Application of the notion of φ-object to the study of p-class groups and p-ramified torsion groups of abelian extensions

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APPLICATION OF THE NOTION OF φ-OBJECT TO THE STUDY OF p-CLASS GROUPS AND p-RAMIFIED TORSION GROUPS OF ABELIAN EXTENSIONS

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Abstract. We revisit, in an elementary way, the *statement* of the "Main Conjecture" for p-class groups and p-ramified torsion groups in abelian fields K, in the non semi-simple case $p \mid [K : \mathbb{Q}]$; for this, we use an "arithmetic" definition of the p-adic isotopic components, different from the "algebraic" one used in the literature but inappropriate with respect to analytical formulas. The two notions coincide for relative class groups and real torsion groups of p-ramification theory, but not for real class groups. Numerical evidence of the gap between the two notions is given (Examples 3.12, 3.13). It would remain to make use of classical tools for this non semi-simple real context, still unproved as explained § 1.4.

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Foreword and preliminary remarks

This survey provides improvements, new results, numerical illustrations (with PARI programs) and some history, regarding our original articles [Gra1976, Gra1977]. These two papers were written, in French, with illegible fonts due to the use of "typits" on typewriters and hand written characters, for mathematical symbols ! So they were hardly accessible. This survey also mention, in § 1.1, pioneering references, as well as some Leopldt's papers on cyclotomy [Leo1954, Leo1962], written in Germain in the 1950/1960's.

The "Main Conjecture" for abelian number fields (giving analytic expressions of annihilators and orders of the p-adic isotopic components of class groups) that we revisit here, were first stated (*especially in the non semi-simple case*) in the papers mentioned above (but not in [Gra1977^b], as erroneously stated by some authors), and given at the meeting "Journées arithmétiques de Caen" (1976) as it is correctly recalled for instance in [Sol1990, Rib2008]. This gives the occasion to mention that [Gra1977^b] (only recalling the statements in the semi-simple case) is also devoted to a method using formal series, giving non trivial congruences when p-adic $\mathbf{L}$-functions have a trivial zero; for instance we proved the following complement of Ankeny–Artin–Chowla–Kudo congruences:

Let $f \equiv 0 \pmod 3$ be the conductor of a real quadratic field K; we consider the case $f/3 \equiv -1 \pmod 3$ ("special case" when 3 splits in the mirror field $K' := \mathbb{Q}(\sqrt{-f/3})$. Let $\varepsilon = t + u\sqrt{f}$, $t, u > 0$, be the fundamental unit of K and let h and h' be the class numbers of K and K', respectively. Then $h \cdot t \cdot u + h' \equiv 0 \pmod 3$.

Note that ε must be written on the $\mathbb{Q}$-basis $\{1, \sqrt{f}\}$, even if $4 \mid f$, and chosen with positive coefficients. The following program verifies the congruence:

```
{for(m=5,10^4,if(core(m)!=m,next);if(Mod(m,9)!=-3,next);f=m;if(Mod(m,4)!=1,f=4*m);
PP=x^2-f;PM=x^2+f/3;KP=bnfinit(PP,1);KM=bnfinit(PM,1);hP=KP.no;hM=KM.no;
E=lift(KP.fu[1]);t=abs(polcoeff(E,0));u=abs(polcoeff(E,1));X=hP*t*u+hM;
print("f=",f," t=",t," u=",u," h=",hP," h'=",hM," htu+h'=",lift(Mod(X,3))))}
```

The Main Conjecture has been proven in the semi-simple case, then in the non semi-simple one for relative class groups and Iwasawa's theory (a main overview on the precise proofs and classical references are given in Washington's book [Was1997, Chapters 6, 8, 13, 15]).

The *non semi-simple case of even p-adic characters φ*, was less understood because of a problematic definition of p-adic isotopic components and cyclotomic units; but at the time, we proposed another more natural conjectural context, still unproved to our knowledge, for which the definition of "Arithmetic φ-objects" may be necessary since

the distinction between "Algebraic" and "Arithmetic" definitions is crucial regarding analytic formulas (we shall give more comments in Remarks 7.7).

Let $\mathscr{G} := \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ be the Galois group of the maximal abelian extension $\mathbb{Q}^{\mathrm{ab}}$ of $\mathbb{Q}$ and denote by K any subfield of finite degree of $\mathbb{Q}^{\mathrm{ab}}$. The present article is divided into the following three parts, after an Introduction giving a brief description about the story (rather prehistory) that led to the numerous approaches giving, under some assumptions, proofs of a "Main Theorem":

(i) An algebraic part giving a systematic study of families $(\mathbf{M}_K)_K$ of $\mathbb{Z}[\mathscr{G}]$-modules and of the $\mathbb{Z}_p[\mathscr{G}]$-modules $\mathscr{M}_K := \mathbf{M}_K \otimes \mathbb{Z}_p$, including the non semi-simple case $p \mid [K:\mathbb{Q}]$. This study leads to the definition of sub-modules $\mathscr{M}_\varphi^{\mathrm{alg}}$ (algebraic) and $\mathscr{M}_\varphi^{\mathrm{ar}}$ (arithmetic), indexed by the set of irreducible p-adic characters φ of $\mathscr{G}$. The difference between $\mathscr{M}_\varphi^{\mathrm{alg}}$ (used in all the literature) and $\mathscr{M}_\varphi^{\mathrm{ar}}$ is that the first one relates to algebraic norms $\nu_{k/k'} \in \mathbb{Z}[\mathrm{Gal}(k/k')]$ for their properties in relative sub-extensions of $K/\mathbb{Q}$, while the second one uses arithmetic norms $\mathbf{N}_{k/k'}$, the gap being given by the relation:

$$\nu_{k/k'} = \mathbf{J}_{k/k'} \circ \mathbf{N}_{k/k'},$$

where the transfer maps $\mathbf{J}_{k/k'}$ are often non injective in p-extensions (see § 3.3 for examples justifying Definition 3.11 for the statement of the Main Conjecture and § 4.3 for the main properties). Moreover, the "arithmetic" point of view is naturally related to the analytical formula:

$$\#\mathscr{M}_K = \prod_{\varphi \in \Phi_K} \#\mathscr{M}_\varphi^{\mathrm{ar}} \ \text{(Theorems 3.15, 4.5)}.$$

(ii) An arithmetic part where we apply the above results to p-class groups $\mathscr{H}_K$, K real or imaginary, then to torsion groups $\mathscr{T}_K$ of the Galois group of the maximal p-ramified abelian pro-p-extension of K real.

For rational characters χ and p-adic characters $\varphi \mid \chi$, we define the "Class Invariants" $m_\varphi^{\mathrm{alg}}(\mathscr{H})$ (algebraic), $m_\varphi^{\mathrm{ar}}(\mathscr{H})$, $m_\varphi^{\mathrm{ar}}(\mathscr{T})$ (arithmetic) then, in § 8.2, the corresponding "Analytic Invariants" $m_\varphi^{\mathrm{an}}(\mathscr{H})$, $m_\varphi^{\mathrm{an}}(\mathscr{T})$ suggested by the analytic formulas of the arithmetic χ-components deduced from Theorem 2.1 (Theorems 5.10, 6.2, 7.5) and we develop the problem of their comparison. We conjecture a new annihilation theorem for $\mathscr{H}_\varphi^{\mathrm{ar}}$, φ even, in the non semi-simple case (Conjecture 7.9).

In the § 7.6, we shed new light on the proof of the Main Conjecture in the real semi-simple case, in the spirit of Thaine's theorem described in Washington's book, and we give numerical illustrations. It becomes clear that *the knowledge of the sole cyclotomic unit η_K of the base field K* contains, by means of very elementary arithmetic, all the information on annihilation and orders of the φ-components of its p-class group.

(iii) An illustration, in the semi-simple case, is given with cyclic cubic fields for $p \equiv 1 \pmod 3$, as well as a PARI program computing the above invariants, which was not possible in the 1970's. It would remain to give non semi-simple computations to verify the conjecture in the real case.

1. Introduction and brief historical survey

1.1. **Main bibliographic reminders.** It is difficult to give here the full story of such a subject, from Bernoulli, Kummer, Herbrand classical context, the initiating work of Iwasawa, Leopoldt, Greenberg, on the conjecture, then the deep results obtained by Ribet, Mazur, Wiles, Thaine, Rubin, Kolyvagin, Solomon, Greither, Coates, Sinnott, and others, on cyclotomy and p-adic **L**-functions. Several papers also give the Iwasawa formulation of the Main Theorem (see e.g., [Gree1975, Gree1977]); Iwasawa's theory is less general than the expected results for finite extensions, but more conceptual in broader contexts (in fact, describing the similarity with the theory of p-adic **L**-functions, a generalizable feature to many fields). Let's give less known contributions of the beginnings:

We refer, for a very nice story of pioneering works, to Ribet [Rib2008, Rib2008[b]], for detailed proofs of Iwasawa Main Conjecture to Washington [Was1997, Chap. 15] following techniques initiated by Thaine then Kolyvagin, Ribet (exposed by Lang

[Lang1990]). A Bourbaki Seminar, by Perrin-Riou [PeRi1990], gives a significant lecture (with an impressive bibliography) on the works of Kolyvagin, Rubin and others about the Main Conjectures for number fields and elliptic curves.

The story is also given in the famous Mazur–Wiles paper, where the attribution of the various statements of the conjecture (in the semi-simple case) is accurately discussed (see [MaWi1984, § 1 and § 10 (i, ii)] for more comments on the works of Iwasawa, Leopoldt, Greenberg and ours), even if some references are missing.

Finally, proofs of our conjecture for the relative p-class groups $\mathscr{H}^-$ and the real torsion groups $\mathscr{T}$ of the Galois groups of the maximal abelian p-ramified pro-p-extensions were given (Solomon for $\mathscr{H}^-$ and $p \neq 2$ [Sol1990, Theorem II.1], Greither for $\mathscr{H}^-$, $\mathscr{T}$ with $p \geq 2$ and $\mathscr{H}^+$ in a semi-simple context [Grei1992, Theorems A, B, C, 4.14, Corollary 4.15]). Let us mention the proof by Rubin [Rub1990], from the Kolyvagin Euler systems [Kol2007] used in above proofs.

Many complementary works about the order or the annihilation of the $\mathscr{H}_\varphi$'s, for irreducible p-adic characters φ, were published before or after the decisive proofs (e.g., [Gra1977^b, Gil1977, Gra1979, Or1981, Or1986, GrKu2004, BeNg2005, All2013, BeMa2014, GrKu2014, All2017, Gra2018^b, GrKu2021, Jau2021, Jau2022, Jau2022^b]). Let's mention a result of Oriat using reflection theorem [Or1986, Théorème, p. 333].

In the same way, it is hopeless to outline all generalizations giving "Main Conjectures" in other contexts than the absolute abelian case (e.g., [Dar1995, MaRu2011, CoLi2019, DaKa2020, CoLi2020, BBDS21, BDSS21]), using essentially the technique of Kolyvagin's Euler systems; an expository book may be [CoSu2006] for recent works, but excluding the story of the origins of the Main Conjecture as explained in Solomon–Greither papers [Sol1990, Grei1992], Washington's book [Was1997] and Ribet's Lectures [Rib2008, Rib2008^b].

In another direction, we refer to enlargements of the algebraic/arithmetic aspects of p-adic characters in the area of metabelian Galois groups by Jaulent, with applications to class groups and units (see for instance [Jau1981, Théorème 1 and consequences], [Jau1984, Jau1986] in a class field theory context, then [Lec2018, SchS2019] in a geometric or Galois cohomology context).

Due to the huge number of articles dealing with the concept of "Main Conjecture", many recent (or not) articles may have escaped our notice.

1.2. **Introduction of Arithmetic φ-objects.** Nevertheless, all these works deal with an *algebraic definition of the φ-class groups* $\mathscr{H}_\varphi^{\rm alg}$, from the p-class group $\mathscr{H}_K$ (for irreducible p-adic characters φ); that is to say, when $G := {\rm Gal}(K/\mathbb{Q})$ is cyclic, of order g (i.e., K is the fixed field by the kernel of a rational character χ),

$$\mathscr{H}_\varphi^{\rm alg} := \mathscr{H}_K \otimes_{\mathbb{Z}_p[G]} \mathbb{Z}_p[\mu_g], \quad \text{for all } \varphi \mid \chi,$$

with the $\mathbb{Z}_p[\mu_g]$-action $\sigma \in G \mapsto \psi(\sigma)$ ($\psi \mid \varphi$ of order g).

We then prove (Theorem 3.7) that:

$$\mathscr{H}_\varphi^{\rm alg} = \{x \in \mathscr{H}_K,\ \nu_{K/k}(x) = 1,\ \forall\, k \subsetneq K\} \otimes_{\mathbb{Z}_p[G]} \mathbb{Z}_p[\mu_g]$$

(where $\nu_{K/k}$ is the algebraic norm), contrary to our definition:

$$\mathscr{H}_\varphi^{\rm ar} := \{x \in \mathscr{H}_K,\ \mathbf{N}_{K/k}(x) = 1,\ \forall\, k \subsetneq K\} \otimes_{\mathbb{Z}_p[G]} \mathbb{Z}_p[\mu_g],$$

where $\mathbf{N}_{K/k}$ is the arithmetic norm.

See § 2.2 for other equivalent characterizations of $\mathscr{H}_\varphi^{\rm alg}$ and $\mathscr{H}_\varphi^{\rm ar}$ using cyclotomic polynomials, then for a summary of the main properties and results of the paper.

In the non semi-simple case $p \mid g$, the distinction between algebraic and arithmetic φ-components is not done in the literature. This does not matter for relative p-class groups $\mathscr{H}_K^-$ and torsion p-groups $\mathscr{T}_K$ since we will prove that the two notions coincide (Theorems 5.8, 6.1); so the case of these invariants is definitely solved, contrary to that of φ-components of p-class groups of real fields K in the non semi-simple case deduced from the "χ-formulas" given in Theorem 7.5 and the important relation:

$$\#\mathscr{H}_K = \prod_{\varphi \in \Phi_K} \#\mathscr{H}_\varphi^{\rm ar} \ \text{(Theorems 3.15, 4.5)}.$$

We compare the two definitions $\mathscr{H}^{\rm alg}$, $\mathscr{H}^{\rm ar}$ in §§ 3.3, 3.4, with numerical illustration showing the gap between them and involving capitulation phenomenon of p-classes in p-extensions (Examples 3.12, 3.13).

1.3. **Relation between the modules $\mathscr{H}$ and $\mathscr{T}$.** If one considers, in the real case, the $\mathbb{Z}_p[\mathscr{G}]$-modules $\mathscr{T}_K$, one gets, for them, an easier annihilation theorem from the p-adic Mellin transform of Stickelberger elements (see § 6.2). Moreover, the norm maps $\mathbf{N}_{k/k'}$ are surjective and the transfer maps $\mathbf{J}_{k/k'}$ are injective under Leopoldt's conjecture [Gra1982, Théorème I.1], [Jau1986, Ng1986, Jau1998] (collected in [Gra2005, Theorem IV.2.1]); so this family behaves as that of relative class groups, which allows an obvious statement of the Main Conjecture and then its proof with similar techniques, as done for instance in [Grei1992].

The order of the p-group $\mathscr{T}_K$ is closely related to the p-adic $\mathbf{L}$-functions "at $s = 1$" [Coa1977] and a particularity of $\mathscr{T}_K$ is its interpretation by means of the three $\mathbb{Z}_p[\mathscr{G}]$-modules $\mathscr{H}_K^{\rm cyc}$, $\mathscr{R}_K$ and $\mathscr{W}_K$; see [Gra2005, Lemma III.4.2.4] leading to the exact sequence (6.1) and the formula $\#\mathscr{T}_K = \#\mathscr{H}_K^{\rm cyc} \cdot \#\mathscr{R}_K \cdot \#\mathscr{W}_K$, where $\mathscr{W}_K$ is an easy canonical invariant depending on local p-roots of unity, $\mathscr{R}_K$ is the normalized p-adic regulator [Gra2018, Lemma 3.1] and $\mathscr{H}_K^{\rm cyc}$ a subgroup of $\mathscr{H}_K$ (equal to $\mathscr{H}_K$, except "the part" corresponding to the maximal unramified extension contained in the cyclotomic $\mathbb{Z}_p$-extension of K).

The order of the group $\mathscr{R}_K$ is (up to an obvious factor) the classical p-adic regulator which intervenes in the p-adic analytic formulas due to the pioneering works of Kubota–Leopoldt on p-adic $\mathbf{L}$-functions, then that of Amice–Fresnel–Barsky (e.g., [Fre1965]), Coates, Ribet and many other; see a survey in [Gra1978^b] and a lecture in [Rib1979] where is used the beginnings of the concept of p-adic pseudo-measures of Mazur, developed by Serre [Ser1978]). See in [Gra2016, Gra2019] more complete studies and conjectures about $\mathscr{R}_K$ and $\mathscr{T}_K$.

At this time was stated the Iwasawa formalism of the Main Conjecture by Greenberg [Gree1975, Gree1977] after Iwasawa [Iwa1964].

1.4. **Main unsolved problem today.** Let $K/\mathbb{Q}$ be a cyclic real extension with a non trivial maximal p-sub-extension (non semi-simple case). Let $\mathbf{E}_K$ (resp. $\mathbf{F}_K$) be the group of units (resp. of Leopoldt's cyclotomic units) then $\mathscr{E}_K = \mathbf{E}_K \otimes \mathbb{Z}_p$ and $\mathscr{F}_K = \mathbf{F}_K \otimes \mathbb{Z}_p$; let $\mathscr{E}_K^0 \subseteq \mathscr{E}_K$ be the subgroup generated by the $\mathscr{E}_k$ for all $k \subsetneq K$.

It would remain to prove our conjecture [Gra1977, § III] for the even p-adic characters φ of K saying that (see also Remarks 7.7 and 8.2):

$$\#\mathscr{H}_\varphi^{\rm ar} = w_\varphi \cdot \#(\mathscr{E}_K/\mathscr{E}_K^0 \cdot \mathscr{F}_K)_\varphi, \quad w_\varphi \in \{1, p\},$$

where $(\mathscr{E}_K/\mathscr{E}_K^0 \cdot \mathscr{F}_K)_\varphi := \{\widetilde{\varepsilon} \in \mathscr{E}_K/\mathscr{E}_K^0 \cdot \mathscr{F}_K,\ P_\varphi(\sigma) \cdot \widetilde{\varepsilon} = 1\}$ and:

$$\mathscr{H}_\varphi^{\rm ar} := \big\{x \in \mathscr{H}_K,\ P_\varphi(\sigma) \cdot x = 1\ \ \&\ \ \mathbf{N}_{K/k}(x) = 1\ \forall\, k \subsetneq K\big\},$$

where P_φ is the local cyclotomic polynomial attached to φ and σ a generator of $\mathrm{Gal}(K/\mathbb{Q})$.

For the φ-component $(\mathscr{E}_K/\mathscr{E}_K^0 \cdot \mathscr{F}_K)_\varphi$, the two notions (arithmetic and algebraic) obviously coincide, but the φ-class group must be defined in the arithmetic sense.

2. Abelian extensions

The idea of definition of the φ-objects owes a lot to the work of Leopoldt [Leo1954, Leo1962] and their writing, in french, by Oriat in [Or1975, Or1975^b]. Some outdated notations in these papers and ours are modified, after changing ℓ into p (e.g., $\Omega_p \mapsto \overline{\mathbb{Q}}_p$, $\widehat{\Omega}_p \mapsto \mathbb{C}_p$, $\Gamma \mapsto \mathbb{Z}_p$).

2.1. **Characters.** Let $\mathbb{Q}^{\rm ab}$ be the maximal abelian extension of $\mathbb{Q}$ contained in an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$; let $\mathbb{Q}_p$ be the p-adic field and $\overline{\mathbb{Q}}_p$ an algebraic closure of $\mathbb{Q}_p$ containing $\overline{\mathbb{Q}}$. We put $\mathscr{G} := {\rm Gal}(\mathbb{Q}^{\rm ab}/\mathbb{Q})$):

Let Ψ be the set of irreducible characters of $\mathscr{G}$, of degree 1 and finite order, with values in $\overline{\mathbb{Q}}_p$. We define the sets of irreducible p-adic characters Φ, for a prime $p \geq 2$, the set $\mathscr{X}$ of irreducible rational characters and the sets of irreducible characters Ψ_K, Φ_K, $\mathscr{X}_K$, of $K \subset \mathbb{Q}^{\rm ab}$.

The notation $\psi \mid \varphi \mid \chi$ (for $\psi \in \Psi$, $\varphi \in \Phi$, $\chi \in \mathscr{X}$) means that φ is a term of χ and ψ a term of φ.

Let $s_\infty \in \mathscr{G}$ be the complex conjugation and $\psi \in \Psi_K$; if $\psi(s_\infty) = 1$ (resp. $\psi(s_\infty) = -1$), we say that ψ is even (resp. odd) and we denote by Ψ_K^+ (resp. Ψ_K^-) the corresponding subsets of characters. Since $\Psi_K^\pm$ is stable by any conjugation, this defines $\Phi_K^\pm$, $\mathscr{X}_K^\pm$.

Let $\chi \in \mathscr{X}$; we denote by g_χ, K_χ, G_χ, f_χ, $\mathbb{Q}(\mu_{g_\chi})$, the order of any $\psi \mid \chi$, the subfield of K fixed by ${\rm Ker}(\chi) := {\rm Ker}(\psi)$, ${\rm Gal}(K_\chi/\mathbb{Q})$, the conductor of K_χ, the field of values of the characters, respectively. The set $\mathscr{X}$ has the following obvious property to be considered as the "Main theorem" for rational components (e.g., [Leo1954, Chap. I, § 1, 1]):

Theorem 2.1. *Let $K/\mathbb{Q}$ be a finite abelian extension and let $(A_\chi)_{\chi\in\mathscr{X}_K}$, $(A'_\chi)_{\chi\in\mathscr{X}_K}$ be two families of positive numbers, indexed by the set $\mathscr{X}_K$ of irreducible rational characters of K. If for all subfields k of K, one has $\prod_{\chi\in\mathscr{X}_k} A'_\chi = \prod_{\chi\in\mathscr{X}_k} A_\chi$, then $A'_\chi = A_\chi$ for all $\chi \in \mathscr{X}_K$.*

2.2. **Main results of the article.** Let $\mathbf{M} = (\mathbf{M}_K)_{K\in\mathscr{K}}$ be a family of $\mathbb{Z}[\mathscr{G}]$-modules, indexed with the set $\mathscr{K}$ of finite abelian extensions and provided with the arithmetic norms $\mathbf{N}_{K/k}$ and transfer maps $\mathbf{J}_{K/k}$, for any $k \subseteq K$, where $\mathbf{J}_{K/k} \circ \mathbf{N}_{K/k} = \nu_{K/k} \in \mathbb{Z}[{\rm Gal}(K/k)]$ (algebraic norm); we will give more definitions and well-known details in Section 3.1.

We associate with $\mathbf{M}$ the family of $\mathbb{Z}_p[\mathscr{G}]$-modules $\mathscr{M} := \mathbf{M} \otimes \mathbb{Z}_p$.

We define various χ-components $\mathbf{M}_\chi^{\rm alg}$, $\mathbf{M}_\chi^{\rm ar}$, $\mathscr{M}_\chi^{\rm alg}$, $\mathscr{M}_\chi^{\rm ar}$ (for $\chi \in \mathscr{X}$) and the associated φ-components $\mathscr{M}_\varphi^{\rm alg}$, $\mathscr{M}_\varphi^{\rm ar}$ (for $\varphi \in \Phi$), as follows:

Let P_χ be the global g_χth cyclotomic polynomial, let P_φ be the local cyclotomic polynomial associated with $\varphi \mid \chi$ (so that $P_\chi = \prod_{\varphi\mid\chi} P_\varphi$ in $\mathbb{Z}_p[X]$). We define (with group algebras actions x^Ω written $\Omega \cdot x$):

$$\begin{cases} \mathbf{M}_\chi^{\rm alg} := \{x \in \mathbf{M}_{K_\chi}, P_\chi(\sigma_\chi)\cdot x = 1\}, \quad \mathscr{M}_\chi^{\rm alg} := \mathbf{M}_\chi^{\rm alg} \otimes \mathbb{Z}_p, \\ \mathscr{M}_\varphi^{\rm alg} := \{x \in \mathscr{M}_\chi^{\rm alg}, P_\varphi(\sigma_\chi)\cdot x = 1\}, \\ \mathbf{M}_\chi^{\rm ar} := \{x \in \mathbf{M}_\chi^{\rm alg}, \mathbf{N}_{K_\chi/k}(x) = 1,\ \forall\, k \subsetneq K_\chi\}, \quad \mathscr{M}_\chi^{\rm ar} := \mathbf{M}_\chi^{\rm ar} \otimes \mathbb{Z}_p, \\ \mathscr{M}_\varphi^{\rm ar} := \{x \in \mathscr{M}_\varphi^{\rm alg},\ \mathbf{N}_{K_\chi/k}(x) = 1,\ \forall\, k \subsetneq K_\chi\}. \end{cases}$$

Then $\mathscr{M}_\varphi^{\rm ar} = \{x \in \mathscr{M}_{K_\chi},\ P_\varphi(\sigma_\chi)\cdot x = 1\ \ \&\ \ \mathbf{N}_{K_\chi/k}(x) = 1, \forall\, k \subsetneq K_\chi\}$.

Being annihilated by $P_\chi(\sigma_\chi)$ (resp. $P_\varphi(\sigma_\chi)$) $\mathbf{M}_\chi^{\rm alg}$ and $\mathscr{M}_\chi^{\rm alg}$ (resp. $\mathbf{M}_\varphi^{\rm alg}$ and $\mathscr{M}_\varphi^{\rm alg}$) are $\mathbb{Z}[\mu_{g_\chi}]$-modules (resp. $\mathbb{Z}_p[\mu_{g_\chi}]$-modules), for the law defined via $\sigma \in \mathscr{G} \mapsto \psi(\sigma) \in \mu_{g_\chi}$, for $\psi \mid \chi$ (resp. $\psi \mid \varphi$).

(i) Then we have the following results:

- $\mathbf{M}_\chi^{\rm alg} = \{x \in \mathbf{M}_{K_\chi},\ \nu_{K_\chi/k}(x) = 1,\ \forall\, k \subsetneq K_\chi\}$ (Theorem 3.7),
- $\mathscr{M}_\chi^{\rm alg} = \bigoplus_{\varphi\mid\chi} \mathscr{M}_\varphi^{\rm alg}$, $\mathscr{M}_\chi^{\rm ar} = \bigoplus_{\varphi\mid\chi} \mathscr{M}_\varphi^{\rm ar}$ (Theorems 4.1, 4.5).

(ii) Assume that $K/\mathbb{Q}$ is cyclic and $\mathbf{M}_K$ finite:

(ii′) If, for all sub-extensions k/k' of $K/\mathbb{Q}$, the norm maps $\mathbf{N}_{k/k'}$ are surjective, then:

- $\#\mathbf{M}_K = \prod_{\chi\in\mathscr{X}_K} \#\mathbf{M}_\chi^{\rm ar}$ (Theorem 3.15),

(ii″) Let K/K_0 be the maximal p-sub-extension of K; if, for all sub-extensions k/k' of K/K_0, the norm maps $\mathbf{N}_{k/k'}$ are surjective, then:

- $\#\mathscr{M}_\chi^{\rm ar} = \prod_{\varphi\mid\chi} \#\mathscr{M}_\varphi^{\rm ar}$ (Theorem 4.5).

The above conditions of surjectivity of the norms are automatically fulfilled for the families $\mathbf{H}$ (class groups), $\mathscr{H} = \mathbf{H}\otimes\mathbb{Z}_p$ (p-class groups) and $\mathscr{T}$ (torsion groups of abelian p-ramification).

(iii) Applying this to $\mathbf{H}$ and $\mathscr{T}$, we obtain:

(iii′) For all characters $\chi\in\mathscr{X}^-$, we have:

- $\mathbf{H}_\chi^{\rm ar} = \mathbf{H}_\chi^{\rm alg}$ and $\mathscr{H}_\varphi^{\rm ar} = \mathscr{H}_\varphi^{\rm alg}$, $\forall\,\varphi\mid\chi$ (Theorem 5.8);
- $\#\mathbf{H}_\chi^{\rm ar} = \#\mathbf{H}_\chi^{\rm alg} = 2^{\alpha_\chi}\cdot w_\chi\cdot\prod_{\psi\mid\chi}\left(-\frac{1}{2}\,\mathbf{B}_1(\psi^{-1})\right)$ (Theorem 5.10), in terms of Bernoulli numbers.

(iii″) For all characters $\chi\in\mathscr{X}^+$, we have:

- $\mathbf{H}_\chi^{\rm ar}\subseteq\mathbf{H}_\chi^{\rm alg}$ and $\mathscr{H}_\varphi^{\rm ar}\subseteq\mathscr{H}_\varphi^{\rm alg}$, $\forall\,\varphi\mid\chi$ (see Examples 3.12, 3.13 for strict inclusions);
- $\#\mathbf{H}_\chi^{\rm ar} = w_\chi\cdot\left(\mathbf{E}_{K_\chi} : \mathbf{E}^0_{K_\chi}\cdot\mathbf{F}_{K_\chi}\right)$ (Theorem 7.5), in terms of cyclotomic units, where $\mathbf{E}^0_{K_\chi} := \langle\,\mathbf{E}_k\,\rangle_{k\subsetneq K_\chi}$.

(iii‴) For all even characters χ, we have:

- $\mathscr{T}_\chi^{\rm ar} = \mathscr{T}_\chi^{\rm alg}$ and $\mathscr{T}_\varphi^{\rm ar} = \mathscr{T}_\varphi^{\rm alg}$, $\forall\,\varphi\mid\chi$ (Theorem 6.1);
- $\#\mathscr{T}_\chi^{\rm ar} = w_\chi^{\rm cyc}\cdot\prod_{\psi\mid\chi}\frac{1}{2}\,\mathbf{L}_p(1,\psi)$ (Theorem 6.2), in terms of p-adic $\mathbf{L}$-functions.

(iv) The Arithmetic Invariants of finite $\mathbb{Z}_p[\mathscr{G}]$ modules $\mathscr{M}_K$ are defined by means of the obvious algebraic writing of $\mathbb{Z}_p[\mu_{g_\chi}]$-modules:

$$\mathscr{M}_\varphi^{\rm ar}\simeq\prod_{i\geq 1}\left[\mathbb{Z}_p[\mu_{g_\chi}]/\mathfrak{p}_\varphi^{n^{\rm ar}_{\varphi,i}(\mathscr{M})}\right],\qquad m^{\rm ar}_\varphi(\mathscr{M}) := \sum_i n^{\rm ar}_{\varphi,i}(\mathscr{M}),$$

where $\mathfrak{p}_\varphi$ is the maximal ideal of $\mathbb{Z}_p[\mu_{g_\chi}]$; the definition of the Analytic Invariants $m^{\rm an}_\varphi(\mathscr{M})$ comes directly from the formulas of $\#\mathscr{M}_\chi^{\rm ar}$ given above in (iii), taking into account the decompositions $\mathscr{M}_\chi^{\rm ar} = \oplus_{\varphi\mid\chi}\mathscr{M}_\varphi^{\rm ar}$, whence the statement of the Main Conjecture "$m^{\rm ar}_\varphi(\mathscr{M}) = m^{\rm an}_\varphi(\mathscr{M})$, for all $\varphi\in\Phi$" (Section 8, Conjecture 8.1).

3. Definition and study of the φ-objects

We shall give, in this section, the general definition of θ-objects, θ being an irreducible character (rational or p-adic), the Galois modules which intervene in the definition of the θ-objects being not necessarily finite, as it is the case for unit groups; finally, the prime p is arbitrary and we shall emphasize on the non semi-simple framework.

3.1. **The Algebraic and Arithmetic $\mathscr{G}$-families.** Let $\mathscr{K}$ be the family of finite extensions K of $\mathbb{Q}$, contained in $\mathbb{Q}^{\rm ab}$, of Galois group G_K. We assume to have a family $\mathbf{M}$ of (multiplicative) $\mathbb{Z}[\mathscr{G}]$-modules, indexed by $\mathscr{K}$ (called a $\mathscr{G}$-family), $\mathbf{M} = (\mathbf{M}_K)_{K\in\mathscr{K}}$.

In general there exist two families of $\mathscr{G}$-homomorphisms, indexed by the set of sub-extensions K/k, $\mathbf{N}_{K/k}:\mathbf{M}_K\to\mathbf{M}_k$ (arithmetic norms), $\mathbf{J}_{K/k}:\mathbf{M}_k\to\mathbf{M}_K$ (arithmetic transfers). For all sub-extensions K/k, we put $\nu_{K/k} := \sum_{\sigma\in{\rm Gal}(K/k)}\sigma\in\mathbb{Z}[{\rm Gal}(K/k)]$ (algebraic norm).

We consider the three following conditions:

(a) For all $K\in\mathscr{K}$, $\mathbf{M}_K^{{\rm Gal}(\mathbb{Q}^{\rm ab}/K)} = \mathbf{M}_K$ (so, for $x\in\mathbf{M}_K$ and $\sigma\in\mathscr{G}$, $x^\sigma = x^{\sigma_K}$, where $\sigma_K\in G_K$ is the restriction of σ to K).

(b) For all sub-extension K/k, the arithmetic maps $\mathbf{N}_{K/k}$ and $\mathbf{J}_{K/k}$ are $\mathscr{G}$-module homomorphisms fulfilling the transitivity formulas:

$$\mathbf{N}_{K/k}\circ\mathbf{N}_{L/K} = \mathbf{N}_{L/k}\ \text{and}\ \mathbf{J}_{L/K}\circ\mathbf{J}_{K/k} = \mathbf{J}_{L/k},$$

for all $k, K, L \in \mathscr{K}$, $k \subseteq K \subseteq L$.

(c) For all sub-extension K/k, $\mathbf{J}_{K/k} \circ \mathbf{N}_{K/k} = \nu_{K/k}$ on $\mathbf{M}_K$.

Definitions 3.1. *(i) If $\mathbf{M} = (\mathbf{M}_K)_{K\in\mathscr{K}}$ only fulfills condition (a), we shall say that the family $(\mathbf{M}, \nu)$ is an algebraic $\mathscr{G}$-family; one may only use Galois theory in K/k and the algebraic norms $\nu_{K/k} \in \mathbb{Z}[\mathrm{Gal}(K/k)]$.*

(ii) If moreover, there exist two families $(\mathbf{N}_{K/k})$ and $(\mathbf{J}_{K/k})$ (canonically associated with $\mathbf{M}$) fulfilling conditions (b) and (c), we shall say that the family $(\mathbf{M}, \mathbf{N}, \mathbf{J})$ is an arithmetic $\mathscr{G}$-family.

The following properties of $\mathbf{M}_K$ and $\mathscr{M}_K := \mathbf{M}_K \otimes \mathbb{Z}_p$ are elementary:

Proposition 3.2. *(i) For all $K \in \mathscr{K}$, $\nu_{K/K}$, $\mathbf{N}_{K/K}$, $\mathbf{J}_{K/K}$ are the identity,* id, *on $\mathbf{M}_K$.*

(ii) If the map $\mathbf{N}_{K/k}$ is surjective or if the map $\mathbf{J}_{K/k}$ is injective, then $\mathbf{N}_{K/k} \circ \mathbf{J}_{K/k}$ is the elevation to the power $[K : k]$.

Remark 3.3. Note that cohomology is only of algebraic nature since, for instance in the case of a cyclic extension K/k of Galois group $G =: \langle\sigma\rangle$, using the class group $\mathbf{H}_K$, we have:

$$\mathrm{H}^1(G, \mathbf{H}_K) = \mathrm{Ker}(\nu_{K/k})/\mathbf{H}_K^{1-\sigma}, \quad \mathrm{H}^2(G, \mathbf{H}_K) = \mathbf{H}_K^G/\nu_{K/k}(\mathbf{H}_K);$$

in general $\nu_{K/k}(\mathbf{H}_K)$ is not isomorphic to $\mathbf{N}_{K/k}(\mathbf{H}_K) \subseteq \mathbf{H}_k$, even if the arithmetic norm is surjective, since the transfer map $\mathbf{J}_{K/k}$ is often non-injective on class groups.

Examples 3.4. The most straightforward examples of such arithmetic $\mathscr{G}$-families $\mathbf{M}_K$ are the following ones:

(i) the group $\mathbf{E}_K$ of units of K (for which the maps $\mathbf{J}_{K/k}$ are injective);

(ii) the class group $\mathbf{H}_K$ of K, or the p-class group $\mathscr{H}_K$.

(iii) the torsion group $\mathscr{T}_K$ of the Galois group of the maximal p-ramified abelian pro-p-extension of K.

(iv) the group-algebra $A[G_K]$, where A is a commutative ring; then $A[G_K]$ is a $A[\mathscr{G}]$-module if one puts $\sigma\cdot\Omega = \sigma_K\Omega$ (product in $A[G_K]$), for all $\Omega \in A[G_K]$ and $\sigma \in \mathscr{G}$. The maps $\mathbf{N}_{K/k}$ and $\mathbf{J}_{K/k}$ are defined by A-linearity by $\mathbf{N}_{K/k}(\sigma_K) := \sigma_k$ and, for $\sigma_k \in G_k$, by $\mathbf{J}_{K/k}(\sigma_k) := \sum_{\tau\in\mathrm{Gal}(K/k)} \tau\cdot\sigma'_k = \nu_{K/k}\cdot\sigma'_k = \nu_{K/k}\sigma'_k$, where σ'_k is any extension of σ_k in G_K. So, for $\sigma_K \in G_K$, $\nu_{K/k}(\sigma_K) = \big(\sum_{\tau\in\mathrm{Gal}(K/k)}\tau\big)\cdot\sigma_K = \nu_{K/k}\sigma_K$.

3.2. **Definition of the $\mathscr{G}$-modules $\mathbf{M}_\chi^{\mathrm{alg}}$, $\mathbf{M}_\chi^{\mathrm{ar}}$, $\mathscr{M}_\varphi^{\mathrm{alg}}$, $\mathscr{M}_\varphi^{\mathrm{ar}}$.** We shall assume in the sequel that $A \in \{\mathbb{Z},\ \mathbb{Z}_p\}$.

3.2.1. *The Γ_κ-conjugation.* Let $\chi \in \mathscr{X}$. Let $P_\chi(X) \in \mathbb{Z}[X]$ be the g_χth global cyclotomic polynomial. Let κ_A be the field of quotients of A and let $\kappa_A(\mu_{g_\chi})/\kappa_A$ be the extension by the g_χth roots of unity; so, $\Gamma_{\kappa_A,\chi} := \mathrm{Gal}(\kappa_A(\mu_{g_\chi})/\kappa_A)$ is isomorphic to a subgroup of $(\mathbb{Z}/g_\chi\mathbb{Z})^\times$.

