An arbitrary-order fully discrete Stokes complex on general polygonal meshes

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Abstract

In this paper we present an arbitrary-order fully discrete Stokes complex on general polygonal meshes. Based upon the recent construction of the de Rham fully discrete complex [14] we extend it using the same principle. We complete it with other polynomial spaces related to vector calculus operators and to the Koszul complex required to accommodate the increased smoothness of the Stokes complex. This complex is especially well suited for problem involving Jacobian, divergence and curl, like e.g. the Stokes system or magnetohydrodynamics. We show a complete set of results on the novelties of this complex complementing those of [14]: exactness properties, uniform Poincaré inequalities and primal and adjoint consistency. We use our new complex on the Stokes system and validate the expected convergence rates with various numerical tests.

Keywords: Discrete Stokes complex, Discrete de Rham complex, compatible discretization, polytopal methods

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1 Introduction.

The exactness of the divergence free condition plays an important role in the numerical resolution of incompressible fluid equations, [6] provides a detailed review. This kind of conservation requires the discrete spaces to reproduce relevant algebraic properties of the continuous spaces. This exactness can be expressed as a differential complex.

$$\mathbb{R} \xrightarrow{i_\Omega} H^1(\Omega) \xrightarrow{\mathbf{rot}} \boldsymbol{H}(\mathrm{div},\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{0} \{0\} \tag{1.1}$$

Many discrete counterparts of the complex (1.1) have been developed. See [7] for a thorough exposition and an extensive bibliography. Although many partial differential equations can be expressed using the de Rham complex, the lack of smoothness can cause issues for some equations. In particular with the Stokes equations (see [3]). So a smoother variant more suited to the Stokes equations and called Stokes complex has been considered. It is written in two dimensions:

$$\mathbb{R} \xrightarrow{i_\Omega} H^2(\Omega) \xrightarrow{\mathbf{rot}} \boldsymbol{H}^1(\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{0} \{0\} \tag{1.2}$$

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The development of discrete counterparts of this smoother complex is much more complicated. See [[7] Chapter 8.7] for a history. Although such construction exits (for example [5]) they often have drawbacks. Recurrent problems can be a large minimal degree and thus numerous unknowns as well as difficulties to enforce Dirichlet boundary conditions. The subject is very active with many recent advances: [11, 12]. Another issue of these constructions is that they are frequently constrained to conformal simplicial meshes, which is limiting for some geometries as well as on the possibility of refinement or agglomeration. A construction of the Stokes complex in virtual finite elements on general meshes has also been recently developed (see [13]).

Our construction works on general polygonal meshes and for arbitrary degrees. The discrete spaces consist of polynomial spaces on the elements of all dimensions: faces, edges and vertices. Compared to the virtual finite element method, the basis functions are explicitly known but do not live in a subspace of continuous functions. The discrete differential operators are therefore necessarily different from the continuous operators. They are constructed according to integration by parts formulas and in a sense converge with the discrete spaces to the continuous operators (see the consistency results of Section 5). A discretization of the de Rham complex (1.1) has been developed in detail by D. A. Di Pietro and J. Droniou [14]. One can find in the introduction a very complete comparison of the different methods leading to discrete de Rham complex on polytopal meshes. Our paper is a continuation of [14]: Our construction is based upon it, and we add the necessary basis functions required by the increased smoothness of the Stokes complex. We define and analyze in detail the Jacobian operator while checking its compatibility with the complex.

More precisely we show the exactness of the complex, the existence of uniform Poincare inequality and many consistency results as well as a discrete version of the right inverse for the divergence for the discrete norm $\boldsymbol{H}^1$. This complex aims to be a building block for a three-dimensional variation, but it is also perfectly usable for two-dimensional problems. Finally, we apply this to the Stokes equations: we show the well-posedness, give an error estimate and find an optimal convergence rate of order $\mathcal{O}(h^{k+1})$, h being the size of the mesh and $k \geq 0$ the chosen polynomial degree. We also explore other choices of boundary conditions and validate numerically every result.

The remaining of the paper is organized as follows. In Section 2 we establish the general setting. We define the discrete spaces and operators (interpolators, differential operators and norms) in Section 3. In Section 4 we show that we do indeed belong to a complex which is exact for simply connected domains. Section 5 is dedicated to consistency properties. Including primal and dual consistency. The Stokes equations are defined in Section 6 and other boundary conditions are studied in 7. We display our numerical results in Section 8. Finally we prove technical propositions in the appendices: on polynomial spaces in appendix A and on various lift in appendix B.

2 Setting.

This section is dedicated to the introduction of the setting and various notations that will be used throughout the paper.

2.1 Mesh and orientation.

In the following we consider a polytopal domain $\Omega \subset \mathbb{R}^2$ and keeping the notation of [14], for any set $Y \subset \mathbb{R}^2$, we write $h_Y := \sup\{|\boldsymbol{x} - \boldsymbol{y}| : \boldsymbol{x}, \boldsymbol{y} \in Y\}$ and $|Y|$ its Hausdorff measure. We consider on this domain a mesh sequence $\mathcal{M}_h = \mathcal{F}_h \cup \mathcal{E}_h \cup \mathcal{V}_h$ parameterized by a positive

real parameter $h \in \mathcal{H}$. Where $\mathcal{F}_h$ is a finite collection of open convex polygon such that $\overline{\Omega} = \cup_{F\in\mathcal{F}_h}\overline{F}$ and $h = \max_{F\in\mathcal{F}_h} h_F > 0$, $\mathcal{E}_h$ is the collection of open polygonal edges of the faces, $\mathcal{V}_h$ the collection of edge vertices. This sequence must be regular in the sense of [[10] Definition 1.9] with the regularity constant ρ.

We take $k \geq 0$ a fixed polynomial degree. In the following most inequalities are true up to a positive constant. This constant depends only on some parameters, here on the chosen polynomial degree k, on the regularity parameter of the mesh sequence ρ and on the domain Ω.

We denote the inequality up to a positive constant by

$$A \lesssim B$$

meaning there exists $C \in \mathbb{R}_+^*$ depending only on some parameters (here usually only on k, ρ and Ω) such that $A \leq CB$. We also write

$$A \approx B$$

meaning that $A \lesssim B$ and $B \lesssim A$.

For a fixed h, we choose an orientation of the plane, and we fix for each edge $E \in \mathcal{E}_h$ an orientation $\boldsymbol{t}_E$. We note by $^\perp$ the rotation of angle $\pi/2$ in the oriented plane and $\boldsymbol{n}_E = \boldsymbol{t}_E^\perp$. For any face $F \in \mathcal{F}_h$ we fix a counter-clockwise orientation of its boundary ∂F. And for any edge of this face $E \in \mathcal{E}_F$ we note $\omega_{FE} \in \{-1, 1\}$ the value such that $\omega_{FE}\boldsymbol{t}_E$ is oriented in the opposite direction of ∂F. We can then check that $\omega_{FE}\boldsymbol{n}_E$ is the outgoing normal unit vector of F. We also define $\boldsymbol{n}_\Omega$ as the outward pointing unit normal vector on the boundary $\partial\Omega$.

2.2 Polynomial spaces.

For any entity $X \in \{E, F\}$, we denote by $\mathcal{P}^k(X)$ the set of polynomials of total degree at most k on X, by $\boldsymbol{\mathcal{P}}^k(F)$ the set of polynomials with vector value in $\mathbb{R}^2$ on F, and by $(\boldsymbol{\mathcal{P}}^k(X))^2$ the set of pairs of polynomials on X forming the rows of a matrix. We use the conventions $\mathcal{P}^{-1}(X) := \{0\}$ and $\mathcal{P}^{0,k}(X) := \{P \in \mathcal{P}^k(X) : \int_X P = 0\}$. We also define the broken polynomial space

$$\mathcal{P}^k(\mathcal{X}_h) := \{P_h \in L^2(\mathcal{X}_h) : \forall X \in \mathcal{X}_h, P_{h|X} \in \mathcal{P}^k(X)\}. \tag{2.1}$$

As well as its continuous counterpart

$$\mathcal{P}_c^k(\mathcal{X}_h) := \{P_h \in C^0(\mathcal{X}_h) : \forall X \in \mathcal{X}_h, P_{h|X} \in \mathcal{P}^k(X)\}. \tag{2.2}$$

Remark 1. Continuous polynomials can be characterized by their values at the interface and their lower order moments on the elements. An explicit construction is deduced from 39, in the context of edges we can see the isomorphism between $\mathcal{P}_c^{k+2}(\mathcal{E}_h)$ and $\mathcal{P}^k(\mathcal{E}_h) \times \mathbb{R}^{\mathcal{V}_h}$.

For the sake of readability we recall here two lemmas on the discrete spaces which will often be used in the following, they are respectively the [[10] Lemma 1.28 and Lemma 1.32] (in a slightly more restrictive setting):

Lemma 2. *Let X be an element of $\mathcal{F}_h \cup \mathcal{E}_h$. Let l be a positive integer and a real number $p \in [1, \infty]$ be fixed. Then, the following inequality holds: For all $v \in \mathcal{P}^l(X)$,*

$$\|\nabla v\|_{L^p(X)} \lesssim h_X^{-1}\|v\|_{L^p(X)}, \tag{2.3}$$

with hidden constant depending only on ρ, l and p.

Lemma 3. *Let $p \in [1,\infty]$ be a fixed real number and $l \geq 0$ be a fixed integer. Then for all $h \in \mathcal{H}$, all $F \in \mathcal{F}_h$, all $E \in \mathcal{E}_h$, all $v \in \mathcal{P}^l(F)$,*

$$\|v\|_{L^p(E)} \lesssim h_F^{-\frac{1}{p}} \|v\|_{L^p(F)} \tag{2.4}$$

with hidden constant depending only on ρ, l and p.

We will also use Koszul complements (see [[14] Section 2.4]). We consider for any face $F \in \mathcal{F}_h$ a point $\boldsymbol{x}_F$ such that $B(\boldsymbol{x}_F, \rho h_F) \subset F$. Then we define the following subspace of $\boldsymbol{\mathcal{P}}^k(F)$:

$$\begin{aligned} \boldsymbol{\mathcal{G}}^k(F) &:= \mathbf{grad}\, \mathcal{P}^{k+1}(F), \quad \boldsymbol{\mathcal{G}}^{c,k}(F) := (\boldsymbol{x} - \boldsymbol{x}_F)^{\perp} \mathcal{P}^{k-1}(F), \\ \boldsymbol{\mathcal{R}}^k(F) &:= \mathbf{rot}\, \mathcal{P}^{k+1}(F), \qquad \boldsymbol{\mathcal{R}}^{c,k}(F) := (\boldsymbol{x} - \boldsymbol{x}_F) \mathcal{P}^{k-1}(F). \end{aligned} \tag{2.5}$$

These spaces are such that:

$$\boldsymbol{\mathcal{P}}^k(F) = \boldsymbol{\mathcal{G}}^k(F) \oplus \boldsymbol{\mathcal{G}}^{c,k}(F) = \boldsymbol{\mathcal{R}}^k(F) \oplus \boldsymbol{\mathcal{R}}^{c,k}(F), \tag{2.6}$$

however the sum is not orthogonal for the L^2 scalar product. We also have the following isomorphisms:

$$\mathbf{rot} \,:\, \mathcal{P}^{0,k}(F) \to \boldsymbol{\mathcal{R}}^{k-1}(F), \tag{2.7}$$

$$\mathrm{div} \,:\, \boldsymbol{\mathcal{R}}^{c,k}(F) \to \mathcal{P}^{k-1}(F). \tag{2.8}$$

We may deduce from the discrete Poincare inequality 2 that $|||\mathbf{rot}||| \lesssim h^{-1}$, $|||\mathrm{div}||| \lesssim h^{-1}$ and from [[14] Lemma 46] that $\left|\left|\left|(\mathbf{rot})^{-1}\right|\right|\right| \lesssim h$, $\left|\left|\left|(\mathrm{div})^{-1}\right|\right|\right| \lesssim h$.

We define the local spaces of Nedelec and of Raviart-Thomas respectively by:

$$\boldsymbol{\mathcal{N}}^k(F) := \boldsymbol{\mathcal{G}}^{k-1}(F) \oplus \boldsymbol{\mathcal{G}}^{c,k}(F), \quad \boldsymbol{\mathcal{RT}}^k(F) := \boldsymbol{\mathcal{R}}^{k-1}(F) \oplus \boldsymbol{\mathcal{R}}^{c,k}(F). \tag{2.9}$$

These spaces are strictly contained between $\boldsymbol{\mathcal{P}}^{k-1}(F)$ and $\boldsymbol{\mathcal{P}}^k(F)$. Another important property given in [[14] Proposition 45] is that for any face $F \in \mathcal{F}_h$ and any edge of this face $E \in \mathcal{E}_F$:

$$\begin{aligned} \forall \boldsymbol{v}_F \in \boldsymbol{\mathcal{N}}^k(F), (\boldsymbol{v}_F)_{|E} \cdot \boldsymbol{t}_E \in \mathcal{P}^{k-1}(E), \\ \forall \boldsymbol{w}_F \in \boldsymbol{\mathcal{RT}}^k(F), (\boldsymbol{w}_F)_{|E} \cdot \boldsymbol{n}_E \in \mathcal{P}^{k-1}(E). \end{aligned} \tag{2.10}$$

In order to fix the notation we write

$$(\boldsymbol{\mathcal{R}}^{c,k}(F))^2 = \begin{pmatrix} \boldsymbol{\mathcal{R}}^{c,k}(F) \\ \boldsymbol{\mathcal{R}}^{c,k}(F) \end{pmatrix}. \tag{2.11}$$

And we take differential operators to be acting row-wise on matrix valued functions. We define the space $\overline{\boldsymbol{\mathcal{R}}}^{c,k}(F)$ by

$$\overline{\boldsymbol{\mathcal{R}}}^{c,k}(F) := \{W \in (\boldsymbol{\mathcal{R}}^{c,k}(F))^2 \,:\, \mathrm{Tr}\, W = 0\}. \tag{2.12}$$

An explicit description of this space is given by (2.16). Let us now construct a complement to this space. First noticing that $\mathrm{Tr}((\boldsymbol{\mathcal{R}}^{c,k}(F))^2) = \mathcal{P}^{0,k}(F)$ we can consider the inverse operator $\boldsymbol{P}_{\mathrm{Tr}}{}^k : \mathcal{P}^{0,k}(F) \to (\boldsymbol{\mathcal{R}}^{c,k}(F))^2$:

$$\boldsymbol{P}_{\mathrm{Tr}}{}^k := \begin{pmatrix} \mathrm{div}^{-1} \\ \mathrm{div}^{-1} \end{pmatrix} \circ \mathbf{grad}\,. \tag{2.13}$$

Where div is the isomorphism from $\boldsymbol{\mathcal{R}}^{c,k}(F)$ into $\mathcal{P}^{k-1}(F)$ given by (2.8). Then we define the space:

$$\overline{\boldsymbol{\mathcal{R}}}^k(F) := \boldsymbol{P}_{\mathrm{Tr}}{}^k \mathcal{P}^{0,k}(F). \tag{2.14}$$

Lemma 36 shows that the spaces $\overline{\boldsymbol{\mathcal{R}}}^{c,k}(F)$ and $\overline{\boldsymbol{\mathcal{R}}}^k(F)$ are complementary.

Remark 4. By construction, we have: $\nabla\cdot\overline{\boldsymbol{\mathcal{R}}}^k(F) = \nabla\cdot\boldsymbol{P}_{\mathrm{Tr}}{}^k\,\mathcal{P}^{0,k}(F) = \mathbf{grad}\ \mathcal{P}^k(F)$.

Remark 5. These spaces are sequential since $\overline{\boldsymbol{\mathcal{R}}}^{c,k} \subset \overline{\boldsymbol{\mathcal{R}}}^{c,k+1}, \overline{\boldsymbol{\mathcal{R}}}^k \subset \overline{\boldsymbol{\mathcal{R}}}^{k+1}$.

