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We consider the problem of recovering elements of a low-dimensional model from under-determined linear measurements. To perform recovery, we consider the minimization of a convex regularizer subject to a data fit constraint. Given a model, we ask ourselves what is the "best" convex regularizer to perform its recovery. To answer this question, we define an optimal regularizer as a function that maximizes a compliance measure with respect to the model. We introduce and study several notions of compliance. We give analytical expressions for compliance measures based on the best-known recovery guarantees with the restricted isometry property. These expressions permit to show the optimality of the ℓ^1-norm for sparse recovery and of the nuclear norm for low-rank matrix recovery for these compliance measures. We also investigate the construction of an optimal convex regularizer using the examples of sparsity in levels and of sparse plus low-rank models.

Keywords: inverse problems, convex regularization, low dimensional modeling, sparse recovery, low rank matrix recovery

1. Introduction

In a finite-dimensional Hilbert space $\mathcal{H}$ (with associated inner product $\langle \cdot, \cdot \rangle$, and norm $\| \cdot \|_{\mathcal{H}}$), we consider the observation model:

$$y = Mx_0 \tag{1.1}$$

where y is an m-dimensional vector of measurements, M is an under-determined linear operator (from $\mathcal{H} = \mathbb{C}^n$, or $\mathbb{R}^n$, to $\mathbb{C}^m$), and $x_0 \in \mathcal{H}$ is the unknown vector we want to recover. The problem of recovering x_0 from y is typically ill-posed. It is thus necessary to use prior information on x_0 to recover it with a guarantee of success.

In this work, we suppose that x_0 belongs to a low-dimensional cone Σ (a positively homogeneous set, *i.e.,* for every $x \in \Sigma$ and $\lambda \geqslant 0$, $\lambda x \in \Sigma$) that models known properties of the unknown. Examples of such models include sparse as well as low-rank models and many of their generalizations. Note that in these examples the models belong to the slightly less general class of models that are (finite or infinite) unions of subspaces (homogeneous sets).

To recover x_0, a classical method is to solve the constrained minimization problem

$$x^* \in \arg\min_{Mx=y} R(x) \tag{1.2}$$

where R is a function – the regularizer – that aims to enforce some structure on the solution x^*.

Many works [11, 12, 18, 31] give practical regularizers ensuring that $x^* = x_0$ for several low-dimensional models (in particular sparse and low-rank models, see [22] for a most complete review

of these results). A practical regularizer is a function that permits the effective calculation of x^*. Without computational constraint, the best possible regularizer would be $R = \iota_\Sigma$: the characteristic function of Σ defined by $\iota_\Sigma(x) = 0$ if $x \in \Sigma$, $\iota_\Sigma(x) = +\infty$ otherwise (see Section 2 for a review of this fact). Unfortunately, this function is generally not convex (unless Σ itself is a convex set) and can lead to an intractable optimization problem in general, even though recent works show that using $R = \iota_\Sigma$ and a dedicated minimization technique is a possible route for certain particular low-dimensional models that can be smoothly embedded in $\mathbb{R}^n$ [16, 36, 37].

In this work, we focus on continuous convex regularizers that guarantee the existence of a minimizer x^* and the existence of practical optimization algorithms to perform minimization (1.2) such as the primal-dual method [13] (provided their proximity operators can be calculated). Note that convexity in itself is not sufficient to guarantee the practical feasibility of minimization (1.2) ($R(x)$ could be NP-hard to calculate, *e.g.,* the nuclear norm for tensors [23], and/or the proximal operator of R could be NP-hard to compute).

Under conditions on the measurement operator M that typically involve the number of measurements and its structure (*e.g.,* random for compressed sensing), the fact that $x_0 \in \Sigma$ permits to give recovery guarantees when the convex regularizer R is well-chosen. For example, when $\Sigma = \Sigma_k$ is the set of k-sparse vectors in $\mathbb{R}^n$ and $R(\cdot) = \|\cdot\|_1$ (ℓ^1-norm), or when $\Sigma = \Sigma_r$ is the set of matrices of rank lower than r in $\mathbb{R}^{p\times p}$ and $R(\cdot) = \|\cdot\|_*$ (nuclear norm), x_0 can be recovered as long as the number of measurements is of the order of the dimension of the model (up to some log factors) : $m \geqslant O(k\log(n/k))$ for sparse recovery or $m \geqslant O(rp)$ for low rank recovery.

The conventional approach to provide these results is to exhibit a regularizer R for a given model set Σ and to give the best possible recovery guarantees for the pair (R,Σ). Recent works aim at giving guidelines to obtain guarantees as tight as possible for general sparse models and convex regularizers [2, 3, 14, 27, 38, 43]. With such frameworks, it becomes possible to compare the performance of different regularizers. This leads naturally to the following question which we address in this work: **what is the "best" convex regularizer to recover a given low-dimensional model Σ?**

To tackle this problem, it is necessary to define the notion of "best" based on recovery guarantees. We propose different possibilities and follow one route that permits us to give optimality results in the sparse and low-rank cases and shows the difficulties that arise when considering more complex generalized sparsity models. This work can be viewed as a way to give meaning to the expression "convexification" of a low-dimensional model, that is often used and rarely defined.

1.1 *Related works*

LOW-COMPLEXITY MODELS INDUCED BY CONVEX REGULARIZATION. Many regularizers encountered in signal processing and machine learning are known to induce a specific type of model. Without aiming for exhaustivity, the use of the ℓ^1 norm [15] induces a sparse pattern in the solution, while group regularization with mixed $\ell^1 - \ell^2$ norms restricts this sparse pattern to satisfy a specific block structure [44]. More advanced model sets, such as low-rank matrices are linked to the use of the nuclear norm [20]. For a wide class of regularizers, including decomposable norms [10], decomposable M-estimator [28], atomic norms [14] and partly smooth functions [41, 42], the connection between non-smooth convexity and model space can be made explicit. Note that all these works take the following stance: given a convex regularizer R, what is the model set Σ induced by minimizing $R(x)$? Conversely, in this paper, we study the question of finding the best regularizer for a given low-dimensional model Σ.

CONVEXIFICATION OF COMBINATORIAL FUNCTIONS. Given a real function f, it is known that its biconjugate f^{**} is a convex and closed function, whatever the initial properties of f. For instance, if f is the constant function equal to 1 except in 0 – that is the counting function ℓ^0 in dimension 1 – restricted to $[-1,1]$, *i.e.,*

$$f(x) = \begin{cases} 1 & \text{if } x \in [-1,1] \setminus \{0\}, \\ 0 & \text{if } x = 0, \\ +\infty & \text{otherwise,} \end{cases}$$

then its biconjugate is the absolute value $|\cdot|$ restricted to $[-1,1]$. Unfortunately, this construction is harder to generalize on an unbounded domain or in higher dimension. For instance, the biconjugate of the ℓ^0 counting function not restricted to a bounded set is the constant 0. Of interest, we can mention convex closures of submodular functions (functions of $\{0,1\}^p$) that can be calculated explicitly using the Lovász extension [5] and convex closure of almost convex functions [25].

CONVEXIFICATION OF OBJECTIVE FUNCTION Many works intent to find a convex proxy to a non-convex objective function. In [7], adding a Lagrangian term to the regularization of a constrained non-convex minimization permits to build an equivalent minimization problem that is convex locally. Another possibility is to try to perform a regularization by infimal regularization [8] for lower semicontinuous objective functionals. In [29], in a function space setting, Pock *et al.* propose a high dimensional lifting of the Lagrangian formulation of (1.2) where the data-fit functional is non-convex. In the context of non-convex polynomial optimization, Lasserre's hierarchies [26] are used to recast the original problem in a hierarchy of convex semi-definite positive problems which provide global convergence results. The drawback of this method is the computational cost that makes it impractical for high-dimensional problems. Finally, convex closure of submodular functions also permits to cast sparsity inducing objective functions (where the regularizer is a submodular function of the support) into convex problems [5]. Note that if one aims to find a non-convex, but continuous, regularization, several works of interest may be cited, such as the use of ℓ^p minimization [21], SCAD [19], or CEL0 [33]. Nevertheless, in this paper, we focus on convex functions.

1.2 *Contributions*

In this paper, we define notions of *compliance measures* between a low-dimensional model and a regularizer in finite dimension. The compliance of a function R for a model Σ is a function

$$R \mapsto A_\Sigma(R) \tag{1.3}$$

that quantifies the recovery capabilities of Σ with R and minimization (1.2).

An optimal regularizer for a model Σ is defined as a regularizer that maximizes the compliance measure. In this article, we focus on the maximization of compliance measures on the set $\mathcal{C}$ of coercive continuous convex regularizers over $\mathcal{H}$. Note that this idea was first mentioned in the preliminary work [40] where optimal regularizers for sparse recovery were considered among weigthed ℓ^1-norms.

- We introduce compliance measures in **Section 2** using tight recovery guarantees: a good regularizer is a regularizer that permits the recovery of Σ as often as possible. We discuss the elementary properties of these measures and show that optimal coercive continuous convex regularizers can be found in the smaller class of atomic norms with atoms included in the model set. While such compliance measures can be optimized in basic toy examples, they require to be approximated in order to deal with sparse and low-rank models.

- We propose in **Section 3** compliance measures exploiting best known uniform recovery guarantees based on the restricted isometry property (RIP). We give explicit formulations of such recovery guarantees and show that, indeed, the ℓ^1-norm and the nuclear norm are optimal for sparse and low-rank recovery (respectively) among coercive continuous convex regularizers.
- We study the case of two generalized sparsity models in **Section 4**: sparsity in levels and sparse plus low-rank models. We show how an optimal regularizer can be explicitly constructed in a small family of convex regularizers (ℓ^1-norm weighted by levels and mixed weighted ℓ^1-nuclear norm respectively). While giving optimal weighting schemes for mixed regularizations, these examples also show the difficulty of calculating optimal regularizers for new low-dimensional models and opens many questions for the study of optimal regularizers.

We give an overview of the different compliance measures and the nature of results considered in this paper in Table 1.

1.3 *Notations*

In $\mathcal{H}$, we denote $S(1) := \{z \in \mathcal{H} : \|z\|_{\mathcal{H}} = 1\}$ the unit sphere with respect to $\|\cdot\|_{\mathcal{H}}$. Given a linear operator $M : \mathcal{H} \to \mathbb{C}^m$, we denote M^H its Hermitian adjoint.

For $\Sigma \subseteq \mathcal{H}$ an arbitrary set, we denote ι_Σ its characteristic function defined by $\iota_\Sigma(x) = 0$ if $x \in \Sigma$, $\iota_\Sigma(x) = +\infty$ otherwise. We denote $\mathcal{E}(\Sigma) := \mathbb{R}_+ \cdot \overline{\text{conv}}(\Sigma)$, where $\overline{\text{conv}}(\Sigma)$ is the closure of the convex hull of Σ. We define $\bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$. Given a function $f : \mathcal{H} \to \bar{\mathbb{R}}$, we denote by $\texttt{dom}(f)$ its domain, *i.e.,* the set $\texttt{dom}(f) := \{x \in \mathcal{H} : f(x) < +\infty\}$.

2. Optimal regularizer for a low dimensional model

In this section, starting from the characterization of exact recovery of a model Σ, we introduce the notion of compliance measure and associated optimal convex regularizer.

2.1 *Characterization of exact recovery using descent cones*

Before defining an optimal regularizer, we must characterize when Σ can be recovered by solving (1.2). The fact that a given $x_0 \in \Sigma$ is recovered by solving (1.2) with regularizer R (*i.e.,* that the solution x^* of (1.2) is unique and satisfies $x^* = x_0$ when $y := Mx_0$) is equivalent to the fact that $R(x_0 + z) > R(x_0)$ for every $z \in \ker(M) \setminus \{0\}$ (see *e.g.,* [14]). To summarize this, we use the following definition of symmetrized descent cones.

DEFINITION 2.1 ((Symmetrized) descent cones.) Consider a regularizer $R : \mathcal{H} \to \bar{\mathbb{R}}$. For any $x \in \texttt{dom}(R)$, the **descent cone** of R at x is

$$\mathcal{T}_R(x) := \{\gamma z : \gamma \in \mathbb{R}, z \in \mathcal{H}, R(x+z) \leqslant R(x)\}. \tag{2.1}$$

For any set $\Sigma \subset \texttt{dom}(R)$, we define $\mathcal{T}_R(\Sigma) := \bigcup_{x \in \Sigma} \mathcal{T}_R(x)$.

Other definitions of descent cones (*e.g.,* non-symmetric like in [14]) could be used. The symmetrization facilitates technical derivations in the following and permits to characterize recovery as well. For ease of reading, in the following, symmetrized descent cones will be referred to as descent cones.
Recovery guarantees with a regularizer R for a linear operator M generally come in two flavors (recall that x^* is the result of minimization (1.2)):

Compliance (type of recovery)	Definition	Section	Results
Based on descent cone (uniform)	$f(\mathcal{T}_R(\Sigma))$	Sec. 2	Optimality of atomic norms (Th 2.4); Equivariance (Lem. 2.5); Invariance (Cor. 2.1)
$A_\Sigma^U(R)$: **Volume** (uniform)	$1-\frac{\mathrm{vol}(\mathcal{T}_R(\Sigma)\cap S(1))}{\mathrm{vol}(S(1))}$	Sec. 2	Monotonicity (Lem. 2.2); Invariance (Cor. 2.2)
$\delta_\Sigma^{\mathtt{suff}}(R)$: **Sufficient RIP** (uniform)	$\frac{1}{\sqrt{\sup_{z\in\mathcal{T}_R(\Sigma)\setminus\{0\}}\frac{\|z-P_\Sigma(z)\|_\Sigma^2}{\|P_\Sigma(z)\|_2^2}+1}}$	Sec. 3	Characterization (Lem. 3.5, Cor. 3.2); Optimal sparse reg. and optimal low-rank reg. (Th. 3.8, Th. 3.9); Sharp bound for near-optimal reg. for sparsity in levels (Th. 4.3)
$\delta_\Sigma^{\mathtt{nec}}(R)$: **Necessary RIP** (uniform)	$\inf_{z\in\mathcal{T}_R(\Sigma)\setminus\{0\}}\delta_\Sigma(I-\Pi_z)$	Sec. 3	Characterization (Lem. 3.3, Cor. 3.1); Optimal sparse reg. and optimal low-rank reg. (Th. 3.6, Th. 3.7); Opt. weights for sparsity in levels (Th. 4.2); Opt. weights for sparse+low-rank (Th. 4.4)
$\delta_\Sigma^{\mathtt{sharp}}(R)$: **Sharp RIP** (uniform)	$\inf_{M:\ker M\cap\mathcal{T}_R(\Sigma)\neq\{0\}}\delta_\Sigma(M)$	Sec. 3	Characterization (Prop. 3.4); Invariance (Lem. 3.2); Bound by $\delta^{\mathtt{nec}}$ and $\delta_\Sigma^{\mathtt{suff}}(R)$ (Eq. 3.5)
$A_\Sigma^{NU}(R)$: Volume (non-uniform)	$1-\sup_{x\in\Sigma}\frac{\mathrm{vol}(\mathcal{T}_R(x)\cap S(1))}{\mathrm{vol}(S(1))}$.	Sec. 2	-
Kinematic formula (non-uniform)	$\sup_{x\in\Sigma}\mathbb{P}\big(\ker M\cap\mathcal{T}_R(x)\neq\{0\}\big)$, M Gaussian	Sec. 2	-
Statistical dimension (non-uniform)	$\sup_{x\in\Sigma}\mathtt{statdim}(\mathcal{T}_R(x))$	Sec. 2	-

Table 1. A summary of different compliances measures and results. Compliances for which some results are given in this article are in bold (we focused on uniform recovery guarantees). Explicit maximization is performed on compliances based on necessary and sufficient conditions with the restricted isometry property (RIP) yielding bounds on sharp RIP-based compliances.

- **Non-uniform recovery**: If $x_0\in\Sigma$, then $x^*=x_0$ is equivalent to $\mathcal{T}_R(x_0)\cap\ker M=\{0\}$.
- **Uniform recovery**: "For all $x_0\in\Sigma$, $x^*=x_0$" is equivalent to

$$\mathcal{T}_R(\Sigma)\cap\ker M=\{0\}. \tag{2.2}$$

In the literature, recovery guarantees are obtained when the measurement operator M fulfills sufficient conditions that imply these characterizations. Distinguishing these two types of recovery guarantees especially makes sense in the context of compressed sensing when M is chosen at random. Typical results are then of the form:

- **Non-uniform recovery**: Given $x_0\in\Sigma$, with high probability on the draw of M, $x^*=x_0$.
- **Uniform recovery**: With high probability on the draw of M, $x^*=x_0$ for all $x_0\in\Sigma$.

The main techniques to obtain recovery guarantees using a condition on the number of measurements differ largely between these two cases (see Section 3). In this work, we mostly focus on uniform recovery guarantees to stay in a fully deterministic setting. For such uniform recovery guarantees, we see that the only interactions that matter between the model set Σ, the regularizer R, and the measurement operator M are summarized by equation (2.2).

2.2 *Compliance measures and optimal regularization*

To define a notion of optimal regularizer, we simply propose to say that an optimal regularizer is a function that optimizes a (hopefully well-chosen) criterion. We call such a criterion, a *compliance measure* and specifically aim at defining it such that it should be maximized. The objective is to define a compliance measure that *quantifies* the recovery capabilities of a given regularizer R given a model set Σ.

Starting from the characterization of exact recovery, we can make the following remark. If the descent sets of a regularizer R_1 are included in the descent sets of another regularizer R_2, then the recovery capability of R_1 are greater in the following way: success of recovery with R_2 implies success of recovery with R_1. Any "reasonable" compliance measure quantifying recovery capabilities *needs* to fulfill the following axiom:

A compliance measure must be monotonously decreasing with respect to the inclusion of descent sets.

We also see that the kernel of M heavily influences the recovery capability of R. If we had some knowledge that $M \in \mathcal{M}$ where $\mathcal{M}$ is a set of linear operators, we would want to define a compliance measure $A_{\Sigma,\mathcal{M}}(R)$ that tells us how good is a regularizer in these situations and to maximize it. Such maximization might yield a function R^* that depends on $\mathcal{M}$ (*e.g.,* in [33], when looking for tight continuous relaxation of the ℓ^0 penalty a dependency on M appears). In the following, we propose a more *universal* notion of optimal convex regularizer that does not depend on a particular class of linear operators $\mathcal{M}$: we propose compliance measures $A_\Sigma(R)$ that depend only on the set Σ and on the regularizer R, and consider their maximization on some set of convex functions $\mathcal{C}$ (that are coercive and continuous, see Section 2.4):

$$\sup_{R \in \mathcal{C}} A_\Sigma(R). \tag{2.3}$$

Of course, the existence of a maximizer of $A_\Sigma(R)$ is in itself a general question of interest: we could ask ourselves what conditions on $A_\Sigma(R)$ and $\mathcal{C}$ are necessary and sufficient for the existence of a maximizer, which is out of the scope of this article – we notably expect potential difficulties when normalized atoms defining the model set are not a compact set. In the sparse recovery and low-rank matrix recovery examples studied in this article, the existence of a maximizer of the considered compliance measures will be verified.

To build a compliance measure that does not depend on M, we define the optimal regularizer as the regularizer which guarantees recovery of Σ in as many situations as possible, *i.e.,* for "as many linear operators M as possible". Intuitively, a regularizer R is "good" if the set $\mathcal{T}_R(\Sigma)$ "leaves a lot of space" for $\ker M$ to not intersect it (trivially), see Figure 1. Among non-convex regularizers, the optimal one is the characteristic function of the model set Σ.

LEMMA 2.1 (Optimality of the characteristic function.) Consider an arbitrary non-empty set $\Sigma \subseteq \mathcal{H}$ and denote ι_Σ its characteristic function. The corresponding descent cone is

$$\mathcal{T}_{\iota_\Sigma}(\Sigma) = \{\gamma z : \gamma \in \mathbb{R}, z \in \Sigma - \Sigma\} \supseteq \Sigma - \Sigma$$

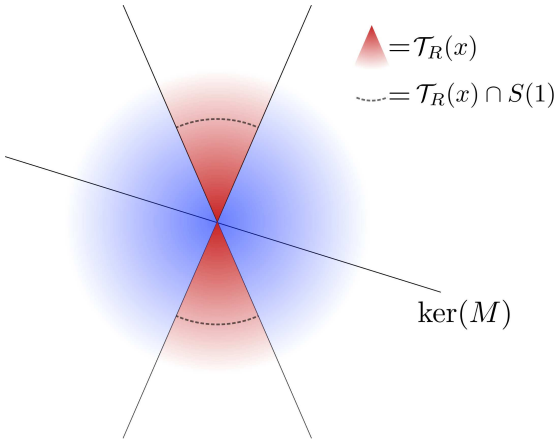

FIG. 1. A representation of recovery guarantees based on descent cones of a convex function. Recovery of $x \in \Sigma$ fails if $\ker(M)$ intersects $\mathcal{T}_R(x)$ non trivially. The bigger is the descent cone (red) the more likely recovery will fail. The bigger the space left by the descent cone (blue), the more likely recovery will succeed

where $\Sigma - \Sigma$ is the so-called secant set of Σ. For any regularizer R such that $\Sigma \subseteq \mathtt{dom}(R)$ we have $\mathcal{T}_{\iota_\Sigma}(\Sigma) \subseteq \mathcal{T}_R(\Sigma)$. Finally, if Σ is positively homogeneous then $\mathcal{T}_{\iota_\Sigma}(\Sigma) = \Sigma - \Sigma$.

Proof. See Appendix A.2 □

This Lemma shows that ι_Σ is at least as successful as any regularizer R for the exact recovery of Σ (without any consideration of compliance measure). Moreover, $\mathcal{T}_{\iota_\Sigma}(\Sigma)$ is the smallest possible descent cone with respect to inclusion. Hence ι_Σ can be considered as the ideal regularizer [9] and indeed the optimal one with respect to any compliance measure defined as $A_\Sigma(R) = f(\mathcal{T}_R(\Sigma))$ where f is some function on subsets of $\mathcal{H}$ that is monotonic with respect to set inclusion. This is why the search for optimal regularizers only makes sense under some constraint on R.

2.3 *A first compliance measure*

As a first concrete example, we define here a theoretical compliance measure that reflects the idea that smaller descent cones are better. However, this compliance measure does not lead to analytical expressions for the general study of sparse recovery. Our core results in the next sections rely on compliance measures based on best known uniform recovery guarantees using the restricted isometry property (RIP).

For convex functions, first, observe that, as only the *directions* of the descent cones and the kernel play a role in recovery guarantees, the size of descent cones can be measured by considering only their intersection with the unit sphere $S(1)$. Choosing the norm $\|\cdot\|_{\mathcal{H}}$ to define the unit sphere is natural (although also somewhat arbitrary) as this is the only metric introduced so far in the considered setting. It will also appear to define RIP constants soon. Second, if we want to consider a measure that is invariant by rotation, the uniform measure on the unit sphere $S(1)$ comes somewhat naturally. It is indeed the unique Haar measure. The uniqueness is essentially guaranteed when it is a measure in the sense of measure theory (additive, non-negative function over a σ-algebra). In our setting, using this measure is a way of considering that we do not have prior information on the orientation of the kernel of M, or on the orientation of the model set Σ.

Using this measure, given a convex function R, the "amount of space left for the kernel of M" can be quantified by the "volume" of the intersection $\mathcal{T}_R(\Sigma) \cap S(1)$ of the descent cone with the unit sphere. Hence, a compliance measure for *uniform recovery* can be defined as

$$A^U_\Sigma(R) := 1 - \frac{\mathrm{vol}\left(\mathcal{T}_R(\Sigma) \cap S(1)\right)}{\mathrm{vol}(S(1))}. \tag{2.4}$$

Here, the volume $\mathrm{vol}(E)$ of a set E is the measure of E with respect to the uniform measure on the sphere $S(1)$ (*i.e.,* the $n-1$-dimensional Hausdorff measure of $\mathcal{T}_R(\Sigma) \cap S(1)$, when $\mathcal{H}$ is n-dimensional). This measure is well-defined as the descent cones of convex functions are symmetrized convex cones.

When looking at *non-uniform recovery* for random Gaussian measurements, the quantity defined by $\frac{\mathrm{vol}(\mathcal{T}_R(x_0) \cap S(1))}{\mathrm{vol}(S(1))}$ represents the probability that a randomly oriented kernel of dimension 1, defined as the span of a random vector uniformly distributed on the sphere $S(1)$, intersects (non trivially) $\mathcal{T}_R(x_0)$. The highest probability of intersection with respect to x_0 quantifies the lack of compliance of R, hence we could define:

$$A^{NU}_\Sigma(R) := 1 - \sup_{x \in \Sigma} \frac{\mathrm{vol}\left(\mathcal{T}_R(x) \cap S(1)\right)}{\mathrm{vol}(S(1))}. \tag{2.5}$$

This can be linked with the Gaussian width and statistical dimension theory of non-uniform sparse recovery [2, 14]. Indeed, if M is a random Gaussian matrix of size $(n-1) \times n$, we have

$$\mathbb{P}\left(\ker M \cap \mathcal{T}_R(x_0) \neq \{0\}\right) = \frac{\mathrm{vol}\left(\mathcal{T}_R(x_0) \cap S(1)\right)}{\mathrm{vol}(S(1))}. \tag{2.6}$$

As shown in [2], for a random Gaussian matrix M of size $m \times n$ with any number of measurements m, the probability $\mathbb{P}\left(\ker M \cap \mathcal{T}_R(x_0) \neq \{0\}\right)$ can be guaranteed to be small if m is greater than the statistical dimension of the descent cones. The kinematic formula (Crofton's formula in this case) gives the exact value

$$\mathbb{P}\left(\ker M \cap \mathcal{T}_R(x_0) \neq \{0\}\right) = \sum_{j=m+1,\ j \text{ even}}^{n} v_j(\mathcal{T}_R(x_0)) \tag{2.7}$$

where $v_j(K)$ is the j-th intrinsic volume of a cone K. For a polyhedral cone it is the probability that the orthogonal projection on K of a Gaussian vector lies in a j-dimensional face of K. The statistical dimension of a descent cone $\mathcal{T}$ is defined by [2, Definition 2.2]

$$\mathtt{statdim}(K) = \sum_{j=0}^{n} j v_j(K). \tag{2.8}$$

As it is used to bound the number of measurements in the non uniform case, its supremum over all the descent cones $K = \mathcal{T}_R(x_0), x_0 \in \Sigma$ could be used as a compliance measure. Moreover, it was shown that the statistical dimension is a measure of the "size" of the convex cones that is additive, invariant by rotation, and monotonous.

The above compliance measures are completely dependent on the metric defining $S(1)$ (here the Hilbert norm $\|\cdot\|_{\mathcal{H}}$), other choices could be considered especially if measurement operators M showing a particular structure were considered.

In this article, we concentrate on compliance measures based on uniform recovery guarantees.

REMARK 2.1 These compliance measures implicitly force $\Sigma \subset \mathtt{dom}(R)$, unless $A_\Sigma(R) = 0$. Indeed, suppose there exists $x \in \Sigma$ such that $R(x) = +\infty$, then for all $z \in \mathcal{H}$, we have $R(x+z) \leqslant +\infty = R(x)$. This implies $\mathcal{T}_R(x) = \mathcal{H}$ and $A^U_\Sigma(R) = A^{NU}_\Sigma(R) = 0$.

REMARK 2.2 When studying convex regularization for low dimensional recovery in infinite dimensional separated Hilbert spaces, the noiseless recovery only depends on the behavior of the regularizer R on $\mathcal{E}(\Sigma)$ (defined in Section 1.3). The behavior of R outside $\mathcal{E}(\Sigma)$ is only studied to obtain properties of robustness to modeling error [38]. In many examples of generalized sparsity and low-dimensional modeling in infinite dimension, the space $\mathcal{E}(\Sigma)$ has a finite dimension [1]. Our framework still applies in this case.

It is an open question to generalize our framework for low-dimensional recovery in more general settings such as Banach spaces (*e.g.,* for off-the-grid super-resolution).

REMARK 2.3 In the uniform recovery case, the compliance measure A^U_Σ defined in (2.4) is monotonous with respect to the partial ordering of descent cones defined by the inclusion property. However, it does not (at least explicitly) take into account potential effects of the dimension of the kernel of M, which may be higher than one. For a given dimension ℓ of the kernel of M, the uniform measure on the corresponding Grassmanian manifold (of all subspaces of dimension ℓ) would be more natural as it would directly quantify the probability of intersection with a random kernel of fixed dimension. This measure for kernels of dimension ℓ and a descent cone K is the following:

$$V_\ell(K) := \mu_{O(n)}\left(\{Q \in O(n) : (QE) \cap K \neq \{0\}\}\right) \tag{2.9}$$

where $\mu_{O(n)}$ is the uniform measure on the orthogonal group and E is an arbitrary fixed ℓ-dimensional subspace. The measure V_ℓ is invariant by rotation and for $\ell = 1$ it matches the Haar measure used in (2.4)-(2.5).

Given a set Σ, and assuming the existence of a maximizer R^* of A^U_Σ (within a prescribed family of possible regularizers), there are only two possibilities: either all maximizers of $A^U_\Sigma(R)$ also minimize $V_\ell(\mathcal{T}_R(\Sigma))$, or not. In this last case, it would mean that there is R^* maximizing A^U_Σ and not minimizing V_ℓ. It is an interesting challenge, left to future work, to understand whether this case can indeed happen.

We remind the reader that compliances considered in this article are summarized in Section 1.2.

2.4 *Coercive continuous convex functions*

As mentioned before we look for practical regularizers. We define $\mathcal{C}$ the set of all functions $R : \mathcal{H} \to \mathbb{R}$ (*i.e.,* with $\mathtt{dom}(R) = \mathcal{H}$) that are convex, continuous, and coercive.

The coercivity condition is typical in the context of convex regularization in order to avoid constant functions.

With any proper lower semi-continuous regularizer (hence, with any regularizer in $\mathcal{C}$) the convergence of the primal dual algorithm is guaranteed [13]. This guarantees the existence of practical algorithms (for the problem $\min_x \frac{1}{2}\|Mx - y\|^2 + \lambda R(x)$) when the proximity operator

$$y \mapsto \mathtt{prox}_{\lambda R}(y) := \arg\min \tfrac{1}{2}\|u - y\|^2_{\mathcal{H}} + \lambda R(u) \tag{2.10}$$

can be approximated efficiently.

2.5 *Elementary properties and reduction to atomic "norms"*

As compliance measures based on uniform recovery guarantees can be written as functions of descent cones $\mathcal{T}_R(\Sigma)$, we have the following immediate Lemma.

LEMMA 2.2 (The compliance measure A^U_Σ is monotonic.) Let R_1, R_2 be two functions such that $\mathcal{T}_{R_1}(\Sigma) \subset \mathcal{T}_{R_2}(\Sigma)$ then $A^U_\Sigma(R_1) \geqslant A^U_\Sigma(R_2)$.

In other words, the compliance measure is decreasing with respect to the inclusion of descent cones. We also verify that multiplying a regularizer by a scalar does not change the compliance measure which is consistent with recovery guarantees.

LEMMA 2.3 (The compliance measures A_Σ^U and A_Σ^{NU} are 0-homogeneous.) Let $\lambda > 0$. Then,

$$\begin{aligned} A_\Sigma^U(\lambda R) &= A_\Sigma^U(R), \\ A_\Sigma^{NU}(\lambda R) &= A_\Sigma^{NU}(R). \end{aligned} \tag{2.11}$$

Proof. Let $x \in \Sigma$. We remark that, the tangent cone is invariant by scalar multiplication:

$$\begin{aligned} \mathcal{T}_{\lambda R}(x) &= \{\gamma z : \gamma \in \mathbb{R}, \lambda R(x+z) \leqslant \lambda R(x)\} \\ &= \{\gamma z : \gamma \in \mathbb{R}; R(x+z) \leqslant R(x)\} \\ &= \mathcal{T}_R(x). \end{aligned} \tag{2.12}$$

This shows directly that $A_\Sigma^{NU}(\lambda R) = A_\Sigma^{NU}(R)$. This also implies that $\mathcal{T}_{\lambda R}(\Sigma) = \mathcal{T}_R(\Sigma)$ and $A_\Sigma^U(\lambda R) = A_\Sigma^U(R)$.

□

More generally, any operation on R that leaves $\mathcal{T}_R(\Sigma)$ invariant does not change the compliance measure. This is typically the case of the post-composition of R with an increasing function.

We now recall the notion of atomic "norm" and show that optimal regularizers can be found in a set of atomic norms.

DEFINITION 2.2 (Atomic norm.) The **atomic "norm"** induced by a set $\mathcal{A}$ is defined as:

$$\|x\|_{\mathcal{A}} := \inf\{t \in \mathbb{R}_+ : x \in t \cdot \overline{\text{conv}}(\mathcal{A})\} \tag{2.13}$$

where $\overline{\text{conv}}(\mathcal{A})$ is the closure of the convex hull $\text{conv}(\mathcal{A})$ in $\mathcal{H}$. This "norm" is finite only on

$$\mathcal{E}(\mathcal{A}) := \mathbb{R}_+ \cdot \overline{\text{conv}}(\mathcal{A}) = \{x = t \cdot y, t \in \mathbb{R}_+, y \in \overline{\text{conv}}(\mathcal{A})\} \subset \mathcal{H}. \tag{2.14}$$

It is extended to $\mathcal{H}$ by setting $\|x\|_{\mathcal{A}} = +\infty$ if $x \notin \mathcal{E}(\mathcal{A})$.

Classical convex regularizer for sparse and low rank models are atomic norms:

- The ℓ^1-norm $\|\cdot\|_1$ is the atomic norm induced by signed canonical basis vectors.
- The nuclear norm $\|\cdot\|_*$ is the atomic norm induced by unitary rank-one matrices.

Atomic norms are convex gauges induced by the convex hull of atoms. Their properties can be linked with the properties of the set $\mathcal{A}$ with classical results on convex gauge functions (see Appendix A.1).

It is possible to define an atomic norm, denoted $\|\cdot\|_\Sigma$, specifically induced by the model Σ.

DEFINITION 2.3 (Atomic norm induced by the model.) Given a cone Σ, we define the **atomic norm induced** by Σ as

$$\|\cdot\|_\Sigma := \|\cdot\|_{\Sigma \cap S(1)}. \tag{2.15}$$

This norm is known as the k-support norm for sparse model, it is not adapted to perform uniform recovery. In particular, it cannot recover 1-sparse vectors.

In [38, Lemma 2.1], it was shown that there is always an atomic norm with smaller descent cones than the descent sets of a proper coercive continuous function with convex level sets. We give a version of this result for our definition of cones and specify the properties of the obtained atomic norm.

LEMMA 2.4 (Optimality of atomic norms for a given model.) Let Σ be a cone such that $\mathcal{E}(\Sigma) = \mathcal{H}$ and R be a coercive continuous convex function. Then there exists $\mathcal{A} \subset \Sigma$ such that:

$$\mathcal{T}_{\|\cdot\|_{\mathcal{A}}}(\Sigma) \subseteq \mathcal{T}_R(\Sigma). \tag{2.16}$$

and $\|\cdot\|_{\mathcal{A}}$ is a coercive, continuous, positively homogeneous convex function.

Proof. See Appendix A.2.2. □

With Lemma 2.4, for all coercive continuous convex functions R (i.e. elements of $\mathcal{C}$), it is possible to find an atomic norm R' with atoms included in Σ such that $\mathcal{T}_{R'}(\Sigma) \subset \mathcal{T}_R(\Sigma)$. We define $\mathcal{C}_\Sigma$ as the set of coercive continuous positively homogeneous atomic "norms" whose atoms $\mathcal{A}$ are included in Σ:

$$\mathcal{C}_\Sigma := \{\|\cdot\|_{\mathcal{A}} : \mathcal{A} \subset \Sigma, \|\cdot\|_{\mathcal{A}} \in \mathcal{C}, \forall x \in \mathcal{H}, \lambda > 0, \|\lambda x\|_{\mathcal{A}} = \lambda \|x\|_{\mathcal{A}}\}. \tag{2.17}$$

Note that positive homogeneity is guaranteed if 0 is in the interior of $\overline{\text{conv}}(\mathcal{A})$ (see Appendix A.1). As a consequence of this Lemma, we have the following immediate result.

THEOREM 2.4 (Optimization of compliance measures over $\mathcal{C}_\Sigma$.) Let Σ be a cone such that $\mathcal{E}(\Sigma) = \mathcal{H}$. Suppose A_Σ is a compliance measure that is a decreasing function of $\mathcal{T}_R(\Sigma)$ with respect to set inclusion. Then

$$\sup_{R \in \mathcal{C}} A_\Sigma(R) = \sup_{R \in \mathcal{C}_\Sigma} A_\Sigma(R). \tag{2.18}$$

In particular

$$\sup_{R \in \mathcal{C}} A^U_\Sigma(R) = \sup_{R \in \mathcal{C}_\Sigma} A^U_\Sigma(R). \tag{2.19}$$

Proof. Let $R \in \mathcal{C}$, with Lemma 2.4, there exists $\|\cdot\|_{\mathcal{A}} \in \mathcal{C}_\Sigma$ such that $\mathcal{T}_{\|\cdot\|_{\mathcal{A}}}(\Sigma) \subset \mathcal{T}_R(\Sigma)$. This implies $\mathcal{T}_{\|\cdot\|_{\mathcal{A}}}(\Sigma) \cap S(1) \subset \mathcal{T}_R(\Sigma) \cap S(1)$ and $A_\Sigma(R) \leqslant A_\Sigma(\|\cdot\|_{\mathcal{A}})$.