One defines, following [Ser1998], the Γ_{κ_A}-conjugation on Ψ by putting, for all $\tau \in \Gamma_{\kappa_A,\chi}$ and $\psi \in \Psi$, $\psi \mid \chi$, $\psi^\tau := \psi^a$, where $a \in \mathbb{Z}$ is a representative of τ in $(\mathbb{Z}/g_\chi\mathbb{Z})^\times$. If σ_χ is a generator of $G_\chi := G_{K_\chi}$, then the $\psi^\tau(\sigma_\chi)$ are the conjugates of $\psi(\sigma_\chi)$ in $\kappa_A(\mu_{g_\chi})/\kappa_A$. This defines the irreducible characters over κ_A (with values in A), $\theta = \sum_{\tau\in\Gamma_{\kappa_A,\chi}}\psi^\tau$.

3.2.2. *Correspondence between characters and cyclotomic polynomials.* Let $\chi \in \mathscr{X}$. In $\kappa_A[X]$, P_χ splits into a product of irreducible distinct polynomials $P_{\chi,i}$; each $P_{\chi,i}$ splits into degree 1 polynomials over $\kappa_A(\mu_{g_\chi})$ and is of degree $[\kappa_A(\mu_{g_\chi}) : \kappa_A]$.

If $\zeta_i \in \mu_{g_\chi}$ is a root of $P_{\chi,i}$, the other roots are the ζ_i^τ for $\tau \in \Gamma_{\kappa_A,\chi}$; thus, these sets of roots are in one by one correspondence with the sets of the form $(\psi^\tau(\sigma_\chi))_{\tau\in\Gamma_{\kappa_A,\chi}}$, $\psi^\tau \mid \chi$, $\psi^\tau \in \Psi$ of order g_χ, describing a representative set of characters for the Γ_{κ_A}-conjugation. One may index, *non-canonically*, the irreducible divisors of P_χ in $\kappa_A[X]$

by means of the characters θ obtained from the characters $\psi \in \Psi$ of orders g_χ and by choosing a generator σ_χ of G_χ. Put:

$$P_\theta := \prod_{\psi|\theta} (X - \psi(\sigma_\chi)) \in A[X]. \tag{3.1}$$

Thus $P_\chi = \prod_{\theta|\chi} P_\theta$; for $A = \mathbb{Z}_p$ we get $P_\chi = \prod_{\varphi\in\Phi,\,\varphi|\chi} P_\varphi$, for $A = \mathbb{Z}$, P_χ is irreducible. So, $A[G_\chi]/(P_\theta(\sigma_\chi)) \simeq A[X]/(X^{g_\chi} - 1, P_\theta(X)) \simeq A[\mu_{g_\chi}]$; then any module annihilated by $P_\theta(\sigma_\chi)$ is a $A[\mu_{g_\chi}]$-module; the law is realized, for $\psi \mid \theta$, via $\sigma \in G_\chi \mapsto \psi(\sigma) \in \mu_{g_\chi}$.

3.2.3. *The $\mathbb{Z}[\mu_{g_\chi}]$-modules $\mathbf{M}_\chi^{\rm alg}$ and the $\mathbb{Z}_p[\mu_{g_\chi}]$-modules $\mathscr{M}_\varphi^{\rm alg}$.* We fix a prime p and consider the set Φ of irreducible p-adic characters of $\mathscr{G}$.

Definition 3.5. *Let $\mathbf{M} = (\mathbf{M}_K)_{K\in\mathscr{K}}$ be a family of $\mathbb{Z}[\mathscr{G}]$-modules and let $\mathscr{M} := \mathbf{M}\otimes\mathbb{Z}_p = (\mathscr{M}_K)_{K\in\mathscr{K}}$. Put, for $\chi \in \mathscr{X}$ and $\varphi \mid \chi$, $\varphi \in \Phi$:*

$$\begin{cases} \mathbf{M}_\chi^{\rm alg} := \big\{x \in \mathbf{M}_{K_\chi},\ P_\chi(\sigma_\chi)\cdot x = 1\big\}, \\ \mathscr{M}_\chi^{\rm alg} := \mathbf{M}_\chi^{\rm alg}\otimes\mathbb{Z}_p = \big\{x \in \mathscr{M}_{K_\chi},\ P_\chi(\sigma_\chi)\cdot x = 1\big\}, \\ \mathscr{M}_\varphi^{\rm alg} := \big\{x \in \mathscr{M}_{K_\chi},\ P_\varphi(\sigma_\chi)\cdot x = 1\big\} = \big\{x \in \mathscr{M}_\chi^{\rm alg},\ P_\varphi(\sigma_\chi)\cdot x = 1\big\}. \end{cases}$$

So, $\mathscr{M}_\varphi^{\rm alg}$ is a sub-$\mathbb{Z}_p[\mu_{g_\chi}]$-module of $\mathscr{M}_{K_\chi}$ (or of $\mathscr{M}_\chi^{\rm alg}$), for the law $\sigma \in G_{K_\chi} \mapsto \psi(\sigma)$, $\psi \mid \varphi$, and the elements of $\mathscr{M}_\varphi^{\rm alg}$ are called algebraic φ-objects.

From relation (3.1), the polynomials P_φ depend on the choice of the generator σ_χ of G_χ, but we have the following canonical property:

Lemma 3.6. *The Definitions 3.5, of the $\mathbb{Z}[\mu_{g_\chi}]$-modules $\mathbf{M}_\chi^{\rm alg}$ and the $\mathbb{Z}_p[\mu_{g_\chi}]$-modules $\mathscr{M}_\varphi^{\rm alg}$, do not depend on the choice of σ_χ.*

Proof. Let $\varphi \mid \chi$. We have $P_\varphi(\sigma_\chi) = \prod_{\psi|\varphi}(\sigma_\chi - \psi(\sigma_\chi))$ and, for $a > 0$ with $\gcd(a, g_\chi) = 1$, let $\sigma'_\chi =: \sigma_\chi^a$ another generator of G_χ giving the relation $P'_\varphi(\sigma'_\chi) = \prod_{\psi|\varphi}(\sigma'_\chi - \psi(\sigma'_\chi))$; one must compare $P_\varphi(\sigma_\chi)$ and $P'_\varphi(\sigma'_\chi)$. Then:

$$P'_\varphi(\sigma_\chi^a) = \prod_{\psi|\varphi}(\sigma_\chi^a - \psi(\sigma_\chi^a)) = \prod_{\psi|\varphi}\big[(\sigma_\chi - \psi(\sigma_\chi))\times(\sigma_\chi^{a-1} + \cdots + \psi^{a-1}(\sigma_\chi))\big],$$

and similarly, writing $1 \equiv a\,a^* \pmod{g_\chi}$, where $a^* > 0$ represents an inverse of a modulo g_χ, we have, from $\sigma_\chi = (\sigma_\chi^a)^{a^*}$:

$$P_\varphi(\sigma_\chi) = \prod_{\psi|\varphi}\big[(\sigma_\chi^a - \psi(\sigma_\chi^a))\times(\sigma_\chi^{a(a^*-1)} + \cdots + \psi^{a(a^*-1)}(\sigma_\chi))\big].$$

Since $P'_\varphi(\sigma'_\chi) \in P_\varphi(\sigma_\chi)\mathbb{Z}_p[G_\chi]$ and $P_\varphi(\sigma_\chi) \in P'_\varphi(\sigma'_\chi)\mathbb{Z}_p[G_\chi]$ the invariance of the definition of the φ-objects follows, as well as that of χ-objects since $P_\chi = \prod_{\varphi|\chi} P_\varphi$. □

3.2.4. *Characterization of $\mathbf{M}_\chi^{\rm alg}$, $\mathscr{M}_\chi^{\rm alg}$, with algebraic norms.* For any $\chi \in \mathscr{X}$, we have defined $\mathbf{M}_\chi^{\rm alg}$ and $\mathscr{M}_\chi^{\rm alg}$. We then have the following characterization, only valid for rational characters, but which will allow another definition of χ and φ-objects (that of "Arithmetic" objects):

Theorem 3.7. *Let $\mathbf{M}$ be a $\mathscr{G}$-family of $\mathbb{Z}[\mathscr{G}]$-modules and define, for any $\chi \in \mathscr{X}$, $\mathbf{M}_\chi^{\rm alg} := \big\{x \in \mathbf{M}_{K_\chi},\ P_\chi(\sigma_\chi)\cdot x = 1\big\}$. Then:*

$$\begin{cases} \mathbf{M}_\chi^{\rm alg} = \big\{x \in \mathbf{M}_{K_\chi},\ \ \nu_{K_\chi/k}(x) = 1,\ \textit{for all } k \subsetneq K_\chi\big\}, \\ \mathscr{M}_\chi^{\rm alg} = \big\{x \in \mathscr{M}_{K_\chi},\ \ \nu_{K_\chi/k}(x) = 1,\ \textit{for all } k \subsetneq K_\chi\big\} \end{cases}$$

(one may limit the norm conditions to $\nu_{K_\chi/k_\ell}(x) = 1$ for all prime divisors ℓ of $[K_\chi : \mathbb{Q}]$, where $k_\ell \subset K_\chi$ is such that $[K_\chi : k_\ell] = \ell$).

Proof. With a contribution of a personal communication from Jacques Martinet (October 1968). We need three preliminary lemmas:

Lemma 3.8. *Let $n \geq 1$ and let q be an arbitrary prime number. Denote by P_n the nth cyclotomic polynomial in $\mathbb{Z}[X]$; then:*

(i) $P_n(X^q) = P_{nq}(X)$, if $q \mid n$;

(ii) $P_n(X^q) = P_{nq}(X)\,P_n(X)$, if $q \nmid n$.

Proof. Obvious for (i), (ii) by means of comparison of the sets of roots of these polynomials. □

Lemma 3.9. *Let $n = \ell_1 \cdots \ell_t$, $t \geq 2$, the ℓ_i's being distinct prime numbers. Then for all pair (i,j), $i \neq j$, there exist A_i^j and A_j^i in $\mathbb{Z}[X]$, such that $A_i^j P_{\frac{n}{\ell_i}} + A_j^i P_{\frac{n}{\ell_j}} = 1$.*

Proof. This can be proved by induction on $t \geq 2$.

If $t = 2$, $n = \ell_1\ell_2$ and:

$$P_{\frac{n}{\ell_2}} = P_{\ell_1} = X^{\ell_1-1} + \cdots + X + 1, \;\; P_{\frac{n}{\ell_1}} = P_{\ell_2} = X^{\ell_2-1} + \cdots + X + 1.$$

Let's call "geometric polynomial" any polynomial in $\mathbb{Z}[X]$ of the form $X^d + X^{d-1} + \cdots + X + 1$, $d \geq 0$ (including the polynomial 0).

Then if P and $Q \neq 0$ are geometric, the residue R of P modulo Q is geometric with residue $(P-R)Q^{-1} \in \mathbb{Z}[X]$; indeed, if $m \geq n$ and $m+1 = q(n+1)+r$, $0 \leq r < n$, we get:

$$X^m + \cdots + X + 1 =$$
$$(X^n + \cdots + X + 1) \times \left[X^{m+1-(n+1)} + X^{m+1-2(n+1)} + \cdots + X^{m+1-q(n+1)}\right]$$
$$+ 1 + X + \cdots + X^{r-1}$$

(if $r \geq 1$, otherwise the residue R is 0). In particular, the gcd algorithm gives geometric polynomials; as the unique non-zero constant geometric polynomial is 1, it follows that if P and Q are co-prime polynomials in $\mathbb{Q}[X]$, $\gcd(P,Q) = 1$ and the Bézout relation takes place in $\mathbb{Z}[X]$, which is the case for the geometric polynomials P_{ℓ_1} and P_{ℓ_2}.

Suppose $t \geq 3$. Let ℓ_i, ℓ_j, q, be three distinct primes dividing n; put $n' := \dfrac{n}{q}$; by induction, since ℓ_i and ℓ_j divide n', there exist polynomials $A_i'^j, A_j'^i$ in $\mathbb{Z}[X]$, such that $A_i'^j(X)P_{\frac{n'}{\ell_i}}(X) + A_j'^i(X)P_{\frac{n'}{\ell_j}}(X) = 1$, thus, $A_i'^j(X^q)P_{\frac{n'}{\ell_i}}(X^q) + A_j'^i(X^q)P_{\frac{n'}{\ell_j}}(X^q) = 1$. But Lemma 3.8 (ii) gives:

$$P_{\frac{n'}{\ell_i}}(X^q) = P_{\frac{n}{\ell_i}}(X)P_{\frac{n'}{\ell_i}}(X) \quad \& \quad P_{\frac{n'}{\ell_j}}(X^q) = P_{\frac{n}{\ell_j}}(X)P_{\frac{n'}{\ell_j}}(X),$$

which yields $A_i'^j(X^q)P_{\frac{n}{\ell_i}}(X)P_{\frac{n'}{\ell_i}}(X) + A_j'^i(X^q)P_{\frac{n}{\ell_j}}(X)P_{\frac{n'}{\ell_j}}(X) = 1$.

We have proved the co-maximality, in $\mathbb{Z}[X]$, of any pair of ideals $(P_{\frac{n}{\ell_i}}(X))$, $(P_{\frac{n}{\ell_j}}(X))$, $i \neq j$ (the case $n = \ell$ giving the prime ideal $(P_\ell(X)\mathbb{Z}[X])$). □

Lemma 3.10. *Let $n = \prod_{i=1}^{t} \ell_i^{a_i} > 1$, $a_i \geq 1$; put $N_{n,\ell}(X) := \sum_{i=0}^{\ell-1} X^{\frac{n}{\ell}i}$ for any prime ℓ dividing n. Then there exist polynomials $A_\ell(X) \in \mathbb{Z}[X]$ such that $P_n(X) = \sum_{\ell \mid n} A_\ell(X)N_{n,\ell}(X)$ and $\big\langle N_{n,\ell}(X),\ \ell \mid n\big\rangle_{\mathbb{Z}[X]} = P_n(X)\mathbb{Z}[X]$.*

Proof. Assume by induction on n that $P_n(X) = \sum_{\ell \mid n} A_\ell(X)N_{n,\ell}(X)$ (with t fixed), and let $q \mid n$; we have, from Lemma 3.8 (i):

$$P_{nq}(X) = P_n(X^q) = \sum_{\ell \mid n} A_\ell(X^q)N_{n,\ell}(X^q).$$

Since we have $N_{n,\ell}(X^q) = \sum_{i=0}^{\ell-1} X^{\frac{n}{\ell}qi} = N_{nq,\ell}(X)$, we obtain that if the lemma is true for n, it is true for nq for all $q \mid n$. It follows that if the property is true for all square-free integers n, it is true for all $n > 1$. So we may assume n square-free to prove the lemma by induction on t.

If $n=\ell_1$, $P_{\ell_1}(X)=X^{\ell_1-1}+\cdots+X+1=N_{\ell_1,\ell_1}(X)$ and the claim is obvious. If $n=\ell_1\ell_2\cdots\ell_t$, $t\geq 2$, with distinct primes, put $n_k=\frac{n}{\ell_k}$ for all k; by assumption, $P_{n_k}(X)=\sum_{1\leq s\leq t,\ s\neq k} A_s^k(X)N_{n_k,\ell_s}(X)$, hence:

$$\begin{aligned}P_{n_k}(X^{\ell_k})&=P_{n_k\ell_k}(X)\cdot P_{n_k}(X)\\&=P_n(X)P_{n_k}(X)=\sum_{1\leq s\leq t\ s\neq k} A_s^k(X^{\ell_k})N_{n,\ell_s}(X),\end{aligned}$$

whence $P_n(X)P_{n_k}(X)\in\langle N_{n,\,\ell}(X),\ \ell\mid n\rangle_{\mathbb{Z}[X]}$, for all k; since $t\geq 2$, Lemma 3.9 applies; a Bézout relation in $\mathbb{Z}[X]$ between any two of the P_{n_k} (say P_{n_i} and P_{n_j}) yields $P_n(X)\times 1\in\langle N_{n,\,\ell}(X),\,\ell\mid n\rangle_{\mathbb{Z}[X]}$, giving the result.

We have proved that the ideal generated, in $\mathbb{Z}[X]$, by the $N_{n,\ell}(X)$, $\ell\mid n$, contains $P_n(X)\mathbb{Z}[X]$. Let's see that $P_n(X)$ contains that ideal; it is sufficient to see that for all $\ell\mid n$, $N_{n,\ell}(X)=P_\ell(X^{\frac{n}{\ell}})$; any root of unity ζ_n of order n (i.e., root of $P_n(X)$), is a root of $N_{n,\ell}(X)$ since $\zeta_n^{\frac{n}{\ell}}=\zeta_\ell\neq 1$ and $\sum_{i=0}^{\ell-1}\zeta_\ell^i=0$; then $P_n(X)\mid N_{n,\ell}(X)$ in $\mathbb{Z}[X]$ (monic polynomials). □

We apply this to $P_\chi(\sigma_\chi)=P_{g_\chi}(\sigma_\chi)$ and to $N_{g_\chi,\ell}(\sigma_\chi)=\nu_{K_\chi/k_\ell}$, where k_ℓ is, for all $\ell\mid g_\chi$, the unique sub-extension of K_χ such that $[K_\chi:k_\ell]=\ell$. The theorem immediately follows. □

3.2.5. *Application to the definition of* $\mathbf{M}_\chi^{\rm ar}$. Let $\mathbf{M}$ be an arithmetic $\mathscr{G}$-family, provided with norms $\mathbf{N}$ and transfer maps $\mathbf{J}$ with $\mathbf{J}\circ\mathbf{N}=\nu$.

Definition 3.11. *By analogy with Theorem 3.7 giving, for χ-objects, the definition $\mathbf{M}_\chi^{\rm alg}:=\{x\in\mathbf{M}_{K_\chi},\ \nu_{K_\chi/k}(x)=1$, for all $k\subsetneq K_\chi\}$ and $\mathscr{M}_\chi^{\rm alg}=\mathbf{M}_\chi^{\rm alg}\otimes\mathbb{Z}_p$, we define the modules of arithmetic χ-objects:*

$$\begin{cases}\mathbf{M}_\chi^{\rm ar}:=\{x\in\mathbf{M}_{K_\chi},\ \ \mathbf{N}_{K_\chi/k}(x)=1,\ \text{for all } k\subsetneq K_\chi\}\subseteq\mathbf{M}_\chi^{\rm alg}\\ \mathscr{M}_\chi^{\rm ar}:=\mathbf{M}_\chi^{\rm ar}\otimes\mathbb{Z}_p.\end{cases}$$

Then $\mathbf{M}_\chi^{\rm ar}$ is a sub-$\mathbb{Z}[\mu_{g_\chi}]$-module of $\mathbf{M}_\chi^{\rm alg}$ and $\mathscr{M}_\chi^{\rm ar}$ is a sub-$\mathbb{Z}_p[\mu_{g_\chi}]$-module of $\mathscr{M}_\chi^{\rm alg}$, with laws defined via the choice of $\psi\mid\chi$ (resp. $\psi\mid\varphi$).

We have $\mathbf{M}_\chi^{\rm ar}=\mathbf{M}_\chi^{\rm alg}$ as soon as the $\mathbf{J}_{K_\chi/k}$'s are injective (for all $k\subsetneq K_\chi$ or simply the k_ℓ's). One verifies easily that if the norms $\mathbf{N}_{K_\chi/k_\ell}$ are surjective for all $\ell\mid g_\chi$, then $\mathbf{M}_\chi^{\rm alg}/\mathbf{M}_\chi^{\rm ar}$ has exponent a divisor of $\prod_{\ell\mid g_\chi}\ell$, whence $\mathscr{M}_\chi^{\rm alg}/\mathscr{M}_\chi^{\rm ar}$ of exponent 1 or p.

3.3. **Comparison with classical definitions of θ-components.** In all classical papers, the θ-components $\mathbf{M}_\theta$ (θ rational or p-adic, above $\psi\in\Psi$) is defined, in an abelian field K of Galois group G, by:

$$\mathbf{M}_\theta:=\mathbf{M}\otimes_{A[G]}A[\theta],$$

where $A[\theta]:=A[\psi]$ is the ring of values of θ over A; the action being defined via $(\sigma,x)\in G\times\mathbf{M}_\theta\mapsto\psi(\sigma)\cdot x\in\mathbf{M}_\theta$. As for the example of cohomology groups, this definition is only algebraic and not arithmetic. We shall compare this definition with Definition 3.11 considering irreducible p-adic characters φ. We have the classical algebraic definition of φ-objects attached to $\mathscr{M}$, that is to say, the largest quotient such that G_χ acts by ψ ([Grei1992, Definition, p. 451], [PeRi1990, § 1.3]):

$$\widehat{\mathscr{M}_\varphi}:=\mathscr{M}\otimes_{\mathbb{Z}_p[G_\chi]}\mathbb{Z}_p[\mu_{g_\chi}]\simeq\mathscr{M}/P_\varphi(\sigma_\chi)\cdot\mathscr{M}$$

Another viewpoint [Sol1990, § II.1, pp. 469–471], is to define $\widehat{\mathscr{M}^\varphi}$ as the largest sub-$\mathbb{Z}_p[G_\chi]$-module of $\mathscr{M}$, such that G_χ acts by ψ. Whence:

$$\widehat{\mathscr{M}^\varphi}:=\{x\in\mathscr{M},\ \ P_\varphi(\sigma_\chi)\cdot x=1\}=\mathscr{M}_\varphi^{\rm alg},$$

with the exact sequence $1\to\widehat{\mathscr{M}^\varphi}=\mathscr{M}_\varphi^{\rm alg}\longrightarrow\mathscr{M}\longrightarrow P_\varphi(\sigma_\chi)\cdot\mathscr{M}\to 1$ giving the equalities $\#\widehat{\mathscr{M}_\varphi}=\#\widehat{\mathscr{M}^\varphi}=\#\mathscr{M}_\varphi^{\rm alg}$ for finite modules.

Moreover, our forthcoming Definition 4.3 of $\mathscr{M}_\varphi^{\rm ar}$:

$$\mathscr{M}_\varphi^{\rm ar} := \mathscr{M}_\chi^{\rm ar} \cap \mathscr{M}_\varphi^{\rm alg} \ \ \text{(with Definition 3.11 of } \mathscr{M}_\chi^{\rm ar}),$$

introduces another kind of computations. Indeed, the Main Theorem on abelian fields in the literature is concerned by algebraic definitions similar to $\widehat{\mathscr{M}_\varphi}$ or $\widehat{\mathscr{M}^\varphi}$, but our conjecture given in the 1970's used $\mathscr{M}_\varphi^{\rm ar}$ and new analytic expressions giving $\#\mathscr{M}_\chi^{\rm ar}$, justifying the conjectural values of $\#\mathscr{M}_\varphi^{\rm ar}$ for finite $\mathscr{M}_K$'s.

It is immediate to verify that, in the non semi-simple case $p \mid g_\chi$, $(\mathscr{M}_\varphi^{\rm alg} : \mathscr{M}_\varphi^{\rm ar})$ is equal to the order of the capitulation kernel of $\mathbf{J}_{K_\chi/k_p}$, where k_p is the subfield of K_χ such that $[K_\chi : k_p] = p$. In the semi-simple case $p \nmid \#G_\chi$, $\mathscr{M} \simeq \mathscr{M}_\varphi \oplus \big[P_\varphi(\sigma_\chi)\cdot\mathscr{M}\big]$ whatever the definitions.

3.4. **Numerical examples.** Let $k = \mathbb{Q}(\sqrt{m})$ be a real quadratic field and let K be the compositum of k with a cyclic extension L of $\mathbb{Q}$ of p-power degree; the field K is of the form K_χ for $\chi \in \mathscr{X}^+$ which is also irreducible p-adic. We have given in [Gra2021^b] many examples of capitulations of the p-class group of k in K, giving $\mathscr{H}_\chi^{\rm ar} \subsetneqq \mathscr{H}_\chi^{\rm alg}$.

3.4.1. *General PARI program* [Pari2016]. For the program, one must precise the prime $p > 2$, the minimal required p-rank rpmin of $\mathbf{H}_k$, the length N of the sub-tower considered and the interval for m (the program uses primes ℓ (in ell) congruent to 1 modulo $2p^N$; the class group (resp. the p-class group) is computed in Ck (resp. Ckp):

```
{p=3;rpmin=1;N=2;bm=2;Bm=10^4;for(m=bm,Bm,if(core(m)!=m,next);P=x^2-m;
k=bnfinit(P,1);Ck=k.clgp;r=matsize(Ck[2])[2];Ckp=List;Ekp=List;rp=0;
for(i=1,r,ei=Ck[2][i];vi=valuation(ei,p);if(vi>0,rp=rp+1;
ai=idealpow(k,Ck[3][i],ei/p^vi);listput(Ckp,ai,rp);
listput(Ekp,p^vi,rp)));if(rp<rpmin,next);L0=List;
for(i=1,rp,listput(L0,0,i));forprime(ell=2,10^4,
if(Mod(ell-1,2*p^N)!=0 || Mod(m,ell)==0,next);
Lq=List;for(i=1,rp,A=Ckp[i];forprime(q=2,10^5,if(q==ell,next);
if(kronecker(m,q)!=1 || Mod((ell-1)/znorder(Mod(q,ell)),p)==0,next);
F=idealfactor(k,q);qi=component(F,1)[1];cij=qi;for(j=1,Ekp[i]-1,
cij=idealmul(k,cij,A);if(Mod(j,p)==0,next);
if(List(bnfisprincipal(k,cij)[1])==L0,listput(Lq,q,i);break(2)))));
print("____");print();print("m=",m," ell=",ell," Lq=",Lq);
for(n=0,N,R=polcompositum(P,polsubcyclo(ell,p^n))[1];K=bnfinit(R,1);
print();print("C",n,"=",K.cyc);for(i=1,rp,Fi=idealfactor(K,Lq[i]);
Qi=component(Fi,1)[1];print(bnfisprincipal(K,Qi)[1])))))}
```

We shall consider the base field $k = \mathbb{Q}(\sqrt{4409})$ (i.e., $m = 4409$ in the program) with $\ell = 19$.

Example 3.12. Let L be the degree 9 subfield of $\mathbb{Q}(\mu_{19})$; for convenience, put $k_0 := k$, $k_1 := L_1k_0$ (resp. $k_2 := L_2k_0$), where L_1 (resp. L_2) is the degree 3 (resp. 9) subfield of $\mathbb{Q}(\mu_{19})$. The prime 2 splits in k_0, is inert in k_2/k_0 and such that $\mathfrak{Q}_0 \mid 2$ in k_0 generates $\mathscr{H}_{k_0}$ (cyclic of order 9); considering the extensions $\mathfrak{Q}_i = \mathbf{J}_{k_i/k_0}(\mathfrak{Q}_0)$ of $\mathfrak{Q}_0$ in k_i, we test its order in $\mathscr{H}_{k_i}$, $i = 1, 2$ (we are going to see that $\mathscr{H}_{k_i} \simeq \mathbb{Z}/9\mathbb{Z}$ for all i, which is supported by the fact that $\mathbf{N}_{k_2/k_0}(\mathfrak{Q}_2) = \mathfrak{Q}_0^9$ but $\mathbf{N}_{k_2/k_0}(\mathscr{H}_{k_2}) = \mathscr{H}_{k_0}$ since k_2/k_0 is totally ramified at 19):

```
C0=[9]   [4]~              C1=[9]   [6]~              C2=[9]   [0]~
```

More precisely, C0 = [9] denotes the class group of k_0 and, using the instruction bnfisprincipal, $[4]^\sim$ means that the class of $\mathfrak{Q}_0 \mid 2$ is h_0^4, where h_0 is the generator (of order 9) given in kn.cyc by PARI; then C1 = [9], $[6]^\sim$, is the similar data for k_1 in which we see a partial capitulation since the class of $\mathfrak{Q}_1 = \mathbf{J}_{k_1/k_0}(\mathfrak{Q}_0)$ becomes of order 3. Finally, C2 = [9], $[0]^\sim$ shows the complete capitulation in k_2; the 18 large integers below are the coefficients, over an integral basis, of a generator of $\mathfrak{Q}_2$ in k_2:

```
[[0]~,[-270476874595642910,323533824277028894,-236208800298303000,
       119737461690335806,-255607858779215282,-198423813102857420,
       410588865020870414,-110028179006577678,-449600797918214026,
       -4906665437527948,10274048566854232,4319852458093887,
       13258715755947394,-6817941144899095,-15448507867705832,
       2623003974789062,-3264916449440532,-16606126998680345]~]
```

We use obvious notations for the characters defining the fields k_i, $i=0,1,2$. Since arithmetic norms are surjective (here, they are isomorphisms), the above computations prove that:

$$\nu_{k_2/k_1}(\mathscr{H}_{k_2}) = \mathbf{J}_{k_2/k_1}\circ\mathbf{N}_{k_2/k_1}(\mathscr{H}_{k_2}) = \mathbf{J}_{k_2/k_1}(\mathscr{H}_{k_1}) \simeq \mathbb{Z}/3\mathbb{Z},$$

since $\mathbf{N}_{k_2/k_1}\circ\mathbf{J}_{k_2/k_1}(\mathscr{H}_{k_1}) = \mathscr{H}_{k_1}^3$, or simply $\mathbf{J}_{k_2/k_1}(\mathscr{H}_{k_1}) = \mathscr{H}_{k_2}^3$ (partial capitulation of $\mathscr{H}_{k_1}\simeq\mathbb{Z}/9\mathbb{Z}$). Whence:

$$\begin{cases} \mathscr{H}_{\chi_2}^{\rm ar} = \{x\in\mathscr{H}_{k_2},\ \mathbf{N}_{k_2/k_1}(x)=1\} = 1,\\ \mathscr{H}_{\chi_2}^{\rm alg} = \{x\in\mathscr{H}_{k_2},\ P_{\chi_2}(\sigma_{\chi_2})\cdot x = 1\}\\ \qquad\quad = \{x\in\mathscr{H}_{k_2},\ \nu_{k_2/k_1}(x)=1\} = \mathscr{H}_{k_2}^3\simeq\mathbb{Z}/3\mathbb{Z}. \end{cases}$$

We have $P_{\chi_2}(\sigma_{\chi_2}) = \sigma_{\chi_2}^6+\sigma_{\chi_2}^3+1 = \nu_{k_2/k_1}$ (since L is principal, the norms ν_{k_i/L_i} does not intervene in the definition of the $\mathscr{H}_{\chi_i}^{\rm alg}$'s).

Similarly, we have:

$$\nu_{k_1/k_0}(\mathscr{H}_{k_1}) = \mathbf{J}_{k_1/k_0}\circ\mathbf{N}_{k_1/k_0}(\mathscr{H}_{k_1}) = \mathbf{J}_{k_1/k_0}(\mathscr{H}_{k_0}) \simeq \mathbb{Z}/3\mathbb{Z}$$

(partial capitulation of $\mathscr{H}_{k_0}\simeq\mathbb{Z}/9\mathbb{Z}$); whence:

$$\begin{cases} \mathscr{H}_{\chi_1}^{\rm ar} = \{x\in\mathscr{H}_{k_1},\ \mathbf{N}_{k_1/k_0}(x)=1\} = 1,\\ \mathscr{H}_{\chi_1}^{\rm alg} = \{x\in\mathscr{H}_{k_1},\ \nu_{k_1/k_0}(x)=1\} = \mathscr{H}_{k_1}^3\simeq\mathbb{Z}/3\mathbb{Z}. \end{cases}$$

Thus, the forthcoming formula of Theorem 3.15 giving:

$$\#\mathscr{H}_{k_2} = \#\mathscr{H}_{\chi_0}^{\rm ar}\cdot\#\mathscr{H}_{\chi_1}^{\rm ar}\cdot\#\mathscr{H}_{\chi_2}^{\rm ar}$$

is of the form $\#\mathscr{H}_{k_2} = 9\times1\times1$, then $\#\mathscr{H}_{k_1} = 9\times1$ since $\mathscr{H}_{\chi_0}^{\rm ar} = \mathscr{H}_{k_0}$.

These formulas are not fulfilled in the algebraic sense, because:

$$\#\mathscr{H}_{\chi_0}^{\rm alg}\cdot\#\mathscr{H}_{\chi_1}^{\rm alg} = 9\times3 = 3^3 \text{ and } \#\mathscr{H}_{\chi_0}^{\rm alg}\cdot\#\mathscr{H}_{\chi_1}^{\rm alg}\cdot\#\mathscr{H}_{\chi_2}^{\rm alg} = 9\times3\times3 = 3^4.$$

Now we intend to compute $\#\mathscr{H}_{\chi_1}^{\rm ar} = \#(\mathscr{E}_{k_1}/\mathscr{E}_{k_1}^0\cdot\mathscr{F}_{k_1})$ (analytic formula of Theorem 7.5); in the general definition, $\mathscr{F}_k$ denotes the Leopoldt group of cyclotomic units of k, $\mathscr{E}_k^0$ the group of units generated by the units of the strict subfields of k.

We give numerical values of the units $|\mathsf{e0}|$ of k_0, $|\mathsf{ei}|$ of L_1, $|\mathsf{Ej}|$ of k_1, and their logarithms; they are, respectively (standard PARI programs):

```
Units                                              Logarithms
e0=664.00150602068057486397714386165380      6.49828441757729630972016
e1=0.28514248182978536439411987353062 74     -1.25476628739511494204754
e2=4.50701864409297629866079992371567 80      1.50563588039686576534798
E1=0.28514248182978536439411987353062 74     -1.25476628739511494204754
E2=0.22187616226319093426668005018505 06     -1.50563588039686576534798
E3=664.00150602068057486397714386165380      6.49828441757729630972016
E4=945628377316488.87204143428389231544     34.48287077198255819743 18
E5=0.00257365190752746549299934631279 51     -5.96242941301396593243487
```

```
Cyclotomic units:
{f=19*4409;z=exp(I*Pi/f);g1=lift(Mod(74956,f)^2);g2=lift(Mod(4410,f)^3);
frob=1;for(s=1,6,frob=lift(Mod(3*frob,f));Eta=1;for(k=1,(4409-1)/2,
for(j=1,(19-1)/3,as=lift(Mod(g1^k*g2^j*frob,f));if(as>f/2,next);
Eta=Eta*(z^as-z^-as)));print("Eta^s",s,"=",Eta," ",log(abs(Eta))))}

Eta^s1=945628377316488.87204143428       34.48287077198255819743 18471
Eta^s2=2433718277092.6834663091300       28.52044135896859226499 69695
Eta^s3=0.00257365190752746549299 93      -5.96242941301396593243487762
Eta^s4=1.05749787547388046520 63 E-15   -34.48287077198255819743 18471
Eta^s5=4.10893902310911119828 24 E-13   -28.52044135896859226499 69695
Eta^s6=388.55293409150677930552135        5.96242941301396593243487762
```

One obtains easily the following relations:

```
E1=e1,   E2=e2^-1,   E3=e0,   E_4^2=Eta^s,   E5^2=Eta^-1,
Eta^{s^3+1}=1,   Eta^{s^2-s+1}=1, giving: Eta^(s^2)=E4^2.E5^2.
```

Then, one gets $(\mathscr{E}_{k_1} : \mathscr{E}^0_{k_1} \cdot \mathscr{F}_{k_1}) = (\mathscr{E}_{k_1} : \mathscr{E}_{k_0} \cdot \mathscr{E}_{L_1} \cdot \mathscr{F}_{k_1}) = 1$ as expected since $\mathscr{H}^{\rm ar}_{\chi_1} = 1$. Moreover, we see that the conjugates of the cyclotomic units are not independent (due, from Lemma 5.17, to norm relations in k_i/k_0 and k_i/L_i since 19 splits in k_0 and 4409 splits in the L_i's), but, with our point of view, this does not matter since $\mathscr{E}^0_{k_1}$ is of $\mathbb{Z}_3$-rank 3 and $\mathscr{F}_{k_1}$ is of $\mathbb{Z}_3$-rank 2. Indeed, these relations lead to some difficulties in χ-formulas of the literature *using larger groups of cyclotomic units* like Sinnott's cyclotomic units (see Remark 7.7).

To be complete, compute the classical index of $\mathscr{F}_{k_0} =: \langle \eta_0 \rangle$ in $\mathscr{E}_{k_0}$:

```
{f=4409;z=exp(I*Pi/f);Eta0=1;g=znprimroot(f)^2;for(k=1,(f-1)/2,
a=lift(g^k);if(a>f/2,next);Eta0=Eta0*(z^a-z^-a)/(z^(3*a)-z^-(3*a)));
print("Eta0=",Eta0," log(Eta0)=",log(abs(Eta0)))}
Eta0=3.985459685929 E-26     log(Eta0)=-58.484559758195
```

giving immediately $\mathsf{log(Eta0)} = -9 * \mathsf{log(e0)}$ from the above computation of $\mathsf{log(e0)}$; whence $\#\mathscr{H}^{\rm ar}_{\chi_0} = (\mathscr{E}_{k_0} : \mathscr{E}^0_{k_0} \cdot \mathscr{F}_{k_0}) = (\mathscr{E}_{k_0} : \mathscr{F}_{k_0}) = 9$; obviously, 9 is the annihilator of $\mathscr{E}_{k_0}/\mathscr{F}_{k_0}$ and $\mathscr{H}^{\rm ar}_{\chi_0}$ (Conjecture 7.9).

The verification of $(\mathscr{E}_{k_2} : \mathscr{E}^0_{k_2} \cdot \mathscr{F}_{k_2}) = 1$ is analogous since $\mathscr{F}_{k_2}$ is of $\mathbb{Z}_3$-rank 8 ($\mathbf{N}_{k_2/k_1}(\mathscr{F}_{k_2}) = \mathscr{F}_{k_1}$, $\mathbf{N}_{k_2/k_0}(\mathscr{F}_{k_2}) = 1$, $\mathbf{N}_{k_2/L_2}(\mathscr{F}_{k_2}) = 1$).