We define a matrix valued equivalent to Raviart-Thomas space by

$$\overline{\boldsymbol{\mathcal{RT}}}^k(F) := \overline{\boldsymbol{\mathcal{R}}}^{c,k}(F) \oplus \overline{\boldsymbol{\mathcal{R}}}^{k-1}(F) \oplus (\boldsymbol{\mathcal{R}}^{k-1}(F))^2. \tag{2.15}$$

Remark 6. For $q \in \mathcal{P}^k(F)$, we have $qI \in \overline{\boldsymbol{\mathcal{R}}}^k(F) \oplus (\boldsymbol{\mathcal{R}}^k(F))^2$. Indeed $\nabla\cdot(\boldsymbol{P}_{\mathrm{Tr}}\, q - qI) = 0$ so $\boldsymbol{P}_{\mathrm{Tr}}\, q - qI \in (\boldsymbol{\mathcal{R}}^k(F))^2$ by the isomorphism (2.8) and (2.6).

Lemma 7. *For $\boldsymbol{x}_F \in F$ the point given by* (2.5) *we have* $\nabla\cdot\overline{\boldsymbol{\mathcal{R}}}^{c,k+1}(F) = (\boldsymbol{x} - \boldsymbol{x}_F)^\perp\mathcal{P}^{k-1}(F)$.

Proof. Indeed we have

$$\overline{\boldsymbol{\mathcal{R}}}^{c,k+1}(F) = \begin{pmatrix} -(x-x_F)(y-y_F)Q & -(y-y_F)^2Q \\ (x-x_F)^2Q & (x-x_F)(y-y_F)Q \end{pmatrix}, Q \in \mathcal{P}^{k-1}(F), \tag{2.16}$$

$$\nabla\cdot\overline{\boldsymbol{\mathcal{R}}}^{c,k+1}(F) = \begin{pmatrix} -(y-y_F)(3Q + (x-x_F)\partial_x Q + (y-y_F)\partial_y Q) \\ (x-x_F)(3Q + (x-x_F)\partial_x Q + (y-y_F)\partial_y Q) \end{pmatrix}, Q \in \mathcal{P}^{k-1}(F).$$

□

3 Discrete complex.

We define now the discrete complex. We start by giving all the elements composing it with the locations of their degrees of freedom. Then we define the discrete differential operators and give some basic properties on them.

3.1 Complex definition.

We define four discrete spaces $\underline{X}^k_{\mathbf{rot},F}$, $\underline{\boldsymbol{X}}^k_{\nabla,F}$, $\underline{\boldsymbol{X}}^{k+1}_{\boldsymbol{L}^2,F}$ and $\underline{\boldsymbol{X}}^k_{L^2,F}$. The diagram 3.1 summarize their connection with each other and with their continuous counterpart.

$$\begin{array}{ccccc} & & \boldsymbol{L}^2(\Omega) & & \\ & & \nabla\uparrow & & \\ H^2(\Omega) & \xrightarrow{\mathbf{rot}} & \boldsymbol{H}^1(\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) \\ \downarrow{\scriptstyle \underline{I}^k_{\mathbf{rot},h}} & & \downarrow{\scriptstyle \underline{\boldsymbol{I}}^k_{\nabla,h}} & & \downarrow{\scriptstyle \pi^k_{\mathcal{P},\mathcal{F}_h}} \quad {\scriptstyle \underline{\boldsymbol{I}}^k_{\boldsymbol{L}^2,h}} \\ \underline{X}^k_{\mathbf{rot},h} & \xrightarrow{\underline{\boldsymbol{C}}^k_h} & \underline{\boldsymbol{X}}^k_{\nabla,h} & \xrightarrow{\underline{D}^k_h} & \underline{\boldsymbol{X}}^k_{L^2,h} \\ & & \downarrow{\scriptstyle \underline{\nabla}^{k+1}_h} & & \\ & & \underline{\boldsymbol{X}}^{k+1}_{\boldsymbol{L}^2,h} & & \end{array} \tag{3.1}$$

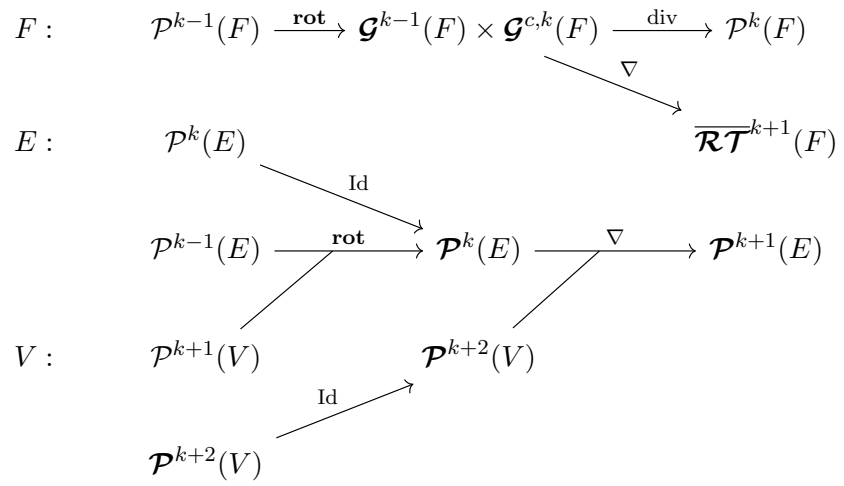


Figure 1: Usage of the local degrees of freedom for the discrete differential operators.

Notice that the interpolators (defined in 3.2) require more smoothness than the spaces shown in 3.1. Discrete spaces are defined by:

$$\begin{aligned}\underline{X}^k_{\mathbf{rot},h} :=\{\underline{q}_h = ((q_E, q_{E'})_{E\in\mathcal{E}_h}, (\boldsymbol{q}_{V'})_{V\in\mathcal{V}_h}, (q_F)_{F\in\mathcal{F}_h}) \,:\, & q_E \in \mathcal{P}^{k+1}_c(\mathcal{E}_h), \\ & q_{E'} \in \mathcal{P}^k(E), \forall E \in \mathcal{E}_h, \\ & \boldsymbol{q}_{V'} \in \mathbb{R}^2, \forall V \in \mathcal{V}_h, \\ & q_F \in \mathcal{P}^{k-1}(F), \forall F \in \mathcal{F}_h\},\end{aligned} \tag{3.2}$$

$$\begin{aligned}\underline{\boldsymbol{X}}^k_{\nabla,h} :=\{\underline{\boldsymbol{v}}_h = ((\boldsymbol{v}_E)_{E\in\mathcal{E}_h}, (\boldsymbol{v}_{\boldsymbol{\mathcal{G}},F}, \boldsymbol{v}^c_{\boldsymbol{\mathcal{G}},F})_{F\in\mathcal{F}_h}) \,:\, & \boldsymbol{v}_E \in \boldsymbol{\mathcal{P}}^{k+2}_c(\mathcal{E}_h), \\ & \boldsymbol{v}_{\boldsymbol{\mathcal{G}},F} \in \boldsymbol{\mathcal{G}}^{k-1}(F), \boldsymbol{v}^c_{\boldsymbol{\mathcal{G}},F} \in \boldsymbol{\mathcal{G}}^{c,k}(F), \forall F \in \mathcal{F}_h\},\end{aligned} \tag{3.3}$$

$$\begin{aligned}\underline{\boldsymbol{X}}^{k+1}_{\boldsymbol{L}^2,h} :=\{\underline{\boldsymbol{W}}_h = ((\boldsymbol{W}_E)_{E\in\mathcal{E}_h}, (\boldsymbol{W}_F)_{F\in\mathcal{F}_h}) \,:\, & \boldsymbol{W}_E \in \boldsymbol{\mathcal{P}}^{k+1}(E), \forall E \in \mathcal{E}_h, \\ & \boldsymbol{W}_F \in \overline{\boldsymbol{\mathcal{RT}}}^{k+1}(F), \forall F \in \mathcal{F}_h\},\end{aligned} \tag{3.4}$$

$$\underline{\boldsymbol{X}}^k_{L^2,h} :=\{\underline{w}_h = ((w_F)_{F\in Fh}) \,:\, w_F \in \mathcal{P}^k(F), \forall F \in \mathcal{F}_h\}. \tag{3.5}$$

Figure 1 summarize the involvement of the various degrees of freedom with the differential operators.

For a given face F we define the local discrete spaces $\underline{X}^k_{\mathbf{rot},F}$, $\underline{\boldsymbol{X}}^k_{\nabla,F}$, $\underline{\boldsymbol{X}}^k_{L^2,F}$ and $\underline{\boldsymbol{X}}^{k+1}_{\boldsymbol{L}^2,F}$ as the restriction of the global one to F, i.e. containing only the components attached to F and those attached to the edges and vertices lying on its boundary. We define in the same way the local discrete spaces attached to an edge E.

3.2 Interpolators.

In this section we define the interpolator linking discrete spaces to their continuous counterpart. Since we project on objects of lower dimension (edges and vertices) we will need a somewhat high smoothness for the continuous functions. For a vertex $V \in \mathcal{V}_h$ we define $\boldsymbol{x}_V \in \mathbb{R}^2$ to be its coordinate. The interpolator on the space $\underline{X}^k_{\mathbf{rot},h}$ is defined for any $q \in C^1(\overline{\Omega})$ by

$$\underline{I}^k_{\mathbf{rot},h} q = ((q_E, \pi^k_{\mathcal{P},E}(\mathbf{rot}\, q \cdot \boldsymbol{t}_E))_{E\in\mathcal{E}_h}, (\mathbf{rot}\, q(V))_{V\in\mathcal{V}_h}, (\pi^{k-1}_{\mathcal{P},F}(q))_{F\in\mathcal{F}_h}), \tag{3.6}$$

where for any edge $E \in \mathcal{E}_h$, q_E is such that $\pi^{k-1}_{\mathcal{P},E}(q_E) = \pi^{k-1}_{\mathcal{P},E}(q)$ and for any vertex $V \in \mathcal{V}_E$, $q_E(\boldsymbol{x}_V) = q(\boldsymbol{x}_V)$.

The interpolator on the space $\underline{\boldsymbol{X}}^k_{\nabla,h}$ is defined for any $\boldsymbol{v} \in \boldsymbol{C}^0(\overline{\Omega})$ by

$$\underline{\boldsymbol{I}}^k_{\nabla,h}\boldsymbol{v} = ((\boldsymbol{v}_E)_{E\in\mathcal{E}_h}, (\boldsymbol{\pi}^{k-1}_{\boldsymbol{\mathcal{G}},F}(\boldsymbol{v}), \boldsymbol{\pi}^{c,k}_{\boldsymbol{\mathcal{G}},F}(\boldsymbol{v}))_{F\in\mathcal{F}_h}), \tag{3.7}$$

where for any edge $E \in \mathcal{E}_h$, $\boldsymbol{v}_E$ is such that $\boldsymbol{\pi}^k_{\boldsymbol{\mathcal{P}},E}(\boldsymbol{v}_E) = \boldsymbol{\pi}^k_{\boldsymbol{\mathcal{P}},E}(\boldsymbol{v})$ and for any vertex $V \in \mathcal{V}_E$, $\boldsymbol{v}_E(\boldsymbol{x}_V) = \boldsymbol{v}(\boldsymbol{x}_V)$.

The interpolator on the space $\underline{\boldsymbol{X}}^{k+1}_{\boldsymbol{L}^2,h}$ is defined for any $\boldsymbol{W} \in (\boldsymbol{C}^0(\overline{\Omega}))^2$ by

$$\underline{\boldsymbol{I}}^k_{\boldsymbol{L}^2,h}\boldsymbol{W} = ((\boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{P}},E}(\boldsymbol{W}\cdot\boldsymbol{t}_E))_{E\in\mathcal{E}_h}, (\boldsymbol{\pi}^{k+1}_{\overline{\boldsymbol{\mathcal{R}}}\boldsymbol{\mathcal{T}},F}(\boldsymbol{W}))_{F\in\mathcal{F}_h}). \tag{3.8}$$

The interpolator on the space $\underline{\boldsymbol{X}}^k_{L^2,h}$ is just $\pi^k_{\mathcal{P},\mathcal{F}_h}$, the piecewise L^2-orthogonal projection on spaces $\mathcal{P}^k(F), F \in \mathcal{F}_h$.

3.3 Curl.

In the following sections we define the discrete operators starting from the discrete curl operator $\underline{\boldsymbol{C}}^k_h$. The operator $\underline{\boldsymbol{C}}^k_h$ is the collection of the local discrete operators (3.11) acting on the edges and faces. For any edge $E \in \mathcal{E}_h$ we define the operator $\boldsymbol{C}^k_E : \underline{X}^k_{\mathbf{rot},E} \to \boldsymbol{\mathcal{P}}^{k+2}_c(\mathcal{E}_h)$ such that $\forall \underline{q}_E = (q_E, q_{E'}, (q_{V'})_{V\in\mathcal{V}_E}) \in \underline{X}^k_{\mathbf{rot},E}$

$$\boldsymbol{C}^k_E\underline{q}_E = \boldsymbol{v}_E. \tag{3.9}$$

Where $\boldsymbol{v}_E$ is such that $\boldsymbol{\pi}^k_{\boldsymbol{\mathcal{P}},E}(\boldsymbol{v}_E) = q_{E'}\boldsymbol{t}_E - \dot{q_E}\boldsymbol{n}_E$ and $\forall V \in \mathcal{V}_E$, $\boldsymbol{v}_E(\boldsymbol{x}_V) = q_{V'}$. With $\dot{q_E}$ the derivative of q_E along the edge E (oriented by $\boldsymbol{t}_E$).

For any face $F \in \mathcal{F}_h$ we define the operator $\boldsymbol{C}^k_F : \underline{X}^k_{\mathbf{rot},F} \to \boldsymbol{\mathcal{P}}^k(F)$ such that $\forall \underline{q}_F = ((q_E, q_{E'})_{E\in\mathcal{E}_F}, (q_{V'})_{V\in\mathcal{V}_F}, q_F) \in \underline{X}^k_{\mathbf{rot},F}$, $\forall \boldsymbol{r}_F \in \boldsymbol{\mathcal{P}}^k(F)$

$$\int_F \boldsymbol{C}^k_F\underline{q}_F \cdot \boldsymbol{r}_F = \int q_F \operatorname{rot}\boldsymbol{r}_F + \sum_{E\in EF}\omega_{FE}\int_E q_E\boldsymbol{r}_F\cdot\boldsymbol{t}_E. \tag{3.10}$$

The full operator $\underline{\boldsymbol{C}}^k_F : \underline{X}^k_{\mathbf{rot},F} \to \underline{\boldsymbol{X}}^k_{\nabla,F}$ is defined as the collection and projection of the local operators. Explicitly for all $\underline{q}_h \in \underline{X}^k_{\mathbf{rot},h}$

$$\underline{\boldsymbol{C}}^k_F\underline{q}_F = ((\boldsymbol{C}^k_E\underline{q}_E)_{E\in\mathcal{E}_F}, (\boldsymbol{\pi}^{k-1}_{\boldsymbol{\mathcal{G}},F}(\boldsymbol{C}^k_F\underline{q}_F), \boldsymbol{\pi}^{c,k}_{\boldsymbol{\mathcal{G}},F}(\boldsymbol{C}^k_F\underline{q}_F))). \tag{3.11}$$

The global operator $\underline{\boldsymbol{C}}^k_h$ is obtained by gathering the local operators $\underline{\boldsymbol{C}}^k_F, F \in \mathcal{F}_h$.

3.4 Jacobian.

Likewise, we begin by defining the local operator on edges: $\nabla^{k+1}_E : \underline{\boldsymbol{X}}^k_{\nabla,E} \to \boldsymbol{\mathcal{P}}^{k+1}(E)$ such that $\forall \boldsymbol{v}_E \in \underline{\boldsymbol{X}}^k_{\nabla,E}$

$$\nabla^{k+1}_E\,\boldsymbol{v}_E = \dot{\boldsymbol{v}}_E. \tag{3.12}$$

Where the derivative is taken along the tangent $\boldsymbol{t}_E$ of the edge E.

