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Rational solutions to the mKdV equation associated to particular polynomials

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Abstract

Rational solutions to the modified Korteweg-de Vries (mKdV) equation are given in terms of a quotient of determinants involving certain particular polynomials. This gives a very efficient method to construct solutions. We construct very easily explicit expressions of these rational solutions for the first orders $n = 1$ until 10.

Key Words : mKdV equation, rational solutions.

PACS numbers :

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1 Introduction

We consider the modified Korteweg-de Vries (mKdV) equation in the following normalization

$$4u_t + 6u^2u_x - u_{xxx} = 0, \quad (1)$$

with as usual the subscript meaning the partial derivatives.

The mKdV equation has many applications in various fields as in nonlinear optics for example to model supercontinuum generation in optical fibres [1] or to describe pulses consisting of a few optical cycles [2].

A lot of methods have been used to construct solutions to the mKdV equation. We can recall some works about this topic. In 1972, Hirota [4] constructed the exact soliton for the mKdV equation. Tanaka [5] was the first to solve the mKdV equation by using the inverse scattering technique. In 1976, Ono constructed rational solutions to the mKdV equation [6] by using Bäcklund transformation. Tanaka constructed rational solutions to this equation using recurrence relations

in [7].

In 2012, solutions to the mKdV equation has been constructed via bilinear Bäcklund transformation in [8] and rational solutions in terms of wronskians were obtained.

Periodic solutions and rational solution of first and second order were presented in [9] in 2016, by using a Darboux transformation.

Here, we construct rational solutions by using particular types of polynomials. To prove the results, we use the bilinear Hirota method. The solution can be expressed as the derivative in x of the logarithm of the quotient of two determinants. So rational solutions as a quotient of polynomials in x and t are given and we construct explicit rational solutions in the simplest cases $n = 1$ until 10.

2 Rational solutions to the mKdV equation

We consider the following polynomials $p_n(x, t)$ defined by

$$\begin{cases} p_n(x, t) = \sum_{l=0}^n \frac{x^k t^{\binom{n-k}{3}}}{k! \left(\frac{n-k}{3}\right)!} \left(1 - \left[\frac{1}{2}(n-k+1-3\left[\frac{n-k}{3}\right])\right]\right), & k \geq 0, \\ p_n(x, t) = 0, & n < 0, \end{cases} \quad (2)$$

with $[x]$ denoting the largest integer less or equal to x .

We denote $A_n(x, t)$ the determinant defined by

$$A_n(x, t) = \det(p_{n+1-2i+j}(x, t))_{\{1 \leq i \leq n, 1 \leq j \leq n\}} \quad (3)$$

With these notations we have the following result

Theorem 2.1 *The function $v_n(x, t)$ defined by*

$$v_n(x, t) = \partial_x \left(\ln \frac{A_{n+1}(x, t)}{A_n(x, t)} \right) \quad (4)$$

is a rational to the mKdV equation (1)

$$4u_t + 6u^2u_x - u_{xxx} = 0.$$

Proof : We know that $v_n(x, t) = \partial_x \left(\ln \frac{f(x, t)}{g(x, t)} \right)$ is a solution to the mKdV equation if f and g check the following equations

$$(D_x^3 - 4D_t)f \cdot g = 0, \quad (5)$$

$$(D_x^2)f \cdot g = 0, \quad (6)$$

where D is the bilinear differential operator.

We have to check (5) for $f = \det(p_{n+2-2i+j}(x, t))_{\{1 \leq i \leq n+1, 1 \leq j \leq n+1\}}$ and $g =$

$\det(p_{n+1-2i+j}(x, t))_{\{1 \leq i \leq n, 1 \leq j \leq n\}}$. We denote C_j the column j of A_{n+1} , $1 \leq j \leq n+1$ and \tilde{C}_j the column j of A_n , $1 \leq j \leq n$:

$$C_j = \begin{pmatrix} p_{n+j} \\ p_{n-2+j} \\ \vdots \\ p_{-n+j} \end{pmatrix}, \quad \tilde{C}_j = \begin{pmatrix} p_{n-1+j} \\ p_{n-3+j} \\ \vdots \\ p_{-n+1+j} \end{pmatrix}. \quad (7)$$

With these notations, $A_{n+1}(x, t)$ and $A_n(x, t)$ can be written as :

$$A_{n+1}(x, t) = |C_1, \dots, C_{n+1}|, \quad A_n(x, t) = |\tilde{C}_1, \dots, \tilde{C}_n|.$$

We denote H the expression $H = (D_x^3 - 4D_x D_t)f \cdot g$. We have to evaluate H .

The polynomials p_k verify $\partial_x(p_k) = p_{k-1}$ and $\partial_t(p_k) = p_{k-3}$.

So H can be written as a sum of 16 terms

$$\begin{aligned} H = & [|C_{-2}, C_2, C_3, \dots, C_{n+1}| + 2|C_{-1}, C_1, C_3, \dots, C_{n+1}| \\ & + |C_0, C_1, C_2, C_4, \dots, C_{n+1}| \times |\tilde{C}_1, \tilde{C}_2, \dots, \tilde{C}_n| \\ & - 3[|C_{-1}, C_2, \dots, C_{n+1}| + |C_0, C_1, C_3, \dots, C_{n+1}|] \times |\tilde{C}_0, \tilde{C}_2, \tilde{C}_3, \dots, \tilde{C}_n| \\ & + 3[|\tilde{C}_{-1}, \tilde{C}_2, \dots, \tilde{C}_n| + |\tilde{C}_0, \tilde{C}_1, \tilde{C}_3, \dots, \tilde{C}_n|] \times |C_0, C_2, \dots, C_{n+1}| \\ & - [|C_{-2}, \tilde{C}_2, \dots, \tilde{C}_n| + 2|C_{-1}, \tilde{C}_1, \tilde{C}_3, \dots, \tilde{C}_n| + |C_0, \tilde{C}_1, \tilde{C}_2, \tilde{C}_4, \dots, \tilde{C}_n|] \times |C_1, C_2, \dots, C_{n+1}| \\ & - 4[|C_{-2}, C_2, \dots, C_{n+1}| + |C_1, C_{-1}, C_3, \dots, C_{n+1}| + |C_1, C_2, C_0, C_4, \dots, C_{n+1}|] \times \\ & |\tilde{C}_1, \tilde{C}_2, \dots, \tilde{C}_n| \\ & + 4[|\tilde{C}_{-2}, \tilde{C}_2, \dots, \tilde{C}_n| + |\tilde{C}_1, \tilde{C}_{-1}, \tilde{C}_3, \dots, \tilde{C}_n| + |\tilde{C}_1, \tilde{C}_2, \tilde{C}_0, \tilde{C}_4, \dots, \tilde{C}_n|] \times |C_1, C_2, \dots, C_{n+1}| \end{aligned}$$

