The Lagrange-D’Alembert Principle in Banach Space
Oleg Zubelevich

To cite this version:
| Oleg Zubelevich. The Lagrange-D’Alembert Principle in Banach Space. 2021. hal-03471116

HAL Id: hal-03471116
https://hal.science/hal-03471116
Preprint submitted on 8 Dec 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers. L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
The Lagrange-D’Alembert Principle in Banach Space

Oleg Zubelevich

Steklov Mathematical Institute of Russian Academy of Sciences
Mechanics and Mathematics Faculty,
M. V. Lomonosov Moscow State University
Russia, 119899, Moscow, MGU
ozubel@yandex.ru

Abstract. The Lagrange-D’Alembert Principle is one of the fundamental tools of classical mechanics. We generalize this principle to mechanics-like ODE in Banach spaces. As an application we discuss geodesics in infinite dimensional manifolds.

1. The Main Theorems. Discussion

In this short note we extend the Lagrange-D’Alembert formalism to an infinite dimensional ODE of classical mechanics type. Particularly we develop an infinite dimensional theory of ideal holonomic and nonholonomic constraints.

Other approach based on the Hamel equations is contained in [5].

Let $X, Y$ be Banach spaces. Let $I = (t_1, t_2) \subset \mathbb{R}$ stand for the fixed open interval. A set $D \subset X$ is open; $M = I \times D \times X$.

We use $\mathcal{B}(X', X)$ to denote the space of bounded linear operators $u : X' \to X$. Here and below the symbol prime denotes duality.

Being equipped with the norm

$$\|u\|_{\mathcal{B}(X', X)} = \sup\{\|u(x)\|_X \mid \|x\|_{X'} \leq 1\}$$

2000 Mathematics Subject Classification. 37K06, 34A34, 34G20, 58D25, 46N20.

Key words and phrases. Lagrangian systems, infinite-dimensional ODE, infinite-dimensional dynamical systems.

The research was funded by a grant from the Russian Science Foundation (Project No. 19-71-30012).
the space $B(X',X)$ becomes a Banach space.

Introduce the following functions

$$f \in C^1(M,X'), \ P \in C^1(M,B(X',X)).$$

This means that the function $f : M \to X'$ is continuously differentiable in $M$ in the sense of Fréchet and for the function $P$ the corresponding assumption is true.

This condition can slightly be relaxed but to simplify the narration we do not do that.

**Hypothesis 1.** Assume also that for each $v \in X'$, $z \in M$ it follows that

$$(v, P(z)v) \geq K(z)\|v\|_{X'}^2.$$  

Here $K$ is a positive valued function.

**Remark 1.** By lemma 1 (see below) this hypothesis implies that for each $z \in M$ the operator $P(z) : X' \to X$ is an isomorphism.

The main object of our study is the following problem

$$\ddot{x} = P(z)(f(z) + N(z)), \ z = (t, x, \dot{x}) \in M. \quad (1.1)$$

Here the function $f$ is given and a function $N$ must be determined such that $N \in C^1(M,X')$ and any solution $x(t)$ to (1.1) satisfies the following equation of constraints:

$$\varphi(z) = 0. \quad (1.2)$$

The function $\varphi$ belongs to $C^2(M,Y)$.

We surely assume that a set

$$S = \{z \in M \mid \varphi(z) = 0\}$$

is non void.

**Hypothesis 2.** For any $z \in M$ the derivative

$$\varphi_\dot{x}(z) : X \to Y, \ \varphi_\dot{x} \in C^1(M, B(X,Y))$$

is a mapping onto.

If the function $N \in C^1(M,X')$ then system (1.1) satisfies all the conditions of the existence and uniqueness theorem [2].

**Remark 2.** Finite dimensional systems of type (1.1) with constraints (1.2) are the main object of classical mechanics [1]. Indeed, if $X = \mathbb{R}^m$, $Y = \mathbb{R}^n$, $n < m$ then a system

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = Q(t, x, \dot{x}) + \lambda \varphi_\dot{x}, \quad L = \frac{1}{2} \dot{x}^T G(t, x) \dot{x} - V(t, x)$$

has shape (1.1) with $P = G^{-1}$, $N = \lambda \varphi_\dot{x}$. 
In classical mechanics the function $\varphi$ is as a rule linear in $\dot{x}$.