We define the local operator on faces: $\nabla^{k+1}_F : \underline{\boldsymbol{X}}^k_{\nabla,F} \to \overline{\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{T}}}^{k+1}(F)$ such that $\forall \underline{\boldsymbol{v}}_F \in \underline{\boldsymbol{X}}^k_{\nabla,F}$, $\forall \boldsymbol{W}_F = \boldsymbol{W}^c_{\overline{\boldsymbol{\mathcal{R}}},F} + \boldsymbol{W}_{\overline{\boldsymbol{\mathcal{R}}},F} + \boldsymbol{W}_{\boldsymbol{\mathcal{R}},F} \in \overline{\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{T}}}^{k+1}(F)$,

$$\int_F \nabla^{k+1}_F(\underline{\boldsymbol{v}}_F) : \boldsymbol{W}_F = -\int_F \boldsymbol{v}^c_{\boldsymbol{\mathcal{G}},F}\cdot\nabla\cdot(\boldsymbol{W}^c_{\overline{\boldsymbol{\mathcal{R}}},F}) - \int_F \boldsymbol{v}_{\boldsymbol{\mathcal{G}},F}\cdot\nabla\cdot(\boldsymbol{W}_{\overline{\boldsymbol{\mathcal{R}}},F}) + \sum_{E\in\mathcal{E}_F}\omega_{FE}\int_E \boldsymbol{v}_E\boldsymbol{W}_F\boldsymbol{n}_E. \tag{3.13}$$

Remark 8. Since $\nabla\cdot(\boldsymbol{\mathcal{R}}^k(F))^2 = 0$ by 6 we see that $\forall q \in \mathcal{P}^k(F)$,

$$\int_F \nabla_F^{k+1}(\underline{\boldsymbol{v}}_F) : (q\ \mathbf{I}_{2,2}) = \int_F \boldsymbol{v}_{\boldsymbol{\mathcal{G}},F} \cdot \mathbf{grad}\, q + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E q \boldsymbol{v}_E \boldsymbol{n}_E$$

The full and global operators $\underline{\nabla}_F^{k+1}$ and $\underline{\nabla}_h^{k+1}$ are merely collections of local operator:

$$\underline{\nabla}_F^{k+1}\underline{\boldsymbol{v}}_F = ((\nabla_E^{k+1}\boldsymbol{v}_E)_{E\in\mathcal{E}_F}, \nabla_F^{k+1}\underline{\boldsymbol{v}}_F), \tag{3.14}$$

$$\underline{\nabla}_h^{k+1}\underline{\boldsymbol{v}}_h = ((\nabla_E^{k+1}\boldsymbol{v}_E)_{E\in\mathcal{E}_h}, (\nabla_F^{k+1}\underline{\boldsymbol{v}}_F)_{F\in\mathcal{F}_h}). \tag{3.15}$$

We prove a first commutative property:

Lemma 9. *For all $E \in \mathcal{E}_h$ and all $F \in \mathcal{F}_h$ the following relations hold:*

$$\nabla_E^{k+1}(\underline{\boldsymbol{I}}_{\nabla,E}^k \boldsymbol{v}) = \boldsymbol{\pi}_{\boldsymbol{\mathcal{P}},E}^{k+1}(\dot{\boldsymbol{v}}), \qquad \forall \boldsymbol{v} \in \boldsymbol{H}^1(E). \tag{3.16}$$

$$\nabla_F^{k+1}(\underline{\boldsymbol{I}}_{\nabla,F}^k \boldsymbol{v}) = \boldsymbol{\pi}_{\overline{\boldsymbol{\mathcal{R}}}\overline{\boldsymbol{\mathcal{T}}},F}^{k+1}(\nabla \boldsymbol{v}), \qquad \forall \boldsymbol{v} \in \boldsymbol{H}^1(F) \cap \boldsymbol{C}^0(\overline{F}). \tag{3.17}$$

Proof. The equation (3.16) is deduced exactly as it is done for [[14] Equation 3.8]. Let prove (3.17): For all $\boldsymbol{v} \in \boldsymbol{H}^1(F) \cap \boldsymbol{C}^0(\overline{F})$ and all $\boldsymbol{W}_F \in \overline{\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{T}}}^{k+1}(F)$,

$$\begin{aligned}
\int_F \nabla_F^{k+1}(\underline{\boldsymbol{I}}_{\nabla,F}^k \boldsymbol{v}) : \boldsymbol{W}_F &= -\int_F \boldsymbol{\pi}_{\boldsymbol{\mathcal{G}},F}^{c,k}(\boldsymbol{v}) \cdot \nabla\cdot(\boldsymbol{W}_{\overline{\boldsymbol{\mathcal{R}}},F}^c) - \int_F \boldsymbol{\pi}_{\boldsymbol{\mathcal{G}},F}^{k-1}\boldsymbol{v} \cdot \nabla\cdot(\boldsymbol{W}_{\overline{\boldsymbol{\mathcal{R}}},F})) \\
&\qquad + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E \boldsymbol{\pi}_{\boldsymbol{\mathcal{P}},E}^{c,k+2}(\boldsymbol{v}) \boldsymbol{W}_F \boldsymbol{n}_E \\
&= -\int_F \boldsymbol{v} \cdot \nabla\cdot(\boldsymbol{W}_{\overline{\boldsymbol{\mathcal{R}}},F}^c) - \int_F \boldsymbol{v} \cdot \nabla\cdot(\boldsymbol{W}_{\overline{\boldsymbol{\mathcal{R}}},F})) \\
&\qquad + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E \boldsymbol{v} \boldsymbol{W}_F \boldsymbol{n}_E \\
&= -\int_F \boldsymbol{v} \cdot \nabla\cdot(\boldsymbol{W}_F) + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E \boldsymbol{v} \boldsymbol{W}_F \boldsymbol{n}_E \\
&= \int_F \nabla \boldsymbol{v} : \boldsymbol{W}_F.
\end{aligned}$$

Where we used the definitions 7 and 2.14 to remove the first two projections ($\boldsymbol{\pi}_{\boldsymbol{\mathcal{G}},F}^{c,k}$ and $\boldsymbol{\pi}_{\boldsymbol{\mathcal{G}},F}^{k-1}$), the property (2.10) to remove the last projection and the integration by parts to conclude. □

Any face $F \in \mathcal{F}_h$ has several polynomials attached to it (on the face itself and on its edges). In order to combine all these polynomials into a single one defined on F we introduce a reconstruction operator $\gamma_{\nabla,F}^{k+1} : \underline{\boldsymbol{X}}_{\nabla,F}^k \to (\boldsymbol{\mathcal{P}}^{k+1}(F))^2$ implicitly defined by the relation: $\forall \boldsymbol{W}_F \in (\boldsymbol{\mathcal{R}}^{c,k+2}(F))^2$, $\forall \underline{\boldsymbol{v}}_F \in \underline{\boldsymbol{X}}_{\nabla,F}^k$,

$$\int_F \gamma_{\nabla,F}^{k+1}(\underline{\boldsymbol{v}}_F) \cdot \nabla\cdot \boldsymbol{W}_F = -\int_F \nabla_F^{k+1}\underline{\boldsymbol{v}}_F : \boldsymbol{W}_F + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E \boldsymbol{v}_E \boldsymbol{W}_F \boldsymbol{n}_E. \tag{3.18}$$

The isomorphism (2.8) ensure the well-posedness.

Remark 10. The relation (3.18) also holds for all $\boldsymbol{W}_F \in (\boldsymbol{\mathcal{P}}^k(F))^2$. Indeed if $\boldsymbol{W}_F$ belongs to $(\boldsymbol{\mathcal{R}}^k(F))^2$ then $\nabla\cdot \boldsymbol{W}_F = 0$ and the left-hand side of (3.18) is null. And since $(\boldsymbol{\mathcal{R}}^k(F))^2 \subset (\boldsymbol{\mathcal{P}}^k(F))^2 \subset \overline{\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{T}}}^{k+1}(F)$ we can apply (3.13) to show that the right-hand side is also zero. Hence, the relation holds for all $(\boldsymbol{\mathcal{R}}^k(F))^2 \oplus (\boldsymbol{\mathcal{R}}^{c,k+2}(F))^2 \supset (\boldsymbol{\mathcal{P}}^k(F))^2$.

Lemma 11 (Consistency properties)**.** *For all $F \in \mathcal{F}_h$ the following relations hold:*

$$\gamma^{k+1}_{\nabla,F}(\underline{\boldsymbol{I}}^k_{\nabla,F}\boldsymbol{v}) = \boldsymbol{v}, \qquad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{P}}^{k+1}(F). \tag{3.19}$$

$$\begin{aligned} \boldsymbol{\pi}^{c,k}_{\boldsymbol{\mathcal{G}},F}(\gamma^{k+1}_{\nabla,F}\underline{\boldsymbol{v}}_F) &= \boldsymbol{v}^c_{\boldsymbol{\mathcal{G}},F}, \\ \boldsymbol{\pi}^{k-1}_{\boldsymbol{\mathcal{G}},F}(\gamma^{k+1}_{\nabla,F}\underline{\boldsymbol{v}}_F) &= \boldsymbol{v}_{\boldsymbol{\mathcal{G}},F}, \end{aligned} \qquad \forall \underline{\boldsymbol{v}}_F \in \underline{\boldsymbol{X}}^k_{\nabla,F}. \tag{3.20}$$

Proof. Let us show (3.19): For any $\boldsymbol{v} \in \boldsymbol{\mathcal{P}}^{k+1}(F)$, since $\nabla \boldsymbol{v} \in (\boldsymbol{\mathcal{P}}^k(F))^2 \subset \overline{\boldsymbol{\mathcal{RT}}}^{k+1}$ the equation (3.17) gives $\nabla^{k+1}_F(\underline{\boldsymbol{I}}^k_{\nabla,F}\boldsymbol{v}) = \nabla \boldsymbol{v}$. Moreover $\boldsymbol{v}_{|E}$ is continuous of degree $k+1 < k+2$ so by the definition (3.18) we have for all $\boldsymbol{W}_F \in (\boldsymbol{\mathcal{RT}}^{c,k+2}(F))^2$:

$$\begin{aligned} \int_F \gamma^{k+1}_{\nabla,F}(\underline{\boldsymbol{I}}^k_{\nabla,F}\boldsymbol{v}) &= -\int_F \nabla \boldsymbol{v} : \boldsymbol{W}_F + \sum_{E\in\mathcal{E}_F} \omega_{FE}\int_E \boldsymbol{v}\boldsymbol{W}_F\boldsymbol{n}_E \\ &= \int_F \boldsymbol{v}\cdot\nabla\cdot\boldsymbol{W}_F. \end{aligned}$$

Now to prove (3.20) we show that for all $\boldsymbol{W}_F \in \overline{\boldsymbol{\mathcal{R}}}^{c,k+1}(F) \subset (\boldsymbol{\mathcal{R}}^{c,k+2}(F))^2$,

$$\begin{aligned} \int_F \boldsymbol{\pi}^{c,k}_{\boldsymbol{\mathcal{G}},F}(\gamma^{k+1}_{\nabla,F}\underline{\boldsymbol{v}}_F)\cdot\nabla\cdot\boldsymbol{W}_F &= \int_F (\gamma^{k+1}_{\nabla,F}\underline{\boldsymbol{v}}_F)\cdot\nabla\cdot\boldsymbol{W}_F \\ &= -\int_F \nabla^{k+1}_F(\underline{\boldsymbol{v}}_F) : \boldsymbol{W}_F + \sum_{E\in\mathcal{E}_F}\omega_{FE}\int_E \boldsymbol{v}_E\boldsymbol{W}_F\boldsymbol{n}_E \\ &= \int_F \boldsymbol{v}^c_{\boldsymbol{\mathcal{G}},F}\cdot\nabla\cdot\boldsymbol{W}_F + \sum_{E\in\mathcal{E}_F}\omega_{FE}\int_E (\boldsymbol{v}_E - \boldsymbol{v}_E)\boldsymbol{W}_F\boldsymbol{n}_E. \end{aligned}$$

Where we used 7 to show the first equality, (3.18) for the second and (3.13) and (2.12) for the last hence, $\boldsymbol{\pi}^{c,k}_{\boldsymbol{\mathcal{G}},F}(\gamma^{k+1}_{\nabla,F}\underline{\boldsymbol{v}}_F) = \boldsymbol{v}^c_{\boldsymbol{\mathcal{G}},F}$. Likewise for all $\boldsymbol{W}_F \in \mathcal{P}^k(F)\,\mathbf{I}_{2,2}$: we have $\nabla\cdot\mathcal{P}^k(F)\,\mathbf{I}_{2,2} = \boldsymbol{\mathcal{G}}^{k-1}(F)$. Since the relation (3.18) holds thanks to 10 and since $\mathcal{P}^k(F) \subset \overline{\boldsymbol{\mathcal{RT}}}^{k+1}(F)$ we can follow the same steps and show that $\boldsymbol{\pi}^{k-1}_{\boldsymbol{\mathcal{G}},F}(\gamma^{k+1}_{\nabla,F}\underline{\boldsymbol{v}}_F) = \boldsymbol{\pi}^{k-1}_{\boldsymbol{\mathcal{G}},F}$. □

3.5 Divergence.

Finally, we define the discrete divergence operator, for all $F \in \mathcal{F}_h$ by:

$$D^k_F := \operatorname{Tr}\nabla^{k+1}_F \in \mathcal{P}^k(F).$$

As in the continuous case the divergence is the trace of the gradient, but we can also define it by a formula mimicking the integration by parts:

$$\begin{aligned} \int_F D^k_F\underline{\boldsymbol{v}}_F w_F &= \int_F \operatorname{Tr}(\nabla^{k+1}_F\,\underline{\boldsymbol{v}}_F)w_F \\ &= \int_F \nabla^{k+1}_F\,\underline{\boldsymbol{v}}_F : w_F\,\mathbf{I}_{2,2} \\ &= -\int_F \boldsymbol{v}_{\boldsymbol{\mathcal{G}},F}\,\mathbf{grad}\,w_F + \sum_{E\in\mathcal{E}_F}\omega_{FE}\int_E \boldsymbol{v}_E\cdot\boldsymbol{n}_E w_F \end{aligned} \tag{3.21}$$

Where we used that $\mathcal{P}^k(F)\,\mathbf{I}_{2,2} \subset \overline{\boldsymbol{\mathcal{RT}}}^{k+1}(F)$ by 6. We get the same definition as the one of the de Rham complex of [14].

