

Des conditions impliquant la commutativité d'opérateurs auto-adjoints non-bornés

Mohammed Hichem Mortad, Université Oran 1, Ahmed Ben Bella
(Algérie)

Journées de Théorie des Opérateurs

22/09/2016, Université Claude Bernard Lyon1 (France)

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Introduction

In [12], the following result, which generalized a result by Albrecht-Spain, was proved :

Theorem

Let A be an unbounded self-adjoint operator and let B be a positive (or negative) bounded operator. If AB (respectively BA) is normal, then AB (respectively BA) is self-adjoint.

It is worth mentioning that W. Rehder [20] already proved in 1982 the same result for (only) bounded operators. And neither Mortad nor Albrecht-Spain were aware of this.

The proof is based on the Fuglede-Putnam theorem. Here it is :
Clearly,

$$B(AB) = (BA)B = (AB)^*B.$$

Since AB and $(AB)^*$ are normal, Fuglede-Putnam Theorem yields

$$B(AB)^* = (AB)B$$

so that

$$B^2A = AB^2.$$

Finally, since $B \geq 0$, we obtain

$$BA = AB,$$

as required.

The condition of positivity is essential. For instance, consider

Digression

In [14], I found a yet simpler proof (even in the unbounded case) with a rather stronger condition. First, recall a result by Embry

Theorem 1 *If A and B are two commuting normal operators and $HA = BH$, where $0 \notin W(H)$ (where $W(H)$ is the numerical range of H), then $A = B$.*

The previous then led to :

Corollary 1 *Assume that A and B are two self-adjoint operators such that A is unbounded and $0 \notin W(B)$. If AB is normal, then it is self-adjoint, i.e. B commutes with A .*

The " AB case" was generalized in [12] to the case of two unbounded self-adjoint operators A and B . Later in [13] it was shown that under the same conditions, the normality of BA does not imply anymore its self-adjointness. But, there are still two cases to look at, namely : Keeping B bounded, but taking A to be positive (both self-adjoint) :

1. Does AB normal imply AB self-adjoint ?
2. Does BA normal imply BA self-adjoint ?

The answer is yes to both questions. Details are given next.

Main Results

The two problems are tackled differently. We recall some other results (needed in the sequel).

The first is the well-known Fuglede-Putnam theorem :

Theorem

[for a proof, see [5]] If T is a bounded operator and N and M are unbounded and normal, then

$$TN \subset MT \implies TN^* \subset M^*T.$$

Corollary

If T is a bounded operator and N and M are unbounded and normal, then

$$TN = MT \implies TN^* = M^*T.$$

The next is a generalization of the Fuglede theorem.

Theorem

[Mortad, [12]] Let T be an unbounded self-adjoint operator with domain $D(T)$. If N is an unbounded normal operator such that $D(N) \subset D(T)$, then

$$TN \subset N^*T \implies TN^* \subset NT.$$

We also note

Lemma 1 *If A and B are densely defined with inverse B^{-1} in $B(H)$. Then $(AB)^* = B^*A^*$. In particular, if U is unitary, then*

$$(UAU)^* = U^*(UA)^* = U^*A^*U^*.$$

Lemma 2 *If A and B are densely defined and closed such that either A is invertible or B is bounded, then AB is closed.*

Here is the first result :

Theorem

Let A and B be two self-adjoint operators where only B is bounded. Assume further that A is positive and that BA is normal. Then both BA and AB are self-adjoint. Besides one has $AB = BA$.

Proof.

We may write

$$A(BA) = (AB)A = (BA)^*A.$$

Since BA is normal, $(BA)^*$ is normal too. Since $D(BA) = D(A)$, Theorem 5 applies and yields

$$A(BA)^* \subset BAA$$

or

$$A^2B \subset BA^2. \quad (1)$$

Let us transform the previous into a commutativity between B and A^2 (i.e. $BA^2 \subset A^2B$).