**Theorem 1.** Under assumptions above there exists a unique function $N \in C^1(M, X')$ such that

1) the function $\varphi$ is a first integral of (1.1) and
2) for all $z \in M$ the following inclusion holds

$$\ker \varphi \dot{x}(z) \subset \ker N(z). \quad (1.3)$$

**Theorem 2.** An operator

$$b(z) = \varphi \dot{x}(z) P(z) \varphi' \dot{x}(z) : Y' \to Y$$

is an isomorphism and

$$b \in C^1(M, \mathcal{B}(Y', Y)), \quad b^{-1} \in C^1(M, \mathcal{B}(Y, Y')).$$

The reaction $N$ given by theorem 1 is expressed as follows

$$N(z) = -\varphi' \dot{x}(z)b^{-1}(z)(\varphi \dot{x}(z) + \varphi \dot{x}(z)\dot{x} + \varphi \dot{x}(z)P(z)f(z)). \quad (1.4)$$

**Remark 3.** Actually the restriction $N \mid_S$ does not depend on which function $\varphi$ determines the manifold $S$. We discuss it in details in section 5.

We prove theorems 1, 2 in section 3.

In the sequel we assume that the function $N$ is chosen in accordance with theorem 1.

The space $\ker \varphi \dot{x}(z)$ is referred to as the space of virtual displacements.

Under conditions of theorem 1 equation (1.2) is called the equation of ideal constraints and the function $N$ is called the reaction of ideal constraints.

Formula (1.3) means that the work done by the reaction of ideal constraint $N$ at a virtual displacement $\xi$, $\varphi \dot{x} \xi = 0$ vanishes: $N\xi = 0$.

Condition 1) of theorem 1 implies in particular that the manifold $S$ is invariant under the phase flow of system (1.1) that is if $x(t) \in C^2(I, X)$ is a solution to (1.1) and for some $t_0 \in I$ one has

$$(t_0, x(t_0), \dot{x}(t_0)) \in S$$

then

$$(t, x(t), \dot{x}(t)) \in S \quad \forall t \in I.$$
Theorem 3. Let $x \in C^2(I, X)$ be a solution to (1.1). Then for all $t \in I$ the following inclusion holds
\[
\ker \varphi_x(t, x(t), \dot{x}(t)) \subset \ker \left( P^{-1}(t, x(t), \dot{x}(t)) \ddot{x}(t) - f(t, x(t), \dot{x}(t)) \right).
\]
(1.5)

Formula (1.5) is called the Lagrange-D’Alembert equation. The Lagrange-D’Alembert equation is free from the reaction of ideal constraints $N$.

The converse assertion holds.

Theorem 4. Assume that a function $x \in C^2(I, X)$ satisfies (1.2) and (1.5). Then $x$ satisfies (1.1).

Theorem 4 is proved in section 4.

1.1. Holonomic Constraints.

Definition 1. We shall say that constraint (1.2) is holonomic iff there exists a function
\[
g \in C^3(I \times D, Y), \quad g = g(t, x)
\]
such that the manifold $S$ can equivalently be determined as follows
\[
S = \{ z = (t, x, \dot{x}) \in M \mid g_t(t, x) + g_x(t, x)\dot{x} = 0 \}.
\]
The mapping $g_x(t, x) : X \to Y$ is onto for each $(t, x) \in I \times D$.

The function $g$ is evidently defined up to an additive constant function $(t, x) \mapsto \text{const} \in Y$.

The problem whether a constraint is holonomic or not is solved by means of the infinite dimensional version of the Frobenius theorem [2]. This situation is analogous to the finite dimensional one.

Theorem 5. A manifold
\[
\Sigma = \{ z \in S \mid g(t, x) = 0 \}
\]
is an invariant manifold of system (1.1).