3.6 Discrete L^2-product.

We build scalar product on discrete spaces. They are made of the sum of the L^2 scalar product on each face and of a stabilization term taking the lower dimensional objects (edges and vertices) into account. First we define them locally for all $F \in \mathcal{F}_h$: For all $\underline{\boldsymbol{v}}_F, \underline{\boldsymbol{w}}_F \in \underline{\boldsymbol{X}}^k_{\nabla,F}$ we set

$$(\underline{\boldsymbol{v}}_F, \underline{\boldsymbol{w}}_F)_{\nabla,F} = \int_F \gamma^{k+1}_{\nabla,F}\underline{\boldsymbol{v}}_F \cdot \gamma^{k+1}_{\nabla,F}\underline{\boldsymbol{w}}_F + \mathrm{s}_{\nabla,F}(\underline{\boldsymbol{v}}_F, \underline{\boldsymbol{w}}_F), \tag{3.22}$$

$$\mathrm{s}_{\nabla,F}(\underline{\boldsymbol{v}}_F, \underline{\boldsymbol{w}}_F) = \sum_{E\in\mathcal{E}_F} h_E(\gamma^{k+1}_{\nabla,F}\underline{\boldsymbol{v}}_F - \boldsymbol{v}_E)\cdot(\gamma^{k+1}_{\nabla,F}\underline{\boldsymbol{w}}_F - \boldsymbol{w}_F). \tag{3.23}$$

For all $\underline{\boldsymbol{V}}_F, \underline{\boldsymbol{W}}_F \in \underline{\boldsymbol{X}}^{k+1}_{\boldsymbol{L}^2,F}$ we set

$$(\underline{\boldsymbol{V}}_F, \underline{\boldsymbol{W}}_F)_{\boldsymbol{L}^2,F} = \int_F \boldsymbol{V}_F : \boldsymbol{W}_F + \mathrm{s}_{\boldsymbol{L}^2,F}(\underline{\boldsymbol{V}}_F, \underline{\boldsymbol{W}}_F). \tag{3.24}$$

$$\mathrm{s}_{\boldsymbol{L}^2,F}(\underline{\boldsymbol{V}}_F, \underline{\boldsymbol{W}}_F) = \sum_{E\in\mathcal{E}_F} h_E(\boldsymbol{V}_F\cdot\boldsymbol{t}_E - \boldsymbol{V}_E)\cdot(\boldsymbol{W}_F\cdot\boldsymbol{t}_E - \boldsymbol{W}_E). \tag{3.25}$$

Global scalar products are then merely the sum of local scalar product over every face $F \in \mathcal{F}_h$. For all $\underline{\boldsymbol{v}}_F \in \underline{\boldsymbol{X}}^k_{\nabla,F}$ and $\underline{\boldsymbol{W}}_F \in \underline{\boldsymbol{X}}^{k+1}_{\boldsymbol{L}^2,F}$ the norm induced by this scalar product is denoted by:

$$\|\underline{\boldsymbol{v}}_F\|_{\nabla,F} = (\underline{\boldsymbol{v}}_F, \underline{\boldsymbol{v}}_F)^{1/2}_{\nabla,F}, \quad \|\underline{\boldsymbol{W}}_F\|_{\boldsymbol{L}^2,F} = (\underline{\boldsymbol{W}}_F, \underline{\boldsymbol{W}}_F)^{1/2}_{\boldsymbol{L}^2,F}.$$

We also define norms built from the sum over the objects of every dimension. For all $\underline{q}_F \in \underline{X}^k_{\mathbf{rot},F}$ we define

$$\left|\!\left|\!\left|\underline{q}_F\right|\!\right|\!\right|^2_{\mathbf{rot},F} = \|q_F\|^2_F + \sum_{E\in\mathcal{E}_F} h_E\left(\|q_E\|^2_E + \|q_{E'}\|^2_E + \sum_{V\in\mathcal{V}_E} h_E|\boldsymbol{q}_{V'}|^2\right). \tag{3.26}$$

For all $\underline{\boldsymbol{v}}_F \in \underline{\boldsymbol{X}}^k_{\nabla,F}$ we define

$$|\!|\!|\underline{\boldsymbol{v}}_F|\!|\!|^2_{\nabla,F} = \|\boldsymbol{v}_{\boldsymbol{G},F}\|^2_F + \|\boldsymbol{v}^c_{\boldsymbol{G},F}\|^2_F + \sum_{E\in\mathcal{E}_F} h_E\|\boldsymbol{v}_E\|^2_E. \tag{3.27}$$

For all $\underline{\boldsymbol{W}}_F \in \underline{\boldsymbol{X}}^{k+1}_{\boldsymbol{L}^2,F}$ we define

$$|\!|\!|\underline{\boldsymbol{W}}_F|\!|\!|^2_{\boldsymbol{L}^2,F} = \|\boldsymbol{W}_F\|^2_F + \sum_{E\in\mathcal{E}_F} h_E\|\boldsymbol{W}_E\|^2_E. \tag{3.28}$$

And for all $p_F \in \underline{\boldsymbol{X}}^k_{L^2,F}$ we define

$$\|p_F\|^2_{L^2,F} = \|p_F\|^2_F. \tag{3.29}$$

We show the equivalence between the norm induced by (3.22) and (3.27) in lemma 14 and the equivalence between those induced by (3.24) and (3.28) in lemma 15.

We define the global norms over Ω as the sum of the local norms over every face $F \in \mathcal{F}_h$, i.e. $|\!|\!|\underline{\boldsymbol{v}}_h|\!|\!|^2_{\nabla,h} = \sum_{F\in\mathcal{F}_h} |\!|\!|\underline{\boldsymbol{v}}_F|\!|\!|^2_{\nabla,F}$.

Lemma 12 (Inverse Poincaré inequality)**.** *For all $F \in \mathcal{F}_h$ and all $\underline{\boldsymbol{v}}_F \in \underline{\boldsymbol{X}}^k_{\nabla,F}$ it holds:*

$$\|\nabla^{k+1}_F \underline{\boldsymbol{v}}_F\| \lesssim h^{-1}|\!|\!|\underline{\boldsymbol{v}}_F|\!|\!|_{\nabla,F}.$$

Proof. Let $F \in \mathcal{F}_h$ and $\underline{\boldsymbol{v}}_F \in \underline{\boldsymbol{X}}^k_{\nabla,F}$. We use the discrete inverse inequality [[10] Lemma 1.28] to show that:

$$\begin{aligned}\| \nabla_F^{k+1} \underline{\boldsymbol{v}}_F\|^2 &= \int_F \nabla_F^{k+1} \underline{\boldsymbol{v}}_F : \nabla_F^{k+1} \underline{\boldsymbol{v}}_F \\ &\lesssim \|\boldsymbol{v}^c_{\boldsymbol{\mathcal{G}},F}\| h^{-1} \| \nabla_F^{k+1} \underline{\boldsymbol{v}}_F\| + \|\boldsymbol{v}_{\boldsymbol{\mathcal{G}},F}\| h^{-1} \| \nabla_F^{k+1} \underline{\boldsymbol{v}}_F\| + \|\boldsymbol{v}_E\| h^{-\frac{1}{2}} \| \nabla_F^{k+1} \underline{\boldsymbol{v}}_F\|_F \\ &\lesssim h^{-1} \| \nabla_F^{k+1} \underline{\boldsymbol{v}}_F\| (\|\boldsymbol{v}^c_{\boldsymbol{\mathcal{G}},F}\| + \|\boldsymbol{v}_{\boldsymbol{\mathcal{G}},F}\| + \sum_{E\in\mathcal{E}_F} h^{\frac{1}{2}} \|\boldsymbol{v}_E\|).\end{aligned}$$

□

Lemma 13 (Boundedness of local potential)**.** *For all $F \in \mathcal{F}_h$ and all $\underline{\boldsymbol{v}}_F \in \underline{\boldsymbol{X}}^k_{\nabla,F}$ it holds:*

$$\|\gamma^{k+1}_{\nabla,F} \underline{\boldsymbol{v}}_F\| \lesssim |||\underline{\boldsymbol{v}}_F|||_{\nabla,F} \tag{3.30}$$

Proof. Let $F \in \mathcal{F}_h$, $\underline{\boldsymbol{v}}_F \in \underline{\boldsymbol{X}}^k_{\nabla,F}$, since $\gamma^{k+1}_{\nabla,F} \underline{\boldsymbol{v}}_F \in \boldsymbol{\mathcal{P}}^{k+1}(F)$ there is $\boldsymbol{W}_F \in (\boldsymbol{\mathcal{R}}^{c,k+2}(F))^2$ such that $\nabla\cdot \boldsymbol{W}_F = \gamma^{k+1}_{\nabla,F} \underline{\boldsymbol{v}}_F$ hence:

$$\begin{aligned}\|\gamma^{k+1}_{\nabla,F} \underline{\boldsymbol{v}}_F\|^2 &= \int_F \gamma^{k+1}_{\nabla,F} \underline{\boldsymbol{v}}_F \cdot \gamma^{k+1}_{\nabla,F} \underline{\boldsymbol{v}} \\ &= \int_F \gamma^{k+1}_{\nabla,F} \underline{\boldsymbol{v}}_F \cdot \nabla\cdot \boldsymbol{W}_F \\ &= -\int_F \nabla_F^{k+1} \underline{\boldsymbol{v}}_F : \boldsymbol{W}_F + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E \boldsymbol{v}_E \boldsymbol{W}_F \boldsymbol{n}_E \\ &\leq \| \nabla_F^{k+1} \underline{\boldsymbol{v}}_F\| \, \|\boldsymbol{W}_F\| + \sum_{E\in\mathcal{E}_F} \|\boldsymbol{v}_E\| \|\boldsymbol{W}_F\|_E \\ &\lesssim h^{-1} |||\underline{\boldsymbol{v}}_F|||_{\nabla,F} \|\boldsymbol{W}_F\| + h^{-1} \sum_{E\in\mathcal{E}_F} h^{\frac{1}{2}} \|\boldsymbol{v}_E\| \|\boldsymbol{W}_F\|_F \\ &\lesssim h^{-1} \|\boldsymbol{W}_F\| |||\underline{\boldsymbol{v}}_F|||_{\nabla,F}.\end{aligned}$$

Where we used the Cauchy-Schwarz inequality to get the fourth line, 12 for the fifth, the discrete trace inequality [[10] Lemma 1.32] for the sixth. We can conclude since $\|\boldsymbol{W}_F\| \lesssim h \|\gamma^{k+1}_{\nabla,F} \underline{\boldsymbol{v}}_F\|$ thanks to the upper bound on the operator norm of the isomorphism (2.8). □

Lemma 14. *It holds, for all $F \in \mathcal{F}_h$ and all $\underline{\boldsymbol{v}}_F \in \underline{\boldsymbol{X}}^k_{\nabla,F}$,*

$$\|\underline{\boldsymbol{v}}_F\|_{\nabla,F} \approx |||\underline{\boldsymbol{v}}_F|||_{\nabla,F}, \forall \underline{\boldsymbol{v}}_F \in \underline{\boldsymbol{X}}^k_{\nabla,F}.$$

Proof. From the definitions (3.22) and (3.23) we have:

$$\begin{aligned}\|\underline{\boldsymbol{v}}_F\|^2_{\nabla,F} &= \int_F \gamma^{k+1}_{\nabla,F} \underline{\boldsymbol{v}}_F \cdot \gamma^{k+1}_{\nabla,F} \underline{\boldsymbol{v}}_F + \sum_{E\in\mathcal{E}_F} h_E \|\gamma^{k+1}_{\nabla,F} \underline{\boldsymbol{v}}_F - \boldsymbol{v}_E\|^2 \\ &\lesssim \|\gamma^{k+1}_{\nabla,F} \underline{\boldsymbol{v}}_F\|^2 + \sum_{E\in\mathcal{E}_F} h_E (\|\gamma^{k+1}_{\nabla,F} \underline{\boldsymbol{v}}_F\|^2_E + \|\boldsymbol{v}_E\|^2) \\ &\lesssim \|\gamma^{k+1}_{\nabla,F} \underline{\boldsymbol{v}}_F\|^2 + \sum_{E\in\mathcal{E}_F} h_E (h_F^{-1} \|\gamma^{k+1}_{\nabla,F} \underline{\boldsymbol{v}}_F\|^2_F + h_E^{-1} |||\underline{\boldsymbol{v}}_F|||^2_{\nabla,F}) \\ &\lesssim |||\underline{\boldsymbol{v}}_F|||^2_{\nabla,F}.\end{aligned}$$

Where we used (3.30) on the first line and the trace inequality 3 on the second.

Conversely, $|||\underline{\boldsymbol{v}}_F|||_{\nabla,F}^2 = \|\boldsymbol{v}_{\mathcal{G},F}^c\|_F^2 + \|\boldsymbol{v}_{\mathcal{G},F}\|_F^2 + \sum_{E\in\mathcal{E}_F} h_E\|\boldsymbol{v}_E\|_E^2$ by (3.27). We bound each term of the right-hand side:

$$\begin{aligned}\|\boldsymbol{v}_E\|_E^2 \leq& \|\boldsymbol{v}_E - \gamma_{\nabla,F}^{k+1}\,\underline{\boldsymbol{v}}_F\|_E^2 + \|\gamma_{\nabla,F}^{k+1}\underline{\boldsymbol{v}}_F\|_E^2\\ \lesssim& \|\boldsymbol{v}_E - \gamma_{\nabla,F}^{k+1}\,\underline{\boldsymbol{v}}_F\|_E^2 + h_F^{-1}\|\gamma_{\nabla,F}^{k+1}\underline{\boldsymbol{v}}_F\|_F^2.\\ \sum_{E\in\mathcal{E}_F} h_E\|\boldsymbol{v}_E\|^2 \lesssim& \|\gamma_{\nabla,F}^{k+1}\underline{\boldsymbol{v}}_F\|_F^2 + \sum_{E\in\mathcal{E}_F} h_E\|\boldsymbol{v}_E - \gamma_{\nabla,F}^{k+1}\underline{\boldsymbol{v}}_F\|_E^2\end{aligned} \tag{3.31}$$

The consistency property (3.20) allows us to write:

$$\|\boldsymbol{v}_{\mathcal{G},F}^c\|^2 + \|\boldsymbol{v}_{\mathcal{G},F}\|^2 = \|\boldsymbol{\pi}_{\mathcal{G},F}^{c,k}\gamma_{\nabla,F}^{k+1}\underline{\boldsymbol{v}}_F\|^2 + \|\boldsymbol{\pi}_{\mathcal{G},F}^{k-1}\gamma_{\nabla,F}^{k+1}\underline{\boldsymbol{v}}_F\|^2 \leq 2\|\gamma_{\nabla,F}^{k+1}\underline{\boldsymbol{v}}_F\|^2 \lesssim \|\underline{\boldsymbol{v}}_F\|_{\nabla,F}^2. \tag{3.32}$$

We conclude by combining (3.31) and (3.32) to show that $|||\underline{\boldsymbol{v}}_F|||_{\nabla,F}^2 \lesssim \|\underline{\boldsymbol{v}}_F\|_{\nabla,F}^2$. □

Lemma 15. *It holds, for all $F \in \mathcal{F}_h$ and all $\underline{\boldsymbol{W}}_F \in \underline{\boldsymbol{X}}_{\boldsymbol{L}^2,F}^{k+1}$,*

$$\forall \underline{\boldsymbol{W}}_F \in \underline{\boldsymbol{X}}_{\boldsymbol{L}^2,F}^{k+1}, \|\underline{\boldsymbol{W}}_F\|_{\boldsymbol{L}^2,F} \approx |||\underline{\boldsymbol{W}}_F|||_{\boldsymbol{L}^2,F}.$$

Proof. Let $\underline{\boldsymbol{W}}_F \in \underline{\boldsymbol{X}}_{\boldsymbol{L}^2,F}^{k+1}$, we have:

$$\begin{aligned}\|\underline{\boldsymbol{W}}_F\|_{\boldsymbol{L}^2,F}^2 =& \int_F \boldsymbol{W}_F : \boldsymbol{W}_F + \sum_{E\in\mathcal{E}_F} h_E \int_E (\boldsymbol{W}_F\cdot\boldsymbol{t}_E - \boldsymbol{W}_E)\cdot(\boldsymbol{W}_F\cdot\boldsymbol{t}_E - \boldsymbol{W}_E)\\ \lesssim& \|\boldsymbol{W}_F\|_F^2 + \sum_{E\in\mathcal{E}_F} h_E\left(\|\boldsymbol{W}_F\|_E^2 + \|\boldsymbol{W}_E\|_E^2\right)\\ \lesssim& |||\underline{\boldsymbol{W}}_F|||_{\boldsymbol{L}^2,F}^2\end{aligned}$$

Where we use the triangular inequality and the discrete trace inequality [[10] Lemma 1.32]. Conversely, we have:

$$\begin{aligned}|||\underline{\boldsymbol{W}}_F|||_{\boldsymbol{L}^2,F}^2 =& \int_F \boldsymbol{W}_F : \boldsymbol{W}_F + \sum_{E\in\mathcal{E}_F} h_E \int_E \boldsymbol{W}_E\cdot\boldsymbol{W}_E\\ \lesssim& \int_F \boldsymbol{W}_F : \boldsymbol{W}_F + \sum_{E\in\mathcal{E}_F} h_E\|\boldsymbol{W}_F\cdot\boldsymbol{t}_E - \boldsymbol{W}_E\|_E^2 + h_E\|\boldsymbol{W}_F\cdot\boldsymbol{t}_E\|_E^2\\ \lesssim& \|\underline{\boldsymbol{W}}_F\|_{\boldsymbol{L}^2,F}^2.\end{aligned}$$

□

4 Complex property.

In this section we regard the following sequence:

$$\underline{X}_{\mathbf{rot},h}^k \xrightarrow{\underline{\boldsymbol{C}}_h^k} \underline{\boldsymbol{X}}_{\nabla,h}^k \xrightarrow{\underline{D}_h^k} \underline{\boldsymbol{X}}_{L^2,h}^k. \tag{4.1}$$

We will show in 17 that (4.1) is indeed a complex, but first we show that the interpolators form a cochain morphism from a continuous de Rham complex into the sequence (4.1).