Since BA and $(BA)^*$ are normal, Corollary 5 allows us to write

$$B(BA)^* = B(AB) = (BA)B \implies B(BA) = (BA)^*B$$

or

$$B^2A = AB^2. \quad (2)$$

This tells us that both B^2A and AB^2 are self-adjoint. Continuing we note that

$$B^2A^2 = AB^2A = A^2B^2$$

and

$$B^2A^4 = B^2A^2A^2 = A^2B^2A^2 = A^4B^2. \quad (3)$$

To prove B commutes with A^2 , we first show that $\overline{BA^2}$ is normal. We may obtain e

$$(BA^2)^* BA^2 \supset B^2 A^4.$$

Passing to adjoints gives

$$\overline{(BA^2)^* BA^2} = (BA^2)^* \overline{BA^2} \subset [(BA^2)^* BA^2]^* \subset (B^2 A^4)^* = A^4 B^2.$$

But $A^4 B^2$ is symmetric by Equation (3) (it is even self-adjoint). Since $\overline{BA^2}$ is closed, $(BA^2)^* \overline{BA^2}$ is self-adjoint, and since self-adjoint operators are maximally symmetric, we immediately obtain

$$\overline{(BA^2)^* BA^2} = A^4 B^2. \quad (4)$$

Similarly, we may obtain

$$B^2 A^4 = A^4 B^2 = A^2 A^2 B B \subset A^2 B A^2 B \subset B A^2 A^2 B = B A^2 (B A^2)^*$$

and passing to adjoints yields

$$\overline{BA^2(BA^2)^*} = \overline{BA^2}(BA^2)^* \subset [BA^2(BA^2)^*]^* \subset (B^2A^4)^* = A^4B^2.$$

Similar arguments as above imply that

$$\overline{BA^2(BA^2)^*} = A^4B^2. \quad (5)$$

By Equations (4) & (5), we see that $\overline{BA^2}$ is normal and hence we deduce as

$$(\overline{BA^2})^* = (BA^2)^* = A^2B$$

that A^2B is closed, in fact normal.

Since A^2B is densely defined, we may adjoint Relation (1) to obtain

$$(A^2B)^* \supset (BA^2)^* = A^2B$$

from which A^2B is symmetric. Since we have just seen that A^2B is normal, we infer that A^2B is self-adjoint. Thus, we have arrived at

the basic inclusion and commutativity relation

$$BA^2 \subset \overline{BA^2} = (A^2B)^* = A^2B = (BA^2)^*.$$

In particular, we then know from Theorem 10 in [3] (or [10]) and the positivity of A that B commutes with A , that is,

$$BA \subset AB (= (BA)^*).$$

But both BA and $(BA)^*$ are normal. Since normal operators are maximally normal, we obtain $BA = AB$.

Accordingly,

$$BA = AB = (BA)^* = (AB)^*,$$

and this completes the proof.

Now, we turn to the case of AB normal (keeping all the other assumptions, except BA normal, as those of Theorem 5). The proof is very simple if we impose the very strong condition of the closedness of BA . We have

Corollary 2 *Let A and B be two self-adjoint operators where only B is bounded. Assume further that A is positive, AB is normal and that BA is closed. Then both AB and BA are self-adjoint. Besides one has $AB = BA$.*

Proof.

Since AB is normal, and B is bounded, $(BA)^*$ is clearly normal. Hence so is

$$(BA)^{**} = \overline{BA} = BA.$$

By Theorem 5, BA is self-adjoint. Therefore,

$$BA = (BA)^* = AB,$$

that is, AB is self-adjoint.

One may wonder that there are so many assumptions that AB normal would certainly imply that BA is closed. This is not the case as seen just below :

Example

Let A be a self-adjoint, positive and boundedly invertible unbounded operator. Let B be its (bounded) inverse. So B too is self-adjoint. It is then clear

$$AB = I, \quad BA \subset I.$$

Hence AB is self-adjoint (hence normal!) but BA is not closed.

So, what happens when BA is not necessarily closed? Can we still show that the normality of AB implies its self-adjointness? As in Theorem 5, to show that AB is self-adjoint, it suffices to show that $BA \subset AB$. One of the ways of obtaining this is via $BA^2 \subset A^2B$ which may be obtained if for instance we have an intertwining result of the type

$$NA \subset AN^* \implies N^*A \subset AN$$

where N is an unbounded normal operator playing the role of AB and A (and also B) is self-adjoint. Such an intertwining relation is

however, not true in general as seen in the next example (we also note that none of the existing unbounded versions of the Fuglede-Putnam theorem, as [12], [15], [18], [19] and [24], allows us to get this desired "inclusion").