Indeed, let $x(t)$ be a solution to (1.1) such that for some $t_0 \in I$ one has
\[
(t_0, x(t_0), \dot{x}(t_0)) \in \Sigma.
\]
The manifold $S$ is invariant:
\[
\frac{d}{dt}g(t, x(t)) = g_x(t, x(t))\dot{x}(t) + g_t(t, x(t)) = 0 \quad \forall t \in I.
\]
Integrating this equality from $t_0$ to $t$ we obtain $g(t, x(t)) = 0$.

This proves the theorem.
2. Geodesics on an Infinite Dimensional Manifold

Assume that constraint (1.2) is holonomic and has the following form
\[ \varphi(t, x, \dot{x}) = g_x(x) \dot{x} = 0. \] (2.1)
The function \( g \in C^3(D,Y) \) is such that the operator \( g_x(x) : X \to Y \) is onto for each \( x \in D \).

Introduce a manifold \( \Gamma = \{ x \in D \mid g(x) = 0 \} \).

Let the operator \( P \) be independent on \( z \).

**Definition 2.** A curve \( \gamma = \{ x(t) \in D \mid t \in I \} \) is said to be a geodesic in \( \Gamma \) if the function \( x \in C^2(I,X) \) is a solution to the following initial value problem:
\[ \ddot{x} = -Pg_x'(x)(g_x(x)Pg_x'(x))^{-1}g_{xx}(x)[\dot{x}, \dot{x}], \] (2.2)
\[ (t_0, x(t_0), \dot{x}(t_0)) \in \Sigma = \{ (t, x, \dot{x}) \in M \mid g(x) = 0, \quad g_x(x)\dot{x} = 0 \}. \] (2.3)

Here \( g_{xx}(x)[\cdot, \cdot] \in \mathcal{B}(X, \mathcal{B}(X,Y)) \).

Equation (2.2) corresponds to equation (1.1) with constraint (2.1) and \( f = 0 \).

By theorem 5 initial condition (2.3) guarantees that \( \gamma \subset \Gamma \).

**Remark 4.** If the spaces \( X, Y \) are finite dimensional and the operator \( P \) is symmetric: \( P' = P \) then \( P^{-1} \) is a Riemann metric in \( X \). This metric endows the manifold \( \Gamma \subset X \) with a Riemann metric in usual way. Equation (2.2) describes dynamics of a particle that slides along the surface \( \Gamma \) freely. Trajectories of such a particle are the geodesics in \( \Gamma \).

2.1. Geodesics on Quadric in Hilbert Space. Let \( X = \ell_2 \).

Recall that \( \ell_2 \) is a Hilbert space with respect to an inner product
\[ x = (x_1, x_2, \ldots), \quad y = (y_1, \ldots), \quad (x, y) = \sum_{i=1}^{\infty} x_i y_i. \]
We consider a real version of \( \ell_2 : x_i, y_i \in \mathbb{R} \). By the Riesz representation theorem this space is identified with its dual: \( \ell_2 = \ell_2' \).

Let us put
\[ P = \text{id}, \quad g(x) = (x, Wx) - 1, \]
where the operator \( W \in \mathcal{B}(\ell_2, \ell_2) \), \( W' = W \neq 0 \). So that \( \Gamma \) is a quadric,
\[ D = \ell_2 \setminus \ker W. \]
Then the equation of geodesics takes the form
\[ \ddot{x} = \frac{(\dot{x}, W\dot{x})}{\|Wx\|_{\ell_2}^2} Wx. \] (2.4)

If \( \{e^{\Omega s} \mid s \in \mathbb{R}\}, \quad A \in \mathcal{B}(\ell_2, \ell_2), \quad \Omega' = -\Omega \)
is a symmetry group of the quadric \( \Gamma: \)
\[ W\Omega - \Omega W = 0 \]
then a function \((x, \Omega \dot{x})\) is a first integral of (2.4).

It is not hard to show that if \( x(t) \) is a solution to (2.4) and \( g(x(t)) = 1, \quad t \in I \) then the "kinetic energy" is conserved:
\[ T = \frac{1}{2}\|x(t)\|_{\ell_2}^2 = \text{const}. \]

3. Proof of Theorems 1, 2

We need the following facts.