□

Theorem 2.4 shows that we can limit ourselves to atomic norms if our only objective is to maximize the compliance measure.

With such measures, we can calculate optimal regularizers for elementary linear transformations of models.

LEMMA 2.5 (Compliance measures as functions of descent cones are equivariant to linear transformations.) Consider a compliance measure defined as: $A_\Sigma(R) := f(\mathcal{T}_R(\Sigma))$ with f some scalar valued function defined on non-empty subsets of $\mathcal{H}$. For any invertible linear map F on $\mathcal{H}$, any model set Σ and any regularizer R, we have

$$\mathcal{T}_R(F\Sigma) = F(\mathcal{T}_{R \circ F}(\Sigma)) \tag{2.20}$$

$$A_{F\Sigma}(R) = f[F(\mathcal{T}_{R \circ F}(\Sigma))]. \tag{2.21}$$

Proof. First $\gamma z \in \mathcal{T}_R(F\Sigma)$ if, and only if, there exists $x \in \Sigma$ such that $R(Fx + z) \leqslant R(Fx)$, *i.e.,* such that $(R \circ F)(x + F^{-1}z) \leqslant (R \circ F)(x)$. This is in turn equivalent to $\gamma F^{-1} z \in \mathcal{T}_{R \circ F}(\Sigma)$, *i.e.,* $\gamma z \in F(\mathcal{T}_{R \circ F}(\Sigma))$. Second, it follows that $A_{F\Sigma}(R) = f(\mathcal{T}_R(F\Sigma)) = f[F(\mathcal{T}_{R \circ F}(\Sigma))]$. □

Thanks to Lemma 2.5, we can build optimal regularizers from other optimal regularizers when the model set is obtained from another one by applying a linear isometry.

COROLLARY 2.1 (Compliance measures as functions of descent cones are invariant under invariant maps.) Consider a compliance measure defined as: $A_\Sigma(R) := f(\mathcal{T}_R(\Sigma))$ with f some scalar valued

function on subsets of $\mathcal{H}$. Assume that f is invariant to a family $\mathcal{F}$ of invertible linear maps on $\mathcal{H}$, *i.e.,* for any $F \in \mathcal{F}$ and any non-empty set $\mathcal{T} \subseteq \mathcal{H}$, $f(F\mathcal{T}) = f(\mathcal{T})$. Then, for any $F \in \mathcal{F}$, any regularizer R and any model set Σ, we have

$$A_{F\Sigma}(R \circ F^{-1}) = A_\Sigma(R). \tag{2.22}$$

Proof. By Lemma 2.5, $A_{F\Sigma}(R \circ F^{-1}) = f[F(\mathcal{T}_{(R\circ F^{-1})\circ F}(\Sigma))] = f(F\mathcal{T}_R(\Sigma)) = f(\mathcal{T}_T(\Sigma)) = A_\Sigma(R)$. □

COROLLARY 2.2 (Compliance measures A^U_Σ are invariant by isometries.) Consider F an isometry on $\mathcal{H}$, R a regularizer and Σ a model set. We have

$$A^U_{F\Sigma}(R \circ F^{-1}) = A^U_\Sigma(R). \tag{2.23}$$

Proof. The volume is invariant to isometries, hence $A^U_\Sigma(R) = f^U(\mathcal{T}_R(\Sigma))$ where $f^U(\cdot)$ is invariant to isometries.

□

2.6 *An exact result in 3D: the most we can do?*

Compliance measures $A^U_\Sigma(R)$ and $A^{NU}_\Sigma(R)$ were effectively optimized [40] in the case of 1-sparse recovery in dimension 3, *i.e.,* for $\Sigma = \Sigma_1$ the set of 1-sparse vectors in $\mathbb{R}^3$. In this case, $\mathcal{C}_\Sigma = \{\|\cdot\|_\mathcal{A} : \mathcal{A} \subset \Sigma_1\}$. It was shown that for the set $\mathcal{C}'_\Sigma = \{\|\cdot\|_\mathcal{A} : \mathcal{A} \subset \Sigma_1, \mathcal{A} = -\mathcal{A}\}$ (which is the set of weighted ℓ^1-norms) that

$$\arg\max_{R\in\mathcal{C}'_\Sigma} A^U_\Sigma(R) = \arg\max_{R\in\mathcal{C}'_\Sigma} A^{NU}_\Sigma(R) = \{\lambda\|\cdot\|_1 : \lambda > 0\}. \tag{2.24}$$

To show this, the solid angles of all descent cones of arbitrary weighted ℓ^1-norms were calculated exactly, and their size minimized with respect to the weights.

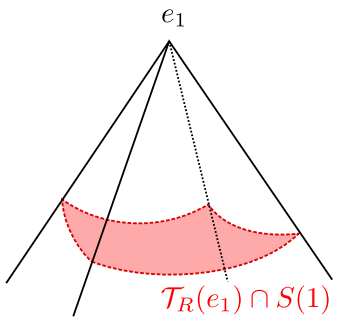

FIG. 2. Solid angle of a half descent cone of a weighted ℓ^1-norm

Unfortunately, calculating exactly these solid angles in dimension d seems out of reach for any sparsity and atomic norm in $\mathcal{C}_\Sigma$ even if some progress in bounds of these quantities [27] in some particular cases (non-uniform recovery with ℓ^1-norm in probability with random matrices). To the best of our knowledge, no general bound is available for the volume of descent cones of arbitrary atomic norms in $\mathcal{C}_\Sigma$ for sparse recovery. To build a compliance measure that we could optimize, we would need to first to establish such general bounds with some tightness.

In the next section, we propose to build compliance measures based on best-known uniform recovery guarantees that have some "tightness" properties. This will enable us to manipulate analytical expressions and give results for sparse recovery and low-rank recovery.

3. Compliance measures based on the restricted isometry property

The most used tool for the study of uniform recovery is the restricted isometry property (RIP). This property is adequate for multiple models [38], to be tight in some sense [17] for sparse and low-rank recovery, to be necessary in some sense [9], and to be well adapted to the study of random operators [30]. In [38], given a regularizer R, an explicit constant $\delta_\Sigma(R)$ is given, such that $\delta_\Sigma(M) < \delta_\Sigma(R)$ guarantees the exact recovery of elements of Σ by minimization (1.2). Hence, using $\delta_\Sigma(R)$ as a compliance measure, the higher the value of $\delta_\Sigma(R)$, the less stringent the RIP condition on M to ensure recovery of all elements of Σ using R as a regularizer.

To formalize further this idea, we start by recalling definitions and results about RIP recovery guarantees then apply our methodology. We also give results that emphasize the relevant quantity (depending on the geometry of the regularizer and the model) to optimize.

DEFINITION 3.1 (RIP constant.) Consider an arbitrary non-empty set $\Sigma \subset \mathcal{H}$ and M a linear map from $\mathcal{H}$ to $\mathbb{C}^m$. The **RIP constant** of M is defined as

$$\delta_\Sigma(M) = \sup_{x \in \Sigma - \Sigma} \left| \frac{\|Mx\|_2^2}{\|x\|_{\mathcal{H}}^2} - 1 \right|, \tag{3.1}$$

where $\Sigma - \Sigma$ (differences of elements of Σ) is called the secant set. For brevity, we will simply denote $\delta(M)$ when the model set Σ is clear from context.

This coincides with the usual notion of RIP for sparse recovery when $\Sigma = \Sigma_k$ is the set of vectors with at most k nonzero entries (and $\Sigma - \Sigma = \Sigma_{2k}$); and with the RIP for low-rank recovery when $\Sigma = \Sigma_r$ is the set of matrices of rank at most r (then, $\Sigma - \Sigma = \Sigma_{2r}$).

A natural and sharp RIP-based compliance measure is $A_\Sigma^{RIP,\mathtt{sharp}}(R) = \delta_\Sigma^{\mathtt{sharp}}(R)$ defined as:

$$\delta_\Sigma^{\mathtt{sharp}}(R) := \inf_{M:\ker M \cap \mathcal{T}_R(\Sigma) \neq \{0\}} \delta_\Sigma(M). \tag{3.2}$$

This is the smallest RIP constant of measurement operators where uniform recovery fails [17], hence the following immediate theorem.

THEOREM 3.2 (The compliance measure $\delta_\Sigma^{\mathtt{sharp}}(R)$ is sharp.) Consider an arbitrary non-empty set $\Sigma \subseteq \mathcal{H}$. Suppose M has RIP with constant $\delta_\Sigma(M) < \delta_\Sigma^{\mathtt{sharp}}(R)$, then for all $x_0 \in \Sigma$ and x^* the result of minimization (1.2) satisfies

$$x^* = x_0. \tag{3.3}$$

Conversely, there exists M with $\delta_\Sigma(M) \geqslant \delta_\Sigma^{\mathtt{sharp}}(R)$ and $x_0 \in \Sigma$ such that $x^* \neq x_0$.

Despite this sharp property with respect to recovery, $\delta_\Sigma^{\mathtt{sharp}}(R)$ is not endowed with any known analytic expression more explicit than its definition, and it is an open question to derive closed-form formulations of this constant for a general R, even for the particular case of sparse or low-rank models. This limits the possibility to conduct an exact optimization with respect to R, and motivates the investigation of more explicit RIP-based compliance measures, with two flavors:

- Compliance measures $\delta_\Sigma^{\mathtt{nec}}(R)$ based on necessary RIP conditions [17] which yield sharp recovery constants for particular set of operators M, *e.g.,*

$$\delta_\Sigma^{\mathtt{nec}}(R) := \inf_{z \in \mathcal{T}_R(\Sigma)\setminus\{0\}} \delta_\Sigma(I - \Pi_z). \tag{3.4}$$

  where Π_z is the orthogonal projection onto the one-dimensional subspace span(z) (other intermediate necessary RIP constants can be defined). Such measures upper bound $\delta_\Sigma^{\mathtt{sharp}}(R)$ ($\delta_\Sigma^{\mathtt{nec}}(R)$ characterizes RIP recovery guarantees of measurement operators having the shape $I - \Pi_z$).

- Compliance measures $\delta_\Sigma^{\mathtt{suff}}(R)$ based on sufficient RIP constants for recovery (*e.g.,* the explicit sufficient RIP constant $\delta_\Sigma(R)$ from [38], which is explained in Section 3.3), which are lower bounds to $\delta_\Sigma^{\mathtt{sharp}}(R)$.

Note that we have the relation

$$\delta_\Sigma^{\mathtt{suff}}(R) \leqslant \delta_\Sigma^{\mathtt{sharp}}(R) \leqslant \delta_\Sigma^{\mathtt{nec}}(R). \tag{3.5}$$

The next sections aim at showing that the ℓ^1-norm (resp. the nuclear norm) maximizes the (best known) upper and lower bounds of $\delta_\Sigma^{\mathtt{sharp}}(R)$ for k-sparse model (resp. low rank models).

First, in Section 3.1, we recall that when Σ is a non-trivial cone, the compliance measures associated to RIP constants can be cast to equivalent compliance measures associated to a restricted conditioning (RC), which can be written more explicitly.

Second, in Section 3.2, we use the expression of the RC-based compliance measure associated to $\delta_\Sigma^{\mathtt{nec}}(\cdot)$ (from Equation (3.4)) to show that the ℓ^1 norm (resp. the trace-norm) optimizes $\delta_\Sigma^{\mathtt{nec}}(\cdot)$ for k-sparse vectors (resp. for matrices of rank at most r), with $\delta_\Sigma^{\mathtt{nec}}(R^\star) \approx 1/\sqrt{2}$ when n is large enough.

Finally, in Section 3.3, we give a characterization of $\delta_\Sigma^{\mathtt{suff}}(R)$ and show the optimality of the ℓ^1-norm (resp. the nuclear norm) with $\delta_\Sigma^{\mathtt{suff}}(R^\star) = 1/\sqrt{2}$.

3.1 *Restricted conditioning as a compliance measure*

When working with a model set Σ that is a cone, instead of working with actual RIP constants, it is easier to use (equivalently) the restricted conditioning.

DEFINITION 3.3 (Restricted conditioning.) Consider a cone $\Sigma \subset \mathcal{H}$ and a linear operator M from $\mathbb{R}^n$ to $\mathbb{C}^m$. We define the **restricted conditioning** of M as

$$\gamma_\Sigma(M) := \frac{\sup_{x \in (\Sigma-\Sigma)\cap S(1)} \|Mx\|_2^2}{\inf_{x \in (\Sigma-\Sigma)\cap S(1)} \|Mx\|_2^2} \in [1, \infty] \tag{3.6}$$

where by convention here $a/0 = +\infty$ for any $a \geqslant 0$. For brevity we will simply denote $\gamma(M)$ when Σ is clear from context.

As shown below, the RIP constant $\delta_\Sigma(M)$ is a monotonically increasing function of $\gamma_\Sigma(M)$. The advantage of the latter is that it is invariant by rescaling M (rescaling leaves unchanged the recovery properties when measuring x_0 with M).

LEMMA 3.1 (The RIP constant $\delta_\Sigma(M)$ is monotone in $\gamma_\Sigma(M)$.) Consider a cone $\Sigma \subseteq \mathcal{H}$. For any M such that $\gamma_\Sigma(M) < \infty$, there is a unique $\lambda > 0$ such that

$$\gamma_\Sigma(M) = \frac{1 + \delta_\Sigma(\lambda M)}{1 - \delta_\Sigma(\lambda M)} \tag{3.7}$$

or equivalently

$$\delta_\Sigma(\lambda M) = \frac{\gamma_\Sigma(M)-1}{\gamma_\Sigma(M)+1}. \tag{3.8}$$

Proof. See Appendix A.3. □

Thus, for cones, RIP-based compliance measures have equivalent RC-based compliance measures such that

$$\gamma_\Sigma(R) = \frac{1+\delta_\Sigma(R)}{1-\delta_\Sigma(R)} \qquad \text{and} \qquad \delta_\Sigma(R) = \frac{\gamma_\Sigma(R)-1}{\gamma_\Sigma(R)+1}. \tag{3.9}$$

The sharp RIP constant (3.2), the necessary RIP constant (3.4) and the sufficient RIP constant $\delta_\Sigma^{\mathtt{suff}}(R)$ of [38] are associated to

$$\gamma_\Sigma^{\mathtt{sharp}}(R) := \inf_{M:\ker M\cap\mathcal{T}_R(\Sigma)\neq\{0\}} \gamma_\Sigma(M) = \frac{1+\delta_\Sigma^{\mathtt{sharp}}(R)}{1-\delta_\Sigma^{\mathtt{sharp}}(R)}, \tag{3.10}$$

$$\gamma_\Sigma^{\mathtt{nec}}(R) := \inf_{z\in\mathcal{T}_R(\Sigma)\setminus\{0\}} \gamma_\Sigma(I-\Pi_z) = \frac{1+\delta_\Sigma^{\mathtt{nec}}(R)}{1-\delta_\Sigma^{\mathtt{nec}}(R)}, \tag{3.11}$$

$$\gamma_\Sigma^{\mathtt{suff}}(R) := \frac{1+\delta_\Sigma^{\mathtt{suff}}(R)}{1-\delta_\Sigma^{\mathtt{suff}}(R)}. \tag{3.12}$$

We deduce from (3.5) the inequalities

$$\gamma_\Sigma^{\mathtt{suff}}(R) \leqslant \gamma_\Sigma^{\mathtt{sharp}}(R) \leqslant \gamma_\Sigma^{\mathtt{nec}}(R). \tag{3.13}$$

The following result shows that $\gamma_\Sigma^{\mathtt{sharp}}(R)$ can be simplified.

PROPOSITION 3.4 (Explicit expression of $\gamma_\Sigma^{\mathtt{sharp}}(R)$.) Consider a cone $\Sigma\subseteq\mathcal{H}$. Let $\mathcal{P}$ be the set of symmetric positive semi-definite (PSD) linear operators on $\mathcal{H}$: $N\in\mathcal{P}$ if and only if $N^H=N$ and $N\succeq 0$. For $z\in\mathcal{H}\setminus\{0\}$ define

$$f_\Sigma^{RC}(z) := \inf_{N\in\mathcal{P}:\ker N=\mathrm{span}(z)} \gamma_\Sigma(N) \tag{3.14}$$

and for any non-empty set $\mathcal{T}\subseteq\mathcal{H}$ such that $\mathcal{T}\neq\{0\}$ define

$$f_\Sigma^{RC}(\mathcal{T}) := \inf_{z\in\mathcal{T}\setminus\{0\}} f_\Sigma^{RC}(z). \tag{3.15}$$

We have

$$\inf_{M:\ker M\cap\mathcal{T}\neq\{0\}} \gamma_\Sigma(M) = f_\Sigma^{RC}(\mathcal{T}). \tag{3.16}$$

Proof. This is an immediate consequence of Lemma A.1 in Appendix A.3. □

Using $\mathcal{T}=\mathcal{T}_\Sigma(R)$, the sharp RC (or RIP) constant is the smallest RC constant of positive symmetric definite PSD operators with kernels of dimension 1 for which recovery of Σ fails:

$$\gamma_\Sigma^{\mathtt{sharp}}(R) = f_\Sigma^{RC}(\mathcal{T}_R(\Sigma)). \tag{3.17}$$

Since $I-\Pi_z\in\mathcal{P}$ for any nonzero z, we have $f_\Sigma^{RC}(z)\leqslant\gamma_\Sigma(I-\Pi_z)$ hence we recover the inequality

$$\gamma_\Sigma^{\mathtt{sharp}}(R) \leqslant \inf_{z\in\mathcal{T}_R(\Sigma)\setminus\{0\}} \gamma_\Sigma(I-\Pi_z) = \gamma_\Sigma^{\mathtt{nec}}(R),$$

however it is an open question to determine whether this is an equality in particular cases or in general. The constant $\gamma_\Sigma^{\mathtt{nec}}$ is already sharp in the following sense: for sparse recovery with the ℓ^1-norms, as well as for low-rank recovery with the nuclear norm, it is the biggest possible RIP constant ($\delta_\Sigma^{\mathtt{suff}}(R) = \frac{1}{\sqrt{2}}$) that guarantees uniform recovery with $\|\cdot\|_1$ (respectively with the nuclear norm) for *all* sparsities k (for all rank levels r respectively) [17].

Similarly, to the compliance measures from Section 2, we can deduce an optimal regularizer after an isometric linear transformation of the model.

LEMMA 3.2 (Invariance of $\gamma_\Sigma^{\mathtt{sharp}}(R)$ under linear isometries.) Consider a cone $\Sigma \subseteq \mathcal{H}$, an arbitrary regularizer R such that $\Sigma \subseteq \mathtt{dom}(R)$, and a (linear) isometry F. We have

$$\gamma_{F\Sigma}^{\mathtt{sharp}}(R \circ F^{-1}) = \gamma_\Sigma^{\mathtt{sharp}}(R). \tag{3.18}$$

Hence, for any class $\mathcal{C}'$ of regularizers,

$$R^* \in \arg\max_{R \in \mathcal{C}'} \gamma_\Sigma^{\mathtt{sharp}}(R) \Leftrightarrow R^* \circ F^{-1} \in \arg\max_{R' \in \mathcal{C}'} \gamma_{F\Sigma}^{\mathtt{sharp}}(R'). \tag{3.19}$$

Proof. See Appendix A.3. □

3.2 *Compliance measures based on necessary RC conditions*

In this section, we characterize the compliance measure

$$\gamma_\Sigma^{\mathtt{nec}}(R) = \inf_{z \in \mathcal{T}_R(\Sigma)\setminus\{0\}} \gamma_\Sigma(I - \Pi_z). \tag{3.20}$$

To show optimality of the ℓ^1-norm for sparse recovery and of the nuclear norm for low-rank recovery, we will use the following characterization of $\gamma_\Sigma^{\mathtt{nec}}(R)$ when Σ is a cone.

LEMMA 3.3 (Characterization of $\gamma_\Sigma^{\mathtt{nec}}(R)$ for a cone.) Consider a cone $\Sigma \subseteq \mathcal{H}$ such that $\Sigma \neq \{0\}$ and R an arbitrary regularizer such that $\Sigma \subseteq \mathtt{dom}(R)$.

1. If there is $x \in \mathcal{H}$ such that $\Sigma \subseteq \mathrm{span}(x)$, then

$$\gamma_\Sigma^{\mathtt{nec}}(R) = \begin{cases} +\infty & \text{if } \mathcal{T}_R(\Sigma) \subseteq \Sigma, \\ 1 & \text{otherwise.} \end{cases} \tag{3.21}$$

2. If $\Sigma \subsetneq \mathrm{span}(x)$ for every $x \in \mathcal{H}$, then

$$\gamma_\Sigma^{\mathtt{nec}}(R) = \frac{1}{1 - \inf_{z \in \mathcal{T}_R(\Sigma)\setminus\{0\}} \sup_{x \in (\Sigma - \Sigma) \cap S(1)} \frac{\langle x, z\rangle^2}{\|z\|_{\mathcal{H}}^2}}. \tag{3.22}$$

Proof. See Appendix A.4. □

To go further, we exploit two assumptions related to orthogonal projections on certain sets.

DEFINITION 3.5 (Orthogonal projection.) For any set E we define, for all $z \in \mathcal{H}$

$$P_E(z) = \arg\min_{y \in E} \|z - y\|_{\mathcal{H}}. \tag{3.23}$$

This is a set-valued operator is called the **orthogonal projection**, and $P_E(z)$ may be empty if the minimum is not achieved.

Some assumptions on E ensure that $P_E(z)$ is not empty for any z.

LEMMA 3.4 (Existence of the projection.) Consider a union of subspaces $E \subseteq \mathcal{H}$, and assume that $E \cap S(1)$ is compact. Then for every $z \in \mathcal{H}$, $P_E(z) \neq \emptyset$. Moreover, for every $x, x' \in P_E(z)$ we have $\|z-x\|_{\mathcal{H}}^2 = \|z-x'\|_{\mathcal{H}}^2$ and $\langle z, x\rangle = \|x\|_{\mathcal{H}}^2 = \|x'\|_{\mathcal{H}}^2 = \langle z, x'\rangle$, hence the notations $\|z - P_E(z)\|_{\mathcal{H}}^2$, $\langle z, P_E(z)\rangle$ and $\|P_E(z)\|_{\mathcal{H}}^2$ are unambiguous. We also have $\|z\|_{\mathcal{H}}^2 = \|z - P_E(z)\|_{\mathcal{H}}^2 + \|P_E(z)\|_{\mathcal{H}}^2$ and

$$\langle z, P_E(z)\rangle = \|P_E(z)\|_{\mathcal{H}}^2 = \sup_{x \in E \cap S(1)} |\langle x, z\rangle|^2.$$

Proof. See Appendix A.4. □

Even if E is a union of subspaces and $E \cap S(1)$ is compact, $P_E(z)$ may not always be a singleton. For example, consider E the set of k-sparse vectors and z the vector with all entries equal to one.

Thanks to Lemma 3.4, we have the following characterization of the maximizers of $\delta_\Sigma^{\mathtt{nec}}$.

COROLLARY 3.1 (Characterization of $\delta_\Sigma^{\mathtt{nec}}$.) Consider a cone $\Sigma \subset \mathcal{H}$ and assume that $\Sigma - \Sigma$ is a union of subspaces, $(\Sigma - \Sigma) \cap S(1)$ is compact, and $\Sigma \neq \mathrm{span}(x)$ for each $x \in \Sigma$. For any class $\mathcal{C}'$ of regularizers such that $\Sigma \subseteq \mathtt{dom}(R)$ for every $R \in \mathcal{C}'$, the set of maximizers of $\delta_\Sigma^{\mathtt{nec}}(\cdot)$ satisfies (whether this set of maximizers is empty)

$$\arg\max_{R \in \mathcal{C}'} \delta_\Sigma^{\mathtt{nec}}(R) = \arg\min_{R \in \mathcal{C}'} B_\Sigma(R) \qquad \text{with} \qquad B_\Sigma(R) := \sup_{z \in \mathcal{T}_R(\Sigma) \setminus \{0\}} \frac{\|z - P_{\Sigma-\Sigma}(z)\|_{\mathcal{H}}^2}{\|P_{\Sigma-\Sigma}(z)\|_{\mathcal{H}}^2}. \tag{3.24}$$

For any regularizer R we have

$$\delta_\Sigma^{\mathtt{nec}}(R) = (1 + 2B_\Sigma(R))^{-1}. \tag{3.25}$$

Proof. See Appendix A.4. □

We now have the tools to study optimality for sparse and low rank models.

OPTIMAL REGULARIZATION FOR SPARSE RECOVERY AND FOR LOW-RANK RECOVERY Consider now $\Sigma = \Sigma_k$ the set of k-sparse vectors in $\mathcal{H} = \mathbb{R}^n$ (associated with the canonical scalar product $\langle \cdot, \cdot \rangle$ and the ℓ^2-norm $\|\cdot\|_{\mathcal{H}} = \|\cdot\|_2$), where $1 \leqslant k \leqslant n/2$, $n \geqslant 3$. We have $\Sigma - \Sigma = \Sigma_{2k}$ (for $n \leqslant 2k$, in particular for $n \leqslant 2$ and any $k \geqslant 1$, uniform recovery is not possible for non-invertible M: as $\Sigma - \Sigma = \mathbb{R}^n$, by Lemma 2.1 we have $\mathcal{T}_R(\Sigma) = \mathbb{R}^n$ for any regularizer, thus $\mathcal{T}_R(\Sigma) \cap \ker M = \{0\}$ implies $\ker M = \{0\}$). It is well-known that Σ and $\Sigma - \Sigma$ are both unions of subspaces (hence Σ is a cone), and it is straightforward that $(\Sigma - \Sigma) \cap S(1)$ is compact and Σ is not reduced to a single line. Moreover, for any nonzero $z \in \mathbb{R}^n$, $P_{\Sigma-\Sigma}(z)$ contains the restriction z_{T_2} of z to any set $T_2 = T_2(z) \subseteq \{1, \ldots, n\}$ of size $2k$ such that $\min_{i \in T_2} |z_i| \geqslant \max_{j \in T_2^c} |z_j|$. Hence, we can invoke Corollary 3.1 to replace the maximization of $\delta_\Sigma^{\mathtt{nec}}(R)$ by the minimization of

$$B_\Sigma(R) = \sup_{z \in \mathcal{T}_R(\Sigma) \setminus \{0\}} \frac{\|z_{T_2^c}\|_2^2}{\|z_{T_2}\|_2^2}. \tag{3.26}$$

Similarly, We consider $\Sigma = \Sigma_r$ the set of matrices of rank at most r in the Hilbert space $\mathcal{H}$ of $n \times n$ symmetric matrices (we study the symmetric case for simplicity, but conjecture that our result can be extended to the non-symmetric case) with $\|\cdot\|_{\mathcal{H}} = \|\cdot\|_F$ (the Frobenius norm). We have again $\Sigma - \Sigma = \Sigma_{2r}$ and all conditions are satisfied such that Corollary 3.1 can be invoked. Denoting $\Delta = \mathtt{eig}(z)$ the vector of eigenvalues of matrix $z \in \mathcal{H}$ sorted by decreasing absolute value, so that $z = U^T \Delta U$ for

some unitary matrix U, and defining $z_T := z = U^T \Delta_T U$ for every index set T, we have $P_{\Sigma-\Sigma}(z) = z_{T_2}$ and $z - P_{\Sigma-\Sigma}(z) = z_{T_2^c}$ where $T_2 = T_2(z) \subseteq [1,n]$ is any index set containing the $2k$ largest eigenvalues (in magnitude) of z, *i.e.,* such that $\min_{i\in T_2} |\Delta_i| \geqslant \max_{j\in T_2^c} |\Delta_j|$. With these observations and notations, we are again left to optimize (3.26).

Specializing to the class $\mathcal{C}$ of convex, coercive, continuous regularizers, we obtain the following theorems.

THEOREM 3.6 (Optimality of ℓ^1-norm for k-sparse vectors for $\delta_\Sigma^{\mathtt{nec}}$.) With k-sparse vectors, $\Sigma = \Sigma_k \subseteq \mathcal{H} = \mathbb{R}^n$, $k < \frac{n}{2}$, and $R^\star(\cdot) = \|\cdot\|_1$, we have

$$\delta_\Sigma^{\mathtt{nec}}(R^\star) = \sup_{R\in\mathcal{C}} \delta_\Sigma^{\mathtt{nec}}(R) = (2B_{k,n}^\star + 1)^{-1} \qquad \text{with} \qquad B_{k,n}^\star := \max_{1\leqslant L\leqslant n-2k} \frac{L/k}{(L/k+1)^2+1}. \tag{3.27}$$

Moreover, for $k = 1$, the unique functions $R \in \mathcal{C}_\Sigma$ maximizing $\delta_\Sigma^{\mathtt{nec}}$ are of the form $R(\cdot) = \lambda\|\cdot\|_1$ with $\lambda > 0$.

THEOREM 3.7 (Optimality of the nuclear norm for rank-r matrices for $\delta_\Sigma^{\mathtt{nec}}$.) With the set of rank-$r$ matrices, $\Sigma = \Sigma_r$, in the space $\mathcal{H}$ of symmetric $n \times n$ matrices, $r < \frac{n}{2}$, and with $R^\star(\cdot) = \|\cdot\|_*$ (the nuclear norm), we have

$$\delta_\Sigma^{\mathtt{nec}}(R^\star) = \sup_{R\in\mathcal{C}} \delta_\Sigma^{\mathtt{nec}}(R) = (2B_{r,n}^\star + 1)^{-1} \qquad \text{with} \qquad B_{r,n}^\star := \max_{1\leqslant L\leqslant n-2r} \frac{L/r}{(L/r+1)^2+1}. \tag{3.28}$$

The proofs of these two theorems exploits technical lemmas that we detail in Appendix A.4.1 and Appendix A.4.2.

Proof. We give the proof for the case of sparse recovery. To express it for low-rank recovery simply replace the notation k by r. For $1 \leqslant s \leqslant n$ and any regularizer R we define

$$B_\Sigma^s(R) := \sup_{z\in\mathcal{T}_R(\Sigma)\setminus\{0\}, z\in\Sigma_s} \frac{\|z_{T_2^c}\|_2^2}{\|z_{T_2}\|_2^2}. \tag{3.29}$$

For $s \leqslant 2k$ and any $z \in \Sigma_s$ we have $z_{T_2^c} = 0$ hence $B_\Sigma^s(R) = 0$, thus $B_\Sigma(R) = \max_{1\leqslant L\leqslant n-2k} B_\Sigma^{2k+L}(R)$. First consider $R \in \mathcal{C}_\Sigma$. Since R is positively homogeneous and subadditive, by Lemma A.4 for Σ_k / Lemma A.8 for Σ_r,

$$B_\Sigma^{2k+L}(R) \geqslant \frac{\frac{L}{k}}{\left(\frac{L}{k}+1\right)^2+1}, \qquad \text{for each } 1 \leqslant L \leqslant n-2k.$$

For $R^\star$ and $1 \leqslant L \leqslant n-2k$ we also have (Lemma A.6 / Lemma A.9, inspired by [17]) that

$$B_\Sigma(R^\star) = \max_{1\leqslant L\leqslant n-2k} \frac{\frac{L}{k}}{\left(\frac{L}{k}+1\right)^2+1}.$$

As a result,

$$B_\Sigma(R) \geqslant B_\Sigma(R^\star) = \max_{1\leqslant L\leqslant n-2k} \frac{\frac{L}{k}}{\left(\frac{L}{k}+1\right)^2+1} =: B_{k,n}^\star$$

Finally, remark that $B_\Sigma(R)$ is an increasing function of $\mathcal{T}_R(\Sigma)$. Using Lemma 2.4, for any $R \in \mathcal{C}$ there is $R' \in \mathcal{C}_\Sigma$ such that

$$B_\Sigma(R) \geqslant B_\Sigma(R') \geqslant B^\star_{k,n}.$$

For $k = 1$, uniqueness comes from the fact that on a given orthant for $R \in \mathcal{C}_\Sigma$, R is a weighted ℓ^1 norm: $R((x_1,\dots,x_n)) = \sum_i w_i|x_i|$ and the equality case in Lemma A.4 forces $w_i = \max_i w_i$.

□

Because of the uniqueness result for $k = 1$, the ℓ^1-norm is the unique convex regularizer in $\cap \mathcal{C}_{\Sigma_k}$ that jointly maximizes $\delta^{\mathtt{nec}}_{\Sigma_k}$ for all $k < \frac{n}{2}$ (the proof of Theorem 3.6 is valid for $\mathcal{C}_{\Sigma_{k'}}$, with $k \leqslant k' < \frac{n}{2}$). It is an open question to determine if we have uniqueness model by model. As the result might change for tighter compliance measures, we leave this question for future work.

In the next section, we will see that the optimization of the sufficient RIP constant leads to very similar expressions.

3.3 *Compliance measures based on sufficient RC conditions*

When Σ is a union of subspaces and R is an arbitrary regularizer, an "explicit" RIP constant $\delta^{\mathtt{suff}}_\Sigma(R)$ is sufficient to guarantee reconstruction [38]. The expression of this constant [38][Eq. (5)] is recalled in the Appendix (Equation (A.42)) and can be used as a compliance measure. It gives rise to the following lemma, which is proved in Appendix A.5.

LEMMA 3.5 (Equality case of the sufficient conditions.) Assume that $\Sigma = \cup_{V\in\mathcal{V}} V$ is a union of subspaces and that $\Sigma \cap S(1)$ is compact. Consider R any regularizer such that $\Sigma \subseteq \mathtt{dom}(R)$. We have

$$\delta^{\mathtt{suff}}_\Sigma(R) \geqslant \frac{1}{\sqrt{\sup\limits_{z\in\mathcal{T}_R(\Sigma)\setminus\{0\}} \frac{\|z-P_\Sigma(z)\|^2_\Sigma}{\|P_\Sigma(z)\|^2_2} + 1}} =: \delta^{\mathtt{suff2}}_\Sigma(R). \tag{3.30}$$

Further, assume that $P_\Sigma(z) \subseteq \arg\min_{x\in\Sigma} \|x-z\|_\Sigma$ for every $z \in \mathcal{H}$ and that, for every $V \in \mathcal{V}$ and every $u \in \Sigma$, $P_{V^\perp}(u) \in \Sigma$. Then, there is equality in (3.30).

Proof. See Appendix A.5. Note that the assumption $P_\Sigma(z) \subseteq \arg\min_{x\in\Sigma} \|x-z\|_\Sigma$ could be replaced by the slightly weaker $P_\Sigma(z) \cap \arg\min_{x\in\Sigma} \|x-z\|_\Sigma/\|x\|_2 \neq \emptyset$.

□

We get an immediate corollary of the first claim in the above lemma.

COROLLARY 3.2 (Expression of a sufficient condition.) Assume that $\Sigma = \cup_{V\in\mathcal{V}} V$ is a union of subspaces and that $\Sigma \cap S(1)$ is compact. For any class $\mathcal{C}'$ of regularizers such that $\Sigma \subseteq \mathtt{dom}(R)$ for every $R \in \mathcal{C}'$, the set of maximizers of $\delta^{\mathtt{suff2}}_\Sigma(\cdot)$ satisfies (whether this set of maximizers is empty)

$$\arg\max_{R\in\mathcal{C}'} \delta^{\mathtt{suff2}}_\Sigma(R) = \arg\min_{R\in\mathcal{C}'} D_\Sigma(R) \qquad \text{with} \qquad D_\Sigma(R) := \sup_{z\in\mathcal{T}_R(\Sigma)\setminus\{0\}} \frac{\|z-P_\Sigma(z)\|^2_\Sigma}{\|P_\Sigma(z)\|^2_{\mathcal{H}}}. \tag{3.31}$$

For any optimal regularizer $R^\star$ we have

$$\delta^{\mathtt{suff2}}_\Sigma(R^\star) = (1 + D_\Sigma(R^\star))^{-1/2}. \tag{3.32}$$

Note the subtle difference in the norm at the numerator in $B_\Sigma(R)$ and $D_\Sigma(R)$.

OPTIMAL REGULARIZATION FOR SPARSE RECOVERY AND LOW-RANK RECOVERY When considering sparse recovery or low-rank recovery, there is indeed equality $\delta_\Sigma^{\mathtt{suff}}(R) = \delta_\Sigma^{\mathtt{suff2}}(R)$ thanks to the following Lemma.

LEMMA 3.6 The assumptions for the equality case of Lemma 3.5 hold for $\Sigma = \Sigma_k$ the set of k-sparse vectors in $\mathcal{H} = \mathbb{R}^n$, as well as for the set $\Sigma = \Sigma_r$ of symmetric matrices of rank at most r in $\mathcal{H}$ the set of symmetric $n \times n$ matrices.