Lemma 16 (Local commutation properties)**.** *It holds for all $F \in \mathcal{F}_h$,*

$$\underline{\boldsymbol{C}}^k_F(\underline{I}^k_{\mathbf{rot},F} q) = \underline{\boldsymbol{I}}^k_{\nabla,F} \mathbf{rot}\, q, \quad \forall q \in C^1(\overline{F}), \tag{4.2a}$$

$$\underline{\nabla}^{k+1}_F(\underline{\boldsymbol{I}}^k_{\nabla,F} \boldsymbol{v}) = \underline{\boldsymbol{I}}^k_{\boldsymbol{L}^2(F)} \nabla\, \boldsymbol{v}, \quad \forall \boldsymbol{v} \in \boldsymbol{C}^1(\overline{F}), \tag{4.2b}$$

$$D^k_F(\underline{\boldsymbol{I}}^k_{\nabla,F} \boldsymbol{v}) = \pi^k_{\mathcal{P},F} \operatorname{div} \boldsymbol{v}, \quad \forall \boldsymbol{v} \in \boldsymbol{C}^0(\overline{F}) \cap \boldsymbol{H}^1(F). \tag{4.2c}$$

Proof. Proof of (4.2a). Let $q \in C^1(\overline{F})$ and $\underline{q}_F = \underline{I}^k_{\mathbf{rot},F} q$. We set $\underline{\boldsymbol{v}}_F = \underline{\boldsymbol{C}}^k_F \underline{q}_F$, and we see that for all $E \in \mathcal{F}_E$ of vertices with coordinates $\boldsymbol{x}_{V_1}$ and $\boldsymbol{x}_{V_1}$, for all $r \in \mathcal{P}^k(E)$,

$$\begin{aligned}
\int_E \dot{q}_E r &= -\int_E q_E \dot{r} + q_E(\boldsymbol{x}_{V_1}) r(\boldsymbol{x}_{V_1}) - q_E(\boldsymbol{x}_{V_2}) r(\boldsymbol{x}_{V_2}) \\
&= -\int_E q \dot{r} + q(\boldsymbol{x}_{V_1}) r(\boldsymbol{x}_{V_1}) - q(\boldsymbol{x}_{V_2}) r(\boldsymbol{x}_{V_2}) \\
&= \int_E \dot{q} r.
\end{aligned}$$

Where we used the continuity of p_E and the fact that $\dot{r} \in \mathcal{P}^{k-1}(F)$ thus:

$$\pi^k_{\mathcal{P},E}(\boldsymbol{v}_E \cdot (-\boldsymbol{n}_E)) = \dot{q_E} = \pi^k_{\mathcal{P},E}(\mathbf{grad}\, q \cdot \boldsymbol{t}_E) = \pi^k_{\mathcal{P},E}(\mathbf{rot}\, q \cdot (-\boldsymbol{n}_E)).$$

Moreover, by definition $\pi^k_{\mathcal{P},E}(\boldsymbol{v} \cdot \boldsymbol{t}_E) = \pi^k_{\mathcal{P},E}(\mathbf{rot}\, q \cdot \boldsymbol{t}_E)$, $\boldsymbol{v}_E(\boldsymbol{x}_V) = \mathbf{rot}\, q(\boldsymbol{x}_V)$. and for all $\boldsymbol{r}_F \in \boldsymbol{\mathcal{N}}^k(F)$, using (2.10) we see that:

$$\begin{aligned}
\int_F \boldsymbol{v}_F \cdot \boldsymbol{r}_F &= \int_F q_F \operatorname{rot} \boldsymbol{r}_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_E \boldsymbol{r}_F \cdot\ \boldsymbol{t}_E \\
&= \int_F q \operatorname{rot} \boldsymbol{r}_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q \boldsymbol{r}_F \cdot\ \boldsymbol{t}_E \\
&= \int_F \mathbf{rot}\, q \cdot \boldsymbol{r}_F.
\end{aligned}$$

Proof of (4.2b). Immediate consequence of (3.16) and (3.17).
Proof of (4.2c). Let $\boldsymbol{v} \in \boldsymbol{C}^0(\overline{F}) \cap \boldsymbol{H}^1(F)$, we set $\underline{\boldsymbol{v}}_F = \underline{\boldsymbol{I}}^k_{\nabla,F} \boldsymbol{v}$ and $q_F = D^k_F \underline{\boldsymbol{v}}_F$. For all $w_F \in \mathcal{P}^k(F)$, since $\mathbf{grad}\, w_F \in \boldsymbol{\mathcal{G}}^{k-1}(F)$ we have:

$$\begin{aligned}
\int_F q_F w_F &= -\int_F \boldsymbol{v}_{\boldsymbol{\mathcal{G}},F} \mathbf{grad}\, w_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \boldsymbol{v}_E \cdot \boldsymbol{n}_E w_F \\
&= -\int_F \boldsymbol{v}\, \mathbf{grad}\, w_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \boldsymbol{v} \cdot \boldsymbol{n}_E w_F \\
&= \int_F \operatorname{div} \boldsymbol{v} w_F.
\end{aligned}$$

□

Theorem 17 (Complex property)**.** *It holds:*

$$\underline{I}^k_{\mathbf{rot},h} \mathbb{R} = \operatorname{Ker} \underline{\boldsymbol{C}}^k_h, \tag{4.3a}$$

$$\operatorname{Im} \underline{\boldsymbol{C}}^k_h \subset \operatorname{Ker} \underline{D}^k_h, \tag{4.3b}$$

$$\operatorname{Im} \underline{D}^k_h = \mathcal{P}^k(\mathcal{F}_h). \tag{4.3c}$$

Proof. Proof of (4.3a). The inclusion $\underline{I}^k_{\mathbf{rot},h}\mathbb{R} \subset \operatorname{Ker} \underline{\boldsymbol{C}}^k_h$ is immediate since for all $F \in \mathcal{F}_h$ and all $\boldsymbol{r}_F \in \boldsymbol{\mathcal{P}}^k(F)$, the continuous integration by parts gives
$\int_F \operatorname{rot} \boldsymbol{r}_F + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E \boldsymbol{r}_F \cdot \boldsymbol{t}_E = 0$. Conversely if $\underline{q}_h \in \underline{X}^k_{\mathbf{rot},h}$ is such that $\underline{\boldsymbol{C}}^k_h \underline{q}_h = 0$ then for all $E \in \mathcal{E}_h$, $q_{E'} = 0$ and for all vertex $V \in \mathcal{V}_h$, $\boldsymbol{q}_{V'} = 0$. Moreover $\dot{q_E} = 0$ so q_E is constant on each edge and since it is continuous on vertices and Ω has a single connected component there is $C \in \mathbb{R}$ such that $\forall E \in \mathcal{E}_h$, $q_E = C$. By (3.11) and (3.10) we must have for all $F \in \mathcal{F}_h$, $\forall \boldsymbol{r}_F \in \boldsymbol{\mathcal{N}}^k(F)$, $\int_F q_F \operatorname{rot} \boldsymbol{r}_F + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E q_E \boldsymbol{r}_F \cdot \boldsymbol{t}_E = 0$. Substituting q_E by C and doing an integration by parts we get $\int_F (q_F - C) \operatorname{rot} \boldsymbol{r}_F = 0$. We can conclude that $q_F = C$ since $\operatorname{rot} \boldsymbol{\mathcal{N}}^k(F) \to \mathcal{P}^{k-1}(F)$ is onto.
Proof of (4.3b). Let $\underline{p}_h \in \underline{X}^k_{\mathbf{rot},h}$ and $\underline{\boldsymbol{v}}_h = \underline{\boldsymbol{C}}^k_h \underline{p}_h$. For all $F \in \mathcal{F}_h$ and all $w_F \in \mathcal{P}^k(F)$ we have:

$$\int_E \boldsymbol{v}_E \cdot \boldsymbol{n}_E w_F = \int_E \boldsymbol{\pi}^k_{\boldsymbol{\mathcal{P}},E} \boldsymbol{v}_E \cdot \boldsymbol{n}_E w_F = -\int_E \dot{q}_E w_F.$$

If we write $\boldsymbol{x}_{E1}$ and $\boldsymbol{x}_{E1}$ the coordinates of the vertices of the edge E, we have

$$-\int_E \dot{q}_E w_F = \int_E q_E \dot{w_F} - q_E w_F(\boldsymbol{x}_{E1}) + q_E w_F(\boldsymbol{x}_{E2}).$$

Where $\dot{w_F}$ is the derivative along E so $\dot{w_F} = \mathbf{grad}\, w_F \cdot \boldsymbol{t}_E$. Moreover $\sum_{E\in\mathcal{E}_F} \omega_{FE}(q_E w_F(\boldsymbol{x}_{E2}) - q_E w_F(\boldsymbol{x}_{E1})) = 0$ thanks to the continuity of q_E so:

$$\sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E \boldsymbol{v}_E \cdot \boldsymbol{n}_E w_F = \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E q_E \,\mathbf{grad}\, w_F \cdot \boldsymbol{t}_E. \tag{4.4}$$

On the other hand we have:

$$\begin{aligned} -\int_F \boldsymbol{v}_{\boldsymbol{\mathcal{G}},F} \,\mathbf{grad}\, w_F &= -\int_F q_F \operatorname{rot} \mathbf{grad}\, w_F - \sum_{E\in\mathcal{F}_E} \omega_{FE} \int_E q_E \,\mathbf{grad}\, w_F \cdot \boldsymbol{t}_E \\ &= -\sum_{E\in\mathcal{F}_E} \omega_{FE} \int_E q_E \,\mathbf{grad}\, w_F \cdot \boldsymbol{t}_E. \end{aligned}$$

Summing with (4.4) we find $D^k_F \underline{\boldsymbol{v}}_F = 0$, for all $F \in \mathcal{F}_h$.
Proof of (4.3c) See lemma 24. □

The complex is exact if and only is the inclusion (4.3b) is in fact an equality. We can show that this the same as asking for Ω to be contractible. The proof is a slight adaptation of [[9] Section 4.3]. and will not be duplicated here.

5 Consistency results.

The last things we need to show in order to efficiently use this complex are consistency results. First we show primal consistency results, controlling the error made when we use the interpolators. Then we show some Poincare type results useful to show stability, including a discrete counterpart to the right inverse for the divergence 24. Finally we show adjoint consistency results, which control the error made when we perform a discrete integration by parts.

We begin by recalling a result from [[1] Lemma 4.3.4]: $\forall p \in (1,\infty)$, $\forall q \in \mathbb{N}$ such that $pq > 2$, $\forall w \in W^{q,p}(F)$,

$$\|w\|_{C(\overline{F})} \lesssim h_F^{-\frac{1}{2}} \sum_{r=0}^{q} h_F^r |w|_{W^{r,p}(F)}. \tag{5.1}$$

Lemma 18 (Primal consistency)**.** *For all $F \in \mathcal{F}_h$ it holds:*

$$\|\gamma^{k+1}_{\nabla,F}(\underline{\boldsymbol{I}}^k_{\nabla,F}\boldsymbol{v}) - \boldsymbol{v}\| \lesssim h^{k+2}|\boldsymbol{v}|_{\boldsymbol{H}^{k+2}}, \quad \forall \boldsymbol{v} \in \boldsymbol{H}^{k+2}(F) \cap \boldsymbol{C}^0(F). \tag{5.2}$$

Proof. For all $F \in \mathcal{F}_h$, (3.19) shows that $\gamma^{k+1}_{\nabla,F}\underline{\boldsymbol{I}}^k_{\nabla,F}$ is a projection on $\mathcal{P}^{k+1}(F)$. Thus we just have to show that $\|\gamma^{k+1}_{\nabla,F}(\underline{\boldsymbol{I}}^k_{\nabla,F}\boldsymbol{v})\| \lesssim \|\boldsymbol{v}\| + h\|\boldsymbol{v}\|_{\boldsymbol{H}^1} + h^2\|\boldsymbol{v}\|_{\boldsymbol{H}^2}$ to conclude with the lemma on approximation properties of bounded projector [[10] Lemma 1.43]. And starting from (3.30) we have

$$\begin{aligned}
\|\gamma^{k+1}_{\nabla,F}(\underline{\boldsymbol{I}}^k_{\nabla,F}\boldsymbol{v})\| \lesssim& \left|\!\left|\!\left| \underline{\boldsymbol{I}}^k_{\nabla,F}\boldsymbol{v} \right|\!\right|\!\right|_{\nabla,F} \\
\lesssim& \|\boldsymbol{\pi}^{c,k}_{\boldsymbol{\mathcal{G}},F}\boldsymbol{v}\|_F + \|\boldsymbol{\pi}^{k-1}_{\boldsymbol{\mathcal{G}},F}\boldsymbol{v}\|_F + \sum_{E\in\mathcal{E}_F} h_E^{\frac{1}{2}}\|\boldsymbol{\pi}^{k+2}_{\boldsymbol{\mathcal{P}},E}\boldsymbol{v}\|_E \\
\lesssim& \|\boldsymbol{v}\|_F + h_F|\boldsymbol{v}|_{\boldsymbol{H}^1(F)}.
\end{aligned}$$

Where we used the continuous trace inequality [[10] Lemma 1.51] and the boundedness of L^2 projectors. □

Lemma 19 (Stabilization forms consistency)**.** *For all $F \in \mathcal{F}_h$ it holds:*

$$s_{\nabla,F}(\ \underline{\boldsymbol{I}}^k_{\nabla,F}\ \boldsymbol{v}, \underline{\boldsymbol{I}}^k_{\nabla,F}\ \boldsymbol{v})^{1/2} \lesssim h^{k+2}|\boldsymbol{v}|_{\boldsymbol{H}^{k+2}(F)},\ \forall \boldsymbol{v} \in \boldsymbol{H}^{k+2}(F) \cap \boldsymbol{C}^0(\overline{F}), \tag{5.3}$$

$$s_{\boldsymbol{L}^2,F}\left(\underline{\boldsymbol{I}}^k_{\boldsymbol{L}^2(F)}\ \boldsymbol{W}, \underline{\boldsymbol{I}}^k_{\boldsymbol{L}^2(F)}\ \boldsymbol{W}\right)^{1/2} \lesssim h^{k+1}|\boldsymbol{W}|_{\boldsymbol{H}^{k+1}(F)},\ \forall \boldsymbol{W} \in \boldsymbol{H}^{k+1}(F) \cap \boldsymbol{C}^0(\overline{F}). \tag{5.4}$$