H can be reduced to a sum of 10 terms

$$\begin{aligned} H = & -3|C_{-2}, C_2, C_3, \dots, C_{n+1}| \times |\tilde{C}_1, \tilde{C}_2, \dots, \tilde{C}_n| \\ & + 6|C_{-1}, C_1, C_3, \dots, C_{n+1}| \times |\tilde{C}_1, \tilde{C}_2, \dots, \tilde{C}_n| \\ & - 3|C_0, C_1, C_2, C_4, \dots, C_{n+1}| \times |\tilde{C}_1, \tilde{C}_2, \dots, \tilde{C}_n| \\ & + 3|\tilde{C}_{-2}, \tilde{C}_2, \tilde{C}_3, \dots, \tilde{C}_n| \times |C_1, C_2, \dots, C_{n+1}| \\ & - 6|\tilde{C}_{-1}, \tilde{C}_1, \tilde{C}_3, \tilde{C}_4, \dots, \tilde{C}_n| \times |C_1, C_2, \dots, C_{n+1}| \\ & + 3[|\tilde{C}_0, \tilde{C}_1, \tilde{C}_2, \tilde{C}_4, \dots, \tilde{C}_n| \times |C_1, C_2, \dots, C_{n+1}| \\ & - 3|C_{-1}, C_2, C_3, \dots, C_{n+1}| \times |\tilde{C}_0, \tilde{C}_2, \tilde{C}_3, \dots, \tilde{C}_n| \\ & - 3|C_0, C_1, C_3, C_4, \dots, C_{n+1}| \times |\tilde{C}_0, \tilde{C}_2, \tilde{C}_3, \dots, \tilde{C}_n| \\ & + 3|C_0, C_2, C_3, \dots, C_{n+1}| \times |\tilde{C}_{-1}, \tilde{C}_2, \tilde{C}_3, \dots, \tilde{C}_n| \\ & + 3|C_0, C_2, C_3, C_4, \dots, C_{n+1}| \times |\tilde{C}_0, \tilde{C}_1, \tilde{C}_3, \dots, \tilde{C}_n| \end{aligned}$$

The following relations

$$\begin{aligned} |C_{-2}, C_2, C_3, \dots, C_{n+1}| &= |C_0, C_1, C_2, C_4, \dots, C_{n+1}|, \\ |C_{-1}, C_2, C_3, \dots, C_{n+1}| &= |C_0, C_1, C_3, C_4, \dots, C_{n+1}| \\ |\tilde{C}_{-2}, \tilde{C}_2, \dots, \tilde{C}_n| &= |\tilde{C}_0, \tilde{C}_1, \tilde{C}_2, \tilde{C}_4, \dots, \tilde{C}_n|, \text{ and} \\ |\tilde{C}_{-1}, \tilde{C}_2, \dots, \tilde{C}_n| &= |\tilde{C}_0, \tilde{C}_1, \tilde{C}_3, \tilde{C}_4, \dots, \tilde{C}_n|, \end{aligned}$$

reduces the expression of H .

It can be written as the following sum $H = 6(H_1 + H_2)$ with

$$\begin{aligned} H_1 = & -|C_1, C_{-1}, C_3, \dots, C_{n+1}| \times |\tilde{C}_1, \tilde{C}_2, \dots, \tilde{C}_n| \\ & + |C_1, C_2, \dots, C_{n+1}| \times |\tilde{C}_{-2}, \tilde{C}_2, \tilde{C}_3, \dots, \tilde{C}_n| \\ & - |C_{-1}, C_2, C_3, \dots, C_{n+1}| \times |\tilde{C}_0, \tilde{C}_2, \tilde{C}_3, \dots, \tilde{C}_n|, \end{aligned}$$

and

$$\begin{aligned}
H_2 &= -|C_{-2}, C_2, C_3, \dots, C_{n+1}| \times |\tilde{C}_1, \tilde{C}_2, \dots, \tilde{C}_n| \\
&+ |C_1, C_2, \dots, C_{n+1}| \times |\tilde{C}_1, \tilde{C}_{-1}, \tilde{C}_3, \dots, \tilde{C}_n| \\
&+ |C_0, C_2, C_3, \dots, C_{n+1}| \times |\tilde{C}_{-1}, \tilde{C}_2, \tilde{C}_3, \dots, \tilde{C}_n|.
\end{aligned}$$

The last expressions H_1 and H_2 can be rewritten as the following determinants of order $2n+1$

$$H_1 = \left| \begin{array}{cccc|cccc} C_1 & C_2 & C_3 & C_4 & \dots & \dots & C_{n+1} & \\ \hline \tilde{C}_0 & \tilde{C}_1 & 0 & 0 & \dots & \dots & 0 & \end{array} \right| \begin{array}{cccc|cccc} C_{-1} & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ \hline \tilde{C}_{-2} & \tilde{C}_2 & \tilde{C}_3 & \tilde{C}_4 & \tilde{C}_5 & \dots & \tilde{C}_n & \end{array} \quad (8)$$

and

$$H_2 = \left| \begin{array}{cccc|cccc} C_{-2} & C_2 & C_3 & C_4 & \dots & \dots & C_{n+1} & \\ \hline \tilde{C}_{-1} & 0 & 0 & 0 & \dots & \dots & 0 & \end{array} \right| \begin{array}{cccc|cccc} C_0 & C_1 & \dots & \dots & \dots & \dots & \dots & 0 \\ \hline \tilde{C}_1 & \tilde{C}_2 & \tilde{C}_3 & \tilde{C}_4 & \tilde{C}_5 & \dots & \tilde{C}_n & \end{array} \quad (9)$$