Example 3.13. Consider the same framework, replacing 19 by the prime 1747; one obtains the data showing, as before with $\mathfrak{Q}_0 \mid 2$, a partial capitulation of $\mathscr{H}_{k_0}$ in k_1 (but $\mathscr{H}_{k_1}$ is not cyclic):

```
C0=[9]    [4]~            C1=[9,3,3]    [6,0,0]~
```

One verifies that the ideal Q_1, extending Q_0 in k_1, is non-principal and such that its class is $\mathsf{h}_1^6\,\mathsf{h}_2^0\,\mathsf{h}_3^0$ on the PARI basis $\{\mathsf{h}_1, \mathsf{h}_2, \mathsf{h}_3\}$:

```
bnfisprincipal(K,[2, [-1,0,0,1,0,0]~,1,3,[0,0,0,1,0,0]~]) = [[6,0,0]~
```

but its 6-power gives as expected the principality and a generator:

```
bnfisprincipal(K,[64,0,0,21,0,0;0,64,0,0,0,42;0,0,64,0,21,0;0,0,0,1,0,0;
                                                0,0,0,0,1,0;0,0,0,0,0,1])
=[[0,0,0]~,[821719075630487115396921 3,52602828277952742913 8218,
           -68778602907559567659413 4,25130170977215548291 7577,
           -21032376402967976888126,-15609327127430752932511]~]
```

The kernel of the arithmetic norm is isomorphic to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, thus:

$$\begin{cases} \mathscr{H}^{\rm ar}_{\chi_1} = \{x \in \mathscr{H}_{k_1},\ \mathbf{N}_{k_1/k_0}(x) = 1\} \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}, \\ \mathscr{H}^{\rm alg}_{\chi_1} = \{x \in \mathscr{H}_{k_1},\ \nu_{k_1/k_0}(x) = 1\} \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}. \end{cases}$$

since the transfer map applies $\mathscr{H}^{\rm ar}_{\chi_0} \simeq \mathbb{Z}/9\mathbb{Z}$ onto $\langle h_1^6 \rangle$.

Formula of Theorem 3.15 is of the form $\#\mathscr{H}_{k_1} = \#\mathscr{H}^{\rm ar}_{\chi_0} \cdot \#\mathscr{H}^{\rm ar}_{\chi_1} = 9 \times 9$, since we have $\mathscr{H}^{\rm ar}_{\chi_0} = \mathscr{H}_{k_0}$ of order 9; of course a same formula with the $\mathscr{H}^{\rm alg}$'s does not exist since $\#\mathscr{H}^{\rm alg}_{\chi_0} \cdot \#\mathscr{H}^{\rm alg}_{\chi_1} = 9 \times 27$.

Remark 3.14. The program gives the following other results, varying only ell, where q is the prime split in k_0 and inert in k_2:

```
ell=37  q=2    C0=[9]  [4]~    C1=[18]   [6]~      C2=[18]     [0]~
ell=73  q=2    C0=[9]  [4]~    C1=[9]    [6]~      C2=[171]    [0]~
ell=109 q=5    C0=[9]  [1]~    C1=[9]    [6]~      C2=[9]      [0]~
ell=127 q=23   C0=[9]  [4]~    C1=[9]    [6]~      C2=[9]      [0]~
ell=163 q=2    C0=[9]  [4]~    C1=[54]   [12]~     C2=[54]     [18]~
ell=181 q=2    C0=[9]  [4]~    C1=[27]   [12]~     C2=[81]     [63]~
ell=199 q=2    C0=[9]  [4]~    C1=[9,3]  [6,0]~    C2=[27,3]   [9,0]~
```

The image of $\mathscr{H}_{k_0}$ in k_1 is of order 3, except for $\ell \in \{163, 181\}$; then $\mathscr{H}_{k_0}$ capitulates in k_2, except for $\ell \in \{163, 181, 199\}$. One verifies that formula of Theorem 3.15 holds with the $\#\mathscr{H}^{\rm ar}_{k_i}$ but not for the $\#\mathscr{H}^{\rm alg}_{k_i}$.

3.5. **Arithmetic factorization of $\#\mathbf{M}_K$ and $\#\mathscr{M}_K$.** Let $\mathbf{M}$ be an arithmetic $\mathscr{G}$-family where all the $\mathbb{Z}[\mathscr{G}]$-modules $\mathbf{M}_K$, $K \in \mathscr{K}$, are finite; then we can state:

Theorem 3.15. *Let $K/\mathbb{Q}$ be a cyclic extension and assume that for all sub-extension k/k' of $K/\mathbb{Q}$, the maps $\mathbf{N}_{k/k'}$ are surjective. Then:*

$$\#\mathbf{M}_K = \prod_{\chi \in \mathscr{X}_K} \#\mathbf{M}_\chi^{\rm ar},$$

where $\mathbf{M}_\chi^{\rm ar} := \{x \in \mathbf{M}_{K_\chi},\ \mathbf{N}_{K_\chi/k}(x) = 1,\ \forall\, k \subsetneqq K_\chi\}$ (Definition 3.11).

Assuming only the cyclicity of the p-Sylow subgroup of G_K, one obtains, $\#\mathscr{M}_K = \prod_{\chi \in \mathscr{X}_K} \#\mathscr{M}_\chi^{\rm ar}$.

Proof. One may replace the $\mathbf{M}_k$, $k \subseteq K$, by the finite $\mathbb{Z}_p[G_K]$-modules $\mathscr{M}_k := \mathbf{M}_k \otimes \mathbb{Z}_p$, for all primes dividing $\#\mathbf{M}_K$, using the previous results, then globalizing at the end. Two classical lemmas are necessary.

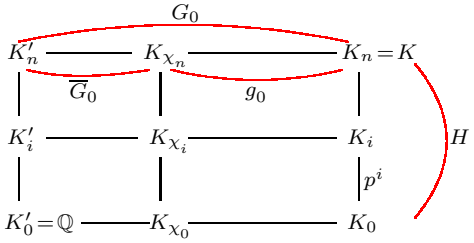
Lemma 3.16. *Assume that $p \nmid [k : k']$. If $\mathbf{N}_{k/k'} : \mathscr{M}_k \longrightarrow \mathscr{M}_{k'}$ is surjective (resp. if $\mathbf{J}_{k/k'} : \mathscr{M}_{k'} \longrightarrow \mathscr{M}_k$ is injective), then $\mathbf{J}_{k/k'}$ is injective (resp. $\mathbf{N}_{k/k'}$ is surjective).*

Proof. From Proposition 3.2, we know that $\mathbf{N}_{k/k'} \circ \mathbf{J}_{k/k'} = [k : k']$; whence the proofs since $[k : k']$ is invertible modulo p. □

Put $G_K = G_0 \oplus H$, where G_0 is a subgroup of prime-to-p order and H (cyclic of order p^n) is the p-Sylow subgroup of G_K. Let K_0 (resp. K'_n) be the field fixed by H (resp. G_0).

The set of subfields of K is of the form $\{K_{\chi_i},\ \chi_i \in \mathscr{X}_K,\ 0 \le i \le n\}$, where χ_i is the rational character above $\psi_i := \psi_0\, \psi_p^{p^{n-i}}$, where $\psi_p \in \Psi_{K'_n}$ is of order p^n and $\psi_0 \in \Psi_{K_0}$; thus K_{χ_i} is the compositum $K_{\chi_0} K'_i$:

3.5.1. *Schema I.*



Let $\mathscr{M}^*_{K_{\chi_i}} := \mathrm{Ker}(\mathbf{N}_{K_{\chi_i}/K_{\chi_{i-1}}})$, $1 \le i \le n$, then put $\mathscr{M}^*_{K_{\chi_0}} := \mathscr{M}_{K_{\chi_0}}$. By assumption, we have the exact sequences of $\mathbb{Z}_p[G_K]$-modules:

$$1 \longrightarrow \mathscr{M}^*_{K_{\chi_i}} \longrightarrow \mathscr{M}_{K_{\chi_i}} \xrightarrow{\mathbf{N}_{K_{\chi_i}/K_{\chi_{i-1}}}} \mathscr{M}_{K_{\chi_{i-1}}} \longrightarrow 1,\ \ 1 \le i \le n. \tag{3.2}$$

One considers them as exact sequences of $\mathbb{Z}_p[G_0]$-modules. The idempotents of this algebra are, for all $\chi_0 \in \mathscr{X}_{K_0}$, of the form:

$$e_{\chi_0} = \frac{1}{\#G_0} \sum_{\sigma \in G_0} \chi_0(\sigma^{-1})\, \sigma \in \mathbb{Z}_p[G_0].$$

From Leopoldt [Leo1954], [Leo1962, Chap. V, § 2], as the norm maps are surjective and the transfer maps injective, regarding the sub-extensions k/k' of prime-to-p degrees in $K/\mathbb{Q}$, we get the following canonical identifications:

Lemma 3.17. *Let $\mathscr{M}$ be an arithmetic $\mathscr{G}$-family whose elements $\mathscr{M}_K$ are $\mathbb{Z}_p[G_0 \oplus H]$-modules in the above sense. Then $\mathscr{M}_{K_i}^{e_{\chi_0}} \simeq \mathscr{M}_{K_{\chi_i}}^{e_{\chi_0}}$ and $(\mathscr{M}^*_{K_i})^{e_{\chi_0}} \simeq (\mathscr{M}^*_{K_{\chi_i}})^{e_{\chi_0}}$.*

Proof. For all i, we identifie $\mathrm{Gal}(K_i/K'_i)$ with G_0 acting by restriction and put $\overline{G}_0 := G_0/g_0$, where $g_0 := \mathrm{Gal}(K_n/K_{\chi_n})$. Thus, by abuse of notation, we identify $\nu_{K_i/K_{\chi_i}}$ with $\nu_{K_n/K_{\chi_n}} =: \nu_{g_0}$; moreover, since the degrees of these extensions are prime to p, we may identify $\mathbf{N}_{K_i/K_{\chi_i}}$ with $\mathbf{N}_{K_n/K_{\chi_n}} =: \mathbf{N}_{g_0}$ and $\mathbf{J}_{K_i/K_{\chi_i}}$ with $\mathbf{J}_{K_n/K_{\chi_n}} =: \mathbf{J}_{g_0}$.

Thus $\mathbf{N}_{g_0}$ is surjective and $\mathbf{J}_{g_0}$ injective. One computes that $e_{\chi_0} = \frac{\nu_{g_0}}{\#g_0}\,\overline{e}_{\chi_0}$, where $\overline{e}_{\chi_0} := \frac{1}{\#\overline{G}_0}\sum_{\overline{\sigma}\in\overline{G}_0}\chi_0(\overline{\sigma}^{-1})\,\sigma \in \mathbb{Z}_p[G_0]$; but we have:

$$\nu_{g_0}(\mathscr{M}_{K_i}) = \mathbf{J}_{g_0}\circ\mathbf{N}_{g_0}(\mathscr{M}_{K_i}) \simeq \mathbf{N}_{g_0}(\mathscr{M}_{K_i}) \simeq \mathscr{M}_{K_{\chi_i}}; \tag{3.3}$$

whence $\mathscr{M}_{K_i}^{e_{\chi_0}} \simeq \mathscr{M}_{K_{\chi_i}}^{\overline{e}_{\chi_0}}$. To get $(\mathscr{M}^*_{K_i})^{e_{\chi_0}} \simeq \mathbf{N}_{g_0}(\mathscr{M}^*_{K_i})^{\overline{e}_{\chi_0}} \simeq (\mathscr{M}^*_{K_{\chi_i}})^{\overline{e}_{\chi_0}}$, it suffices to verify that, for all $i \geq 1$, $\mathbf{N}_{g_0}(\mathscr{M}^*_{K_i}) = \mathscr{M}^*_{K_{\chi_i}}$. The inclusion $\mathbf{N}_{g_0}(\mathscr{M}^*_{K_i}) \subseteq \mathscr{M}^*_{K_{\chi_i}}$ being obvious, let $x \in \mathscr{M}^*_{K_{\chi_i}}$; we have $x = \mathbf{N}_{g_0}(y)$, $y \in \mathscr{M}_{K_i}$, then $1 = \mathbf{N}_{K_{\chi_i}/K_{\chi_{i-1}}}\circ\mathbf{N}_{g_0}(y) = \mathbf{N}_{g_0}\circ\mathbf{N}_{K_i/K_{i-1}}(y)$. Let $z := \mathbf{N}_{K_i/K_{i-1}}(y)$, we have $\mathbf{N}_{g_0}(z) = 1$; applying $\mathbf{J}_{K_{i-1}/K_{\chi_{i-1}}}$, one gets $\nu_{g_0}(z) = 1$; but we have, as for (3.3), $\nu_{g_0}(\mathscr{M}_{K_{i-1}}) \simeq \mathscr{M}_{K_{\chi_{i-1}}}$; whence $z = 1$, $y \in \mathscr{M}^*_{K_i}$ and $x \in \mathbf{N}_{g_0}(\mathscr{M}^*_{K_i})$. □

From [Leo1954, Chap. I, § 1, 2; formula (6), p. 21] or our previous norm computations since $p \nmid \#G_0$, we have the relations (surjectivity of the norms and Lemma 3.16):

$$\begin{cases} \mathscr{M}_{K_{\chi_i}}^{\overline{e}_{\chi_0}} = \{x \in \mathscr{M}_{K_{\chi_i}},\ \mathbf{N}_{K_{\chi_i}/k}(x) = 1 \text{ for all } k,\quad K'_i \subseteq k \subsetneqq K_{\chi_i}\},\\ \mathscr{M}^{*\,\overline{e}_{\chi_0}}_{K_{\chi_i}} = \{x \in \mathscr{M}^*_{K_{\chi_i}},\ \mathbf{N}_{K_{\chi_i}/k}(x) = 1 \text{ for all } k,\quad K'_i \subseteq k \subsetneqq K_{\chi_i}\}. \end{cases}$$

From the norm definitions of $(\mathscr{M}^{\rm ar}_{K_{\chi_i}})_{\chi_0}$ and from:

$$\mathscr{M}^*_{K_{\chi_i}} := \{x \in \mathscr{M}_{K_{\chi_i}},\ \mathbf{N}_{K_{\chi_i}/K_{\chi_{i-1}}}(x) = 1\},$$

it follows that $\mathscr{M}^{*\,\overline{e}_{\chi_0}}_{K_{\chi_i}} = \mathscr{M}^{\rm ar}_{\chi_i}$, for all $i \geq 1$. In the finite case, this yields, using the above, the exact sequence (3.2) and $\mathscr{M}^*_{K_0} = \mathscr{M}_{K_0}$:

$$\begin{cases} \displaystyle\prod_{i=0}^{n} \#\mathscr{M}^{*\,\overline{e}_{\chi_0}}_{K_{\chi_i}} = \#\mathscr{M}^{*\,\overline{e}_{\chi_0}}_{K_0} \prod_{i=1}^{n} \frac{\#\mathscr{M}^{\overline{e}_{\chi_0}}_{K_i}}{\#\mathscr{M}^{\overline{e}_{\chi_0}}_{K_{i-1}}} = \#\mathscr{M}^{\overline{e}_{\chi_0}}_{K},\\ \displaystyle\prod_{\chi\in\mathscr{X}_K} \#\mathscr{M}^{\rm ar}_{\chi} = \prod_{\chi_0} \#\mathscr{M}^{\overline{e}_{\chi_0}}_{K} = \#\mathscr{M}_K. \end{cases} \tag{3.4}$$

Which ends the proof of the theorem and gives useful relations. □

The assumption on the surjectivity of the norms is fulfilled for class groups $\mathbf{H}$ (resp. p-class groups $\mathscr{H}$ and p-torsion groups $\mathscr{T}$), as soon as $K/\mathbb{Q}$ (resp. the maximal p-subextension of $K/\mathbb{Q}$) is cyclic, whence totally ramified, class field theory implying the claim.

4. Semi-simple decomposition of $\mathscr{A}_\chi := \mathbb{Z}_p[G_\chi]/(P_\chi(\sigma_\chi))$

Let $\mathscr{M}$ be a $\mathscr{G}$-family of $\mathbb{Z}_p[\mathscr{G}]$-modules provided with norms and transfer maps as usual. From $\psi \in \Psi$ given, there exist unique ψ_0, $\psi_p \in \Psi$ such that $\psi = \psi_0\,\psi_p$, ψ_0 of prime-to-p order and ψ_p of p-power order. We restrict the study to $K := K_\chi$ for the rational character χ above ψ, so that, from the previous § 3.5, G_K becomes $G_\chi = G_0 \oplus H$ of order $g_\chi = g_{\chi_0}\cdot p^n$.

We shall use what we call the "semi-simple idempotents" of $\mathbb{Z}_p[G_\chi]$:

$$e^{\varphi_0} := \frac{1}{g_{\chi_0}}\sum_{\sigma\in G_0}\varphi_0(\sigma^{-1})\,\sigma \in \mathbb{Z}_p[G_0], \tag{4.1}$$

where φ_0 is the p-adic character over ψ_0. In the semi-simple case $\psi_p = 1$, then $\varphi_0 = \varphi$.

4.1. Semi-simple decomposition of the $\mathscr{A}_\chi$-modules $\mathscr{M}_\chi^{\rm alg}$.

The algebra $\mathscr{A}_\chi$ occurs naturally because the $\mathscr{M}_\chi^{\rm alg}$ are, by definition, $\mathbb{Z}_p[G_\chi]$-modules annihilated by $P_\chi(\sigma_\chi)$, then modules over $\mathscr{A}_\chi$; this algebra is an integral domain if and only if p does not split in $\mathbb{Q}(\mu_{g_\chi})/\mathbb{Q}$. We shall see that it is semi-simple even when G_χ is not of prime-to-p order.

Theorem 4.1. *Let $\mathscr{M}$ be a $\mathscr{G}$-family of $\mathbb{Z}_p[\mathscr{G}]$-modules.*

(i) For all $\chi \in \mathscr{X}$ we get, by means of the irreducible p-adic characters $\varphi \in \Phi$, the decompositions $\mathscr{M}_\chi^{\rm alg} = \bigoplus_{\varphi|\chi} \mathscr{M}_\varphi^{\rm alg}$ (cf. Definition 3.5).

More generally, if $\mathscr{M}'_\chi$ is a sub-$\mathscr{A}_\chi$-module of $\mathscr{M}_\chi^{\rm alg}$, then $\mathscr{M}'_\chi = \oplus_{\varphi|\chi} \mathscr{M}'_\varphi$, where $\mathscr{M}'_\varphi = \{x' \in \mathscr{M}'_\chi,\ P_\varphi(\sigma_\chi)\cdot x' = 1\} \subseteq \mathscr{M}_\varphi^{\rm alg}$.

(ii) The sub-$\mathscr{A}_\chi$-modules $\mathscr{M}_\varphi^{\rm alg}$, $\varphi \mid \chi$, coincide with the $(\mathscr{M}_\chi^{\rm alg})^{e^{\varphi_0}}$'s, where e^{φ_0} is the semi-simple idempotent (4.1) associated to φ_0 above the component ψ_0 of prime-to-p order of $\psi \mid \varphi \mid \chi$.

(iii) These modules $\mathscr{M}_\varphi^{\rm alg}$, $\mathscr{M}'_\varphi$ are canonically $\mathbb{Z}_p[\mu_{g_\chi}]$-modules by means of the choice of $\psi \mid \varphi$ and the action $\sigma \in G_\chi \mapsto \psi(\sigma) \in \mu_{g_\chi}$.

Proof. One may suppose that $g_\chi \equiv 0 \pmod p$, otherwise we are in the semi-simple case and the proof is obvious [Or1975, Part II].

Let φ_1 and φ_2 be two distinct p-adic characters dividing χ (if $\chi = \varphi$ is p-adic irreducible, the result is trivial). Put $P_{\varphi_1} =: Q_1$, $P_{\varphi_2}(X) =: Q_2$ (cf. § 3.2.2 for the definition of P_φ). The following lemma is probably clear for abelian polynomials, but it is not general (e.g., for $p = 5$, take $P = x^4 - 2x^3 + 55x^2 - 54x + 379$, irreducible in $\mathbb{Z}[X]$, giving, in $\mathbb{Z}_5[X]$, $P \equiv (x^2 + 24x + 12)\cdot(x^2 + 24x + 17) \pmod{5^2}$ and the PARI relation $\mathsf{bezout}(\mathsf{x}^2 + 24 * \mathsf{x} + 12,\ \mathsf{x}^2 + 24 * \mathsf{x} + 17) = [-1/5, 1/5, 1]$).

Lemma 4.2. *There exist $U_1, U_2 \in \mathbb{Z}_p[X]$ such that $U_1 Q_1 + U_2 Q_2 = 1$.*

Proof. Since the distinct polynomials Q_1 and Q_2 are irreducible in $\mathbb{Q}_p[X]$, one may write a Bézout relation in $\mathbb{Z}_p[X]$ of the form:

$$U_1 Q_1 + U_2 Q_2 = p^k,\ \ k \geq 1,$$

choosing U_1 (resp. U_2) of degree less than the degree of Q_2 (resp. Q_1); moreover, since Q_1 and Q_2 are monic, one may suppose that (for instance) the coefficients of U_2 are not all divisible by p, otherwise, necessarily $U_1 \equiv 0 \pmod p$ and one can decrease k.

Let D_χ be the decomposition group of p in $\mathbb{Q}(\mu_{g_\chi})/\mathbb{Q}$ and let $\zeta \in \mu_{g_\chi}$ be a root of Q_1 (ζ is of order g_χ and the other roots are the ζ^a for Artin symbols $\sigma_a \in D_\chi$); we then have:

$$U_2(\zeta)\, Q_2(\zeta) = p^k \text{ in } \mathbb{Z}[\mu_{g_\chi}]; \tag{4.2}$$

but $Q_2(X) = \prod_{\sigma_a \in D_\chi} (X - \zeta_1^a)$, where $\zeta_1 =: \zeta^c$, for some $\sigma_c \notin D_\chi$; thus:

$$Q_2(\zeta) = \prod_{\sigma_a \in D_\chi} (\zeta - \zeta_1^a) = \prod_{\sigma_a \in D_\chi} (\zeta - \zeta^{ac}) = \prod_{\sigma_a \in D_\chi} \left[\zeta(1 - \zeta^{ac-1})\right].$$

Recall that $g_\chi = g_{\chi_0} p^n$, $n \geq 1$. Then $1 - \zeta^{ac-1}$ is non invertible in $\mathbb{Z}_p[\mu_{g_\chi}]$ if and only if $ac - 1 \equiv 0 \pmod{g_{\chi_0}}$, which implies $\sigma_a \sigma_c \in D_\chi$ since $\mathrm{Gal}(\mathbb{Q}(\mu_{g_\chi})/\mathbb{Q}(\mu_{g_{\chi_0}})) \subseteq D_\chi$ because of the total ramification of p in the p-extension, but $\sigma_a \in D_\chi$ implies $\sigma_c \in D_\chi$ (absurd). So $Q_2(\zeta)$ is a p-adic unit, whence, from (4.2), $U_2(\zeta) \equiv 0 \pmod{p^k}$, $k \geq 1$.

Denote by $\mathfrak{p}$ the maximal ideal of $\mathbb{Z}_p[\mu_{g_\chi}]$ and let $\overline{F}_p := \mathbb{Z}_p[\mu_{g_\chi}]/\mathfrak{p}$ be the residue field; for any $P \in \mathbb{Z}_p[X]$, let $\overline{P}$ be its image in $\mathbb{F}_p[X]$ and let $\overline{\zeta}$ be the image of ζ in $\overline{F}_p$. We have, in $\mathbb{F}_p[X]$:

$$\overline{Q}_1 = (\overline{Q}_0)^e, \tag{4.3}$$

where $e = p^{n-1}(p-1)$ (ramification index of p in $\mathbb{Q}(\mu_{g_\chi})/\mathbb{Q}$) and where $\overline{Q}_0$ is irreducible in $\mathbb{F}_p[X]$ (i.e., the irreducible polynomial of $\overline{\zeta}$, in fact that of the image of a generator of $\mu_{g_{\chi_0}}$).

With these notations, any polynomial $P \in \mathbb{Z}_p[X]$ such that $P(\zeta) \equiv 0 \pmod{\mathfrak{p}}$ is such that $\overline{P} \in \overline{Q}_0\, \mathbb{F}_p[X]$; in particular, it is the case of $\overline{U}_2$, so we will have, in $\mathbb{F}_p[X]$ (since $\overline{U}_2 \neq 0$ in $\mathbb{F}_p[X]$ by assumption), $\overline{U}_2 = \overline{A}\,(\overline{Q}_0)^\alpha$, $\alpha \geq 1$, $\overline{A} \neq 0$, $\overline{Q}_0 \nmid \overline{A}$. We may assume that A, $Q_0 \in \mathbb{Z}_p[X]$ have same degrees as their images in $\mathbb{F}_p[X]$. This yields:

$$U_2 = A\, Q_0^\alpha + pB,\ B \in \mathbb{Z}_p[X],$$

thus $U_2(\zeta) = A(\zeta)\,Q_0^\alpha(\zeta) + p\,B(\zeta) \equiv 0 \pmod{p^k}$, whence $A(\zeta)\,Q_0^\alpha(\zeta) \equiv 0 \pmod p$. But $A(\zeta)$ is a p-adic unit (since $\overline{Q_0} \nmid \overline{A}$), which gives:

$$Q_0^\alpha(\zeta) \equiv 0 \pmod p. \tag{4.4}$$

Let's show that $\alpha \geq e$; the unique case where, possibly, $p \mid g_\chi$ and $e = 1$ is the case $p = 2$, $n = 1$; this case trivially gives $\alpha \geq e$. Consider the g_{χ_0}th cyclotomic polynomial. Assuming $e > 1$, we have:

$$P_{g_{\chi_0}}(\zeta) = \prod_{a \in (\mathbb{Z}/g_{\chi_0}\mathbb{Z})^*} (\zeta - \zeta^{p^n a}) = \prod_a [\,\zeta(1 - \zeta^{p^n a - 1})\,];$$

$\zeta^{p^n a-1}$ is of p-power order if and only if $p^n a \equiv 1 \pmod{g_{\chi_0}}$; taking into account the domain of a, this defines a_0 such that $p^n a_0 \equiv 1 \pmod{g_{\chi_0}}$, whence $p^n a_0 \not\equiv 1 \pmod{p g_{\chi_0}}$ and $1 - \zeta^{p^n a_0 - 1} \in \mathfrak{p} \setminus \mathfrak{p}^2$, thus the fact that $P_{g_{\chi_0}}(\zeta) \in \mathfrak{p} \setminus \mathfrak{p}^2$; it follows, from $P_{g_{\chi_0}} = C\,Q_0^\beta + pD$, $\beta \geq 1$, $C, D \in \mathbb{Z}_p[X]$, $C(\zeta) \not\equiv 0 \pmod{\mathfrak{p}}$, that $P_{g_{\chi_0}}(\zeta) \equiv C(\zeta)\,Q_0^\beta(\zeta) \pmod{\mathfrak{p}^e}$, thus $Q_0^\beta(\zeta) \in \mathfrak{p} \setminus \mathfrak{p}^2$ since $e > 1$. This implies $\beta = 1$ and $Q_0(\zeta) \in \mathfrak{p} \setminus \mathfrak{p}^2$.

The congruence (4.4), written $Q_0^\alpha(\zeta) \equiv 0 \pmod{\mathfrak{p}^e}$, implies $\alpha \geq e$ and $U_2 = A'\,Q_0^e + p\,B$, where $A' := A\,Q_0^{\alpha - e}$; but we also have from (4.3):

$$Q_1 = Q_0^e + p\,T,\ T \in \mathbb{Z}_p[X],$$

hence $U_2 = A'\,(Q_1 - p\,T) + p\,B = A'\,Q_1 + p\,S$, $S \in \mathbb{Z}_p[X]$. Since $A \neq 0$ may be chosen monic by assumption, $A' \neq 0$ is monic, U_2 is of degree larger or equal to that of Q_1 (absurd). In conclusion, $\overline{U}_2 = 0$, contrary to the assumption $k \geq 1$ in (4.2). □

Give now some properties of the system of idempotents of $\mathscr{A}_\chi = \mathbb{Z}_p[G_\chi]/(P_\chi(\sigma_\chi))$.

Let $\{\varphi_1, \ldots, \varphi_{g_p}\}$ be the set of distinct p-adic characters dividing χ (thus, $g_p \mid \phi(g_{\chi_0})$ is the number of prime ideals dividing p in $\mathbb{Q}(\mu_{g_{\chi_0}})/\mathbb{Q}$, so that, only the case $g_p = 1$ is trivial for the Main Conjecture); from the property of co-maximality, given by Lemma 4.2, one may write:

$$\mathbb{Z}_p[X]/(P_\chi(X)) \simeq \prod_{u=1}^{g_p} \mathbb{Z}_p[X]/\big(Q_u(X)\big) \simeq (\mathbb{Z}_p[\mu_{g_\chi}])^{g_p}. \tag{4.5}$$

There exist elements $e_{\varphi_u}(X) \in \mathbb{Z}_p[X]$, whose images modulo $P_\chi(X)$ constitute an exact system of orthogonal idempotents of $\mathbb{Z}_p[X]/(P_\chi(X))$. Whence the system of orthogonal idempotents $e_{\varphi_u}(\sigma_\chi)$ of $\mathbb{Z}_p[G_\chi]$.

Since $(\mathscr{M}_\chi^{\rm alg})^{P_\chi(\sigma_\chi)} = 1$, we obtain (in the algebraic meaning):

$$\mathscr{M}_\chi^{\rm alg} = \bigoplus_{u=1}^{g_p} (\mathscr{M}_\chi^{\rm alg})^{e_{\varphi_u}(\sigma_\chi)}. \tag{4.6}$$

It remains to verify that:

$$(\mathscr{M}_\chi^{\rm alg})^{e_{\varphi_u}(\sigma_\chi)} = \mathscr{M}_{\varphi_u}^{\rm alg} = \{x \in \mathscr{M}_\chi^{\rm alg},\ P_{\varphi_u}(\sigma_\chi) \cdot x = 1\}.$$

If $x \in (\mathscr{M}_\chi^{\rm alg})^{e_{\varphi_u}(\sigma_\chi)}$, $x = y^{e_{\varphi_u}(\sigma_\chi)}$ with $y \in \mathscr{M}_\chi^{\rm alg}$; then $x^{P_{\varphi_u}(\sigma_\chi)} = y^{e_{\varphi_u}(\sigma_\chi) P_{\varphi_u}(\sigma_\chi)}$, but $e_{\varphi_u}(\sigma_\chi)\,P_{\varphi_u}(\sigma_\chi)) \equiv 0 \pmod{P_\chi(\sigma_\chi)}$, whence $y^{e_{\varphi_u}(\sigma_\chi)\,P_{\varphi_u}(\sigma_\chi)} = 1$ since $y \in \mathscr{M}_\chi^{\rm alg}$ and $x \in \mathscr{M}_{\varphi_u}^{\rm alg}$.

If $x \in \mathscr{M}_{\varphi_u}^{\rm alg}$, then $x^{P_{\varphi_u}(\sigma_\chi)} = 1$; writing $x = \prod_{j=1}^{g_p} x^{e_{\varphi_v}(\sigma_\chi)}$, we get $e_{\varphi_v}(\sigma_\chi) \equiv \delta_{u,v} \pmod{P_{\varphi_u}(\sigma_\chi)}$, thus $e_{\varphi_v}(\sigma_\chi) \equiv 0 \pmod{P_{\varphi_u}(\sigma_\chi)}$ for $v \neq u$ and $x^{e_{\varphi_v}(\sigma_\chi)} = 1$, for $v \neq u$. Whence $x = x^{e_{\varphi_u}(\sigma_\chi)}$.

In the algebra $\mathscr{A}_\chi = \mathbb{Z}_p[G_\chi]/(P_\chi(\sigma_\chi))$, we obtain two systems of idempotents, that is to say, the images in $\mathscr{A}_\chi$ of the $e_{\varphi_{u,0}} \in \mathbb{Z}_p[G_0]$, where $\varphi_{u,0}$ is above the component $\psi_{u,0}$, of prime-to-p order, of ψ_u, and that of the $e_{\varphi_u}(\sigma_\chi)$ corresponding to φ_u. Fixing the character $\varphi_u =: \varphi$ above $\psi =: \psi_0\,\psi_p$ and its non p-part φ_0 above ψ_0, we consider both:

$$e^{\varphi_0} := \frac{1}{g_{\chi_0}} \sum_{\sigma \in G_0} \varphi_0(\sigma^{-1})\,\sigma \tag{4.7}$$

and $e_{\varphi_0}(\sigma_\chi)$ defined as follows by means of polynomial relations in $\mathbb{Z}[X]$ deduced from (4.5):

$$\begin{cases} e_{\varphi_0}(\sigma_\chi) = \Lambda_\varphi(\sigma_\chi) \cdot \prod\limits_{\varphi' \neq \varphi} P_{\varphi'}(\sigma_\chi), \text{ such that:} \\ \qquad\qquad \Lambda_\varphi(X) \cdot \prod\limits_{\varphi' \neq \varphi} P_{\varphi'}(X) \equiv 1 \pmod{P_\varphi(X)}; \end{cases} \tag{4.8}$$

we denote $e_{\varphi_0}(\sigma_\chi)$ simply by e_{φ_0}, which is legitimate by Lemma 3.6.

To verify that $(\mathscr{M}_\chi^{\rm alg})^{e^{\varphi_0}} = (\mathscr{M}_\chi^{\rm alg})^{e_{\varphi_0}}$, it suffices to show that e^{φ_0} and e_{φ_0} correspond to the same simple factor of the algebra $\mathscr{A}_\chi$. For this, we remark that the homomorphism defined, for the fixed character φ, by $\sigma_\chi \mapsto \psi(\sigma_\chi)$, $\psi \mid \varphi$, induces a surjective homomorphism $\mathscr{A}_\chi \longrightarrow \mathbb{Z}_p[\mu_{g_\chi}]$ whose kernel is equal to $\bigoplus\limits_{\varphi \neq \varphi} \mathscr{A}_\chi\, e_{\varphi'_0}$.
Thus, to show that $\mathscr{A}_\chi e^{\varphi_0} = \mathscr{A}_\chi e_{\varphi_0}$, it suffices to show that $\psi(e^{\varphi_0}) \neq 0$; but, from (4.7), e^{φ_0} is a sum of the idempotents $e_{\psi'_0} = \frac{1}{g_{\chi_0}} \sum\limits_{\sigma \in G_0} \psi'_0(\sigma)\sigma^{-1}$ where $\psi'_0 \mid \varphi_0$. It follows, since $\psi = \psi_0\, \psi_p$, that $\psi(\sigma) = \psi_0(\sigma)$ and then:

$$\psi(e_{\psi'_0}) = \frac{1}{g_{\chi_0}} \sum_{\sigma \in G_0} \psi'_0(\sigma)\psi(\sigma)^{-1} = \frac{1}{g_{\chi_0}} \sum_{\sigma \in G_0} \psi'_0(\sigma)\psi_0(\sigma)^{-1},$$

which is zero for all ψ'_0 except $\psi'_0 = \psi_0$ where $\psi(e_{\psi_0}) = 1$. Whence $\psi(e^{\varphi_0}) \neq 0$. Let $\mathscr{M}_\chi^{\rm alg}$ as $\mathscr{A}_\chi$-module; on may write $\mathscr{M}_\chi^{\rm alg} = \bigoplus\limits_{\varphi \mid \chi} (\mathscr{M}_\chi^{\rm alg})^{e_{\varphi_0}}$ (from (4.6)) but $(\mathscr{M}_\chi^{\rm alg})^{e_{\varphi_0}}$ coincides with $(\mathscr{M}_\chi^{\rm alg})^{e^{\varphi_0}} = \mathscr{M}_\varphi^{\rm alg}$ (Definition (4.7)); then, due to the properties of the e_{φ_0} (defined by (4.8)):

$$(\mathscr{M}_\chi^{\rm alg})^{e_{\varphi_0}} = \{x \in \mathscr{M}_\chi^{\rm alg},\ P_\varphi(\sigma_\chi) \cdot x = 1\} = \mathscr{M}_\varphi^{\rm alg}.$$

Denote by e_{φ_0} any of these two semi-simple idempotents e^{φ_0} or e_{φ_0}.

If $\mathscr{M}'_\chi$ is a sub-$\mathscr{A}_\chi$-module of $\mathscr{M}_\chi^{\rm alg}$, then:

$$\mathscr{M}'_\varphi := (\mathscr{M}'_\chi)^{e_{\varphi_0}} = \{x' \in \mathscr{M}'_\chi,\, P_\varphi(\sigma_\chi) \cdot x' = 1\}.$$

Since $\mathscr{A}_\chi\, e_{\varphi_0} \simeq \mathbb{Z}_p[\mu_{g_\chi}]$, $\mathscr{M}_\varphi^{\rm alg}$ and $\mathscr{M}'_\varphi$ are canonically $\mathbb{Z}_p[\mu_{g_\chi}]$-modules.

This finishes the proof of Theorem 4.1. □

4.2. **Semi-simple decomposition of the $\mathscr{A}_\chi$-modules $\mathscr{M}_\chi^{\rm ar}$.** From Definition 3.11, $\mathscr{M}_\chi^{\rm ar} := \{x \in \mathscr{M}_{K_\chi},\ \mathbf{N}_{K_\chi/k}(x) = 1,\ \text{for all } k \subsetneqq K_\chi\}$. This invites to give the following arithmetic definition:

Definition 4.3. *Let $\mathscr{M}$ be an arithmetic family of $\mathbb{Z}_p[\mathscr{G}]$-modules. For any $\varphi \mid \chi$, $\chi \in \mathscr{X}$, $\varphi \in \Phi$, we define the arithmetic $\mathbb{Z}_p[\mu_{g_\chi}]$-module:*

$$\mathscr{M}_\varphi^{\rm ar} := \mathscr{M}_\varphi^{\rm alg} \cap \mathscr{M}_\chi^{\rm ar} = \{x \in \mathscr{M}_\varphi^{\rm alg},\ \mathbf{N}_{K_\chi/k}(x) = 1,\ \textit{for all } k \subsetneqq K_\chi\}.$$

Note that if $p \mid g_\chi$, then the norm conditions may be limited to $\mathbf{N}_{K_\chi/k_p}(x) = 1$, with $[K_\chi : k_p] = p$.

Remark 4.4. Of course $\mathscr{M}_\varphi^{\rm ar} = (\mathscr{M}_\chi^{\rm ar})^{e_{\varphi_0}}$, e_{φ_0} being defined by (4.7) or (4.8), and $\mathscr{M}_\varphi^{\rm ar}$ is a sub-$\mathbb{Z}_p[\mu_{g_\chi}]$-module of $\mathscr{M}_\varphi^{\rm alg}$. In the sequel, we shall privilege the notation $\mathscr{M}_\varphi^{\rm ar}$ rather than $(\mathscr{M}_\chi^{\rm ar})^{e_{\varphi_0}}$ since the definition of $\mathscr{M}_\varphi^{\rm ar} = \{x \in \mathscr{M}_\chi^{\rm ar},\, P_\varphi(\sigma_\chi) \cdot x = 1\}$ is more convenient in any viewpoint.