Proof. Proof of (5.3). For all $\boldsymbol{z}_F \in \boldsymbol{\mathcal{P}}^{k+1}(F)$ we have $\gamma^{k+1}_{\nabla,F}(\underline{\boldsymbol{I}}^k_{\nabla,F}\boldsymbol{z}_F) = \boldsymbol{z}_F$ by (3.19) so for all $\underline{\boldsymbol{w}}_F \in \underline{\boldsymbol{X}}^k_{\nabla,F}$,

$$\mathrm{s}_{\nabla,F}(\ \underline{\boldsymbol{I}}^k_{\nabla,F}\boldsymbol{z}_F, \underline{\boldsymbol{w}}_F) = \sum_{E\in\mathcal{E}_F} h_E(\gamma^{k+1}_{\nabla,F}\ \underline{\boldsymbol{I}}^k_{\nabla,F}\boldsymbol{z}_F - \boldsymbol{z}_F)\cdot(\gamma^{k+1}_{\nabla,F}\underline{\boldsymbol{w}}_F - \boldsymbol{w}_F) = 0.$$

Hence

$$\mathrm{s}_{\nabla,F}(\ \underline{\boldsymbol{I}}^k_{\nabla,F}\boldsymbol{v}_F, \underline{\boldsymbol{I}}^k_{\nabla,F}\boldsymbol{v}_F) = \mathrm{s}_{\nabla,F}(\ \underline{\boldsymbol{I}}^k_{\nabla,F}(\boldsymbol{v}_F - \boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{P}},F}), \underline{\boldsymbol{I}}^k_{\nabla,F}(\boldsymbol{v}_F - \boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{P}},F})) \lesssim \|\underline{\boldsymbol{I}}^k_{\nabla,F}(\boldsymbol{v}_F - \boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{P}},F})\|^2_{\nabla,F}.$$

We conclude by the norm equivalence 14 and [[10] Theorem 1.45].
Proof of (5.4). Let $\boldsymbol{W} \in \boldsymbol{H}^{k+1}(F) \cap \boldsymbol{C}^0(\overline{F})$, we have:

$$\begin{aligned}
\mathrm{s}_{\boldsymbol{L}^2,F}\left(\underline{\boldsymbol{I}}^k_{\boldsymbol{L}^2(F)}\boldsymbol{W}, \underline{\boldsymbol{I}}^k_{\boldsymbol{L}^2(F)}\boldsymbol{W}\right) =& \sum_{E\in\mathcal{E}_F} h_E\|\boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{P}},E}\boldsymbol{W}\cdot\boldsymbol{t}_E - \boldsymbol{\pi}^{k+1}_{\overline{\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{T}}},F}\boldsymbol{W}\cdot\boldsymbol{t}_E\|^2_E \\
\leq& \sum_{E\in\mathcal{E}_F} h_E\|\boldsymbol{W} - \boldsymbol{\pi}^{k+1}_{\overline{\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{T}}},F}\boldsymbol{W}\|^2_E \\
\leq& \sum_{E\in\mathcal{E}_F} h_E(\|\boldsymbol{W} - \boldsymbol{\pi}^k_{\boldsymbol{\mathcal{P}},F}\boldsymbol{W}\|^2_E + \|\boldsymbol{\pi}^k_{\boldsymbol{\mathcal{P}},F}\boldsymbol{W} - \boldsymbol{\pi}^{k+1}_{\overline{\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{T}}},F}\boldsymbol{W}\|^2_E) \\
\lesssim& h^{2(k+1)}|\boldsymbol{W}|_{\boldsymbol{H}^{k+1}} + \|\boldsymbol{W} - \boldsymbol{\pi}^k_{\boldsymbol{\mathcal{P}},F}\boldsymbol{W}\|^2_F.
\end{aligned}$$

Where the second equality comes from $\overline{\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{T}}}^{k+1}(F)\cdot\boldsymbol{t}_E \subset (\boldsymbol{\mathcal{P}}^{k+1}(E))^2$ and where we used the approximation properties on traces [[10] Theorem 1.45 Equation 1.75] on the first term and the discrete trace inequality [[10] Lemma 1.32] on the second term to get the last equality. We conclude with [[10] Theorem 1.45 Equation 1.74]. □

5.1 Poincaré inequality.

We begin by stating two lemmas which will be useful to prove the Poincare inequality.

Lemma 20. *For all $F \in \mathcal{F}_h$ and all $\underline{\boldsymbol{v}}_F \in \underline{\boldsymbol{X}}^k_{\nabla,F}$ it holds that*

$$\sum_{E\in\mathcal{E}_F} h_E^{-1}\|\gamma^{k+1}_{\nabla,F}\underline{\boldsymbol{v}}_F - \boldsymbol{v}_E\|_E^2 \lesssim \left|\!\left|\!\left|\underline{\nabla}^{k+1}_F\underline{\boldsymbol{v}}_F\right|\!\right|\!\right|^2_{\boldsymbol{L}^2,F}. \tag{5.5}$$

Proof. For any $F \in \mathcal{F}_h$ and $\underline{\boldsymbol{v}}_F \in \underline{\boldsymbol{X}}^k_{\nabla,F}$ set $\mathbb{R}^2 \ni A_{\boldsymbol{v},\partial F} = \frac{1}{\sum_{E\in\mathcal{E}_F}\int_E 1}\sum_{E\in\mathcal{E}_F}\int_E \boldsymbol{v}_E$ then (3.19) gives $A_{\boldsymbol{v},\partial F} = \gamma^{k+1}_{\nabla,F}(\underline{\boldsymbol{I}}^k_{\nabla,F}A_{\boldsymbol{v},\partial F})$ and

$$\sum_{E\in\mathcal{E}_F} h_E^{-1}\|\gamma^{k+1}_{\nabla,F}\underline{\boldsymbol{v}}_F - \boldsymbol{v}_E\|_E^2 \le \sum_{E\in\mathcal{E}_F} h_E^{-1}\left(\|\gamma^{k+1}_{\nabla,F}\underline{\boldsymbol{v}}_F - A_{\boldsymbol{v},\partial F}\|_E^2 + \|A_{\boldsymbol{v},\partial F} - \boldsymbol{v}_E\|_E^2\right).$$

Invoking a Poincare-Wirtinger inequality on the boundary ∂F we show that, since the Poincare constant is bounded by the diameter:

$$\sum_{E\in\mathcal{E}_F} h_E^{-1}\|A_{\boldsymbol{v},\partial F} - \boldsymbol{v}_E\|_E^2 \lesssim h_F^2\sum_{E\in\mathcal{E}_F} h_E^{-1}\|\nabla^{k+1}_E\boldsymbol{v}_E\|^2 \lesssim \left|\!\left|\!\left|\underline{\nabla}^{k+1}_F\underline{\boldsymbol{v}}_F\right|\!\right|\!\right|^2_{\boldsymbol{L}^2,F}. \tag{5.6}$$

Take $\boldsymbol{W}_F \in (\boldsymbol{\mathcal{R}}^{c,k+2}(F))^2$ such that $\nabla\cdot\boldsymbol{W}_F = \gamma^{k+1}_{\nabla,F}(\underline{\boldsymbol{v}}_F - \underline{\boldsymbol{I}}^k_{\nabla,F}A_{\boldsymbol{v},\partial F})$. Then by the estimate on the operator norm of (2.8) we have $\|\boldsymbol{W}_F\| \lesssim h_F\|\gamma^{k+1}_{\nabla,F}(\underline{\boldsymbol{v}}_F - \underline{\boldsymbol{I}}^k_{\nabla,F}A_{\boldsymbol{v},\partial F})\|$. Moreover (3.17) states that $\nabla^{k+1}_F\,\underline{\boldsymbol{I}}^k_{\nabla,F}A_{\boldsymbol{v},\partial F} = 0$ so

$$\begin{aligned}
\|\gamma^{k+1}_{\nabla,F}(\underline{\boldsymbol{v}}_F - \underline{\boldsymbol{I}}^k_{\nabla,F}A_{\boldsymbol{v},\partial F})\|^2 =& -\int_F \nabla^{k+1}_F\,\underline{\boldsymbol{v}}_F : \boldsymbol{W}_F + \sum_{E\in\mathcal{E}_F}\omega_{FE}\int_E(\boldsymbol{v}_E - A_{\boldsymbol{v},\partial F})\boldsymbol{W}_F\boldsymbol{n}_E\\
\lesssim& \|\boldsymbol{W}_F\|\,\|\nabla^{k+1}_F\,\underline{\boldsymbol{v}}_F\| + \|\boldsymbol{W}_F\|_F\sum_{E\in\mathcal{E}_F} h_E^{-\frac12}\|\boldsymbol{v}_E - A_{\boldsymbol{v},\partial F}\|_E\\
\lesssim& h_F\|\gamma^{k+1}_{\nabla,F}(\underline{\boldsymbol{v}}_F - \underline{\boldsymbol{I}}^k_{\nabla,F}A_{\boldsymbol{v},\partial F})\|\left(\|\nabla^{k+1}_F\,\underline{\boldsymbol{v}}_F\| + \left|\!\left|\!\left|\underline{\nabla}^{k+1}_F\underline{\boldsymbol{v}}_F\right|\!\right|\!\right|_{\boldsymbol{L}^2,F}\right).
\end{aligned}$$

Where we used the discrete trace inequality 3 then the inequality (5.6). We conclude since for any $E \in \mathcal{E}_F$,

$$h_E^{-\frac12}\|\gamma^{k+1}_{\nabla,F}\underline{\boldsymbol{v}}_F - A_{\boldsymbol{v},\partial F}\|_E \lesssim h_E^{-1}\|\gamma^{k+1}_{\nabla,F}(\underline{\boldsymbol{v}}_F - \underline{\boldsymbol{I}}^k_{\nabla,F}A_{\boldsymbol{v},\partial F})\|_F \lesssim \|\nabla^{k+1}_F\,\underline{\boldsymbol{v}}_F\| + \left|\!\left|\!\left|\underline{\nabla}^{k+1}_F\underline{\boldsymbol{v}}_F\right|\!\right|\!\right|_{\boldsymbol{L}^2,F}.$$

□

Lemma 21. *For all $F \in \mathcal{F}_h$ and all $\underline{\boldsymbol{v}}_F \in \underline{\boldsymbol{X}}^k_{\nabla,F}$ it holds that*

$$\|\nabla\gamma^{k+1}_{\nabla,F}\underline{\boldsymbol{v}}_F\|^2 + \sum_{E\in\mathcal{E}_F} h_E^{-1}\|\gamma^{k+1}_{\nabla,F}\underline{\boldsymbol{v}}_F - \boldsymbol{v}_E\|_E^2 \lesssim \left|\!\left|\!\left|\underline{\nabla}^{k+1}_F\ \underline{\boldsymbol{v}}_F\right|\!\right|\!\right|^2_{\boldsymbol{L}^2,F}. \tag{5.7}$$

Proof. Let $\boldsymbol{W}_F = \nabla \gamma_{\nabla,F}^{k+1} \underline{\boldsymbol{v}}_F$, we have

$$\begin{aligned}
\| \nabla \gamma_{\nabla,F}^{k+1} \underline{\boldsymbol{v}}_F \|^2 =& - \int_F \gamma_{\nabla,F}^{k+1} \underline{\boldsymbol{v}}_F \cdot \nabla \cdot \boldsymbol{W}_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \gamma_{\nabla,F}^{k+1} \underline{\boldsymbol{v}}_F \boldsymbol{W}_F \boldsymbol{n}_E \\
=& \int_F \nabla_F^{k+1} \underline{\boldsymbol{v}}_F : \boldsymbol{W}_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E (\gamma_{\nabla,F}^{k+1} \underline{\boldsymbol{v}}_F - \boldsymbol{v}_E) \boldsymbol{W}_F \boldsymbol{n}_E \\
\leq& \| \nabla_F^{k+1} \underline{\boldsymbol{v}}_F \| \| \boldsymbol{W}_F \| + \sum_{E \in \mathcal{E}_F} h_E^{-\frac{1}{2}} \| \gamma_{\nabla,F}^{k+1} \underline{\boldsymbol{v}}_F - \boldsymbol{v}_E \|_E h_E^{\frac{1}{2}} \| \boldsymbol{W}_F \|_E \\
\lesssim& \| \nabla_F^{k+1} \underline{\boldsymbol{v}}_F \| \| \boldsymbol{W}_F \| + \| \boldsymbol{W}_F \| (\sum_{E \in \mathcal{E}_F} h_E^{-1} \| \gamma_{\nabla,F}^{k+1} \underline{\boldsymbol{v}}_F - \boldsymbol{v}_E \|_E^2)^{\frac{1}{2}} \\
\lesssim& \| \nabla \gamma_{\nabla,F}^{k+1} \underline{\boldsymbol{v}}_F \| \| \boldsymbol{W}_F \|.
\end{aligned}$$

Where we used $\boldsymbol{W}_F \in (\boldsymbol{\mathcal{P}}^k(F))^2$ with 10 on the second line, the discrete trace inequality 3 on the third, and we concluded with 20. □

Lemma 22. *For all $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}_{\nabla,h}^k$ such that $\sum_{F \in \mathcal{F}_h} \int_F \gamma_{\nabla,F}^{k+1} \underline{\boldsymbol{v}}_F = 0$ it holds that*

$$|||\underline{\boldsymbol{v}}_h|||_{\nabla,h} \lesssim \left| \left| \left| \underline{\nabla}_h^{k+1} \underline{\boldsymbol{v}}_h \right| \right| \right|_{\boldsymbol{L}^2,h}. \tag{5.8}$$

Proof. For any $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}_{\nabla,h}^k$ we apply [[10] Theorem 6.5] to $((\gamma_{\nabla,F}^{k+1} \underline{\boldsymbol{v}}_F)_{F \in \mathcal{F}_h}, (\boldsymbol{\pi}_{\boldsymbol{\mathcal{P}},E}^{k+1} \boldsymbol{v}_E)_{E \in \mathcal{E}_h})$ to get:

$$\sum_{F \in \mathcal{F}_h} \| \gamma_{\nabla,F}^{k+1} \underline{\boldsymbol{v}}_F \|^2 \lesssim \sum_{F \in \mathcal{F}_h} \left(\| \nabla \gamma_{\nabla,F}^{k+1} \underline{\boldsymbol{v}}_F \|^2 + \sum_{E \in \mathcal{E}_F} h_E^{-1} \| \gamma_{\nabla,F}^{k+1} \underline{\boldsymbol{v}}_F - \boldsymbol{\pi}_{\boldsymbol{\mathcal{P}},E}^{k+1} \boldsymbol{v}_E \|_E^2 \right).$$

Moreover, since $\gamma_{\nabla,F}^{k+1} \underline{\boldsymbol{v}}_F \in \boldsymbol{\mathcal{P}}^{k+1}(F)$ we have:

$$\begin{aligned}
\| \gamma_{\nabla,F}^{k+1} \underline{\boldsymbol{v}}_F - \boldsymbol{\pi}_{\boldsymbol{\mathcal{P}},E}^{k+1} \boldsymbol{v}_E \|_E^2 =& \| \gamma_{\nabla,F}^{k+1} \underline{\boldsymbol{v}}_F \|_E^2 - 2 \int_E \gamma_{\nabla,F}^{k+1} \underline{\boldsymbol{v}}_F \cdot \boldsymbol{v}_E + \| \boldsymbol{\pi}_{\boldsymbol{\mathcal{P}},E}^{k+1} \boldsymbol{v}_E \|_E^2 \\
\leq& \| \gamma_{\nabla,F}^{k+1} \underline{\boldsymbol{v}}_F - \boldsymbol{v}_E \|_E^2,
\end{aligned}$$

Hence by lemma 21

$$\sum_{F \in \mathcal{F}_h} \| \gamma_{\nabla,F}^{k+1} \underline{\boldsymbol{v}}_F \|^2 \lesssim \left| \left| \left| \underline{\nabla}_h^{k+1} \underline{\boldsymbol{v}}_h \right| \right| \right|_{\boldsymbol{L}^2,F}^2.$$

We conclude with 14 and 20 that respectively states:

$$\begin{aligned}
|||\underline{\boldsymbol{v}}_h|||_{\nabla,h} \approx \| \underline{\boldsymbol{v}}_h \|_{\nabla,h} =& \left(\sum_{F \in \mathcal{F}_h} \| \gamma_{\nabla,F}^{k+1} \underline{\boldsymbol{v}}_F \|^2 + \mathrm{s}_{\nabla,F}(\underline{\boldsymbol{v}}_F, \underline{\boldsymbol{v}}_F) \right)^{\frac{1}{2}}, \\
\mathrm{s}_{\nabla,F}(\underline{\boldsymbol{v}}_F, \underline{\boldsymbol{v}}_F) =& \sum_{E \in \mathcal{E}_F} h_E \| \gamma_{\nabla,F}^{k+1} \underline{\boldsymbol{v}}_F - \boldsymbol{v}_E \|_E^2 \lesssim h_F^2 \left| \left| \left| \underline{\nabla}_F^{k+1} \underline{\boldsymbol{v}}_F \right| \right| \right|_{\boldsymbol{L}^2,F}^2.
\end{aligned}$$

□

Remark 23. When $k \geq 1$ the assumption $\sum_{F \in \mathcal{F}_h} \int_F \gamma_{\nabla,F}^{k+1} \underline{\boldsymbol{v}}_F = 0$ translates to $\sum_{F \in \mathcal{F}_h} \int_F \boldsymbol{v}_F = 0$ by (3.20). However this does not hold when $k = 0$.