Proof. See Appendix A.5. □

Consider $\Sigma := \Sigma_k$, and regularizers in $\mathcal{C}_\Sigma$. Similarly to the necessary case, from Lemma 3.5, we have (when Σ is a union of subspace and $\Sigma \cap S(1)$ is closed)

$$D_\Sigma(R) = \sup_{z \in \mathcal{T}_R(\Sigma)\setminus\{0\}} \frac{\|z_{T^c}\|_\Sigma^2}{\|z_T\|_2^2} \tag{3.33}$$

where T denotes the support of the k largest coordinates of z.

We obtain similar results as in the necessary RIP constant case.

THEOREM 3.8 (Optimality of ℓ^1-norm for k-sparse vectors for $\delta_\Sigma^{\mathtt{suff}}$.) With k-sparse vectors, $\Sigma = \Sigma_k \subseteq \mathcal{H} = \mathbb{R}^n$, $k < \frac{n}{2}$, and $R^\star(\cdot) = \|\cdot\|_1$, we have

$$\delta_\Sigma^{\mathtt{suff}}(R^\star) = \sup_{R \in \mathcal{C}} \delta_\Sigma^{\mathtt{suff}}(R) = \frac{1}{\sqrt{2}}. \tag{3.34}$$

Moreover, for $k = 1$, the unique functions $R \in \mathcal{C}_\Sigma$ maximizing $\delta_\Sigma^{\mathtt{suff}}$ are of the form $R(\cdot) = \lambda\|\cdot\|_1$ with $\lambda > 0$.

THEOREM 3.9 (Optimality of the nuclear norm for rank-r matrices for $\delta_\Sigma^{\mathtt{suff}}$.) With the set of rank-$r$ matrices, $\Sigma = \Sigma_r$, in the space $\mathcal{H}$ of symmetric $n \times n$ matrices, $r < \frac{n}{2}$, and with $R^\star(\cdot) = \|\cdot\|_*$ (the nuclear norm), we have

$$\delta_\Sigma^{\mathtt{suff}}(R^\star) = \sup_{R \in \mathcal{C}} \delta_\Sigma^{\mathtt{suff}}(R) = \frac{1}{\sqrt{2}}. \tag{3.35}$$

Proof. We give the proof for the case of sparse recovery. To express it for low-rank recovery simply replace the notation k by r. For $1 \leqslant s \leqslant n$ and any regularizer R we define

$$D_\Sigma^s(R) := \sup_{z \in \mathcal{T}_R(\Sigma)\setminus\{0\}, z \in \Sigma_s} \frac{\|z_{T^c}\|_\Sigma^2}{\|z_T\|_2^2}.$$

For $s \leqslant k$ and any $z \in \Sigma_s$ we have $z_{T^c} = 0$ hence $D_\Sigma^s(R) = 0$, thus $D_\Sigma(R) = \max_{1 \leqslant L \leqslant n-k} D_\Sigma^{k+L}(R)$.

First consider $R \in \mathcal{C}_\Sigma$. Since R is positively homogeneous and subadditive, by Lemma A.13 for Σ_k / Lemma A.15 for Σ_r,

$$D_\Sigma^{k+L}(R) \geqslant \min(1, \frac{L}{k}), \qquad \text{for each } 1 \leqslant L \leqslant n-k.$$

For $R^\star$ and $1 \leqslant L \leqslant n-k$ we also have (with Lemma A.12 / Lemma A.14) that

$$D_\Sigma^{k+L}(R^\star) = \min(1, \frac{L}{k}).$$

As a result,

$$D_\Sigma(R) \geqslant D_\Sigma(R^\star) = \max_{1\leqslant L\leqslant n-k} \min(1, \frac{L}{k}) = 1.$$

Finally, remark that $D_\Sigma(R)$ is an increasing function of $\mathcal{T}_R(\Sigma)$. Using Lemma 2.4, for any $R \in \mathcal{C}$ there is $R' \in \mathcal{C}_\Sigma$ such that

$$D_\Sigma(R) \geqslant D_\Sigma(R') \geqslant 1.$$

□

3.4 *Discussion*

Even without an analytic expression of the sharp RIP constant, it would have been possible to show that $R^\star$ optimizes $\delta_\Sigma^{\mathtt{sharp}}$ if it were simultaneously optimizing its lower and upper bound, *i.e.,* if we had

$$\sup_{R\in\mathcal{C}} \delta_\Sigma^{\mathtt{suff}}(R) = \delta_\Sigma^{\mathtt{suff}}(R^\star) = \delta_\Sigma^{\mathtt{nec}}(R^\star) = \sup_{R\in\mathcal{C}} \delta_\Sigma^{\mathtt{nec}}(R). \tag{3.36}$$

Unfortunately, this is not the case in the sparse and low rank examples. We observe that for fixed k, n we have in both cases

$$\frac{1}{\sqrt{2}} = \delta_\Sigma^{\mathtt{suff}}(R^\star) < \delta_\Sigma^{\mathtt{nec}}(R^\star). \tag{3.37}$$

Because of this fact, we cannot conclude on the optimality of $R^\star$ for $\delta_\Sigma^{\mathtt{sharp}}$. However, indexing all objects of the problem by n the dimension of $\mathcal{H}$ (respectively the dimension of the diagonals): the set of regularizers $\mathcal{C}^{(n)}$, the models $\Sigma_k^{(n)}$ and the corresponding $R^{\star,(n)}$ (independent of k for $k < n/2$ as we saw previously). We have (see Remark A.1)

$$\inf_{n\geqslant 3} \inf_{k\in\{1,\dots,\lfloor n/2\rfloor\}} \sup_{R\in\mathcal{C}^{(n)}} \delta_{\Sigma_k^{(n)}}^{\mathtt{nec}}(R) = \frac{1}{\sqrt{2}} = \delta_{\Sigma_k^{(n)}}^{\mathtt{suff}}(R^{\star,(n)}). \tag{3.38}$$

We deduce

$$\inf_{n\geqslant 3} \inf_{k\in\{1,\dots,\lfloor n/2\rfloor\}} \sup_{R\in\mathcal{C}^{(n)}} \delta_{\Sigma_k^{(n)}}^{\mathtt{sharp}}(R) = \frac{1}{\sqrt{2}}. \tag{3.39}$$

and

$$\inf_{n\geqslant 3} \inf_{k\in\{1,\dots,\lfloor n/2\rfloor\}} \left| \delta_{\Sigma_k^{(n)}}^{\mathtt{sharp}}(R^{\star,(n)}) - \left[\sup_{R\in\mathcal{C}^{(n)}} \delta_{\Sigma_k^{(n)}}^{\mathtt{sharp}}(R) \right] \right| = 0. \tag{3.40}$$

This shows that the functions $R^{\star,(n)}$ are optimal as a family with respect to a family of models $\Sigma_k^{(n)}$ and the worst case of their associated compliance measures $\delta_{\Sigma_k^{(n)}}^{\mathtt{sharp}}(R)$.

These results can be interpreted in terms of number of measurements needed to recover uniformly sparse or low rank objects with convex regularization. Under the *best known* (RIP-based) uniform recovery conditions, it is guaranteed that using the optimal regularization with respect to RIP-based compliance measures will enable the use of fewer measurements. In particular in the case of an operator M built from m random Gaussian measurements, it has been proven (see e.g. [22]) that there is a universal constant C such that if $m \geqslant C\frac{k\log(k/n)}{\delta^2}$ then M satisfies a prescribed RIP constant δ with high probability. Hence, the larger the required RIP constant is, the lower the number of measurement needs to be. Such results on the required number of measurement to verify the RIP have been extended to more general low dimensional models (see e.g. [30]), making RIP-based optimal regularizers tools of choice to optimize the number of random measurements of elements of a given low dimensional model.

4. Towards the construction of optimal convex regularizers? The examples of sparsity in levels and beyond.

In the previous Section, optimality was shown by exhibiting the optimal regularizer (ℓ^1-norm and nuclear norm). The only constructive part in these results is given by Theorem 2.4 that shows that we can look for optimal regularizers in the set of atomic norms $\mathcal{C}_\Sigma$ constructed using the model set Σ. Ideally, given a compliance measure, we would like to be able to construct for any model Σ, an optimal regularizer $R^\star \in \mathcal{C}_\Sigma$. As such an objective seems out of reach with the tools we have developed so far, we study on an example (the case of *sparsity in levels*) the simpler problem of looking for the optimal regularizer in a smaller set of regularizers. We consider a set of weighted ℓ^1-norms and explore the explicit construction of an optimal regularizer for the compliance measure $\delta_\Sigma^{\mathtt{nec}}$. We then extend this result to the similar setting of Cartesian product of sparse and low-rank models.

4.1 *Sparsity in levels*

Sparsity *in levels* is a possible extension of plain sparsity, which is useful for the precise modeling of signals such as medical images [1, 6]. It is also useful for simultaneous modeling of sparse signal and sparse noise [34, 39].

DEFINITION 4.1 (Sparsity in levels.) In $\mathcal{H} = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \ldots \times \mathbb{R}^{n_L}$, given sparsity levels $k_1, \ldots, k_L$, we define the **sparsity in levels** model with

$$\Sigma_{k_1,\ldots,k_L} := \{x = (x_1, \ldots, x_L) : x_i \in \Sigma_{k_i}\} \tag{4.1}$$

where Σ_{k_i} is the k_i-sparse model in $\mathbb{R}^{n_i}$.

While the ℓ^1-norm was shown to be is a candidate to perform regularization for models that are sparse in levels [1], it was also shown that it is possible to obtain better sufficient RIP recovery guarantees when weighting the ℓ^1 norm by $\sqrt{k_i}$ in each level [38]. The following theorem permits to show that this weighting scheme is close to optimal for the compliance measure $\delta_\Sigma^{\mathtt{nec}}$ by giving explicit expressions for the calculation of optimal weights.

Given weights $w = (w_1, \ldots, w_L) \in \mathbb{R}_+^L$, we define the ℓ^1-norm weighted by levels.

$$\|(x_1, \ldots, x_L)\|_w = \sum_{i=1}^{L} w_i \|x_i\|_1. \tag{4.2}$$

We have the following result.

THEOREM 4.2 (Optimal weighted ℓ^1 norms for $\delta_\Sigma^{\mathtt{nec}}$ for sparsity in levels.) Consider two integers $k_1, k_2 \geqslant 2$ and the model set $\Sigma = \Sigma_{k_1,k_2}$ in $\mathcal{H} = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ where we assume that $n_1 \geqslant 4k_1$, $n_2 \geqslant 4k_2$. Let $\tilde{a} = 2\sqrt{3} - 3$. We define

$$B_\Sigma^\star := \min_{\substack{\nu_1 \in [\tilde{a}, 1-\tilde{a}] \\ \nu_2 = 1-\nu_1}} \max_{i \in \{1,2\}} \max_{x_i \in \{\lfloor k_i \sqrt{1+1/\nu_i} \rfloor; \lceil k_i \sqrt{1+1/\nu_i} \rceil\}} \frac{x_i/k_i}{\nu_i (x_i/k_i + 1)^2 + 1} \tag{4.3}$$

where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote the lower and upper integer part and (ν_1^*, ν_2^*) minimizing this expression.

Then

$$w^* \in \arg\max_w \delta_\Sigma^{\mathtt{nec}}(\|\cdot\|_w) \tag{4.4}$$

if and only if $w^* = (w_1^*, w_2^*)$ where $w_1^*, w_2^* > 0$ satisfy

$$\frac{w_2^*}{w_1^*} = \sqrt{\frac{k_1}{k_2}(1/\nu_1^* - 1)}. \tag{4.5}$$

Moreover, denoting $w_0 = w_0(k_1,k_2) := (1/\sqrt{k_1}, 1/\sqrt{k_2})$ we have

$$\begin{aligned} B_\Sigma(\|\cdot\|_{w^*}) = B_\Sigma^\star \leqslant B_\Sigma(\|\cdot\|_{w_0}) \leqslant (\sqrt{3}-1)/2 \\ \delta_\Sigma^{\mathtt{nec}}(\|\cdot\|_{w^*}) = (1+2B_\Sigma^\star)^{-1} \geqslant \delta_\Sigma^{\mathtt{nec}}(\|\cdot\|_{w_0}) \geqslant 1/\sqrt{3}. \end{aligned} \tag{4.6}$$

Finally, we have

$$\inf_{k_1,k_2\geqslant 1} \inf_{n_1\geqslant 4k_1, n_2\geqslant 4k_2} \delta_\Sigma^{\mathtt{nec}}(\|\cdot\|_{w_0(k_1,k_2)}) = 1/\sqrt{3}. \tag{4.7}$$

Proof. See Appendix A.6. □

This theorem comes from the fact that (see proof) the quantity defined in (3.26) satisfies

$$B_{\Sigma_{k_1,k_2}}(\|\cdot\|_{(w_1,w_2)}) = \max_{L_1,L_2} B^{L_1,L_2}_{\Sigma_{k_1,k_2}}((w_1,w_2)) \tag{4.8}$$

where $B^{L_1,L_2}_{\Sigma_{k_1,k_2}}(\|\cdot\|_{(w_1,w_2)})$ can be computed explicitly (similarly to B_Σ^{2k+L} from (3.29) for sparsity).

Thanks to the expression of $B_\Sigma(\|\cdot\|_{w^*})$ from Theorem 4.2, it becomes tractable to evaluate numerically optimal weights. We simply perform the minimization over $\nu_1 \in [\tilde{a}, 1-\tilde{a}]$ over a regular grid (of 10^6 points in our experiment) and take the minimum. The value of w_1^*/w_2^* is returned according to (4.5). Let $w_0 = w_0(k_1,k_2) = (1/\sqrt{k_1}, 1/\sqrt{k_2})$. In Figure 3, we show a representation of the two criteria $C_1(k_1,k_2) = |1 - \frac{\langle w^*, w_0\rangle}{\|w^*\|_2\|w_0\|_2}|$ and $C_2(k_1,k_2) = |\delta_\Sigma^{\mathtt{nec}}(\|\cdot\|_{w^*}) - \delta_\Sigma^{\mathtt{nec}}(\|\cdot\|_{w_0})|$ for different pairs (k_1,k_2). The case $C_1(k_1,k_2) = C_2(k_1,k_2) = 0$ occurs if and only if w_0 is optimal).

We observe numerically that for $2 \leqslant k_1,k_2 \leqslant 200$, $C_1(k_1,k_2) \leqslant 10^{-5}$ and $C_2(k_1,k_2) \leqslant 5\cdot 10^{-3}$ and that the error tends to decrease for greater k_1,k_2. This comes from the fact that the result of the discrete optimization over the integers L_i in (4.8) gets closer to the result of a continuous optimization that yields $w_2^*/w_1^* = \sqrt{k_1}/\sqrt{k_2}$ (obtained by dropping the integer parts in Theorem 4.2).

For the "asymptotically optimal" weighting scheme $w_0 = w_0(k_1,k_2) = \left(\frac{1}{\sqrt{k_1}}, \frac{1}{\sqrt{k_2}}\right)$, we find

$$\inf_{k_1',k_2'\geqslant 1, n_1'\geqslant 4k_1', n_2'\geqslant 4k_2'} \delta^{\mathtt{nec}}_{\Sigma_{k_1',k_2'}}(\|\cdot\|_{w_0}) \overset{(4.7)}{=} \frac{1}{\sqrt{3}} \overset{(*)}{\leqslant} \delta^{\mathtt{suff}}_{\Sigma_{k_1,k_2}}(\|\cdot\|_{w_0}) \leqslant \delta^{\mathtt{sharp}}_{\Sigma_{k_1,k_2}}(\|\cdot\|_{w_0}) \leqslant \delta^{\mathtt{nec}}_{\Sigma_{k_1,k_2}}(\|\cdot\|_{w_0}). \tag{4.9}$$

The inequality (*) is shown in Theorem 4.3 below (improving for $L=2$ the lower bound $\frac{1}{\sqrt{2+L}} = 1/\sqrt{4} = 1/2$ for sparsity in L levels previously given in [38, Theorem 4.2]), and the last inequalities are generic, cf (3.5).

The double-sided bound (4.9) confirms that the weighting scheme $\left(\frac{1}{\sqrt{k_1}}, \frac{1}{\sqrt{k_2}}\right)$ is close to an optimal choice (w.r.t the maximization of $\delta^{\mathtt{sharp}}_{\Sigma_{k_1,k_2}}$) when the sparsities are known.

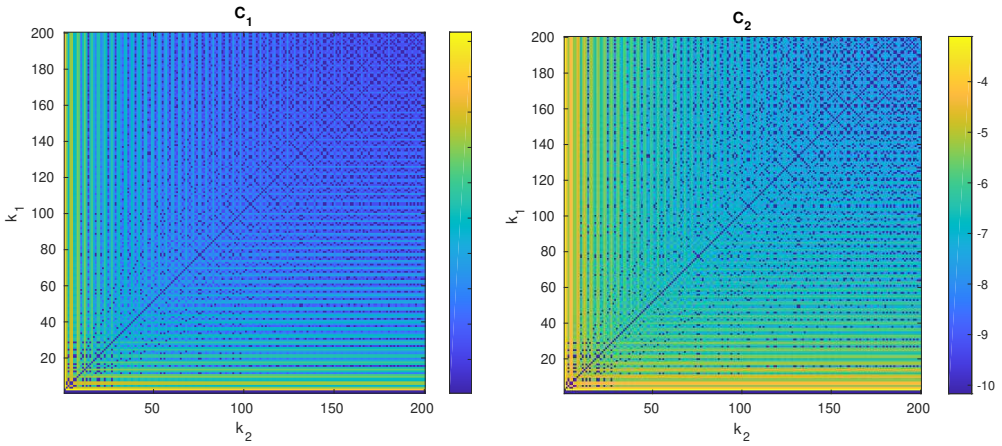

FIG. 3. Then quantities $\log_{10}(C_1(k_1,k_2)) := \log_{10}\left(|1-\frac{\langle w^*,w_0\rangle}{\|w^*\|_2\|w_0\|_2}|\right)$ (left) and $\log_{10}(C_2(k_1,k_2)) := \log_{10}(|\delta_\Sigma^{\mathtt{nec}}(\|\cdot\|_{w^*}) - \delta_\Sigma^{\mathtt{nec}}(\|\cdot\|_{w_0})|)$ (right) where $w^* = (w_1^*, w_2^*)$ is obtained from Theorem 4.2 and $w_0 = (1/\sqrt{k_1}, 1/\sqrt{k_2})$ for different $k_1, k_2 \geqslant 2$.

THEOREM 4.3 (Sufficient RIP condition for near-optimal ℓ^1 norms for sparsity in levels.) Consider two integers $k_1, k_2 \geqslant 2$ and the model set $\Sigma = \Sigma_{k_1,k_2}$ in $\mathcal{H} = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with $n_i \geqslant k_i$, $i = 1,2$, and the norm $\|(x_1,x_2)\|_w = \sum_{i=1}^2 \frac{1}{\sqrt{k_i}}\|x_i\|_1$. Then

$$\delta_{\Sigma_{k_1,k_2}}^{\mathtt{suff}}(\|\cdot\|_w) \geqslant \frac{1}{\sqrt{3}}. \tag{4.10}$$

Proof. See Appendix A.6. □

4.2 *Beyond sparsity in levels*

Beyond sparsity in levels, we obtain exactly the same result for the Cartesian product of a sparse model and a low-rank model. Consider $\Sigma_{k,r} = \Sigma_k \times \Sigma_r \subset \mathbb{R}^n \times \mathcal{H}_p$ where $\mathcal{H}_p$ is the set of symmetric matrices of size $p \times p$. This model with $n = p^2$ can be used to model sums of sparse and low rank matrices. To address associated matrix reconstruction problems it was suggested in [35] to use a weighted sum of the ℓ^1-norm and the nuclear norm with weights ratio $\frac{\sqrt{k}}{\sqrt{r}}$, ie $\|(z_1,z_2)\|_w = \frac{1}{\sqrt{k}}\|z_1\|_1 + \frac{1}{\sqrt{r}}\|z_2\|_*$. The following Theorem guarantees that the previous numerical experiments hold with this model (by replacing k_1 by k and k_2 by r). It thus confirms that this is a near optimal choice of weights.

THEOREM 4.4 (Optimal mixed norms for $\delta_\Sigma^{\mathtt{nec}}$ for sparse plus low-rank models.) Consider two integers $k, r \geqslant 2$ and the model set $\Sigma = \Sigma_k \times \Sigma_r$ in $\mathcal{H} = \mathbb{R}^n \times \mathcal{H}_p$ where we assume that $n \geqslant 4k$, $p \geqslant 4r$. Consider $\tilde{a} = 2\sqrt{3} - 3$, $B_\Sigma^\star$ and (v_1^*, v_2^*) from Theorem 4.2 with $k_1 = k$ and $k_2 = r$. Then, with $\|(z_1,z_2)\|_w := w_1\|z_1\|_1 + w_2\|z_2\|_*$, we have:

$$w^* \in \arg\max_w \delta_\Sigma^{\mathtt{nec}}(\|\cdot\|_w) \tag{4.11}$$

if and only if $w^* = (w_1^*, w_2^*)$ where $w_1^*, w_2^* > 0$ satisfy

$$\frac{w_2^*}{w_1^*} = \sqrt{\frac{k}{r}(1/v_1^* - 1)}. \tag{4.12}$$

Moreover, denoting $w_0 = w_0(k,r) := (1/\sqrt{k}, 1/\sqrt{r})$ we have

$$\begin{aligned} B_\Sigma(\|\cdot\|_{w^*}) = B^\star_\Sigma \leqslant B_\Sigma(\|\cdot\|_{w_0}) \leqslant (\sqrt{3}-1)/2 \\ \delta^{\mathtt{nec}}_\Sigma(\|\cdot\|_{w^*}) = (1+2B^\star_\Sigma)^{-1} \geqslant \delta^{\mathtt{nec}}_\Sigma(\|\cdot\|_{w_0}) \geqslant 1/\sqrt{3}. \end{aligned} \tag{4.13}$$

Finally, we have

$$\inf_{k,r\geqslant 1} \inf_{n\geqslant 4k, p\geqslant 4r} \delta^{\mathtt{nec}}_\Sigma(\|\cdot\|_{w_0(k,r)}) = 1/\sqrt{3}. \tag{4.14}$$

Proof. See Appendix A.6. □

The resulting weighting scheme for the sparse + low-rank model has been used practically in [24]. In the context of structure-texture decomposition of images, the structure is modeled by images with sparse gradients and the texture is modeled by images having patches with low-rank structure. In this work, a heuristic based on the weigthing scheme $\frac{w^*_2}{w^*_1} = \sqrt{\frac{k}{r}}$ is proposed and permits to automatically set regularization parameters for local sparse + low rank models. These results for sparsity in levels and beyond show that even with a simple model and parametrized family of functions, optimization might lead to complicated expressions. We also remark that we could perform the optimization because restricting to weighted atomic norms leads to an analytical description of atoms generating the regularizers. This in turn leads to an analytical description of descent cones. The question of optimality within more general sets of atomic norms remains. Unfortunately the lack of analytical description of descent cones in the general case makes the direct extension of our proof technique difficult.

5. Discussion and future work

We gave a general way of defining compliance measures between a regularizer R and a low dimensional model set Σ, and described some elementary properties of such measures. This opens questions on conditions on compliance measures that guarantee the existence of an optimal convex regularizer. Also, the question of manipulating compliance measures for transformations and combinations of models (intersections, unions, sums, ...) is a wide and challenging potential area of research.

We considered noiseless observations instead of the classical noisy model $y = Mx_0 + e$ where e is a measurement noise with finite energy $\|e\|_2$ because of the following remark. Suppose we define an optimal regularizer for a given noise level $\|e\|_2$. There are two possible cases, either the regularizer is also optimal for $\|e\|_2 = 0$ or it is not. In the second case, it means that we would need to trade exact recovery capability for improved stability to noise. This is a possible route to follow especially if there is some latitude on the design of the measurement operator M, *i.e.,* it is possible to increase measurements to improve stability to noise. The analysis of such trade-offs is out of the scope of this article and left for future work.

We have shown that the ℓ^1-norm is optimal among coercive continuous convex functions for sparse recovery for compliance measures based on necessary and sufficient RIP conditions. This result had to be expected due to the symmetries of the problem. The important point is that we could explicitly quantify the notion of good regularizer. We obtained the same expected results with the nuclear norm for low-rank matrix recovery.

It must be noted that we did not use constructive proofs (we *exhibited* the candidate maximum of the compliance measure) for the sparsity and low-rank cases. A full constructive proof, *i.e.,* an exact calculation and optimization of the quantities $B_\Sigma(R)$ and $D_\Sigma(R)$ would be intellectually more satisfying as it would not require the prior knowledge of the candidate optimum, which is our ultimate objective. We saw in the case of sparsity in levels and beyond that we can construct the regularizer that achieved

optimality among a simple parametrized family of convex functions (weighted ℓ^1-norms in levels). It is an open question to obtain more general constructions.

We used compliance measures based on (uniform) RIP recovery guarantees to give results for the sparse recovery case, it would be interesting to do such analysis using (non-uniform) recovery guarantees based on the statistical dimension or on the Gaussian width of the descent cones [2, 14]. One would need to precisely lower and upper bound these quantities, similarly to our approach with the RIP, to get satisfying results.

Finally, while these compliance measures are designed to make sense with respect to known results in the area of sparse recovery, one might design other compliance measures tailored for particular needs, in this search for optimal regularizers.

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A. Appendices

This section describes the tools and proofs used to obtain our results.

A.1 *Summary of properties used in proofs*

From [38, Table 1] (which summarizes results from [32]), the function $x \in \mathcal{E}(\mathcal{A}) \mapsto \|x\|_{\mathcal{A}}$ is always non-negative, lower semi-continuous and subadditive (*i.e.,* it satisfies the triangle inequality). It is furthermore positively homogeneous as soon as $0 \in \overline{\text{conv}}(\mathcal{A})$, continuous as soon as 0 is in the interior of $\overline{\text{conv}}(\mathcal{A})$, and coercive as soon as $\overline{\text{conv}}(\mathcal{A})$ is bounded. Finally, it is indeed a norm if $\overline{\text{conv}}(\mathcal{A}) = -\overline{\text{conv}}(\mathcal{A})$.

We refer the reader to [38][Section 2.2] and [4] for properties of the atomic norm $\|\cdot\|_\Sigma$ (cf Theorem 2.3). We will use the following two properties of $\|\cdot\|_\Sigma$ (defined in Section 2.5).

Fact A.1 (From e.g. [38]) For all $x \in \Sigma$, $\|x\|_\Sigma = \|x\|_{\mathcal{H}}$.

Fact A.2 (From [38][Fact 2.1] applied to $\|\cdot\|_\Sigma$) For all $z \in \mathcal{H}$

$$\|z\|_\Sigma = \inf\left\{ \sqrt{\sum \lambda_i \|u_i\|_{\mathcal{H}}^2} : \lambda_i \in \mathbb{R}_+, \sum \lambda_i = 1, u_i \in \Sigma, z = \sum \lambda_i u_i \right\}. \tag{A.1}$$

A.2 *Proofs for Section 2*

A.2.1 *Proof of Lemma 2.1.* Consider $x \in \Sigma$, and $z \in \mathcal{H}$. We have $\iota_\Sigma(x+z) \leqslant \iota_\Sigma(x) = 0$ if and only if $x+z \in \Sigma$, *i.e.,* if there is $x' \in \Sigma$ such that $z = x' - x$. Hence, $\mathcal{T}_{\iota_\Sigma}(x) = \{\gamma(x'-x) : \gamma \in \mathbb{R}, x' \in \Sigma\}$. It follows that $\mathcal{T}_{\iota_\Sigma}(\Sigma) = \{\gamma z : \gamma \in \mathbb{R}, z \in \Sigma - \Sigma\} \supseteq \Sigma - \Sigma$. When Σ is positively homogeneous, for any $z = x' - x \in \Sigma - \Sigma$ and $\gamma \in \mathbb{R}$ we have: if $\gamma > 0$ then $\gamma z = \gamma x' - \gamma x \in \Sigma - \Sigma$; if $\gamma < 0$ then $\gamma z = (-\gamma)x - (-\gamma)x' \in \Sigma - \Sigma$; if $\gamma = 0$ then $\gamma z = 0 = x - x \in \Sigma - \Sigma$, hence indeed $\mathcal{T}_{\iota_\Sigma}(\Sigma) \subseteq \Sigma - \Sigma$.

Now consider $y \in \mathcal{T}_{\iota_\Sigma}(\Sigma)$ and write it as $y = \gamma(x_1 - x_2)$ where $x_1, x_2 \in \Sigma$ and $\gamma \in \mathbb{R}$. Since $\Sigma \subseteq \mathtt{dom}(R)$ we have $\max(R(x_1), R(x_2)) < \infty$. We will prove that $y \in \mathcal{T}_R(\Sigma)$. We distinguish two cases: if $R(x_1) \leqslant R(x_2)$ then $R(x_2 + (x_1 - x_2)) = R(x_1) \leqslant R(x_2)$ hence $y = \gamma(x_1 - x_2) \in \mathcal{T}_R(x_2)$, and as $x_2 \in \Sigma$ it follows that $y \in \mathcal{T}_R(\Sigma)$; otherwise $R(x_2) < R(x_1)$ hence $R(x_1 + (x_2 - x_1)) = R(x_2) < R(x_1)$ hence $y = (-\gamma)(x_2 - x_1) \in \mathcal{T}_R(x_1)$ and therefore $y \in \mathcal{T}_R(\Sigma)$. □

A.2.2 *Proof of Lemma 2.4* . Given $t > R(0)$, the level set $\mathcal{L}(R,t) = \{y \in \mathcal{H} : R(y) \leqslant t\}$ is nonempty, convex and closed (by convexity and lower semi-continuity of R), and bounded (by coercivity of R). We define $\mathcal{A} := \mathcal{L}(R,t) \cap \Sigma = \{x \in \Sigma : R(x) \leqslant t\}$.

Consider $z \in \mathcal{T}_{\|\cdot\|_{\mathcal{A}}}(\Sigma)$. If $z = 0$ then clearly $z \in \mathcal{T}_R(\Sigma)$. Let us prove that the same holds when $z \neq 0$. By definition, there exists $\gamma \in \mathbb{R} \setminus \{0\}$ and $x \in \Sigma$ such that

$$\|x + z/\gamma\|_{\mathcal{A}} \leqslant \|x\|_{\mathcal{A}}.$$

On the one hand we have $R(0 \cdot x) = R(0) < t$. On the other hand, since R is coercive, we have $R(\lambda x) \underset{\lambda \to +\infty}{\to} +\infty$. Since R is continuous, by the mean value theorem, there is $\lambda_0 > 0$ such that

$$R(\lambda_0 x) = t.$$

Since Σ is a cone, the vector $x' = \lambda_0 x$ belongs to Σ and, since $R(x') = t$, by definition of $\mathcal{A}$ we have indeed $x' \in \mathcal{A}$, hence $\|x'\|_{\mathcal{A}} \leqslant 1$. Furthermore, since $\|\cdot\|_{\mathcal{A}}$ is positively homogeneous (because $0 \in \overline{\text{conv}}(\mathcal{A})$), we have

$$\|x' + \lambda_0 z/\gamma\|_{\mathcal{A}} = \lambda_0 \|x + z/\gamma\|_{\mathcal{A}} \leqslant \lambda_0 \|x\|_{\mathcal{A}} = \|x'\|_{\mathcal{A}}.$$

We now observe that, on the one hand, the level set $\mathcal{L}(\|\cdot\|_{\mathcal{A}}, 1) = \overline{\text{conv}}(\mathcal{A})$ is the smallest closed convex set containing $\mathcal{A}$; on the other hand $\mathcal{A} \subset \mathcal{L}(R,t)$ and $\mathcal{L}(R,t)$ is convex and closed. Thus $\mathcal{L}(\|\cdot\|_{\mathcal{A}}, 1) \subset \mathcal{L}(R,t)$ and the fact that $\|x' + \lambda_0 z/\gamma\|_{\mathcal{A}} \leqslant \|x'\|_{\mathcal{A}} \leqslant 1$ therefore implies

$$R(x' + \lambda_0 z/\gamma) \leqslant t = R(x'). \tag{A.2}$$

This shows that $z \in \mathcal{T}_R(\Sigma)$ and establishes that $\mathcal{T}_{\|\cdot\|_{\mathcal{A}}}(\Sigma) \subseteq \mathcal{T}_R(\Sigma)$.

Let us now prove that $\|\cdot\|_{\mathcal{A}}$ is continuous, convex, coercive and positively homogeneous. First, from the property of gauges (see Appendix A.1), $\|\cdot\|_{\mathcal{A}}$ is always convex and lower semi-continuous. Second, since R is coercive, its level sets are bounded, hence $\overline{\text{conv}}(\mathcal{A})$ is bounded and $\|\cdot\|_{\mathcal{A}}$ is coercive. Finally, as $R(0) < t$ and R is continuous, 0 is in the interior of $\mathcal{L}(R,t)$. There exists $\varepsilon > 0$ such that an open ball O of radius ε centered on 0 is included in $\mathcal{L}(R,t)$. This implies $O \cap \Sigma \subset \mathcal{L}(R,t) \cap \Sigma = \mathcal{A}$ which in turns imply $\text{conv}(O \cap \Sigma) \subset \text{conv}(\mathcal{A}) \subset \overline{\text{conv}(\mathcal{A})}$. Remark that $\mathcal{E}(O \cap \Sigma) = \mathcal{E}(\Sigma) = \mathcal{H}$. Now we need to find O' an open ball of radius ε' such that $O' \subset \text{conv}(O \cap \Sigma)$. In each orthant Ω_r, we can find a normalized basis $E = (e_i) \in \Sigma$ such that $\Omega_r \subset \mathcal{E}(E)$. We define the norm $\|\sum_i \mu_i e_i\|_E = \sum \mu_i$. This norm is equivalent to $\|\cdot\|_{\mathcal{H}}$. This implies there is a constant c_r depending on the orthant Ω_r, such that for $x = \sum_i \mu_i e_i \in O' \cap \Omega_r$, $\max_i \mu_i < c_r \varepsilon'$. This implies

$$x = t \sum_i \frac{\mu_i}{\sum_j \mu_j} \varepsilon e_i \tag{A.3}$$

with $t = \frac{\sum_j \mu_j}{\varepsilon} \leqslant n c_r \frac{\varepsilon'}{\varepsilon}$. Taking $\varepsilon' < \varepsilon/(n c_r)$ implies $t < 1$ and $x \in \text{conv}(O \cap \Sigma)$. As there is a finite number of orthants we can chose ε' such that we always have $x \in O'$ implies $x \in \text{conv}(O \cap \Sigma)$. □

A.3 *Proofs for Section 3.1*

Proof of Lemma 3.1.

Denote $\alpha = \inf_{x \in (\Sigma - \Sigma) \cap S(1)} \|Mx\|_2^2$ and $\beta = \sup_{x \in (\Sigma - \Sigma) \cap S(1)} \|Mx\|_2^2$, so that $\gamma(M) = \beta/\alpha$. Since Σ is a cone, we have for every $x \in \Sigma - \Sigma$,

$$\alpha \|x\|_{\mathcal{H}}^2 \leqslant \|Mx\|_2^2 \leqslant \beta \|x\|_{\mathcal{H}}^2 = \gamma(M) \alpha \|x\|_{\mathcal{H}}^2, \tag{A.4}$$

Multiplying x in (A.4) by any $\lambda > 0$, we have

$$\lambda^2\alpha\|x\|_{\mathcal{H}}^2 \leqslant \|\lambda Mx\|_2^2 \leqslant \lambda^2\gamma(M)\alpha\|x\|_{\mathcal{H}}^2.$$

We look for $\lambda > 0$, $\delta \neq 1$ such that λM satisfies a symmetric RIP with constant δ, *i.e.,*

$$\lambda^2\alpha = 1-\delta \quad \text{and} \quad \lambda^2\gamma(M)\alpha = 1+\delta.$$

Adding these two equalities yields $\lambda^2\alpha(1+\gamma(M)) = 1$, hence $\lambda^2 = \frac{1}{\alpha(1+\gamma(M))}$. Dividing them yields

$$\frac{1-\delta}{1+\delta} = \gamma(M) \iff \delta = \frac{\gamma(M)-1}{\gamma(M)+1}.$$

We have shown that for any M, there exists $\lambda > 0$ such that

$$\delta(\lambda M) \leqslant \frac{\gamma(M)-1}{\gamma(M)+1}.$$

Remark that the value of λ that makes the RIP bounds symmetrical is unique, and that no better symmetrical RIP bound can be obtained, otherwise we could construct a better restricted conditioning (which is impossible by definition of $\gamma(M)$). We deduce

$$\delta(\lambda M) = \frac{\gamma(M)-1}{\gamma(M)+1}.$$

□

LEMMA A.1 Consider a cone $\Sigma \subseteq \mathcal{H}$ and $\mathcal{T} \subseteq \mathcal{H}$ a non-empty set, and denote $\mathcal{P}$ the set of symmetric positive semi-definite linear operators on $\mathcal{H}$, *i.e.,* $N \in \mathcal{P}$ if and only if $N^H = N$ and $N \succeq 0$. Then

$$\inf_{M:\ker M\cap\mathcal{T}\neq\{0\}} \gamma_\Sigma(M) = \inf_{N\in\mathcal{P}:\dim\ker N=1,\ker N\cap\mathcal{T}\neq\{0\}} \gamma_\Sigma(N). \tag{A.5}$$

Proof. The infimum on the r.h.s. of (A.5) is over a more constrained set than on the l.h.s., hence

$$\inf_{M:\ker M\cap\mathcal{T}\neq\{0\}} \gamma_\Sigma(M) \leqslant \inf_{N\in\mathcal{P}:\dim\ker N=1,\ker N\cap\mathcal{T}\neq\{0\}} \gamma_\Sigma(N).$$

If the l.h.s. is infinite, then the right-hand side must also be infinite, and we are done.