We consider the determinant H_1 . It can be rewritten as four blocks

$$H_1 = \left| \begin{array}{cccc|cccc} p_{n+1} & p_{n+2} & p_{n+3} & p_{n+4} & \dots & p_{2n+1} & p_{n-1} & 0 & 0 & \dots & 0 \\ p_{n-1} & p_n & p_{n+1} & p_{n+2} & \dots & p_{2n-1} & p_{n-3} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline p_{-n+1} & p_{-n+2} & p_{-n+3} & p_{-n+4} & \dots & p_1 & p_{-n-1} & 0 & 0 & \dots & 0 \\ p_{n-1} & p_n & 0 & 0 & \dots & 0 & p_{n-3} & p_{n+1} & p_{n+2} & \dots & p_{2n-1} \\ p_{n-3} & p_{n-2} & 0 & 0 & \dots & 0 & p_{n-5} & p_{n-1} & p_n & \dots & p_{2n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{-n+1} & p_{-n+2} & 0 & 0 & \dots & 0 & p_{-n-1} & p_{-n+3} & p_{-n+4} & \dots & p_1 \end{array} \right| \quad (10)$$

We denote by \mathcal{L} the rows and by \mathcal{C} the columns of this determinant of order $2n+1$.

We combine the rows of the previous determinant in the following way : we replace \mathcal{L}_{n+1+j} by $\mathcal{L}_{n+1+j} - \mathcal{L}_{j+1}$ for $1 \leq j \leq n$, then we obtain the following determinant

$$H_1 = \left| \begin{array}{cccc|cccc} p_{n+1} & p_{n+2} & p_{n+3} & p_{n+4} & \dots & p_{2n+1} & p_{n-1} & 0 & 0 & \dots & 0 \\ p_{n-1} & p_n & p_{n+1} & p_{n+3} & \dots & p_{2n-1} & p_{n-3} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline p_{-n+1} & p_{-n+2} & p_{-n+3} & p_{-n+4} & \dots & p_1 & p_{-n-1} & 0 & 0 & \dots & 0 \\ 0 & 0 & -p_{n+1} & -p_{n+3} & \dots & -p_{2n-1} & 0 & p_{n+1} & p_{n+2} & \dots & p_{2n-1} \\ 0 & 0 & -p_{n-1} & -p_{n+1} & \dots & -p_{2n-3} & 0 & p_{n-1} & p_n & \dots & p_{2n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & -p_{-n+3} & -p_{-n+1} & \dots & -p_1 & 0 & p_{-n+3} & p_{-n+4} & \dots & p_1 \end{array} \right| \quad (11)$$

Then replacing \mathcal{C}_j by $\mathcal{C}_j + \mathcal{C}_{n+j}$ for $3 \leq j \leq n$, when we obtain the following determinant

$$H_1 = \left| \begin{array}{cccccc|cccc} p_{n+1} & p_{n+2} & p_{n+3} & p_{n+4} & \cdots & p_{2n+1} & p_{n-1} & 0 & 0 & \cdots & 0 \\ p_{n-1} & p_n & p_{n+1} & p_{n+3} & \cdots & p_{2n-1} & p_{n-3} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{-n+1} & p_{-n+2} & p_{-n+3} & p_{-n+1} & \cdots & p_1 & p_{-n-1} & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 & \cdots & 0 & 0 & p_{n+1} & p_{n+2} & \cdots & p_{2n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & p_{n-1} & p_n & \cdots & p_{2n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & p_{-n+3} & p_{-n+4} & \cdots & p_1 \end{array} \right| \quad (12)$$

This last determinant is clearly equal to 0.

For the expression H_2 , we use the same strategy : we write as a determinant of order $2n+1$ and we prove by combining rows and columns that it is equal to 0.

$$H_2 = \left| \begin{array}{cccccc|cccc} p_{n-2} & p_{n+2} & p_{n+3} & p_{n+4} & \cdots & p_{2n+1} & p_n & p_{n+1} & 0 & \cdots & 0 \\ p_{n-4} & p_n & p_{n+1} & p_{n+2} & \cdots & p_{2n-1} & p_{n-2} & p_{n-1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{-n-2} & p_{-n+2} & p_{-n+3} & p_{-n+4} & \cdots & p_1 & p_{-n} & p_{-n+1} & 0 & \cdots & 0 \\ \hline p_{n-2} & 0 & 0 & 0 & \cdots & 0 & p_n & p_{n+1} & p_{n+2} & \cdots & p_{2n-1} \\ p_{n-4} & 0 & 0 & 0 & \cdots & 0 & p_{n-2} & p_{n-1} & p_n & \cdots & p_{2n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{-n} & 0 & 0 & 0 & \cdots & 0 & p_{-n+2} & p_{-n+3} & p_{-n+4} & \cdots & p_1 \end{array} \right| \quad (13)$$