Example

Define the following operators A and N by

$$Af(x) = (1 + |x|)f(x)$$

and

$$Nf(x) = -i(1 + |x|)f'(x)$$

(with $i^2 = -1$) respectively on the domains

$$D(A) = \{f \in L^2(\mathbb{R}) : (1 + |x|)f \in L^2(\mathbb{R})\}$$

and

$$D(N) = \{f \in L^2(\mathbb{R}) : (1 + |x|)f' \in L^2(\mathbb{R})\}.$$

Then A is self-adjoint and positive (admitting even an everywhere defined inverse) and N is normal. We then find that

$$AN^*f(x) = N Af(x) = -i(1 + |x|)\operatorname{sgn}(x)f(x) - i(1 + |x|)^2f'(x)$$

for any f in the *equal* domains

$$D(AN^*) = D(NA) = \{f \in L^2(\mathbb{R}) : (1+|x|)f \in L^2(\mathbb{R}), (1+|x|)^2f' \in L^2(\mathbb{R})\}$$

and thus

$$AN^* = NA.$$

However

$$AN \not\subset N^*A$$

and

$$AN \not\supset N^*A$$

for

$$ANf(x) = -i(1 + |x|)^2 f'(x)$$

whereas

$$N^* Af(x) = -2i \operatorname{sgn}(x)(1 + |x|)f(x) - i(1 + |x|)^2 f'(x).$$

Thus the method of proof of Theorem 5 could not be applied to the case AB and the approach then had to be different. After some investigation of possible counterexamples we were able instead to establish the affirmative result as follows :

Theorem 2 *Let A and B be two self-adjoint operators where only B is bounded. Assume further that A is positive and that AB is normal. Then both \overline{BA} and AB are self-adjoint. Besides one has $AB = \overline{BA}$.*

To prove it, we need a few lemmas which are also interesting in their own right.

Lemma 3 *Let A, B be self-adjoint and $B \in B(H)$. If AB is densely defined, then we have :*

$$\overline{BA} = (AB)^*.$$

Proof.

This easily follows from

$$(BA)^* = AB \implies (AB)^* = (BA)^{**} = \overline{BA}.$$

In all the coming lemmas we assume that A and B are two self-adjoint operators such that $B \in B(H)$ and that AB is normal.

Lemma 4 *We have :*

$$|B|A \subset A|B|.$$

Proof.

We may write

$$B(AB) = BAB \subset \overline{BA}B.$$

Since both AB and \overline{BA} are normal, Theorem 5 yields

$$B(AB)^* \subset (\overline{BA})^* B = (BA)^* B$$

or merely

$$B^2 A \subset AB^2.$$

Finally, by [9] (or [16]), we obtain

$$|B|A \subset A|B|.$$

Before giving the next lemmas, let

$$B = U|B| = |B|U$$

be the polar decomposition of the self-adjoint B , where U is unitary (cf. [21]). Hence

$$B = U^*|B| = |B|U^*.$$

One of the major points is that U is even self-adjoint. To see this, just re-do the proof of Theorem 12.35 (b) in [21] in the case of a self-adjoint operator. Then use the (self-adjoint!) Functional Calculus to get that U is self-adjoint. Another proof may be found in [2]. Therefore, $U = U^*$ and $U^2 = I$.

Let us also agree that any U which appears from now on is the U involved in this polar decomposition of B .

Lemma 5 *We have :*

$$(AB)^* = UABU$$

so that

$$(AB)^*U = UAB$$

and

$$(AB)U = U(AB)^*.$$

Proof.

Since $|B|A \subset A|B|$, we have $UBA \subset ABU$. Hence

$$UBAU \subset AB$$

or

$$(AB)^* \subset (UBAU)^*.$$

Since U is bounded, self-adjoint and invertible, we clearly have (by Lemma 1)

$$(UBAU)^* = U(BA)^*U = UABU.$$

Since AB is normal, so are $UABU$ and $(AB)^*$ so that

$$(AB)^* \subset UABU \implies (AB)^* = UABU$$

because normal operators are maximally normal.

Lemma 6 Assume also that $A \geq 0$. Then $A|B|$ is positive, self-adjoint and we have :

$$|AB| = A|B|.$$

Proof.