Assume that the l.h.s. is finite. We now prove the reverse inequality. For this, consider M a linear operator with $\ker M\cap\mathcal{T} \neq \{0\}$ and $\gamma_\Sigma(M) < \infty$. There exists a nonzero vector $t \in \ker M\cap\mathcal{T}$. We build an operator $N \in \mathcal{P}$ such that $\ker N = \operatorname{span}(t)$ and with $\gamma_\Sigma(N)$ arbitrarily close to $\gamma_\Sigma(M)$.

Since $\gamma_\Sigma(M) < \infty$, M is nonzero hence a singular value decomposition allows writing $M = \sum_{i=1}^r \sigma_i u_i v_i^H$ where $(u_i)_{i=1}^r$ and $(v_i)_{i=1}^r$ are orthonormal families and $\min_{1\leqslant i\leqslant r}\sigma_i > 0$. First we deal with the case where $\dim\ker M = 1$. We set $N = \sum_{i=1}^r \sigma_i v_i v_i^H$ so that $N \in \mathcal{P}$ and $\dim\ker N = 1$ too. Since $\|Nx\|_2^2 = \sum_{i=1}^r \sigma_i^2\langle v_i,x\rangle^2 = \|Mx\|_2^2$ for any vector x we have $\gamma(N) = \gamma(M)$, and we are done. Assume now that $k := \dim\ker M \geqslant 2$. Observe that $\operatorname{span}(t) \subset \ker M$ and let $(e_1,\dots,e_{k-1})$ be an orthonormal basis of the orthogonal complement of $\operatorname{span}(t)$ in $\ker M$, so that $(v_1,\dots,v_r,e_1,\dots,e_{k-1})$ is an orthonormal family. For each $\varepsilon > 0$, define $N_\varepsilon = \sum_{i=1}^r \sigma_i v_i v_i^H + \varepsilon\sum_{j=1}^{k-1} e_j e_j^H$. Again, $N_\varepsilon \in \mathcal{P}$ and $\operatorname{span}(t) = \ker N_\varepsilon$ so that $\dim\ker N_\varepsilon = 1$, and for each $x \in \mathcal{H}$ we have

$$\|N_\varepsilon x\|_2^2 = \sum_{i=1}^r \sigma_i^2\langle v_i,x\rangle^2 + \varepsilon^2\sum_{j=1}^{k-1}\langle e_j,x\rangle^2 = \|Mx\|_2^2 + \varepsilon^2\sum_{j=1}^{k-1}\langle e_j,x\rangle^2,$$

hence $\|Mx\|_2^2 \leqslant \|N_\varepsilon x\|_2^2 \leqslant \|Mx\|_2^2 + \varepsilon^2\|x\|_2^2$. Since $\gamma_\Sigma(M) < \infty$, we get

$$0 < \inf_{x\in(\Sigma-\Sigma)\cap S(1)} \|Mx\|_2^2 \leqslant \inf_{x\in(\Sigma-\Sigma)\cap S(1)} \|N_\varepsilon x\|_2^2 \leqslant \sup_{x\in(\Sigma-\Sigma)\cap S(1)} \|N_\varepsilon x\|_2^2 \leqslant \sup_{x\in(\Sigma-\Sigma)\cap S(1)} \|Mx\|_2^2 + \varepsilon^2$$

which implies

$$\gamma_\Sigma(N_\varepsilon) \leqslant \frac{\sup_{x\in(\Sigma-\Sigma)\cap S(1)} \|Mx\|_2^2 + \varepsilon^2}{\inf_{x\in(\Sigma-\Sigma)\cap S(1)} \|Mx\|_2^2} = \gamma_\Sigma(M) + \frac{\varepsilon^2}{\inf_{x\in(\Sigma-\Sigma)\cap S(1)} \|Mx\|_2^2}.$$

This implies that $\inf_{\varepsilon>0} \gamma_\Sigma(N_\varepsilon) \leqslant \gamma_\Sigma(M)$ as claimed.

□

Proof of Lemma 3.2. We define

$$G(\Sigma, E, M) := \frac{\sup_{y\in(\Sigma-\Sigma)\cap E} \|My\|_2^2}{\inf_{y\in(\Sigma-\Sigma)\cap E} \|My\|_2^2}. \tag{A.6}$$

For any nonzero M, we have

$$\gamma_{F\Sigma}(M) = \frac{\sup_{x\in(F\Sigma-F\Sigma)\cap S(1)} \|Mx\|_2^2}{\inf_{x\in(F\Sigma-F\Sigma)\cap S(1)} \|Mx\|_2^2} = \frac{\sup_{y\in(\Sigma-\Sigma)\cap F^{-1}S(1)} \|MFy\|_2^2}{\inf_{y\in(\Sigma-\Sigma)\cap F^{-1}S(1)} \|MFy\|_2^2}. \tag{A.7}$$

Hence,

$$\begin{aligned} A_{F\Sigma}^{RC}(R\circ F^{-1}) &= \inf_{M:\ker M\cap \mathcal{T}_{R\circ F^{-1}}(F\Sigma)\neq\{0\}} \gamma_{F\Sigma}(M) \\ &= \inf_{M:\ker M\cap \mathcal{T}_{R\circ F^{-1}}(F\Sigma)\neq\{0\}} G(\Sigma, F^{-1}S(1), MF). \end{aligned}$$

By Lemma 2.5 with $R' = R\circ F^{-1}$, $\mathcal{T}_{R\circ F^{-1}}(F\Sigma) = \mathcal{T}_{R'}(F\Sigma) = F(\mathcal{T}_{R'\circ F}(\Sigma)) = F(\mathcal{T}_R(\Sigma))$. Also, $\ker M \cap \mathcal{T}_{R\circ F^{-1}}(F\Sigma) \neq \{0\}$ is equivalent to the existence of $z \in \ker M$ such that $z' := F^{-1}z \in \mathcal{T}_R(\Sigma)$, *i.e.,* of $z' \in \mathcal{T}_R(\Sigma)$ such that $z := Fz' \in \ker M$. As a result,

$$\inf_{M:\ker M\cap \mathcal{T}_{R\circ F^{-1}}(F\Sigma)\neq\{0\}} \gamma_{F\Sigma}(M) = \inf_{M:F^{-1}\ker M\cap \mathcal{T}_R(\Sigma)\neq\{0\}} G(\Sigma, F^{-1}S(1), MF). \tag{A.8}$$

Rewriting $M' = MF$, we have $\ker M' = F^{-1}\ker M$ and

$$\inf_{M:\ker M\cap \mathcal{T}_{R\circ F^{-1}}(F\Sigma)\neq\{0\}} \gamma_{F\Sigma}(M) = \inf_{M':\ker M'\cap \mathcal{T}_R(\Sigma)\neq\{0\}} G(\Sigma, F^{-1}S(1), M') \tag{A.9}$$

which gives the desired result using the fact that $F^{-1}S(1) = S(1)$ since F is a linear isometry.

□

A.4 *Proofs for Section 3.2*

Proof of Lemma 3.3.

Consider $z \in \mathcal{H} \setminus \{0\}$ and $M = I - \Pi_z$. For every $x \in S(1)$, we have

$$\|Mx\|_2^2 = 1 - \frac{\langle x,z\rangle^2}{\|z\|_{\mathcal{H}}^2} \tag{A.10}$$

hence

$$\gamma_\Sigma(M) = \frac{\sup_{x\in(\Sigma-\Sigma)\cap S(1)} \|Mx\|_2^2}{\inf_{x\in(\Sigma-\Sigma)\cap S(1)} \|Mx\|_2^2} = \frac{1 - \inf_{x\in(\Sigma-\Sigma)\cap S(1)} \frac{\langle x,z\rangle^2}{\|z\|_{\mathcal{H}}^2}}{1 - \sup_{x\in(\Sigma-\Sigma)\cap S(1)} \frac{\langle x,z\rangle^2}{\|z\|_{\mathcal{H}}^2}}$$

Case 1: By assumption there is x_0 such that $\|x_0\|_{\mathcal{H}} = 1$ and $\Sigma \subseteq \operatorname{span}(x_0)$. Since $\Sigma \neq \{0\}$ is a cone, it follows that $(\Sigma-\Sigma)\cap S(1) = \operatorname{span}(x_0)\cap S(1) = \{-x_0, +x_0\}$ and

$$\inf_{x\in(\Sigma-\Sigma)\cap S(1)} \frac{\langle x,z\rangle^2}{\|z\|_{\mathcal{H}}^2} = \sup_{x\in(\Sigma-\Sigma)\cap S(1)} \frac{\langle x,z\rangle^2}{\|z\|_{\mathcal{H}}^2} = \frac{\langle x_0,z\rangle^2}{\|z\|_{\mathcal{H}}^2}. \tag{A.11}$$

Hence, if $z \in \Sigma = \operatorname{span}(x_0)$ we have $\gamma_\Sigma(M) = +\infty$, otherwise $\frac{\langle x_0,z\rangle^2}{\|z\|_{\mathcal{H}}^2} < 1$ and $\gamma_\Sigma(M) = 1$. Thus, if $\mathcal{T}_R(\Sigma) \subseteq \Sigma$ we have $A_\Sigma^{RIP,\mathtt{nec}}(R) = +\infty$, otherwise there is $z \in \mathcal{T}_R(\Sigma)\setminus\Sigma$, and $A_\Sigma^{RIP,\mathtt{nec}}(R) = 1$.
Case 2: Let us show that for any $z \neq 0$ there is some $x \in (\Sigma-\Sigma)\setminus\{0\}$ such that $\langle x,z\rangle = 0$. This implies $\inf_{x\in(\Sigma-\Sigma)\cap S(1)} \frac{\langle x,z\rangle^2}{\|z\|_{\mathcal{H}}^2} = 0$ and yields the result. Indeed, by assumption, given any $x_1 \in \Sigma\setminus\{0\}$ there is $x_2 \in \Sigma$ such that $x_2 \notin \operatorname{span}(x_1)$ (hence $x_2 \neq 0$). If $\langle x_1,z\rangle = 0$ we take $x = x_1 = x_1 - \lambda x_2$ with $\lambda = 0$. Otherwise, with $\lambda = \frac{\langle x_2,z\rangle}{\langle x_1,z\rangle}$ we set $x = \lambda x_1 - x_2$. In both cases we have $x \neq 0$ and, since Σ is a cone, $x \in \Sigma - \Sigma$ and $\langle \lambda x_1 - x_2, z\rangle = 0$.

□

Proof of Lemma 3.4. Since $E \cap S(1)$ is compact, for any z there exists $\tilde{x} \in E\cap S(1)$ such that

$$|\langle \tilde{x},z\rangle|^2 = \max_{\tilde{y}\in E\cap S(1)} |\langle \tilde{y},z\rangle|^2. \tag{A.12}$$

Since E is a union of subspaces, it is homogeneous. Thus, as $\tilde{x} \in E$, we have $x := \langle \tilde{x},z\rangle\tilde{x} \in E$. If $y \in E\setminus\{0\}$, we have $\tilde{y} := y/\|y\|_{\mathcal{H}} \in E\cap S(1)$, $\langle z,\tilde{y}\rangle\tilde{y}$ is the orthogonal projection of z on $\tilde{y}$ and

$$\begin{aligned} \|z-y\|_{\mathcal{H}}^2 = \big\|z - \|y\|_{\mathcal{H}}\cdot\tilde{y}\big\|_{\mathcal{H}}^2 &\geqslant \|z - \langle z,\tilde{y}\rangle\tilde{y}\|_{\mathcal{H}}^2 = \|z\|_{\mathcal{H}}^2 - |\langle z,\tilde{y}\rangle|^2 \\ &\overset{(A.12)}{\geqslant} \|z\|_{\mathcal{H}}^2 - |\langle z,\tilde{x}\rangle|^2 \end{aligned} \tag{A.13}$$

Since $\|z-x\|_{\mathcal{H}}^2 = \|z\|_{\mathcal{H}}^2 - 2\mathcal{R}e\langle z,x\rangle + \|x\|_{\mathcal{H}}^2 = \|z\|_{\mathcal{H}}^2 - |\langle z,\tilde{x}\rangle|^2$, we conclude

$$\|z-y\|_{\mathcal{H}}^2 \geqslant \|z-x\|_{\mathcal{H}}^2 \tag{A.14}$$

and $x \in P_E(z)$ by definition of P_E.

If $x' \in P_E(z)$, we have $\|z-x'\|_{\mathcal{H}}^2 = \|z-x\|_{\mathcal{H}}^2 = \min_{y\in E}\|z-y\|_{\mathcal{H}}^2$ hence the notation $\|z-P_E(z)\|_{\mathcal{H}}^2$ is unambiguous. Since $x' \in P_E(z)$, there is equality in the above equation with $y = x'$, hence $\|y\|_{\mathcal{H}} = \langle z,\tilde{y}\rangle$ and $|\langle z,\tilde{y}\rangle|^2 = |\langle z,\tilde{x}\rangle|^2$, therefore $\langle z,y\rangle = \langle z,\|y\|_{\mathcal{H}}\tilde{y}\rangle = \|y\|_{\mathcal{H}}\langle z,\tilde{y}\rangle = \|y\|_{\mathcal{H}}^2 = \langle z,\tilde{y}\rangle^2 = \langle z,\tilde{x}\rangle^2 = \|x\|_{\mathcal{H}}^2$. This shows that the notations $\|P_E(z)\|_{\mathcal{H}}^2$ and $\langle z,P_E(z)\rangle$ are unambiguous and that $\|P_E(z)\|_{\mathcal{H}}^2 = \langle z,P_E(z)\rangle$.

We also have $\|z\|_{\mathcal{H}}^2 = \|x\|_{\mathcal{H}}^2 + \|z-x\|_{\mathcal{H}}^2 = \|x'\|^2$, and $\langle z,y\rangle = \|y\|_{\mathcal{H}}$ hence the notations $\|z-P_E(z)\|_{\mathcal{H}}^2$ and $\|P_E(z)\|_{\mathcal{H}}^2$ are unambiguous. □

Proof of Corollary 3.1. Since $\Sigma-\Sigma$ is a union of subspaces and $(\Sigma-\Sigma)\cap S(1)$ is compact, by Lemma 3.4, $\sup_{x\in(\Sigma-\Sigma)\cap S(1)} \frac{\langle x,z\rangle^2}{\|z\|_{\mathcal{H}}^2} = \frac{\|P_{\Sigma-\Sigma}(z)\|_{\mathcal{H}}^2}{\|z\|_{\mathcal{H}}^2}$, hence we have

$$(B_\Sigma(R)+1)^{-1} = \left(\sup_{z\in\mathcal{T}_R(\Sigma)\setminus\{0\}} \frac{\|z-P_{\Sigma-\Sigma}(z)\|_{\mathcal{H}}^2}{\|P_{\Sigma-\Sigma}(z)\|_{\mathcal{H}}^2}+1\right)^{-1} = \inf_{z\in\mathcal{T}_R(\Sigma)\setminus\{0\}} \frac{\|P_{\Sigma-\Sigma}(z)\|_{\mathcal{H}}^2}{\|z-P_{\Sigma-\Sigma}(z)\|_{\mathcal{H}}^2+\|P_{\Sigma-\Sigma}(z)\|_{\mathcal{H}}^2}$$
$$= \inf_{z\in\mathcal{T}_R(\Sigma)\setminus\{0\}} \frac{\|P_{\Sigma-\Sigma}(z)\|_{\mathcal{H}}^2}{\|z\|_{\mathcal{H}}^2}.$$

Since Σ is a cone and $\Sigma \neq \mathrm{span}(x)$ for each $x\in\Sigma$, by Lemma 3.3, using (3.9) we have $\gamma_\Sigma^{\mathtt{nec}}(R) = \frac{1}{1-(1+B_\Sigma(R))^{-1}} = 1+1/B_\Sigma(R)$ hence $\delta_\Sigma^{\mathtt{nec}}(R) = \frac{\gamma_\Sigma^{\mathtt{nec}}(R)-1}{\gamma_\Sigma^{\mathtt{nec}}(R)+1} = (2B_\Sigma(R)+1)^{-1}$.

We conclude using that $b\mapsto 1/(1+2b)$ is decreasing.

□

A.4.1 *Lemmas for the proof of Theorem 3.6 (sparse recovery).* We begin by some technical lemmas. We recall that $T_2 = T_2(z) \subseteq \{1,\dots,n\}$ denotes a set indexing any $2k$ largest components (in magnitude) of vector z, while $T = T(z) \subseteq \{1,\dots,n\}$ will denote a set indexing k largest components (in magnitude). Given an index set $\emptyset \neq H \subseteq \{1,\dots,n\}$, Q_H is the "cube" of all vectors $v\in\mathbb{R}^n$ such that $\mathtt{supp}(v) = H$ and $|v_i| = 1$ for every $i\in H$. The restriction of v to H, $v_H\in\mathbb{R}^n$, is such that $(v_H)_i = v_i$, $i\in H$ and $\mathtt{supp}(v_H)\subseteq H$.

LEMMA A.2 Let $\Sigma = \Sigma_k$. Let $\|\cdot\|_w$ be a weighted ℓ^1-norm (for $w = (w_i)_{i=1}^n$ with $w_i > 0$, $\|x\|_w = \sum w_i\|x\|_1$). Let $z\in\mathcal{T}_{\|\cdot\|_w}(\Sigma)$. There is a support H of size $\leqslant k$ such that

$$\|z_{H^c}\|_w - \|z_H\|_w = \inf_{x\in\Sigma}\{\|x+z\|_w - \|x\|_w\} \leqslant 0, \tag{A.15}$$

i.e., the infimum is achieved at $x^* = -z_H$.

Moreover, if $\|\cdot\|_w = \|\cdot\|_1$, $H = T(z)$.

Proof. The result is trivial for $z=0$, so we prove it for $z\in\mathcal{T}_{\|\cdot\|_w}(\Sigma)\setminus\{0\}$. Consider $H\in\arg\min_{T:|T|\leqslant k}\{\|z_{T^c}\|_w - \|z_T\|_w\}$. By definition of $\mathcal{T}_{\|\cdot\|_w}(\Sigma)$, since $z\in\mathcal{T}_{\|\cdot\|_w}(\Sigma)\setminus\{0\}$, there are $x'\in\Sigma$, $\lambda\in\mathbb{R}\setminus\{0\}$ such that $\|x'+\lambda z\|_w \leqslant \|x'\|_w$. By homogeneity of Σ, $x := x'/\lambda\in\Sigma$ and $\|x+z\|_w\leqslant\|x\|_w$. This shows that $\inf_{x\in\Sigma}\{\|x+z\|_w-\|x\|_w\}\leqslant 0$ as claimed. For any such $x\in\Sigma$, consider $T = \mathtt{supp}(x)$.

By the reverse triangle inequality $|x_i+z_i|-|x_i| \geqslant -|z_i|$, we have

$$\|x+z_T\|_w - \|x\|_w = \sum_{i\in T} w_i(|x_i+z_i|-|x_i|) \geqslant -\sum_{i\in T} w_i|z_i| = -\|z_T\|_w \tag{A.16}$$

Hence $\|x+z\|_w - \|x\|_w = \|x+z_T\|_w + \|z_{T^c}\|_w - \|x\|_w \geqslant \|z_{T^c}\|_w - \|z\|_w \geqslant \|z_{H^c}\|_w - \|z_H\|_w$.

If $\|\cdot\|_w = \|\cdot\|_1$, let $T = T(z)$ and remark that $\|z_{H^c}\|_1 - \|z_H\|_1 \geqslant \|z_{T^c}\|_1 - \|z_T\|_1$

□

The following Lemma permits to construct and to characterize elements of descent cones.

LEMMA A.3 Assume that R and Σ are positively homogeneous. For every $v_0\in\Sigma$ such that $R(v_0)>0$ and any $v_1\in\mathcal{H}$, we have that $z := v_1 - \alpha v_0\in\mathcal{T}_R(\Sigma)$ where $\alpha = \max(R(v_1)/R(v_0),1)$. If, in addition, Σ is homogeneous and R is even, we have conversely that any $z\in\mathcal{T}_R(\Sigma)$ can be written as $z = v_1 - v_0$ where $v_0\in\Sigma$, $v_1\in\mathcal{H}$, and $R(v_1)\leqslant R(v_0)$.

Proof. Since Σ is positively homogeneous, $x := \alpha v_0 \in \Sigma$, and $R(x+z) = R(\alpha v_0 + z) = R(v_1)$. If $R(v_1) > R(v_0)$ then $\alpha > 1$ and $R(x+z) = R(v_1) = \alpha R(v_0) = R(\alpha v_0) = R(x)$. Otherwise, $\alpha = 1$ and $R(x+z) = R(v_1) \leqslant R(v_0) = R(x)$. In both cases we obtain that $z \in \mathcal{T}_R(x) \subseteq \mathcal{T}_R(\Sigma)$.

Regarding the second claim, when $z \in \mathcal{T}_R(\Sigma)$, by definition there exists $x \in \Sigma$, $u \in \mathcal{H}$ and $\gamma \in \mathbb{R}$ such that $z = \gamma u$ where $R(x+u) \leqslant R(x)$. Denote $v_0 := \gamma x$ and $v_1 := v_0 + z$. Since Σ is homogeneous, we have $v_0 \in \Sigma$. Since R is even and positively homogeneous, $R(v_1) = R(\gamma x + \gamma u) = |\gamma| R(x+u) \leqslant |\gamma| R(x) = R(\gamma x) = R(v_0)$.

□

The next lemma permits to compare $B^s_\Sigma(R)$ with $B^s_\Sigma(\|\cdot\|_1)$ (see definition in (3.29)) which was calculated in [17] to characterize the necessary RIP condition for sparse recovery.

LEMMA A.4 Let $\Sigma = \Sigma_k$ be the set of k-sparse vectors in $\mathbb{R}^n$ with $k < n/2$ and $1 \leqslant L \leqslant n-2k$. Assume that R is positively homogeneous, subadditive, and nonzero.

Consider

$$(H_0, v_0) \in \arg \max_{\substack{H \subseteq \{1,\dots,n\}:\ |H|=k \\ v \in Q_H}} R(v) \tag{A.17}$$

$$(H_1, v_1) \in \arg \min_{\substack{H \subseteq \{1,\dots,n\} \setminus H_0, |H|=k+L \\ v \in Q_H}} R(v). \tag{A.18}$$

1. We have $R(v_0) > 0$, and for any H of size $k' \geqslant k$ and any $v \in Q_H$, we have

$$R(v) \leqslant \frac{k'}{k} R(v_0). \tag{A.19}$$

If $R = R^\star = \|\cdot\|_1$ then we have indeed equality $R^\star(v) = \frac{k'}{k} R^\star(v_0)$.

2. We have

$$B^{2k+L}_\Sigma(R) := \sup_{z \in \mathcal{T}_R(\Sigma) \setminus \{0\} : |\mathrm{supp}(z)| = 2k+L} \frac{\|z_{T_2^c}\|_2^2}{\|z_{T_2}\|_2^2} \geqslant \frac{\frac{L}{k}}{\max\left(\left(\frac{R(v_1)}{R(v_0)}\right)^2, 1\right) + 1} \geqslant \frac{\frac{L}{k}}{\left(\frac{L}{k}+1\right)^2 + 1}. \tag{A.20}$$

Proof. As a preliminary observe that if $R^\star = \|\cdot\|_1$ then $R^\star(v) = |H|$ for any $H, v \in Q_H$, hence H_0, H_1 can be any pair of disjoint sets of respective sizes $k, k+L$, and $v_i \in Q_{H_i}$ can be arbitrary, for example $v_i = 1_{H_i}$. This yields $R^\star(v_0) = k$, $R^\star(v_1) = k+L$, hence $R^\star(v_1) = (1+L/k) R^\star(v_0)$.

To prove the first claim, consider $\{G_i\}_{1 \leqslant i \leqslant \binom{k'}{k}}$ the collection of all subsets $G_i \subseteq H$ of size exactly k. Since $v \in Q_H$, we have $v_{G_i} \in Q_{G_i}$ for each i. Also, since $|G_i| = k$ for every i, by definition of H_0, v_0 we obtain $\max_i R(v_{G_i}) \leqslant R(v_0)$. Notice that given a coordinate $j \in H$, there are $\binom{k'-1}{k-1}$ sets G_i such that $j \in G_i$. With $\lambda := \frac{1}{\binom{k'-1}{k-1}}$ we get $v = \lambda \sum_i v_{G_i}$ hence by positive homogeneity and subadditivity of R (which imply convexity)

$$R(v) = R(\lambda \sum_{i=1}^{\binom{k'}{k}} v_{G_i}) \leqslant \sum_{i=1}^{\binom{k'}{k}} R(\lambda v_{G_i}) = \lambda \sum_{i}^{\binom{k'}{k}} R(v_{G_i}) \leqslant \frac{\binom{k'}{k}}{\binom{k'-1}{k-1}} R(v_0) = \frac{k'}{k} R(v_0). \tag{A.21}$$

This establishes (A.19). With $R = R^\star$, we have $R^\star(v) = \|v\|_1 = k'$ for $v \in Q_H$, hence $R^\star(v) = (k'/k)R^\star(v_0)$ as claimed.

For the sake of contradiction, assume that $R(v_0) \leqslant 0$. As we have just proved, this implies $R(v) \leqslant (n/k)R(v_0) \leqslant 0$ for every $v \in \{-1,+1\}^n = Q_H$ with $H = \{1,\dots,n\}$. By convexity of R it follows that $R(v) \leqslant 0$ for each $v \in [-1,1]^n = \mathrm{conv}(Q_H)$, and by positive homogeneity,

$$R(v) \leqslant 0, \ \forall v \in \mathcal{H}. \tag{A.22}$$

Positive homogeneity and subadditivity also imply

$$0 = 0 \cdot R(v_0) = R(0 \cdot v_0) = R(0) = R(-v+v) \leqslant R(-v) + R(v) \overset{(A.22)}{\leqslant} R(-v)$$

for every $v \in \mathcal{H}$, hence $R(v) = 0$ on $\mathcal{H}$, which yields the desired contradiction since we assume that R is nonzero.

Regarding the second claim, since $2k+L \leqslant n$ there is indeed some H of size $k+L$ such that $H \cap H_0 = \emptyset$, hence H_1 is well defined. By construction, $H_1 \cap H_0 = \emptyset$. Since $R(v_0) > 0$, R is positively homogeneous and Σ is homogeneous, by Lemma A.3, $z = -\alpha v_0 + v_1 \in \mathcal{T}_R(\Sigma)$ with $\alpha := \max(R(v_1)/R(v_0), 1)$. Observe that $|\mathtt{supp}(z)| = |H_0| + |H_1| = 2k+L$. Since $\alpha \geqslant 1$ and all nonzero entries of v_0, v_1 have magnitude one, a set of $2k$ largest components of z is $T_2 = H_0 \cup T_1'$ with T_1' any subset of H_1 with k components, and we obtain (A.20). once we observe that

$$\frac{\|z_{T_2^c}\|_2^2}{\|z_{T_2}\|_2^2} = \frac{L}{k\alpha^2 + k} = \frac{L/k}{\alpha^2 + 1}.$$

□

LEMMA A.5 Consider $c_\infty, c_1 > 0$, an integer $n \geqslant 2$, and the optimization problem

$$\sup_{x \in \mathbb{R}^n_+ : \|x\|_\infty \leqslant c_\infty ; \|x\|_1 \leqslant c_1} \|x\|_2^2. \tag{A.23}$$

If $c_1 \geqslant c_\infty$ then there exists $1 \leqslant L \leqslant n-1$ and $0 \leqslant \theta \leqslant 1$ such that

$$x^* := c_\infty (\underbrace{1,\dots,1}_{L \geqslant 1}, \theta, \ \underbrace{0,\dots,0}_{n-(L+1) \geqslant 0})$$

is a maximizer. Otherwise, a maximizer is $x^* = (c_1, 0, \dots, 0)$.

Proof. Standard compactness arguments show the existence of a maximizer x^*. We distinguish two cases:

- If $\|x^*\|_\infty < c_\infty$ then x^* is indeed a maximizer of the Euclidean norm under an ℓ^1 constraint, hence x^* is a Dirac: without loss of generality, $x^* = (c_1, 0, \dots, 0)$ so that $c_1 = \|x^*\|_\infty < c_\infty$.
- Otherwise $\|x^*\|_\infty = c_\infty$, in which case we show that all entries of x^*, except at most one, are either zero or equal to c_∞. For the sake of contradiction, assume that x^* contains two distinct entries with values $0 < a < b < c_\infty$, then for small enough $t > 0$, replacing these entries with $0 < a-t < b+t < c_\infty$ and keeping all other entries unchanged would lead to a vector x satisfying $\|x\|_\infty = \|x^*\|_\infty = c_\infty$, $\|x\|_1 = \|x^*\|_1$. However, since $\|x\|_2^2 - \|x^*\|_2^2 = (a-t)^2 + (b+t)^2 - (a^2+b^2) = 2t^2 + 2(b-a)t > 0$. Since x^* has optimal objective value, this yields the desired contradiction. Since the objective value and the constraints are invariant to index permutations, there is thus a maximizer with the claimed shape, and we have $c_1 \geqslant \|x^*\|_1 \geqslant \|x^*\|_\infty = c_\infty$.

The two cases respectively correspond to $c_1 < c_\infty$ or $c_1 \geqslant c_\infty$, which are mutually exclusive, hence the conclusion.

□

LEMMA A.6 ([17]) Consider $\Sigma = \Sigma_k \subseteq \mathbb{R}^n$. We have

$$B_\Sigma(\|\cdot\|_1) = \max_{1\leqslant L\leqslant n-2k} \frac{\frac{L}{k}}{\left(\frac{L}{k}+1\right)^2+1}. \tag{A.24}$$

Proof. With $B^s_\Sigma(R)$ defined in (3.29), and recalling the expression (3.26) of $B_\Sigma(R)$, we have

$$B_\Sigma(\|\cdot\|_1) = \max_{1\leqslant L\leqslant n-2k} B^{2k+L}_\Sigma(\|\cdot\|_1)$$

By Lemma A.4, $\frac{R^\star(v_1)}{R^\star(v_0)} = (L/k)+1 > 1$ and $B^{2k+L}_\Sigma(\|\cdot\|_1) \geqslant \frac{\frac{L}{k}}{\left(\frac{R^\star(v_1)}{R^\star(v_0)}\right)^2+1} = \frac{\frac{L}{k}}{\left(\frac{L}{k}+1\right)^2+1}$. This implies

$$B_\Sigma(\|\cdot\|_1) \geqslant \max_{1\leqslant L\leqslant n-2k} \frac{\frac{L}{k}}{\left(\frac{L}{k}+1\right)^2+1}, \tag{A.25}$$

and there only remains to show there is indeed equality. We isolate this result from [17] for completeness. This will also help understand the case of sparsity in levels in Appendix A.6.

First, we show we can restrict the maximization used to express $B_\Sigma(\|\cdot\|_1)$ (cf (3.26)) over vectors z having constant amplitude $\alpha > 0$ on $T(z)$.

Indeed, consider $z \neq 0$ such that $z \in \mathcal{T}_{\Sigma_k}(\|\cdot\|_1)$. By Lemma A.2, we have $\|z_{T^c}\|_1 \leqslant \|z_T\|_1$ with $T = T(z)$ a set of k indices of components of largest magnitude of z. Assume that there are $i \neq j$ in T such that $|z_i| \neq |z_j|$. Let y such that $y_l = z_l$ for $l \notin \{i,j\}$ and $y_i = y_j = (|z_i|+|z_j|)/2$. The set T remains a support of the k largest amplitudes in y, and $T_2 = T_2(z)$ remains a support of the $2k$ largest amplitudes in y. Moreover, we have $\|y_T\|_1 = \|z_T\|_1 \geqslant \|z_{T^c}\|_1 = \|y_{T^c}\|_1 = \|-y_T+y\|_1$ hence we have $y \in \mathcal{T}_{\Sigma_k}(\|\cdot\|_1)$. Since $\|y_{T_2}\|_2^2 - \|z_{T_2}\|_2^2 = \|y_T\|_2^2 - \|z_T\|_2^2 = 2[(|z_i|+|z_j|)/2]^2 - |z_i|^2 - |z_j|^2 = -(|z_i|-|z_j|)^2/2 < 0$ and $\|y_{T_2^c}\|_2^2 = \|z_{T_2^c}\|_2^2$ we have $\|y_{T_2^c}\|_2^2/\|y_{T_2}\|_2^2 > \|z_{T_2^c}\|_2^2/\|z_{T_2}\|_2^2$.

Second, the same reasoning on $T' = T_2 \setminus T$, shows that we can further restrict the maximization used to define $B_\Sigma(\|\cdot\|_1)$ to vectors having constant amplitude $0 \leqslant \beta \leqslant \alpha$ over T'. This leads to

$$B_\Sigma(\|\cdot\|_1) = \sup_{z\neq 0:\|z_{T^c}\|_1\leqslant\|z_T\|_1} \frac{\|z_{T_2^c}\|_2^2}{\|z_{T_2}\|_2^2} = \sup_{\alpha,\beta:\alpha\geqslant\beta>0} \sup_{x\in\mathbb{R}^{n-2k}:\|x\|_\infty\leqslant\beta,\|x\|_1\leqslant k(\alpha-\beta)} \frac{\|x\|_2^2}{k(\alpha^2+\beta^2)}. \tag{A.26}$$

Using Lemma A.5, the supremum with respect to x is reached with vectors with the shape

$$(\underbrace{\beta,\ldots,\beta}_{L},\theta,\ \underbrace{0,\ldots,0}_{n-2k-(L+1)\geqslant 0})$$

with $0 \leqslant \theta \leqslant \beta$ and $0 \leqslant L \leqslant n-2k-1$. We deduce

$$\begin{aligned} B_\Sigma(\|\cdot\|_1) &= \sup_{\alpha,\beta:\alpha\geqslant\beta>0} \sup_{\substack{L,\theta:0\leqslant L\leqslant n-2k-1,0\leqslant\theta\leqslant\beta\\ \theta\leqslant k\alpha-(k+L)\beta}} \frac{L\beta^2+\theta^2}{k(\alpha^2+\beta^2)} \\ &= \max_{0\leqslant L\leqslant n-2k-1} \sup_{\alpha,\beta:\alpha\geqslant\beta>0} \sup_{\substack{\theta:0\leqslant\theta\leqslant\beta\\ \theta\leqslant k\alpha-(k+L)\beta}} \frac{L\beta^2+\theta^2}{k(\alpha^2+\beta^2)} \end{aligned} \tag{A.27}$$

When $0 \leqslant \beta \leqslant k\alpha - (k+L)\beta$ we have

$$\sup_{\theta:0\leqslant\theta\leqslant\beta,\theta\leqslant k\alpha-(k+L)\beta} \frac{L\beta^2+\theta^2}{k(\alpha^2+\beta^2)} = \frac{(L+1)\beta^2}{k(\alpha^2+\beta^2)} \tag{A.28}$$

while when $\beta \geqslant k\alpha - (k+L)\beta \geqslant 0$ we have

$$\sup_{\theta:0\leqslant\theta\leqslant\beta,\theta\leqslant k\alpha-(k+L)\beta} \frac{L\beta^2+\theta^2}{k(\alpha^2+\beta^2)} = \frac{L\beta^2+(k\alpha-(k+L)\beta)^2}{k(\alpha^2+\beta^2)}. \tag{A.29}$$

On the one hand, when $0 < \beta \leqslant \alpha$ satisfies $\beta \leqslant k\alpha - (k+L)\beta$ we have $\alpha \geqslant (1+(L+1)/k)\beta$ hence

$$\sup_{\theta:0\leqslant\theta\leqslant\beta,\theta\leqslant k\alpha-(k+L)\beta} \frac{L\beta^2+\theta^2}{k(\alpha^2+\beta^2)} = \frac{(L+1)\beta^2}{k(\alpha^2+\beta^2)} = \frac{(L+1)/k}{(\alpha/\beta)^2+1} \leqslant \frac{(L+1)/k}{[1+(L+1)/k]^2+1}. \tag{A.30}$$

On the other hand, when $0 < \beta \leqslant \alpha$ satisfies $\beta \geqslant k\alpha - (k+L)\beta \geqslant 0$ we have $(1+(L+1)/k))\beta \geqslant \alpha \geqslant (1+L/k)\beta$ and, denoting $g(t) := \frac{L/k+kt^2}{(1+L/k+t)^2+1}$ for $t \geqslant 0$, we get

$$\sup_{\theta:0\leqslant\theta\leqslant\beta,\theta\leqslant k\alpha-(k+L)\beta} \frac{L\beta^2+\theta^2}{k(\alpha^2+\beta^2)} = \frac{L\beta^2+(k\alpha-(k+L)\beta)^2}{k(\alpha^2+\beta^2)} = \frac{L/k+k[\alpha/\beta-(1+L/k)]^2}{(\alpha/\beta)^2+1} = g(\alpha/\beta-(1+L/k)). \tag{A.31}$$

A simple function study shows that $g'(t)$ is positively proportional to a second degree polynomial $P(t)$ with positive leading coefficient and such that $P(0) < 0$. It follows that there is $t_0 > 0$ such that $g'(t) \leqslant 0$ for $0 \leqslant t \leqslant t_0$ and $g'(t) \geqslant 0$ for $t \geqslant t_0$. Hence, g is decreasing on $[0,t_0]$ and increasing on $[t_0,+\infty)$, so that

$$g(\alpha/\beta-(1+L/k)) \leqslant \sup_{0\leqslant t\leqslant 1/k} g(t) = \max(g(0),g(1/k)) = \max\left(\frac{L/k}{(1+L/k)^2+1}, \frac{(L+1)/k}{(1+(L+1)/k)^2+1}\right).$$

As all the above bounds also hold if $\beta = 0$, we obtain the claimed result.