We combine the rows $\tilde{\mathcal{L}}$ and the columns $\tilde{\mathcal{C}}$ as the following way; we replace $\tilde{\mathcal{L}}_j$ by $\tilde{\mathcal{L}}_j - \tilde{\mathcal{L}}_{n+1+j}$ for $1 \leq j \leq n$ and we obtain

$$H_2 = \left| \begin{array}{cccccc|cccc} 0 & p_{n+2} & p_{n+3} & p_{n+4} & \cdots & p_{2n+1} & 0 & 0 & -p_{n+2} & \cdots & -p_{2n+1} \\ 0 & p_n & p_{n+1} & p_{n+3} & \cdots & p_{2n-1} & 0 & 0 & -p_n & \cdots & -p_{2n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & p_{-n+4} & p_{-n+5} & p_{-n+6} & \cdots & p_3 & 0 & 0 & -p_{-n+6} & \cdots & -p_3 \\ p_{-n-2} & p_{-n+2} & p_{-n+3} & p_{-n+4} & \cdots & p_1 & p_{-n} & p_{-n+1} & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 & \cdots & 0 & p_n & p_{n+1} & p_{n+2} & \cdots & p_{2n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & p_{n-2} & p_{n-1} & p_n & \cdots & p_{2n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{-n} & 0 & 0 & 0 & \cdots & 0 & p_{-n+2} & p_{-n+3} & p_{-n+4} & \cdots & p_1 \end{array} \right| \quad (14)$$

Then we replace \tilde{C}_{n+1+j} by $\tilde{C}_{n+j+1} + \tilde{C}_{j-1}$ for $3 \leq j \leq n$ and we obtain using that $p_{-n-2} = 0$, $p_{-n} = 0$ and $p_{-n+1} = 0$ for $n \geq 0$

$$H_2 = \left(\begin{array}{cccccc|ccccc} 0 & p_{n+2} & p_{n+3} & p_{n+4} & \cdots & p_{2n+1} & 0 & 0 & 0 & \cdots & 0 \\ 0 & p_n & p_{n+1} & p_{n+3} & \cdots & p_{2n-1} & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & p_{-n+4} & p_{-n+5} & p_{-n+6} & \cdots & p_3 & 0 & 0 & 0 & \cdots & 0 \\ 0 & p_{-n+2} & p_{-n+3} & p_{-n+4} & \cdots & p_1 & 0 & 0 & p_{-n+2} & \cdots & p_1 \\ \hline 0 & 0 & 0 & 0 & \cdots & 0 & p_n & p_{n+1} & p_{n+2} & \cdots & p_{2n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & p_{n-2} & p_{n-1} & p_n & \cdots & p_{2n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & p_{-n+2} & p_{-n+3} & p_{-n+4} & \cdots & p_1 \end{array} \right) \quad (15)$$

It can be easily seen that this last determinant is clearly equal to 0. This proves that

$$H = (D_x^3 - 4D_x D_t) f \cdot \cdot$$

2. The same arguments can be used to prove the second relation (6) $D_x^2 f \cdot g = 0$.

We denote G the expression $G = D_x^2 f \cdot g$. We evaluate G .

$G = G_1 + G_2$ can be written as a sum of 6 terms where

$$\begin{aligned} G_1 &= |C_0, C_1, C_3, \dots, C_{n+1}| \times |\tilde{C}_1, \tilde{C}_2, \dots, \tilde{C}_n| \\ &+ |C_1, C_2, C_3, \dots, C_{n+1}| \times |\tilde{C}_{-1}, \tilde{C}_2, \tilde{C}_3, \dots, \tilde{C}_n| \\ &- |C_0, C_2, \dots, C_{n+1}| \times |\tilde{C}_0, \tilde{C}_2, \tilde{C}_3, \dots, \tilde{C}_n| \end{aligned}$$

and

$$\begin{aligned} G_2 &= |C_{-1}, C_2, C_3, \dots, C_{n+1}| \times |\tilde{C}_1, \tilde{C}_2, \dots, \tilde{C}_n| \\ &+ |C_1, C_2, C_3, \dots, C_{n+1}| \times |\tilde{C}_0, \tilde{C}_1, \tilde{C}_3, \dots, \tilde{C}_n| \\ &- |C_0, C_2, \dots, C_{n+1}| \times |\tilde{C}_0, \tilde{C}_2, \tilde{C}_3, \dots, \tilde{C}_n| \end{aligned}$$

The idea of the proof of this second part is the same that this used in the first part.

We write G_1 and G_2 as the determinants of order $2n + 1$

$$G_1 = \left| \begin{array}{cccccc|cccccc} C_1 & C_2 & C_3 & C_4 & \cdots & \cdots & C_{n+1} & C_0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \tilde{C}_0 & \tilde{C}_1 & 0 & 0 & \cdots & \cdots & 0 & \tilde{C}_{-1} & \tilde{C}_2 & \tilde{C}_3 & \tilde{C}_4 & \tilde{C}_5 & \cdots & \tilde{C}_n \end{array} \right| \quad (16)$$

and

$$G_2 = \left| \begin{array}{cccccc|cccccc} C_{-1} & C_2 & C_3 & C_4 & \cdots & \cdots & C_{n+1} & C_0 & C_1 & \cdots & \cdots & \cdots & \cdots & 0 \\ \tilde{C}_0 & 0 & 0 & 0 & \cdots & \cdots & 0 & \tilde{C}_1 & \tilde{C}_2 & \tilde{C}_3 & \tilde{C}_4 & \tilde{C}_5 & \cdots & \tilde{C}_n \end{array} \right| \quad (17)$$

G_1 can be rewritten as

$$G_1 = \left| \begin{array}{cccccc|ccccc} p_{n+1} & p_{n+2} & p_{n+3} & p_{n+4} & \cdots & p_{2n+1} & p_n & 0 & 0 & \cdots & 0 \\ p_{n-1} & p_n & p_{n+1} & p_{n+2} & \cdots & p_{2n-1} & p_{n-2} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{-n+1} & p_{-n+2} & p_{-n+3} & p_{-n+1} & \cdots & p_1 & p_{-n} & 0 & 0 & \cdots & 0 \\ \hline p_{n-1} & p_n & 0 & 0 & \cdots & 0 & p_{n-2} & p_{n+1} & p_{n+2} & \cdots & p_{2n-1} \\ p_{n-3} & p_{n-2} & 0 & 0 & \cdots & 0 & p_{n-4} & p_{n-1} & p_n & \cdots & p_{2n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{-n+1} & p_{-n+2} & 0 & 0 & \cdots & 0 & p_{-n} & p_{-n+3} & p_{-n+4} & \cdots & p_1 \end{array} \right| \quad (18)$$