We show that the fully discrete divergence is onto $\underline{D}_h^k : \underline{\boldsymbol{X}}_{\nabla,h}^k \to \underline{\boldsymbol{X}}_{L^2,h}^k$. The main difficulty is to show the boundedness of the inverse with the discrete norms.

Lemma 24 (Right-inverse for the divergence)**.** *For all $\underline{p}_h \in \underline{\boldsymbol{X}}_{L^2,h}^k$ there is $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}_{\nabla,h}^k$ such that $\underline{D}_h^k \underline{\boldsymbol{v}}_h = \underline{p}_h$ and $|||\underline{\boldsymbol{v}}_h|||_{\nabla,h} + \left|\left|\left|\nabla_h^{k+1} \underline{\boldsymbol{v}}_h\right|\right|\right|_{\boldsymbol{L}^2,h} \lesssim \|\underline{p}_h\|_{L^2,h}$.*

Proof. Existence. Let $\underline{p}_h = (p_F)_{F\in\mathcal{F}_h} \in \underline{\boldsymbol{X}}_{L^2,h}^k$ and apply 39 to find $\tilde{p} \in C^0(\overline{\Omega})$ such that $\forall F \in \mathcal{F}_h$, $\tilde{p}_{|F} \in \mathcal{P}^{k+\max_{h,E\in\mathcal{E}_h}(|\mathcal{E}_F|)}(F)$, $\pi_{\mathcal{P},F}^k \tilde{p} = p_F$ and $\|\tilde{p}\|_{L^2(\Omega)} \approx \|\underline{p}_h\|_{L^2,h}$. Under the assumption on the regularity of the mesh we have $\max_{h,E\in\mathcal{E}_h}(|\mathcal{E}_F|) \lesssim 1$ ([[10] Lemma 1.12]) so that the maximum degree is bounded independently of h. Since $\tilde{p}$ is a piecewise polynomial, continuous and of trace zero on the boundary, $\tilde{p} \in H_0^1(\Omega)$. We apply 42 to find $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$ such that $\operatorname{div} \boldsymbol{u} = \tilde{p}$, $\|\boldsymbol{u}\|_{\boldsymbol{H}^2} \lesssim \|\tilde{p}\|_{H^1}$ and $\|\boldsymbol{u}\|_{\boldsymbol{H}^1} \lesssim \|\tilde{p}\|_{L^2}$. We build $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}_{\nabla,h}^k$ in such a way that on each edge $E \in \mathcal{E}_h$, $\boldsymbol{\pi}_{\boldsymbol{\mathcal{P}},E}^k \boldsymbol{v}_E = \boldsymbol{\pi}_{\boldsymbol{\mathcal{P}},E}^k \boldsymbol{u}$ and on each vertex $V \in \mathcal{V}_E$ of coordinate x_V, $\boldsymbol{v}_E(x_V) = 0$.

Then on each face $F \in \mathcal{F}_h$ since $\mathbf{grad} : \mathcal{P}^{0,k}(F) \to \boldsymbol{\mathcal{G}}^{k-1}(F)$ is an isomorphism, we can choose $\boldsymbol{v}_{\boldsymbol{\mathcal{G}},F}$ such that $\forall w_F \in \mathcal{P}^{0,k}(F)$,

$$-\int_F \boldsymbol{v}_{\boldsymbol{\mathcal{G}},F} \mathbf{grad}\, w_F + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E \boldsymbol{v}_E \cdot \boldsymbol{n}_E w_F = \int_F p_F w_F.$$

Finally we set $\boldsymbol{v}_{\boldsymbol{\mathcal{G}},F}^c = \boldsymbol{\pi}_{\boldsymbol{\mathcal{G}},F}^{c,k} \boldsymbol{u}$ so that $\forall w_F \in \mathcal{P}^{0,k}(F)$, $\int_F D_F^k \underline{\boldsymbol{v}}_F w_F = \int_F p_F w_F$ and $\forall w_F \in \mathcal{P}^0(F)$,

$$\begin{aligned}\int_F D_F^k \underline{\boldsymbol{v}}_F w_F &= \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E \boldsymbol{v}_E \cdot \boldsymbol{n}_E w_F = w_F \sum_{E\in EF} \omega_{FE} \int_E \boldsymbol{u} \cdot \boldsymbol{n}_E \\ &= w_F \int_F \operatorname{div} \boldsymbol{u} = w_F \int_F \tilde{p} = \int_F w_F p_F.\end{aligned}$$

Where the second equality comes from $\forall E \in \mathcal{E}_h$, $\boldsymbol{\pi}_{\boldsymbol{\mathcal{P}},E}^0 \boldsymbol{v}_E = \boldsymbol{\pi}_{\boldsymbol{\mathcal{P}},E}^0 \boldsymbol{u}$ and the last equality from $\forall F \in \mathcal{F}_h$, $\pi_{\mathcal{P},F}^0 \tilde{p} = \pi_{\mathcal{P},F}^0 p_F$. Thus we have $\underline{D}_h^k \underline{\boldsymbol{v}}_h = \underline{p}_h$.
Boundedness. It remains to show that $|||\underline{\boldsymbol{v}}_h|||_{\nabla,h} \lesssim \|\underline{p}_h\|_{L^2,h}$. For any face $F \in \mathcal{F}_h$ remind that

$$|||\underline{\boldsymbol{v}}_F|||_{\nabla,F}^2 = \|\boldsymbol{v}_{\boldsymbol{\mathcal{G}},F}\|_F^2 + \|\boldsymbol{v}_{\boldsymbol{\mathcal{G}},F}^c\|_F^2 + \sum_{E\in\mathcal{E}_F} h_E \|\boldsymbol{v}_E\|_E^2. \tag{5.9}$$

We estimate the last term of (5.9) with:

$$\|\boldsymbol{v}_E\|_E \approx \ \|\boldsymbol{\pi}_{\boldsymbol{\mathcal{P}},E}^k \boldsymbol{v}_E\|_E = \|\boldsymbol{\pi}_{\boldsymbol{\mathcal{P}},E}^k \boldsymbol{u}\|_E \lesssim \|\boldsymbol{u}\|_E.$$

Where the first equality comes from 39 applied in one dimension. The continuous trace inequality [[10] Lemma 1.31] gives

$$h_F^{\frac{1}{2}} \|\boldsymbol{u}\|_E \lesssim \|\boldsymbol{u}\|_{\boldsymbol{L}^2(F)} + h_F |\boldsymbol{u}|_{\boldsymbol{H}^1(F)} \lesssim \|\boldsymbol{u}\|_{\boldsymbol{H}^1(F)}.$$

To estimate the first term of (5.9) we take $w_F \in \mathcal{P}^{0,k}(F)$ such that $\mathbf{grad}\, w_F = \boldsymbol{v}_{\boldsymbol{\mathcal{G}},F}$ so

$\|w_F\| \lesssim h_F\|\boldsymbol{v}_{\boldsymbol{\mathcal{G}},F}\|$ and by construction:

$$\begin{aligned}\int_F \boldsymbol{v}_{\boldsymbol{\mathcal{G}},F}\cdot\boldsymbol{v}_{\boldsymbol{\mathcal{G}},F} =& -\int_F p_F w_F + \sum_{E\in\mathcal{E}_F}\omega_{FE}\int_E \boldsymbol{v}_E\cdot\boldsymbol{n}_E w_F\\ \leq& \|p_F\|_F\|w_F\|_F + \sum_{E\in\mathcal{E}_F} h_E^{-1}h_E^{\frac12}\|\boldsymbol{v}_E\|_E h_E^{\frac12}\|w_F\|_E\\ \lesssim& \|p_F\|_F\|w_F\|_F + h_F^{-1}\|w_F\|_F\sum_{E\in\mathcal{E}_F} h_E^{\frac12}\|\boldsymbol{v}_E\|_E\\ \lesssim& \|\boldsymbol{v}_{\boldsymbol{\mathcal{G}},F}\|_F(\|p_F\|_F + \sum_{E\in\mathcal{E}_F} h_E^{\frac12}\|\boldsymbol{v}_E\|_E).\end{aligned}$$

Applying the same estimate on the boundary we find

$$\|\boldsymbol{v}_{\boldsymbol{\mathcal{G}},F}\|_F^2 \lesssim \|p_F\|_F^2 + \|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(F)}.$$

Lastly for the middle term of (5.9):

$$\|\boldsymbol{v}^c_{\boldsymbol{\mathcal{G}},F}\|_F = \|\boldsymbol{\pi}^{c,k}_{\boldsymbol{\mathcal{G}},F}\boldsymbol{u}\|_F \leq \|\boldsymbol{u}\|_{\boldsymbol{L}^2(F)} \leq \|\boldsymbol{u}\|_{\boldsymbol{H}^1(F)},$$

hence, summing over every face $F\in\mathcal{F}_h$ gives

$$|||\underline{\boldsymbol{v}}_h|||^2_{\nabla,h} \lesssim \|\underline{p}_h\|^2_{L^2,h} + \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \lesssim \|\underline{p}_h\|^2_{L^2,h}.$$

Now to estimate $\left|\left|\left|\underline{\nabla}_h^{k+1}\underline{\boldsymbol{v}}_h\right|\right|\right|_{\boldsymbol{L}^2,F}$ let $\boldsymbol{W}_F = \nabla_F^{k+1}\,\underline{\boldsymbol{v}}$ and $q_F\in\mathcal{P}^{0,k}(F)$ such that $\nabla\cdot\boldsymbol{W}_{\overline{\boldsymbol{\mathcal{R}}},F} = \mathbf{grad}\,q_F$ and $\|q_F\|\approx\|\boldsymbol{W}_{\overline{\boldsymbol{\mathcal{R}}},F}\|$. We have:

$$\begin{aligned}\int_F \nabla_F^{k+1}\,\underline{\boldsymbol{v}}_F : \nabla_F^{k+1}\,\underline{\boldsymbol{v}}_F =& -\int_F \boldsymbol{v}^c_{\boldsymbol{\mathcal{G}},F}\cdot\nabla\cdot\boldsymbol{W}^c_{\overline{\boldsymbol{\mathcal{R}}},F} - \int_F \boldsymbol{v}_{\boldsymbol{\mathcal{G}},F}\cdot\nabla\cdot\boldsymbol{W}_{\overline{\boldsymbol{\mathcal{R}}},F} + \sum_{E\in\mathcal{E}_F}\omega_{FE}\int_E \boldsymbol{v}_E\boldsymbol{W}_F\cdot\boldsymbol{n}_E\\ =& -\int_F \boldsymbol{v}^c_{\boldsymbol{\mathcal{G}},F}\cdot\nabla\cdot\boldsymbol{W}^c_{\overline{\boldsymbol{\mathcal{R}}},F} + \int_F p_F q_F + \sum_{E\in\mathcal{E}_F}\omega_{FE}\int_E \boldsymbol{v}_E(\boldsymbol{W}_F - q_F\,\mathbf{I}_{2,2})\cdot\boldsymbol{n}_E.\end{aligned}$$

Applying (2.10) gives

$$\int_E \boldsymbol{v}_E(\boldsymbol{W}_F - q_F\,\mathbf{I}_{2,2})\cdot\boldsymbol{n}_E = \int_E \boldsymbol{\pi}^k_{\boldsymbol{\mathcal{P}},E}\boldsymbol{v}_E(\boldsymbol{W}_F - q_F\,\mathbf{I}_{2,2})\cdot\boldsymbol{n}_E = \int_E \boldsymbol{u}(\boldsymbol{W}_F - q_F\,\mathbf{I}_{2,2})\cdot\boldsymbol{n}_E$$

so that after an integration by parts:

$$\sum_{E\in\mathcal{E}_F}\omega_{FE}\int_E \boldsymbol{v}_E(\boldsymbol{W}_F - q_F\,\mathbf{I}_{2,2})\cdot\boldsymbol{n}_E = \int_F \nabla\,\boldsymbol{u} : (\boldsymbol{W}_F - q_F\,\mathbf{I}_{2,2}) + \int_F \boldsymbol{u}\cdot\nabla\cdot(\boldsymbol{W}_F - q_F\,\mathbf{I}_{2,2}).$$

Since $\nabla\cdot(\boldsymbol{W}_F - q_F\,\mathbf{I}_{2,2}) = \nabla\cdot\boldsymbol{W}^c_{\overline{\boldsymbol{\mathcal{R}}},F}$ by (2.15) we have:

$$\begin{aligned}\int_F \nabla_F^{k+1}\,\underline{\boldsymbol{v}}_F : \nabla_F^{k+1}\,\underline{\boldsymbol{v}}_F =& -\int_F \boldsymbol{v}^c_{\boldsymbol{\mathcal{G}},F}\cdot\nabla\cdot\boldsymbol{W}^c_{\overline{\boldsymbol{\mathcal{R}}},F} + \int_F \nabla\,\boldsymbol{u} : (\boldsymbol{W}_F - q_F\,\mathbf{I}_{2,2}) + \int_F \boldsymbol{u}\cdot\nabla\cdot\boldsymbol{W}^c_{\overline{\boldsymbol{\mathcal{R}}},F}\\ =& \int_F \nabla\,\boldsymbol{u} : (\boldsymbol{W}_F - q_F\,\mathbf{I}_{2,2}),\end{aligned}$$

thus

$$\|\nabla_F^{k+1}\,\underline{\boldsymbol{v}}_F\|_F^2 \lesssim \|\nabla\,\boldsymbol{u}\|_F\|\nabla_F^{k+1}\,\underline{\boldsymbol{v}}_F\|_F. \tag{5.10}$$

Let us now focus on the estimate over edges. The Poincare-Wirtinger inequality and 2 show that for all polynomial r defined on $E \in \mathcal{E}_h$, $\|r - \frac{1}{h_E}\int_E r\|_E \approx h_E^{-1}\|\dot{r}\|_E$. Since $\|\boldsymbol{v}_E - \frac{1}{h_E}\int_E \boldsymbol{v}_E\|_E^2 = \|\boldsymbol{v}_E\|_E^2 - \frac{1}{h_E}\left(\int_E \boldsymbol{v}_E\right)^2$, $\|\boldsymbol{\pi}^k_{\boldsymbol{\mathcal{P}},E}\boldsymbol{v}_E\|_E \approx \|\boldsymbol{v}_E\|_E$ and $\int_E \boldsymbol{v}_E = \int_E \boldsymbol{\pi}^k_{\boldsymbol{\mathcal{P}},E}\boldsymbol{v}_E$ we have $\|\dot{\boldsymbol{v}}_E\|_E \approx \|\dot{\boldsymbol{\pi}}^k_{\boldsymbol{\mathcal{P}},E}\boldsymbol{v}_E\|_E$, hence

$$\|\dot{\boldsymbol{v}}_E\|_E \approx \|\dot{\boldsymbol{\pi}}^k_{\boldsymbol{\mathcal{P}},E}\boldsymbol{v}_E\|_E = \|\dot{\boldsymbol{\pi}}^k_{\boldsymbol{\mathcal{P}},E}\boldsymbol{v}_E - \dot{\boldsymbol{u}} + \dot{\boldsymbol{u}}\|_E \leq \|\dot{\boldsymbol{\pi}}^k_{\boldsymbol{\mathcal{P}},E}\boldsymbol{u} - \dot{\boldsymbol{u}}\|_E + \|\nabla \boldsymbol{u}\|_E.$$