So, we have the arithmetic version of Theorem 4.1:

Theorem 4.5. *Let $\mathscr{M}$ be a $\mathscr{G}$-family of $\mathbb{Z}_p[\mathscr{G}]$-modules. Then we get, for all $\chi \in \mathscr{X}$, the decomposition $\mathscr{M}_\chi^{\rm ar} = \bigoplus\limits_{\varphi \mid \chi} \mathscr{M}_\varphi^{\rm ar}$.*

4.3. **Summary of the properties of the $\mathscr{G}$-families $\mathscr{M}^{\rm alg}$, $\mathscr{M}^{\rm ar}$.** From Theorems 3.15, 4.1, 4.5, Definitions 3.5, 3.11, 4.3. Let $\chi \in \mathscr{X}$ and let $\varphi \mid \chi$, $\varphi \in \Phi$. Let σ_χ be a generator of $G_\chi = {\rm Gal}(K_\chi/\mathbb{Q})$; put $g_\chi := \#G_\chi$.

(i) Recall that P_χ (resp. $P_\varphi \mid P_\chi$) is the g_χth global cyclotomic polynomial (resp. the local φ-cyclotomic polynomial); define:

$$\begin{cases} \mathscr{M}_\chi^{\rm alg} := \big\{x \in \mathscr{M}_{K_\chi},\ P_\chi(\sigma_\chi)\cdot x = 1\big\},\\ \mathscr{M}_\varphi^{\rm alg} := \big\{x \in \mathscr{M}_{K_\chi},\ P_\varphi(\sigma_\chi)\cdot x = 1\big\},\\ \mathscr{M}_\chi^{\rm ar} := \{x \in \mathscr{M}_\chi^{\rm alg},\ \mathbf{N}_{K_\chi/k}(x) = 1,\ \forall\, k \subsetneq K_\chi\},\\ \mathscr{M}_\varphi^{\rm ar} := \big\{x \in \mathscr{M}_\varphi^{\rm alg},\ \mathbf{N}_{K_\chi/k}(x) = 1,\ \forall\, k \subsetneq K_\chi\big\} = \mathscr{M}_\varphi^{\rm alg} \cap \mathscr{M}_\chi^{\rm ar}. \end{cases}$$

Then $\mathscr{M}_\chi^{\rm alg} = \bigoplus\limits_{\varphi\mid\chi} \mathscr{M}_\varphi^{\rm alg}$ and $\mathscr{M}_\chi^{\rm ar} = \bigoplus\limits_{\varphi\mid\chi} \mathscr{M}_\varphi^{\rm ar}$. All these components are $\mathbb{Z}_p[\mu_{g_\chi}]$-modules via $\sigma \in G_\chi \mapsto \psi(\sigma)$, for $\psi \mid \chi$, $\psi \mid \varphi$, respectively.

(ii) Assume that the maximal p-sub-extension of $K/\mathbb{Q}$ is cyclic and such that for all its sub-extensions k/k', the norms $\mathbf{N}_{k/k'}$ are surjective. Then, if $\mathscr{M}_K$ is finite, $\#\mathscr{M}_K = \prod_{\chi\in\mathscr{X}_K} \#\mathscr{M}_\chi^{\rm ar} = \prod_{\varphi\in\Phi_K} \#\mathscr{M}_\varphi^{\rm ar}$.

5. Application to relative class groups

5.1. **Arithmetic definition of relative class groups.** We will apply the previous results using first odd characters χ giving $\mathbf{H}_\chi^{\rm alg}$ and $\mathbf{H}_\chi^{\rm ar}$. The case of even characters requires some deepening of Leopoldt's results [Leo1954]; it will be considered in the next section.

For $K \in \mathscr{K}$, we denote by $\mathbf{H}_K$ the class group of K in the ordinary sense. If K is imaginary, with maximal real subfield K^+, we define the relative class group of K:

$$(\mathbf{H}_K^{\rm ar})^- := \{h \in \mathbf{H}_K,\ \mathbf{N}_{K/K^+}(h) = 1\} \tag{5.1}$$

(the notation $\mathbf{H}^{\rm ar}$ recalls that the definition of the minus part uses the arithmetic norm and not the algebraic one ν_{K/K^+}).

It is classical to put $\mathbf{H}_K^+ := \mathbf{H}_{K^+}$; since K/K^+ is ramified for the real infinite places of K^+, class field theory implies that $\mathbf{N}_{K/K^+}$ is surjective for class groups in the ordinary sense, giving the exact sequence:

$$1 \to (\mathbf{H}_K^{\rm ar})^- \longrightarrow \mathbf{H}_K \xrightarrow{\mathbf{N}_{K/K^+}} \mathbf{H}_{K^+} = \mathbf{H}_K^+ \to 1$$

and the formula:

$$\#\mathbf{H}_K = \#(\mathbf{H}_K^{\rm ar})^- \cdot \#\mathbf{H}_K^+. \tag{5.2}$$

We denote by $\mathscr{H}_K$ (resp. $(\mathscr{H}_K^{\rm ar})^-$ and $\mathscr{H}_K^+ := \mathscr{H}_{K^+}$), the p-Sylow subgroup of $\mathbf{H}_K$ (resp. $(\mathbf{H}_K^{\rm ar})^-$ and $\mathbf{H}_K^+$). For the $\mathbb{Z}_p[\mathscr{G}]$-modules $\mathscr{H}_K$, we introduce the $\mathscr{A}_\chi$-modules $\mathscr{H}_\chi^{\rm alg}$ and $\mathscr{H}_\chi^{\rm ar}$ for $\chi \in \mathscr{X}$, then their φ-components (Definitions 3.5, 3.11, 4.3) which are $\mathbb{Z}_p[\mu_{g_\chi}]$-modules.

5.2. **Proof of the equality $\mathbf{H}_\chi^{\rm ar} = \mathbf{H}_\chi^{\rm alg}$, for all $\chi \in \mathscr{X}^-$.** To prove this equality and then the equalities $\mathscr{H}_\varphi^{\rm ar} = \mathscr{H}_\varphi^{\rm alg}$, $\varphi \mid \chi$, it is sufficient to consider, for any $p \geq 2$, the p-Sylow subgroups $\mathscr{H}_{K_\chi}$ and to prove the equality of the χ-components $\mathscr{H}_\chi^{\rm alg}$, $\mathscr{H}_\chi^{\rm ar}$.

Lemma 5.1. *Assume that $\mathscr{H}_\chi^{\rm ar} \subsetneq \mathscr{H}_\chi^{\rm alg}$. Then there exists a unique sub-extension $K_{\chi'}$ of K_χ, such that $[K_\chi : K_{\chi'}] = p$ (i.e., if $\psi \mid \chi$ then χ' is above $\psi' = \psi^p$), and a class $h \in \mathscr{H}_\chi^{\rm alg}$ such that $h' := \mathbf{N}_{K_\chi/K_{\chi'}}(h)$ fulfills the following properties:*

(i) For all prime $\ell \neq p$ dividing g_χ, $\nu_{K_{\chi'}/k'_\ell}(h') = 1$, where k'_ℓ is the unique sub-extension of $K_{\chi'}$ such that $[K_{\chi'} : k'_\ell] = \ell$;

(ii) $\mathbf{J}_{K_\chi/K_{\chi'}}(h') = 1$;

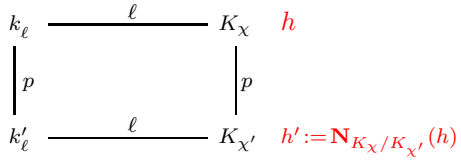
(iii) h' is of order p in $\mathscr{H}_{K_{\chi'}}$.

Proof. Indeed, if $[K_\chi : \mathbb{Q}]$ is prime to p, we are in the semi-simple case and $\mathscr{H}_\chi^{\rm alg} = \mathscr{H}_\chi^{\rm ar}$. So we assume that $p \mid [K_\chi : \mathbb{Q}]$, whence the existence and unicity of $K_{\chi'}$.

Let $h \in \mathscr{H}_\chi^{\rm alg}$, $h \notin \mathscr{H}_\chi^{\rm ar}$, and let $h' := \mathbf{N}_{K_\chi/K_{\chi'}}(h)$. Let $\ell \mid g_\chi$, $\ell \neq p$.

(i) We have the following diagram where k_ℓ is the unique sub-extension of K_χ such that $[K_\chi : k_\ell] = \ell$ and then $k'_\ell = k_\ell \cap K_{\chi'}$:

5.2.1. *Schema II.*



We have $\nu_{K_\chi/k_\ell}(h) = 1$ since $h \in \mathscr{H}_\chi^{\rm alg}$; applying $\mathbf{N}_{K_\chi/K_{\chi'}}$, we get $\nu_{K_{\chi'}/k'_\ell}(h') = 1$.

(ii) We have $\mathbf{J}_{K_\chi/K_{\chi'}}(h') = \mathbf{J}_{K_\chi/K_{\chi'}} \circ \mathbf{N}_{K_\chi/K_{\chi'}}(h) = \nu_{K_\chi/K_{\chi'}}(h) = 1$ since $h \in \mathscr{H}_\chi^{\rm alg}$.

(iii) Since the class h' capitulates in K_χ, its order is 1 or p. Suppose that $h' = 1$; for $\ell \neq p$, the maps $\mathbf{J}_{K_\chi/k_\ell}$ and $\mathbf{J}_{K_{\chi'}/k'_\ell}$ are injective, so $\mathbf{N}_{K_\chi/k_\ell}(h) = 1$, for all $\ell \neq p$ dividing g_χ; since moreover $h' = \mathbf{N}_{K_\chi/K_{\chi'}}(h) = 1$, this yields by definition $h \in \mathscr{H}_\chi^{\rm ar}$ (absurd). □

Lemma 5.2. *Let K/k be a cyclic extension of degree p and Galois group $G =: \langle\sigma\rangle$. Let $\mathbf{E}_k$ and $\mathbf{E}_K$ be the unit groups of k and K, respectively. Consider the transfer map $\mathbf{J}_{K/k} : \mathscr{H}_k \to \mathscr{H}_K$; then $\mathrm{Ker}(\mathbf{J}_{K/k})$ is isomorphic to a subgroup of $\mathrm{H}^1(G, \mathbf{E}_K) \simeq \mathbf{E}_K^*/\mathbf{E}_K^{1-\sigma}$ (where $\mathbf{E}_K^* = \mathrm{Ker}(\nu_{K/k})$). The group $\mathbf{E}_K^*/\mathbf{E}_K^{1-\sigma}$ is of exponent 1 or p.*

Proof. Let $\mathbf{Z}_k$ and $\mathbf{Z}_K$ be the rings of integers of k and K, respectively; let $\mathscr{A}_k(\mathfrak{a}) \in \mathscr{H}_k$, with $\mathfrak{a}\mathbf{Z}_K =: (\alpha)\mathbf{Z}_K$, $\alpha \in K^\times$. We then have $\alpha^{1-\sigma} =: \varepsilon \in \mathbf{E}_K^*$. The map, which associates with $\mathscr{A}_k(\mathfrak{a}) \in \mathrm{Ker}(\mathbf{J}_{K/k})$ the class of ε modulo $\mathbf{E}_K^{1-\sigma}$, is obviously injective.

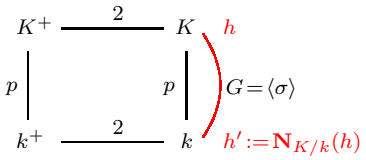
If $\varepsilon \in \mathbf{E}_K^*$, then $1 = \varepsilon^{1+\sigma+\cdots+\sigma^{p-1}} = \varepsilon^{p+(\sigma-1)\Omega}$, $\Omega \in \mathbb{Z}[G]$; whence $\varepsilon^p \in \mathbf{E}_K^{1-\sigma}$. □

5.2.2. *Study of the case $p \neq 2$.* We are in the context of Lemma 5.1. Put $K := K_\chi$ and $k := K_{\chi'}$; then K/k is of degree p and the class $h' = \mathbf{N}_{K/k}(h) \in \mathscr{H}_k$ is of order p and capitulates in K.

Assume that K is imaginary (i.e., χ is odd, thus $h \in (\mathscr{H}_K^{\rm ar})^-$); since K/k is of degree $p \neq 2$, k is also imaginary and $h' \in (\mathscr{H}_k^{\rm ar})^-$.

We introduce the maximal real subfields, giving the diagram:

5.2.3. *Schema III.*



Lemma 5.3. *Let μ_K^* be the p-torsion sub-group of $\mathbf{E}_K^*$, that is to say the set of p-roots of unity ζ of K such that $\mathbf{N}_{K/k}(\zeta) = 1$. Then the image of $(\mathscr{H}_k^{\rm ar})^- \cap \mathrm{Ker}(\mathbf{J}_{K/k})$, by the map $\mathrm{Ker}(\mathbf{J}_{K/k}) \to \mathbf{E}_K^*/\mathbf{E}_K^{1-\sigma}$ of Lemma 5.2, is contained in the image of μ_K^* modulo $\mathbf{E}_K^{1-\sigma}$.*

Proof. Let q be the map $\mathbf{E}_K^* \to \mathbf{E}_K^*/\mathbf{E}_K^{1-\sigma}$. Denote by $x \mapsto \overline{x}$ the complex conjugation in K. If $h' \in (\mathscr{H}_k^{\rm ar})^- \cap \mathrm{Ker}(\mathbf{J}_{K/k})$, then $\mathbf{N}_{k/k^+}(h') = 1$ and $\nu_{k/k^+}(h') = h'\overline{h'} = 1$; if $h' = \mathscr{A}_k(\mathfrak{a})$ we then have $\mathfrak{a}\overline{\mathfrak{a}} = a\mathbf{Z}_k$, $a \in k^\times$, and $\mathfrak{a}\mathbf{Z}_K\overline{\mathfrak{a}}\mathbf{Z}_K = a\mathbf{Z}_K$, with $\mathfrak{a}\mathbf{Z}_K = (\alpha)\mathbf{Z}_K$ and $\overline{\mathfrak{a}}\mathbf{Z}_K = (\overline{\alpha})\mathbf{Z}_K$, $\alpha \in K^\times$ (since $\mathfrak{a}$ and $\overline{\mathfrak{a}}$ become principal in K), which yields relations of the form $\alpha^{1-\sigma} = \varepsilon$, $\overline{\alpha}^{1-\sigma} = \overline{\varepsilon}$, $\varepsilon, \overline{\varepsilon} \in \mathbf{E}_K^*$. From the relation $\mathfrak{a}\overline{\mathfrak{a}} = a\mathbf{Z}_k$, one obtains, in K, $\alpha\overline{\alpha} = \eta a$, $\eta \in \mathbf{E}_K$, then $\alpha^{1-\sigma}\overline{\alpha}^{1-\sigma} = \eta^{1-\sigma}$, giving $\varepsilon\overline{\varepsilon} = \eta^{1-\sigma}$.

From [Has1952, Satz 24], $\varepsilon = \varepsilon^+\zeta$, $\varepsilon^+ \in \mathbf{E}_{K^+}$, $\zeta \in \mu_K$. So $q(\varepsilon\overline{\varepsilon}) = q(\varepsilon^{+2}) = 1$. Since p is odd and $\mathbf{E}_K^*/\mathbf{E}_K^{1-\sigma}$ of exponent divisor of p, $\varepsilon^+ \in \mathbf{E}_K^{1-\sigma}$; since $\varepsilon \in \mathbf{E}_K^*$, we have $\zeta \in \mathbf{E}_K^*$, whence $q(\varepsilon) = q(\zeta) \in q(\mu_K^*) = \mu_K^*/(\mathbf{E}_K^{1-\sigma} \cap \mu_K^*)$. □

Lemma 5.4. *The group $q(\mu_K^*)$ (of order 1 or p) is of order p if and only if $\mu_K^* = \langle\zeta_1\rangle$ and $\mathbf{E}_K^{1-\sigma} \cap \langle\zeta_1\rangle = 1$, where ζ_1 is of order p.*

Proof. A direction being obvious, assume that $q(\mu_K^*) = \mu_K^*/(\mathbf{E}_K^{1-\sigma} \cap \mu_K^*)$ is of order p and let ζ be a generator of μ_K^* (necessarily, $\zeta \neq 1$). If $\zeta \in k$, then $\mathbf{N}_{K/k}(\zeta) = \zeta^p$, so $\zeta^p = 1$ and $\zeta = \zeta_1 \in k$.

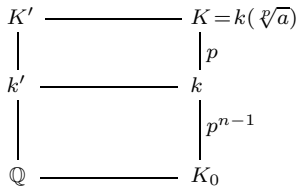
If $\zeta \notin k$, $K = k(\zeta)$; it follows that $\zeta_1 \in k$ and that $\zeta^p \in k$ (since $[K : k] = [\mathbb{Q}(\zeta) : k \cap \mathbb{Q}(\zeta)] = p$), thus K/k is a Kummer extension of the form $K = k(\sqrt[p]{\zeta_r})$, ζ_r of order p^r, $r \geq 1$, $\zeta = \zeta_{r+1}$, and $\zeta^{1-\sigma} = \zeta_1$, giving $\mathbf{N}_{K/k}(\zeta) = \zeta^p = 1$, hence $\zeta = \zeta_1 \in k$ (absurd). So we have $\zeta = \zeta_1 \in k$ and $\mathbf{E}_K^{1-\sigma} \cap \mu_K^* \subseteq \langle\zeta_1\rangle$. Thus, $q(\mu_K^*)$ being of order p, necessarily $\mathbf{E}_K^{1-\sigma} \cap \mu_K^* = 1$. □

Lemma 5.5. *If $(\mathscr{H}_k^{\rm ar})^- \cap \mathrm{Ker}(\mathbf{J}_{K/k}) \neq 1$, this group is of order p and K/k is a Kummer extension of the form $K = k(\sqrt[p]{a})$, $a \in k^\times$, $a\mathbf{Z}_k = \mathfrak{a}^p$, the ideal $\mathfrak{a}$ of k being non-principal (such a Kummer extension is said to be "of class type").*

Proof. If $h' \in (\mathscr{H}_k^{\rm ar})^- \cap \mathrm{Ker}(\mathbf{J}_{K/k})$, $h' := \mathscr{C}\ell_k(\mathfrak{a}) \neq 1$, this means that $\mathfrak{a}\mathbf{Z}_K = \alpha\mathbf{Z}_K$, $\alpha \in K^\times$; so $\alpha^{1-\sigma} = \varepsilon$, $\varepsilon \in \mathbf{E}_K^*$; from Lemma 5.4, $q(\varepsilon) = q(\zeta_1)^\lambda$, hence $\varepsilon = \zeta_1^\lambda \eta^{1-\sigma}$, $\eta \in \mathbf{E}_K$, whence $\alpha^{1-\sigma} = \zeta_1^\lambda \eta^{1-\sigma}$ and in the equality $\mathfrak{a}\mathbf{Z}_K = \alpha\mathbf{Z}_K$ one may suppose α chosen modulo $\mathbf{E}_K$ such that $\alpha^{1-\sigma} = \zeta_1^\lambda$; moreover we have $\lambda \not\equiv 0 \pmod p$, otherwise α should be in k and $\mathfrak{a}$ should be principal. Thus $\alpha^{1-\sigma} = \zeta_1'$ of order p and $\alpha^p = a \in k^\times$, whence $K = k(\alpha)$ is the Kummer extension $k(\sqrt[p]{a})$; we have $a\mathbf{Z}_K = \mathfrak{a}^p\mathbf{Z}_K$, hence $a\mathbf{Z}_k = \mathfrak{a}^p$, since extension of ideals is injective. □

We shall show now that the context of Lemma 5.5 is not possible for a cyclic extension $K/\mathbb{Q}$, which will apply to $K_\chi/\mathbb{Q}$:

5.2.4. *Schema IV.*



Since $K = k(\sqrt[p]{a})$, with $a\mathbf{Z}_k = \mathfrak{a}^p$, only the prime ideals dividing p can ramify in K/k. Consider the above decomposition of the extension $K/\mathbb{Q}$ for $p \neq 2$, with K/K_0 and $K'/\mathbb{Q}$ cyclic of p-power degree p^n, K/K' and $K_0/\mathbb{Q}$ of prime-to-p degree, and let ℓ be a prime number totally ramified in $K'/\mathbb{Q}$ (such a prime does exist since $G_{K'} \simeq \mathbb{Z}/p^n\mathbb{Z}$); this prime is then totally ramified in K/K_0, hence in K/k, which implies $\ell = p$ and p is the unique ramified prime in $K'/\mathbb{Q}$.

This identifies the extension $K'/\mathbb{Q}$. Its conductor is p^{n+1}, $n \geq 1$, since $p \neq 2$; thus K' is the unique sub-extension of degree p^n of $\mathbb{Q}(\mu_{p^{n+1}})$ and k' is the unique sub-extension of degree p^{n-1} of $\mathbb{Q}(\mu_{p^n})$ (in other words, K' is contained in the cyclotomic $\mathbb{Z}_p$-extension). Since $\zeta_1 \in k$, one has $\mu_{p^n} \subset k$, $\mu_{p^{n+1}} \subset K$ and $\mu_{p^{n+1}} \not\subset k$, so $K = k(\zeta) = k(\sqrt[p]{\zeta^p})$, with ζ of order p^{n+1}.

It suffices to apply Kummer theory which shows that $k(\sqrt[p]{a}) = k(\sqrt[p]{\zeta^p})$ implies $a = \zeta^{\lambda p} b^p$, with $p \nmid \lambda$ and $b \in k^\times$; so $a\mathbf{Z}_k = b^p\mathbf{Z}_k = \mathfrak{a}^p$, whence $\mathfrak{a} = b\mathbf{Z}_k$ principal (absurd).

So in the case $p \neq 2$, for $K/\mathbb{Q}$ imaginary cyclic and K/k cyclic of degree p, we have the relation $(\mathscr{H}_k^{\rm ar})^- \cap \mathrm{Ker}(\mathbf{J}_{K/k}) = 1$ (injectivity of $\mathbf{J}_{K/k}$ on the relative p-class group).

5.2.5. *Case $p = 2$.* The extension $K/\mathbb{Q}$ is still imaginary cyclic, k is necessarily equal to K^+ and σ is the complex conjugation s_∞.

From [Has1952, Satz 24] the "index of units" Q_K^- is trivial in the cyclic case; thus for all $\varepsilon \in \mathbf{E}_K^*$, $\varepsilon = \varepsilon^+\zeta$, $\varepsilon^+ \in k$, ζ root of unity of 2-power order; then $\mathbf{N}_{K/k}(\varepsilon) = 1$ yields $\varepsilon^{+2} = 1$, thus $\varepsilon^+ = \pm 1$ and $\varepsilon = \zeta' = \pm\zeta$; since $K/\mathbb{Q}$ is cyclic (whence $\mathbb{Q}(\zeta)/\mathbb{Q}$ cyclic), we shall have $\varepsilon \in \{1, -1, i, -i\}$. Recall that $h' = \mathbf{N}_{K/k}(h) \in \mathrm{Ker}(\mathbf{J}_{K/k})$, $h' = \mathscr{C}\ell_k(\mathfrak{a}) \neq 1$, with $\mathfrak{a}\mathbf{Z}_K = \alpha\mathbf{Z}_K$ and $\alpha^{1-\sigma} = \varepsilon \in \mathbf{E}_K^*$. One may assume $\varepsilon \in \{-1, i, -i\}$ ($\varepsilon \neq 1$ since $\alpha \notin k^\times$):

(i) Case $\varepsilon=-1$. Then $\alpha^{1-\sigma}=-1$, $\alpha^2=:a\in k^\times$, $\alpha\notin k^\times$, and we get the Kummer extension $K=k(\sqrt{a})$ with $a\mathbf{Z}_k=\mathfrak{a}^2$, $\mathfrak{a}$ non-principal (Kummer extension of class type).

(ii) Case $\varepsilon=\pm i$. Then $\alpha^{1-\sigma}=\pm i$ with $-1=(\pm i)^{1-\sigma}$; one may assume $\alpha^{1-\sigma}=i$. This yields $\alpha^2 i^{-1}\in k^\times$. Put $\alpha^2=ic$, $c\in k^\times$; it follows $\mathfrak{a}^2\mathbf{Z}_K=\alpha^2\mathbf{Z}_K=c\mathbf{Z}_K$, hence $\mathfrak{a}^2=c\mathbf{Z}_k$.

Let τ be a generator of G_K; one has $\alpha^{2\tau}=i^\tau c^\tau=-ic^\tau=-c^{\tau-1}\alpha^2$, hence $\alpha^{2\tau}=\alpha^2 d$, $d:=-c^{\tau-1}\in k^\times$; we obtain $(\alpha\mathbf{Z}_K)^{2\tau}=(\alpha\mathbf{Z}_K)^2 d\mathbf{Z}_K$, thus $\mathfrak{a}^{2\tau}\mathbf{Z}_K=\mathfrak{a}^2\mathbf{Z}_K d\mathbf{Z}_K$ giving $\mathfrak{a}^{2\tau}=\mathfrak{a}^2 d\mathbf{Z}_k$.

If $d\in k^{\times 2}$, $d=e^2$, $e\in k^\times$, and $\mathfrak{a}^\tau\sim\mathfrak{a}$ saying that h' is an invariant class in $k/\mathbb{Q}$.

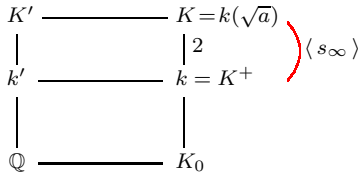
If $d\notin k^{\times 2}$, the relation $\alpha^{2\tau}=\alpha^2 d$ shows that $d=(\alpha^{\tau-1})^2\in K^{\times 2}$; from Kummer theory, since $K=k(\sqrt{d})=k(i)$, one obtains $d=-\delta^2$, $\delta\in k^\times$, and $\mathfrak{a}^{2\tau}=\mathfrak{a}^2\delta^2\mathbf{Z}_K$, still giving $\mathfrak{a}^\tau=\mathfrak{a}\cdot\delta\mathbf{Z}_k$ and an invariant class in $k/\mathbb{Q}$.

But K is the direct compositum over $\mathbb{Q}$ of $k=K^+$ and $\mathbb{Q}(i)$ and must be cyclic, so $[k:\mathbb{Q}]$ is necessarily odd and an invariant class in $k/\mathbb{Q}$ is of odd order giving the principality of $\mathfrak{a}$ in k (absurd).

So, only case (i) is a priori possible.

Consider the following diagram, with K/K_0 and $K'/\mathbb{Q}$ cyclic of 2-power order, then K/K' and $K_0/\mathbb{Q}$ of odd degree, where we recall that $a\mathbf{Z}_k=\mathfrak{a}^2$ with $\mathfrak{a}$ non-principal and $\mathfrak{a}\mathbf{Z}_K=\alpha\mathbf{Z}_K$, $\alpha\in K^\times$. Similarly, since K/k is only ramified at 2, then K/K_0 and $K'/\mathbb{Q}$ are totally ramified at 2, the conductor of K' is a power of 2, say 2^{r+1}, $r\geq 1$ (K' is an imaginary cyclic subfield of $\mathbb{Q}(\mu_{2^{r+1}})$):

5.2.6. *Schema V.*



The Kummer extension K'/k' is 2-ramified of the form $K'=k'(\sqrt{a'})$, $a'\in k'^\times$. So we have $a'\mathbf{Z}_{k'}=\mathfrak{a}'^2$ or $a'\mathbf{Z}_{k'}=\mathfrak{a}'^2\mathfrak{p}'$, where $\mathfrak{p}'\mid 2$ in k'. But all the subfields of $\mathbb{Q}(\mu_{2^\infty})$ have a trivial 2-class group; thus, one may suppose that a' is, up to $k'^{\times 2}$, a unit or an uniformizing parameter of k'. Then $K=k(\sqrt{a'})$ is not of class type (absurd); so $h'=1$. Whence:

Proposition 5.6. *For any imaginary cyclic extension $K/\mathbb{Q}$ and any relative extension K/k of prime degree, $(\mathscr{H}_k^{\rm ar})^-\cap\mathrm{Ker}(\mathbf{J}_{K/k})=1$ if $p\neq 2$ (the relative classes of k do not capitulate in K), then $\mathrm{Ker}(\mathbf{J}_{K/K^+})=1$ if $p=2$ (the real 2-classes of $k=K^+$ do not capitulate in K).*

Using the order formula (5.2), we get:

Corollary 5.7. *We get $\mathbf{J}_{K/K^+}(\mathscr{H}_{K^+})\simeq\mathscr{H}_K^+:=\mathscr{H}_{K^+}=\mathbf{N}_{K/K^+}(\mathscr{H}_K)$ and the direct sum $\mathscr{H}_K=(\mathscr{H}_K^{\rm ar})^-\oplus\mathbf{J}_{K/K^+}(\mathscr{H}_{K^+})$.*

We have obtained the following result about relative class groups:

Theorem 5.8. *Let K be an imaginary cyclic field of maximal real subfield K^+. Let p be any prime number and set $\mathscr{H}=\mathbf{H}\otimes\mathbb{Z}_p$. Define:*

$$\text{(5.3)}\qquad\begin{cases}(\mathscr{H}_K^{\rm ar})^-:=\{h\in\mathscr{H}_K,\ \mathbf{N}_{K/K^+}(h)=1\}\\(\mathscr{H}_K^{\rm alg})^-:=\{h\in\mathscr{H}_K,\ \nu_{K/K^+}(h)=1\}.\end{cases}$$

Then $\mathscr{H}_K^{\rm ar}=\mathscr{H}_K^{\rm alg}$, $\mathscr{H}_\varphi^{\rm ar}=\mathscr{H}_\varphi^{\rm alg}$ for all $\varphi\in\Phi_K^-$, $(\mathbf{H}_K^{\rm ar})^-=(\mathbf{H}_K^{\rm alg})^-$.

Proof. For all subfield k of K with $[K:k]=p$, $\mathbf{J}_{K/k}$ is injective on $(\mathscr{H}_k^{\rm ar})^-$ if $p\neq 2$ and $\mathbf{J}_{K/K^+}$ is injective on $\mathscr{H}_{K^+}$ for $p=2$; so $\nu_{K/k}=\mathbf{J}_{K/k}\circ\mathbf{N}_{K/k}$ yields $(\mathscr{H}_K^{\rm ar})^-=(\mathscr{H}_K^{\rm alg})^-$ from Definition 3.11, then $(\mathbf{H}_K^{\rm ar})^-=(\mathbf{H}_K^{\rm alg})^-$ by globalization. □

We shall write simply $\mathbf{H}_K^-$ for the two notions "alg" and "ar" in the cyclic case. Using Theorem 4.1 we may write, for all $\chi \in \mathscr{X}^-$, $\#\mathscr{H}_\chi^{\rm alg} = \#\mathscr{H}_\chi^{\rm ar} = \prod\limits_{\varphi\mid\chi} \#\mathscr{H}_\varphi^{\rm ar}$.

Corollary 5.9. *Let $K/\mathbb{Q}$ be an imaginary cyclic extension. Then:*

$$\#\mathbf{H}_K^+ = \textstyle\prod_{\chi\in\mathscr{X}_K^+} \#\mathbf{H}_\chi^{\rm ar} \quad \& \quad \#\mathbf{H}_K^- = \prod_{\chi\in\mathscr{X}_K^-} \#\mathbf{H}_\chi^{\rm ar}.$$

Proof. To apply Theorem 3.15, we shall prove that all the arithmetic norms are surjective in any sub-extension k/k' of $K/\mathbb{Q}$; we do this for each p-class group; so the proof of the surjectivity is only necessary in the sub-extensions k/k' of p-power degree; then we use the fact that this property holds as soon as k/k' is totally ramified at some place.

Consider K as direct compositum $K'K_0$, over $\mathbb{Q}$, where K/K_0 and $K'/\mathbb{Q}$ are cyclic of p-power degree and where K/K' and $K_0/\mathbb{Q}$ are of prime-to-p degree. Let ℓ be a prime number totally ramified in $K'/\mathbb{Q}$; thus ℓ is totally ramified in any sub-extension k/k' of $K'/\mathbb{Q}$ (and in K/K_0). So Theorem 3.15 implies $\#\mathbf{H}_K = \prod\limits_{\chi\in\mathscr{X}_K} \#\mathbf{H}_\chi^{\rm ar}$.

From (5.2), $\#\mathbf{H}_K = \#\mathbf{H}_K^- \cdot \#\mathbf{H}_K^+$ and we can also apply Theorem 3.15 to the maximal real subfield K^+ of K, giving $\#\mathbf{H}_K^+ = \prod\limits_{\chi\in\mathscr{X}_K^+} \#\mathbf{H}_\chi^{\rm ar}$, whence the formulas taking into account the relation $\mathbf{H}_\chi^{\rm ar} = \mathbf{H}_\chi^{\rm alg}$ for odd characters (Theorem 5.8). □

5.3. **Computation of $\#\mathbf{H}_\chi^{\rm ar}$ for $\chi \in \mathscr{X}^-$.** For an arbitrary imaginary extension $K/\mathbb{Q}$, we have (e.g., from [Has1952, p. 12] or [Was1997, Theorem 4.17]) the formula:

$$\#\mathbf{H}_K^- = Q_K^- w_K^- \prod_{\psi\in\Psi_K^-} \big(-\tfrac{1}{2}\,\mathbf{B}_1(\psi^{-1})\big), \quad \mathbf{B}_1(\psi^{-1}) := \frac{1}{f_\chi} \sum_{a\in[1,f_\chi[} \psi^{-1}(\sigma_a)\, a,$$

where w_K^- is the order of the group of roots of unity of K and Q_K^- the index of units; from [Has1952, Satz 24], $Q_K^- = 1$ when $K/\mathbb{Q}$ is cyclic. Recall that $\mathbf{H}_\chi^{\rm ar} := \{h \in \mathbf{H}_{K_\chi},\ \mathbf{N}_{K_\chi/k}(x) = 1, \text{ for all } k \subsetneq K_\chi\}$; then:

Theorem 5.10. *Let $\chi \in \mathscr{X}^-$, let g_χ be the order of χ, f_χ its conductor; then $\#\mathbf{H}_\chi^{\rm ar} = \#\mathbf{H}_\chi^{\rm alg} = 2^{\alpha_\chi} \cdot w_\chi \cdot \prod\limits_{\psi\mid\chi} \big(-\frac{1}{2}\,\mathbf{B}_1(\psi^{-1})\big)$, where $\alpha_\chi = 1$ (resp. $\alpha_\chi = 0$) if g_χ is a 2-power (resp. if not) and:*

(i) $w_\chi = 1$ if K_χ is not an imaginary cyclotomic field;
(ii) $w_\chi = p$ if $K_\chi = \mathbb{Q}(\mu_{p^n})$, $p \neq 2$ prime, $n \geq 1$;
(iii) $w_\chi = 2$ if $K_\chi = \mathbb{Q}(\mu_4)$ for $p = 2$.

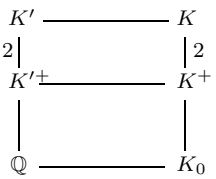
Proof. We use [Or1975[b], Proposition III (g)] or [Leo1954, Chap. I, § 1 (4)] recalled in Theorem 2.1; it is sufficient to prove that for any imaginary cyclic extension $K/\mathbb{Q}$, $\#\mathbf{H}_K^- = \prod\limits_{\chi\in\mathscr{X}_K^-} \big(2^{\alpha_\chi} \cdot w_\chi \cdot \prod\limits_{\psi\mid\chi} \big(-\frac{1}{2}\,\mathbf{B}_1(\psi^{-1})\big)\big)$, the expected equality will come from Theorem 5.8 and the relation:

$$\#\mathbf{H}_K^- = \prod_{\chi\in\mathscr{X}_K^-} \#\mathbf{H}_\chi^{\rm ar}.$$

So, it remains to prove that $\prod_{\chi\in\mathscr{X}_K^-} \big(2^{\alpha_\chi} \cdot w_\chi\big) = w_K^-$.

Consider the following diagram, where K/K_0 and $K'/\mathbb{Q}$ are cyclic of 2-power degree and where K/K' and $K_0/\mathbb{Q}$ are of odd degree. :

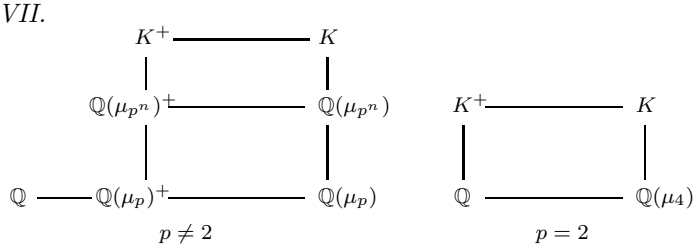
5.3.1. *Schema VI.*



As K^+ and K'^+ are real, $\alpha_\chi = 0$, except when g_χ is a 2-power, hence for the unique χ_0 defining K' for which $\alpha_\chi = 1$; whence $\prod_{\chi\in\mathscr{X}_K^-} 2^{\alpha_\chi} = 2$.

If K does not contain any cyclotomic field (different from $\mathbb{Q}$), then $w_K^- = 2$, moreover, all the w_χ are trivial and the required equality holds in that case. So, let $\mathbb{Q}(\mu_{p^n})$, $n \geq 1$, be the largest cyclotomic field contained in K; this yields two possibilities:

5.3.2. *Schema VII.*



If $p \neq 2$, $\prod_{\chi \in \mathscr{X}_K^-} w_\chi = p^n$ (due to the n odd characters defined by the $\mathbb{Q}(\mu_{p^i})$, $1 \leq i \leq n$) and, for $p = 2$, this gives $\prod_{\chi \in \mathscr{X}_K^-} w_\chi = 2$; whence the result (cf. [Has1952, Chap. III, § 33, Theorem 34 and others]). □

Remark 5.11. We have $\#\mathbf{H}_K^- = \dfrac{Q_K^- w_K^-}{2^{n_K^-}} \displaystyle\prod_{\chi \in \mathscr{X}_K^-} \#\mathbf{H}_\chi^{\rm alg}$, for any imaginary extension K, where n_K^- is the number of imaginary cyclic sub-extensions of K of 2-power degree and w_K^- is the 2-part of w_K (resp. $\frac{1}{2}w_K$) if $\mathbb{Q}(\mu_4) \not\subset K$ (resp. $\mathbb{Q}(\mu_4) \subset K$). See [Gra1976, Remarque II 2, p. 32].

5.4. **Annihilation theorem for $\mathscr{H}_K^-$.** Before significant improvements by means of Stickelberger's elements (leading to the construction of p-adic measures, to index formulas and annihilators of various invariants), Iwasawa [Iwa1962] proves the following formula for the cyclotomic fields $K = \mathbb{Q}(\mu_{p^n})$, $p \neq 2$, $n \geq 1$, of Galois group G_K:

$$\#\mathbf{H}_K^- = \big(\mathbb{Z}[G_K]^- : \mathbf{B}_K \mathbb{Z}[G_K] \cap \mathbb{Z}[G_K]^-\big),$$

where $\mathbb{Z}[G_K]^- := \{\Omega \in \mathbb{Z}[G_K],\ (1 + s_\infty) \cdot \Omega = 0\}$, s_∞ being the complex conjugation, and $\mathbf{B}_K := \dfrac{1}{p^n} \displaystyle\sum_{a \in [1,\, p^n[,\, p \nmid a} a\, \sigma_a^{-1}$ where $\sigma_a \in G_K$ denotes the corresponding Artin automorphism.