denoting \mathcal{L} the rows and \mathcal{C} the columns of this determinant of order $2n + 1$, we combine the rows of the previous determinant in the following way :
we replace \mathcal{L}_{n+1+j} by $\mathcal{L}_{n+1+j} - \mathcal{L}_{j+1}$ for $1 \leq j \leq n$, then we obtain the following determinant

$$G_1 = \left| \begin{array}{cccccc|ccccc} p_{n+1} & p_{n+2} & p_{n+3} & p_{n+4} & \cdots & p_{2n+1} & p_n & 0 & 0 & \cdots & 0 \\ p_{n-1} & p_n & p_{n+1} & p_{n+3} & \cdots & p_{2n-1} & p_{n-2} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{-n+1} & p_{-n+2} & p_{-n+3} & p_{-n+1} & \cdots & p_1 & p_{-n} & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & -p_{n+1} & -p_{n+3} & \cdots & -p_{2n-1} & 0 & p_{n+1} & p_{n+2} & \cdots & p_{2n-1} \\ 0 & 0 & -p_{n-1} & -p_{n+1} & \cdots & -p_{2n-3} & 0 & p_{n-1} & p_n & \cdots & p_{2n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & -p_{-n+3} & -p_{-n+1} & \cdots & -p_1 & 0 & p_{-n+3} & p_{-n+4} & \cdots & p_1 \end{array} \right| \quad (19)$$

Then replacing \mathcal{C}_j by $\mathcal{C}_j + \mathcal{C}_{n+j}$ for $3 \leq j \leq n$, when we obtain the following determinant

$$G_1 = \left| \begin{array}{cccccc|ccccc} p_{n+1} & p_{n+2} & p_{n+3} & p_{n+4} & \cdots & p_{2n+1} & p_n & 0 & 0 & \cdots & 0 \\ p_{n-1} & p_n & p_{n+1} & p_{n+3} & \cdots & p_{2n-1} & p_{n-1} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{-n+1} & p_{-n+2} & p_{-n+3} & p_{-n+1} & \cdots & p_1 & p_{-n} & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 & \cdots & 0 & 0 & p_{n+1} & 0 & \cdots & p_{2n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & p_{n-1} & 0 & \cdots & p_{2n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & p_{-n+3} & \cdots & \cdots & p_1 \end{array} \right| \quad (20)$$

It is clear that this last determinant is clearly equal to 0.

For the expression G_2 , we prove by combining rows and columns that it is equal to 0.

$$G_2 = \left(\begin{array}{cccccc|cccc} p_{n-1} & p_{n+2} & p_{n+3} & p_{n+4} & \cdots & p_{2n+1} & p_n & p_{n+1} & 0 & \cdots & 0 \\ p_{n-3} & p_n & p_{n+1} & p_{n+2} & \cdots & p_{2n-1} & p_{n-2} & p_{n-1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{-n-1} & p_{-n+2} & p_{-n+3} & p_{-n+4} & \cdots & p_1 & p_{-n} & p_{-n+1} & 0 & \cdots & 0 \\ \hline p_{n-1} & 0 & 0 & 0 & \cdots & 0 & p_n & p_{n+1} & p_{n+2} & \cdots & p_{2n-1} \\ p_{n-3} & 0 & 0 & 0 & \cdots & 0 & p_{n-2} & p_{n-1} & p_n & \cdots & p_{2n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{-n+1} & 0 & 0 & 0 & \cdots & 0 & p_{-n+2} & p_{-n+3} & p_{-n+4} & \cdots & p_1 \end{array} \right) \quad (21)$$

We combine the rows $\tilde{\mathcal{L}}$ and the columns $\tilde{\mathcal{C}}$ as the following way; we replace $\tilde{\mathcal{L}}_j$ by $\tilde{\mathcal{L}}_j - \tilde{\mathcal{L}}_{n+1+j}$ for $1 \leq j \leq n$ and we obtain

$$G_2 = \left(\begin{array}{cccccc|cccc} 0 & p_{n+2} & p_{n+3} & p_{n+4} & \cdots & p_{2n+1} & 0 & 0 & -p_{n+2} & \cdots & -p_{2n+1} \\ 0 & p_n & p_{n+1} & p_{n+3} & \cdots & p_{2n-1} & 0 & 0 & -p_n & \cdots & -p_{2n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & p_{-n+4} & p_{-n+5} & p_{-n+6} & \cdots & p_3 & 0 & 0 & -p_{-n+4} & \cdots & -p_3 \\ \hline p_{-n-1} & p_{-n+2} & p_{-n+3} & p_{-n+4} & \cdots & p_1 & p_{-n} & p_{-n+1} & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 & \cdots & 0 & p_n & p_{n+1} & p_{n+2} & \cdots & p_{2n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & p_{n-2} & p_{n-1} & p_n & \cdots & p_{2n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{-n+1} & 0 & 0 & 0 & \cdots & 0 & p_{-n+2} & p_{-n+3} & p_{-n+4} & \cdots & p_1 \end{array} \right) \quad (22)$$

Then we replace $\tilde{\mathcal{C}}_{n+1+j}$ by $\tilde{\mathcal{C}}_{n+j+1} + \tilde{\mathcal{C}}_{j-1}$ for $3 \leq j \leq n$ and we obtain using that $p_{-n-2} = 0$, $p_{-n} = 0$ and $p_{-n+1} = 0$ for $n \geq 0$

$$G_2 = \left(\begin{array}{cccccc|cccc} 0 & p_{n+2} & p_{n+3} & p_{n+4} & \cdots & p_{2n+1} & 0 & 0 & 0 & \cdots & 0 \\ 0 & p_n & p_{n+1} & p_{n+3} & \cdots & p_{2n-1} & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & p_{-n+4} & p_{-n+5} & p_{-n+6} & \cdots & p_3 & 0 & 0 & -0 & \cdots & 0 \\ \hline 0 & p_{-n+2} & p_{-n+3} & p_{-n+4} & \cdots & p_1 & 0 & 0 & p_{-n+2} & \cdots & p_1 \\ \hline 0 & 0 & 0 & 0 & \cdots & 0 & p_n & p_{n+1} & p_{n+2} & \cdots & p_{2n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & p_{n-2} & p_{n-1} & p_n & \cdots & p_{2n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & p_{-n+2} & p_{-n+3} & p_{-n+4} & \cdots & p_1 \end{array} \right) \quad (23)$$

It is also clear that this last determinant is equal to 0.