**Theorem 6 ([4]).** Let \( E, F, H \) be Banach spaces. And operators
\[ A \in \mathcal{B}(E, H), \quad B \in \mathcal{B}(E, F) \]
be such that \( B \) is onto and \( \ker B \subset \ker A \).

Then there exists an operator \( \mathcal{L} \in \mathcal{B}(F, H) \) such that the following diagram
\[
\begin{array}{c}
F \\
\downarrow \\
E \\
\downarrow \\
H \\
\end{array}
\begin{array}{c}
\mathcal{L} \\
\downarrow \\
A \\
\downarrow \\
B \\
\end{array}
\begin{array}{c}
F \\
\downarrow \\
E \\
\downarrow \\
H \\
\end{array}
\]
is commutative: \( A = \mathcal{L}B \).

**Theorem 7 ([3]).** An operator \( \psi \in \mathcal{B}(E, F) \) is onto iff the operator \( \psi' : F' \to E' \) is a strong isomorphism between \( F' \) and \( \psi'(F') \).
The operator \( \psi' \in \mathcal{B}(F', E') \) is onto iff the operator \( \psi : E \to F \) is an isomorphism between \( E \) and \( \psi(E) \).

In the sequel \( c_1, c_2, \ldots \) denote inessential positive constants.

**Lemma 1.** Assume that an operator \( J \in \mathcal{B}(E', E) \) is such that
\[ (w, Jw) \geq c_1\|w\|_{E'}^2, \quad \forall w \in E'. \] (3.1)

Then the operator \( J \) is an isomorphism between \( E' \) and \( E \).
Proof of lemma 1. By the Hahn-Banach theorem we have
\[ \|Jw\|_E = \sup \{(\xi, Jw) \mid \|\xi\|_{E'} \leq 1\} \]
\[ \geq \frac{1}{\|w\|_{E'}} (w, Jw) \geq c_1 \|w\|_{E'}. \]
Thus \( J \) is an isomorphism between \( E' \) and \( J(E') \). By theorem 7 the operator
\[ J' : E' \to E'' \]
is onto.

The operator \( J' \) is one-to-one. This follows by formula \((w, Jw) = (J'w, w)\) from inequality (3.1).

From the Open Mapping theorem we conclude that \( J' \) is an isomorphism.

By theorem 7 the operator \( J \) is an isomorphism.

Lemma 1 is proved.

Theorem 8. Let \( E, F \) be Banach spaces;
\[ A \in \mathcal{B}(E', E), \quad B \in \mathcal{B}(E, F). \]
The operator \( B \) is onto. The operator \( A \) satisfies the following inequality
\[ (w, Aw) \geq c_2 \|w\|_{E'}^2, \quad \forall w \in E'. \]

Then the operator
\[ R = BAB' : F' \to F \]
is an isomorphism.

Proof of Theorem 8. From theorem 7 it follows that \( B' : F' \to B'(F') \subset E' \) is an isomorphism. We consequently have
\[ \|B'w\|_{E'} \geq c_3 \|u\|_{F'}. \]

It follows that
\[ (w, Rw) = (B'w, AB'w) \geq c_2 \|B'w\|_{E'}^2 \geq c_4 \|w\|_{F'}^2. \] (3.2)

Now the assertion follows from lemma 1.

Theorem 8 is proved.

We are ready to prove theorems 1, 2.

First suppose that the reaction \( N \) exists and prove its uniqueness.