First, remember by Lemma 4 that $|B|A \subset A|B|$. Hence $A|B|$ is positive and self-adjoint as both $|B|$ and A are commuting and positive (see e.g. Exercise 23, Page 113 of [23]). Now, by Lemma 5 we have

$$AB(AB)^* = AB|A|B| = (A|B|)^2.$$

Since AB is normal, we have

$$|AB|^2 = (AB)^*AB = (A|B|)^2$$

so that (for instance by Theorem 11 of [3])

$$|AB| = A|B|.$$

Lemma 7 *The operator UAB is normal.*

Proof.

First, UAB is closed as U is invertible and AB is closed. Now,

$$UAB(UAB)^* = UAB(AB)^*U = (AB)^*(AB).$$

On the other hand,

$$(UAB)^*UAB = (AB)^*U^2AB = (AB)^*AB,$$

establishing the normality of UAB .

Lemma 8 *We have :*

$$U|AB| = |AB|U.$$

Proof.

Since UAB is normal, we clearly have

$$UAB(AB)^*U = (AB)^*AB$$

which entails

$$UAB(AB)^* = U(AB)^*AB = (AB)^*ABU,$$

i.e.

$$U|AB|^2 = |AB|^2U.$$

Hence (by [3]), we are sure at least that $U|AB| \subset |AB|U$. Since $|AB|$ is self-adjoint, a similar argument to that used in the proof of Lemma 5 gives us

$$U|AB| = |AB|U.$$

Lemma 9 *Assume also that $A \geq 0$. Then B commutes with A , i.e. $BA \subset AB$.*

Proof.

We have by Lemmas 6 & 8

$$U|AB| = |AB|U \iff UA|B| = A|B|U \iff UA|B| = AB.$$

Using Lemma 4

$$U|B|A \subset AB$$

or

$$BA \subset AB.$$

We are now ready to prove Theorem 2.

Proof.

By Lemma 9, $BA \subset AB$ so that

$$(AB)^* \subset AB.$$

Therefore, AB is self-adjoint as we already know that $D(AB) = D[(AB)^*]$. Finally, Lemma 3 gives

$$AB = \overline{BA}.$$

The question of the essential self-adjointness of a product of two self-adjoint operators is not easy. In [12], a three page counterexample was constructed to show that if A and B are two unbounded self-adjoint operators such that $B \geq 0$, then the normality of \overline{AB} does not entail its self-adjointness. Related to the question of essential self-adjointness of products, the reader may consult [11]. Having said this, now we may rephrase the result of Theorem 2 as follows :

Corollary 3 *Let A and B be two self-adjoint operators where only B is bounded. Assume further that A is positive and that \overline{BA} is normal. Then BA is essentially self-adjoint.*

Proof.

Since \overline{BA} is normal, so is $(\overline{BA})^*$ or AB . Then by Theorem 2, AB is self-adjoint. By Lemma 3, $\overline{BA} = (AB)^*$ so that \overline{BA} is self-adjoint.

Conclusion

In this conclusion, we summarize all the related results to the problem considered in this paper. These are gathered from the present paper, [12] and [13] :

Theorem

Let A and B be two self-adjoint operators. Set $N = AB$ and $M = BA$.

1. If $A, B \in B(H)$ (one of them is positive) and N (resp. M) is normal, then N (resp. M) is self-adjoint. In either case, we also have $AB = BA$.
2. If only $B \in B(H)$, $B \geq 0$ and N (resp. M) is normal, then N (resp. M) is self-adjoint. Also $BA \subset AB$ (resp. $BA = AB$).
3. If $B \in B(H)$, $A \geq 0$ and N (resp. M) is normal, then N (resp. M) is self-adjoint. Also $BA \subset AB$ (resp. $BA = AB$).
4. If $B \in B(H)$ and either A or B is positive, then \overline{M} normal gives the essential self-adjointness of M .

5. If both A and B are unbounded and N is normal, then it is self-adjoint whenever $B \geq 0$.
6. If both A and B are unbounded and \overline{N} is normal, then N need not be essentially self-adjoint even if $B \geq 0$.
7. If both A and B are unbounded and M is normal, then it not necessarily self-adjoint even when $B \geq 0$.

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