So we get the result, $v_n(x, t) = \partial_x(\ln \frac{f(x, t)}{g(x, t)})$ is a solution to the mKdV equation.

□

3 Explicit rational solutions to the mKdV equation for the first orders

In the following, we will call

$$v_k(x, t) = \partial_x \left(\ln \frac{A_{k+1}(x, t)}{A_k(x, t)} \right)$$

a rational solution to the mKdV equation of order k .

In this section we will give some explicit examples of rational solutions to the mKdV equation.

3.1 Rational solutions of order 1 to the mKdV equation

Example 3.1 *The function $v_k(x, t)$ defined by*

$$v_k(x, t) = \frac{-(2x^3 + 3t)x}{(-x^3 + 3t)x^2} \quad (24)$$

is a rational to the KdV equation.

3.2 Rational solutions of order 2 to the mKdV equation

Example 3.2 *The function $v_k(x, t)$ defined by*

$$v_k(x, t) = \frac{3(x^6 - 6tx^3 + 90t^2)(-x^3 + 3t)x^2}{(-x^3 + 3t)^2(-x^6 + 15tx^3 + 45t^2)} \quad (25)$$

is a rational to the mKdV equation.

3.3 Rational solutions of order 3 to the mKdV equation

Example 3.3 *The function $v_k(x, t)$ defined by*

$$v_k(x, t) = \frac{(4x^{15} - 150tx^{12} + 2250t^2x^9 + 37800t^3x^6 - 141750t^4x^3 + 212625t^5)(-x^6 + 15tx^3 + 45t^2)}{(-x^6 + 15tx^3 + 45t^2)^2(-x^9 + 45tx^6 + 4725t^3)x} \quad (26)$$

is a rational to the mKdV equation.

3.4 Rational solutions of order 4 to the mKdV equation

Example 3.4 *The function $v_k(x, t)$ defined by*

$$v_k(x, t) = \frac{n(x, t)}{d(x, t)} \quad (27)$$

with

$$n(x, t) = -5(x^{24} - 114tx^{21} + 4410t^2x^{18} - 15120t^3x^{15} + 2182950t^4x^{12} - 50604750t^5x^9 + 250047000t^6x^6 + 1875352500t^7x^3 + 1406514375t^8)(-x^9 + 45tx^6 + 4725t^3)x$$

and

$$d(x, t) = (-x^9 + 45tx^6 + 4725t^3)^2(x^{15} - 105tx^{12} + 1575t^2x^9 - 33075t^3x^6 - 992250t^4x^3 + 1488375t^5)x^2$$

is a rational to the mKdV equation.

3.5 Rational solutions of order 5 to the mKdV equation

Example 3.5 The function $v_k(x, t)$ defined by

$$v_k(x, t) = \frac{n(x, t)}{d(x, t)} \quad (28)$$

with

$$n(x, t) = 3(2x^{33} - 525tx^{30} + 50400t^2x^{27} - 1984500t^3x^{24} + 65488500t^4x^{21} + 62511750t^5x^{18} - 111583473750t^6x^{15} + 2644247025000t^7x^{12} + 590736037500t^8x^9 + 406131025781250t^9x^6 - 682300123312500t^{10}x^3 + 10234501849687500t^{11})(x^{15} - 105tx^{12} + 1575t^2x^9 - 33075t^3x^6 - 992250t^4x^3 + 1488375t^5)x^2$$

and

$$d(x, t) = (x^{15} - 105tx^{12} + 1575t^2x^9 - 33075t^3x^6 - 992250t^4x^3 + 1488375t^5)^2(x^{21} - 210tx^{18} + 10395t^2x^{15} - 264600t^3x^{12} - 5457375t^4x^9 - 343814625t^5x^6 + 3438146250t^6x^3 + 5157219375t^7)$$

is a rational to the mKdV equation.

3.6 Rational solutions of order 6 to the mKdV equation

Example 3.6 The function $v_k(x, t)$ defined by

$$v_k(x, t) = \frac{n(x, t)}{d(x, t)} \quad (29)$$

with

$$n(x, t) = 7(x^{48} - 516tx^{45} + 104760t^2x^{42} - 10697400t^3x^{39} + 664438950t^4x^{36} - 20691389250t^5x^{33} + 308450835000t^6x^{30} - 62918076375000t^7x^{27} + 5078958570984375t^8x^{24} - 107131794804168750t^9x^{21} + 2646690914052281250t^{10}x^{18} + 19530938501266500000t^{11}x^{15} - 766159925718531093750t^{12}x^{12} + 13402310448457399218750t^{13}x^9 + 91280600892196340625000t^{14}x^6 - 228201502230490851562500t^{15}x^3 + 171151126672868138671875t^{16})(x^{21} - 210tx^{18} + 10395t^2x^{15} - 264600t^3x^{12} - 5457375t^4x^9 - 343814625t^5x^6 + 3438146250t^6x^3 + 5157219375t^7)$$

and

$$d(x, t) = (x^{21} - 210tx^{18} + 10395t^2x^{15} - 264600t^3x^{12} - 5457375t^4x^9 - 343814625t^5x^6 + 3438146250t^6x^3 + 5157219375t^7)^2(x^{27} - 378tx^{24} + 42525t^2x^{21} - 2182950t^3x^{18} + 19646550t^4x^{15} - 3094331625t^5x^{12} - 147496474125t^6x^9 + 6637341335625t^7x^6 + 232306946746875t^9)x$$

is a rational to the mKdV equation.

