.e  $G$  has an edge of multiplicity 2 or not), we see that it only remains to compare the cases for which we have found two possibilities: when  $n$  is either 13 we need to compare  $\tau(PP_{4,13})$  and  $\tau(F_{4,13})$ , and when  $n$  is even and at most 14, we need to compare  $PP_{4,n}$  and  $PC_{4,n}$ . We see that  $\tau(PP_{4,13}) < \tau(F_{4,13})$ , and also that  $\tau(PP_{4,2k}) < \tau(PC_{4,2k})$  if  $k \leq 5$  while the inequality is reversed if  $k \geq 6$ , which concludes the proof.  $\square$

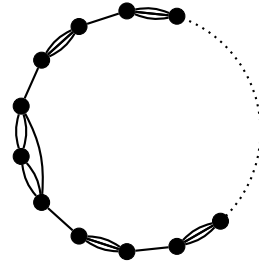


Figure 5: A 4-regular  $n$ -vertex multigraph with  $(16n - 12) \cdot 3^{(n-5)/2}$  spanning trees.

## References

- [1] ALON, N. The number of spanning trees in regular graphs. *Random Structures & Algorithms* 1, 2 (1990), 175–181.
- [2] BOGDANOWICZ, Z. R. On the minimum number of spanning trees in cubic multigraphs. *Disc. Math. Graph Theory* 40 (2020), 149–159.
- [3] EGORYCHEV, G. P. The solution of the van der Waerden problem for permanents. *Dokl. Akad. Nauk SSSR* 258, 5 (1981), 1041–1044.
- [4] FALIKMAN, D. I. Proof of the van der Waerden conjecture on the permanent of a doubly stochastic matrix. *Mat. Zametki* 29, 6 (1981), 931–938, 957.
- [5] GURVITS, L. Van der Waerden/Schrijver-Valiant like conjectures and stable (aka hyperbolic) homogeneous polynomials: one theorem for all. *Electron. J. Combin.* 15, 1 (2008), Research Paper 66, 26. With a corrigendum.

- [6] OK, S., AND THOMASSEN, C. On the minimum number of spanning trees in  $k$ -edge-connected graphs. *Journal of Graph Theory* 84, 3 (2017), 286–296.
- [7] PEKÁREK, J., SERENI, J.-S., AND YILMA, Z. B. The minimum number of spanning trees in regular multigraphs I: the odd-degree case. Manuscript, 2021.