From formula (1.3) and theorem 6 it follows that for each \( z \in M \) there exists an operator \( \Lambda(z) \in Y' \) such that
\[ N(z) = \Lambda(z)\phi_x(z) = \phi'_x(z)\Lambda(z). \] (3.3)
Condition 1) of theorem 1 is equivalent to the following equation
\[ \varphi_t(z) + \varphi_x(z) \dot{x} + \varphi_z(z) P(z) (f(z) + N(z)) = 0. \] (3.4)

Substituting (3.3) to (3.4) and by using theorem 8 we obtain
\[ \Lambda(z) = -b^{-1}(z) (\varphi_t(z) + \varphi_x(z) \dot{x} + \varphi_z(z) P(z) f(z)). \]

Since \( b \in C^1(M, \mathcal{B}(Y', Y)) \) it follows that \( b^{-1} \in C^1(M, \mathcal{B}(Y, Y')) \)
[2]. Therefore we yield (1.4) and \( N \in C^1(M, X'). \)
Formula (1.4) proves the existence as well.

Theorems 1, 2 are proved.

4. Proof of Theorem 4

By theorem 6 inclusion (1.5) implies that there exists a function
\[ \tilde{\Lambda} : I \to Y' \]
such that
\[ P^{-1}(t, x(t), \dot{x}(t)) \ddot{x}(t) - f(t, x(t), \dot{x}(t)) = \varphi_x'(t, x(t), \dot{x}(t)) \tilde{\Lambda}(t) \]

or
\[ \ddot{x}(t) = P(t, x(t), \dot{x}(t)) \left( f(t, x(t), \dot{x}(t)) + \varphi_x'(t, x(t), \dot{x}(t)) \tilde{\Lambda}(t) \right). \] (4.1)

Differentiate the equality
\[ \varphi(t, x(t), \dot{x}(t)) = 0 \]
to have
\[ \varphi_t(t, x(t), \dot{x}(t)) + \varphi_x(t, x(t), \dot{x}(t)) \dot{x}(t) + \varphi_z(t, x(t), \dot{x}(t)) \ddot{x}(t) = 0. \]

Substituting here \( \ddot{x} \) from (4.1) we make sure that
\[ \tilde{\Lambda}(t) = \Lambda(t, x(t), \dot{x}(t)). \]
This completes the proof.

5. Independence \( N \mid_S \) on \( \varphi \)

Formula (1.4) looks like \( N \mid_S \) depends on \( \varphi \). Actually it is not so. Indeed, consider a function
\[ U \in C^1(M \times Y, Y), \quad U = U(t, x, \dot{x}, y) \]
such that for all \( z \in M \) the mapping
\[ U_y(z, 0) : Y \to Y \]
is an isomorphism and
\[ U(t, x, \dot{x}, y) = 0 \iff y = 0. \]
This particularly implies
\[ U_t(z, 0) = 0, \quad U_x(z, 0) = 0, \quad U_{\dot{x}}(z, 0) = 0. \]  \hfill (5.1)
Equation (1.2) is equivalent to the following one
\[ \sigma(z) = 0, \]  \hfill (5.2)
where
\[ \sigma(z) = U(z, \varphi(z)). \]
In other words equation (5.2) determines the same manifold \( S \).
Then one yields
\[ N \big|_{z \in S} = -\varphi_{x}^\prime(z) b^{-1}(z) \left( \varphi_t(z) + \varphi_x(z) \dot{x} + \varphi_{\dot{x}}(z) P(z) f(z) \right) \big|_{z \in S} \]
\[ = -\sigma_x^\prime(z) (\sigma_{\dot{x}}(z) P(z) \sigma_{x}^\prime(z))^{-1} \big( \sigma_t(z) + \sigma_x(z) \dot{x} + \sigma_{\dot{x}}(z) P(z) f(z) \big) \big|_{z \in S}. \]  \hfill (5.3)
Indeed, due to (5.1) one has:
\[ \sigma_t \big|_{z \in S} = (U_t + U_y \varphi_t) \big|_{z \in S} = U_y \varphi_t, \]
\[ \sigma_x \big|_{z \in S} = (U_x + U_y \varphi_x) \big|_{z \in S} = U_y \varphi_x, \]
\[ \sigma_{\dot{x}} \big|_{z \in S} = (U_{\dot{x}} + U_y \varphi_{\dot{x}}) \big|_{z \in S} = U_y \varphi_{\dot{x}}. \]
Now formula (5.3) follows from direct calculation.

References

Email address: ozubel@yandex.ru