3.7 Rational solutions of order 7 to the mKdV equation

Example 3.7 The function $v_k(x, t)$ defined by

$$v_k(x, t) = \frac{n(x, t)}{d(x, t)} \quad (30)$$

with

$$\begin{aligned} n(x, t) = & -(8x^{63} - 7308tx^{60} + 2770740t^2x^{57} - 573388200t^3x^{54} + 73517390100t^4x^{51} - 6018268721850t^5x^{48} + \\ & 335265468060600t^6x^{45} - 15896126159991000t^7x^{42} + 112751070091852500t^8x^{39} + 109151834921265386250t^9x^{36} - \\ & 9193362965791861106250t^{10}x^{33} + 421668079507718418375000t^{11}x^{30} - 6155017110530839175625000t^{12}x^{27} + \\ & 48652560275540649553125000t^{13}x^{24} - 7698733672089088443189375000t^{14}x^{21} + 432691544810520766800717187500t^{15}x^{18} + \\ & 275075049234943040535014062500t^{16}x^{15} + 49670782713309725940554613281250t^{17}x^{12} \\ & - 1126712952508127399160038261718750t^{18}x^9 + 3889474849754083624497666328125000t^{19}x^6 \\ & + 14585530686577813591866248730468750t^{20}x^3 + 7292765343288906795933124365234375t^{21})(x^{27} - \\ & 378tx^{24} + 42525t^2x^{21} - 2182950t^3x^{18} + 19646550t^4x^{15} - 3094331625t^5x^{12} - 147496474125t^6x^9 + \\ & 6637341335625t^7x^6 + 232306946746875t^9)x \end{aligned}$$

and

$$\begin{aligned} d(x, t) = & (x^{27} - 378tx^{24} + 42525t^2x^{21} - 2182950t^3x^{18} + 19646550t^4x^{15} - 3094331625t^5x^{12} - \\ & 147496474125t^6x^9 + 6637341335625t^7x^6 + 232306946746875t^9)^2(-x^{36} + 630tx^{33} - 135135t^2x^{30} + \\ & 13825350t^3x^{27} - 638512875t^4x^{24} + 29499294825t^5x^{21} + 1179971793000t^6x^{18} + 54426198952125t^7x^{15} - \\ & 12683959292379375t^8x^{12} + 132879573539212500t^9x^9 - 2092853283242596875t^{10}x^6 - 31392799248638953125t^{11}x^3 + \\ & 31392799248638953125t^{12})x^2 \end{aligned}$$

is a rational to the mKdV equation.

3.8 Rational solutions of order 8 to the mKdV equation

Example 3.8 The function $v_k(x, t)$ defined by

$$v_k(x, t) = \frac{n(x, t)}{d(x, t)} \quad (31)$$

with

$$\begin{aligned} n(x, t) = & 9(x^{78} - 1500tx^{75} + 970200t^2x^{72} - 358835400t^3x^{69} + 85386089250t^4x^{66} - 13839887719350t^5x^{63} + \\ & 1590995300895000t^6x^{60} - 133885499492745000t^7x^{57} + 7772156345589260625t^8x^{54} - 299229467351821256250t^9x^{51} + \\ & 40195977979910303801250t^{10}x^{48} - 7885034029944807986250000t^{11}x^{45} + 860331027602278517691656250t^{12}x^{42} - \\ & 49042436051036490839057531250t^{13}x^{39} + 1751863511574639624252341250000t^{14}x^{36} \\ & - 60341491647562717548686917687500t^{15}x^{33} + 728485213859962379636955201562500t^{16}x^{30} \\ & + 205090979865932365476475041337500000t^{17}x^{27} - 6110692198752612939851069319445312500t^{18}x^{24} + \\ & 183798097133187256020153764614828125000t^{19}x^{21} - 2884871725550740774926862886632781250000t^{20}x^{18} - \\ & 98745174585313076455269232726174921875000t^{21}x^{15} + 234189201294341927147297983264065000000000t^{22}x^{12} + \\ & 232317049249110996043649525549017968750000t^{23}x^9 + 131767326370980143068507465272333629150390625t^{24}x^6 - \\ & 100622321956021200161405700753418407714843750t^{25}x^3 + 1509334829340318002421085511301276115722656250t^{26})(-x^{36} + \\ & 630tx^{33} - 135135t^2x^{30} + 13825350t^3x^{27} - 638512875t^4x^{24} + 29499294825t^5x^{21} + 1179971793000t^6x^{18} + \\ & 54426198952125t^7x^{15} - 12683959292379375t^8x^{12} + 132879573539212500t^9x^9 - 2092853283242596875t^{10}x^6 - \\ & 31392799248638953125t^{11}x^3 + 31392799248638953125t^{12})x^2 \end{aligned}$$

and

$$\begin{aligned}
d(x, t) = & (-x^{36} + 630 tx^{33} - 135135 t^2 x^{30} + 13825350 t^3 x^{27} - 638512875 t^4 x^{24} + 29499294825 t^5 x^{21} + \\
& 1179971793000 t^6 x^{18} + 54426198952125 t^7 x^{15} - 12683959292379375 t^8 x^{12} + 132879573539212500 t^9 x^9 - \\
& 2092853283242596875 t^{10} x^6 - 31392799248638953125 t^{11} x^3 + 31392799248638953125 t^{12})^2 (-x^{45} + \\
& 990 tx^{42} - 363825 t^2 x^{39} + 67567500 t^3 x^{36} - 6810804000 t^4 x^{33} + 445298879025 t^5 x^{30} - 8210637059625 t^6 x^{27} + \\
& 1121710685720625 t^7 x^{24} - 48406130360713125 t^8 x^{21} - 23885103343673446875 t^9 x^{18} + 1037875841064050113125 t^{10} x^{15} - \\
& 23588549355427309378125 t^{11} x^{12} - 305263579893765180187500 t^{12} x^9 - 9615802766653603175906250 t^{13} x^6 + \\
& 72118520749902023819296875 t^{14} x^3 + 72118520749902023819296875 t^{15})
\end{aligned}$$

is a rational to the mKdV equation.