□

REMARK A.1 The maximum value of $\frac{L/k}{((k+L)/k)^2+1}$ (with respect to L) is reached for L/k maximizing $f(u) = u/((u+1)^2+1)$ (which is maximized at $\sqrt{2}$ over $\mathbb{R}$). We verify that it matches the necessary RIP condition $\frac{1}{\sqrt{2}}$ from [17], $f(\sqrt{2}) = 2\sqrt{2}/(2+\sqrt{2})$ which gives $\gamma_\Sigma(\|\cdot\|_1) = (4+3\sqrt{2})/\sqrt{2} = \frac{\sqrt{2}+1}{\sqrt{2}-1}$.

A.4.2 *Lemmas for the proof of Theorem 3.7* . Given a matrix U, we denote $U_{k:l}$ the restriction of U to its rows $k,\dots,l$. We denote $O(n)$ the orthogonal group. Given a symmetric matrix z, we write $\texttt{eig}(z)$ the vector of eigenvalues ordered decreasingly with respect to their absolute value. Given a vector x of size n, we write $\texttt{diag}(x)$ the diagonal matrix with diagonal equal to x. To match the notations for the case of sparsity, given a matrix $z = U^T\texttt{diag}(w)U$, we write $z_H = U^T\texttt{diag}(w_H)U$ and Q_H as in the previous section. We denote $T = \{1,..,r\}$ and $T_2 = \{1,..,2r\}$. We denote $\|\cdot\|_F$ the Frobenius norm.

Using the same demonstration as Lemma A.2 we characterize the descent cones of the nuclear norm.

LEMMA A.7 Let $\Sigma = \Sigma_r$. Let $\|\cdot\|_w$ be a weighted nuclear-norm. Let $z \in \mathcal{T}_{\|\cdot\|_w}(\Sigma)$. There is a support H of size $\leqslant r$ such that

$$\|z_{H^c}\|_w - \|z_H\|_w = \inf_{x\in\Sigma}\{\|x+z\|_w - \|x\|_w\} \leqslant 0, \tag{A.32}$$

i.e., the infimum is achieved at $x^* = -z_H$. Moreover, if $\|\cdot\|_w = \|\cdot\|_*$, $H = T(z)$.

LEMMA A.8 Let $\Sigma = \Sigma_r$ be the set of $n \times n$ symmetric matrices with rank at most r with $r < n/2$, and $1 \leqslant L \leqslant n-2r$. Assume R is positively homogeneous, subadditive and nonzero. Consider the supports $H_0 = \{1,2,..,r\}$ and $H_1 = \{r+1,\dots,2r+L\}$.

$$(U_0, v_0) \in \arg \max_{U \in O(n), v \in Q_{H_0}} \|U^T \mathtt{diag}(v) U\|_{\mathcal{A}}, \tag{A.33}$$

$$(U_1, v_1) \in \arg \min_{U \in O(n), v \in Q_{H_1} : U_{0,1:r} U^T_{r+1:2r+L} = 0} \|U^T \mathtt{diag}(v) U\|_{\mathcal{A}}. \tag{A.34}$$

1. We have $R(U_0^T v_0 U_0) > 0$, and for any H of size $r' \geqslant r$, $V \in O(n)$ and $w \in Q_H$, we have

$$R(V^T \mathtt{diag}(w) V) \leqslant \frac{r'}{r} R(U_0^T v_0 U_0). \tag{A.35}$$

   If $R = R^\star = \|\cdot\|_*$ then we have indeed equality $R(V^T \mathtt{diag}(w) V) = \frac{r'}{r} R(U_0^T v_0 U_0)$.

2. We have

$$\begin{aligned} B_\Sigma^{L+2r}(R) := \sup_{z \in \mathcal{T}_{\|\cdot\|_{\mathcal{A}}}(\Sigma) \setminus \{0\} : |\mathtt{supp}(\mathtt{eig}(z))| = 2r+L} \frac{\|z_{T_2^c}\|_F^2}{\|z_{T_2}\|_F^2} &\geqslant \frac{\frac{L}{r}}{\left(\max\left(\frac{R(U_1^T \mathtt{diag}(v_1) U_1)}{R(U_0^T \mathtt{diag}(v_0) U_0)}, 1\right)\right)^2 + 1} \\ &\geqslant \frac{\frac{L}{r}}{\left(\frac{L}{r}+1\right)^2 + 1}. \end{aligned} \tag{A.36}$$

Proof. As a preliminary observe that if $R^\star = \|\cdot\|_*$ then $R^\star(V^T w V) = |H|$ for any $H, w \in Q_H, V \in O(n)$, hence $w_i \in Q_{H_i}$ can be arbitrary, for example $w_i = 1_{H_i}$. This yields $R^\star(U_0^T \mathtt{diag}(v_0) U_0) = r$, $R^\star(U_1^T \mathtt{diag}(v_1) U_1) = r + L$, hence $R^\star(U_1^T \mathtt{diag}(v_1) U_1) = (1 + L/r) R^\star((U_0^T \mathtt{diag}(v_0) U_0)$.

To prove the first claim, consider $\{G_i\}_{1 \leqslant i \leqslant \binom{r'}{r}}$ the collection of all subsets $G_i \subseteq H$ of size exactly r. Since $w \in Q_H$, we have $w_{G_i} \in Q_{G_i}$ for each i. Also, since $|G_i| = r$ for every i, by definition of H_0, v_0 and remarking that the maximization over $O(n)$ permits to consider any permutation of the support, we obtain $\max_i R(V^T \mathtt{diag}(v_{G_i}) V) \leqslant R(U_0^T \mathtt{diag}(v_0) U_0)$.

Notice that given a coordinate $j \in H$, there are $\binom{r'-1}{r-1}$ sets G_i such that $j \in G_i$. With $\lambda := \frac{1}{\binom{r'-1}{r-1}}$, we get $V^T \mathtt{diag}(w) V = V^T \lambda \sum_i \mathtt{diag}(w_{G_i}) V$ hence by positive homogeneity and subadditivity of R (which imply convexity)

$$\begin{aligned} R(V^T w V) = R(\lambda V^T \sum_{i=1}^{\binom{r'}{r}} \mathtt{diag}(w_{G_i}) V) &\leqslant \sum_{i=1}^{\binom{r'}{r}} R(V^T \lambda \mathtt{diag}(w_{G_i}) V) = \lambda \sum_{i}^{\binom{r'}{r}} R(V^T \mathtt{diag}(w_{G_i}) V) \\ &\leqslant \frac{\binom{r'}{r}}{\binom{r'-1}{r-1}} R(U_0^T \mathtt{diag}(v_0) U_0) = \frac{r'}{r} R(U_0^T \mathtt{diag}(v_0) U_0). \end{aligned} \tag{A.37}$$

This establishes (A.35). With $R = R^\star$, we have $R^\star(V^T \mathtt{diag}(w) V) = \|w\|_1 = r'$ for $w \in Q_H$, hence $R^\star(V^T \mathtt{diag}(w) V) = (r'/r) R^\star(U_0^T \mathtt{diag}(v_0) U_0)$ as claimed.

For the sake of contradiction, assume that $R(U_0^T \mathtt{diag}(v_0) U_0) \leqslant 0$. As we have just proved, this implies $R(V^T \mathtt{diag}(w) V) \leqslant (n/k) R(U_0^T \mathtt{diag}(v_0) U_0) \leqslant 0$ for every $w \in \{-1,+1\}^n = Q_H$ with $H =$

$\{1,\dots,n\}$ and $V \in O(n)$. By convexity of R it follows that $R(V^T \mathtt{diag}(w)V) \leqslant 0$ for each $w \in [-1,1]^n = \mathrm{conv}(Q_H)$, and by positive homogeneity,

$$R(V^T \mathtt{diag}(w)V) \leqslant 0, \quad \forall w \in \mathbb{R}^n. \tag{A.38}$$

Positive homogeneity and subadditivity also imply

$$\begin{aligned} 0 = 0 \cdot R(U_0^T \mathtt{diag}(v_0)U_0) = R(0 \cdot U_0^T \mathtt{diag}(v_0)U_0) = R(0) &= R(-V^T \mathtt{diag}(w)V + V^T \mathtt{diag}(w)V) \\ &\leqslant R(-V^T \mathtt{diag}(w)V) + R(V^T \mathtt{diag}(w)V) \\ &\overset{(A.38)}{\leqslant} R(-V^T \mathtt{diag}(w)V) \end{aligned}$$

for every $V^T \mathtt{diag}(w)V \in \mathcal{H}$, hence $R(V^T \mathtt{diag}(w)V) = 0$ on $\mathcal{H}$, which yields the desired contradiction since we assume that R is nonzero.

Regarding the second claim, since $2r+L \leqslant n$, by construction, $H_1 \cap H_0 = \emptyset$. Since $R(U_0^T \mathtt{diag}(v_0)U_0) > 0$, R is positively homogeneous and the set Σ is homogeneous, and we have by Lemma A.3, $z = -\alpha U_0^T \mathtt{diag}(v_0)U_0 + U_1^T \mathtt{diag}(v_1)U_1 \in \mathcal{T}_R(\Sigma)$ with $\alpha := \max(R(U_1^T \mathtt{diag}(v_1)U_1)/R(U_0^T \mathtt{diag}(v_0)U_0), 1)$. Observe that $|\mathtt{supp}(\mathtt{eig}(z))| = |H_0| + |H_1| = 2r+L$. Since $\alpha \geqslant 1$ and all nonzero entries of v_0, v_1 have magnitude one, a set of the $2r$ largest components of $eig(z)$ is $T_2 = H_0 \cup T_1'$ with T_1' any subset of H_1 with k components, and we obtain (A.36). once we observe that

$$\frac{\|z_{T_2^c}\|_2^2}{\|z_{T_2}\|_2^2} = \frac{L}{r\alpha^2 + r} = \frac{L/r}{\alpha^2 + 1}. \tag{A.39}$$

□

LEMMA A.9 Let $\Sigma = \Sigma_r$. Then

$$B_\Sigma(\|\cdot\|_*) = \max_{0 \leqslant L \leqslant n-2r} \frac{\frac{L}{r}}{\left(\frac{L}{r}+1\right)^2 + 1}. \tag{A.40}$$

Proof.

We have $z \in \mathcal{T}_{\|\cdot\|_*}(\Sigma_r)$ is equivalent to $\|z_{T_2^c}\|_* + \|z_{T'}\|_* \leqslant \|z_T\|_*$ where $T' = \mathtt{supp}(z) \setminus (T_2^c \cup T)$ (Lemma A.7). Hence,

$$B_\Sigma^{L+2r}(\|\cdot\|_*) = \sup_{z : \|z_{T_2^c}\|_* + \|z_{T'}\|_* \leqslant \|z_T\|_*} \frac{\|z_{T_2^c}\|_F^2}{\|z_{T_2}\|_F^2}. \tag{A.41}$$

Using the fact that $\|z\|_* = \|\mathtt{eig}(z)\|_1$ and $\|z\|_F = \|\mathtt{eig}(z)\|_2$, we fall on the expression of $B_\Sigma^{L+2r}(\|\cdot\|_1)$ and get the result using Lemma A.6.

□

A.5 *Proofs for Section 3.3*

Proof of Lemma 3.5. The constant $\delta_\Sigma^{\mathtt{suff}}(R)$ [38][Eq. (5)] has the following expression:

$$\delta_\Sigma^{\mathtt{suff}}(R) = \inf_{z \in \mathcal{T}_R(\Sigma) \setminus \{0\}} \sup_{x \in \Sigma} \frac{-\mathcal{R}e\langle x, z\rangle}{\|x\|_{\mathcal{H}} \sqrt{\|x+z\|_\Sigma^2 - \|x\|_{\mathcal{H}}^2 - 2\mathcal{R}e\langle x, z\rangle}}. \tag{A.42}$$

Considering any nonzero $z \in \mathcal{H}$, since Σ is a union of subspaces and $\Sigma \cap S(1)$ is compact, by Lemma 3.4 the set $P_\Sigma(z)$ is not empty and $\langle P_\Sigma(z), z\rangle = \|P_\Sigma(z)\|_{\mathcal{H}}^2$ is unambiguous. Choosing an arbitrary $y \in P_\Sigma(z)$ and setting $x = -y$, we obtain

$$\sup_{x\in\Sigma} \frac{-\mathcal{R}e\langle x,z\rangle}{\|x\|_{\mathcal{H}}\sqrt{\|x+z\|_\Sigma^2 - \|x\|_{\mathcal{H}}^2 - 2\mathcal{R}e\langle x,z\rangle}} \geqslant \frac{\|P_\Sigma(z)\|_{\mathcal{H}}^2}{\|P_\Sigma(z)\|_{\mathcal{H}}\sqrt{\|z-P_\Sigma(z)\|_\Sigma^2 - \|P_\Sigma(z)\|_{\mathcal{H}}^2 + 2\|P_\Sigma(z)\|_{\mathcal{H}}^2}}$$
$$= \frac{1}{\sqrt{\sup\limits_{z\in\mathcal{T}_R(\Sigma)\setminus\{0\}} \frac{\|z-P_\Sigma(z)\|_\Sigma^2}{\|P_\Sigma(z)\|_{\mathcal{H}}^2} + 1}}.$$

Considering the infimum over $z \in \mathcal{T}_R(\Sigma)\setminus\{0\}$ yields the first claim. Let us now proceed to the second claim.

Given $z \in \mathcal{T}_R(\Sigma)\setminus\{0\}$, consider an arbitrary $x \in \Sigma$, and $V \in \mathcal{V}$ such that $x \in V$. With Fact A.2, for every $v \in \mathcal{H}$, $\|v\|_\Sigma^2$ is the infimum of $\sum_i \lambda_i \|u_i\|_{\mathcal{H}}^2$ over convex decompositions $v = \sum_i \lambda_i u_i$ over Σ, hence there exists $u_i \in \Sigma$, $\lambda_i \geqslant 0$ such that $\sum_i \lambda_i = 1$, $\sum_i \lambda_i u_i = x + z$ and

$$\|x+z\|_\Sigma^2 = \sum_i \lambda_i \|u_i\|_{\mathcal{H}}^2.$$

Since $V \subset \Sigma$, $u_{i,V} := P_V u_i \in \Sigma$. By the additional assumption, since $u_i \in \Sigma$ we also have and $u_{i,V^\perp} := P_{V^\perp} u_i \in \Sigma$ for each i. Observe also that $P_{V^\perp} x = 0$. Hence, with the notations $z_V = P_V z$, $z_{V^\perp} = P_{V^\perp} z$, we have the convex decompositions

$$z_{V^\perp} = P_{V^\perp}(x+z) = \sum \lambda_i u_{i,V^\perp}$$
$$x + z_V = P_V(x+z) = \sum \lambda_i u_{i,V}.$$

Using Jensen's inequality for the convex functions $\|\cdot\|_\Sigma^2$ and $\|\cdot\|_{\mathcal{H}}^2$ and the identity $\|v\|_\Sigma^2 = \|v\|_{\mathcal{H}}^2$ for $v \in \Sigma$ (Fact A.1), we have

$$\|z_{V^\perp}\|_\Sigma^2 + \|x+z_V\|_{\mathcal{H}}^2 \leqslant \sum_i \lambda_i \|u_{i,V^\perp}\|_\Sigma^2 + \sum_i \lambda_i \|u_{i,V}\|_{\mathcal{H}}^2 = \sum_i \lambda_i \|u_{i,V^\perp}\|_{\mathcal{H}}^2 + \sum_i \lambda_i \|u_{i,V}\|_{\mathcal{H}}^2$$
$$= \sum_i \lambda_i \|u_i\|_{\mathcal{H}}^2 = \|x+z\|_\Sigma^2.$$

Since P_V is the (linear and self-adjoint) orthogonal projection onto V, we have $\mathcal{R}e\langle x, z_V\rangle = \mathcal{R}e\langle x, P_V z\rangle = \mathcal{R}e\langle P_V x, z\rangle = \mathcal{R}e\langle x, z\rangle$, and we obtain

$$\begin{aligned}
\|z_{V^\perp}\|_\Sigma^2 + \|z_V\|_{\mathcal{H}}^2 &\leqslant \|x+z\|_\Sigma^2 - \|x+z_V\|_{\mathcal{H}}^2 + \|z_V\|_{\mathcal{H}}^2 \\
\|z_{V^\perp}\|_\Sigma^2 + \|z_V\|_{\mathcal{H}}^2 &\leqslant \|x+z\|_\Sigma^2 - \|x\|_{\mathcal{H}}^2 - 2\mathcal{R}e\langle x, z_V\rangle \\
\|z_{V^\perp}\|_\Sigma^2 + \|z_V\|_{\mathcal{H}}^2 &\leqslant \|x+z\|_\Sigma^2 - \|x\|_{\mathcal{H}}^2 - 2\mathcal{R}e\langle x, z\rangle.
\end{aligned} \tag{A.43}$$

Using Cauchy-Schwarz inequality, we have $(\mathcal{R}e(\langle x,z\rangle))^2 = (\mathcal{R}e(\langle x, z_V\rangle))^2 \leqslant \|x\|_{\mathcal{H}}^2 \|z_V\|_{\mathcal{H}}^2$. Denoting V_0 such that $P_{V_0}(z) \in P_\Sigma(z)$, we get

$$(\mathcal{R}e\langle x,z\rangle)^2\left(\|z_{V^\perp}\|_\Sigma^2+\|z_V\|_{\mathcal{H}}^2\right)\leqslant\|x\|_{\mathcal{H}}^2\|z_V\|_{\mathcal{H}}^2\left(\|x+z\|_\Sigma^2-\|x\|_{\mathcal{H}}^2-2\mathcal{R}e\langle x,z\rangle\right)$$

$$\frac{(\mathcal{R}e\langle x,z\rangle)^2}{\|x\|_{\mathcal{H}}^2\left(\|x+z\|_\Sigma^2-\|x\|_{\mathcal{H}}^2-2\mathcal{R}e\langle x,z\rangle\right)}\leqslant\frac{\|z_V\|_{\mathcal{H}}^2}{\left(\|z_{V^\perp}\|_\Sigma^2+\|z_V\|_{\mathcal{H}}^2\right)}=\frac{1}{\frac{\|z_{V^\perp}\|_\Sigma^2}{\|z_V\|_{\mathcal{H}}^2}+1}\leqslant\frac{1}{\frac{\|z-P_Vz\|_\Sigma^2}{\|P_\Sigma(z)\|_{\mathcal{H}}^2}+1},$$

where the last inequality (we could use here the weaker alternative assumption $P_\Sigma(z)\cap\arg\min_{x\in\Sigma}\|x-z\|_\Sigma/\|x\|_{\mathcal{H}}\neq\emptyset$) uses that $z_{V^\perp}=z-P_Vz$ and $\|P_{V_0}z\|_{\mathcal{H}}=\|P_\Sigma(z)\|_{\mathcal{H}}\geqslant\|P_V(z)\|_{\mathcal{H}}=\|z_V\|_{\mathcal{H}}$. To conclude, we use the additional hypothesis $P_\Sigma(z)\subseteq\arg\min_{x\in\Sigma}\|x-z\|_\Sigma$, which implies $\|z-P_\Sigma(z)\|_\Sigma\leqslant\|z-P_Vz\|_\Sigma$ since $P_Vz\in\Sigma$

$$\sup_{x\in\Sigma}\frac{-\mathcal{R}e\langle x,z\rangle}{\|x\|_{\mathcal{H}}\sqrt{\|x+z\|_\Sigma^2-\|x\|_{\mathcal{H}}^2-2\mathcal{R}e\langle x,z\rangle}}\leqslant\frac{1}{\sqrt{\sup\limits_{z\in\mathcal{T}_R(\Sigma)\setminus\{0\}}\frac{\|z-P_\Sigma(z)\|_\Sigma^2}{\|P_\Sigma(z)\|_{\mathcal{H}}^2}+1}}.$$

□

To replicate the proof used in the necessary case, we show a monotonicity property of $\|\cdot\|_\Sigma$.

LEMMA A.10 Consider a model set $\Sigma\subset\mathcal{H}$, $\|\cdot\|_\Sigma$ the atomic "norm" induced by Σ, and $D:\mathcal{H}\to\mathcal{H}$ a linear operator. If $D\Sigma\subseteq\Sigma$ and $\|D\|_{op}:=\sup_{\|v\|_{\mathcal{H}}\leqslant1}\|Dv\|_{\mathcal{H}}\leqslant1$ then

$$\|Dv\|_\Sigma\leqslant\|v\|_\Sigma,\quad\forall v\in\mathcal{H}. \tag{A.44}$$

Proof. Let λ_i,u_i such that $u_i\in\Sigma$, $\sum_i\lambda_i=1$, $\sum_i\lambda_iu_i=v$. Denoting $u_i'=Du_i$ we have $u_i'\in\Sigma$ and $Dv=\sum\lambda_iu_i'$. By Jensen's inequality and the fact that $\|u\|_\Sigma=\|u\|_{\mathcal{H}}$ for any $u\in\Sigma$ (Fact A.1), it follows that

$$\|Dv\|_\Sigma^2\leqslant\sum\lambda_i\|u_i'\|_\Sigma^2=\sum\lambda_i\|u_i'\|_{\mathcal{H}}^2=\sum\lambda_i\|Du_i\|_{\mathcal{H}}^2\leqslant\sum\lambda_i\|u_i\|_{\mathcal{H}}^2. \tag{A.45}$$

With Fact A.2, $\|v\|_\Sigma^2$ is the infimum of the right-hand side over all such decompositions $v=\sum\lambda_iu_i$.

□

COROLLARY A.1 With $\Sigma:=\Sigma_k$ the set of k-sparse vectors in $\mathcal{H}=\mathbb{R}^n$, we have:

1. the norm $\|\cdot\|_\Sigma$ is invariant by permutation and coordinate sign changes;
2. for any vectors $v,v'\in\mathcal{H}$ such that $|v_j|\leqslant|v_j'|$ for all j we have $\|v\|_\Sigma\leqslant\|v'\|_\Sigma$;
3. consider any vector z, and T_k a subset indexing k components of the largest magnitude, *i.e.,* $\min_{i\in T}|z_i|\geqslant\max_{j\notin T}|z_j|$, with $|T|=k$. Then

$$\max_{|T|\leqslant k}\|z_T\|_\Sigma = \|z_{T_k}\|_\Sigma \tag{A.46}$$

$$\min_{|T|\leqslant k}\|z-z_T\|_\Sigma = \|z-z_{T_k}\|_\Sigma. \tag{A.47}$$

Proof. We show the three properties separately.

- **Property 1:** Let π be a permutation of $(1,\dots,n)$ and $\varepsilon_1,\dots,\varepsilon_n \in \{\pm 1\}$. Define D by $(Du)_i = \varepsilon_i u_{\pi(i)}$. Observe that $D\Sigma_k \subseteq \Sigma_k$ and $\|D\|_{op} = 1$. Conclude using Lemma A.10 that $\|Du\|_\Sigma \leqslant \|u\|_\Sigma$ for any $u \in \mathcal{H}$. The same holds with $D' = D^{-1}$, hence $\|u\|_\Sigma = \|D^{-1}Du\|_\Sigma \leqslant \|Du\|_\Sigma$ for any u. This shows $\|D\cdot\|_\Sigma = \|\cdot\|_\Sigma$.

- **Property 2:** Given the assumptions on v, v', the linear operator defined by $(Du)_i = v_i u_i / v'_i$ if $v'_i \neq 0$ (and $(Du)_i = 0$ otherwise) satisfies $D\Sigma \subseteq \Sigma$ and $\|D\|_{op} \leqslant 1$ hence, using Lemma A.10 again, $\|v\|_\Sigma = \|Dv'\|_\Sigma \leqslant \|v'\|_\Sigma$.

- **Property 3:** By the invariance by permutation and coordinate sign changes of $\|\cdot\|_\Sigma$, it is sufficient to prove the result when $z_1 \geqslant \dots \geqslant z_n \geqslant 0$ and $T_k = \{1,\dots,k\}$. Given T of size k, there is a permutation ϕ of $(1,\dots,n)$ such that $T = \{\phi(1),\dots,\phi(k)\}$ where $\phi(1) < \dots < \phi(k)$. It follows that $z_{\phi(i)} \leqslant z_i$ for $1 \leqslant i \leqslant k$. Hence by Property 2, we have $\|z_T\|_\Sigma = \|(z_{\phi(1)},\dots,z_{\phi(k)},0,\dots,0)\|_\Sigma \leqslant \|(z_1,\dots,z_k,0,\dots,0)\|_\Sigma = \|z_{T_k}\|_\Sigma$. A similar argument using T^c yields $\|z - z_T\|_\Sigma \geqslant \|z - z_{T_k}\|_\Sigma$.

□

COROLLARY A.2 With $\Sigma := \Sigma_r$ the set of matrices of rank lower than r in $\mathcal{H}$ the set of symmetric matrices in $\mathbb{R}^{n\times n}$, we have:

1. for any matrices $V^T\mathtt{diag}(w)V, V^T\mathtt{diag}(w')V$ with $V \in O(n)$ such that $|w_j| \leqslant |w'_j|$ for all j we have $\|V^T\mathtt{diag}(w)V\|_\Sigma \leqslant \|V^T\mathtt{diag}(w')V\|_\Sigma$;

2. For any symmetric matrix z, and T_r a subset indexing r components of largest magnitude of $\mathtt{eig}(z)$, *i.e.*,
$$\min_{i\in T} |\mathtt{eig}(z)_i| \geqslant \max_{j\notin T} |\mathtt{eig}(z)_j|,$$
with $|T| = r$. Then
$$\max_{|T|\leqslant r} \|z_T\|_\Sigma = \|z_{T_r}\|_\Sigma \tag{A.48}$$
$$\min_{|T|\leqslant r} \|z - z_T\|_\Sigma = \|z - z_{T_r}\|_\Sigma. \tag{A.49}$$

Proof. We show the two properties separately.

- **Property 1:** Given the assumptions on w, w', the linear operator defined by $Dz = V^TWVz$ where W is the diagonal matrix such that $W_{ii} = w_i/w'_i$ if $w'_i \neq 0$ (and $W_{ii} = 0$ otherwise) satisfies $D\Sigma \subseteq \Sigma$ and $\|D\|_{op} \leqslant 1$. We have $D(V^T\mathtt{diag}(w')V) = V^TWw'V = V^TwV$. With Lemma A.10, we get $\|V^T\mathtt{diag}(w)V\|_\Sigma = \|D(V^T\mathtt{diag}(w')V)\|_\Sigma \leqslant \|V^T\mathtt{diag}(w')V\|_\Sigma$.

- **Property 2:** This property is direct using the eigenvalue decomposition
$$z = U^T\mathtt{diag}(\mathtt{eig}(z))U^T = U^T\mathtt{diag}(\mathtt{eig}(z)_T + \mathtt{eig}(z)_{T^c})U^T$$
and Property 1.

□ We now prove Lemma 3.6. *Proof of Lemma 3.6.* Consider first $\Sigma = \Sigma_k$. First, the properties of $\|\cdot\|_\Sigma$ established in Corollary A.1 directly show that the minimum of $\|x - z\|_\Sigma$ with respect to $x \in \Sigma$ is reached at any $x \in P_\Sigma(z)$. Then, we can write $\Sigma = \cup_{V\in\mathcal{V}} V$

where $V \in \mathcal{V}$ if, and only if there is an index set $T \subseteq \{1,\dots,n\}$ such that $|T| \leqslant k$ and $V = \operatorname{span}(e_i)_{i\in T}$. Given $V \in \mathcal{V}$ and $u \in \Sigma_k$, let us show that $P_{V^\perp} u \in \Sigma_k$. Writing $V = \operatorname{span}(e_i)_{i\in T}$ where $|T| \leqslant k$, we have $P_V(u) = u_T$ and $P_{V^\perp}(u) = u_{T^c}$. As $\mathtt{supp}(u_{T^c}) \subseteq \mathtt{supp}(u)$ it follows that $\|u_{T^c}\|_0 \leqslant k$, hence $P_{V^\perp}(u) \in \Sigma_k$.

In the case of low rank matrices $\Sigma = \Sigma_r$. We take $\mathcal{V} = \{\operatorname{span}(U_i)_{i\in I}, |I| \leqslant r, \|U_i\|_F = 1, \mathtt{rank}(U_i) = 1, \langle U_i, U_j\rangle = 0, i \neq j\}$. With Corollary A.2, the minimum of $\|x-z\|_\Sigma$ with respect to $x \in \Sigma$ is reached at any $x \in P_\Sigma(z)$. Let $z \in \Sigma_r$ and $V \in \mathcal{V}$. We have $P_V(z) = V_1^T S_1 V_1$ has rank r' lower than r. We can write $z = V_1^T S_1 V_1 + V_2^T S_2 V_2$ with $V_1 V_2^T = 0$. Hence, $P_{V^\perp}(z)$ has rank at most $r - r' \leqslant r$ and $P_{V^\perp}(z) \in \Sigma_r$ otherwise z would be of rank greater than r.

□ We need the following Lemma to control $\|\cdot\|_\Sigma$.

LEMMA A.11 Let $\Sigma = \Sigma_k \subset \mathbb{R}^n$. Then for any v

$$\|v\|_\Sigma^2 \geqslant \frac{\|v\|_1^2}{k}. \tag{A.50}$$

Let $\Sigma = \Sigma_r$. Then for any v

$$\|v\|_\Sigma^2 \geqslant \frac{\|v\|_*^2}{r}. \tag{A.51}$$

Proof.

Case $\Sigma = \Sigma_k$: Let $\lambda_i \geqslant 0, u_i \in \Sigma$ such that $\|v\|_\Sigma^2 = \sum \lambda_i \|u_i\|_2^2$ and $v = \sum \lambda_i u_i$ from Fact A.2. We have, by convexity

$$\|v\|_1 = \left\| \sum_i \lambda_i u_i \right\|_1 \leqslant \sum_i \lambda_i \|u_i\|_1. \tag{A.52}$$

Using the fact that $\|x\|_1 \leqslant \sqrt{k}\|x\|_2$ if $|\mathtt{supp}(x)| \leqslant k$ and the concavity of the square root,

$$\|v\|_1 \leqslant \sqrt{k} \sum_i \lambda_i \|u_i\|_2 \leqslant \sqrt{k} \sqrt{\sum_i \lambda_i \|u_i\|_2^2} = \sqrt{k}\|v\|_\Sigma. \tag{A.53}$$

Case $\Sigma = \Sigma_r$: Let $\lambda_i \geqslant 0, u_i \in \Sigma$ such that $\|v\|_\Sigma^2 = \sum \lambda_i \|u_i\|_F^2$ and $v = \sum \lambda_i u_i$ from Fact A.2. We have, by convexity

$$\|v\|_1 = \left\| \sum_i \lambda_i u_i \right\|_* \leqslant \sum_i \lambda_i \|u_i\|_*. \tag{A.54}$$

Using the fact that $\|x\|_* \leqslant \sqrt{r}\|x\|_F$ if $\mathtt{rank}(x) \leqslant r$ and the concavity of the square root,

$$\|v\|_* \leqslant \sqrt{k} \sum_i \lambda_i \|u_i\|_F \leqslant \sqrt{k} \sqrt{\sum_i \lambda_i \|u_i\|_F^2} = \sqrt{k}\|v\|_\Sigma. \tag{A.55}$$

□

A.5.1 *Sparsity.* We prove several intermediates lemmas to obtain $D_\Sigma(\|\cdot\|_1)$.

LEMMA A.12 Consider $\Sigma = \Sigma_k$ the set of k-sparse vectors in $\mathcal{H} = \mathbb{R}^n$, and $0 \leqslant L \geqslant n-k$. We have

$$D_\Sigma^{k+L}(\|\cdot\|_1) := \sup_{z \in \mathcal{T}_{\|\cdot\|_1}(\Sigma)\setminus\{0\} : |\mathtt{supp}(z)| = k+L} \frac{\|z_{T^c}\|_\Sigma^2}{\|z_T\|_2^2} = \min\left(1, \frac{L}{k}\right). \tag{A.56}$$

Proof. It was already proven in [38, Theorem 4.1] that $\delta_\Sigma^{\texttt{suff}}(\|\cdot\|_1) \geqslant \frac{1}{\sqrt{2}}$ hence by Lemma 3.5

$$\sup_{z \in \mathcal{T}_{\|\cdot\|_1}(\Sigma)\setminus\{0\}} \frac{\|z_{T^c}\|_\Sigma^2}{\|z_T\|_2^2} = D_\Sigma(\|\cdot\|_1) \leqslant 1. \tag{A.57}$$

Hence, $D_\Sigma^{k+L}(\|\cdot\|_1) \leqslant 1$

Consider H_0 of cardinality k, H_1 of cardinality L such that $H_0 \cap H_1 = \emptyset$ (this is possible as $k+L \leqslant n$), and define $z = \alpha 1_{H_0} + 1_{H_1}$ where $\alpha = \max(1, L/k)$. As $\alpha \geqslant 1$, a set of the k largest components of z is $T = H_0$. Moreover, $\|z_{H_0}\|_1 = \alpha k = \max(k, L) \geqslant L = \|z_{H_1}\|_1 = \|z_{H_0^c}\|_1$.

We distinguish two cases:

- **Case 1:** $L \geqslant k$, from Lemma A.11, $\|z_{T^c}\|_\Sigma^2 \geqslant \frac{1}{k}\|z_{H_1}\|_1^2 = L^2/k$. Moreover, $\|z_T\|_2^2 = k\alpha^2 = L^2/k$, thus $\|z_{T^c}\|_\Sigma^2/\|z_T\|_2^2 \geqslant 1$. Combining with (A.57) yields $D_\Sigma^{k+L}(\|\cdot\|_1) = 1 = \min(1, L/k)$.
- **Case 2:** $L < k$**,** we have $z_{T^c} = z_{H_1} \in \Sigma_k$ hence $\|z_{T^c}\|_\Sigma^2 = \|z_{T^c}\|_2^2 = \|z_{H_1}\|_2^2 = L$ and $\|z_{T^c}\|_\Sigma^2/\|z_T\|_2^2 = L/k$. This shows that $D_\Sigma^{k+L}(\|\cdot\|_1) \geqslant L/k = \min(1, L/k)$. To conclude, we show that $D_\Sigma^{k+L}(\|\cdot\|_1) \leqslant L/k$. Consider *any* $z' \in \mathcal{T}_{\|\cdot\|_1}(\Sigma)$ such that $|\texttt{supp}(z')| = k+L$, with Lemma A.2, there is a support H of size lower than k such that, $\|z'_H\|_1 \geqslant \|z'_{H^c}\|_1$, let T a set of the k largest components of z'. We have $\|z'_T\|_1 - \|z'_{T^c}\|_1 \geqslant \|z'_H\|_1 - \|z'_{H^c}\|_1$. As $\|z'\|_0 \leqslant k+L$ and $L < k$, $z'_{T^c} \in \Sigma_L \subset \Sigma_k$ hence $\|z'_{T^c}\|_\Sigma = \|z'_{T^c}\|_2$. Moreover, $|z'_i| \geqslant \|z'_{T^c}\|_\infty$ for any $i \in T$, hence $\|z'_T\|_2^2 \geqslant k\|z'_{T^c}\|_\infty^2$. As a result

$$\frac{\|z'_{T^c}\|_\Sigma^2}{\|z'_T\|_2^2} = \frac{\|z'_{T^c}\|_2^2}{\|z'_T\|_2^2} \leqslant \frac{L\|z'_{T^c}\|_\infty^2}{k\|z'_{T^c}\|_\infty^2} = L/k.$$

□

LEMMA A.13 Let $\Sigma = \Sigma_k$ be the set of k-sparse vectors in $\mathbb{R}^n$ with $k < n/2$ and $1 \leqslant L \leqslant n-k$. Assume that R is positively homogeneous, subadditive and nonzero.