This formula does not generalize for arbitrary imaginary extension $K/\mathbb{Q}$ (see the counterexample given in [Gra1976, p. 33]). Many contributions have appeared (e.g., [Leo1962, Gil1975, Coa1977, Gra1978, All2013, All2017]; for more precise formulas, see [Sin1980], [Was1997, § 6.2, § 15.1], among many other). Nevertheless, we gave in [Gra1976] another definition in the spirit of the φ-objects which succeeded to give a correct formula (we shall make the same remark for the index formulas given via cyclotomic units in the real case).

5.4.1. *General definition of Stickelberger's elements.* Let $K \in \mathscr{K} \setminus \{\mathbb{Q}\}$. Let $f_K =: f > 1$ be the conductor of K and let $\mathbb{Q}(\mu_f)$ be the corresponding cyclotomic field. Define the more suitable writing of the Stickelberger element defined in [Gra1978, Chap.IV, § 1] or [Gra1978^b, Chap.I, § 1], from the study of partial zêta-functions in [Coa1977, §§ 2.1, 3.2], and that leads to a new normalized definition of Gauss sums (in the summation, integers a are prime to f and Artin symbols are taken over $\mathbb{Q}$):

$$\mathbf{B}_{\mathbb{Q}(\mu_f)} := -\sum_{a=1}^{f} \Big(\frac{a}{f} - \frac{1}{2}\Big) \cdot \Big(\frac{\mathbb{Q}(\mu_f)}{a}\Big)^{-1}.$$

Note that the part $\sum_{a=1}^{f} \big(\frac{\mathbb{Q}(\mu_f)}{a}\big)^{-1}$ is the algebraic norm $\nu_{\mathbb{Q}(\mu_f)/\mathbb{Q}}$ which does not modify the image of $\mathbf{B}_{\mathbb{Q}(\mu_f)}$ by ψ, for $\psi \in \Psi$, $\psi \neq 1$.

We shall use two arithmetic $\mathscr{G}$-families: the $\mathscr{G}$-family $\mathbf{M}$, for which $\mathbf{M}_K = \mathbb{Z}[G_K]$ and the $\mathscr{G}$-family $\mathbf{S}$ defined by:

$$\begin{cases} \mathbf{S}_K := \mathbf{B}_K \mathbb{Z}[G_K] \cap \mathbb{Z}[G_K], \text{ where} \\ \mathbf{B}_K := \mathbf{N}_{\mathbb{Q}(\mu_f)/K}(\mathbf{B}_{\mathbb{Q}(\mu_f)}) = -\displaystyle\sum_{a=1}^{f} \Big(\frac{a}{f} - \frac{1}{2}\Big)\Big(\frac{K}{a}\Big)^{-1}. \end{cases} \tag{5.4}$$

Lemma 5.12. *For any c, prime to $2f$, let $\mathbf{B}_K^c := \left(1 - c\left(\frac{K}{c}\right)^{-1}\right)\cdot \mathbf{B}_K$; then $\mathbf{B}_K^c \in \mathbb{Z}[G_K]$.*

Proof. We have:

$$\mathbf{B}_K^c = \frac{-1}{f}\sum_a \left[a\left(\frac{K}{a}\right)^{-1} - ac\left(\frac{K}{a}\right)^{-1}\left(\frac{K}{c}\right)^{-1}\right] + \frac{1-c}{2}\sum_a \left(\frac{K}{a}\right)^{-1}.$$

Let $a'_c \in [1, f]$ be the unique integer such that $a'_c \cdot c \equiv a \pmod f$; put:

$$a'_c \cdot c = a + \lambda_a(c) f, \ \ \lambda_a(c) \in \mathbb{Z};$$

using the bijection $a \mapsto a'_c$ in the summation of the second term in between $[\ \ ]$ and the relation $\left(\frac{K}{a'_c}\right)\left(\frac{K}{c}\right) = \left(\frac{K}{a}\right)$, this yields:

$$\begin{aligned}\mathbf{B}_K^c &= \frac{-1}{f}\Big[\sum_a a\left(\frac{K}{a}\right)^{-1} - \sum_a a'_c\cdot c\left(\frac{K}{a'_c}\right)^{-1}\left(\frac{K}{c}\right)^{-1}\Big] + \frac{1-c}{2}\sum_a\left(\frac{K}{a}\right)^{-1}\\ &= \frac{-1}{f}\sum_a\Big[a - a'_c\cdot c\Big]\left(\frac{K}{a}\right)^{-1} + \frac{1-c}{2}\sum_a\left(\frac{K}{a}\right)^{-1}\\ &= \sum_a\Big[\lambda_a(c) + \frac{1-c}{2}\Big]\left(\frac{K}{a}\right)^{-1} \in \mathbb{Z}[G_K].\end{aligned}$$

We have $\lambda_{f-a}(c) + \frac{1-c}{2} = -\big(\lambda_a(c) + \frac{1-c}{2}\big)$, which proves that:

$$\mathbf{B}_K^c = \mathbf{B}_K'^c \cdot (1 - s_\infty),\ \ \mathbf{B}_K'^c \in \mathbb{Z}[G_K], \tag{5.5}$$

useful in the case $p = 2$ and giving $\mathbf{N}_{K/K^+}(\mathbf{B}_K^c) = 0$. □

Definition 5.13. *Let K be an imaginary abelian field. Put:*

$$\mathfrak{A}_K := \{\Omega \in \mathbb{Z}[G_K],\ \Omega\,\mathbf{B}_K \in \mathbb{Z}[G_K]\}$$

($\mathfrak{A}_K$ is an ideal of $\mathbb{Z}[G_K]$ and $\mathbf{S}_K := \mathbf{B}_K\cdot\mathfrak{A}_K$ (cf. (5.4)). Denote by $\Lambda_K \in \mathfrak{A}_K$ the least rational integer such that $\Lambda_K\mathbf{B}_K \in \mathbb{Z}[G_K]$ (thus $\Lambda_K \mid 2f$, where f is the conductor of K).
For $K = K_\chi$, $\chi \in \mathscr{X}^-$, we put $\mathfrak{A}_{K_\chi} =: \mathfrak{A}_\chi$ and $\Lambda_{K_\chi} =: \Lambda_\chi$.

Since we will only use images by $\psi \in \Psi^-$ of elements of $\mathbb{Q}[G_K]$, we can neglect, by abuse, the term $\sum_{a=1}^f \frac{1}{2}\left(\frac{K}{a}\right)^{-1}$ in some reasonings and computations, using $\frac{1}{f}\sum_{a=1}^f a\left(\frac{K}{a}\right)^{-1}$ instead of $\mathbf{B}_K$.

Note that for any odd c prime to f, $\left(1 - c\left(\frac{K}{c}\right)^{-1}\right)\cdot\sum\limits_{a=1}^f \frac{1}{2}\left(\frac{K}{a}\right)^{-1}$ is in $\mathbb{Z}[G_K]$ and that such considerations only concerns the case $p = 2$ when f is an odd prime power with $[\mathbb{Q}(\mu_f) : K]$ odd (see Example 5.20 with $K = \mathbb{Q}(\mu_{47})$).

Lemma 5.14. *Let α_σ be the coefficient of $\sigma \in G_K$ in the writing of $\sum\limits_{a=1}^f a\left(\frac{K}{a}\right)^{-1}$ on the canonical basis G_K of $\mathbb{Z}[G_K]$; in particular, we have $\alpha_1 = \sum\limits_{a,\ \sigma_{a\mid K}=1} a$. Then $\alpha_\sigma \equiv c\,\alpha_1 \pmod f$, where c is a representative modulo f such that $\sigma_c = \sigma^{-1}$. Thus, we have $\Lambda_K = \dfrac{f}{\gcd(f, \alpha_1)}$.*

Proof. The first claim is obvious and Λ_K is the least integer Λ such that $\dfrac{\Lambda\cdot\alpha_1}{f} \in \mathbb{Z}$, since $\Lambda\sum\limits_{a=1}^f \frac{a}{f}\left(\frac{K}{a}\right)^{-1} \in \mathbb{Z}[G_K]$ if and only if $\dfrac{\Lambda\cdot\alpha_\sigma}{f} \in \mathbb{Z}$ for all $\sigma \in G_K$, thus, for instance, for $\sigma = 1$. □

Proposition 5.15. *(i) The ideal $\mathfrak{A}_K$ of $\mathbb{Z}[G_K]$ is a free $\mathbb{Z}$-module; a $\mathbb{Z}$-basis is given by the set $\left\{\cdots, \left(\frac{K}{a}\right) - a, \cdots;\ \Lambda_K\right\}$, for the representatives a of $(\mathbb{Z}/f\mathbb{Z})^\times \setminus \{1\}$.*

(ii) If $K/\mathbb{Q}$ is cyclic, then $\mathfrak{A}_K$ is the ideal of $\mathbb{Z}[G_K]$ generated by $\left(\frac{K}{c}\right) - c$ and Λ_K, where $\left(\frac{K}{c}\right)$ is any generator of G_K.

Proof. See [Gra1976, p. 35–36]. □

5.4.2. *Study of the algebraic $\mathscr{G}$-families* $\mathbf{M}_K := \mathbb{Z}[G_K]$, $\mathbf{S}_K := \mathbf{B}_K\mathfrak{A}_K$. We then have (where $\mathbf{M}_\chi$ and $\mathbf{S}_\chi$ are ideals of $\mathbf{M}_{K_\chi}$):

$$\begin{cases} \mathbf{M}_{K_\chi} = \mathbb{Z}[G_\chi], & \mathbf{S}_{K_\chi} = \mathbf{B}_{K_\chi}\,\mathfrak{A}_\chi, \\ \mathbf{M}_\chi \ = \{\Omega \in \mathbb{Z}[G_\chi],\ P_\chi(\sigma_\chi)\cdot\Omega = 0\}, & \mathbf{S}_\chi \ = \mathbf{B}_{K_\chi}\mathfrak{A}_\chi \cap \mathbf{M}_\chi \end{cases}$$

Lemma 5.16. *We have* $\mathbf{M}_\chi = \prod_{\ell\mid g_\chi}(1-\sigma_\chi^{g_\chi/\ell})\,\mathbb{Z}[G_\chi]$, $\mathfrak{a}_\chi := \psi(\mathbf{M}_\chi) = \prod_{\ell\mid g_\chi}\big(1-\psi(\sigma_\chi)^{g_\chi/\ell}\big)$*; then* $\mathbf{S}_\chi$ *gives rise to an ideal* $\mathfrak{b}_\chi$ *multiple of* $\mathfrak{a}_\chi$.

Proof. See [Gra1976, Lemmes II.8 and II.9, pp. 37/39]. □

The computation of $\mathfrak{b}_\chi$ needs to recall the norm action on Stickelberger's elements; because of the similarity of the result for the norm action on cyclotomic numbers, we recall, without proof, the following well-known formulas (see, e.g., [Gra2018^b, Section 4]):

Lemma 5.17. *Let* $f > 1$ *and* $m \mid f$, $m > 1$, *be any modulus; let* $\mathbb{Q}(\mu_f)$, $\mathbb{Q}(\mu_m) \subseteq \mathbb{Q}(\mu_f)$, *be the corresponding cyclotomic fields. Let:*

$$\mathbf{B}_{\mathbb{Q}(\mu_f)} := -\sum_{a=1}^{f}\left(\frac{a}{f}-\frac{1}{2}\right)\cdot\left(\frac{\mathbb{Q}(\mu_f)}{a}\right)^{-1}, \qquad \mathbf{C}_{\mathbb{Q}(\mu_f)} := 1-\zeta_f.$$

We have, where $\mathbf{N}_{\mathbb{Q}(\mu_f)/\mathbb{Q}(\mu_m)}\colon \mathbb{Q}[G_{\mathbb{Q}(\mu_f)}] \longrightarrow \mathbb{Q}[G_{\mathbb{Q}(\mu_m)}]$*:*

$$\mathbf{N}_{\mathbb{Q}(\mu_f)/\mathbb{Q}(\mu_m)}(\mathbf{B}_{\mathbb{Q}(\mu_f)}) = \Omega\cdot\mathbf{B}_{\mathbb{Q}(\mu_m)},\ \mathbf{N}_{\mathbb{Q}(\mu_f)/\mathbb{Q}(\mu_m)}(\mathbf{C}_{\mathbb{Q}(\mu_f)}) = \mathbf{C}^{\Omega}_{\mathbb{Q}(\mu_m)},$$

where $\Omega := \prod_{p\mid f,\ p\nmid m}\big(1-\big(\frac{\mathbb{Q}(\mu_m)}{p}\big)^{-1}\big)$.

We can conclude by the following [Gra1976, Théorèmes II.5, II.6]:

Theorem 5.18. *Let* $\chi \in \mathscr{X}^-$ *and* $\psi \mid \chi$. *The* $\mathbb{Z}[\mu_{g_\chi}]$*-module* $\mathbf{H}^{\mathrm{alg}}_\chi = \mathbf{H}^{\mathrm{ar}}_\chi$ *is annihilated by the ideal* $\mathbf{B}_1(\psi^{-1})\cdot(\psi(\sigma_a)-a, \Lambda_\chi)$ *of* $\mathbb{Z}[\mu_{g_\chi}]$, *where* $\sigma_a := \big(\frac{K}{a}\big)$ *is any generator of* G_K *(Lemma 5.14, Proposition 5.15).*

The ideal $(\psi(\sigma_a)-a, \Lambda_\chi)$ *is the unit ideal except if* $K_\chi \neq \mathbb{Q}(\mu_4)$ *is an extension of* $\mathbb{Q}(\mu_p)$ *of* p*-power degree and if* $\Lambda_\chi \equiv 0 \pmod p$, *in which case, this ideal is a prime ideal* $\mathfrak{p}_\chi \mid p$ *in* $\mathbb{Q}(\mu_{g_\chi})$. *If* $K_\chi = \mathbb{Q}(\mu_4)$, *this ideal is the ideal* (4).

Theorem 5.19. *Let* $\varphi \in \Phi^-$ *and let* $\psi \mid \varphi$. *Then the* $\mathbb{Z}_p[\mu_{g_\chi}]$*-module* $\mathscr{H}^{\mathrm{alg}}_\varphi = \mathscr{H}^{\mathrm{ar}}_\varphi$ *is annihilated by the ideal* $\mathbf{B}_1(\psi^{-1})\cdot(\psi(\sigma_a)-a, \Lambda_\chi)$ *of* $\mathbb{Z}_p[\mu_{g_\chi}]$, *where* σ_a *is any generator of* G_K.

The ideal $(\psi(\sigma_a)-a, \Lambda_\chi)$ *of* $\mathbb{Z}_p[\mu_{g_\chi}]$ *is the unit ideal except if* $K_\chi \neq \mathbb{Q}(\mu_4)$ *is extension of* $\mathbb{Q}(\mu_p)$ *of* p*-power degree, if* $\Lambda_\chi \equiv 0 \pmod p$ *and if* $\lambda = 1$ *in the writing* $\psi = \omega^\lambda\cdot\psi_p$ *(where* ω *is the Teichmüller character and* ψ_p *of* p*-power order), in which case, this ideal is the prime ideal of* $\mathbb{Z}_p[\mu_{g_\chi}]$.

If $K_\chi = \mathbb{Q}(\mu_4)$, *this ideal is the ideal* (4).

Example 5.20. Let $K := K_\chi$ be the field $\mathbb{Q}(\mu_{47})$, of degree $g_\chi = 46$. From Theorem 5.10, we have $\#\mathbf{H}_\chi = 2^{\alpha_\chi}\cdot w_\chi\cdot\prod_{\psi\mid\chi}\big(-\frac{1}{2}\mathbf{B}_1(\psi^{-1})\big)$ with in that case $\alpha_\chi = 0$ and $w_\chi = 47$ and where by definition:

$$-\frac{1}{2}\mathbf{B}_1(\psi^{-1}) = -\frac{1}{2}\sum_{a=1}^{46}\big(\frac{a}{47}-\frac{1}{2}\big)\psi^{-1}(\sigma_a) = -\frac{1}{2}\sum_{a=1}^{46}\frac{a}{47}\,\psi^{-1}(\sigma_a).$$

Let's compute $\#\mathbf{H}_\chi = 47\cdot\mathbf{N}_{\mathbb{Q}(\mu_{46})/\mathbb{Q}}\big(-\frac{1}{2}\sum_{a=1}^{46}\frac{a}{47}\,\psi^{-1}(\sigma_a)\big)$:

```
{P=polcyclo(46);g=lift(znprimroot(47));A=0;for(n=0,45,
a=lift(Mod(g,47)^n);A=A+x^n*(1/47*a-1/2));B=Mod(-1/2*A,P);
print(47*norm(B))}
139
```

Note that $-\frac{47}{2}\mathbf{B}_1(\psi^{-1})$ is, writing $x = \zeta_{46}$, the PARI integer:

```
4*x^21+25*x^20+9*x^19+26*x^18-19*x^17+11*x^16-22*x^15
+x^14-24*x^13+10*x^12+6*x^11+16*x^10-21*x^9+20*x^8
+8*x^7+7*x^6-4*x^5+14*x^4-12*x^3+3*x^2+14*x+27
```

Whence $\#\mathbf{H}_\chi = 139$ and $\mathbf{H}_\chi \simeq \mathbb{Z}[\mu_{46}]/\mathfrak{p}_{139}$. Since $\Lambda_\chi = 47$, the ideal $\mathfrak{A}_K$ is $\big(\sigma_a - a, 47\big)$, with for instance $a = 5$ (Lemma 5.14), and $\mathfrak{A}_K \cdot \frac{1}{2}\mathbf{B}_K$ annihilates $\mathbf{H}_\chi$; since the image of $\mathfrak{A}_K \cdot \frac{1}{2}\mathbf{B}_K$ is the ideal $\big(\frac{1}{2}\mathbf{B}_1(\psi^{-1})\big) = \mathfrak{p}_{139}$, *the annihilator* of $\mathbf{H}_\chi$ is $\mathfrak{p}_{139}$. But this ideal is not principal in $\mathbb{Q}(\mu_{46})$ (from [Gra1979^b]):

```
{L=bnfinit(polcyclo(46));F=idealfactor(L,139);
print(bnfisprincipal(L,component(F,1)[1])[1])}
[2]~
```

showing that its class is the square of the PARI generating class. More precisely, the class group of $\mathbb{Q}(\mu_{46}) = \mathbb{Q}(\mu_{23})$ is equal to 3; then any $\mathfrak{q}_{47} \mid 47$ or $\mathfrak{q}_{139} \mid 139$ generates this class group.

In [Gra1978, Chap. IV, § 2; Théorème IV1], [Gra1979^b, Théorèmes 1, 2, 3], we have given improvements of the annihilation for 2-class groups but it is difficult to say if the case $p = 2$ is optimal or not. By way of example, we cite the following under the above context:

Theorem 5.21. *Let $\chi \in \mathscr{X}^-$ and $\psi \mid \varphi \mid \chi$ with $\psi = \psi_0\,\psi_2$, $\psi_0 \neq 1$ of even order, ψ_2 of 2-power order. Put $K := K_\chi$. The $\mathbb{Z}_2[\mu_{g_\chi}]$-module $\mathscr{H}_\varphi/\mathbf{J}_{K/K^+}(\mathscr{H}^+_{\varphi'})$ is annihilated by $\big(\frac{1}{2}\mathbf{B}_1(\psi^{-1})\big)$, where $\mathscr{H}^+_\varphi := \{h \in \mathscr{H}_{K^+},\ P_{\varphi'}(\sigma_\chi)\cdot x = 1\}$ with $\varphi' \in \Phi^+$ is above $\psi' := \psi_0\,\psi_2^2$.*

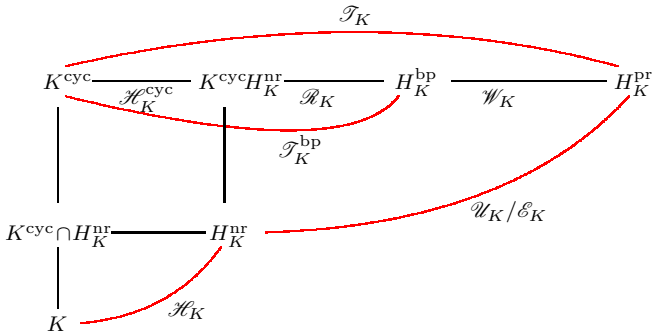
6. Application to torsion groups of abelian p-ramification

Let K be a totally real number field and let $\mathscr{T}_K$ be the torsion group of the Galois group of the maximal p-ramified abelian pro-p-extension $H_K^{\rm pr}$ of K.

Under Leopoldt's conjecture, we have $\mathscr{T}_K = {\rm Gal}(H_K^{\rm pr}/K^{\rm cyc})$, where $K^{\rm cyc}$ is the cyclotomic $\mathbb{Z}_p$-extension of K.

Let $H_K^{\rm nr}$ be the p-Hilbert class field and let $H_K^{\rm bp}$ be the Bertrandias–Payan field; the $\mathbb{Z}_p$-module $\mathscr{T}_K^{\rm bp} := {\rm Gal}(H_K^{\rm bp}/K^{\rm cyc})$ is the Bertrandias–Payan module ([Ng1986, Sec. 4], [Jau1990, Sec. 2 (b)]).

6.0.1. *Schema VIII.*



Let K_v be the completion of K at the place v. The above diagram is related to the exact sequence (where K_v is the completion of K at the place v):

$$1 \to \mathscr{W}_K \longrightarrow {\rm tor}_{\mathbb{Z}_p}\big(\mathscr{U}_K/\mathscr{E}_K\big) \xrightarrow{\log_p} \mathscr{R}_K := {\rm tor}_{\mathbb{Z}_p}\big(\log_p\big(\mathscr{U}_K\big)/\log_p(\mathscr{E}_K)\big) \longrightarrow 0, \tag{6.1}$$

where $\mathscr{W}_K := \big(\oplus_{v\mid p}\,\mu_p(K_v)\big)/\mu_p(K)$, $\mathscr{U}_K$ denotes the group of local units at p and $\mathscr{E}_K = \mathbf{E}_K \otimes \mathbb{Z}_p$ is identified with its diagonal image in $\mathscr{U}_K$ (see [Gra2005, § III.2, (c), Fig. 2.2; Lemma III.4.2.4] and [Gra2018]):

In all the sequel, we assume that K is abelian real.

6.1. **Computation of $\#\mathscr{T}_K$ for $\chi \in \mathscr{X}^+$.** The order of the $\mathbb{Z}_p[G_K]$-module $\mathscr{T}_K$ is well known and given, analytically, by the residue at $s=1$ of the p-adic ζ-function of K, whence by the values at $s=1$ of p-adic $\mathbf{L}$-functions of the non-trivial characters of K (after [Coa1977, Appendix]); see for instance [Gra2019, § 3.4, formula (3.8)] for analytic context.

In conclusion we can write:

$$\#\mathscr{T}_K = \#\mathscr{H}_K^{\rm cyc} \cdot \#\mathscr{R}_K \cdot \#\mathscr{W}_K \sim [K \cap \mathbb{Q}^{\rm cyc} : \mathbb{Q}] \cdot \prod_{\psi \neq 1} \tfrac{1}{2}\, \mathbf{L}_p(1,\psi). \tag{6.2}$$

Since the arithmetic family of these $\mathbb{Z}_p[\mathscr{G}]$-modules $\mathscr{T}_K$, for real fields K, follows the most favorable properties (surjectivity of the norms, injectivity of the transfer maps in relative sub-extensions), we can state, in a similar context as for Theorems 5.8:

Theorem 6.1. *For all $\chi \in \mathscr{X}^+$ (resp. $\varphi \in \Phi^+$, $\varphi \mid \chi$), we have:*

$$\begin{cases} \mathscr{T}_\chi^{\rm ar} = \mathscr{T}_\chi^{\rm alg} = \{x \in \mathscr{T}_{K_\chi},\ P_\chi(\sigma_\chi)\cdot x = 1\} \\ \qquad\qquad\quad = \{x \in \mathscr{T}_{K_\chi},\ \mathbf{N}_{K_\chi/k}(x) = 1,\ \text{for all } k \subsetneq K_\chi\}, \\ \mathscr{T}_\varphi^{\rm ar} = \mathscr{T}_\varphi^{\rm alg} = \{x \in \mathscr{T}_{K_\chi},\ P_\varphi(\sigma_\chi)\cdot x = 1\}. \end{cases}$$

Moreover, if $K/\mathbb{Q}$ is real cyclic, $\#\mathscr{T}_K = \prod\limits_{\chi \in \mathscr{X}_K} \#\mathscr{T}_\chi^{\rm ar} = \prod\limits_{\varphi \in \Phi_K} \#\mathscr{T}_\varphi^{\rm ar}$.

We denote simply $\mathscr{T}_\chi$ (resp. $\mathscr{T}_\varphi$) these components in the algebraic and arithmetic senses. In the analytic point of view, we have the analogue of Theorems 5.10 and 7.5 (see some p-adic formulas about $\mathbf{L}_p$-functions, from classical papers [KuLe1964, AmFr1972, Gra1978^b] and a broad presentation in [Was1997, Theorems 5.18, 5.24]):

Theorem 6.2. *Let $\chi \in \mathscr{X}^+ \setminus \{1\}$. Then $\#\mathscr{T}_\chi = w_\chi^{\rm cyc} \cdot \prod\limits_{\psi \mid \chi} \frac{1}{2}\, \mathbf{L}_p(1,\psi)$, where $w_\chi^{\rm cyc}$ is as follows, from analytic formula (6.2):*

(i) $w_\chi^{\rm cyc} = 1$ if K_χ is not a subfield of $\mathbb{Q}^{\rm cyc}$;

(ii) $w_\chi^{\rm cyc} = p$ if K_χ is a subfield of $\mathbb{Q}^{\rm cyc}$.

6.2. **Annihilation theorem for $\mathscr{T}_K$.** An annihilator of $\mathscr{T}_K$ is given by the following statement [Gra2018^b, Theorem 5.5] which does not assume any hypothesis on K real and p and gives again the known results (e.g., [Gra1979], [Or1981]):

Theorem 6.3. *Let K be a real abelian field of conductor f_K. Let f_n be the conductor of $L_n := K\mathbb{Q}(\mu_{qp^n})$, n large enough, where $q=p$ or 4 as usual. Let $c \in \mathbb{Z}$ be prime to $2pf_K$. For all $a \in [1, f_n]$, prime to f_n, let $a'_c \in [1, f_n]$ be the unique integer such that $a'_c \cdot c \equiv a \pmod{f_n}$ and put $a'_c \cdot c - a = \lambda_a^n(c)\, f_n$, $\lambda_a^n(c) \in \mathbb{Z}$. Then consider:*

$$\mathbf{A}_{K,n}(c) := \sum_{a=1}^{f_n} \lambda_a^n(c)\, a^{-1}\Big(\frac{K}{a}\Big) =: \mathbf{A}'_{K,n}(c)\cdot(1+s_\infty) \in \mathbb{Z}_p[G_K],$$

where s_∞ is the complex conjugation and $\mathbf{A}'_{K,n}(c) = \sum\limits_{a=1}^{f_n/2} \lambda_a^n(c)\, a^{-1}\Big(\frac{K}{a}\Big)$.

Let $\mathbf{A}_K(c) := \lim\limits_{n\to\infty} \Big[\sum\limits_{a=1}^{f_n} \lambda_a^n(c)\, a^{-1}\Big(\frac{K}{a}\Big)\Big] =: \mathbf{A}'_K(c)\cdot(1+s_\infty)$; then:

(i) For $p \neq 2$, $\mathbf{A}'_K(c)$ annihilates the $\mathbb{Z}_p[G_K]$-module $\mathscr{T}_K$.

(ii) For $p=2$, the annihilation is true for $2\cdot\mathbf{A}_K(c)$ and $4\cdot\mathbf{A}'_K(c)$.

Example 6.4. Consider, for $p=7$, the cubic field K of conductor $f=2557$ defined by the polynomial $P = x^3 + x^2 - 852\,x + 9281$; then (using the main program of § 9.2), one obtains:

$$\mathscr{H}_K \simeq \mathbb{Z}[j]/(1-2j)\mathbb{Z}[j] \quad\text{and}\quad \mathscr{E}_K/\mathscr{F}_K \simeq \mathbb{Z}[j]/(1-2j)\mathbb{Z}[j],$$

where $(1-2j)\mathbb{Z}[j]$ is a prime $\mathfrak{p}$ dividing 7, and $\mathscr{T}_K \simeq \mathbb{Z}/7^2\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z}$.

The following program (only valid for prime conductors f) computes the annihilator $\mathbf{A}_K(c)$ of $\mathscr{T}_K$; it defines the classes $\sigma^k \cdot \mathrm{Gal}(\mathbb{Q}(\mu_{fp^N})/K)$, $k=0,1,2$, of Artin symbols, giving $\mathbf{A}_K(c) = A_0 + A_1\sigma + A_2\sigma^2$, then $\beta := A_0 - A_2 + (A_1 - A_2)\, j$, yielding $(\beta) = \mathfrak{p}_1^u \cdot \mathfrak{p}_2^v$ in $\mathbb{Z}[j]$ (up to a prime-to-p ideal):

```
{p=7;f=2557;N=4;pN=p^N;fpN=f*pN;c=lift(znprimroot(f));cm=Mod(c,fpN)^-1;
g=znprimroot(f);lg=lift(Mod((1-lift(g))/f,pN));g=Mod(lift(g)+lg*f,fpN);
g3=g^3;G=znprimroot(pN);lG=lift(Mod((1-lift(G))/pN,f));
G=Mod(lift(G)+lG*pN,fpN);A0=0;A1=0;A2=0;for(k=1,(f-1)/3,
for(j=1,p^(N-1)*(p-1),A=g3^k*G^j;gA=g*A;ggA=g^2*A;
a=lift(A);aa=lift(A*cm);la=(aa*c-a)/fpN;A0=A0+la*Mod(a,pN)^-1;
a=lift(gA);aa=lift(gA*cm);la=(aa*c-a)/fpN;A1=A1+la*Mod(a,pN)^-1;
a=lift(ggA);aa=lift(ggA*cm);la=(aa*c-a)/fpN;A2=A2+la*Mod(a,pN)^-1));
print(A0," ",A1," ",A2)}
Mod(184, 2401) Mod(1526, 2401) Mod(643, 2401)
```

Modulo 7^4, $A_0 = 184$, $A_1 = 1526$ and $A_2 = 643$; this yields the ideal $(1-2j)^3 = \mathfrak{p}^3$. Necessarily, $\mathscr{T}_K \simeq \mathbb{Z}[j]/\mathfrak{p}^2 \oplus \mathbb{Z}[j]/\mathfrak{p}$. We note that the annihilator is $\mathfrak{p}^3$ (and not $\mathfrak{p}^2$) although the structure is not $\mathbb{Z}[j]/\mathfrak{p}^3$.

Remarks 6.5. (i) In practice, when the exponent p^e of $\mathscr{T}_K$ is known, one can take $n = n_0 + e$, where $n_0 \geq 0$ is defined by $[K \cap \mathbb{Q}^{\rm cyc} : \mathbb{Q}] =: p^{n_0}$, and use the annihilators $\mathbf{A}_{K,n}(c)$, $\mathbf{A}'_{K,n}(c)$ (but any $n \gg 0$ is suitable). When $K = K_\chi$, the annihilator limit $\mathbf{A}_{K_\chi}(c)$ is related to p-adic $\mathbf{L}$-functions via the formula:

$$\psi(\mathbf{A}_{K_\chi}(c)) = (1 - \psi(c)) \cdot \mathbf{L}_p(1, \psi), \quad \text{for } \psi \mid \chi.$$

If g_χ is not a p-power, one can choose c such that $1 - \psi(c)$ is invertible giving (up to a p-adic unit) $\psi(\mathbf{A}_{K_\chi}(c)) \sim \mathbf{L}_p(1, \psi)$; if $g_\chi = p^n$, $n \geq 1$, $\psi(\mathbf{A}_{K_\chi}(c)) \sim \pi_\chi \mathbf{L}_p(1, \psi)$, where π_χ is an uniformizing parameter in $\mathbb{Q}_p(\mu_{p^n})$.

This theorem is the analog of Theorem 5.19, using Bernoulli's numbers, linked to $\mathbf{L}_p(0, \omega\psi^{-1})$, instead of $\mathbf{L}_p(1, \psi)$.

(ii) Some other annihilation theorems exist for the Jaulent logarithmic class group (see [Jau2021, Jau2022, Jau2022^b]); [Jau2022^b] is related to Greenberg's conjecture and, when K contains μ_p, [Jau2021] obtains that the Stickelberger ideal annihilates the imaginary component of the logarithmic class group and that its reflection annihilates the real component of the Bertrandias–Payan module. It will be interesting to formulate a "Main Conjecture" about the φ-components of these modules.

7. Application to class groups of real abelian extensions

Denote by $\mathbf{E}$ the $\mathscr{G}$-family for which $\mathbf{E}_K$, $K \in \mathscr{K}$, is the group of absolute value of the global units of K, the Galois action being defined by $|\varepsilon|^\sigma = |\varepsilon^\sigma|$ for any unit ε and any $\sigma \in \mathscr{G}$. The $\mathbf{E}_K$ are free $\mathbb{Z}$-modules of rank $[K : \mathbb{Q}] - 1$ for real fields K.

7.1. **The Leopoldt χ-units.** In [Leo1954] Leopoldt defined unit groups, $\mathbf{E}_\chi$, that we shall call (as in [Or1975^b]) the group of χ-units for rational characters $\chi \in \mathscr{X}^+ \setminus \{1\}$; from the definition of χ-objects and the results of the previous sections we can write (where ν may be replaced by $\mathbf{N}$):

$$\begin{aligned} \mathbf{E}_\chi &= \{|\varepsilon| \in \mathbf{E}_{K_\chi},\ \ P_\chi(\sigma_\chi) \cdot |\varepsilon| = 1\} \\ &= \{|\varepsilon| \in \mathbf{E}_{K_\chi},\ \nu_{K_\chi/k}(|\varepsilon|) = 1, \text{ for all } k \subsetneq K_\chi\}. \end{aligned} \tag{7.1}$$

What follows is also available in [Leo1954, Leo1962, Or1975^b].

Definitions 7.1. *(i) For any cyclic real field K, denote by $\mathbf{E}^0_K$ the subgroup of $\mathbf{E}_K$ generated by the $\mathbf{E}_k$'s for all the subfields $k \subsetneq K$ (or simply by each of the k_ℓ such that $[K : k_\ell] = \ell \mid [K : \mathbb{Q}]$, ℓ prime).*

(ii) Let $Q_K = \big(\mathbf{E}_K : \oplus_{\chi \in \mathscr{X}_K} \mathbf{E}_\chi\big)$ where $\mathbf{E}_\chi$ is the group of χ-units (Definition (7.1)) and, for all $\chi \in \mathscr{X}^+_K$, let $Q_\chi = \big(\mathbf{E}_{K_\chi} : \mathbf{E}^0_{K_\chi} \oplus \mathbf{E}_\chi\big)$.

(iii) Let ϕ be the Euler totient function and put, for $\chi \in \mathscr{X}^+$:

$$\begin{cases} q_\chi = \prod_{\ell \mid g_\chi} \ell^{\frac{\phi(g_\chi)}{\ell - 1}}, \ \text{if } g_\chi \text{ is not the power of a prime number,} \\ q_\chi = \ell^{\frac{\phi(g_\chi)}{\ell-1} - 1} = \ell^{\ell^{n-1} - 1}, \ \text{if } g_\chi \text{ is a prime power } \ell^n,\ n \geq 1, \\ q_1 = 1. \end{cases}$$

Set $q_K = \Big(\dfrac{g^{g-2}}{\prod_{\chi \in \mathscr{X}_K} d_\chi}\Big)^{\frac{1}{2}}$, where $g := [K : \mathbb{Q}]$ and d_χ is the discriminant of $\mathbb{Q}(\mu_{g_\chi})$.

Lemma 7.2. *(i) We have* $\mathbf{E}^0_{K_\chi}\cdot\mathbf{E}_\chi=\mathbf{E}^0_{K_\chi}\oplus\mathbf{E}_\chi$, *for all* $\chi\in\mathscr{X}^+$.

(ii) We have, for all cyclic real field K, $Q_K=\prod_{\chi\in\mathscr{X}_K}Q_\chi$.

(iii) We have, for all cyclic real field K, $q_K=\prod_{\chi\in\mathscr{X}_K}q_\chi$.

Proof. (i) One may find various equivalent definitions of the χ-units and their properties in [Leo1954, Chap. 5, § 4] or [Or1975^b]; but knowing the norm characterization (7.1) of $\mathbf{E}_\chi$, the proof of (i) is obvious.

(ii) This may be proved locally; for this, we use the $\mathscr{G}$-family $\mathscr{E}_K:=\mathbf{E}_K\otimes\mathbb{Z}_p$, for any prime p, and the $\mathscr{E}_\chi$'s as above. Then one uses, inductively, Lemma 7.2 (i) with characters $\psi\mid\varphi\mid\chi$, written as $\psi=\psi_0\,\psi_p$ (ψ_0 of prime-to-p order, ψ_p of order p^n, $n\geq 0$). See [Gra1976, pp. 72–75].

(iii) From [Has1952, § 15, p. 34; (2), p. 35]; see [Gra1976, pp. 76–77] for more details. □

7.2. **The Leopoldt cyclotomic units.** For the main definitions and properties of cyclotomic units, see [Leo1954, § 8 (1)] or [Or1975].

Definitions 7.3. *(i) Let* $\chi\in\mathscr{X}^+$ *of conductor* f_χ; *we define the "cyclotomic numbers"* $\mathbf{C}_\chi:=\prod_{a\in A_\chi}(\zeta_{2f_\chi}^a-\zeta_{2f_\chi}^{-a})$, *with* $\zeta_{2f_\chi}:=\exp\left(\frac{i\pi}{f_\chi}\right)$, *where* A_χ *is a half-system of representatives, in* $(\mathbb{Z}/f_\chi\mathbb{Z})^\times$, *of* $\mathrm{Gal}(\mathbb{Q}(\mu_{f_\chi})/K_\chi)$.

(ii) Let K *be a real abelian field and let* $\mathbf{C}_K$ *be the multiplicative group generated by the conjugates of* $|\mathbf{C}_\chi|$, *for all* $\chi\in\mathscr{X}_K$. *Then we define the group of cyclotomic units* $\mathbf{F}_K:=\mathbf{C}_K\cap\mathbf{E}_K$ *and* $\mathscr{F}_K:=\mathbf{F}_K\otimes\mathbb{Z}_p$.