3.9 Rational solutions of order 9 to the mKdV equation

Example 3.9 The function $v_k(x, t)$ defined by

$$v_k(x, t) = \frac{n(x, t)}{d(x, t)} \quad (32)$$

with

$$\begin{aligned}
n(x, t) = & 5(2x^{99} - 4653 tx^{96} + 4799520 t^2 x^{93} - 2919456540 t^3 x^{90} + 1177028552700 t^4 x^{87} - 334613109981150 t^5 x^{84} + \\
& 69733689267592350 t^6 x^{81} - 10932392980379871000 t^7 x^{78} + 1306220634815585989500 t^8 x^{75} - 121146984881525086233750 t^9 x^{72} + \\
& 9320789025408230752882500 t^{10} x^{69} - 500705890189599166577505000 t^{11} x^{66} - 41832040420750126783836600000 t^{12} x^{63} + \\
& 18338410130868093522520964700000 t^{13} x^{60} - 2730856493733147315608983911000000 t^{14} x^{57} + 242934266578687370437331166460875000 t^{15} x^{54} - \\
& 14969624200681230581860585801011937500 t^{16} x^{51} + 584715350204300649895694060606585156250 t^{17} x^{48} \\
& - 4798448149052652504054108653314125000000 t^{18} x^{45} + 1269212187595244032943536914536686017187500 t^{19} x^{42} - \\
& 179122900491707815151177546562032102442187500 t^{20} x^{39} + 12565064668809514621695666757366189442561718750 t^{21} x^{36} - \\
& 449301113343170109970511345442173092137152343750 t^{22} x^{33} + 573586202683635891089283164387903227397156250000 t^{23} x^{30} - \\
& 78877145899749961380668818349383506289020937500000 t^{24} x^{27} + 22816724570611143207434688336826913771144789531250000 t^{25} x^{24} - \\
& 375334689633655774507281362220239230191992461640625000 t^{26} x^{21} + 7735165716727020590083875357398512721527415017382812500 t^{27} x^{18} + \\
& 13634147508744289200140961784353357911649986469140625000 t^{28} x^{15} \\
& - 625478676595763406255434077653018720492489568895507812500 t^{29} x^{12} \\
& + 11338984947224872923457450178681819821218595816010742187500 t^{30} x^9 \\
& + 48430918757638440283241990593691840592323493993808593750000 t^{31} x^6 \\
& - 90807972670572075531078732363172201110606551238391113281250 t^{32} x^3 \\
& + 45403986335286037765539366181586100555303275619195556640625 t^{33}) (-x^{45} + 990 tx^{42} - 363825 t^2 x^{39} + 67567500 t^3 x^{36} - \\
& 6810804000 t^4 x^{33} + 445298879025 t^5 x^{30} - 8210637059625 t^6 x^{27} + 1121710685720625 t^7 x^{24} - 48406130360713125 t^8 x^{21} - \\
& 23885103343673446875 t^9 x^{18} + 1037875841064050113125 t^{10} x^{15} - 23588549355427309378125 t^{11} x^{12} - 305263579893765180187500 t^{12} x^9 - \\
& 9615802766653603175906250 t^{13} x^6 + 72118520749902023819296875 t^{14} x^3 + 72118520749902023819296875 t^{15})
\end{aligned}$$

and

$$\begin{aligned}
d(x, t) = & (-x^{45} + 990 tx^{42} - 363825 t^2 x^{39} + 67567500 t^3 x^{36} - 6810804000 t^4 x^{33} + 445298879025 t^5 x^{30} - 8210637059625 t^6 x^{27} + \\
& 1121710685720625 t^7 x^{24} - 48406130360713125 t^8 x^{21} - 23885103343673446875 t^9 x^{18} + 1037875841064050113125 t^{10} x^{15} - \\
& 23588549355427309378125 t^{11} x^{12} - 305263579893765180187500 t^{12} x^9 - 9615802766653603175906250 t^{13} x^6 + 72118520749902023819296875 t^{14} x^3 + \\
& 72118520749902023819296875 t^{15})^2 (-x^{54} + 1485 tx^{51} - 868725 t^2 x^{48} + 269594325 t^3 x^{45} - 49165491375 t^4 x^{42} + \\
& 5714434826100 t^5 x^{39} - 396365167823625 t^6 x^{36} + 24418778773764375 t^7 x^{33} - 435655173246418125 t^8 x^{30} - 194269936514328675000 t^9 x^{27} - \\
& 40358223939201682263750 t^{10} x^{24} + 5744505548909944754437500 t^{11} x^{21} - 290076716794050362473171875 t^{12} x^{18} + \\
& 1331101840126763068207593750 t^{13} x^{15} - 209648539819965183242696015625 t^{14} x^{12} - 3997298825900669493827404031250 t^{15} x^9 + \\
& 179878447165530127222233181406250 t^{16} x^6 + 314787282539677226389080674609375 t^{17} x^3) x
\end{aligned}$$

is a rational to the mKdV equation.

We could go on and present more explicit rational solutions, but they become very complicated. For example, in the case of order 10 the numerator includes 59 terms and the denominator 59 terms with big coefficients. It will be relevant to study in detail the structure of these polynomials.

4 Conclusion

Rational solutions to the modified Korteweg de Vries equation have been given in terms of determinants involving particular polynomials. This gives a very efficient method to construct rational solutions of this equation. We have constructed some examples of these rational solutions for the first orders.

We can mention some other recent works about this equation. For example, in [10], first and second-order rational solutions are given as limiting cases of periodic solutions. Explicit periodic and rational solutions of first and second order are given in [11]. The first four exact rational solutions of the set of rational solutions of the mKdV equation are presented in [12]. Multiple periodic solutions of the mKdV equation are given in [13] and in particular first to third-order rational solutions.

The solutions presented in this work are different from these given in [14] to It would be relevant to better understand the structure of polynomials defining the rational solutions of this equation.

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