Consider

$$(H_0, v_0) \in \arg \max_{\substack{H \subseteq \{1,\dots,n\}:\ |H|=k \\ v \in Q_H}} R(v) \tag{A.58}$$

$$(H_1, v_1) \in \arg \min_{\substack{H \subseteq \{1,\dots,n\}\setminus H_0, |H|=L \\ v \in Q_H}} R(v). \tag{A.59}$$

We have

$$D_\Sigma^{k+L}(R) := \sup_{z \in \mathcal{T}_R(\Sigma)\setminus\{0\}:|\texttt{supp}(z)|=k+L} \frac{\|z_{T^c}\|_\Sigma^2}{\|z_T\|_2^2} \geqslant \min\left(1, \frac{L}{k}\right). \tag{A.60}$$

Proof. From Lemma A.4, $R^\star(v_1) = \frac{L}{k}R^\star(v_0)$. Since $k+L \leqslant n$ there is indeed some H of cardinality L such that $H \cap H_0 = \emptyset$, hence H_1 is well-defined. By construction, $H_1 \cap H_0 = \emptyset$. From Lemma A.4, we also have $R(v_0) > 0$ and $R(v_1)/R(v_0) \leqslant L/k$.

Since $R(v_0) > 0$, R is positively homogeneous and Σ is homogeneous, by Lemma A.3, $z = -\alpha v_0 + v_1 \in \mathcal{T}_R(\Sigma)$ with $\alpha := \max(R(v_1)/R(v_0), 1)$. Observe that $|\texttt{supp}(z)| = |H_0| + |H_1| = k+L$. Since $\alpha \geqslant 1$ and all nonzero entries of v_0, v_1 have magnitude one, a set of the k largest components of z is $T = H_0$. We have

$$\frac{\|z_{T^c}\|_\Sigma^2}{\|z_T\|_2^2} = \frac{\|v_1\|_\Sigma^2}{k\alpha^2}. \tag{A.61}$$

With Lemma A.11, $\|v_1\|_\Sigma^2 \geqslant \frac{\|v_1\|_1^2}{k} \geqslant \frac{L^2}{k}$ if $L \geqslant k$ and $\|v_1\|_\Sigma^2 = \|v_1\|_2^2$ otherwise (Fact A.1). If $L \geqslant k$

$$\frac{\|z_{T^c}\|_\Sigma^2}{\|z_T\|_2^2} \geqslant \frac{L^2}{k^2\alpha^2} \geqslant \frac{L^2}{k^2 \max(L/k,1)^2} = 1. \tag{A.62}$$

If $L < k$,

$$\frac{\|z_{T^c}\|_\Sigma^2}{\|z_T\|_2^2} = \frac{L}{k\alpha^2} \geqslant \frac{L}{k} \tag{A.63}$$

which leads to the conclusion.

□

A.5.2 *Low rank.*

LEMMA A.14 Consider $\Sigma = \Sigma_r$ the set of symmetric matrices of rank lower than r. For any $L \geqslant 0$ such that $r+L \leqslant n$ we have,

$$D_\Sigma^{r+L}(\|\cdot\|_*) := \sup_{z \in \mathcal{T}_{\|\cdot\|_*}(\Sigma)\setminus\{0\} : rank(z) = r+L} \frac{\|z_{T^c}\|_\Sigma^2}{\|z_T\|_F^2} = \min\left(1, \frac{L}{r}\right) \tag{A.64}$$

where z_T is z restricted to its r biggest eigenvalues, and $z_{T^c} = z - z_T$

Proof. It was already proven in [38, Theorem 4.1] that $\delta_\Sigma^{\texttt{suff}}(\|\cdot\|_*) \geqslant \frac{1}{\sqrt{2}}$ hence by Lemma 3.5

$$\sup_{z \in \mathcal{T}_{\|\cdot\|_*}(\Sigma)\setminus\{0\}} \frac{\|z_{T^c}\|_\Sigma^2}{\|z_T\|_F^2} = D_\Sigma(\|\cdot\|_*) \leqslant 1. \tag{A.65}$$

Consider $H_0 = \{1,..r\}$, $H_1 = \{r+1,..,r+L\}$, let $U \in O(n)$ and define $z = U^T \texttt{diag}(\alpha 1_{H_0} + 1_{H_1})U$ where $\alpha = \max(1, L/r)$. As $\alpha \geqslant 1$, a set of the r largest components of $\texttt{eig}(z)$ is $T = H_0$. Moreover, $\|z_T\|_* = \alpha r = \max(r, L) \geqslant L = \|z - z_T\|_* = \|z_{T^c}\|_*$.

If $L \geqslant r$, from Lemma A.11, $\|z_{T^c}\|_\Sigma^2 \geqslant \frac{1}{r}(\|z_{T^c}\|_*)^2 = L^2/r$. Moreover, $\|z_T\|_F^2 = r\alpha^2 = L^2/r$, thus $\|z_{T^c}\|_\Sigma^2 / \|z_T\|_F^2 \geqslant 1$. Combining with (A.65) yields $D_L(\|\cdot\|_*) = 1 = \min(1, L/r)$.

If $L < r$, we have $z_{T^c} \in \Sigma_r$ hence $\|z_{T^c}\|_\Sigma^2 = L$ and $\|z_{T^c}\|_\Sigma^2 / \|z_T\|_2^2 = L/r$. This shows that $D_L(\|\cdot\|_*) \geqslant L/r = \min(1, L/r)$. To conclude, we show that $D_L(\|\cdot\|_*) \leqslant L/r$. Consider *any* $z' \in \mathcal{T}_{\|\cdot\|_*}(\Sigma)$ such that $|\texttt{supp}(z')| = r+L$, with Lemma A.7, there is a support r' and $H = 1,..,r'$ such that $\|z'_H\|_* \geqslant \|z'_{H^c}\|_*$, let T a set of r largest components of z'. We have $\|z'_T\|_* - \|z'_{T^c}\|_* \geqslant \|z'_H\|_* - \|z'_{H^c}\|_*$. As $\|\texttt{eig}(z')\|_0 \leqslant r+L$ and $L < r$, $z'_{T^c} \in \Sigma_L \subset \Sigma_r$ hence $\|z'_{T^c}\|_\Sigma = \|z'_{T^c}\|_F$. Moreover, $|\texttt{eig}(z')_i| \geqslant \|\texttt{eig}(z'_{T^c})\|_\infty$ for any $i \in T$, hence $\|z'_T\|_F^2 \geqslant r\|\texttt{eig}(z'_{T^c})\|_\infty^2$. As a result

$$\frac{\|z'_{T^c}\|_\Sigma^2}{\|z'_T\|_F^2} = \frac{\|z'_{T^c}\|_F^2}{\|z'_T\|_F^2} \leqslant \frac{L\|\texttt{eig}(z'_{T^c})\|_\infty^2}{r\|\texttt{eig}(z'_{T^c})\|_\infty^2} = L/r.$$

□

LEMMA A.15 Let $\Sigma = \Sigma_r$ be the set of $n \times n$ symmetric matrices with rank at most r with $r < n/2$, and $1 \leqslant L \leqslant n - r$. Assume R is positively homogeneous, subadditive and nonzero. Consider the supports $H_0 = \{1,2,..,r\}$ and $H_1 = \{r+1,\ldots,r+L\}$.

$$(U_0, v_0) \in \arg\max_{U \in O(n), v \in Q_{H_0}} \|U^T \texttt{diag}(v) U\|_{\mathcal{A}} \tag{A.66}$$

$$(U_1, v_1) \in \arg\min_{U \in O(n), v \in Q_{H_1} : U_{0,1:r} U_{r+1:r+L}^T = 0} \|U^T \texttt{diag}(v) U\|_{\mathcal{A}}. \tag{A.67}$$

We have

$$D_{\Sigma}^{r+L}(R) := \sup_{z \in \mathcal{T}_R(\Sigma)\setminus\{0\} : |\mathtt{supp}(z)|=r+L} \frac{\|z_{T^c}\|_{\Sigma}^2}{\|z_T\|_F^2} \geqslant \min\left(1, \frac{L}{r}\right). \tag{A.68}$$

Proof. From Lemma A.8, $R^\star(U_1^T \mathtt{diag}(v_1) U_1 = \frac{L}{r} R^\star(U_0^T \mathtt{diag}(v_0) U_0)$, $R(U_0^T \mathtt{diag}(v_0) U_0) > 0$ and

$$R(U_1^T \mathtt{diag}(v_1) U_1) / R(U_0^T \mathtt{diag}(v_0) U_0) \leqslant L/r.$$

Since $R(v_0) > 0$, R is positively homogeneous and Σ is homogeneous, by Lemma A.3, $z = -\alpha U_0^T \mathtt{diag}(v_0) U_0 + U_1^T \mathtt{diag}(v_1) U_1 \in \mathcal{T}_R(\Sigma)$ with $\alpha := \max(R(U_1^T \mathtt{diag}(v_1) U_1)/R(U_0^T \mathtt{diag}(v_0) U_0), 1)$. Observe that $|\mathtt{supp}(\mathtt{eig}(z))| = |H_0| + |H_1| = r + L$. Since $\alpha \geqslant 1$ and all nonzero entries of v_0, v_1 have magnitude one, a set of the r largest components of z is $T = H_0$. We have

$$\frac{\|z_{T^c}\|_{\Sigma}^2}{\|z_T\|_F^2} = \frac{\|U_1^T \mathtt{diag}(v_1) U_1\|_{\Sigma}^2}{r\alpha^2}. \tag{A.69}$$

With Lemma A.11, we have

$$\begin{cases} \|U_1^T \mathtt{diag}(v_1) U_1\|_{\Sigma}^2 \geqslant \frac{1}{r} \|U_1^T \mathtt{diag}(v_1) U_1\|_*^2 = \frac{L^2}{r} & \text{if } L \geqslant r \\ \|U_1^T \mathtt{diag}(v_1) U_1\|_{\Sigma}^2 = \|U_1^T \mathtt{diag}(v_1) U_1\|_F^2 & \text{otherwise (Fact A.1).} \end{cases} \tag{A.70}$$

If $L \geqslant r$

$$\frac{\|z_{T^c}\|_{\Sigma}^2}{\|z_T\|_F^2} \geqslant \frac{L^2}{r^2\alpha^2} \geqslant \frac{L^2}{r^2 \max(L/r, 1)^2} = 1. \tag{A.71}$$

If $L < r$,

$$\frac{\|z_{T^c}\|_{\Sigma}^2}{\|z_T\|_F^2} = \frac{L}{r\alpha^2} \geqslant \frac{L}{r} \tag{A.72}$$

which leads to the conclusion.

□

A.6 *Proofs for Section 4*

We extend notations for classical sparsity to sparsity in levels ($\Sigma = \Sigma_{k_1,k_2}$). For $z = (z_1, z_2) \in \mathcal{H}$, we we define the following projections $P_1(z) := z_1$ and $P_2(z) := z_2$ and denote $T = (S_1, S_2) = T(z)$ where for $i \in \{1,2\}$, $S_i \subseteq \{1, \ldots, n_i\}$ is a support containing k_i largest coordinates (in absolute value) of z_i, i.e. $|S_i| = k_i$ and $\min_{j \in S_1} |z_{i,j}| \geqslant \max_{j \in S_i^c} |z_{i,j}|$. For every $U = (U_1, U_2)$ where $U_i \subseteq \{1, \ldots, n_i\}$ and $|U_i| = k_i$, we also have $\|(z_i)_{S_i}\|_1 \geqslant \|(z_i)_{U_i}\|_1$ hence $\|z_T\|_w \geqslant \|z_U\|_w$ and similarly $\|z_{T^c}\|_w \leqslant \|z_{U^c}\|_w$.

We define similarly $T_2 = T_2(z) = (S_1', S_2')$ with S_i' containing the $2k_i$ largest coordinates of z_i. We begin by simplifying the condition $z \in \mathcal{T}_{\|\cdot\|_w}(\Sigma) \setminus \{0\}$.

LEMMA A.16 Let $w = (w_1, w_2) \in \mathbb{R}_+^2$. Let $\|\cdot\|_w = w_1 \|P_1(\cdot)\|_1 + w_2 \|P_2(\cdot)\|_1$ Let $z \in \mathcal{T}_{\|\cdot\|_w}(\Sigma_{k_1,k_2}) \setminus \{0\}$ then

$$\|z_{T^c}\|_w \leqslant \|z_T\|_w. \tag{A.73}$$

Reciprocally,

$$\|z_{T^c}\|_w \leqslant \|z_T\|_w \tag{A.74}$$

implies $z \in \mathcal{T}_{\|\cdot\|_w}(\Sigma_{k_1,k_2})$.

Proof. By definition, if $z \in \mathcal{T}_{\|\cdot\|_w}(\Sigma_{k_1,k_2}) \setminus \{0\}$ then there exists $x \in \Sigma_{k_1,k_2}$ and $\gamma \in \mathbb{R} \setminus \{0\}$ such that $z = \gamma y$ and $\|x+y\|_w \leqslant \|x\|_w$. With $U := \mathtt{supp}(x)$ we have $\|y_{U^c}\|_w + \|(x+y)_U\|_w = \|x+y\|_w \leqslant \|x\|_w = \|x_U\|_w$. By the triangle inequality this implies

$$\|y_{U^c}\|_w \leqslant \|x_U\|_w - \|(x+y)_U\|_w \leqslant \|y_U\|_w. \tag{A.75}$$

As $\gamma \neq 0$, we obtain $\|z_{U^c}\|_w \leqslant \|z_U\|_w$. We have

$$\|z_T\|_w \geqslant \|z_U\|_w \geqslant \|z_{U^c}\|_w \geqslant \|z_{T^c}\|_w. \tag{A.76}$$

□

To calculate $B_\Sigma(\|\cdot\|_w)$ (see definition in Corollary 3.1), we need a few lemmas.

LEMMA A.17 Consider $w_1, w_2, k_1, k_2 > 0$ and $\beta_1, \beta_2, \lambda \geqslant 0$ and

$$V := \min_{\alpha_1,\alpha_2 \geqslant 0} k_1\alpha_1^2 + k_2\alpha_2^2 \quad \text{s.t.} \quad \alpha_1 \geqslant \beta_1,\ \alpha_2 \geqslant \beta_2,\ k_1w_1\alpha_1 + k_2w_2\alpha_2 \geqslant \lambda \tag{A.77}$$

- If $\lambda < k_1w_1\beta_1 + k_2w_2\beta_2$ then $V = k_1\beta_1^2 + k_2\beta_2^2$.
- If $\lambda \geqslant k_1w_1\beta_1 + k_2w_2\beta_2$ then the minimum is achieved at α_1^*, α_2^* such that $k_1w_1\alpha_1^* + k_2w_2\alpha_2^* = \lambda$. Moreover,
  - if $\lambda \geqslant (k_1w_1^2 + k_2w_2^2)\max(\beta_1/w_1, \beta_2/w_2)$ then
  $$V = \min_{\alpha_1,\alpha_2 \geqslant 0, w_1\alpha_1 + k_2w_2\alpha_2 = \lambda} k_1\alpha_1^2 + k_2\alpha_2^2 = \lambda^2/(k_1w_1^2 + k_2w_2^2);$$
  - otherwise
  $$V = \min\left(k_1\beta_1^2 + \frac{(\lambda - k_1w_1\beta_1)^2}{k_2w_2^2}, k_2\beta_2^2 + \frac{(\lambda - k_2w_2\beta_2)^2}{k_1w_1^2}\right) > \lambda^2/(k_1w_1^2 + k_2w_2^2).$$

Proof. Consider the change of variables $x = \sqrt{k_1}\alpha_1$, $y = \sqrt{k_2}\alpha_2$ and denote $x_0 := \sqrt{k_1}\beta_1$, $y_0 := \sqrt{k_2}\beta_2$, $a := \sqrt{k_1}w_1$, $b := \sqrt{k_2}w_2$. This leads to the equivalent problem

$$\min_{x,y \geqslant 0} x^2 + y^2 \ \text{ s.t. } \ x \geqslant x_0, y \geqslant y_0, ax + by \geqslant \lambda$$

which involves a convex objective to be minimized over a polyhedral constraint set. If $ax_0 + by_0 > \lambda$, i.e., if $k_1w_1\beta_1 + k_2w_2\beta_2 > \lambda$, then this problem is equivalent to

$$\min_{x,y \geqslant 0} x^2 + y^2 \ \text{ s.t. } \ x \geqslant x_0, y \geqslant y_0$$

which is minimized at (x_0, y_0), with value $x_0^2 + y_0^2 = k_1\beta_1^2 + k_2\beta_2^2$. Otherwise, the candidate optima must satisfy the constraint $ax + by = \lambda$, hence $y = (\lambda - ax)/b$ and the problem is equivalent to

$$\min_{x_0 \leqslant x \leqslant (\lambda - by_0)/a} x^2 + (ax - \lambda)^2/b^2. \tag{A.78}$$

The unconstrained minimum of (A.78) is at x^* satisfying $2x^* + 2a(ax^* - \lambda)/b^2 = 0$, *i.e.*, , $x^* = \frac{a\lambda}{a^2+b^2}$, leading to $y^* = (\lambda - ax^*)/b = \frac{b\lambda}{a^2+b^2}$ and to an optimal unconstrained problem value

$$(x^*)^2 + (y^*)^2 = \lambda^2/(a^2+b^2) = \lambda^2/(k_1w_1^2 + k_2w_2^2).$$

This is also the value of the constrained minimum of (A.78), provided that $x_0 \leqslant x^* \leqslant (\lambda - by_0)/a$, i.e., that $\lambda \geqslant (a^2+b^2)\max(x_0/a, y_0/b) = (k_1 w_1^2 + k_2 w_2^2)\max(\beta_1/w_1, \beta_2/w_2)$. Otherwise, the constrained minimum is either at $x = x_0$ and $y = (\lambda - ax_0)/b$, so that $x^2+y^2 = x_0^2 + (\lambda - ax_0)^2/b^2$; or at $y = y_0$ and $x = (\lambda - by_0)/a$, so that $x^2+y^2 = y_0^2 + (\lambda - by_0)^2/a^2$. The value at the optimum is then $\min(x_0^2 + (\lambda - ax_0)^2/b^2, y_0^2 + (\lambda - by_0)^2/a^2)$, which is necessarily larger than that of the unconstrained minimum. Once translated in terms of the original variables, this yields the result.

□

LEMMA A.18 Let $\rho \geqslant 0$, $k_1, k_2, L_1, L_2, w_1, w_2, \lambda > 0$

$$\max_{\beta_1 \geqslant 0, \beta_2 \geqslant 0} \frac{L_1\beta_1^2 + L_2\beta_2^2}{\rho + k_1\beta_1^2 + k_2\beta_2^2} \text{ s.t. } w_1(k_1+L_1)\beta_1 + w_2(k_2+L_2)\beta_2 = \lambda \tag{A.79}$$

is equal to

$$\max_{i \in \{1,2\}} \frac{L_i \lambda^2}{\rho w_i^2 (k_i + L_i)^2 + k_i \lambda^2}. \tag{A.80}$$

Denoting i^* the index maximizing this expression, the maximum is reached for $\beta_{i^*} = \frac{\lambda}{w_{i^*}(k_{i^*}+L_{i^*})}$ (and $\beta_j = 0$ for $j \neq i$).

Proof. Let $c \geqslant 0$. Observe that

$$\frac{L_1\beta_1^2 + L_2\beta_2^2}{\rho + k_1\beta_1^2 + k_2\beta_2^2} \geqslant c \tag{A.81}$$

is equivalent to

$$(L_1 - ck_1)\beta_1^2 + (L_2 - ck_2)\beta_2^2 \geqslant c\rho. \tag{A.82}$$

With the change of variable $b_i = w_i(k_i + L_i)\beta_i$ we have $b_1 + b_2 = \lambda$ and (A.82) reads

$$\frac{(L_1 - ck_1)}{w_1^2(k_1+L_1)^2} b_1^2 + \frac{(L_2 - ck_2)}{w_2^2(k_2+L_2)^2}(b_1 - \lambda)^2 \geqslant c\rho. \tag{A.83}$$

The left side is *maximized* (with respect to $0 \leqslant b_1 \leqslant \lambda$) for either $b_1 = 0$ or $b_1 = \lambda$. The initial inequality (A.81) is thus feasible if, and only if, the maximum of the left-hand side of (A.83) over these two values verifies the inequality

$$\max_{i \in \{1,2\}} \frac{(L_i - ck_i)}{w_i^2(k_i+L_i)^2} \lambda^2 \geqslant c\rho \tag{A.84}$$

i.e., if there is $i \in \{1,2\}$ such that $(L_i - ck_i)\lambda^2 \geqslant c\rho w_i^2(k_i+L_i)^2$. This is equivalent to $L_i\lambda^2 \geqslant c(\rho w_i^2(k_i + L_i)^2 + k_i\lambda^2)$ and

$$c \leqslant \frac{L_i\lambda^2}{\rho w_i^2(k_i+L_i)^2 + k_i\lambda^2}. \tag{A.85}$$

□

LEMMA A.19 Consider $w_1, w_2, \beta_1, \beta_2, c \geqslant 0$ and

$$V := \sup_{0 \leqslant \theta_i \leqslant \beta_i, w_1\theta_1 + w_2\theta_2 \leqslant c} \theta_1^2 + \theta_2^2. \tag{A.86}$$

Denoting $(\ell, r) \in \{(1,2),(2,1)\}$ such that $w_\ell \beta_\ell \leqslant w_r \beta_r$, we have

1. if $c < w_\ell \beta_\ell$ then $V = \max_{i\in\{1,2\}} (c/w_i)^2$;
2. if $w_\ell \beta_\ell \leqslant c < w_r \beta_r$ then $V = \max((c/w_r)^2, \beta_\ell^2 + [(c - w_\ell \beta_\ell)/w_r]^2$;
3. if $w_r \beta_r \leqslant c < w_1 \beta_1 + w_2 \beta_2$ then $V = \max_{(i,j)\in\{(1,2),(2,1)\}} \beta_i^2 + [(c - w_i \beta_i)/w_j]^2$;
4. if $c \geqslant w_1 \beta_1 + w_2 \beta_2$ then $V = \beta_1^2 + \beta_2^2$;

Proof. The optimum V is the maximization of a quadratic form within the intersection of a rectangle and a half-space delimited by an affine function. Using standard compactness arguments there exists at least a maximizer (θ_1^*, θ_2^*) of the considered expression. If $\theta_i^* < \beta_i$ for some $i \in \{1,2\}$ then the constraint $c = w_1\theta_1^* + w_2\theta_2^*$ is satisfied (otherwise, we would have $0 \leqslant \theta_i^* < \beta_i$ and $w_1\theta_1 + w_2\theta_2 < c$, and we could exhibit other $\theta_i > \theta_i*$ still satisfying the constraints and such that $\theta_1^2 + \theta_2^2$ is increased), hence $w_1\beta_1 + w_2\beta_2 > w_1\theta_1^* + w_2\theta_2^* = c$.
Vice-versa if $w_1\beta_1 + w_2\beta_2 > c$ then since (θ_1^*, θ_2^*) satisfies all constraints we have $w_1\theta_1^* + w_2\theta_2^* \leqslant c < w_1\beta_1 + w_2\beta_2$, hence there is at least one index $i \in \{1,2\}$ such that $\theta_i^* < \beta_i$. We can thus consider the following cases (depending on the shape of the domain):

- if $w_1\beta_1 + w_2\beta_2 \leqslant c$ then for each $i \in \{1,2\}$, $\theta_i^* = \beta_i$ hence $V = \beta_1^2 + \beta_2^2$ as claimed;
- otherwise, i.e., if $w_1\beta_1 + w_2\beta_2 > c$, we have $w_1\theta_1^* + w_2\theta_2^* = c$ and we distinguish three cases:
  (a) $\theta_1^* < \beta_1$, $\theta_2^* < \beta_2$: then, since $\theta_2^* = (c - w_1\theta_1^*)/w_2$ where θ_1^* is a maximizer of $\theta_1^2 + [(c - w_1\theta_1)/w_2]^2$ under the constraint $0 \leqslant \theta_1$ and $c - w_1\theta_1 \geqslant 0$, there is $(i,j) \in \{(1,2),(2,1)\}$ such that $\theta_j^* = 0$ and $\theta_i^* = c/w_i$. This is feasible provided that $c/w_i < \beta_i$.
  (b) $\theta_1^* = \beta_1$, $\theta_2^* < \beta_2$, hence $\theta_2^* = (c - w_1\beta_1)/w_2$. This satisfies $0 \leqslant \theta_2^* < \beta_2$ if, and only if, $c \geqslant w_1\beta_1$.
  (c) $\theta_1^* < \beta_1$, $\theta_2^* = \beta_2$, hence $\theta_1^* = (c - w_2\beta_2)/w_1$. This is feasible provided that $c \geqslant w_2\beta_2$.

  We now discuss the possible cases depending on the value of c:

  - $c < w_\ell\beta_\ell$: (a) with any $(i,j) \in \{(1,2),(2,1)\}$ is feasible; (b)-(c) are unfeasible, hence $V = \max_{i\in\{1,2\}} (c/w_i)^2$.
  - $c \geqslant w_r\beta_r$: (a) is unfeasible; (b)-(c) are both feasible, hence the claimed value of V for this case.
  - $w_\ell\beta_\ell \leqslant c < w_r\beta_r$: (a) is feasible with (i,j) such that $c < w_i\beta_i$, *i.e.*, , with $(i,j) = (r,\ell)$, leading to a value $(\theta_j^*)^2 + (\theta_i^*)^2 = (c/w_i)^2 = (c/w_r)^2$; (b) is feasible provided that $c \geqslant w_1\beta_1$, i.e., that $(r,\ell) = (2,1)$, leading to a value $(\theta_1^*)^2 + (\theta_2^*)^2 = \beta_1^2 + [(c - w_1\beta_1)/w_2]^2 = \beta_\ell^2 + [(c - w_\ell\beta_\ell)/w_r]^2$; similarly, (c) is feasible provided that $(r,\ell) = (2,1)$, leading to a value $(\theta_2^*)^2 + (\theta_1^*)^2 = \beta_2^2 + [(c - w_2\beta_2)/w_1]^2 = \beta_\ell^2 + [(c - w_\ell\beta_\ell)/w_r]^2$. Overall, this leads to $V = \max((c/w_r)^2, \beta_\ell^2 + [(c - w_\ell\beta_\ell)/w_r]^2$.

□

As in the case of the ℓ^1 norm for sparsity and the nuclear norm for low-rank matrices, we compute $B_\Sigma(\|\cdot\|_w)$ (see definition in Corollary 3.1) via intermediate quantities $B^{L_1,L_2}(w)$ that we now introduce and control. We find an expression consistent with the ℓ^1 case.

LEMMA A.20 Consider weights $w=(w_1,w_2)$ with $w_i>0$ and integers $k_i\geqslant 0$. Denote for any integers $L_1,L_2\geqslant 0$

$$B^{L_1,L_2}(w):=\sup_{\substack{\alpha_i\geqslant\beta_i\geqslant 0,\beta_1+\beta_2>0\\ \sum_{i=1}^2(k_iw_i\alpha_i-w_i(k_i+L_i)\beta_i)=0}}\frac{\sum_{i=1}^2L_i\beta_i^2}{\sum_{i=1}^2k_i(\alpha_i^2+\beta_i^2)}. \tag{A.87}$$

For $m\in\{1,2\}$, consider

$$g_m(L_1,L_2,\alpha_1,\alpha_2,\beta_1,\beta_2):=\frac{L_1\beta_1^2+L_2\beta_2^2+[(\sum_{i=1}^2(k_iw_i\alpha_i-(k_i+L_i)w_i\beta_i)/w_m]^2}{\sum_{i=1}^2k_i(\alpha_i^2+\beta_i^2)}.$$

We have

$$\sup_{\substack{\alpha_i,\beta_i:0\leqslant\beta_i\leqslant\alpha_i;\beta_1+\beta_2>0\\ \sum_{i=1}^2(k_i+L_i)w_i\beta_i\leqslant\sum_{i=1}^2k_iw_i\alpha_i}}g_m(L_1,L_2,\alpha_1,\alpha_2,\beta_1,\beta_2)\leqslant B^{L_1,L_2}(w). \tag{A.88}$$

(A.89)

Proof. First we show that there exist $\alpha_i^*\in\mathbb{R}_+,\beta_i^*\in\mathbb{R}_+$ such that

$$g_m(L_1,L_2,\alpha_1^*,\alpha_2^*,\beta_1^*,\beta_2^*)=\sup_{\substack{\alpha_i,\beta_i:0\leqslant\beta_i\leqslant\alpha_i;\beta_1+\beta_2>0\\ \sum_{i=1}^2(k_i+L_i)w_i\beta_i\leqslant\sum_{i=1}^2k_iw_i\alpha_i}}g_m(L_1,L_2,\alpha_1,\alpha_2,\beta_1,\beta_2) \tag{A.90}$$

with $0\leqslant\beta_i^*\leqslant\alpha_i^*;\beta_1^*+\beta_2^*>0$, and $\sum_{i=1}^2(k_i+L_i)w_i\beta_i^*\leqslant\sum_{i=1}^2k_iw_i\alpha_i^*$. Indeed, given any α_i,β_i satisfying these constraints, setting $\beta_j'=\beta_j/(\beta_1+\beta_2)$, $\alpha_j'=\alpha_j/(\beta_i+\beta_j)$, we have $g_m(L_1,L_2,\alpha_1',\alpha_2',\beta_1',\beta_2')=g_m(L_1,L_2,\alpha_1,\alpha_2,\beta_1,\beta_2)$ hence the supremum is unchanged if we impose $\beta_1'+\beta_2'=1$ instead of $\beta_1+\beta_2>0$. Given any such pair β_1',β_2', Lemma A.17 yields the optimum over α_i satisfying the constraints, and as the resulting expression is continuous with respect to β_j', the existence of a maximizer follows using a compactness argument.

We will soon prove that $\sum_i(k_i+L_i)w_i\beta_i^*=\sum_ik_iw_i\alpha_i^*$. If this equality is verified, since $0\leqslant\beta_i^*\leqslant\alpha_i^*$, we obtain the desired result

$$\begin{aligned}g_m(L_1,L_2,\alpha_1^*,\alpha_2^*,\beta_1^*,\beta_2^*)&=\frac{\sum_{i=1}^2(L_i\beta_i^*)^2}{\sum_{i=1}^2k_i((\alpha_i^*)^2+(\beta_i^*)^2)}\\&\leqslant\sup_{\substack{\alpha_i,\beta_i:0\leqslant\beta_i\leqslant\alpha_i;\beta_1+\beta_2>0\\ \sum_{i=1}^2(k_i+L_i)w_i\beta_i=\sum_{i=1}^2k_iw_i\alpha_i}}\frac{\sum_{i=1}^2L_i\beta_i^2}{\sum_{i=1}^2k_i((\alpha_i)^2+(\beta_i)^2)}=B^{L_1,L_2}(w).\end{aligned} \tag{A.91}$$

For the sake of contradiction, assume that $\sum_i(k_i+L_i)w_i\beta_i^*<\sum_ik_iw_i\alpha_i^*$, then with the shorthand $C:=g_m(L_1,L_2,\alpha_1^*,\alpha_2^*,\beta_1^*,\beta_2^*)$, we have

$$[(\sum_ik_iw_i\alpha_i^*-\sum_i(k_i+L_i)w_i\beta_i^*)/w_m]^2+\sum_i(L_i-Ck_i)(\beta_i^*)^2=C\sum_ik_i(\alpha_i^*)^2. \tag{A.92}$$

Since $g_m(L_1,L_2,\alpha_1,\alpha_2,\beta_1,\beta_2)\leqslant C$ within the constraints of (A.88), (β_1^*,β_2^*) maximize

$$h(\beta_1,\beta_2):=[(\sum_ik_iw_i\alpha_i^*-\sum_i(k_i+L_i)w_i\beta_i)/w_v]^2+\sum_i(L_i-Ck_i)(\beta_i)^2$$

among all β_1, β_2 such that $0 \leqslant \beta_i \leqslant \alpha_i^*$, $\beta_1 + \beta_2 > 0$ and $\sum_{i=1}^2 (k_i + L_i) w_i \beta_i \leqslant \sum_{i=1}^2 k_i w_i \alpha_i^*$.

Consider $j \in \{1,2\}$.

If $C > L_j/k_j$, then h is decreasing with respect to β_j on the considered range, hence $\beta_j^* = 0$. Otherwise $C \leqslant L_j/k_j$, and since h is a second degree polynomial in β_j with positive leading coefficient, its maximum is at one of the extremities of the optimization interval, i.e., since we assumed $\sum_i (k_i + L_i) w_i \beta_i^* < \sum_i k_i w_i \alpha_i^*$, at least one of the constraints $\beta_j^* = 0$, $\beta_j^* = \alpha_j^*$ is reached.

Since the optimum satisfies all constraints of (A.88), we have $\beta_1^* + \beta_2^* > 0$, hence in light of the above observations there is at least one index $j \in \{1,2\}$ such that $C \leqslant L_j/k_j$, and for which we have $\beta_j^* = \alpha_j^* > 0$.

Since $\sum_{i=1}^2 k_i w_i \beta_i^* \leqslant \sum_{i=1}^2 (k_i + L_i) w_i \beta_i^* < \sum_{i=1}^2 k_i w_i \alpha_i^*$, both constraints $\beta_1^* = \alpha_1^*, \beta_2^* = \alpha_2^*$ cannot be reached at the same time hence there is $(i,j) \in \{(1,2),(2,1)\}$ such that $\beta_i^* = 0$, $\beta_j^* = \alpha_j^*$ and

$$C = g_m(L_1, L_2, \alpha_1^*, \alpha_2^*, \beta_1^*, \beta_2^*) = \frac{L_j (\beta_j^*)^2 + [(k_i w_i \alpha_i^* + k_j w_j \alpha_j^* - (k_j + L_j) w_j \beta_j^*)/w_m]^2}{k_i (\alpha_i^*)^2 + k_j (\alpha_j^*)^2 + k_j (\beta_j^*)^2} \tag{A.93}$$

$$= \frac{L_j (\alpha_j^*)^2 + [(k_i w_i \alpha_i^* - L_j w_j \alpha_j^*)/w_m]^2}{k_i (\alpha_i^*)^2 + 2 k_j (\alpha_j^*)^2}. \tag{A.94}$$

This can be rewritten $(L_j - 2Ck_j)(\alpha_j^*)^2 + [(k_i w_i \alpha_i^* - L_j w_j \alpha_j^*)/w_m]^2 = C k_i (\alpha_i^*)^2$. Observe that any $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that $\beta_i = 0$, $\beta_j = \alpha_j > 0$, $\alpha_i = \alpha_i^*$, and $L_j w_j \alpha_j \leqslant k_i w_i \alpha_i^*$ satisfy the constraints of (A.88), hence $g_m(L_1, L_2, \alpha_1, \alpha_2, \beta_1, \beta_2) \leqslant C$, or equivalently

$$(L_j - 2Ck_j)(\alpha_j)^2 + [(k_i w_i \alpha_i^* - L_j w_j \alpha_j)/w_m]^2 \leqslant C k_i (\alpha_i^*)^2 \tag{A.95}$$

Thus, α_j^* maximizes the left hand side of (A.95) under the constraint $0 \leqslant L_j w_j \alpha_j \leqslant k_i w_i \alpha_i^*$. If $L - 2Ck_j \leqslant 0$, then the left hand side of (A.95) is decreasing with respect to α_j in the considered range, hence $\alpha_j^* = 0$, which is not possible since $0 < \beta_1 + \beta_2 = \beta_j^* = \alpha_j^*$. Therefore we must have $L_j - 2Ck_j > 0$, hence the left hand side of (A.95) is a second degree polynomial in α_j with positive leading coefficient. Its maximum is achieved at one extremity of the interval constraint : the case $\alpha_j^* = 0$ was already ruled out as impossible, hence $L_j w_j \alpha_j^* = k_i w_i \alpha_i^*$. This implies $(k_i + L_i) w_i \beta_i^* + (k_j + L_j) w_j \beta_j^* = (k_j + L_j) w_j \alpha_j^* = k_j w_j \alpha_j^* + k_i w_i \alpha_i^*$, which yields the desired contradiction to the assumption that $\sum_i (k_i + L_i) w_i \beta_i^* < \sum_i k_i w_i \alpha_i^*$.

□

LEMMA A.21 Consider weights $w = (w_1, w_2)$ and integers k_i, n_i such that $1 \leqslant 2k_i < n_i$ and $\Sigma = \Sigma_{k_1,k_2} \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, $i \in \{1,2\}$. We have

$$B_\Sigma(\|\cdot\|_w) = \max_{0 \leqslant L_i \leqslant n - 2k_i} B^{L_1,L_2}(w) \tag{A.96}$$

where $B^{L_1,L_2}(w)$ is defined in (A.87).