Recall that $\mathbf{C}_\chi^2\in K_\chi$ and that any conjugate $\mathbf{C}'_\chi$ of $\mathbf{C}_\chi$ is such that $\frac{\mathbf{C}'_\chi}{\mathbf{C}_\chi}\in\mathbf{E}_{K_\chi}$. If f_χ is not a prime power, then $\mathbf{C}_\chi$ is a unit and $\mathbf{F}_K=\mathbf{C}_K$.

7.3. **Arithmetic computation of** $\#\mathbf{H}^{\mathrm{ar}}_\chi$, $\chi\in\mathscr{X}^+$. Using the Leopoldt formula [Leo1954, Satz 21, § 8 (4)] and Lemma 7.2 (ii), (iii), we obtain (see [Gra1976, Théorème III.1]):

Proposition 7.4. *For all* $\chi\in\mathscr{X}^+\setminus\{1\}$, *let* $\Delta_\chi=\prod_{\ell\mid g_\chi}\left(1-\sigma_\chi^{g_\chi/\ell}\right)$; *then* $\#\mathbf{H}^{\mathrm{ar}}_\chi=\frac{Q_\chi}{q_\chi}\cdot(\mathbf{E}_\chi:\mathbf{C}_\chi^{\Delta_\chi})$ *and* $\#\mathbf{H}^{\mathrm{ar}}_\chi=\frac{1}{q_\chi}\big(\mathbf{E}_{K_\chi}:\mathbf{E}^0_{K_\chi}\oplus\mathbf{C}_\chi^{\Delta_\chi}\big)$, *interpreting* Q_χ [Gra1976, Corollaire III.1].

To interpret the coefficient q_χ, we have replaced the Leopoldt group $\mathbf{C}_\chi^{\Delta_\chi}$ of cyclotomic units by the larger group $\mathbf{F}_{K_\chi}:=\mathbf{C}_{K_\chi}\cap\mathbf{E}_{K_\chi}$ (Definition 7.3); see the long proof [Gra1976, Chap. III, § 3] giving the final result interpreting the coefficient q_χ and giving the analog of Theorem 5.10 for real class groups:

Theorem 7.5. *Let* $\mathbf{H}^{\mathrm{ar}}_\chi:=\{x\in\mathbf{H}_{K_\chi},\,\mathbf{N}_{K_\chi/k}(x)=1,\ \text{for all } k\subsetneq K_\chi\}$. *Let* g_χ *be the order of* $\chi\in\mathscr{X}^+\setminus\{1\}$ *and* f_χ *its conductor. Then:*

$$\#\mathbf{H}^{\mathrm{ar}}_\chi=w_\chi\cdot\big(\mathbf{E}_{K_\chi}:\mathbf{E}^0_{K_\chi}\cdot\mathbf{F}_{K_\chi}\big),$$

where w_χ *is defined as follows:*

(i) Case g_χ *non prime power. Then* $w_\chi=1$;

(ii) Case $g_\chi=p^n$, $p\neq 2$ *prime,* $n\geq 1$:

(ii′) Case $f_\chi=\ell^k$, ℓ *prime,* $k\geq 1$. *Then* $w_\chi=1$;

(ii″) Case f_χ *non prime power. Then* $w_\chi=p$;

(iii) Case $g_\chi=2^n$, $n\geq 1$:

(iii′) Case $f_\chi=\ell^k$, ℓ *prime,* $k\geq 1$. *Then* $w_\chi=1$;

(iii″) Case f_χ *non prime power. Then* $w_\chi\in\{1,2\}$.

Proof. For the ugly proof see [Gra1976, Théorème III.2, pp. 78–85]. □

Corollary 7.6. *If* $p\nmid g_\chi$, $\#\mathscr{H}_\chi=(\mathscr{E}_\chi:\mathscr{F}_\chi)=\prod_{\varphi\mid\chi}(\mathscr{E}_\varphi:\mathscr{F}_\varphi)$, *where* $\mathscr{E}_\varphi=\mathscr{E}_{K_\chi}^{e_\varphi}$ *and* $\mathscr{F}_\varphi=\big(\langle\mathbf{C}_\chi\rangle\otimes\mathbb{Z}_p\big)^{e_\varphi}$ *now giving* $\#\mathscr{H}_\varphi=(\mathscr{E}_\varphi:\mathscr{F}_\varphi)$.

Proof. In the semi-simple case $p \nmid g_\chi$, for any $\mathbb{Z}_p[G_K]$-module $\mathscr{M}_K$, $\mathscr{M}_\chi = \mathscr{M}_K^{e_\chi}$ and $\mathscr{M}_\varphi = \mathscr{M}_K^{e_\varphi}$, with the usual semi-simple idempotents; thus, $\widetilde{\mathscr{E}}_\chi = \widetilde{\mathscr{E}}_\chi^{e_\chi} = \mathscr{E}_{K_\chi}^{e_\chi}/\mathscr{E}_{K_\chi}^{0\,e_\chi} \cdot \mathscr{F}_{K_\chi}^{e_\chi} = \mathscr{E}_\chi/\mathscr{F}_\chi$, since $\mathscr{E}_{K_\chi}^{0\,e_\chi} = 1$. The claim for $\varphi \mid \chi$ is the Main Theorem proved in the semi-simple context. □

Remarks 7.7. (i) The viewpoint given by Theorem 7.5, which appears to have been ignored, seems more convenient than formulas trying to use Sinnott's cyclotomic units. Indeed, compare with [Grei1992, Theorem 4.14] using instead $\mathscr{H}_\chi^{\rm alg}$ (in a partial semi-simple context as explained in Remark 8.2) and Sinnott's group of cyclotomic units, larger than classical Leopoldt's group of Definition 7.3, but which give rise to intricate index formulas.

Moreover, as we have mentioned in [Gra1977, Remark III.1], an analytic formula for $\#\mathscr{H}_\chi^{\rm alg}$, $\chi \in \mathscr{X}^+$, does not seem obvious (if any) because of capitulation aspects (see the numerical examples of § 3.3).

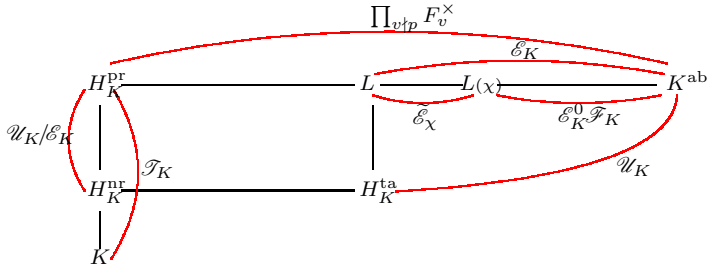
We hope that Theorem 7.5 suggests a new statement of the Main Conjecture for the $\mathscr{H}_\varphi$'s, especially in the non semi-simple real case (see § 8.2 for the corresponding analytic values).

(ii) Since $\mathbf{N}_{K_\chi/k}(\mathbf{E}_{K_\chi}) \subseteq \mathbf{E}_{K_\chi}^0$, $\mathbf{N}_{K_\chi/k}(\mathscr{E}_{K_\chi}) \subseteq \mathscr{E}_{K_\chi}^0$, for all $k \subsetneq K_\chi$, the modules $\widetilde{\mathbf{E}}_\chi := \mathbf{E}_{K_\chi}/\mathbf{E}_{K_\chi}^0 \cdot \mathbf{F}_{K_\chi}$ and $\widetilde{\mathscr{E}}_\chi := \mathscr{E}_{K_\chi}/\mathscr{E}_{K_\chi}^0 \cdot \mathscr{F}_{K_\chi}$ are χ-objects (algebraic and arithmetic). Then $\widetilde{\mathscr{E}}_\chi = \bigoplus_{\varphi|\chi} \widetilde{\mathscr{E}}_\varphi$, where the φ-components are $\widetilde{\mathscr{E}}_\varphi = \{\widetilde{x} \in \widetilde{\mathscr{E}}_\chi,\ \ P_\varphi(\sigma_\chi) \cdot \widetilde{x} = 1\}$.

7.4. **Class field theory and regulators.** Let $K \in \mathscr{K}$ be a real cyclic field defining $\chi \in \mathscr{X}^+$ in what follows. To simplify diagrams and statements, we assume to be in the most common case where $\mathscr{W}_K = 1$ and $K \cap \mathbb{Q}^{\rm cyc} = \mathbb{Q}$, which gives $\mathscr{T}_K = \mathscr{T}_K^{\rm bp}$ (cf. Diagram of Section 6) and $\#\mathscr{T}_K \sim \prod_{\psi|\chi,\,\psi\neq 1} \frac{1}{2}\,\mathbf{L}_p(1,\psi)$ (Formula (6.2)).

The Galois group $\mathscr{R}_K \subseteq \mathscr{T}_K$ may be compared with a larger "cyclotomic regulator" $\mathscr{R}_K^{\rm cyc}$ interpreted as a Galois group only depending of χ. For this purpose, the following diagram of the maximal abelian pro-p-extension $K^{\rm ab}$ of K is necessary (from [Gra2005, III.4 (d) & Diagram III.4.4.1] with our present notations), where $H_K^{\rm ta}$ is the maximal tamely ramified abelian pro-p-extension of K and $F_v^\times$ the p-Sylow subgroup of the multiplicative group of the residue field of the tame place v; let $L := H_K^{\rm pr} H_K^{\rm ta}$:

7.4.1. *Schema IX.*



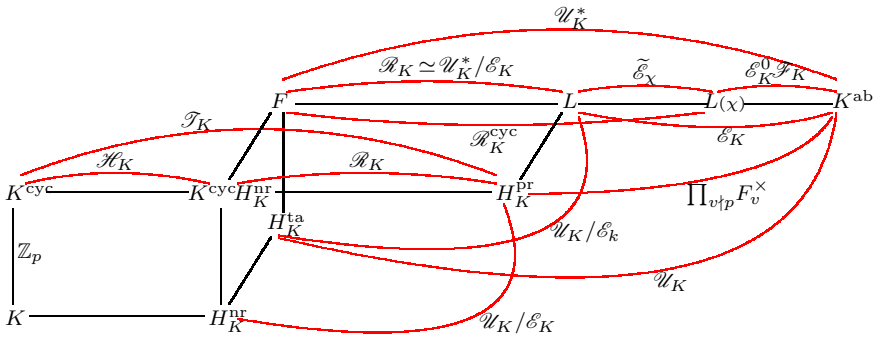
In this diagram, class field theory interprets $\mathrm{Gal}(K^{\rm ab}/H_K^{\rm ta})$ as the $\mathbb{Z}_p$-module $\mathscr{U}_K$ of principal local units at p (isomorphic to the direct product of the inertia groups of the p-places) and $\mathrm{Gal}(K^{\rm ab}/L)$ as the $\mathbb{Z}_p$-module $\mathscr{E}_K := \mathbf{E}_K \otimes \mathbb{Z}_p$ (embedded both in $\mathscr{U}_K$ and the product $\prod_{v\nmid p} F_v^\times$ of the inertia groups of the tame places, with suitable Artin maps described in [Gra2005, § III.4.4.5.1]).

Now, put $\mathscr{U}_K^* := \{u \in \mathscr{U}_K,\ \mathbf{N}_{K/\mathbb{Q}}(u) = \pm 1\}$; since K is real, $\mathscr{E}_K$ is of finite index in $\mathscr{U}_K^*$ and $\mathrm{tor}_{\mathbb{Z}_p}(\mathscr{U}_K/\mathscr{E}_K) = \mathscr{U}_K^*/\mathscr{E}_K \simeq \mathscr{R}_K$.

Assume $K^{\rm cyc} \cap H_K^{\rm nr} = K$ to simplify; so $H_K^{\rm ta} \cap K^{\rm cyc} = H_K^{\rm nr}$ then $F := H_K^{\rm ta}\, K^{\rm cyc} H_K^{\rm nr}$ is fixed by $\mathscr{U}_K^*$ and $F \cap H_K^{\rm pr} = K^{\rm cyc} H_K^{\rm nr}$. Recall the exact sequence $1 \to \mathscr{R}_K^{\rm ram} \to \mathscr{R}_K \to \mathscr{R}_K^{\rm nr} \to 1$ [Gra2021, § 2 & Figure 3], due to genus theory; so, a sub-extension of L/F may be unramified.

We have moreover $\mathrm{Gal}(F/K^{\rm cyc} H_K^{\rm nr}) \simeq \mathrm{Gal}(L/H_K^{\rm pr}) \simeq \big(\prod_{v\nmid p} F_v^\times\big)/\mathscr{E}_K$:

7.4.2. *Schema X.*



Define (under the previous assumptions), $\mathscr{R}_K^{\rm cyc} := \mathscr{U}_K^*/\mathscr{E}_K^0\cdot\mathscr{F}_K$, which yields, for $\chi\neq 1$ and $K=K_\chi$, the $\mathbb{Z}_p[G_K]$-modules isomorphism:

$$\mathscr{R}_K \simeq \mathscr{R}_K^{\rm cyc}/\widetilde{\mathscr{E}}_\chi. \tag{7.2}$$

We then have $\mathscr{R}_K^{\rm cyc}\simeq {\rm Gal}(L_{(\chi)}/F)$, where $L_{(\chi)}$ is the subfield of $K^{\rm ab}$ fixed by the image of $\mathscr{E}_K^0\mathscr{F}_K$.

Remark 7.8. Let $\chi\in\mathscr{X}^+\setminus\{1\}$, $K=K_\chi$; assume to simplify that $\mathscr{W}_K=1$, $w_\chi=1$ in Theorem 7.5, $K\cap\mathbb{Q}^{\rm cyc}=\mathbb{Q}$ and $K^{\rm cyc}\cap H_K^{\rm nr}=K$:

(i) Theorem 7.5 and isomorphism (7.2) give, in terms of χ-components:

$$\#\widetilde{\mathscr{E}}_\chi = \#\mathscr{R}_K^{\rm cyc}/\#\mathscr{R}_K = \#\mathscr{H}_\chi^{\rm ar} \text{ and } \#\mathscr{T}_\chi = \#\mathscr{R}_\chi^{\rm cyc}.$$

The $\mathscr{A}_\chi$-modules $\mathscr{T}_\chi$ and $\mathscr{R}_\chi^{\rm cyc}$ (resp. $\widetilde{\mathscr{E}}_\chi$ and $\mathscr{H}_\chi^{\rm ar}$) are not necessarily isomorphic as shown by the following excerpt giving cyclic cubic fields K such that $\mathscr{R}_\chi$ is of 7-rank 2 and $\mathscr{T}_\chi$ of 7-rank ≥ 3 implying $\mathscr{H}_\chi\neq 1$ with $\mathscr{H}_\chi\simeq\mathbb{Z}/7\mathbb{Z}\oplus\mathbb{Z}/7\mathbb{Z}$ for the followings (no example of 7-rank ≥ 4 exists in the interval considered):

```
x^3+x^2-39666*x-2582719     Structure of the 7-torsion group: [7,7,7]
x^3+x^2-43300*x-3411104     Structure of the 7-torsion group: [7^2,7,7]
x^3+x^2-13226*x-508479      Structure of the 7-torsion group: [7^3,7,7]
x^3+x^2-427660*x-31551829   Structure of the 7-torsion group: [7^4,7,7]
x^3+x^2-2033484*x-966131001 Structure of the 7-torsion group: [7^2,7^2,7]
```

(ii) The sub-diagram given by the extension $K^{\rm ab}/K^{\rm cyc}$, opens an access way for an interpretation of the Main Conjecture for even characters or at least for an annihilation theorem of $\mathscr{H}_\varphi^{\rm ar}$ by $\widetilde{\mathscr{E}}_\varphi$, in the spirit of Thaine's theorem (see § 7.6, Conjectures 7.9, 7.13). Indeed, $\widetilde{\mathscr{E}}_\chi$ has same order as $\mathscr{H}_\chi^{\rm ar}$ and the units may be seen diagonally embedded in the (infinite) product of the places of K. Remark that $\widetilde{\mathscr{E}}_\varphi$ is a sub-module of $\mathscr{R}_\varphi^{\rm cyc}$ (quotient $\mathscr{R}_\varphi$) but $\mathscr{H}_\varphi^{\rm ar}$ is a quotient of $\mathscr{T}_\varphi$ (by $\mathscr{R}_\varphi$).

7.5. **Annihilation conjecture for real p-class groups.** Before any proof of the conjectural equality $\#\mathscr{H}_\varphi^{\rm ar}=\#\widetilde{\mathscr{E}}_\varphi=\#(\mathscr{E}_{K_\chi}/\mathscr{E}_{K_\chi}^0\cdot\mathscr{F}_{K_\chi})_\varphi$ (giving a Main Theorem for $\varphi\in\Phi_K^+$), it will be interesting to prove that any annihilator of $\widetilde{\mathscr{E}}_\varphi$ annihilates $\mathscr{H}_\varphi^{\rm ar}$, which will be more precise than the annihilators of $\mathscr{T}_\varphi$ (see Theorem 6.2, Remarks 6.5, 7.8).

To our knowledge, the best known annihilation theorem of real p-class groups is Thaine's Theorem [Thai1988], [Was1997, Theorem 15.2] saying that any annihilator of $\mathscr{E}_{K_\chi}/\mathscr{F}'_{K_\chi}$ (for a suitable definition of the group of cyclotomic units $\mathscr{F}'_{K_\chi}$) is an annihilator of $\mathscr{H}_{K_\chi}$. But Thaine's Theorem only concerns the semi-simple case.

Mention also annihilation theorems by Solomon [Sol1992], which are not often optimal because of vanishing of Euler factors; this is discussed in [Gra2018^b]. Finally mention the numerous papers of Greither and Kučera (like [GrKu2004, GrKu2014, GrKu2021]) on the annihilation of real class groups, using special units or/and giving information on the Fitting ideals.

Conjecture 7.9. *Let $\chi \in \mathscr{X}^+ \setminus \{1\}$ and let $\varphi \mid \chi$. Any element of $\mathbb{Z}[\mu_{g_\chi}]$ (resp. $\mathbb{Z}_p[\mu_{g_\chi}]$) annihilating $\mathbf{E}_{K_\chi}/\mathbf{E}^0_{K_\chi}\cdot\mathbf{F}_{K_\chi}$ (resp. $(\mathscr{E}_{K_\chi}/\mathscr{E}^0_{K_\chi}\cdot\mathscr{F}_{K_\chi})_\varphi$), annihilates $\mathbf{H}^{\rm ar}_\chi$ (resp. $\mathscr{H}^{\rm ar}_\varphi$).*

In this direction, we state the following lemma, giving some obvious prerequisites on the subject.

Lemma 7.10. *Let $\mathbf{M}_{K_\chi}$ be a torsion-free monogenic $\mathbb{Z}[G_\chi]$-module (i.e., $\mathbb{Z}$-free and $\mathbb{Z}[G_\chi]$-generated by a single element). Let $\mathbf{M}'_{K_\chi}$ be a sub-module of $\mathbf{M}_{K_\chi}$ such that $\mathbf{M}_{K_\chi}/\mathbf{M}'_{K_\chi}$ is annihilated by $P_\chi(\sigma_\chi)\,\mathbb{Z}[G_\chi]$ and finite. Then:*

$$(\mathscr{M}_{K_\chi}/\mathscr{M}'_{K_\chi})_\varphi := ((\mathbf{M}_{K_\chi}/\mathbf{M}'_{K_\chi})\otimes\mathbb{Z}_p)_\varphi \simeq \mathbb{Z}_p[\mu_{g_\chi}]/\mathfrak{p}_\varphi^{\lambda_\varphi},$$

$\lambda_\varphi \geq 0$, for all $\varphi \mid \chi$.

Proof. By assumption, $\mathbf{M}_{K_\chi}/\mathbf{M}'_{K_\chi}$ is a finite monogenic $\mathbb{Z}[\mu_{g_\chi}]$-module, of the form $\mathbb{Z}[\mu_{g_\chi}]/\mathfrak{A}$, $\mathfrak{A}\neq 0$; so $\mathscr{M}_{K_\chi}/\mathscr{M}'_{K_\chi} \simeq (\mathbb{Z}[\mu_{g_\chi}]/\mathfrak{A})\otimes\mathbb{Z}_p$, giving:

$$\mathscr{M}_{K_\chi}/\mathscr{M}'_{K_\chi} \simeq \bigoplus\nolimits_{\varphi\mid\chi}\big[\mathbb{Z}_p[\mu_{g_\chi}]/\mathfrak{p}_\varphi^{\lambda_\varphi}\big],$$

with the usual correspondence between primes $\mathfrak{p}\mid p$ and p-adic characters $\varphi\mid\chi$; whence the claim. □

It is well-known that there exists in $\mathbf{E}_{K_\chi}$ a unit ε generating, with its conjugates, a subgroup $\mathbf{E}$ of $\mathbf{E}_{K_\chi}$ of prime-to-p finite index (Minkowski unit). Then $\mathbf{M} := \mathbb{Z}[G_\chi]\cdot|\varepsilon|$ is monogenic and torsion-free.

Let $\mathbf{M}'_{K_\chi} := \mathbf{E}^0_{K_\chi}\cdot\mathbf{F}_{K_\chi}$. Taking into account orders, monogenicity and the fact that $(P_\chi(\sigma_\chi))$ annihilates $\mathbf{M}_{K_\chi}/\mathbf{M}'_{K_\chi}$, Lemma 7.10 is coherent with an annihilation theorem of the $\mathscr{H}^{\rm ar}_\varphi$'s from the results of § 7.4.

7.6. **Mysterious link between cyclotomic units and ideal classes.** The brief overview, that we give now, must be completed by technical elements that the reader can find especially in [Was1997, § 15.2, 15.3] (all of them borrow from classical arithmetic) and in the references that we talked about, giving systematic generalizations of "Euler systems".

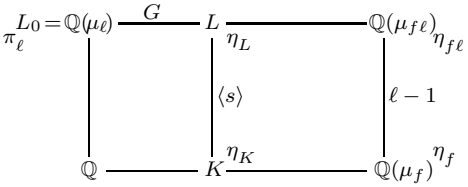
To simplify, consider the real semi-simple case for $p>2$ with $K=K_\chi$ of conductor f; for $\varphi\mid\chi$, we need to establish *arithmetic links* between $\widetilde{\mathscr{E}}_\varphi = \mathscr{E}_\varphi/\mathscr{F}_\varphi$ and $\mathscr{H}_\varphi$, where $\mathscr{E}_\varphi =: \langle\varepsilon_\varphi\rangle_{\mathbb{Z}_p}$ and $\mathscr{F}_\varphi =: \langle\eta_\varphi\rangle_{\mathbb{Z}_p}$ is built from Leopoldt's cyclotomic units (Definitions 7.3). But $\widetilde{\mathscr{E}}_\varphi$ has, a priori, no obvious connection with class groups, except the analytic equality $\prod_{\varphi\mid\chi}\#\mathscr{H}_\varphi = \prod_{\varphi\mid\chi}\#\widetilde{\mathscr{E}}_\varphi$ (Corollary 7.6).

The trick, for the proof of the Main Conjecture, consists in using a classical context of "analytic genus theory", by means of auxiliary cyclic ℓ-ramified extensions $K(\mu_\ell)$ of degree multiple of the exponent $N=\lambda p^e$, $e\geq 1$, of $\mathbf{H}_K$.

Let $\ell\nmid f$, $\ell\equiv 1 \pmod N$, totally split in K; put $L_0=\mathbb{Q}(\mu_\ell)$ and $L := L_0K$:

Let $\eta_{f\ell}=1-\zeta_{f\ell}$, $\eta_f=1-\zeta_f$, $\eta_\ell=1-\zeta_\ell$ and consider the cyclotomic numbers $\eta_L := \mathbf{N}_{\mathbb{Q}(\mu_{f\ell})/L}(\eta_{f\ell})$, $\eta_K := \mathbf{N}_{\mathbb{Q}(\mu_f)/K}(\eta_f)$; by assumption on the total splitting of ℓ in $K/\mathbb{Q}$, $\mathbf{N}_{L/K}(\eta_L)=1$ (cf. Lemma 5.17). We remark that $\eta_{f\ell}\equiv\eta_f \pmod{\pi_\ell}$ where $\pi_\ell := \eta_\ell$ is an uniformizing parameter at the places above ℓ in L_0, so that $\eta_L\equiv\eta_K \pmod{\pi_\ell}$, giving a ℓ-adic link between η_K and η_L which will be fundamental for the congruences (7.6):

7.6.1. *Schema XI.*



A main step is to apply Hilbert's Theorem 90 (Kummer's Theorem [Kum1855, II]), saying that $\eta_L=\alpha_L^{s-1}$, where s is a generator of $\mathrm{Gal}(L/K)$ and $\alpha_L\in L^\times$ is such that

$(\alpha_L) \in \mathbf{I}_L^{\langle s \rangle}$, where $\mathbf{I}$ denotes ideal groups; since α_L is defined modulo $K^\times$, we can take α_L integer in L (or at least ℓ-integer), whence:

$$(\alpha_L) = \mathbf{J}_{L/K}(\mathfrak{a}_K) \cdot \mathfrak{L}_0^{\Omega_\ell}, \tag{7.3}$$

where $\mathfrak{a}_K \in \mathbf{I}_K$ may be taken prime to ℓ, where $\mathfrak{L}_0$ is a fixed prime ideal dividing ℓ in L and:

$$\Omega_\ell = \textstyle\sum_{\sigma \in G} r_\sigma \cdot \sigma^{-1},\ r_\sigma \geq 0; \tag{7.4}$$

thus, since $\mathbf{N}_{L/K}(\mathfrak{L}_0) = \mathfrak{l}_0$, $\mathfrak{L}_0 \mid \mathfrak{l}_0 \mid \ell$ in L/K:

$$(\alpha_K) := (\mathbf{N}_{L/K}(\alpha_L)) = \mathfrak{a}_K^{\ell-1} \cdot \mathfrak{l}_0^{\Omega_\ell}. \tag{7.5}$$

But $\mathfrak{a}_K^{\ell-1}$ is principal, whence $\mathfrak{l}_0^{\Omega_\ell}$ principal.

The following property elucidates the "mysterious link" giving an information that we can "project" on each φ-component and obtain the annihilation of the φ-class of $\mathfrak{l}_0$ by the φ-component of Ω_ℓ:

Lemma 7.11. *Except a finite number of primes ℓ, the ideal $\mathfrak{L}_0^{\Omega_\ell}$ of (7.3) gives a non trivial relation, in the meaning that Ω_ℓ in (7.4) is not of the form $\lambda \cdot \nu_{L/L_0}$, $\lambda \geq 0$, giving $\mathfrak{l}_0^{\Omega_\ell} = (\ell)^\lambda$ in (7.5).*

Proof. Assume that $\Omega_\ell = \lambda \cdot \nu_{L/L_0}$; the character of $\mathfrak{L}_0^{\Omega_\ell} = (\pi_\ell^\lambda)$, as $\mathbb{Z}[G]$-module, is the unit one and any non-trivial φ-component $\alpha_{L,\varphi}$ of α_L is prime to ℓ, thus congruent, modulo any $\mathfrak{L} \mid \ell$, to $\rho_\mathfrak{l} \in \mathbb{Z}$, $\rho_\mathfrak{l} \not\equiv 0 \pmod{\ell}$ (residue degrees 1 in $L/\mathbb{Q}$). Since $\mathfrak{L}^s = \mathfrak{L}$, we obtain $\eta_{L,\varphi} = \alpha_{L,\varphi}^{s-1} \equiv 1 \pmod{\mathfrak{L}}$; but $\eta_{K,\varphi} \equiv \eta_{L,\varphi} \pmod{\pi_\ell}$ leads to $\eta_{K,\varphi} \equiv 1 \pmod{\mathfrak{l}}$, for all $\mathfrak{l} \mid \ell$, giving $\eta_{K,\varphi} \equiv 1 \pmod{\ell}$ (absurd for almost all ℓ). □

Reducing modulo ν_{L/L_0}, one may get $\Omega_\ell \neq 0$ with $r_\sigma \geq 0$ for all $\sigma \in G$, but not all zero. Consider $\frac{\alpha_L^\sigma}{\pi_\ell^{r_\sigma}}$ modulo $\mathfrak{L}_0$ and the conjugations $\alpha_L^s = \alpha_L \cdot \eta_L$ and $\frac{\pi_\ell^s}{\pi_\ell} = \frac{1-\zeta_\ell^{\mathbf{g}_\ell}}{1-\zeta_\ell} \equiv \mathbf{g}_\ell \pmod{\pi_\ell}$ (where $\mathbf{g}_\ell$ is a primitive root modulo ℓ such that $\zeta_\ell^s =: \zeta_\ell^{\mathbf{g}_\ell}$); one gets:

$$\left(\frac{\alpha_L^\sigma}{\pi_\ell^{r_\sigma}}\right)^s = \frac{\alpha_L^{s\sigma}}{\pi_\ell^{sr_\sigma}} \equiv \frac{\eta_L^\sigma \alpha_L^\sigma}{(\mathbf{g}_\ell \pi_\ell)^{r_\sigma}} \equiv \frac{\eta_L^\sigma}{\mathbf{g}_\ell^{r_\sigma}} \cdot \frac{\alpha_L^\sigma}{\pi_\ell^{r_\sigma}} \pmod{\mathfrak{L}_0};$$

since the residue modulo $\mathfrak{L}_0$ of $\frac{\alpha_L^\sigma}{\pi_\ell^{r_\sigma}}$ is non-zero and invariant by s, we obtain:

$$\mathbf{g}_\ell^{r_\sigma} \equiv \eta_L^\sigma \equiv \eta_K^\sigma \pmod{\mathfrak{l}_0}. \tag{7.6}$$

So we have obtained a non-trivial relation between the classes of the conjugates of $\mathfrak{l}_0$; for instance, if $\eta_{K,\varphi} = \varepsilon_{K,\varphi}^{p^h}$, one gets $r_\sigma \equiv 0 \pmod{p^h}$, whence a property of annihilation of the φ-class group. Recall that α_L is given by an explicit Hilbert resolvent allowing explicit computations.

Remark 7.12. (i) In the literature, the properties of the α_L's give rise to an homomorphism $\mathbf{F}_K/\mathbf{F}_K^{p^e} \to \mathbb{Z}/p^e\mathbb{Z}[G]$, of $\mathbb{Z}_p[G]$-modules, allowing reasoning for the φ-components. To get more information, one varies ℓ, using Chebotarev's Theorem and Nakayama's Lemma. Then the problem of the order of the $\mathscr{H}_\varphi$'s needs the knowledge of the whole analytic formula of Theorem 7.5 (see the details in [Was1997, § 15.2, 15.3], from Thaine's theorem).

(ii) We will return elsewhere to the links with genus theory given by the well-known fixed-points exact sequence:

$$1 \to \mathscr{A}_L(\mathbf{I}_L^{\langle s \rangle}) \otimes \mathbb{Z}_p \longrightarrow \mathscr{H}_L^{\langle s \rangle} \longrightarrow \mathscr{E}_K \cap \mathbf{N}_{L/K}(L^\times)/\mathbf{N}_{L/K}(\mathscr{E}_L) \to 1$$

and (in the present context) Chevalley's formula [Che1933, pp. 402-406] in L/K:

$$\#\mathscr{H}_L^{\langle s \rangle} = \#\mathscr{H}_K \cdot \frac{p^{e\,([K:\mathbb{Q}]-1)}}{(\mathscr{E}_K : \mathscr{E}_K \cap \mathbf{N}_{L/K}(L^\times))}$$

(where $p^e \,||\, \ell - 1$) and similar formulas in the sub-extensions of L/K (noting that the exact sequence and Chevalley's formula may be written in terms of φ-objects without any difficulty). The reason of such a link with genus theory is the fact that, assuming $\mathscr{F}_M = \mathscr{E}_M$ for the subfield M of L of degree p over K we know that $\mathbf{N}_{L/M}(\mathscr{F}_L) =$

$\mathscr{F}_M = \mathscr{E}_M$, so that the above exact sequence in L/M reduces to $\mathscr{H}_L^{\langle s^p\rangle} = \mathscr{C}\ell_L(\mathbf{I}_L^{\langle s^p\rangle})\otimes\mathbb{Z}_p$ and $\#\mathscr{H}_L^{\langle s^p\rangle} = \#\mathscr{H}_M \cdot p^{e\,([K:\mathbb{Q}]-1)}$.

(iii) Any "$\mathscr{G}$-family of numbers η" satisfying, in cyclic extensions L/K, relations of the form $\mathbf{N}_{L/K}(\eta_L) = \eta_K^{1-\mathrm{Frob}_{L/K}(\ell)}$ and $\eta_L \equiv \eta_K \pmod{\prod_{\mathfrak{l}|\ell}\mathfrak{l}}$, for suitable primes ℓ, is called an "Euler system" [Kol2007, PeRi1990] and gives rise to similar reasonings in many domains.

(iv) Equations of the general form $\mathbf{N}_{L/K}(y) = \mathbf{N}_{L/K}(\mathfrak{B})$, giving $(y) = \mathfrak{B}\cdot\mathfrak{A}^{s-1}$, are fundamental in various questions, as Greenberg's conjecture, in a genus theory framework (see [Gra2018, § 3, Algorithm]). Such equations are due to some $x \in K^\times$, local norm in L/K at the ℓ-places, such that $(x) = \mathbf{N}_{L/K}(\mathfrak{B})$, giving the relation $x = \mathbf{N}_{L/K}(y)$, for some unknown y (Hasse's norm theorem in L/K). In various papers (as [Gra2019^b, § 7.1]) we have discussed these random aspects by computing some ideals $\mathfrak{A}$, so that we may conjecture the following more precise property (see Schemas 7.4.1, 7.4.2, Lemma 7.11, Relations (7.3)–(7.6)).

Conjecture 7.13. *Let K be a real abelian field of conductor f, of class group of exponent N and let $\eta_K := \mathbf{N}_{\mathbb{Q}(\mu_f)/K}\big(1-\zeta_f\big)$. Consider primes $\ell \equiv 1 \pmod N$, totally split in K; let $\mathfrak{l}_0 \mid \ell$ in K and let $\boldsymbol{g}_\ell$ be a primitive root modulo ℓ. When ℓ varies, η_K provides infinitely many elements $\Omega_\ell = \sum_{\sigma\in G} r_\sigma\cdot\sigma^{-1}$, with $\eta_K^\sigma \equiv \boldsymbol{g}_\ell^{r_\sigma} \pmod{\mathfrak{l}_0}$, yielding annihilators and orders of the φ-components $\mathscr{H}_\varphi^{\mathrm{ar}}$ as $\mathbb{Z}_p[G]$-modules.*

The following program, for cyclic cubic fields, computes the invariants of $\psi(\Omega_\ell) = r_1 - r_2 - (r_1+2\,r_2)\cdot j$ only with the knowledge of η_K; taking a primitive root $\mathbf{g}_\ell$ modulo ℓ, the r_σ's come from the PARI instructions $\mathsf{r = znlog(L[j], g)}$, where the $\mathsf{L[j]}$ are the rationals a_σ such that $\eta_K^\sigma \equiv a_\sigma \pmod{\mathfrak{l}_0}$ in K (we use the results of the § 9.3 (c) to compute $\eta_K = \varepsilon_K^{\alpha+\beta\,\sigma}$ and $\mathbf{H}_K$). The line Orders of components of cl(Lell) of the form $(p^u, p^v, \cdots)$ means that the components of the p-class of $\mathfrak{l}_0$ (on the PARI system of generators of $\mathscr{H}_K$), are of orders p^u, p^v, $\cdots$; one sees that the annihilator Ω_ℓ is independent on these orders, but it is clear that, using Chebotarev's theorem, any set of components may be obtained.

```
{p=7;n=3;P=x^3+x^2-884540*x-393129;alpha=-112;beta=-70;
Q=y^2+y+1;k=bnfinit(Q);J=Mod(y,Q);pi=idealfactor(k,p);
pi1=component(pi,1)[1];pi2=component(pi,1)[2];
K=bnfinit(P,1);G=nfgaloisconj(P);CK=K.cyc;d=matsize(CK)[2];
CKp=List;for(i=1,d,h=p^valuation(CK[i],p);listput(CKp,h,i));
print("P=",P," p-class group=",CKp);
E=K.fu;E1=E[1];E2=nfgaloisapply(K,G[2],E[1]);
F1=E1^alpha*E2^beta;F2=nfgaloisapply(K,G[2],F1);F1=lift(F1);F2=lift(F2);
forprime(ell=1,5*10^5,
if(Mod(ell,p^n)!=1 || matsize(factor(P+O(ell)))[1]!=3,next);
g=znprimroot(ell);Lell=component(idealfactor(K,ell),1)[1];
F10=Mod(polcoeff(F1,0),ell);F11=Mod(polcoeff(F1,1),ell);
F12=Mod(polcoeff(F1,2),ell);Eta1=lift(F12*x^2+F11*x+F10);
F20=Mod(polcoeff(F2,0),ell);F21=Mod(polcoeff(F2,1),ell);
F22=Mod(polcoeff(F2,2),ell);Eta2=lift(F22*x^2+F21*x+F20);
Leta=List;listput(Leta,Eta1,1);listput(Leta,Eta2,2);
L=List;for(i=1,2,A=Mod(Leta[i],P);for(a=1,ell-1,v=idealval(K,A-a,Lell);
if(v>0,listput(L,a,i))));Lr=List;for(i=1,2,r=znlog(L[i],g);listput(Lr,r));
print();print("ell=",ell," Omega=",Lr);
X=Lr[1]-Lr[2]+(-Lr[1]-2*Lr[2])*J;w1=idealval(k,X,pi1);w2=idealval(k,X,pi2);
Y=alpha+beta*J;W1=idealval(k,Y,pi1);W2=idealval(k,Y,pi2);
print("Cyclotomic invariants=",W1,",",W2," Omega invariants=",w1,",",w2);
Exp=List;Order=bnfisprincipal(K,Lell)[1];for(i=1,d,tp=valuation(CK[i],p);
if(Order[i]==0,Or=1);if(Order[i]!=0,t=valuation(Order[i],p);Or=p^(tp-t));
listput(Exp,Or));print("Orders of components of cl(Lell)=",Exp))}

P=x^3+x^2-884540*x-393129 p-class group=List([343,7])
ell=1373 Omega=List([1162, 1246])
Cyclotomic invariants=1,3 Omega invariants=1,3
Orders of components of cl(Lell)=List([343, 7])

ell=7547 Omega=List([6888, 1526])
```

```
Cyclotomic invariants=1,3 Omega invariants=1,3
Orders of components of cl(Lell)=List([343, 7])

ell=8233 Omega=List([6496, 742])
Cyclotomic invariants=1,3 Omega invariants=1,3
Orders of components of cl(Lell)=List([49, 7])

ell=18523 Omega=List([11830, 12586])
Cyclotomic invariants=1,3 Omega invariants=1,3
Orders of components of cl(Lell)=List([343, 1])

ell=22639 Omega=List([4004, 13104])
Cyclotomic invariants=1,3 Omega invariants=1,3
Orders of components of cl(Lell)=List([343, 7])

ell=30871 Omega=List([27734, 5390])
Cyclotomic invariants=1,3 Omega invariants=2,3
Orders of components of cl(Lell)=List([343, 1])

ell=39103 Omega=List([32018, 35812])
Cyclotomic invariants=1,3 Omega invariants=1,3
Orders of components of cl(Lell)=List([49, 7])

ell=42533 Omega=List([1330, 17262])
Cyclotomic invariants=1,3 Omega invariants=1,3
Orders of components of cl(Lell)=List([343, 7])

ell=54881 Omega=List([44366, 18662])
Cyclotomic invariants=1,3 Omega invariants=1,3
Orders of components of cl(Lell)=List([49, 7])

ell=58997 Omega=List([5236, 21938])
Cyclotomic invariants=1,3 Omega invariants=1,3
Orders of components of cl(Lell)=List([343, 7])

ell=72031 Omega=List([24276, 51884])
Cyclotomic invariants=1,3 Omega invariants=1,3
Orders of components of cl(Lell)=List([343, 7])

ell=76147 Omega=List([17066, 25606])
Cyclotomic invariants=1,3 Omega invariants=1,3
Orders of components of cl(Lell)=List([343, 7])

ell=80263 Omega=List([22036, 79352])
Cyclotomic invariants=1,3 Omega invariants=1,3
Orders of components of cl(Lell)=List([343, 7])

ell=93983 Omega=List([69174, 5558])
Cyclotomic invariants=1,3 Omega invariants=1,3
Orders of components of cl(Lell)=List([343, 7])
```

For $P = x^3 - 4792107\,x + 4022175142$ (conductor $f = 3^2 \cdot 1597369$, $\alpha = -7$, $\beta = -21$, the φ-components of the 7-class group $\mathscr{H}_K$ are $\mathscr{H}_{\varphi_1} \simeq \mathbb{Z}_7[j]/\mathfrak{p}_{\varphi_1} \oplus \mathbb{Z}_7[j]/\mathfrak{p}_{\varphi_1}$ and $\mathscr{H}_{\varphi_2} \simeq \mathbb{Z}_7[j]/\mathfrak{p}_{\varphi_2}$; nevertheless, we have $\widetilde{\mathscr{E}}_{\varphi_1} \simeq \mathbb{Z}_7[j]/\mathfrak{p}_{\varphi_1}^2$ (non-isomorphic to $\mathscr{H}_{\varphi_1}$) and $\widetilde{\mathscr{E}}_{\varphi_2} \simeq \mathbb{Z}_7[j]/\mathfrak{p}_{\varphi_2}$.