Proof. We use the same proof method as in Lemma A.6. With the notations $T = T(z), T_2 = T_2(z)$ from the beginning of Appendix A.6, denote $T' = T_2 \setminus T$ so that $\|z_{T_2^c}\|_w + \|z_{T'}\|_w = \|z_{T^c}\|_w$. By Lemma A.16, we have

$$B_\Sigma(\|\cdot\|_w) = \sup_{z : z \neq 0, \|z_{T_2^c}\|_w + \|z_{T'}\|_w \leqslant \|z_T\|_w} \frac{\|z_{T_2^c}\|_2^2}{\|z_{T_2}\|_2^2}. \tag{A.97}$$

We now show that this expression can be simplified by maximizing over vectors z with a particular shape. Consider z a vector satisfying the constraint in (A.97). Replacing each entry z_i of z with its magnitude $|z_i|$ leaves the constraint (as well as the maximized quantity) unchanged, hence without loss of generality we can assume that z has nonnegative entries $z_i \geqslant 0$. Similarly, we can assume without loss of generality that for each $i \in \{1,2\}$, the index set $S_i = [1,k_i]$ indexes k_i largest entries of $P_i(z)$ and $S_i' = [1,2k_i]$ indexes the $2k_i$ largest entries.

Given some $j \in \{1,2\}$, consider two (equal or distinct) indices in S_j and the vector $\tilde{z}$ obtained by keeping unchanged all entries of z, except those indexed by these indices which are replaced by their average. This has the following effect:

1. Each S_i (resp. S_i'), $i \in \{1,2\}$, is a set of the k_i (resp. $2k_i$) largest coordinates of $P_i(\tilde{z})$, hence $T(\tilde{z}) = T = (S_1, S_2)$, $T_2(\tilde{z}) = T_2 = (S_1', S_2')$, $T'(\tilde{z}) = T' = T_2 \backslash T$, $\tilde{z}_{T_2^c} = z_{T_2^c}$, $\tilde{z}_{T'} = z_{T'}$, and the support of $P_i(\tilde{z})$, $i \in \{1,2\}$ is the same as that of $P_i(z)$.

2. Denoting $a, b \geqslant 0$ the values of the two considered entries, since $(a+b)/2 + (a+b)/2 = a+b$, we have $\|[P_j(\tilde{z})]_{S_j}\|_1 = \|[P_j(z)]_{S_j}\|_1$, and we obtain that $\|\tilde{z}_T\|_w = \|z_T\|_w$, hence $\tilde{z}$ still satisfies the optimization constraint;

3. As $\|\tilde{z}_{T_2^c}\|_2 = \|z_{T_2^c}\|_2$ and $\|\tilde{z}_{T_2}\|_2^2 - \|z_{T_2}\|_2^2 = 2[(a+b)/2]^2 - a^2 - b^2 = -(a-b)^2/2 \leqslant 0$, hence $\|\tilde{z}_{T_2^c}\|_2^2 / \|\tilde{z}_{T_2}\|_2^2 \geqslant \|\tilde{z}_{T_2^c}\|_2^2 / \|\tilde{z}_{T_2}\|_2^2$ where the inequality is strict as soon as $a \neq b$.

All the above imply that, without loss of generality, we can restrict the optimization to vectors z such that, for $i \in \{1,2\}$, all entries of z_{S_i} are equal. We denote $\alpha_i > 0$ their common value. A similar reasoning with $S_j' \backslash S_j$ instead of S_j shows that we can also assume without loss of generality that all entries of $z_{S_i' \backslash S_i}$, $i \in \{1,2\}$, are equal. We denote $\beta_i \geqslant 0$ their common value.

The value of the smallest component of $[P_i(z)]_{S_i}$ is α_i, while the smallest component of $[P_i(z)]_{S_i'}$ is $\min(\alpha_i, \beta_i)$. Denoting $x_i = P_i(z)_{(S_i')^c}$, we have $x_i \in \mathbb{R}_+^{n_i - 2k_i}$ and the largest component of $[P_i(z)]_{(S_i')^c}$ is $\|x_i\|_\infty$. Hence, S_i and S_i' are respectively a set of the k_i and $2k_i$ largest components of $P_i(z)$ if, and only if, $\|x_i\|_\infty \leqslant \beta_i \leqslant \alpha_i$.

Finally, we observe that $\|z_T\|_w - \|z_{T'}\|_w - \|z_{T_2^c}\|_w = w_1 k_1 \alpha_1 + w_2 k_2 \alpha_2 - w_1 k_1 \beta_1 - w_2 k_2 \beta_2 - w_1 \|x_1\|_1 - w_2 \|x_2\|_1$, $\|z_{T_2^c}\|_2^2 = \|x_1\|_2^2 + \|x_2\|_2^2$ and $\|z_{T_2}\|_2^2 = k_1 \alpha_1^2 + k_2 \alpha_2^2 + k_1 \beta_1^2 + k_2 \beta_2^2$. This establishes

$$B_\Sigma(\|\cdot\|_w) = \sup_{\substack{\beta_i : \beta_i \geqslant 0 \\ \beta_1 + \beta_2 > 0}} \sup_{\alpha_i : \alpha_i \geqslant \beta_i} \sup_{\substack{\|x_i\|_\infty \leqslant \beta_i \\ \sum_{i=1}^2 w_i \|x_i\|_1 \leqslant \sum_{i=1}^2 k_i w_i (\alpha_i - \beta_i)}} \frac{\sum_{i=1}^2 \|x_i\|_2^2}{\sum_{i=1}^2 k_i (\alpha_i^2 + \beta_i^2)}, \tag{A.98}$$

where the restriction $\beta_1 + \beta_2 > 0$ simply follows from the fact that when $\beta_1 + \beta_2 = 0$ we have $x_1 = x_2 = 0$ which leads to a sub-optimal objective value. To show that the supremum in (A.98) is achieved, observe that both the constraints on $y := (\alpha_1, \alpha_2, \beta_1, \beta_2, x_1, x_2)$ and the quantity $f(y)$ that is maximized are invariant by multiplication by a positive constant factor. Hence, the supremum is unchanged if we add a scaling constraint. e.g. by fixing $\|y\|_\infty$. This leads to the supremum of a continuous function over a compact set (the unit ℓ_∞ ball), hence there exists $\alpha_i^*, \beta_i^*, x_i^*$ reaching the supremum in (A.98).

Thanks to Lemma A.5, given the constraints (depending on α_i and β_i), the maximization w.r.t x_i is reached with vectors with the shape

$$(\underbrace{\beta_i, \ldots, \beta_i}_{L_i}, \theta_i, \quad \underbrace{0, \ldots, 0}_{n_i - 2k_i - (L_i+1) \geqslant 0})$$

with $0 \leqslant \theta_i \leqslant \beta_i$, $0 \leqslant L_i \leqslant n_i - 2k_i - 1$, including potentially $L_i = 0$ (case of vector x_i with a single nonzero coordinate θ_i). We deduce

$$B_\Sigma(\|\cdot\|_w) = \sup_{\substack{\beta_i:\beta_i\geqslant 0\\ \beta_1+\beta_2>0}} \sup_{\alpha_i:\alpha_i\geqslant\beta_i} \sup_{\substack{L_i,\theta_i:0\leqslant L_i\leqslant n-2k_i-1, 0\leqslant\theta_i\leqslant\beta_i\\ \sum_{i=1}^2 w_i\theta_i\leqslant\sum_{i=1}^2(k_iw_i\alpha_i - w_i(k_i+L_i)\beta_i)}} \frac{\sum_{i=1}^2 L_i\beta_i^2+\theta_i^2}{\sum_{i=1}^2 k_i(\alpha_i^2+\beta_i^2)}. \tag{A.99}$$

Hence, denoting

$$f(L_1,L_2,\alpha_1,\alpha_2,\beta_1,\beta_2) := \sup_{\substack{\theta_i:0\leqslant\theta_i\leqslant\beta_i\\ \sum_{i=1}^2 w_i\theta_i\leqslant\sum_{i=1}^2(k_iw_i\alpha_i - w_i(k_i+L_i)\beta_i)}} \frac{\sum_{i=1}^2 L_i\beta_i^2+\theta_i^2}{\sum_{i=1}^2 k_i(\alpha_i^2+\beta_i^2)}, \tag{A.100}$$

for parameters α_i,β_i,L_i such that $c := \sum_{i=1}^2(k_iw_i\alpha_i - w_i(k_i+L_i)\beta_i) \geqslant 0$, we have

$$B_\Sigma(\|\cdot\|_w) = \max_{0\leqslant L_i\leqslant n_i-2k_i-1} \underbrace{\sup_{\substack{\beta_1,\beta_2\geqslant 0\\ \beta_1+\beta_2>0}} \sup_{\substack{\alpha_i:\alpha_i\geqslant\beta_i\\ \sum_{i=1}^2 k_iw_i\alpha_i\geqslant\sum_{i=1}^2(k_i+L_i)w_i\beta_i}} f(L_1,L_2,\alpha_1,\alpha_2,\beta_1,\beta_2)}_{f(L_1,L_2)}. \tag{A.101}$$

To continue, we bound $f(L_1,L_2)$ via characterizations of $f(L_1,L_2,\alpha_1,\alpha_2,\beta_1,\beta_2)$ in different parameter ranges. The supremum in (A.100) is covered by Lemma A.19 hence we need to primarily distinguish cases depending on relative order of $c = \sum_{i=1}^2(k_iw_i\alpha_i - w_i(k_i+L_i)\beta_i) \geqslant 0$, $w_1\beta_1+w_2\beta_2$, $w_1\beta_1$, and $w_2\beta_2$. This suggests writing $f(L_1,L_2) = \max_{u\in\{0,1\}} f_u(L_1,L_2)$ where

$$f_0(L_1,L_2) := \sup_{\substack{\beta_i,\alpha_i:0\leqslant\beta_i\leqslant\alpha_i,\beta_1+\beta_2>0\\ \sum_{i=1}^2 k_iw_i\alpha_i\geqslant\sum_{i=1}^2(k_i+L_i+1)w_i\beta_i}} f(L_1,L_2,\alpha_1,\alpha_2,\beta_1,\beta_2) \tag{A.102}$$

$$f_1(L_1,L_2) := \sup_{\substack{\beta_i,\alpha_i:0\leqslant\beta_i\leqslant\alpha_i,\beta_1+\beta_2>0\\ \sum_{i=1}^2(k_i+L_i)w_i\beta_i\leqslant\sum_{i=1}^2 k_iw_i\alpha_i<\sum_{i=1}^2(k_i+L_i+1)w_i\beta_i}} f(L_1,L_2,\alpha_1,\alpha_2,\beta_1,\beta_2). \tag{A.103}$$

To express $f_0(L_1,L_2)$ and bound $f_1(L_1,L_2)$, we use the functions g_m, $m\in\{1,2\}$, from Lemma A.20.

Expressing and bounding f_0: if $\sum_{i=1}^2 k_iw_i\alpha_i \geqslant \sum_{i=1}^2(k_i+L_i+1)w_i\beta_i$ then $c \geqslant w_1\beta_1+w_2\beta_2$ hence Lemma A.19, case 4 yields

$$f(L_1,L_2,\alpha_1,\alpha_2,\beta_1,\beta_2) = \frac{\sum_{i=1}^2(L_i+1)\beta_i^2}{\sum_{i=1}^2 k_i(\alpha_i^2+\beta_i^2)} \tag{A.104}$$

$$f_0(L_1,L_2) = \sup_{\substack{0\leqslant\beta_i\leqslant\alpha_i,\beta_1+\beta_2>0\\ \sum_{i=1}^2 k_iw_i\alpha_i\geqslant\sum_{i=1}^2(k_i+L_i+1)w_i\beta_i}} \frac{\sum_{i=1}^2(L_i+1)\beta_i^2}{\sum_{i=1}^2 k_i(\alpha_i^2+\beta_i^2)} \tag{A.105}$$

$$\overset{\text{Lemma A.17}}{=} \sup_{\substack{0\leqslant\beta_i\leqslant\alpha_i,\beta_1+\beta_2>0\\ \sum_{i=1}^2 k_iw_i\alpha_i=\sum_{i=1}^2(k_i+L_i+1)w_i\beta_i}} \frac{\sum_{i=1}^2(L_i+1)\beta_i^2}{\sum_{i=1}^2 k_i(\alpha_i^2+\beta_i^2)} = B^{L_1+1,L_2+1}(w). \tag{A.106}$$

As a result

$$f_0(L_1,L_2) \leqslant \max_{0\leqslant L_i\leqslant n_i-2k_i-1} B^{L_1+1,L_2+1}(w) \leqslant \max_{0\leqslant L'_i\leqslant n_i-2k_i} B^{L'_1,L'_2}(w) \tag{A.107}$$

Bounding f_1: we denote $(\ell,r)\in\{(1,2),(2,1)\}$ a pair such that $w_\ell\beta_\ell = \min_i w_i\beta_i \leqslant \max_i w_i\beta_i = w_r\beta_r$. When $\sum_{i=1}^2 (k_i+L_i)w_i\beta_i \leqslant \sum_{i=1}^2 k_i w_i\alpha_i < \sum_{i=1}^2 (k_i+L_i+1)w_i\beta_i$ we can distinguish three cases.

1. if $(k_\ell+L_\ell)w_\ell\beta_\ell+(k_r+L_r+1)w_r\beta_r \leqslant \sum_{i=1}^2 k_i w_i\alpha_i < \sum_{i=1}^2 (k_i+L_i+1)w_i\beta_i$ then $\max(w_1\beta_1,w_2\beta_2) = w_r\beta_r \leqslant c < w_1\beta_1+w_2\beta_2$ hence Lemma A.19, case 3 yields

$$f(L_1,L_2,\alpha_1,\alpha_2,\beta_1,\beta_2) = \max_{(u,v)\in\{(1,2),(2,1)\}} \underbrace{\frac{(L_u+1)\beta_u^2+L_v\beta_v^2+[(c-w_u\beta_u)/w_v]^2}{\sum_{i=1}^2 k_i(\alpha_i^2+\beta_i^2)}}_{=g_v(L'_1,L'_2,\alpha_1,\alpha_2,\beta_1,\beta_2),\ L'_u=L_u+1,L'_v=L_v}. \tag{A.108}$$

2. if $(k_\ell+L_\ell+1)w_\ell\beta_\ell+(k_r+L_r)w_r\beta_r \leqslant \sum_{i=1}^2 k_i w_i\alpha_i < (k_\ell+L_\ell)w_\ell\beta_\ell+(k_r+L_r+1)w_r\beta_r$ then $\min(w_1\beta_1,w_2\beta_2) = w_\ell\beta_\ell \leqslant c < w_r\beta_r = \max(w_1\beta_1,w_2\beta_2)$ hence Lemma A.19, case 2 yields

$$f(L_1,L_2,\alpha_1,\alpha_2,\beta_1,\beta_2) = \max\left(\underbrace{\frac{L_1\beta_1^2+L_2\beta_2^2+(c/w_r)^2}{\sum_{i=1}^2 k_i(\alpha_i^2+\beta_i^2)}}_{g_r(L_1,L_2,\alpha_1,\alpha_2,\beta_1,\beta_2)}, \underbrace{\frac{(L_\ell+1)\beta_\ell^2+L_r\beta_r^2+[(c-w_\ell\beta_\ell)/w_r]^2}{\sum_{i=1}^2 k_i(\alpha_i^2+\beta_i^2)}}_{g_r(L'_1,L'_2,\alpha_1,\alpha_2,\beta_1,\beta_2),\ L'_\ell=L_\ell+1,L'_r=L_r}\right). \tag{A.109}$$

3. otherwise, $\sum_{i=1}^2 (k_i+L_i)w_i\beta_i \leqslant \sum_{i=1}^2 k_i w_i\alpha_i < (k_\ell+L_\ell+1)w_\ell\beta_\ell+(k_r+L_r)w_r\beta_r$, hence $c<\min(w_1\beta_1,w_2\beta_2)$ and by Lemma A.19, case 1

$$f(L_1,L_2,\alpha_1,\alpha_2,\beta_1,\beta_2) = \max\left(\underbrace{\frac{L_1\beta_1^2+L_2\beta_2^2+(c/w_1)^2}{\sum_{i=1}^2 k_i(\alpha_i^2+\beta_i^2)}}_{g_1(L_1,L_2,\alpha_1,\alpha_2,\beta_1,\beta_2)}, \underbrace{\frac{L_1\beta_1^2+L_2\beta_2^2+(c/w_2)^2}{\sum_{i=1}^2 k_i(\alpha_i^2+\beta_i^2)}}_{g_2(L_1,L_2,\alpha_1,\alpha_2,\beta_1,\beta_2)}\right). \tag{A.110}$$

Thus, in the range of α_i,β_i involved in the definition of $f_1(L_1,L_2)$ as a supremum, there are integers $0\leqslant L'_i\leqslant n_i-2k_i$ and $v\in\{1,2\}$ such that $f(L_1,L_2,\alpha_1,\alpha_2,\beta_1,\beta_2) = g_v(L'_1,L'_2,\alpha_1,\alpha_2,\beta_1,\beta_2)$. We will shortly prove that given the relations between L'_i and the considered range of α_i,β_i we have

$$\sum_{i=1}^2 (k_i+L'_i)w_i\beta_i \leqslant \sum_{i=1}^2 k_i w_i\alpha_i. \tag{A.111}$$

hence using Lemma A.20 we obtain $g_v(L'_1,L'_2,\alpha_1,\alpha_2,\beta_1,\beta_2) \leqslant B^{L'_1,L'_2}(w)$.

This implies

$$f_1(L_1,L_2) \leqslant \max_{0\leqslant L'_i\leqslant n_i-2k_i} B^{L'_1,L'_2}(w)$$

and, combined with (A.101)-(A.107), yields the upper bound

$$B_\Sigma(\|\cdot\|_w) = \max_{0\leqslant L_i\leqslant n_i-2k_i} \max(f_0(L_1,L_2), f_1(L_1,L_2)) \leqslant \max_{0\leqslant L_i'\leqslant n_i-2k_i} B^{L_1',L_2'}(w). \tag{A.112}$$

Proof of (A.111)**.** We treat separately the three cases respectively associated to (A.108), (A.109), (A.110).

1. When $\sum_{i=1}^2 (k_i+L_i)w_i\beta_i \leqslant \sum_{i=1}^2 k_i w_i\alpha_i < (k_\ell+L_\ell+1)w_\ell\beta_\ell + (k_r+L_r)w_r\beta_r$, by (A.110) there is $v\in\{1,2\}$ such that $f(L_1,L_2,\alpha_1,\alpha_2,\beta_1,\beta_2) = g_v(L_1',L_2',\alpha_1,\alpha_2,\beta_1,\beta_2)$ with $(L_1',L_2') = (L_1,L_2)$. We observe that $\sum_{i=1}^2 (k_i+L_i')w_i\beta_i = \sum_{i=1}^2 (k_i+L_i)w_i\beta_i \leqslant \sum_{i=1}^2 k_i w_i\alpha_i$.

2. When $(k_\ell+L_\ell)w_\ell\beta_\ell + (k_r+L_r+1)w_r\beta_r \leqslant \sum_{i=1}^2 k_i w_i\alpha_i < \sum_{i=1}^2 (k_i+L_i+1)w_i\beta_i$, by (A.108), we have $f(L_1,L_2,\alpha_1,\alpha_2,\beta_1,\beta_2) = g_v(L_1',L_2',\alpha_1,\alpha_2,\beta_1,\beta_2)$ where $(L_1',L_2',v)\in\{(L_1+1,L_2,2),(L_1,L_2+1,1)\}$. If $(L_\ell',L_r') = (L_\ell,L_r+1)$ then $\sum_{i=1}^2 (k_i+L_i')w_i\beta_i = (k_\ell+L_\ell)w_\ell\beta_\ell + (k_r+L_r+1)w_r\beta_r$. Otherwise we have $(L_\ell',L_r') = (L_\ell+1,L_r)$, hence $\sum_{i=1}^2 (k_i+L_i')w_i\beta_i = (k_\ell+L_\ell+1)w_\ell\beta_\ell + (k_r+L_r)w_r\beta_r \leqslant (k_\ell+L_\ell)w_\ell\beta_\ell + (k_r+L_r+1)w_r\beta_r$ since $w_\ell\beta_\ell \leqslant w_r\beta_r$ by definition of r,ℓ. In both cases we get $\sum_{i=1}^2 (k_i+L_i')w_i\beta_i \leqslant \sum_{i=1}^2 k_i w_i\alpha_i$.

3. When $(k_\ell+L_\ell+1)w_\ell\beta_\ell + (k_r+L_r)w_r\beta_r \leqslant \sum_{i=1}^2 k_i w_i\alpha_i < (k_\ell+L_\ell)w_\ell\beta_\ell + (k_r+L_r+1)w_r\beta_r$, (A.109) yields $f(L_1,L_2,\alpha_1,\alpha_2,\beta_1,\beta_2) = g_r(L_1',L_2',\alpha_1,\alpha_2,\beta_1,\beta_2)$ with $(L_\ell',L_r')\in\{(L_\ell,L_r),(L_\ell+1,L_r)\}$, hence we have $\sum_{i=1}^2 (k_i+L_i')w_i\beta_i \leqslant (k_\ell+L_\ell+1)w_\ell\beta_\ell + (k_r+L_r)w_r\beta_r \leqslant \sum_{i=1}^2 k_i w_i\alpha_i$.

As these three cases cover all possibilities, we deduce bound (A.111) as claimed.

To conclude, we obtain a lower bound on $B_\Sigma(\|\cdot\|_w)$. Consider any integers $0\leqslant L_i\leqslant n_i-2k_i$ and any scalars α_i,β_i such that $0\leqslant\beta_i\leqslant\alpha_i$, $\beta_1+\beta_2>0$ and $\sum_{i=1}^2 (k_i+L_i)w_i\beta_i = \sum_{i=1}^2 k_i w_i\alpha_i$, and let $z=(z_1,z_2)$ where

$$z_i = (\underbrace{\alpha_i,\dots,\alpha_i}_{k_i},\underbrace{\beta_i,\dots,\beta_i}_{k_i+L_i},\underbrace{0,\dots,0}_{n_i-(2k_i+L_i)}).$$

We have $\|z_T\|_w = k_1w_1\alpha_1 + k_2w_2\alpha_2 = (k_1+L_1)w_1\beta_1 + (k_2+L_2)w_2\beta_2 = \|z_{T^c}\|_w$ hence, by Lemma A.16 and the definition of $B_\Sigma(\|\cdot\|_w)$,

$$B_\Sigma(\|\cdot\|_w) \geqslant \frac{\|z_{T_2^c}\|_2^2}{\|z_{T_2}\|_2^2} = \frac{\sum_{i=1}^2 L_i\beta_i^2}{\sum_{i=1}^2 k_i(\alpha_i^2+\beta_i^2)}. \tag{A.113}$$

Taking the supremum over α_i,β_i under the considered constraints yields $B_\Sigma(\|\cdot\|_w) \geqslant B^{L_1,L_2}(w)$. We deduce

$$B_\Sigma(\|\cdot\|_w) \geqslant \max_{0\leqslant L_i\leqslant n_i-2k_i} B^{L_1,L_2}(w).$$

□

We give a characterization/lower bound (depending on w) of the intermediate $B^{L_1,L_2}(w)$.

LEMMA A.22 Consider $w=(w_1,w_2)$, $0\leqslant L_i\leqslant n_i-2k$, and $B^{L_1,L_2}(w)$ defined as in Lemma A.20. We have

$$\max_{(i,j)\in\{(1,2),(2,1)\}} \frac{L_i/k_i}{\frac{\nu_i}{1-\nu_i}(L_i/k_i)^2+2} \leqslant B^{L_1,L_2}(w) \leqslant \max_{(i,j)\in\{(1,2),(2,1)\}} \frac{L_i/k_i}{\nu_i(L_i/k_i+1)^2+1} \tag{A.114}$$

with $\nu_i = \frac{1}{1+k_j/(k_i\mu_i^2)}$ and $\mu_i = \frac{w_i}{w_j}$ for $(i,j)\in\{(1,2),(2,1)\}$. The rhs is an equality if $\nu_i \geqslant \frac{k_i}{k_i+L_i}$, $\forall i\in\{1,2\}$.

Proof. For L_1, L_2 such that $L_1 + L_2 > 0$, we rewrite B^{L_1,L_2} defined in (A.87) as

$$B^{L_1,L_2}(w) = \sup_{\lambda>0} \sup_{\substack{\beta_i:\beta_1,\beta_2\geqslant 0 \\ \sum_{i=1}^2 w_i(k_i+L_i)\beta_i=\lambda}} \sup_{\substack{\alpha_i:\alpha_i\geqslant\beta_i \\ \sum_{i=1}^2 k_i w_i\alpha_i=\lambda}} \frac{L_1\beta_1^2+L_2\beta_2^2}{k_1\alpha_1^2+k_2\alpha_2^2+k_1\beta_1^2+k_2\beta_2^2}. \tag{A.115}$$

For fixed $\lambda > 0$ and β_1, β_2 such that $\sum_{i=1}^2 w_i(k_i+L_i)\beta_i = \lambda$, we have $\lambda > \sum_{i=1}^2 w_i k_i \beta_i$ hence, by Lemma A.17,

$$B^{L_1,L_2}(w) \leqslant \sup_{\lambda>0} \sup_{\substack{\beta_1\geqslant 0,\beta_2\geqslant 0 \\ w_1(k_1+L_1)\beta_1+w_2(k_2+L_2)\beta_2=\lambda}} \frac{L_1\beta_1^2+L_2\beta_2^2}{\frac{\lambda^2}{k_1w_1^2+k_2w_2^2}+k_1\beta_1^2+k_2\beta_2^2}. \tag{A.116}$$

with equality if the maximizers $\hat\lambda, \hat\beta_i$ of the right side satisfy the constraints $\hat\lambda \geqslant (k_1w_1^2+k_1w_2^2)\max(\hat\beta_1/w_1, \hat\beta_2/w_2)$.

Consider $(i,j) \in \{(1,2),(2,1)\}$. Since $v_i := \frac{1}{1+k_jw_j^2/(k_iw_i^2)} = \frac{k_iw_i^2}{k_iw_i^2+k_jw_j^2}$, we obtain by Lemma A.18

$$B^{L_1,L_2}(w) \leqslant \sup_{\lambda>0} \max_{i\in\{1,2\}} \frac{L_i\lambda^2}{\frac{\lambda^2}{k_1w_1^2+k_2w_2^2}w_i^2(k_i+L_i)^2+k_i\lambda^2} = \max_{(i,j)\in\{(1,2),(2,1)\}} \frac{L_i/k_i}{v_i(L_i/k_i+1)^2+1}. \tag{A.117}$$

This establishes the upper bound in (A.114). Denoting (i^*, j^*) maximizing the right-hand-side expression above, and using the optimal values from Lemma A.18, $\hat\beta_{i^*} = \frac{\hat\lambda}{w_{i^*}(k_{i^*}+L_{i^*})}$ (with $\hat\beta_{j^*} = 0$ and an arbitrary $\hat\lambda > 0$), we have $\max(\hat\beta_1/w_1, \hat\beta_2/w_2) = \hat\beta_{i^*}/w_{i^*} = \frac{\hat\lambda}{w_{i^*}^2(k_{i^*}+L_{i^*})}$ hence equality holds in (A.117) if the following inequality is satisfied

$$(k_1w_1^2+k_1w_2^2)\frac{1}{w_{i^*}^2(k_{i^*}+L_{i^*})} \leqslant 1,$$

or equivalently if $\frac{k_{i^*}}{v_{i^*}(k_{i^*}+L_{i^*})} \leqslant 1$. This is guaranteed as soon as $v_\ell \geqslant \frac{k_\ell}{k_\ell+L_\ell}$ for every $\ell \in \{1,2\}$. This establishes the equality case in the rhs of (A.114).

We now treat the lower bound in (A.114). For fixed $\beta_i \geqslant 0$ and $\lambda > 0$ such that $(k_1+L_1)w_1\beta_1 + (k_2+L_2)w_2\beta_2 = \lambda$, we still have $\lambda > k_1w_1\beta_1 + k_2w_2\beta_2$. By Lemma A.17, letting

$$V = \min_{\substack{\alpha_i:\alpha_i\geqslant\beta_i \\ \sum_{i=1}^2 k_iw_i\alpha_i=\lambda}} k_1\alpha_1^2+k_2\alpha_2^2, \tag{A.118}$$

we either have

- $V = \min\left(k_1\beta_1^2 + k_2(\frac{\lambda-k_1w_1\beta_1}{k_2w_2})^2, k_2\beta_2^2 + k_1(\frac{\lambda-k_2w_2\beta_2}{k_1w_1})^2\right)$;
- or

$$V = \lambda^2/(k_1w_1^2+k_2w_2^2) = \min_{\substack{\alpha_i:\alpha_i\geqslant 0 \\ \sum_{i=1}^2 k_i\alpha_i=\lambda}} k_1\alpha_1^2+k_2\alpha_2^2$$
$$\leqslant \min\left(k_1\beta_1^2 + k_2(\frac{\lambda-k_1w_1\beta_1}{k_2w_2})^2, k_2\beta_2^2 + k_1(\frac{\lambda-k_2w_2\beta_2}{k_1w_1})^2\right) \tag{A.119}$$

where the last inequality was obtained by evaluating $k_1\alpha_1^2+k_2\alpha_2^2$ at $\alpha_1 = \beta_1$ (resp. at $\alpha_2 = \beta_2$) with α_2 (resp. α_1) tuned so that $k_1w_1\alpha_1 + k_2w_2\alpha_2 = \lambda$.

We deduce that $V \leqslant \min\left(k_1\beta_1^2 + k_2(\frac{\lambda - k_1 w_1 \beta_1}{k_2 w_2})^2, k_2\beta_2^2 + k_1(\frac{\lambda - k_2 w_2 \beta_2}{k_1 w_1})^2\right)$ and it follows using (A.115) that

$$\begin{aligned} B^{L_1,L_2}(w) &\geqslant \sup_{\lambda>0} \sup_{\substack{\beta_i:\beta_1,\beta_2\geqslant 0 \\ \sum_{i=1}^2 w_i(k_i+L_i)\beta_i=\lambda}} \frac{L_1\beta_1^2 + L_2\beta_2^2}{\min\left(k_1\beta_1^2 + k_2(\frac{\lambda - k_1 w_1 \beta_1}{k_2 w_2})^2, k_2\beta_2^2 + k_1(\frac{\lambda - k_2 w_2 \beta_2}{k_1 w_1})^2\right) + k_1\beta_1^2 + k_2\beta_2^2} \\ &= \sup_{\substack{\beta_i:\beta_1,\beta_2\geqslant 0 \\ \sum_{i=1}^2 w_i(k_i+L_i)\beta_i=1}} \frac{L_1\beta_1^2 + L_2\beta_2^2}{\min\left(k_1\beta_1^2 + k_2(\frac{1 - k_1 w_1 \beta_1}{k_2 w_2})^2, k_2\beta_2^2 + k_1(\frac{1 - k_2 w_2 \beta_2}{k_1 w_1})^2\right) + k_1\beta_1^2 + k_2\beta_2^2} \\ &= \sup_{\substack{\beta_i:\beta_1,\beta_2\geqslant 0 \\ \sum_{i=1}^2 w_i(k_i+L_i)\beta_i=1}} \max\left(\frac{L_1\beta_1^2 + L_2\beta_2^2}{2k_1\beta_1^2 + k_2\beta_2^2 + k_2(\frac{1 - k_1 w_1 \beta_1}{k_2 w_2})^2}, \frac{L_1\beta_1^2 + L_2\beta_2^2}{k_1\beta_1^2 + 2k_2\beta_2^2 + k_1(\frac{1 - k_2 w_2 \beta_2}{k_1 w_1})^2}\right). \end{aligned} \tag{A.120}$$

For $(i,j) \in \{(1,2),(2,1)\}$, using the values $\tilde{\beta}_i = \frac{1}{w_i(k_i+L_i)}, \tilde{\beta}_j = 0$, we have

$$B^{L_1,L_2}(w) \geqslant \frac{L_1\tilde{\beta}_i^2}{2k_i\tilde{\beta}_i^2 + k_j(\frac{1 - k_i w_i \tilde{\beta}_i}{k_j w_j})^2}. \tag{A.121}$$

Since $1 - k_i w_i \tilde{\beta}_i = w_i(k_i + L_i)\tilde{\beta}_i - k_i w_i \tilde{\beta}_i = w_i L_i \tilde{\beta}_i$, we have

$$\frac{L_i\tilde{\beta}_i^2}{2k_i\tilde{\beta}_i^2 + k_j(\frac{1 - k_i w_i \tilde{\beta}_i}{k_j w_j})^2} = \frac{L_i\tilde{\beta}_i^2}{2k_i\tilde{\beta}_i^2 + k_j(\frac{w_i L_i}{k_j w_j})^2\beta_i^2} = \frac{L_i}{2k_i + k_j(\frac{w_i L_i}{k_j w_j})^2}.$$

Since $\nu_i = \frac{1}{1 + k_j w_j^2/(k_i w_i^2)}$, we have $(1-\nu_i)/\nu_i = 1/\nu_i - 1 = k_j w_j^2/k_i w_i^2$. We deduce

$$B^{L_1,L_2}(w) \geqslant \frac{L_i}{2k_i + \frac{(w_i L_i)^2}{k_j w_j^2}} = \frac{L_i/k_i}{\frac{k_i w_i^2}{k_j w_j^2}(L_i/k_i)^2 + 2} = \frac{L_i/k_i}{\frac{\nu_i}{1-\nu_i}(L_i/k_i)^2 + 2}. \tag{A.122}$$

□

The following function study will be used to deal with the optimization of the $B^{L_1,L_2}(w)$.

LEMMA A.23 Consider a such that $0 < a \leqslant 1$. The function

$$g_1 : u \geqslant 0 \mapsto g_1(u;a) := \frac{u}{a(u+1)^2 + 1} \tag{A.123}$$

is maximized at $u_1^* = \sqrt{1 + 1/a}$, increasing for $u \leqslant u_1^*$, decreasing for $u \geqslant u_1^*$ and

$$g_1(u_1^*;a) = \frac{1}{2}(\sqrt{1 + 1/a} - 1). \tag{A.124}$$

Proof. Since $g_1'(u;a) = \frac{-au^2+a+1}{(a(u+1)^2+1)^2}$, the equality $g_1'(u_1^*;a) = 0$ implies $a(u_1^*)^2 = a+1$ and $u_1^* = \sqrt{1+1/a}$. Given the sign of $g_1'(u;a)$, $g_1(\cdot;a)$ is increasing for $u \leqslant u_1^*$ and decreasing for $u \geqslant u_1^*$. As $a = [(u_1^*)^2 - 1]^{-1}$ we get

$$f_1(a) := g_1(u_1^*, a) = \frac{u_1^*((u_1^*)^2-1)}{(u_1^*+1)^2 + (u_1^*)^2 - 1} = \frac{u_1^*((u_1^*)^2-1)}{2(u_1^*)^2 + 2u_1^*} = \frac{1}{2}(u_1^* - 1) = \frac{1}{2}(\sqrt{1+1/a} - 1) \tag{A.125}$$

which is decreasing with respect to a. □

LEMMA A.24 Consider $0 \leqslant a \leqslant 1$, $g_1(u;a) := \frac{u}{a(u+1)^2+1}$ and $g_2(u;a) := \frac{u}{\frac{a}{1-a}u^2+2}$ and define

$$h_i(u,v;a) := \max(g_i(u;a), g_i(v;1-a)), i \in \{1,2\}. \tag{A.126}$$

1. For $a = 1/2$ we have $h_1(u,v;1/2) \leqslant \frac{\sqrt{3}-1}{2}$ for every $u, v \geqslant 0$.
2. If $a \notin [\tilde{a}, 1-\tilde{a}]$, where $\tilde{a} := 2\sqrt{3} - 3 \approx 0.46$ then $h_2(2,2,a) > \frac{\sqrt{3}-1}{2}$.

Consider integers such that $k_i \geqslant 2$, $1 \leqslant 4k_i \leqslant n_i$ for $i \in \{1,2\}$ and

$$H_1(a) := \max_{0 \leqslant L_i \leqslant n_i - 2k_i} h_1(L_1/k_1, L_2/k_2; a). \tag{A.127}$$

3. There exists $a^* \in [\tilde{a}, 1-\tilde{a}]$ such that $H_1(a^*) = \min_{a \in [\tilde{a}, 1-\tilde{a}]} H_1(a)$.
4. Consider $a \in [\tilde{a}, 1-\tilde{a}]$ then

$$H_1(a) = \max\left(\max_{L_1 \in \{\lfloor k_1\sqrt{1+1/a}\rfloor; \lceil k_1\sqrt{1+1/a}\rceil\}} g_1(L_1/k_1; a), \max_{L_2 \in \{\lfloor k_2\sqrt{1+1/(1-a)}\rfloor; \lceil k_2\sqrt{1+1/(1-a)}\rceil\}} g_1(L_2/k_2; 1-a) \right).$$

where $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ denote the lower and upper integer part. Moreover, the L_i^* maximizing the above expression are also maximizing (A.127) and are such that $L_1^*/k_1 \geqslant 1/a - 1$ and $L_2^*/k_2 \geqslant 1/(1-a) - 1$.

Proof.