But almost all Ω_ℓ give the expected response $(2, 1)$ whatever the order of the p-class of $\mathfrak{l}_0 \mid \ell$:

```
P=x^3 - 4792107*x + 4022175142 p-class group=List([7,7,7])

ell=1373 Omega=List([917, 1267])
Cyclotomic invariants=2,1 Omega invariants=2,1
Orders of components of cl(Lell)=List([7, 7, 7])

ell=8233 Omega=List([1141, 3535])
Cyclotomic invariants=2,1 Omega invariants=2,1
Orders of components of cl(Lell)=List([7, 1, 7])
```

```
ell=49393 Omega=List([41069, 39277])
Cyclotomic invariants=2,1 Omega invariants=2,1
Orders of components of cl(Lell)=List([1, 7, 1])

ell=54881 Omega=List([14357, 31311])
Cyclotomic invariants=2,1 Omega invariants=2,2
Orders of components of cl(Lell)=List([7, 7, 7])

ell=63799 Omega=List([53977, 53767])
Cyclotomic invariants=2,1 Omega invariants=2,1
Orders of components of cl(Lell)=List([7, 7, 7])

ell=76147 Omega=List([44912, 73514])
Cyclotomic invariants=2,1 Omega invariants=2,1
Orders of components of cl(Lell)=List([1, 7, 7])

ell=80263 Omega=List([20328, 16387])
Cyclotomic invariants=2,1 Omega invariants=3,1
Orders of components of cl(Lell)=List([1, 7, 7])
               (...)
ell=329281 Omega=List([311136, 189770])
Cyclotomic invariants=2,1 Omega invariants=2,1
Orders of components of cl(Lell)=List([7, 7, 7])

ell=331339 Omega=List([157696, 276465])
Cyclotomic invariants=2,1 Omega invariants=2,1
Orders of components of cl(Lell)=List([7, 7, 7])

ell=343687 Omega=List([174391, 82173])
Cyclotomic invariants=2,1 Omega invariants=2,2
Orders of components of cl(Lell)=List([7, 7, 7])

ell=363581 Omega=List([204974, 276584])
Cyclotomic invariants=2,1 Omega invariants=2,1
Orders of components of cl(Lell)=List([7, 7, 7])

ell=384847 Omega=List([254100, 68887])
Cyclotomic invariants=2,1 Omega invariants=2,1
Orders of components of cl(Lell)=List([7, 7, 7])

ell=396509 Omega=List([114947, 1540])
Cyclotomic invariants=2,1 Omega invariants=2,1
Orders of components of cl(Lell)=List([7, 7, 7])

ell=403369 Omega=List([11361, 206458])
Cyclotomic invariants=2,1 Omega invariants=2,1
Orders of components of cl(Lell)=List([7, 7, 7])

ell=408857 Omega=List([364287, 259343])
Cyclotomic invariants=2,1 Omega invariants=5,1
Orders of components of cl(Lell)=List([7, 7, 1])

ell=415717 Omega=List([239225, 363657])
Cyclotomic invariants=2,1 Omega invariants=2,1
Orders of components of cl(Lell)=List([7, 1, 7])

ell=417089 Omega=List([327908, 33957])
Cyclotomic invariants=2,1 Omega invariants=3,4
Orders of components of cl(Lell)=List([1, 7, 7])

ell=419147 Omega=List([17059, 339451])
Cyclotomic invariants=2,1 Omega invariants=2,1
Orders of components of cl(Lell)=List([1, 1, 1])

ell=426007 Omega=List([161434, 215859])
Cyclotomic invariants=2,1 Omega invariants=2,1
Orders of components of cl(Lell)=List([7, 7, 7])
```

```
ell=456877 Omega=List([361697, 10010])
Cyclotomic invariants=2,1 Omega invariants=3,1
Orders of components of cl(Lell)=List([7, 7, 7])
```

For $\ell = 419147$ (first example), any prime ideal $\mathfrak{l} \mid \ell$ is principal:

```
bnfisprincipal(K,Lell)=[[0,0,0]~,
                [1311001361541054679,35057663364174,1019317530188062]~]
```

but the invariants of Ω_ℓ are still $(2,1)$ giving $\#\mathscr{H}_{\varphi_1} = 7^2$ and $\#\mathscr{H}_{\varphi_2} = 7$.

The numerical experiments are particularly remarkable and confirm that the Ω_ℓ's define an universal Ω_K which replaces, in the real case, the Stickelberger element of the imaginary case.

For this, we notice that the embeddings (injectivity from [Gra2005, Theorem III.4.4]) of $\mathscr{F}_K$ and $\mathscr{E}_K$ in the direct product $\prod_{v\nmid p}(F_v^\times \otimes \mathbb{Z}_p)$ (see Schemas 7.4.1, 7.4.2) govern the congruences (7.6) giving the relations Ω_ℓ involving only $\mathbf{F}_K$, without any memory of the arithmetic of the auxiliary fields $\mathbb{Q}(\mu_\ell)$. Then, the Schmidt–Chevalley theorem (local–global principle for powers, e.g., [Gra2005, § 6.3, Theorem II.6.3.3]) claims that there are infinitely many primes ℓ (totally split in K) giving the "good" Ω_K.

From Lemma 7.10 giving standard structure of $\mathscr{E}_\varphi$ and $\mathscr{F}_\varphi$, it is then obvious that one obtains equalities of the φ-invariants of $\mathscr{E}_\varphi/\mathscr{F}_\varphi$ and $\mathscr{H}_\varphi$ in the semi-simple case. Are there improvements of these techniques being able to distinguish, for instance, the structures $\mathbb{Z}_p[\mu_{g_\chi}]/\mathfrak{p}_\varphi \oplus \mathbb{Z}_p[\mu_{g_\chi}]/\mathfrak{p}_\varphi$ and $\mathbb{Z}_p[\mu_{g_\chi}]/\mathfrak{p}_\varphi^2$?

To day, it would remain to investigate further the context of real abelian fields K such that $\mathrm{Gal}(K/\mathbb{Q}) \otimes \mathbb{Z}_p \neq 1$ (non semi-simple case with the definition of the Arithmetic φ-components of the p-class group).

8. Invariants (Algebraic, Arithmetic, Analytic)

We fix an irreducible rational character $\chi \in \mathscr{X} = \mathscr{X}^+ \cup \mathscr{X}^-$ and we apply the previous results to the $\mathbb{Z}_p[\mu_{g_\chi}]$-modules $\mathscr{H}_\varphi^{\mathrm{alg}}$, $\mathscr{H}_\varphi^{\mathrm{ar}}$ and $\mathscr{T}_\varphi^{\mathrm{ar}} = \mathscr{T}_\varphi^{\mathrm{alg}} =: \mathscr{T}_\varphi$, for any $\varphi \mid \chi$, $\varphi \in \Phi^+ \cup \Phi^-$ ($\varphi \in \Phi^+$ for $\mathscr{T}_\varphi$).

8.1. **Algebraic and Arithmetic Invariants $m^{\mathrm{alg}}(\mathscr{M})$, $m^{\mathrm{ar}}(\mathscr{M})$.** Write simply that $\mathscr{H}_\varphi^{\mathrm{alg}}$, $\mathscr{H}_\varphi^{\mathrm{ar}}$ and $\mathscr{T}_\varphi$ are finite $\mathbb{Z}_p[\mu_{g_\chi}]$-modules whatever φ; let $\mathfrak{p}_\varphi$ be the maximal ideal of $\mathbb{Z}_p[\mu_{g_\chi}]$:

$$\begin{cases} \mathscr{H}_\varphi^{\mathrm{alg}} \simeq \prod_{i\geq 1} \mathbb{Z}_p[\mu_{g_\chi}]/\mathfrak{p}_\varphi^{n_{\varphi,i}^{\mathrm{alg}}(\mathscr{H})}, \\ \mathscr{H}_\varphi^{\mathrm{ar}} \simeq \prod_{i\geq 1} \mathbb{Z}_p[\mu_{g_\chi}]/\mathfrak{p}_\varphi^{n_{\varphi,i}^{\mathrm{ar}}(\mathscr{H})}, \\ \mathscr{T}_\varphi \simeq \prod_{i\geq 1} \mathbb{Z}_p[\mu_{g_\chi}]/\mathfrak{p}_\varphi^{n_{\varphi,i}^{\mathrm{ar}}(\mathscr{T})}, \end{cases}$$

where the $n_{\varphi,i}$ are decreasing integers up to 0. Put:

$$\begin{cases} m_\varphi^{\mathrm{alg}}(\mathscr{H}) := \sum_{i\geq 1} n_{\varphi,i}^{\mathrm{alg}}(\mathscr{H}), & m_\chi^{\mathrm{alg}}(\mathscr{H}) := \sum_{\varphi\mid\chi} m_\varphi^{\mathrm{alg}}(\mathscr{H}), \\ m_\varphi^{\mathrm{ar}}(\mathscr{H}) := \sum_{i\geq 1} n_{\varphi,i}^{\mathrm{ar}}(\mathscr{H}), & m_\chi^{\mathrm{ar}}(\mathscr{H}) := \sum_{\varphi\mid\chi} m_\varphi^{\mathrm{ar}}(\mathscr{H}), \\ m_\varphi^{\mathrm{ar}}(\mathscr{T}) := \sum_{i\geq 1} n_{\varphi,i}^{\mathrm{ar}}(\mathscr{T}), & m_\chi^{\mathrm{ar}}(\mathscr{T}) := \sum_{\varphi\mid\chi} m_\varphi^{\mathrm{ar}}(\mathscr{T}). \end{cases}$$

Whence the order formulas:

$$\#\mathscr{H}_\varphi^{\mathrm{alg}} = p^{\varphi(1)\, m_\varphi^{\mathrm{alg}}(\mathscr{H})}, \quad \#\mathscr{H}_\varphi^{\mathrm{ar}} = p^{\varphi(1)\, m_\varphi^{\mathrm{ar}}(\mathscr{H})}, \quad \#\mathscr{T}_\varphi = p^{\varphi(1)\, m_\varphi^{\mathrm{ar}}(\mathscr{T})}.$$

8.2. **Analytic Invariants $m^{\mathrm{an}}(\mathscr{M})$.** We define, in view of the statement of the Main Conjecture, the following Analytic Invariants m_φ^{an}, from the expressions given with rational characters, where $\mathrm{val}_p(\bullet)$ denotes the usual p-adic valuation; the purpose is to satisfy the necessary relations implied by Theorems 3.15, 4.1 about arithmetic components:

$$\sum_{\varphi\mid\chi} m_\varphi^{\mathrm{ar}}(\mathscr{M}) = \sum_{\varphi\mid\chi} m_\varphi^{\mathrm{an}}(\mathscr{M}),$$

for any family $\mathscr{M} \in \{\mathscr{H}, \mathscr{T}\}$ and $\chi \in \mathscr{X}$ (cf. Theorems 5.10, 7.5, 6.2).

8.2.1. *Case $\varphi\in\Phi^-$ for class groups.* Then, Algebraic and Arithmetic Invariants coincide. The definitions given in [Gra1976, Gra1977] were:

(i) Case $p\neq 2$ (proven by Solomon [Sol1990, Theorem II.1]).

(i′) K_χ is not of the form $\mathbb{Q}(\mu_{p^n})$, $n\geq 1$; then:

- $m^{\rm an}_\varphi(\mathscr{H}^-):=\mathrm{val}_p\Big(\prod\limits_{\psi\vert\varphi}\big(-\frac{1}{2}\mathbf{B}_1(\psi^{-1})\big)\Big)$,

(i″) $K_\chi=\mathbb{Q}(\mu_{p^n})$, $n\geq 1$; let $\psi=\omega^\lambda\cdot\psi_p$, ψ_p of order p^{n-1} (where ω is the Teichmüller character); then:

- $m^{\rm an}_\varphi(\mathscr{H}^-):=\mathrm{val}_p\Big(\prod\limits_{\psi\vert\varphi}\big(-\frac{1}{2}\mathbf{B}_1(\psi^{-1})\big)\Big)$, if $\lambda\neq 1$,
- $m^{\rm an}_\varphi(\mathscr{H}^-):=0$, if $\lambda=1$.

(ii) Case $p=2$ (proven by Greither [Grei1992, Theorem B], when g_χ is not a 2-power and f_χ odd).

(ii′) g_χ is not a 2-power; then:

- $m^{\rm an}_\varphi(\mathscr{H}^-):=\mathrm{val}_2\Big(\prod\limits_{\psi\vert\varphi}\big(-\frac{1}{2}\mathbf{B}_1(\psi^{-1})\big)\Big)$.

(ii″) g_χ is a 2-power; then:

- $m^{\rm an}_\varphi(\mathscr{H}^-):=\mathrm{val}_2\Big(\prod\limits_{\psi\vert\varphi}\big(-\frac{1}{2}\mathbf{B}_1(\psi^{-1})\big)\Big)+1$, if $K_\chi\neq\mathbb{Q}(\mu_4)$,
- $m^{\rm an}_\varphi(\mathscr{H}^-):=0$, if $K_\chi=\mathbb{Q}(\mu_4)$.

8.2.2. *Case $\varphi\in\Phi^+$, $\varphi\neq 1$, for class groups.* From Definition 7.3 and Theorem 7.5, we consider any real cyclic field K, where we recall that:
$\mathbf{E}^0_K:=\langle\,\mathbf{E}_k\,\rangle_{k\subsetneq K}$, $\mathbf{F}_K:=\mathbf{C}_K\cap\mathbf{E}_K$, $\mathscr{E}_K:=\mathbf{E}_K\otimes\mathbb{Z}_p$, $\mathscr{E}^0_K:=\mathbf{E}^0_K\otimes\mathbb{Z}_p$, $\mathscr{F}_K:=\mathbf{F}_K\otimes\mathbb{Z}_p$, and $\widetilde{\mathscr{E}}_\chi:=\mathscr{E}_{K_\chi}/\mathscr{E}^0_{K_\chi}\cdot\mathscr{F}_{K_\chi}$, for which we have $\widetilde{\mathscr{E}}_\chi=\bigoplus\limits_{\varphi\vert\chi}\widetilde{\mathscr{E}}_\varphi$, where $\widetilde{\mathscr{E}}_\varphi=\{\widetilde{x}\in\widetilde{\mathscr{E}}_\chi,\ P_\varphi(\sigma_\chi)\cdot\widetilde{x}=1\}$.

Consider the relation $\#\mathscr{H}^{\rm ar}_\chi=w_\chi\cdot\big(\mathscr{E}_{K_\chi}:\mathscr{E}^0_{K_\chi}\cdot\mathscr{F}_{K_\chi}\big)=w_\chi\cdot\prod_{\varphi\vert\chi}\#\widetilde{\mathscr{E}}_\varphi$ of Theorem 7.5; we remark that $w_\chi=p$ occurs only when g_χ is a p-power, in which case p is totally ramified in $\mathbb{Q}(\mu_{g_\chi})$ and $\varphi=\chi$ (which defines $w_\varphi:=w_\chi$). So, we may define $m^{\rm an}_\varphi(\mathscr{H}^+)$ and w_φ as follows from $\widetilde{\mathscr{E}}_\varphi\simeq\mathbb{Z}_p[\mu_{g_\chi}]/\mathfrak{p}_\varphi^{m^{\rm an}_\varphi(\mathscr{H}^+)}$, $m^{\rm an}_\varphi(\mathscr{H}^+)\geq 0$:

(i) Case g_χ non prime power. Then $w_\varphi=1$ and:

- $m^{\rm an}_\varphi(\mathscr{H}^+):=\mathrm{val}_p(\#\widetilde{\mathscr{E}}_\varphi)$.

(ii) Case $g_\chi=p^n$, $p\neq 2$ prime, $n\geq 1$:

(ii′) Case $f_\chi=\ell^k$, ℓ prime, $k\geq 1$. Then $w_\varphi=1$ and :

- $m^{\rm an}_\varphi(\mathscr{H}^+):=\mathrm{val}_p(\#\widetilde{\mathscr{E}}_\varphi)$,

(ii″) Case f_χ non prime power. Then $w_\varphi=p$ and

- $m^{\rm an}_\varphi(\mathscr{H}^+):=\mathrm{val}_p(\#\widetilde{\mathscr{E}}_\varphi)+1$.

(iii) Case $g_\chi=2^n$, $n\geq 1$:

(iii′) Case $f_\chi=\ell^k$, ℓ prime, $k\geq 1$. Then $w_\varphi=1$ and:

- $m^{\rm an}_\varphi(\mathscr{H}^+):=\mathrm{val}_p(\#\widetilde{\mathscr{E}}_\varphi)$,

(iii″) Case f_χ non prime power. Then $w_\varphi\in\{1,2\}$ and:

- $m^{\rm an}_\varphi(\mathscr{H}^+)\in\{\mathrm{val}_p(\#\widetilde{\mathscr{E}}_\varphi),\mathrm{val}_p(\#\widetilde{\mathscr{E}}_\varphi)+1\}$.

8.2.3. *Case $\varphi\in\Phi^+$ for p-torsion groups.* From Theorem 6.2, we define $m^{\rm an}_\varphi(\mathscr{T})$ as follows (proven by Greither [Grei1992, Theorem C], when g_χ is not a 2-power):

(i) Case where g_χ and f_χ are not p-powers. Then:

- $m^{\rm an}_\varphi(\mathscr{T}):=\mathrm{val}_p\Big(\prod_{\psi\vert\varphi}\frac{1}{2}\,\mathbf{L}_p(1,\psi)\Big)$.

(ii) Case where $g_\chi\neq 1$ and f_χ are p-powers. Then:

- $m^{\rm an}_\varphi(\mathscr{T}):=\mathrm{val}_p\Big(\prod_{\psi\vert\varphi}\frac{1}{2}\,\mathbf{L}_p(1,\psi)\Big)+1$.

8.3. **Main Conjecture – 1976 original statement.** The conjecture we gave in [Gra1976, Gra1977], especially in the non semi-simple case, where simply equality of Arithmetic and Analytic φ-Invariants. The main justification of such equalities comes from the easy Theorem 2.1 with the arithmetic definitions of § 8.1, the analytic definitions of § 8.2 and the arithmetic expressions of the χ-components that we recall:

(i) Theorem 5.10: $\mathbf{H}^{\rm ar}_\chi = 2^{\alpha_\chi}\cdot w_\chi \cdot \prod_{\psi\vert\chi}\big(-\frac{1}{2}\mathbf{B}_1(\psi^{-1})\big)$, for $\chi\in\mathscr{X}^-$,

(ii) Theorem 6.2: $\#\mathscr{T}_\chi = w^{\rm cyc}_\chi\cdot\prod_{\psi\vert\chi}\frac{1}{2}\,\mathbf{L}_p(1,\psi)$, for $\chi\in\mathscr{X}^+$,

(iii) Theorem 7.5: $\#\mathbf{H}^{\rm ar}_\chi = w_\chi\cdot(\mathbf{E}_{K_\chi}:\mathbf{E}^0_{K_\chi}\cdot\mathbf{F}_{K_\chi})$, for $\chi\in\mathscr{X}^+$;

they satisfy, for any family $\mathscr{M}\in\{\mathscr{H}^-,\mathscr{H}^+,\mathscr{T}\}$, the equalities:

- $\sum_{\varphi\vert\chi} m^{\rm ar}_\varphi(\mathscr{M}) = \sum_{\varphi\vert\chi} m^{\rm an}_\varphi(\mathscr{M})$, for all $\chi\in\mathscr{X}$,

taking into account the decomposition $\mathscr{M}^{\rm ar}_\chi = \oplus_{\varphi\vert\chi}\mathscr{M}^{\rm ar}_\varphi$ (Theorem 4.5).

Moreover, the annihilation properties of Theorems 5.18, 5.19, 5.21, 6.2, enforce the conjecture as well as reflection theorems that were given, after the Leopoldt's Spiegelungsatz, in [Gra1998] or [Gra2005, Theorem II.5.4.5] giving a more suitable comparison, for instance between $\mathscr{H}_\varphi$ and $\mathscr{T}_{\omega\varphi^{-1}}$, $\varphi\in\Phi^-$, where ω is the Teichmüller character. See also [Or1981, Or1986] for similar informations and complements.

Conjecture 8.1. *For any p-adic irreducible character $\varphi\in\Phi$, we have:*

$$\begin{cases} m^{\rm ar}_\varphi(\mathscr{H}) = m^{\rm an}_\varphi(\mathscr{H}) \ (\varphi\in\Phi^+\cup\Phi^-),\\ m^{\rm ar}_\varphi(\mathscr{T}) = m^{\rm an}_\varphi(\mathscr{T}) \quad (\varphi\in\Phi^+). \end{cases}$$

Remark 8.2. Let $K/\mathbb{Q}$ with a maximal p-sub-extension K/K_0 cyclic of degree p^n, $n\geq 1$, and let K_i, $K_0\subseteq K_i\subset K$, be such that $[K_i:K_0]=p^i$. Let $\psi_0\in\Psi_{K_0}$ and let $\psi_p\in\Psi_K$ of order p^n; we put $\psi_i=\psi_0\cdot\psi_p^{p^{n-i}}\in\Psi_{K_i}$ and we consider the p-adic characters φ_i above ψ_i, $0\leq i\leq n$.

The Main Conjecture proven by Greither in [Grei1992, Theorem 4.14, Corollary 4.15], using Sinnott's cyclotomic units, deals with the semi-simple context defined by φ_0 above ψ_0 (it is indeed that of the relations (3.4) which do not give each $\#\mathscr{H}^{\rm ar}_{\varphi_i}$ compared with $\#\widetilde{\mathscr{E}}_{\varphi_i}$).

In other words, in his pioneering work, Greither proves the relation $\sum_{i=0}^{n} m^{\rm ar}_{\varphi_i}(\mathscr{H}^+) = \sum_{i=0}^{n} m^{\rm an}_{\varphi_i}(\mathscr{H}^+)$, for each $\varphi_0\in\Phi_{K_0}$, instead of our conjecture $m^{\rm ar}_{\varphi_i}(\mathscr{H}^+) = m^{\rm an}_{\varphi_i}(\mathscr{H}^+)$ for all $i\in\{0,1,\cdots,n\}$. However see many progress by Greither–Kučera in [GrKu2004, GrKu2014] and some of their other papers.

Remark 8.3. It remains the problem of $\#\mathscr{H}^{\rm alg}_\chi$ and $\#\mathscr{H}^{\rm alg}_\varphi$, for which no analytic formula does exist in the non semi-simple real case. For instance, in Example 3.12 with $p=3$, K is the compositum of $k_0=\mathbb{Q}(\sqrt{4409})$ with the degree 9 field of conductor 19, $\chi_i=\varphi_i$ ($i\in\{1,2\}$) is the character of the field k_i of degree $2\cdot 3^i$; then one gets $\mathscr{H}^{\rm alg}_{\chi_i}\simeq\mathbb{Z}/3\mathbb{Z}$ while $\mathscr{H}^{\rm ar}_{\chi_i}=1$, as predicted by the conjecture and checked numerically. In Example 3.13, one finds $\mathscr{H}^{\rm alg}_{\chi_1}\simeq(\mathbb{Z}/3\mathbb{Z})^3$ while $\mathscr{H}^{\rm ar}_{\chi_1}\simeq(\mathbb{Z}/3\mathbb{Z})^2$.

This phenomenon is due to the capitulation of p-classes in p-extensions and we have given in [Gra2021^b, Conjecture 4.1] a general conjecture justified by means of many computations.

8.4. **Finite Iwasawa's theory in cyclic p-extensions.** For more details and an application to classical Iwasawa's theory for the cyclotomic $\mathbb{Z}_p$-extensions, see [Gra1976, Chap. IV] (the real case being in the spirit of Greenberg's conjecture [Gree1976]); nevertheless, *the results hold in arbitrary totally ramified cyclic p-extensions* of an abelian field, as follows depending of a base field real or imaginary:

8.4.1. *Real case.* Let $\psi \mid \varphi \mid \chi \in \mathscr{X}^+$ and set $\psi = \psi_0 \cdot \psi_p$, where ψ_0 is of order g_0, prime to p, and ψ_p of p-power order; then, $G_\chi = G_0 \oplus H$ in an obvious meaning. We consider, temporarily, the semi-simple idempotents $e_{\varphi_0} := \frac{1}{g_0} \sum_{\sigma \in G_0} \varphi_0(\sigma^{-1})\, \sigma$, for φ_0 above ψ_0. We have:

$$\widetilde{\mathscr{E}}_\chi := \mathscr{E}_{K_\chi}/\mathscr{E}^0_{K_\chi} \cdot \mathscr{F}_{K_\chi} = \bigoplus_{\varphi \mid \chi} \widetilde{\mathscr{E}}_\varphi,$$

with $\widetilde{\mathscr{E}}_\varphi = \widetilde{\mathscr{E}}_\chi^{e_{\varphi_0}}$; we note that $\mathscr{E}^{0\, e_{\varphi_0}}_{K_\chi} \simeq \mathscr{E}_{\varphi'}$ and $\widetilde{\mathscr{E}}_\varphi \simeq \mathscr{E}^{e_{\varphi_0}}_{K_\chi}/\mathscr{E}_{\varphi'} \cdot \mathscr{F}^{e_{\varphi_0}}_{K_\chi}$, where φ' is above $\psi_0 \cdot \psi_p^p$ and χ' above φ'. This yields $(\mathscr{E}_{K_\chi}/\mathscr{E}_{K_{\chi'}})_\varphi \simeq \mathbb{Z}_p[\mu_{g_\chi}]$ ([Gra1976, Lemma IV.1]) and the following principle taking place in the layers of any p-tower K_N/K_0, of degree p^N over an abelian field K_0, totally ramified at a set $S \neq \emptyset$ of finite places of K_0 [Gra1976, Proposition IV.1]:

Theorem 8.4. *Let $\chi \in \mathscr{X}^+$ be such that $g_\chi = g_0 \cdot p^n$, $p \nmid g_0$, $n \geq 2$. Let χ', χ'' be such that $[K_\chi : K_{\chi'}] = [K_{\chi'} : K_{\chi''}] = p$. To simplify, set $K := K_\chi$, $K' := K_{\chi'}$, $K'' := K_{\chi''}$ and assume that $\mathbf{N}_{K/K'}(\mathscr{F}_K) = \mathscr{F}_{K'}$ (see Lemma 5.17 giving the ramification conditions). Let $\mathfrak{p}_\varphi$ be the maximal ideal of $\mathbb{Z}_p[\mu_{g_\chi}]$; put $(\mathscr{F}_K/\mathscr{F}_K \cap \mathscr{E}_{K'})_\varphi \simeq \mathfrak{p}_\varphi^A$, $A \geq 0$ and, in the isomorphism $(\mathscr{E}_{K'}/\mathscr{E}_{K''})_{\varphi'} \simeq \mathbb{Z}_p[\mu_{g_\chi/p}]$, put:*

$$(\mathscr{F}_{K'}/\mathscr{F}_{K'} \cap \mathscr{E}_{K''})_{\varphi'} \simeq \mathfrak{p}_{\varphi'}^a \simeq \mathfrak{p}_\varphi^{pa},\ a \geq 0,$$

$$(\mathbf{N}_{K/K'}(\mathscr{E}_K)/\mathbf{N}_{K/K'}(\mathscr{E}_K) \cap \mathscr{E}_{K''})_{\varphi'} \simeq \mathfrak{p}_{\varphi'}^b \simeq \mathfrak{p}_\varphi^{p\, b},\ b \geq 0.$$

(i) If $a < p^{n-2}(p-1)$, then $A = a - b$.
(ii) If $a \geq p^{n-2}(p-1)$, then $A \geq p^{n-2}(p-1) - b$.

This allows to prove again Iwasawa's formula in the case $\mu = 0$ [Gra1976, Theorems IV.1, IV.2, Remark IV.4] and gives an analytic algorithm to study the p-class groups in the first layers.

Let $k =: k_0$ be real of prime-to-p degree g and let $k^{\mathrm{cyc}} = \bigcup_{n \geq 0} k_n$ be its cyclotomic $\mathbb{Z}_p$-extension. The condition $\mu = 0$ of Iwasawa's theory is here equivalent to the existence of $n_0 \gg 0$ (corresponding to a character χ_{n_0} of order $g\, p^{n_0}$) such that, for each φ_{n_0}-component, $a_{n_0-1} < p^{n_0-2}(p-1)$ (case (i) of Theorem 8.4); then the sequence $\#\mathscr{H}_{\chi_n}$ becomes constant giving the λ-invariant and the relations $\mathscr{E}_{k_{n-1}} = \mathbf{N}_{k_n/k_{n-1}}(\mathscr{E}_{k_n}) \cdot \mathscr{E}_{k_{n-2}}$, for all $n \gg 0$; then $p^\lambda = (\mathscr{E}_{k_n} : \mathscr{E}^0_{k_n} \cdot \mathscr{F}_{k_n})$ for $n \gg 0$. More precisely:

$$p^{\lambda_\varphi} = \#(\mathscr{E}_{k_n}/\mathscr{E}_{k_{n-1}} \cdot \mathscr{F}_{k_n})_\varphi,\ n \gg 0.$$

This methodology does exist in terms of p-adic **L**-functions for abelian fields (see, e.g., [Gra1978^b, Chapitre V]).

Recall that Greenberg's conjecture [Gree1976] for a totally real base field (i.e., $\lambda = \mu = 0$) is equivalent to the property that the norms $\mathbf{N}_{k_m/k_n} : \mathscr{H}_{k_m} \to \mathscr{H}_{k_n}$, $m \geq n \gg 0$ are isomorphisms (see other equivalent conditions in [Gra2019, Corollary 3.4]). Whence the result:

Theorem 8.5. *Let k be a real abelian field of prime-to-p degree. Greenberg's conjecture is equivalent to $\mathscr{E}_{k_n} = \mathscr{E}^0_{k_n} \cdot \mathscr{F}_{k_n}$, for all $n \gg 0$, where $\mathscr{E}^0_{k_n}$ is the subgroup of $\mathscr{E}_{k_n}$ generated by the units of the strict subfields and $\mathscr{F}_{k_n}$ is the group of Leopoldt cyclotomic units (Definitions 7.1 (i), 7.3).*

8.4.2. *Imaginary case.* This part is related to relative p-class groups for $p \neq 2$ [Gra1976, Proposition IV.2, Théorème IV.2]:

Theorem 8.6. *Let $\chi \in \mathscr{X}^-$ be such that $g_\chi = g_0 \cdot p^n$, $p \nmid g_0$, $n \geq 2$. Let χ' be such that $[K_\chi : K_{\chi'}] = p$. Set $K := K_\chi$, $K' := K_{\chi'}$ and assume that the Stickelberger elements $\mathbf{B}_K$, $\mathbf{B}_{K'}$ are p-integers in $\mathbb{Q}[G_K]$ and that $\mathbf{N}_{K/K'}(\mathbf{B}_K) = \mathbf{B}_{K'}$ (see Lemma 5.17). Put:*

$$\mathbf{B}_1(\psi^{-1})\mathbb{Z}_p[\mu_{g_\chi}] = \mathfrak{p}_\varphi^A,\ A \geq 0,$$

$$\mathbf{B}_1(\psi^{-p})\mathbb{Z}_p[\mu_{g_\chi/p}] = \mathfrak{p}_\varphi^{p\, a},\ a \geq 0.$$

(i) If $a < p^{n-2}(p-1)$, then $A = a$.
(ii) If $a \geq p^{n-2}(p-1)$, then $A \geq p^{n-2}(p-1)$.

Remark 8.7. The integers A and a are the Analytic Invariants $m_{\varphi}^{\mathrm{an}}(\mathscr{H}^{-})$ and $m_{\varphi'}^{\mathrm{an}}(\mathscr{H}^{-})$, respectively, defined § 8.2. From [Gra1976, Remark IV.4], the Iwasawa μ-invariant is zero as soon as there exists $n_0 \gg 0$ such that the case (i) of the theorem is satisfied for all φ of K_{n_0}. In a $\mathbb{Z}_p$-extension $\widetilde{k}/k$, this condition implies that the p-rank of the $\mathscr{H}_{k_n}^{\mathrm{ar}}$'s is bounded (a known result of Iwasawa's theory [Was1997, Proposition 13.23]).

9. Numerical illustrations with cyclic cubic fields

For $\chi \in \mathscr{X}^{+}$ and $\widetilde{\mathscr{E}}_{\chi} := \mathscr{E}_{K_\chi}/\mathscr{E}_{K_\chi}^{0} \cdot \mathscr{F}_{K_\chi}$, we have $\#\mathscr{H}_{\chi}^{\mathrm{ar}} = w_\chi \cdot \#\widetilde{\mathscr{E}}_{\chi}$ (Theorem 7.5), and for any $\varphi \mid \chi$ we have (conjecturally):

$$\#\mathscr{H}_{\varphi}^{\mathrm{ar}} = w_\varphi \cdot \#\widetilde{\mathscr{E}}_{\varphi},\ w_\varphi \in \{1, p\},\ \text{where } \widetilde{\mathscr{E}}_{\varphi} = \{\widetilde{x} \in \widetilde{\mathscr{E}}_{\chi},\ P_\varphi(\sigma_\chi) \cdot \widetilde{x} = 1\}.$$

In another way, we have:

$$\begin{cases} \widetilde{\mathscr{E}}_{\varphi} \simeq \mathbb{Z}_p[\mu_{g_\chi}]/\mathfrak{p}_{\varphi}^{m_{\varphi}^{\mathrm{an}}(\mathscr{H})}, & m_{\varphi}^{\mathrm{an}}(\mathscr{H}) \geq 0, \\ \mathscr{H}_{\varphi}^{\mathrm{ar}} \simeq \bigoplus\limits_{i=1}^{r_\varphi} \mathbb{Z}_p[\mu_{g_\chi}]/\mathfrak{p}_{\varphi}^{m_{\varphi,i}^{\mathrm{ar}}(\mathscr{H})}, & r_\varphi \geq 0,\quad m_{\varphi,i}^{\mathrm{ar}}(\mathscr{H}) \geq 0, \end{cases}$$

and $m_{\varphi}^{\mathrm{an}}(\mathscr{H}) := \sum_{i=1}^{r_\varphi} m_{\varphi,i}^{\mathrm{ar}}(\mathscr{H})$ to be compared with $m_{\varphi}^{\mathrm{ar}}(\mathscr{H})$.

We intend to see more precisely what happens for these analytic and arithmetic invariants since the above equality defining $m_{\varphi}^{\mathrm{an}}(\mathscr{H})$ can be fulfilled in various ways (indeed, $\widetilde{\mathscr{E}}_{\varphi}$ is cyclic and $\mathscr{H}_{\varphi}$ may have arbitrary structure). We will examine the case of the cyclic cubic fields $K = K_\chi$ for primes $p \equiv 1 \pmod 3$ giving two p-adic characters $\varphi \mid \chi$; in that case, $\mathscr{E}_K^0 = 1$ and $\#\mathscr{H}_{\varphi}^{\mathrm{ar}} = (\mathscr{E}_K : \mathscr{F}_K)$.

For example, for $p = 7$, the possible structures, for the $\mathbb{Z}[j]$-module $\mathbf{E}_K/\mathbf{F}_K$, are of the form $\mathbb{Z}[j]/\big[(-2+j)^{m_1} \cdot (3+j)^{m_2} \cdot \mathfrak{a}\big]$, ($m_1, m_2 \geq 0$ and $\mathfrak{a}$ prime to 7), giving the two φ-components $\mathbb{Z}_7/7^{m_1}\mathbb{Z}_7$ and $\mathbb{Z}_7/7^{m_2}\mathbb{Z}_7$ (from $\big[\mathbb{Z}[j]/(-2+j)^{m_1}\big] \otimes \mathbb{Z}_7$ and $\big[\mathbb{Z}[j]/(3+j)^{m_2} \otimes \mathbb{Z}_7\big]$), for the $\widetilde{\mathscr{E}}_{\varphi}$'s.

9.1. **Description of the computations.** The PARI program computing all the cyclic cubic fields is that given in [Gra2019, § 6.1].

A crucial fact, without which the checking of the φ-components of the G_K-modules $\mathscr{E}_K/\mathscr{F}_K$ and $\mathscr{H}_K$ could be misleading, is the definition of a generator σ of G_K giving the correct conjugation, both for the fundamental units, the cyclotomic ones and the elements of the class group; this is not so easy even if a conjugation does exist for the data given by $\mathsf{K} = \mathsf{bnfinit(P)}$ from the explicit instructions $\mathsf{G} = \mathsf{nfgaloisconj(P)}$, giving x^σ under the form $g(x)$, $g \in \mathbb{Q}[X]$, for a root x of the defining polynomial P, and $\mathsf{nfgaloisapply(K, G[i], X)}$ acting on any PARI object X.

Thus it is not too difficult to find, from $\mathsf{K.fu}$ giving a $\mathbb{Z}$-basis of $\mathbf{E}_K$, a "Minkowski unit" ε and its conjugate ε^σ such that $\langle \varepsilon, \varepsilon^\sigma \rangle_{\mathbb{Z}} = \mathbf{E}_K$; indeed, for the evaluation of $\varepsilon(x)$ and $\varepsilon(g(x))$, at a root $\rho \in \mathbb{R}$ of P, we only have a set $\{\rho_1, \rho_2, \rho_3\}$ given in a random order by $\mathsf{polroot(P)}$. Any change of root gives an inconsequential permutation $(\varepsilon, \varepsilon^\sigma) \mapsto (\varepsilon^\tau, \varepsilon^{\tau\sigma})$, for some $\tau \in G_K$.

For security, we test $\mathsf{Reg_1/Reg} = 1$ where $\mathsf{Reg_1}$ is the regulator of the units $\varepsilon(\rho)$ and $\varepsilon(g(\rho))$, computed with the root ρ, and where $\mathsf{Reg} = \mathsf{K.reg}$ is the true regulator given by PARI.

Then we must write the Leopoldt cyclotomic unit η of K of conductor f (Definition 7.3) under the form $\eta = \varepsilon^{\alpha + \beta\,\sigma}$, $\alpha, \beta \in \mathbb{Z}$, which is easy as soon as we have η and η^σ. But η is computed by means of the analytic expression of $|\mathbf{C}| = \prod\limits_{a \in [1, f/2[,\, \sigma_{a\,|K} = 1} |\zeta_{2f}^{a} - \zeta_{2f}^{-a}|$, as product of the $|\zeta_{2f}^{a} - \zeta_{2f}^{-a}|$ for the prime-to-f integers $a < f/2$ such that the Artin symbol $\sigma_a = \left(\frac{\mathbb{Q}(\mu_f)/\mathbb{Q}}{a}\right)$ is in $\mathrm{Gal}(\mathbb{Q}(\mu_f)/K)$ (which is tested using a prime $q_a \equiv a \pmod f$ giving $\sigma_{a\,|K} = 1$ if and only if q_a splits in K).

If f is prime, $\zeta_{2f} - \zeta_{2f}^{-1}$ generates the prime ideal above f; thus:

$$\pi := \mathbf{N}_{\mathbb{Q}(\mu_f)/K}(\zeta_{2f} - \zeta_{2f}^{-1}) = \pm \mathbf{C}^2$$

with $\pi^3 = f \cdot \eta'$, $\eta' \in \mathbf{E}_K$, whence $\pi^{3\,(1-\sigma)} = \eta'^{1-\sigma} = \eta^6 := (\mathbf{C}^{1-\sigma})^6$ (Proposition 7.4); the program computes $3\log(\mathbf{C}) - \frac{1}{2}\log(f) = \frac{1}{2}\log(\eta')$, so that, to compute η from $\eta^3 = \sqrt{\eta'}^{1-\sigma}$, we must divide the regulator RegC by 3 and multiply $\alpha + j\,\beta$ by $\frac{1-j}{3}$ in that case where $w_\chi = 1$.

If f is composite, we have $\eta = \mathbf{C}$ obtained via the half-system and the class number is the product of the index of units by $w_\chi = 3$, so this appear in the results (e.g., for the first example $f = 13 \cdot 97$, $P = x^3 + x^2 - 420x - 1728$, $\mathsf{classgroup} = [21]$ and $\mathsf{Index}\ [\mathsf{E_K} : \mathsf{C_K}] = 7$, but $\alpha + j\,\beta = -3 - 2j$ of norm 7; for $f = 3^2 \cdot 307$, $P = x^3 - 921x - 10745$, $\mathsf{classgroup} = [21, 3]$ and $\mathsf{Index}\ [\mathsf{E_K} : \mathsf{C_K}] = 21$, but $\alpha + j\,\beta = -5 - j$ of norm 21).

To define the correct conjugation, $\zeta_{2f} \mapsto \zeta_{2f}^{\sigma} =: \zeta_{2f}^{q}$, for some prime q, we use the fundamental property of Frobenius automorphisms giving $y^{\mathrm{Frob}(q)} \equiv y^q \pmod q$, for any q-integer y of K, if q is inert in $K/\mathbb{Q}$; using $x^\sigma = g(x)$, we test the congruence $g(x) - x^q \pmod q$ to decide if $\sigma = \mathrm{Frob}(q)$ or $\mathrm{Frob}(q)^2$, in which case $\zeta_{2f}^{\sigma} = \zeta_{2f}^{q}$ or $\zeta_{2f}^{q^2}$, giving easily the conjugate η^σ.

9.2. **The general PARI program.** The program is the following and we explain, with some examples, how to use the numerical results checking the Main Conjecture (of course, now, the Main Theorem); $\mathsf{hmin} = \mathsf{p^{vp}}$ means that the program only computes fields with p-class groups CKp of order at least $\mathsf{p^{vp}}$; then bf, Bf define an interval for the conductors f.

Other indications are given in the text of the program (if necessary, the program can be copy and past at (file "Program.tex"): https://www.dropbox.com/s/t8f4jj5v9sp629j/Program%20Phi-objects.tex?dl=

```
\p 50
{p=7; \\ Take any prime p congruent to 1 modulo 3
bf=2;Bf=10^6;hmin=p^2;
\\ Arithmetic of Q(j), j^2+j+1=0:
S=y^2+y+1;kappa=bnfinit(S);Y=idealfactor(kappa,p);
 \\ Decomposition (p)=P1*P2 in Z[j]:
P1=component(Y,1)[1];P2=component(Y,1)[2];
\\ Iteration over the conductors f in [bf,Bf]:
for(f=bf,Bf,vf=valuation(f,3);if(vf!=0 & vf!=2,next);
F=f/3^vf;if(core(F)!=F,next);F=factor(F);Div=component(F,1);
d=matsize(F)[1];for(j=1,d,D=Div[j];if(Mod(D,3)!=1,break));
\\ Computation of solutions a and b such that f=(a^2+27*b^2)/4:
\\ Iteration over b, then over a:
for(b=1,sqrt(4*f/27),if(vf==2 & Mod(b,3)==0,next);A=4*f-27*b^2;
if(issquare(A,&a)==1,
\\ computation of the corresponding defining polynomial P:
if(vf==0,if(Mod(a,3)==1,a=-a);P=x^3+x^2+(1-f)/3*x+(f*(a-3)+1)/27);
if(vf==2,if(Mod(a,9)==3,a=-a);P=x^3-f/3*x-f*a/27);
K=bnfinit(P,1); \\ PARI definition of the cubic field K
\\ Test on the p-class number #CKp regarding hmin:
if(Mod(K.no,hmin)==0,print();
G=nfgaloisconj(P); \\ Definition of the Galois group G
\\ Frob = Artin symbol defining the PARI generator sigma=G[2]:
forprime(q=2,10^4,if(Mod(f,q)==0,next);
Pq=factor(P+O(q));if(matsize(Pq)[1]==1,Frob=q;break));X=x^Frob-G[2];
if(valuation(norm(Mod(X,P)),Frob)==0,Frob=lift(Mod(Frob^2,f)));
E=K.fu;Reg=K.reg; \\ Group of units, Regulator
\\ We certify that a suitable PARI unit is a Z[G]-generator of E_K:
E1=lift(E[1]);E2=lift(nfgaloisapply(K,G[2],E[1]));
Root=polroots(P);Rho=real(Root[1]); \\  Selecting a root of P
e1= abs(polcoeff(E1,0)+polcoeff(E1,1)*Rho+polcoeff(E1,2)*Rho^2);
e2= abs(polcoeff(E2,0)+polcoeff(E2,1)*Rho+polcoeff(E2,2)*Rho^2);
l1=log(e1);l2=log(e2);Reg1=l1^2+l1*l2+l2^2;quot=Reg1/Reg;
print(quot); \\ This quotient must be equal to 1
\\ Computation of the cyclotomic units C1,C2=sigma(C1):
z=exp(I*Pi/f);C1=1;C2=1;
\\ Case of a prime conductor f using (Z/fZ)^* cyclic):
if(isprime(f)==1,g=znprimroot(f)^3;
\\ Description of a half-system:
for(k=1,(f-1)/6,gk=lift(g^k);sgk=lift(Mod(gk*Frob,f));
C1=C1*(z^gk-z^-gk);C2=C2*(z^sgk-z^-sgk));
```

```
 \\ Logarithms of C1,C2:
L1=3*log(abs(C1))-log(f)/2;L2=3*log(abs(C2))-log(f)/2;
\\ computation of the cyclotomic regulator and of the index Quot=(E:F):
RegC=L1^2+L1*L2+L2^2;Quot=1/3*RegC/Reg); \\ Division by 3 of RegC
\\ Case of a composite conductor:
if(isprime(f)==0,for(aa=1,(f-1)/2,if(gcd(aa,f)!=1,next);
\\ Search of a prime qa congruent to a modulo f, split in K:
qa=aa;while(isprime(qa)==0,qa=qa+f);
if(matsize(idealfactor(K,qa))[1]==1,next);
\\ The Artin symbol of aa fixes K:
C1=C1*(z^aa-z^-aa);C2=C2*(z^(Frob*aa)-z^-(Frob*aa)));
L1=log(abs(C1));L2=log(abs(C2)); \\ Logarithms of C1,C2
\\ computation of the cyclotomic regulator and the index Quot=(E:F):
RegC=L1^2+L1*L2+L2^2;Quot=RegC/Reg);
\\ printing of the basic data of K:
print("P=",P," f=",f,"=",factor(f)," (a,b)=","(",a,",",b")",
" class group=",K.cyc,"  sigma=",Frob);print("Index [E_K:C_K]=",Quot);
\\ Annihilator alpha+sigma.beta of the quotient E/C:
alpha=((log(e1)+log(e2))*L1+log(e2)*L2)/Reg;
beta=(log(e2)*L1-log(e1)*L2)/Reg;
 \\ In the prime case one multiply alpha+j.beta by (1-j)/3:
if(isprime(f)==1,
alpha0=(alpha+beta)/3;
beta0=(-alpha+2*beta)/3;alpha=alpha0;beta=beta0);
\\ Writing of alpha and beta as reals for checking:
print("(alpha,beta)=","(",alpha,", ",beta,")");
\\ Computation of alpha and beta as integers:
alpha=sign(alpha)*floor(abs(alpha)+10^-6);
beta=sign(beta)*floor(abs(beta)+10^-6);
\\ Class group (r = global rank;rp = p-rang;expo = exposant of CKp)
\\ vp = valuations of CKp, ve = valuation of the exponent expo of CKp:
CK=K.clgp;r=matsize(CK[2])[2];CKp=List;EKp=List;rp=0;vp=0;ve=0;
for(i=1,r,ei=CK[2][i];vi=valuation(ei,p);
if(vi>0,rp=rp+1;vp=vp+vi;ve=max(ve,vi));expo=p^ve;
\\ The rp following ideals Ai generate the p-class group CKp:
Ai=idealpow(K,CK[3][i],ei/p^vi);listput(CKp,Ai,i);listput(EKp,p^vi,i));
\\ Matrices h and sh of Ai and sAi on the PARI basis of CK
L0=List;for(i=1,r,listput(L0,0,i));LH=List;LsH=List;
for(i=1,rp,Ai=CKp[i];h=bnfisprincipal(K,Ai)[1];
sAi=nfgaloisapply(K,G[2],Ai);sh=bnfisprincipal(K,sAi)[1];
print("h=",h,", ","sigma(h)=",sh);listput(LH,h,i);listput(LsH,sh,i));
\\ Determination of the Pi-valuations of (alpha+j.beta), i=1,2:
Z=Mod(alpha+y*beta,S);w1=idealval(kappa,Z,P1);w2=idealval(kappa,Z,P2);
print(w1,"  ",w2,"  P1 and P2-valuations for alpha+j*beta");
\\ Galois structure of CKp; computation of the phi-components:
if(rp==1,
u=lift(LsH[1][1]*Mod(LH[1][1],expo)^-1);
YY=Mod(y-u,S);v1=idealval(kappa,YY,P1);v2=idealval(kappa,YY,P2);
v1=min(v1,ve);v2=min(v2,ve);
print(v1,"  ",v2,"  P1 and P2-valuations for H"));
if(rp==2,
\\ Computation of ci(mod expo) such that Pi=(ci+j),i=1,2:
Sp=lift(factor(S+O(p^ve)));Sp1=component(Sp,1)[1];Sp2=component(Sp,1)[2];
c1=polcoeff(Sp1,0);c2=polcoeff(Sp2,0);
\\ Coefficients of LH[1],LsH[1],LH[2],LsH[2], on the PARI basis of CK
H1=LH[1];A1=H1[1];B1=H1[2];sH1=LsH[1];C1=sH1[1];D1=sH1[2];
H2=LH[2];A2=H2[1];B2=H2[2];sH2=LsH[2];C2=sH2[1];D2=sH2[2];
\\ Computation of the determinants of the relations:
Delta1=((C1+c1*A1)*(D2+c1*B2)-(D1+c1*B1)*(C2+c1*A2));
Delta1=lift(Mod(Delta1,expo));
Delta2=((C1+c2*A1)*(D2+c2*B2)-(D1+c2*B1)*(C2+c2*A2));
Delta2=lift(Mod(Delta2,expo));
print(Delta1,"  ",Delta2,"   Determinants: Delta1,Delta2");
\\ Computation of the relations defining the phi-components:
r11x=C1+c1*A1;r11y=C2+c1*A2;r12x=D1+c1*B1;r12y=D2+c1*B2;
r11x=lift(Mod(r11x,expo));r11y=lift(Mod(r11y,expo));
r12x=lift(Mod(r12x,expo));r12y=lift(Mod(r12y,expo));
r21x=C1+c2*A1;r21y=C2+c2*A2;r22x=D1+c2*B1;r22y=D2+c2*B2;
```

```
r21x=lift(Mod(r21x,expo));r21y=lift(Mod(r21y,expo));
r22x=lift(Mod(r22x,expo));r22y=lift(Mod(r22y,expo));
print("R11=",r11x,"*X+",r11y,"*Y","  R12=",r12x,"*X+",r12y,"*Y");
print("R21=",r21x,"*X+",r21y,"*Y","  R22=",r22x,"*X+",r22y,"*Y"));
\\ Structure of the torsion group Tp of p-ramification:
n=6; \\ Choose any n, large enough, such that p^(n+1) annihilates Tp:
LTp=List;Kpn=bnrinit(K,p^n);Hpn=Kpn.cyc;
dim=component(matsize(Hpn),2);for(k=2,dim,c=component(Hpn,k);
if(Mod(c,p)==0,listput(LTp,p^valuation(c,p),k)));
print("Structure of the ",p,"-torsion group: ",LTp)))))}
```

9.3. **Numerical examples.** Since the approximations are in general very good (with precision \p 50), we have suppressed useless decimals in the numerical results for integers computed and given as real numbers. But for some conductors, the precision \p 100 may be necessary, because of a fundamental unit close to 0 (e.g., $f = 21193, 30223$). For $f = 42667$, \p 100 does not compute correctly and \p 150 gives a nice result for α and β; but we see that, for this example,

```
e_1=3062171948818717694.348000505806  &  e_2=1.221295564694 E-69.
```

Note that, according to the PARI version used, numerical data for generators of class groups may vary and propagate in other computations, but without any trouble for final results.

9.3.1. *Galois structure of $\mathscr{E}_K/\mathscr{F}_K$.* Let ε be the $\mathbb{Z}[G]$-generator of $\mathbf{E}_K$ and let η that of the subgroup $\mathbf{F}_K$ of Leopoldt's cyclotomic units; thus we have $\eta = \varepsilon^{\alpha+\beta\,\sigma}$ and obtain the isomorphism:

$$\mathbf{E}_K/\mathbf{F}_K \simeq \mathbb{Z}[j]/(\alpha + j\,\beta)\mathbb{Z}[j],$$

where j is a root of $\mathsf{S} := \mathsf{y}^2 + \mathsf{y} + 1$.

In all the sequel, from a factorization $p = (r_1 + j\,r_1') \cdot (r_2 + j\,r_2')$ giving the ideal product $(p) = \mathfrak{p}_1\mathfrak{p}_2$ in $\mathbb{Z}[j]$, we associate, for the exponent p^e, the two annihilators $c_i + \sigma$ such that $(c_i + j) = \mathfrak{p}_i^e$ (up to a prime-to-p ideal); this preserves the definition of the φ_1 and φ_2-components.

For instance, for $p = 7$, $\mathfrak{p}_1 := (-2+j)\mathbb{Z}[j]$ and $\mathfrak{p}_2 := (3+j)\mathbb{Z}[j]$; writing $(\alpha+j\,\beta) =: \mathfrak{p}_1^u \cdot \mathfrak{p}_2^v \cdot \mathfrak{a}$, $\mathfrak{a}$ prime to 7, we get immediately the two φ-components of $\widetilde{\mathscr{E}}_K = \mathscr{E}_K/\mathscr{F}_K$ (e.g., if $e = 2$, the two annihilators are $19 + j$ and $-18 + j$, respectively; for $p = 13$, we get $23 + j$ and $-22 + j$).

9.3.2. *Galois structure of $\mathscr{H}_K$.* Recall that bnfisprincipal(K, A)[1] gives the matrix of components of the class of A on the basis $\{h_1, \ldots, h_r\}$ given by K.clgp (in CK) and the fact that 0 at the place i means that the corresponding component of cl(A) on h_i is trivial.

We first replace the generators of $\mathbf{H}_K$ by generators Ai of $\mathscr{H}_K$ (where $r_p \le r$ is the p-rank). The Galois action on the Ai is computed using the instructions (where G[2] gives the σ-conjugate, G[1] being the identity):

```
h=bnfisprincipal(K,Ai)[1];sAi=nfgaloisapply(K,G[2],Ai);
sh=bnfisprincipal(K,sAi)[1]};
```

so the Galois structure of $\mathscr{H}_K$ becomes linear algebra from the matrices given by the program, via the relations:

$$h = \textstyle\prod_{i=1}^{r_p} h_i^{a_i} \text{ (in h)} \quad \& \quad h^\sigma = \prod_{i=1}^{r_p} h_i^{b_i} \text{ (in sh)}.$$

(a) **Case of 7-rank** $r_7 = 1$. This case is obvious, writing $h = h_1^a$, $h^\sigma = h_1^b$; we put $P_{\varphi_1} \equiv c_1 + y \pmod{7^e}$ and $P_{\varphi_2} \equiv c_2 + y \pmod{7^e}$, where 7^e is the exponent of $\mathscr{H}_K$; we obtain $h^{c_1+\sigma} = h_1^{c_1 a+b}$ and $h^{c_2+\sigma} = h_1^{c_2 a+b}$; so $\mathscr{H}_K = \mathscr{H}_{\varphi_1}$ (resp. $\mathscr{H}_{\varphi_2}$) if and only if $c_1 a + b \equiv 0 \pmod{7^e}$ (resp. $c_2 a + b \equiv 0 \pmod{7^e}$). In fact one computes $-a^* b + j$, where a^* is inverse of a modulo 7^e, and write $(-a^* b + j) = \mathfrak{p}_i^u$ for the suitable $i \in \{1, 2\}$.

The Galois actions are to be read in columns; for instance, the valuations in the two lines:

```
v   0     P1 and P2 − valuations for alpha + j * beta
v   0     P1 and P2 − valuations for   H
```

give the structures $\mathbb{Z}[j]/\mathfrak{p}_1^v \cdot \mathfrak{p}_2^0$ for "$\mathscr{M} = \widetilde{\mathscr{E}} = \mathscr{E}/\mathscr{F}$ and $\mathscr{H}$", respectively, whence $\mathscr{M}_{\varphi_1} \simeq \mathbb{Z}[j]/\mathfrak{p}_1^v$, $\mathscr{M}_{\varphi_2} = 1$, and so on. First examples:

```
P=x^3+x^2-104*x+371  f=313=Mat([313,1])  (a,b)=(35,1)
Class group=[7]  sigma=4
(alpha,beta)=(-3.000000000,-2.000000000)  Index [E_K:C_K]=7.000000000
h=[1]~,  sigma(h)=[2]~
1  0   P1 and P2-valuations for alpha+j*beta
1  0   P1 and P2-valuations for  H
Structure of the 7-torsion group: List([7,7])
```

We have $\widetilde{\mathscr{E}}_{\varphi_1} \simeq \mathscr{H}_{\varphi_1} \simeq (\mathbb{Z}[j]/\mathfrak{p}_1) \otimes \mathbb{Z}_7 \simeq \mathbb{Z}/7\mathbb{Z}$ and the conjugation $h^\sigma = h^2$, giving the annihilator $(-2+j) = \mathfrak{p}_1$ as expected; whence the two columns given by the program. We deduce that $\mathscr{T}_K = \mathscr{H}_K \oplus \mathscr{R}_K$.

```
P=x^3+x^2-2450*x-1089  f=7351=Mat([7351,1])  (a,b)=(-1,33)
Class group=[49]  sigma=4
(alpha,beta)=(5.000000000,8.000000000)  Index [E_K:C_K]=49.000000000
h=[1]~,  sigma(h)=[30]~
2  0   P1 and P2-valuations for alpha+j*beta
2  0   P1 and P2-valuations for  H
Structure of the 7-torsion group: List([2401])
```

We have $(\alpha + j\,\beta) = (5+8j)$, thus the annihilator $(19+j) = \mathfrak{p}_1^2$; then $h^\sigma = h^{30}$ gives (modulo 7^2) the same annihilator. The φ_2-components are trivial. Since $\mathscr{T}_K \simeq \mathbb{Z}/7^4\mathbb{Z}$, $\mathscr{R}_K = \mathscr{T}_K^{7^2}$, $\mathscr{H}_K \simeq \mathscr{T}_K/\mathscr{R}_K \simeq \mathbb{Z}/7^2\mathbb{Z}$.

The first field such that $\mathscr{H}_K \simeq \mathbb{Z}/7^3\mathbb{Z}$ is the following:

```
P=x^3+x^2-77006*x-34225  f=231019=Mat([231019,1])  (a,b)=(-1,185)
Class group=[343]  sigma=4
(alpha,beta)=(19.000000000,18.000000000)  Index [E_K:C_K]=343.000000000
h=[1]~,  sigma(h)=[18]~
0  3   P1 and P2-valuations for alpha+j*beta
0  3   P1 and P2-valuations for  H
Structure of the 7-torsion group: List([343,7])
```

The annihilator of $\mathscr{H}_K$ is $(-18+j) = \mathfrak{p}_2^3$. The structures are similar with the φ_2-components since $(19+18j) = \mathfrak{p}_2^3$. In that case, $\mathscr{T}_K = \mathscr{H}_K \oplus \mathscr{R}_K$ with $\mathscr{H}_K \simeq \mathbb{Z}/7^3\mathbb{Z}$ and $\mathscr{R}_K \simeq \mathbb{Z}/7\mathbb{Z}$.

(b) **Case of 7-rank** $r_7 = 2$ This case depends on the matrices giving:

$$\mathsf{h} = [\mathsf{a}, \mathsf{b}],\ \mathsf{sigma}(\mathsf{h}) = [\mathsf{c}, \mathsf{d}] \quad \& \quad \mathsf{h}' = [\mathsf{a}', \mathsf{b}'],\ \mathsf{sigma}(\mathsf{h}') = [\mathsf{c}', \mathsf{d}'];$$

this means that the corresponding generating classes h, h', fulfill the relations (regarding the basis $\{h_1, h_2\}$ of the class group) $h = h_1^a \cdot h_2^b$ and $h^\sigma = h_1^c \cdot h_2^d$, then $h' = h_1^{a'} \cdot h_2^{b'}$ and $h^\sigma = h_1^{c'} \cdot h_2^{d'}$. Thus we compute the conditions $H^{c_i+\sigma} = 1$, $i = 1, 2$, for $H := h^x \cdot h'^y$; this gives the relations R11, R21 (R12, R22 are checked by security since they must be proportional to the previous ones); whence the arrangement of lines when the conjecture holds. The program computes the corresponding determinants of the relation (Determinants Delta1 Delta2); this is superfluous but have been computed (but not printed) for verification.

```
P=x^3+x^2-3422*x-1521  f=10267=Mat([10267,1])  (a,b)=(-1,39)
Class group=[7,7]  sigma=2
(alpha,beta)=(-7.000000000,-7.000000000)  Index [E_K:C_K]=49.000000000
h=[1,0]~,  sigma(h)=[0,1]~
h'=[0,1]~, sigma(h')=[6,6]~
1  1   P1 and P2-valuations for alpha+j*beta
R11=3*X+6*Y  R12=1*X+2*Y
R21=5*X+6*Y  R22=1*X+4*Y
Structure of the 7-torsion group: List([49,7])
```

This case means that $\widetilde{\mathscr{E}}_K \simeq \mathbb{Z}[j]/(7)$, giving the two non trivial φ-components of order 7. The relations, for $\mathscr{H}_K$, reduce to R11 and R21 Thus $\mathscr{H}_K = \mathscr{H}_{\varphi_1} \oplus \mathscr{H}_{\varphi_2} \simeq \mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z}$, $\mathscr{R}_K = \mathscr{T}_K^7 \simeq \mathbb{Z}/7\mathbb{Z}$.

```
P=x^3+x^2-55296*x-1996812  f=165889=[19,1;8731,1]  (a,b)=(-322,144)
Class group=[294,2,2,2] sigma=25
(alpha,beta)=(-32.000000000,-20.000000000) Index [E_K:C_K]=784.000000000
h=[6,0,0,0]~, sigma(h)=[108,1,0,0]~
0  2   P1 and P2-valuations for alpha+j*beta
0  2   P1 and P2-valuations for  H
Structure of the 7-torsion group: List([49])
```

Here $\mathscr{R}_K = 1$ and $\mathscr{T}_K = \mathscr{H}_K \simeq (\mathbb{Z}[j]/\mathfrak{p}_2^2) \otimes \mathbb{Z}_7 \simeq \mathbb{Z}_7/7^2\mathbb{Z}_7$.

```
P=x^3+x^2-453576*x+117425873  f=1360729=Mat([1360729,1])  (a,b)=(2333,1)
Class group=[98,14]  sigma=2
(alpha,beta)=(42.000000000,28.000000000)  Index [E_K:C_K]=1372.000000000
h=[1,0]~,  sigma(h)=[44,11]~
h'=[0,1]~, sigma(h')=[7,11]~
2  1   P1 and P2-valuations for alpha+j*beta
R11=14*X+7*Y  R12=11*X+30*Y
R21=26*X+7*Y  R22=11*X+42*Y
Structure of the 7-torsion group: List([49,7,7])
```

We have $(\alpha + \beta j) = 2 \cdot 7(3 + 2j)$ giving the annihilator $\mathfrak{p}_1^2\mathfrak{p}_2$ which is also the annihilator of $\mathscr{H}_K$. The structure is $\mathscr{T}_K = \mathscr{H}_K \oplus \mathscr{R}_K$.

```
P=x^3+x^2-884540*x-393129  f=2653621=Mat([2653621,1])  (a,b)=(-1,627)
Class group=[686,14]  sigma=2
(alpha,beta)=(-112.00000000,-70.00000000)  Index [E_K:C_K]=9604.00000000
h=[2,0]~,  sigma(h)=[36,2]~
h'=[0,2]~, sigma(h')=[0,4]~
1   3   P1 and P2-valuations for alpha+j*beta
R11=74*X+0*Y  R12=2*X+42*Y
R21=0*X+0*Y  R22=2*X+311*Y
Structure of the 7-torsion group: List([343,49])
```

In that case, $\mathscr{T}_K \simeq Z/7^3\mathbb{Z} \oplus \mathbb{Z}/7^2\mathbb{Z}$ and $\mathscr{R}_K \simeq (\mathbb{Z}/7^3\mathbb{Z})^0 \oplus (7\,\mathbb{Z}/7^2\mathbb{Z})$.

(c) **Larger 7-ranks**. If the order 7^3, with 7-rank 1 or 2, is rather frequent for the 7-class group, we find, after several days of computer, only three examples of 7-rank 3 in the interval $f \in [7, 50071423]$; they are obtained with the conductors $f = 14376321, 39368623, 43367263$, giving interesting structures (use precision \p 100). The least cubic field with 7-rank 3 is the following:

```
P=x^3-4792107*x+4022175142 f=14376321=[3,2;1597369,1] (a,b)=(-7554,128)
Class group=[21,7,7] sigma=5
(alpha,beta)=(-7.000000000,-21.000000000) Index [E_K:C_K]=343.000000000
h =[3,0,0]~, sigma(h) =[15, 4, 0]~
h'=[0,1,0]~, sigma(h')=[3, 1, 0]~
h"=[0,0,1]~, sigma(h")=[6, 5, 2]~
2   1   P1 and P2-valuations for alpha+j*beta
Structure of the 7-torsion group: List([7,7,7])
```

Using the information on α and β, we obtain, for $\widetilde{\mathscr{E}}_K = \mathscr{E}_K/\mathscr{F}_K$:

$$\widetilde{\mathscr{E}}_K \simeq (\mathbb{Z}[j]/7\mathfrak{p}_2) \otimes \mathbb{Z}_7 \simeq (\mathbb{Z}[j]/\mathfrak{p}_1^2\,\mathfrak{p}_2) \otimes \mathbb{Z}_7 \simeq (\mathbb{Z}[j]/\mathfrak{p}_1^2 \oplus \mathbb{Z}[j]/\mathfrak{p}_2) \otimes \mathbb{Z}_7,$$

where $\mathfrak{p}_1 = (-2 + j)$ and $\mathfrak{p}_2 = (3 + j)$. We get the φ-components:

$$\widetilde{\mathscr{E}}_{\varphi_1} \simeq (\mathbb{Z}[j]/\mathfrak{p}_1^2) \otimes \mathbb{Z}_7 \simeq \mathbb{Z}/7^2\mathbb{Z} \text{ and } \widetilde{\mathscr{E}}_{\varphi_2} \simeq (\mathbb{Z}[j]/\mathfrak{p}_2) \otimes \mathbb{Z}_7 \simeq \mathbb{Z}/7\mathbb{Z}.$$

To obtain the two φ-components of $\mathscr{H}_K = \mathscr{T}_K$, we put $H = h^x h'^y h''^z$ and we determine the solutions of the two relations$H^{P_{\varphi_i}(\sigma)} = 1$, $i = 1, 2$, that is to say, $H^{-2+\sigma} = 1$ and $H^{3+\sigma} = 1$, respectively.

We then obtain the systems (considered modulo 7 since the exponent of $\mathscr{H}_K$ is 7) of ranks 1 and 2, respectively:

$$\begin{cases} 2x + 3y + 6z = 0 \\ 4x + 6y + 5z = 0 \end{cases} (H^{-2+\sigma} = 1) \quad \& \quad \begin{cases} 3x + 3y + 6z = 0 \\ 4x + 4y + 5z = 0 \\ z = 0, \end{cases} (H^{3+\sigma} = 1).$$

They are equivalent to:

$$2x + 3y + 6z = 0 \ (H^{-2+\sigma} = 1) \quad \& \quad \big[x + y = 0 \ \ \& \ \ z = 0\big] \ (H^{3+\sigma} = 1).$$

Which gives, considering the $\mathbb{F}_7$-dimensions given by the systems:

$$\mathscr{H}_{\varphi_1} \simeq \big[(\mathbb{Z}[j]/\mathfrak{p}_1)\otimes\mathbb{Z}_7\big] \oplus \big[(\mathbb{Z}[j]/\mathfrak{p}_1)\otimes\mathbb{Z}_7\big] \quad \& \quad \mathscr{H}_{\varphi_2} \simeq (\mathbb{Z}[j]/\mathfrak{p}_2)\otimes\mathbb{Z}_7.$$

We have indeed equalities for the orders of the φ-components relative to $\widetilde{\mathscr{E}}_K$ and $\mathscr{H}_K$, respectively, but of course with different structures of $\mathbb{Z}_7[j]$-modules since $\widetilde{\mathscr{E}}_{\varphi_1} \simeq \mathbb{Z}/7^2\mathbb{Z}$ and $\mathscr{H}_{\varphi_1} \simeq \big[\mathbb{Z}/7\mathbb{Z}\big]^2$.

The two other examples are similar:

```
P=x^3+x^2-13122874*x-7765825411
f=39368623=[7,1;79,1;71191,1] (a,b)=(-5323,2187)
class group=[21,21,7] sigma=4
(alpha,beta)=(28.000000000,-7.000000000) Index [E_K:C_K]=1029.000000000
h =[3,0,0]~,  sigma(h) =[3,9,0]~
h'=[0,3,0]~,  sigma(h')=[18,15,0]~
h"=[0,0,1]~,  sigma(h")=[15,6,4]~
1   2  P1 and P2-valuations for alpha+j*beta
Structure of the 7-torsion group: List([7,7,7])

P=x^3+x^2-14455754*x-16977480367
f=43367263=[43,1;1008541,1] (a,b)=(-10567,1513)
class group=[273,7,7] sigma=2
(alpha,beta)=(42.000000000,77.000000000) Index [E_K:C_K]=4459.000000000
h =[39,0,0]~,  sigma(h) =[0,5,1]~
h'=[0,1,0]~,   sigma(h')=[156,6,5]~
h"=[0,0,1]~,   sigma(h")=[0,0,2]~
2   1  P1 and P2-valuations for alpha+j*beta
Structure of the 7-torsion group: List([49,7,7])
```

(d) **Larger primes** p. Let's give, without comments, some examples:

```
p=13 P=x^3+x^2-15196*x-726047 f=45589=Mat([45589,1]) (a,b)=(-427,1)
Class group=[169] sigma=2
(alpha,beta)=(15.000000000,8.000000000) Index [E_K:C_K]=169.000000000
h=[1]~, sigma(h)=[146]~
2  0   P1 and P2-valuations for alpha+j*beta
2  0   P1 and P2-valuations for H
Structure of the 13-torsion group: List([169])
p=13 P=x^3+x^2-238516*x-7579519 f=715549=Mat([715549,1]) (a,b)=(-283,321)
Class group=[13,13] sigma=2
(alpha,beta)=(7.000000000,-8.000000000) Index [E_K:C_K]=169.000000000
h =[1,0]~, sigma(h) =[9,0]~
h'=[0,1]~, sigma(h')=[0,9]~
0  2   P1 and P2-valuations for alpha+j*beta
R11=0*X+0*Y  R12=0*X+0*Y
R21=6*X+0*Y  R22=0*X+6*Y
Structure of the 13-torsion group: List([13,13])

p=19 P=x^3-137271*x+45757 f=411813=[3,2;45757,1] (a,b)=(-3,247)
Class group=[1083] sigma=2
(alpha,beta)=(-21.000000000,-5.000000000) Index [E_K:C_K]=361.000000000
h=[3]~, sigma(h)=[204]~
0  2   P1 and P2-valuations for alpha+j*beta
0  2   P1 and P2-valuations for H
Structure of the 19-torsion group: List([361])
p=19 P=x^3+x^2-162636*x+25190561 f=487909=[31,1;15739,1] (a,b)=(1397,1)
Class group=[57,19] sigma=2
(alpha,beta)=(19.00000000,4.19514516 E-69) Index [E_K:C_K]=361.00000000
h =[3,0]~,  sigma(h) =[51,16]~
h'=[0,1]~,  sigma(h')=[3,1]~
1  1   P1 and P2-valuations for alpha+j*beta
R11=18*X+3*Y  R12=16*X+9*Y
R21=11*X+3*Y  R22=16*X+13*Y
Structure of the 19-torsion group: List([19,19])

p=31 P=x^3+x^2-63804*x+6181931 f=191413=Mat([191413,1]) (a,b)=(875,1)
class group=[31,31] sigma=4
(alpha,beta)=(31.00000000,-4.10842850 E-69) Index [E_K:C_K]=961.00000000
h=[1,0]~,  sigma(h) =[30,30]~
```

```
h'=[0,1]~, sigma(h')=[1,0]~
1  1  P1 and P2-valuations for alpha+j*beta
R11=5*X+1*Y  R12=30*X+6*Y
R21=25*X+1*Y  R22=30*X+26*Y
Structure of the 31-torsion group: List([31,31])

p=31 P=x^3+x^2-76004*x-8090239 f=228013=Mat([228013,1]) (a,b)=(-955,1)
class group=[961] sigma=2
(alpha,beta)=(-11.000000000,-35.000000000) Index [E_K:C_K]=961.000000000
h=[1]~, sigma(h)=[439]~
2  0  P1 and P2-valuations for alpha+j*beta
2  0  P1 and P2-valuations for H
Structure of the 31-torsion group: List([961])
```

Conclusion

Standard probabilistic approaches may confirm (or not) the classical Cohen–Lenstra–Malle–Martinet heuristics on p-class groups. Indeed, heuristics on the whole p-class group are given by that of the components $\mathscr{H}_\varphi^{\rm ar}$'s which must be compatible with that obtained for the $\widetilde{\mathscr{E}}_\varphi$'s.

Then, the main problem remains *a proof of the Main Conjecture in the non semi-simple real case* using the statement with Arithmetic φ-objects, especially a proof that for all abelian real field K, with a cyclic maximal p-sub-extension, we have, for all $\varphi \in \Phi_K$ (cf. § 8.2.2):

$$\#\mathscr{H}_\varphi^{\rm ar} = w_\varphi \cdot \#(\mathscr{E}_K/\mathscr{E}_K^0 \cdot \mathscr{F}_K)_\varphi, \quad w_\varphi \in \{1, p\}.$$

where $(\mathscr{E}_K/\mathscr{E}_K^0 \cdot \mathscr{F}_K)_\varphi := \{\widetilde{\varepsilon} \in (\mathscr{E}_K/\mathscr{E}_K^0 \cdot \mathscr{F}_K),\ P_\varphi(\sigma_\chi) \cdot \widetilde{\varepsilon} = 1\}$.

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