Item 1. By Lemma A.23, with $a = 1/2$ and any $u, v \geqslant 0$ we have $g_1(u;a) \leqslant g_1(u_1^*;a) = \frac{1}{2}(\sqrt{1+1/a} - 1) = (\sqrt{3}-1)/2$ and similarly $g_1(v;1-a) = (\sqrt{3}-1)/2$ hence $h_1(u,v;a) = (\sqrt{3}-1)/2$.

Item 2. We prove the inequality for $a < \tilde{a}$. Since $h_2(2,2;1-a) = h_2(2,2;a)$ by definition of h_2, the same inequality holds if $a > 1-\tilde{a}$. For $a < \tilde{a}$ since $a < 1/2$ we have $a/(1-a) < (1-a)/a$ hence using the definition of g_2 we have $g_2(2,a) > g_2(2,1-a)$. By monotonicity of $a \mapsto (1-a)/(1+a)$ we get

$$h_2(2,2;a) = \max(g_2(2,a), g_2(2,1-a)) = g_2(2,a) = \frac{2}{4\frac{a}{1-a}+2} = \frac{1-a}{1+a} > \frac{1-\tilde{a}}{1+\tilde{a}} = g_2(2,\tilde{a}).$$

Finally, we compute

$$g_2(2,\tilde{a}) = \frac{1-(2\sqrt{3}-3)}{1+2\sqrt{3}-3} = \frac{4-2\sqrt{3}}{2\sqrt{3}-2} = \frac{(4-2\sqrt{3})(2\sqrt{3}+2)}{(2\sqrt{3})^2-2^2} = \frac{8\sqrt{3}-12+8-4\sqrt{3}}{8} = \frac{4\sqrt{3}-4}{8} = \frac{\sqrt{3}-1}{2}.$$

Item 3. The function H_1 is defined as the maximum of a finite number of continuous functions of a. By continuity of the maximum, H_1 is continuous and its minimum on the compact set $[\tilde{a}, 1-\tilde{a}]$ is reached.

Item 4. Consider $a \in [\tilde{a}, 1-\tilde{a}]$. By Lemma A.23 the function $u \mapsto g_1(u;a)$ is maximized at a $u_1^* = \sqrt{1+1/a}$. Similarly, $v \mapsto g_1(v;1-a)$ is maximized at $u_2^* = \sqrt{1+1/(1-a)}$. Since $1/3 \leqslant \tilde{a} \leqslant 1/2$, with $\nu_1 = a$, $\nu_2 = 1-a$ we have $\nu_i \geqslant 1/3$ hence $u_i^* = \sqrt{1+1/\nu_i} \leqslant 2$. Moreover, since we assume $n_i \geqslant 4k_i$, we have $n_i - 2k_i \geqslant 2k_i \geqslant k_i\sqrt{1+1/\nu_i} = k_i u_i^*$. It follows that for $i \in \{1,2\}$ we have

$$\max_{0 \leqslant L_i \leqslant n_i - 2k_i} g_1(L_i/k_i; \nu_i) = \max_{L_i \in \{\lfloor k_i u_i^* \rfloor; \lceil k_i u_i^* \rceil\}} g_1(L_i/k_i; \nu_i).$$

As a result, the maximizers L_i^* of both sides are identical, and we have $H_1(a) = \max(g_1(L_1^*/k_1; a), g_1(L_2^*/k_2; 1-a))$ with

$$L_1^* \in \arg \max_{L_1 \in \{\lfloor k_1 u_1^* \rfloor; \lceil k_1 u_1^* \rceil\}} g_1(L_1/k_1; a)$$
$$L_2^* \in \arg \max_{L_2 \in \{\lfloor k_2 u_2^* \rfloor; \lceil k_2 u_2^* \rceil\}} g_1(L_2/k_2; 1-a).$$

There remains to show that $L_i^*/k_i \geqslant 1/\nu_1 - 1$. For this, we first observe that since $k_i \geqslant 2$ we have

$$L_i^*/k_i \geqslant \lfloor k_i u_i^* \rfloor / k_i \geqslant (k_i u_i^* - 1)/k_i = u_i^* - 1/k_i \geqslant u_i^* - 1/2 = \sqrt{1+1/\nu_i} - 1/2.$$

The derivative of $x \to \sqrt{1+x} - 1/2 - (x-1) = \sqrt{1+x} - x + 1/2$ at any $x \geqslant 0$ is $1/(2\sqrt{1+x}) - 1 \leqslant -1/2$ hence this function is monotonically decreasing. Since $a \in [\tilde{a}, 1-\tilde{a}]$ and $\nu_1 = a$, $\nu_2 = 1-a$ we have $\nu_i \geqslant \tilde{a}$ hence $1/\nu_i \leqslant 1/\tilde{a}$ for $i \in \{1,2\}$, hence

$$\sqrt{1+1/\nu_i} - 1/2 - (1/\nu_i - 1) \geqslant \sqrt{1+1/\tilde{a}} - 1/2 - (1/\tilde{a} - 1) \approx 0.12 > 0.$$

We deduce that $L_i^*/k_i \geqslant 1/\nu_i - 1$ as claimed.

□

We can conclude with the proof of Theorem 4.2.

Proof of Theorem 4.2. The proof starts from the fact (Corollary 3.1) that

$$\arg\max_{R \in \mathcal{C}'} \delta_\Sigma^{\mathtt{nec}}(R) = \arg\min_{R \in \mathcal{C}'} B_\Sigma(R) \tag{A.128}$$

with $\mathcal{C}' = \{R(\cdot) = \|\cdot\|_w : w = (w_1, w_2), w_1 > 0, w_2 > 0\}$. Using Lemma A.21, for each w we have

$$B_\Sigma(\|\cdot\|_w) = \max_{0 \leqslant L_i \leqslant n - 2k_i, i \in \{1,2\}} B^{L_1,L_2}(w). \tag{A.129}$$

With the notations of Lemma A.22 we have $\mu_1 = w_1/w_2$ and $\mu_2 = w_2/w_1$ hence $\mu_1 = 1/\mu_2$, and one can check that $\nu_1 + \nu_2 = 1$ where $\nu_1 = \nu_1(w) := (1 + k_2 w_2^2/(k_1 w_1^2))^{-1}$. Hence, by Lemma A.22 (taking $u = L_1/k_1, v = L_2/k_2, a = \nu_1$ and using (A.126)) and with the notation of Lemma A.24, for all integers $0 \leqslant L_i \leqslant n_i - 2k_i$ we have

$$h_2(L_1/k_1, L_2/k_2; \nu_1) \leqslant B^{L_1,L_2}(w) \leqslant h_1(L_1/k_1, L_2/k_2; \nu_1) \tag{A.130}$$

with equality in the right hand s if for each $i \in \{1,2\}$ we have $\nu_i \geqslant L_i/(k_i + L_i)$, i.e., $L_i/k_i \geqslant 1/\nu_i - 1$. Using (A.129) we get

$$\max_{0 \leqslant L_i \leqslant n-2k_i, i \in \{1,2\}} h_2(L_1/k_1, L_2/k_2; \nu_1) \leqslant B_\Sigma(\|\cdot\|_w) \leqslant \max_{0 \leqslant L_i \leqslant n-2k_i, i \in \{1,2\}} h_1(L_1/k_1, L_2/k_2; \nu_1), \tag{A.131}$$

and if the maximizers L_i^* of the right-hand side of (A.131) satisfy $L_i^*/k_i \geqslant 1/\nu_i - 1$ for each $i \in \{1,2\}$ then in fact

$$B_\Sigma(\|\cdot\|_w) = H_1(\nu_1) := h_1(L_1^*/k_1, L_2^*/k_2; \nu_1) \tag{A.132}$$

where H_1 is defined as the maximum of h_1 over the L_i/k_i (Lemma A.24). In particular for $\nu_i = 1/2$, this is verified if $L_i^* \geqslant k_i$. Next we proceed in three steps. We set $\tilde{a} := 2\sqrt{3} - 3 \approx 0.46$.

Step 1. We show that if w', w'' are such that $\nu_1(w') \notin [\tilde{a}, 1-\tilde{a}]$ and $\nu_1(w'') = 1/2 \in [\tilde{a}, 1-\tilde{a}]$ then

$$B_\Sigma(\|\cdot\|_{w'}) > \frac{\sqrt{3}-1}{2} \geqslant B_\Sigma(\|\cdot\|_{w''}).$$

A first consequence is to establish (4.6), using Corollary 3.1 to convert the bound on $B_\Sigma(\|\cdot\|_w)$, for $w \in \{w^*, w_0\}$, to a bound on $\delta_\Sigma^{\mathtt{nec}}(\|\cdot\|_w)$. Indeed, since $\nu_1(w'') = k_1(w_1'')^2/(k_1(w_1'')^2 + k_2(w_2'')^2)$, the fact that $\nu_1(w'') = 1/2$ corresponds to $w'' \propto (1/\sqrt{k_1}, 1/\sqrt{k_2}) =: w_0(k_1,k_2)$, hence $B_\Sigma^* \leqslant B_\Sigma(\|\cdot\|_{w_0}) \leqslant (\sqrt{3}-1)/2$.

A second consequence is that the optimization of $w = (w_1, w_2)$ can be restricted to a range corresponding to $\nu_1 = \nu_1(w) \in [\tilde{a}, 1-\tilde{a}]$.

Indeed, on the one hand, for $\nu_1(w'') = 1/2$, by Lemma A.24-Item 4 we have ,

$$H_1(1/2) = \max\left(\max_{L_1 \in \{\lfloor k_1\sqrt{3}\rfloor; \lceil k_1\sqrt{3}\rceil\}} g_1(L_1/k_1; 1/2), \max_{L_2 \in \{\lfloor k_2\sqrt{3}\rfloor; \lceil k_2\sqrt{3}\rceil\}} g_1(L_2/k_2; 1/2) \right)$$

where g_1 is defined in Lemma A.24.

Hence, $L_i^* \geqslant k_i$ so that (A.132) holds, and we deduce that $B_\Sigma(\|\cdot\|_{w''}) = H_1(1/2)$.

From Lemma A.24-Item 1, we have

$$H_1(1/2) \leqslant \frac{\sqrt{3}-1}{2},$$

and we obtain $B_\Sigma(\|\cdot\|_w) \leqslant (\sqrt{3}-1)/2$ as claimed.

We can also establish the conclusion of the theorem (4.7) by considering $\sup_{k_1', k_2' \geqslant 1, n_1' \geqslant 4k_1', n_2' \geqslant 4k_2'} B_\Sigma(\|\cdot\|_{w''})$. Using the expression of $H(1/2)$ we have that

$$\begin{aligned} \sup_{k_1', k_2' \geqslant 1, n_1' \geqslant 4k_1', n_2' \geqslant 4k_2'} B_\Sigma(\|\cdot\|_{w''}) &= \sup_{k_1' \geqslant 1, n_1' \geqslant 4k_1'} \max_{L_1 \in \{\lfloor k_1'\sqrt{3}\rfloor; \lceil k_1'\sqrt{3}\rceil\}} g_1(L_1/k_1'; 1/2) \\ &= \sup_{k \geqslant 1} \max_{L \in \{\lfloor k\sqrt{3}\rfloor; \lceil k\sqrt{3}\rceil\}} g_1(L/k; 1/2) \end{aligned} \tag{A.133}$$

Using Lemma A.23, as g_1 is continuous and $\lfloor k\sqrt{3}\rfloor/k \to_{k\to\infty} \sqrt{3} = u_1^*$ the maximizer of $g_1(\cdot; 1/2)$ (because $(k\sqrt{3}-1)/k \leqslant \lfloor k\sqrt{3}\rfloor/k \leqslant \sqrt{3}$), we have $\sup_k \max_{L \in \{\lfloor k\sqrt{3}\rfloor; \lceil k\sqrt{3}\rceil\}} g_1(L/k; 1/2) = g_1(u_1^*; 1/2) = (\sqrt{3}-1)/2$. Again using Corollary 3.1 to link $\delta_\Sigma^{\mathtt{nec}}$ and B_Σ yields (4.7).

On the other hand, if $\nu_1 \notin [\tilde{a}, 1-\tilde{a}]$ then by Lemma A.24-Item 2 we have $h_2(2,2;\nu_1) > (\sqrt{3}-1)/2$. Since $n_i \geqslant 4k_i$, the integers $L_i := 2k_i$, $i \in \{1,2\}$ satisfy $0 < L_i \leqslant n_i - 2k_i$ hence, by the left-hand side in (A.131),

$$B_\Sigma(\|\cdot\|_{w'}) \geqslant h_2(L_1/k_1, L_2/k_2, \nu_1) = h_2(2,2;\nu_1) > \frac{\sqrt{3}-1}{2}.$$

Step 2. We show that if w satisfies $\nu_1 = \nu_1(w) \in [\tilde{a}, 1-\tilde{a}]$ then $B_\Sigma(\|\cdot\|_w) = H_1(\nu_1(w))$.

Since $k_i \geqslant 2$ and $n_i \geqslant 4k_i$, by Lemma A.24-Item 4, we have the equality $H_1(\nu_1) = h_1(L_1^*/k_1, L_2^*/k_2, \nu_1)$ where $L_1^* \in \{\lfloor k_1\sqrt{1+1/\nu_1}\rfloor; \lceil k_1\sqrt{1+1/\nu_1}\rceil\}$, $L_2^* \in \{\lfloor k_2\sqrt{1+1/(1-\nu_1)}\rfloor; \lceil k_2\sqrt{1+1/(1-\nu_1)}\rceil\}$ and $L_i^*/k_i \geqslant 1/\nu_i - 1$. By (A.131)- (A.132) we deduce that the equality $B_\Sigma(\|\cdot\|_w) = H_1(\nu_1(w))$ holds.

Step 3. By Lemma A.24-Item 3, there is $a^* \in [\tilde{a}, 1-\tilde{a}]$ such that $H_1(a^*) = \min_{\tilde{a}\leqslant a\leqslant 1-\tilde{a}} H_1(a)$. In light of Steps 1 and 2, the infimum over w of $B_\Sigma(\|\cdot\|_w)$ is thus achieved, and a weight vector w^* satisfies

$$B_\Sigma(\|\cdot\|_{w^*}) = \min_w B_\Sigma(\|\cdot\|_w)) = H_1(a^*) \tag{A.134}$$

if, and only if $H_1(\nu_1(w^*)) = H_1(a^*)$. Since $\nu_1(w) = \left(1 + \frac{k_2}{k_1}(w_2/w_1)^2\right)^{-1}$, combining all the above yields

$$\frac{w_2^*}{w_1^*} = \sqrt{\frac{k_1}{k_2}(1/\nu_1^* - 1)}$$

where ν_1^* is an optimum of

$$B_\Sigma(\|\cdot\|_{w^*}) = \min_{\nu_1\in[\tilde{a},1-\tilde{a}],\nu_2=1-\nu_1} \max_{i\in\{1,2\}} \max_{x_i\in\{\lfloor k_i\sqrt{1+1/\nu_i}\rfloor;\lceil k_i\sqrt{1+1/\nu_i}\rceil\}} g_1(x_i/k_i;\nu_i).$$

□

The following Lemma is needed for the proof of Theorem 4.3.

LEMMA A.25 Consider integers $n \geqslant k \geqslant 1$, a nonzero vector $z \in \mathbb{R}^n$, S a set of the k largest entries of z. There exists $0 \leqslant r \leqslant k-1$, $\beta \in \mathbb{R}^n$, $\gamma \geqslant 0$ such that

$$\|\beta\|_0 = k - r - 1 \leqslant L := |\mathtt{supp}(z_{S^c})| \leqslant n-1 \tag{A.135}$$

$$\|\beta\|_\infty \leqslant \min_{l\in S} |z(l)| \tag{A.136}$$

$$\|z_{S^c}\|_1 = \|\beta\|_1 + \gamma \tag{A.137}$$

$$\|z_{S^c}\|_{\Sigma_k}^2 = \|\beta\|_2^2 + \frac{1}{r+1}\gamma^2 \tag{A.138}$$

Moreover if $k-r-1 \geqslant 1$ then

$$\gamma < (r+1) \min_{l\in\mathtt{supp}(\beta)} |\beta(l)|\,; \tag{A.139}$$

Proof. We use the fact that for any integer k the norm $\|\cdot\|_{\Sigma_k}$ coincides with the so-called k-support norm [4, Definition 2.1], that is invariant by permutation of the coordinates and has the following expression for each $y \in \mathbb{R}^n$ sorted in descending order:

$$\|y\|_{\Sigma_k}^2 = \sum_{l=1}^{k-r-1} |y(l)|^2 + \frac{1}{r+1}\left(\sum_{l=k-r}^{n} |y(l)|\right)^2 \tag{A.140}$$

where r is the unique integer in $\{0,\dots,k-1\}$ such that

$$|y(k-r-1)| > \frac{1}{r+1}\sum_{l=k-r}^{n} |y(l)| \geqslant |y(k-r)|, \tag{A.141}$$

with the convention $|y(0)| = +\infty$, see [4, Proposition 2.1].

We apply the above characterization with $y \in \mathbb{R}^n$ the sorting of z_{S^c} by descending order of absolute values. Notice that $k-r-1 \leqslant L := |\mathtt{supp}(z_{S^c})| = |\mathtt{supp}(y_i)| \leqslant n-1$ (since $z \neq 0$, $|S| \geqslant 1$), which establishes the rhs inequality in (A.135): otherwise we would have $y(k-r-1) = y(k-r) = 0$ which would contradict (A.141).

We define $\beta \in \mathbb{R}^n$ and $\gamma \geqslant 0$ as

$$\beta(l) := |y(l)|,\ 1 \leqslant l \leqslant k-r-1; \quad \beta(l) := 0,\ k-r \leqslant l \leqslant n; \tag{A.142}$$

$$\gamma := \sum_{l=k-r}^{n} |y(l))| \tag{A.143}$$

Since $\beta(l)$ is non-increasing with l, with $\beta(k-r-1) = |y(k-r-1)| > 0$ and $\beta(k-r) = 0$ we have $\|\beta\|_0 = k-r-1$ hence (A.135) holds. Moreover, by definition of S, we also have $\min_{l\in S}|z(l)| \geqslant \|z_{S^c}\|_\infty = \|y\|_\infty = \|\beta\|_\infty$ hence (A.136) holds. We also obviously have $\|z_{S^c}\|_1 = \|y\|_1 = \|\beta\|_1 + \gamma$, i.e. the required identity (A.137).

Since y is a decreasing rearrangement of z_{S^c} and $\|\cdot\|_{\Sigma_k}$ is invariant by permutation we have

$$\|z_{S^c}\|_{\Sigma_k}^2 = \|y\|_{\Sigma_k}^2 \overset{\text{(A.140)+(A.142)+(A.143)}}{=} \|\beta\|_2^2 + \frac{1}{r+1}\gamma^2.$$

This establishes (A.138). Finally when $k-r-1 \geqslant 1$ we have $\min_{l\in\mathtt{supp}(\beta)}\beta(l) = \beta(k-r-1) = |y(k-r-1)|$ hence (A.139) is a direct consequence of (A.141).

□

We now give the proof of Theorem 4.3.

Proof of Theorem 4.3. The assumptions of Lemma 3.5 hold, so we can rely on expression (3.30): to lower bound $\delta^{\mathtt{suff}}_{\Sigma_{k_1,k_2}}(\|\cdot\|_w)$ by $1/\sqrt{3}$, we thus upper bound $\|z-P_\Sigma(z)\|_\Sigma^2$ by $2\|P_\Sigma(z)\|_2^2$ for every $z \in \mathcal{T}_\Sigma(\|\cdot\|_w)$. First we characterize $P_\Sigma(z)$ for any z. With $T = T(z) = (S_1, S_2)$ defined as in the beginning of Appendix A.6 we have $P_\Sigma(z) = z_T$ because for $z = (z_1, z_2) \in \mathcal{H}$ we have

$$\min_{y\in\Sigma}\|z-y\|_2^2 = \min_{y_1\in\Sigma_{k_1}, y_2\in\Sigma_{k_2}}\left(\|z_1-y_1\|_2^2 + \|z_2-y_2\|_2^2\right) = \|z_1-(z_1)_{S_1}\|^2 + \|z_2-(z_2)_{S_2}\|_2^2 = \|z-z_T\|_2^2.$$

We will use that, by Lemma A.16, we have

$$z \in \mathcal{T}_\Sigma(\|\cdot\|_w) \Leftrightarrow \sum_{i=1}^{2}\|(z_i)_{S_i^c}\|_1/\sqrt{k_i} \leqslant \sum_{i=1}^{2}\|(z_i)_{S_i}\|_1/\sqrt{k_i}. \tag{A.144}$$

Now, using the fact that $\|(u_1,u_2)\|_\Sigma^2 = \|u_1\|_{\Sigma_{k_1}}^2 + \|u_2\|_{\Sigma_{k_2}}^2$ (from [38, Lemma 4.2]) we obtain

$$\|z-P_\Sigma(z)\|_\Sigma^2 = \|z_{T^c}\|_\Sigma^2 = \sum_{i=1}^{2}\|(z_i)_{S_i^c}\|_{\Sigma_{k_i}}^2 \tag{A.145}$$

With Lemma A.25, we obtain an explicit expression of the ratio $\|z-P_\Sigma(z)\|_\Sigma^2/\|P_\Sigma(z)\|_2^2$. For $i \in \{1,2\}$, let $L_i = |\mathtt{supp}((z_i)_{S_i^c})|$. There exists $0 \leqslant r_i \leqslant k_i - 1$ and $\beta_i \in \mathbb{R}^{n_i}$, $\gamma_i \geqslant 0$ such that $\|\beta_i\|_\infty \leqslant \min_{l\in S_i}|z_i(l)|$, $\|\beta_i\|_0 = k_i - r_i - 1 \leqslant L_i$ and

$$\|(z_i)_{S_i}\|_{\Sigma_{k_i}}^2 = \|\beta_i\|_2^2 + \frac{1}{r_i+1}\gamma_i^2 \tag{A.146}$$

$$\|(z_i)_{S_i^c}\|_1 = \|\beta_i\|_1 + \gamma_i. \tag{A.147}$$

where, if $k_i - r_i - 1 \geqslant 1$, we further have

$$\gamma_i < (r_i+1) \min_{l \in \mathrm{supp}(\beta_i)} |\beta_i(l)|. \tag{A.148}$$

Consider $i \in \{1,2\}$. Depending on the value of $k_i - r_i - 1$, we have the following properties

- If $k_i - r_i - 1 = 0$ then $\|\beta_i\|_0 = 0$ and $\beta_i = 0$. Hence by (A.147) we have $\gamma = \|(z_i)_{S_i^c}\|_1$ and

$$\|\beta_i\|_2^2 + \frac{1}{r_i+1}\gamma_i^2 = \frac{\|(z_i)_{S_i^c}\|_1^2}{k_i}. \tag{A.149}$$

- If $k_i - r_i - 1 \geqslant 1$ then, since $k_i\|\beta_i\|_\infty^2 \leqslant k_i \min_{l\in S}|z_i(l)|^2 \leqslant \|(z_i)_{S_i}\|_2^2$ and $\|\beta_i\|_0 = k_i - r_i - 1$, we have

$$\begin{aligned} \|\beta_i\|_2^2 + \frac{1}{r_i+1}\gamma_i^2 &\overset{(A.148)}{\leqslant} \|\beta_i\|_2^2 + (r_i+1)\cdot\left(\min_{l\in\mathrm{supp}(\beta_i)}|\beta_i(l)|\right)^2 \\ &\leqslant [(k_i - r_i - 1) + (r_i+1)]\cdot\|\beta_i\|_\infty^2 = k_i\|\beta_i\|_\infty^2 \leqslant \|(z_i)_{S_i}\|_2^2. \end{aligned} \tag{A.150}$$

Thanks to these properties, we distinguish two easy cases to bound $\|z - P_\Sigma(z)\|_\Sigma^2 / \|P_\Sigma(z)\|_2^2$.

1. If $k_i - r_i - 1 \geqslant 1$ for each $i \in \{1,2\}$ then

$$\|z - P_\Sigma(z)\|_\Sigma^2 \overset{(A.138)}{=} \sum_{i=1}^2 \left(\|\beta_i\|_2^2 + \frac{1}{r_i+1}\gamma_i^2\right) \overset{(A.150)}{\leqslant} \sum_{i=1}^2 \|(z_i)_{S_i}\|_2^2 = \|P_\Sigma(z)\|_2^2.$$

2. If $k_i - r_i - 1 = 0$ for each $i \in \{1,2\}$ then

$$\begin{aligned} \|z - P_\Sigma(z)\|_\Sigma^2 \overset{(A.138)}{=} \sum_{i=1}^2 \left(\|\beta_i\|_2^2 + \frac{1}{r_i+1}\gamma_i^2\right) &\overset{(A.149)}{=} \sum_i \|z_{S_i^c}\|_1^2/k_i \\ &\overset{|a|^2+|b|^2 \leqslant (|a|+|b|)^2}{\leqslant} (\sum_i \|z_{S_i^c}\|_1/\sqrt{k_i})^2 \\ &\overset{(A.144)}{\leqslant} (\sum_i \|z_{S_i}\|_1/\sqrt{k_i})^2 \\ &\leqslant (\sum_i \|z_{S_i}\|_2)^2 \\ &\overset{(|a|+|b|)^2 \leqslant 2(|a|^2+|b|^2)}{\leqslant} 2\sum_i \|z_{S_i}\|_2^2 = 2\|P_\Sigma(z)\|_2^2. \end{aligned}$$

In both cases we obtain $\|z - P_\Sigma(z)\|_\Sigma^2 / \|P_\Sigma(z)\|_2^2 \leqslant 2$.

When these easy cases do not hold we have e.g. $0 = k_1 - r_1 - 1$ and $k_2 - r_2 - 1 \geqslant 1$ (the same reasoning holds if $k_2 - r_2 - 1 = 0$ and $k_1 - r_1 - 1 \geqslant 1$), $\|z_{S_1^c}\|_{\Sigma_{k_1}}^2 = \|z_{S_1^c}\|_1^2/k_1$ and $\|z_{S_2^c}\|_{\Sigma_{k_2}}^2 = \|\beta_2\|_2^2 + \gamma^2/(r+1)$.

This gives

$$\begin{aligned}\|z-P_\Sigma(z)\|_\Sigma^2 &= \|z_{S_1^c}\|_1^2/k_1+\|\beta_2\|_2^2+\gamma^2/(r+1)\\ &\overset{(A.144)}{\leqslant} (\|z_{S_1}\|_1/\sqrt{k_1}+\|z_{S_2}\|_1/\sqrt{k_1}-\|z_{S_2^c}\|_1/\sqrt{k_2})^2+\|\beta_2\|_2^2+\gamma^2/(r+1)\\ &\overset{(A.147)}{=} (\|z_{S_1}\|_1/\sqrt{k_1}+\|z_{S_2}\|_1/\sqrt{k_1}-(\|\beta_2\|_1+\gamma)/\sqrt{k_2})^2+\|\beta_2\|_2^2+\gamma^2/(r+1).\end{aligned}$$

We have $0\leqslant \|z_{S_1}\|_1/\sqrt{k_1}+\|z_{S_2}\|_1/\sqrt{k_1}-(\|\beta_2\|_1+\gamma)/\sqrt{k_2})^2 \leqslant \|z_{S_1}\|_2+\|z_{S_2}\|_2-(\|\beta_2\|_1+\gamma)/\sqrt{k_2})^2$ and

$$\|z-P_\Sigma(z)\|_\Sigma^2 \leqslant (\|z_{S_1}\|_2+\|z_{S_2}\|_2-(\|\beta_2\|_1+\gamma)/\sqrt{k_2})^2+\|\beta_2\|_2^2+\gamma^2/(r+1). \tag{A.151}$$

As this a quadratic function of $\gamma\geqslant 0$ with positive leading coefficient it is maximized, at either bound of the range of γ, i.e $\gamma=0$ or $\gamma\to(r_2+1)\min_{l\in\mathtt{supp}(\beta_2)}|\beta_2(l)|=:(r_2+1)\tilde\beta_2$.

For the case, $\gamma\to(r_2+1)\tilde\beta_2$,

$$\|z-P_\Sigma(z)\|_\Sigma^2 \leqslant (\|z_{S_1}\|_2+\|z_{S_2}\|_2-(\|\beta_2\|_1+(r_2+1)\tilde\beta_2)/\sqrt{k_2})^2+\|\beta_2\|_2^2+(r_2+1)\tilde\beta_2^2. \tag{A.152}$$

Let us call $f(\beta_2)$ the numerator. For fixed $\tilde\beta_2,\|\beta_2\|_\infty$, consider β_2^* the maximizer of f under the constraint $\tilde\beta_2\leqslant|\beta_2(l)|\leqslant\|\beta_2\|_\infty$. We remark that given $l\in\mathtt{supp}(\beta_2)$, we have that $f(\beta_2)$ where we fixed $\beta_2(l')=\beta_2^*(l')$ for $l'\in\mathtt{supp}(\beta_2)\setminus\{l\}$ is a quadratic function of $|\beta_2(l)|$ with positive leading coefficient. Under the constraint $\tilde\beta_2\leqslant|\beta_2(l)|\leqslant\|\beta_2\|_\infty$, it is maximized at either of the two bounds on $|\beta_2(l)|$, we deduce that $\beta_2^*(l)=\tilde\beta_2$ or $\beta_2^*(l)=\|\beta_2\|_\infty$.

This implies that there is (an integer – but we relax this constraint) $0\leqslant s\leqslant\|\beta_2\|_0=k_2-r_2-1$ such that

$$\begin{aligned}\|z-P_\Sigma(z)\|_\Sigma^2 \leqslant (&\|z_{S_1}\|_2+\|z_{S_2}\|_2-(s\|\beta_2\|_\infty+(k_2-r_2-1-s)\tilde\beta_2+(r_2+1)\tilde\beta_2)/\sqrt{k_2})^2\\ &+s\|\beta_2\|_\infty^2+(k_2-r_2-1-s)\tilde\beta_2^2+(r_2+1)\tilde\beta_2^2.\end{aligned} \tag{A.153}$$

Again the right side is a quadratic function of $\tilde\beta_2$ under the constraint $0<\tilde\beta_2\leqslant\|\beta\|_\infty$, and bounded at either $\tilde\beta_2\to 0$ or $\tilde\beta_2=\|\beta_2\|_\infty$:

$$\begin{aligned}&\|z-P_\Sigma(z)\|_\Sigma^2\\ &\leqslant \max\left((\|z_{S_1}\|_2+\|z_{S_2}\|_2-s\|\beta_2\|_\infty/\sqrt{k_2})^2+s\|\beta_2\|_\infty^2,(\|z_{S_1}\|_2+\|z_{S_2}\|_2-\sqrt{k_2}\|\beta_2\|_\infty))^2+k_2\|\beta_2\|_\infty^2\right)\\ &\leqslant \max_{0\leqslant s'\leqslant k_2}\left((\|z_{S_1}\|_2+\|z_{S_2}\|_2-s'\|\beta_2\|_\infty/\sqrt{k_2})^2+s'\|\beta_2\|_\infty^2\right).\end{aligned} \tag{A.154}$$

For each $0\leqslant s'\leqslant k_2$ the denominator in the last line is a quadratic function of of $\|\beta_2\|_\infty$ with $0<\|\beta_2\|_\infty\leqslant\min_{l\in S_2}|z(l)|$ (from Lemma A.25) hence

$$\begin{aligned}&\|z-P_\Sigma(z)\|_\Sigma^2\\ &\leqslant \max_{0\leqslant s'\leqslant k_2}\max\left((\|z_{S_1}\|_2+\|z_{S_2}\|_2)^2,(\|z_{S_1}\|_2+\|z_{S_2}\|_2-s'\min_{l\in S_2}|z(l)|]/\sqrt{k_2})^2+s'[\min_{l\in S_2}|z(l)|]^2\right) &\text{(A.155)}\\ &= \max_{0\leqslant s'\leqslant k_2}\left((\|z_{S_1}\|_2+\|z_{S_2}\|_2-s'\min_{l\in S_2}|z(l)|]/\sqrt{k_2})^2+s'[\min_{l\in S_2}|z(l)|]^2\right). &\text{(A.156)}\end{aligned}$$

Similarly, for $\gamma = 0$

$$\|z - P_\Sigma(z)\|_\Sigma^2 \leqslant (\|z_{S_1}\|_2 + \|z_{S_2}\|_2 - \|\beta_2\|_1/\sqrt{k_2})^2 + \|\beta_2\|_2^2 \tag{A.157}$$

which implies using the same argument used to obtain (A.156)

$$\|z - P_\Sigma(z)\|_\Sigma^2 \leqslant \max_{0 \leqslant s' \leqslant k_2} (\|z_{S_1}\|_2 + \|z_{S_2}\|_2 - s' \min_{l \in S} |z(l)|]/\sqrt{k_2})^2 + s'[\min_{l \in S} |z(l)|]^2. \tag{A.158}$$

Still using the same argument about quadratic functions the right side of the maximum is bounded by either of the cases $s' = 0$ or $s' = k_2$. For $s' = 0$

$$\|z - P_\Sigma(z)\|_\Sigma^2 \leqslant (\|z_{S_1}\|_2 + \|z_{S_2}\|_2)^2 \leqslant 2(\|z_{S_1}\|_2^2 + \|z_{S_2}\|_2^2) = 2\|P_\Sigma(z)\|_2^2.$$

For $s' = k_2$

$$\frac{\|z - P_\Sigma(z)\|_\Sigma^2}{\|P_\Sigma(z)\|_2^2} \leqslant V := \frac{(\|z_{S_1}\|_2 + \|z_{S_2}\|_2 - \sqrt{k_2}\min_{l \in S}|z(l)|])^2 + k_2[\min_{l \in S}|z(l)|]^2}{\sum_{i=1}^2 \|z_{S_i}\|_2^2}.$$

Denote $c = 1 - \sqrt{k_2}\frac{[\min_{l \in S}|z(l)|]}{\|z_{S_2}\|_2} \in [0,1)$ and consider $r > 0$, $\theta \in [0, \pi/2]$ such that $\|z_{S_1}\|_2 = r\cos\theta$, $\|z_{S_2}\|_2 = r\sin\theta$

$$\begin{aligned} V &= \frac{(r\cos\theta + cr\sin\theta)^2 + (1-c)^2 r^2 \sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta} \\ &= \frac{\cos^2\theta + c^2\sin^2\theta + (1-c)^2\sin^2\theta + 2c\sin\theta\cos\theta}{\cos^2\theta + \sin^2\theta} \\ &= 1 + (2c^2 - 2c)\sin^2\theta + 2c\sin\theta\cos\theta. \end{aligned}$$

This is a quadratic function of c with positive leading coefficient hence it is maximized at $c = 0$ or $c = 1$. Hence

$$\begin{aligned} V &\leqslant \max(1, 1 + 2\sin\theta\cos\theta) \\ &= \max(1, 1 + \sin(2\theta)) \\ &\leqslant 2. \end{aligned}$$

□

Proof of Theorem 4.4.

We follow the same proof structure (cf Appendix A.6) as for Theorem 4.2, with analog definitions of $P_1(z) = z_1$ and $P_2(z) = z_2$ for $z = (z_1, z_2) \in \mathcal{H}$. We also denote $T = (S_1, S_2) = T(z)$ where S_1 denotes a support of the k largest coordinates in absolute values and $S_2 = \{1, \ldots, r\}$ the set indexing the first r (largest) eigenvalues collected in vector $\texttt{eig}(u)$ (this index set was denoted T in Appendix A.4.2). We modify accordingly the notation T_2 for the $2k$ (resp. $2r$) largest coordinates (resp eigenvalues).

Remark that Lemma A.16 is still valid with $\|\cdot\|_w = w_1\|P_1(\cdot)\|_1 + w_2\|P_2(\cdot)\|_*$.

This permits in turns to obtain the expression (with $T' = T_2 \setminus T$)

$$B_\Sigma(\|\cdot\|_w) = \sup_{z:z\neq 0,\|z_{T_2^c}\|_w+\|z_{T'}\|_w\leqslant\|z_T\|_w} \frac{\|z_{T_2^c}\|_{\mathcal{H}}^2}{\|z_{T_2}\|_{\mathcal{H}}^2} \tag{A.159}$$

from the proof of Lemma A.21. Now remark that this is exactly the same expression as in the sparsity in levels case using the vector of ordered eigenvalues for the part in $\mathcal{H}_p$: $\|(u_1,u_2)\|_{\mathcal{H}}^2 = \|u_1\|_2^2 + \|\mathtt{eig}(u_2)\|_2^2$ and $\|(u_1,u_2)\|_w = w_1\|u_1\|_1 + w_2\|\mathtt{eig}(u_2)\|_1$. This in turns show that (using the same proof as Lemma A.21)

$$B_\Sigma(\|\cdot\|_w) = \max_{0\leqslant L_1\leqslant n-2k, 0\leqslant L_2\leqslant p-2r} B^{L_1,L_2}(w) \tag{A.160}$$

where $B^{L_1,L_2}(w)$ is defined in (A.87). This is the first step of Theorem 4.2.

The rest of the proof then exactly matches the next steps of the proof of Theorem 4.2.

□