By [[10] Theorem 1.45] we see that $h_E^{\frac{1}{2}}\|\dot{\boldsymbol{\pi}}^k_{\boldsymbol{\mathcal{P}},E}\boldsymbol{u} - \dot{\boldsymbol{u}}\|_E \lesssim \|\boldsymbol{u}\|_{\boldsymbol{H}^1(F)}$ and by the continuous trace inequality [[10] Lemma 1.31] that

$$h_F^{\frac{1}{2}}\|\nabla \boldsymbol{u}\|_E \lesssim \|\nabla \boldsymbol{u}\|_F + h_F\|\nabla\nabla \boldsymbol{u}\|_F,$$

so

$$h_E\|\dot{\boldsymbol{v}_E}\|_E^2 \lesssim \|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(F)} + h_F^2\|\boldsymbol{u}\|^2_{\boldsymbol{H}^2(F)}. \tag{5.11}$$

Combining (5.10), (5.11) and summing over every face we get:

$$\left|\left|\left|\underline{\nabla}_h^{k+1}\underline{\boldsymbol{v}}_h\right|\right|\right|^2_{\boldsymbol{L}^2,h} \lesssim \|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} + h^2\|\boldsymbol{u}\|^2_{\boldsymbol{H}^2(\Omega)}.$$

We can conclude since $\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \lesssim \|\underline{p}_h\|_{L^2,h}$ and

$$h\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} \lesssim h\|\tilde{p}\|_{\boldsymbol{H}^1(\Omega)} \lesssim \|\tilde{p}\|_{L^2(\Omega)} \lesssim \|\underline{p}_h\|_{L^2,h}.$$

Where we used the inverse Poincare inequality on $h^2\|\tilde{p}\|^2_{\boldsymbol{H}^1(\Omega)} = \sum_{F\in\mathcal{F}_h} h^2\|\tilde{p}\|_{\boldsymbol{H}^1(F)}$. □

Remark 25. We can easily adapt lemma 24 to require $\sum_{F\in\mathcal{F}_h}\int_F \gamma^{k+1}_{\nabla,F}\underline{\boldsymbol{v}}_F = 0$. Simply define $\underline{\boldsymbol{v}'}_h = \underline{\boldsymbol{v}}_h - \underline{\boldsymbol{I}}^k_{\nabla,h}\left(\frac{1}{\int_\Omega 1}\sum_{F\in\mathcal{F}_h}\int_F \gamma^{k+1}_{\nabla,F}\underline{\boldsymbol{v}}_F\right)$. It is clear from (3.19) that $\sum_{F\in\mathcal{F}_h}\int_F \gamma^{k+1}_{\nabla,F}\underline{\boldsymbol{v}'}_F = 0$, from 16 that $\underline{D}^k_h\underline{\boldsymbol{v}'}_h = \underline{p}_h$ and from (3.30) that the estimate of 24 on the norm of $\underline{\boldsymbol{v}'}_h$ still holds.

5.2 Adjoint consistency.

We define the adjoint consistency error for all $\boldsymbol{W} \in \boldsymbol{C}^0(\overline{\Omega}) \cap \boldsymbol{H}^1_0(\Omega)$ and all $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\nabla,h}$ by:

$$\tilde{\mathcal{E}}_{\nabla,h}(\boldsymbol{W},\underline{\boldsymbol{v}}_h) = \sum_{F\in\mathcal{F}_h}\left(\left(\underline{\boldsymbol{I}}^k_{\boldsymbol{L}^2(F)}\boldsymbol{W}, \underline{\nabla}^{k+1}_F\underline{\boldsymbol{v}}_F\right)_{\boldsymbol{L}^2,F} + \int_F \nabla\cdot\boldsymbol{W}\cdot\gamma^{k+1}_{\nabla,F}\underline{\boldsymbol{v}}_F\right). \tag{5.12}$$

Theorem 26 (Adjoint consistency for the gradient)**.** *For all $\boldsymbol{W} \in \boldsymbol{C}^0(\overline{\Omega}) \cap \boldsymbol{H}^1_0(\Omega)$ such that $\boldsymbol{W} \in \boldsymbol{H}^{k+2}(\mathcal{F}_h)$ and all $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\nabla,h}$,*

$$|\tilde{\mathcal{E}}_{\nabla,h}(\boldsymbol{W},\underline{\boldsymbol{v}}_h)| \lesssim h^{k+1}\left(|\boldsymbol{W}|_{\boldsymbol{H}^{k+1}} + |\boldsymbol{W}|_{\boldsymbol{H}^{k+2}}\right)\left(|||\underline{\boldsymbol{v}}_h|||_{\nabla,h} + \left|\left|\left|\underline{\nabla}^{k+1}_h\underline{\boldsymbol{v}}_h\right|\right|\right|_{\boldsymbol{L}^2,h}\right). \tag{5.13}$$

Proof. Remarks 10 and 5 show that $\forall \boldsymbol{W}_h \in (\boldsymbol{\mathcal{RT}}^{k+1}(\mathcal{F}_h))^2$,

$$\int_F \gamma^{k+1}_{\nabla,F}\underline{\boldsymbol{v}}_F\cdot\nabla\cdot\boldsymbol{W}_F + \int_F \nabla^{k+1}_F\underline{\boldsymbol{v}}_F : \boldsymbol{W}_F - \sum_{E\in\mathcal{E}_F}\omega_{FE}\int_E \boldsymbol{v}_E\boldsymbol{W}_F\boldsymbol{n}_E = 0.$$

Moreover, since $\boldsymbol{W}\cdot\boldsymbol{n}_\Omega = 0$ on $\partial\Omega$ and since the $\boldsymbol{v}_E$ are single valued we have

$$\sum_{F\in\mathcal{F}_h}\sum_{E\in\mathcal{E}_F}\omega_{FE}\int_E \boldsymbol{v}_E\boldsymbol{W}\boldsymbol{n}_E = 0. \tag{5.14}$$

Hence we can write:

$$\begin{aligned}\tilde{\mathcal{E}}_{\nabla,h}(\boldsymbol{W},\underline{\boldsymbol{v}}_h) = \sum_{F\in\mathcal{F}_h}\Bigg(&\int_F(\boldsymbol{W}-\boldsymbol{W}_F):\nabla_F^{k+1}\underline{\boldsymbol{v}}_F + \int_F\nabla\cdot(\boldsymbol{W}-\boldsymbol{W}_F)\cdot\gamma_{\nabla,F}^{k+1}\underline{\boldsymbol{v}}_F\\ &+\sum_{E\in\mathcal{E}_F}\omega_{FE}\int_E\boldsymbol{v}_E(\boldsymbol{W}-\boldsymbol{W}_F)\boldsymbol{n}_E + \mathrm{s}_{\boldsymbol{L}^2,F}\left(\underline{\boldsymbol{I}}^k_{\boldsymbol{L}^2(F)}\boldsymbol{W},\nabla_F^{k+1}\underline{\boldsymbol{v}}_F\right)\Bigg)\\ \lesssim\sum_{F\in\mathcal{F}_h}&\left(\|\boldsymbol{W}-\boldsymbol{W}_F\|+\|\nabla\cdot(\boldsymbol{W}-\boldsymbol{W}_F)\|\right)\left(\|\nabla_F^{k+1}\underline{\boldsymbol{v}}_F\|+\|\gamma_{\nabla,F}^{k+1}\underline{\boldsymbol{v}}_F\|\right)\\ &+\mathrm{s}_{\boldsymbol{L}^2,F}\left(\underline{\boldsymbol{I}}^k_{\boldsymbol{L}^2(F)}\boldsymbol{W},\underline{\boldsymbol{I}}^k_{\boldsymbol{L}^2(F)}\boldsymbol{W}\right)^{1/2}\mathrm{s}_{\boldsymbol{L}^2,F}\left(\nabla_F^{k+1}\underline{\boldsymbol{v}}_F,\nabla_F^{k+1}\underline{\boldsymbol{v}}_F\right)^{1/2}\\ &+\sum_{E\in\mathcal{E}_F}\omega_{FE}\int_E\boldsymbol{v}_E(\boldsymbol{W}-\boldsymbol{W}_F)\boldsymbol{n}_E.\end{aligned}$$

Applying (5.4) and 15 gives:

$$\mathrm{s}_{\boldsymbol{L}^2,F}\left(\underline{\boldsymbol{I}}^k_{\boldsymbol{L}^2(F)}\boldsymbol{W},\underline{\boldsymbol{I}}^k_{\boldsymbol{L}^2(F)}\boldsymbol{W}\right)^{1/2}\mathrm{s}_{\boldsymbol{L}^2,F}\left(\nabla_F^{k+1}\underline{\boldsymbol{v}}_F,\nabla_F^{k+1}\underline{\boldsymbol{v}}_F\right)^{1/2}\lesssim h^{k+1}|\boldsymbol{W}|_{\boldsymbol{H}^{k+1}(F)}\left|\!\left|\!\left|\underline{\nabla}_F^{k+1}\underline{\boldsymbol{v}}_F\right|\!\right|\!\right|_{\boldsymbol{L}^2,F}.$$

Using the approximation properties of the spaces $\boldsymbol{\mathcal{RT}}^{k+1}(F)$ given by a slight adaptation of [[14] Lemma 43] we can find $\boldsymbol{W}_F\in\boldsymbol{\mathcal{RT}}^{k+1}(F)$ such that

$$\|\boldsymbol{W}-\boldsymbol{W}_F\|+\|\nabla\cdot(\boldsymbol{W}-\boldsymbol{W}_F)\|\lesssim h^{k+1}\left(|\boldsymbol{W}|_{\boldsymbol{H}^{k+1}(F)}+|\boldsymbol{W}|_{\boldsymbol{H}^{k+2}(F)}\right)$$

By (3.30) we see that

$$\|\nabla_F^{k+1}\underline{\boldsymbol{v}}_F\|+\|\gamma_{\nabla,F}^{k+1}\underline{\boldsymbol{v}}_F\|\lesssim\left|\!\left|\!\left|\underline{\nabla}_F^{k+1}\underline{\boldsymbol{v}}_F\right|\!\right|\!\right|_{\boldsymbol{L}^2,F}+|||\underline{\boldsymbol{v}}_F|||_{\nabla,F}.$$

Lastly we use 40 to find $\boldsymbol{R}_{\underline{\boldsymbol{v}}_F}\in\boldsymbol{H}^1(F)$ such that

$$\begin{aligned}\sum_{E\in\mathcal{E}_F}\omega_{FE}\int_E\boldsymbol{v}_E(\boldsymbol{W}-\boldsymbol{W}_F)\boldsymbol{n}_E&=\sum_{E\in\mathcal{E}_F}\omega_{FE}\int_E\boldsymbol{R}_{\underline{\boldsymbol{v}}_F}(\boldsymbol{W}-\boldsymbol{W}_F)\boldsymbol{n}_E\\ &=\int_F\nabla\boldsymbol{R}_{\underline{\boldsymbol{v}}_F}:(\boldsymbol{W}-\boldsymbol{W}_F)+\int_F\boldsymbol{R}_{\underline{\boldsymbol{v}}_F}\cdot\nabla\cdot(\boldsymbol{W}-\boldsymbol{W}_F).\end{aligned}$$

Hence

$$\left|\sum_{E\in\mathcal{E}_F}\omega_{FE}\int_E\boldsymbol{v}_E(\boldsymbol{W}-\boldsymbol{W}_F)\boldsymbol{n}_E\right|\lesssim\left(\|\boldsymbol{W}-\boldsymbol{W}_F\|+\|\nabla\cdot(\boldsymbol{W}-\boldsymbol{W}_F)\|\right)\left(\|\nabla\boldsymbol{R}_{\underline{\boldsymbol{v}}_F}\|+\|\boldsymbol{R}_{\underline{\boldsymbol{v}}_F}\|\right)$$

and we conclude with 40 which gives the boundedness of $\boldsymbol{R}_{\underline{\boldsymbol{v}}_F}$. □

We can sharpen the estimate (5.12) when $\boldsymbol{W}$ is the gradient of some field. Indeed, if were to take $\boldsymbol{W}=\nabla w$ in 26, we would see that a norm over $\boldsymbol{H}^{k+3}$ appears in the estimate, which is suboptimal.

We define the adjoint consistency error for all $\boldsymbol{w}\in\boldsymbol{H}^2(\Omega)$ such that $\nabla\boldsymbol{w}\cdot\boldsymbol{n}_\Omega=0$ and all $\underline{\boldsymbol{v}}_h\in\underline{\boldsymbol{X}}^k_{\nabla,h}$ by:

$$\tilde{\mathcal{E}}_{\boldsymbol{\Delta},h}(\boldsymbol{w},\underline{\boldsymbol{v}}_h)=\sum_{F\in\mathcal{F}_h}\left(\int_F\boldsymbol{\Delta}\boldsymbol{w}\cdot\gamma_{\nabla,F}^{k+1}\underline{\boldsymbol{v}}_F+\left(\underline{\nabla}_F^{k+1}\underline{\boldsymbol{I}}^k_{\nabla,F}\boldsymbol{w},\underline{\nabla}_F^{k+1}\underline{\boldsymbol{v}}_F\right)_{\boldsymbol{L}^2,F}\right).\tag{5.15}$$

Remark 27. The assumption $\boldsymbol{w} \in \boldsymbol{H}^2(\Omega)$ imply that $\boldsymbol{w} \in \boldsymbol{C}^0(\overline{\Omega})$ in two dimensions (see e.g. [[2] 12.60]).

Theorem 28 (Adjoint consistency for the Laplacian)**.** *For all* $\boldsymbol{w} \in \boldsymbol{H}^2(\Omega)$ *such that* $\nabla \boldsymbol{w} \cdot \boldsymbol{n}_\Omega = 0$ *and* $\boldsymbol{w} \in \boldsymbol{H}^{k+2}(\mathcal{F}_h)$ *and for all* $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\nabla,h}$,

$$|\tilde{\mathcal{E}}_{\boldsymbol{\Delta},h}(\boldsymbol{w}, \underline{\boldsymbol{v}}_h)| \lesssim h^{k+1} |\boldsymbol{w}|_{\boldsymbol{H}^{k+2}} \left|\!\left|\!\left| \underline{\nabla}_h^{k+1} \underline{\boldsymbol{v}}_h \right|\!\right|\!\right|_{\boldsymbol{L}^2,h}. \tag{5.16}$$

Proof. For any $F \in \mathcal{F}_h$, (4.2b) gives:

$$\left(\underline{\nabla}_F^{k+1} \underline{\boldsymbol{I}}^k_{\nabla,F} \boldsymbol{w}, \underline{\nabla}_F^{k+1} \underline{\boldsymbol{v}}_G\right)_{\boldsymbol{L}^2,F} = \int_F \boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{RT}},F} \nabla \boldsymbol{w} : \nabla_F^{k+1} \underline{\boldsymbol{v}}_F + \mathrm{s}_{\boldsymbol{L}^2,F}\left(\underline{\boldsymbol{I}}^k_{\boldsymbol{L}^2(F)} \nabla \boldsymbol{w}, \underline{\nabla}_F^{k+1} \underline{\boldsymbol{v}}\right).$$

With an integration by parts and since $\int_F \boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{RT}},F} \nabla \boldsymbol{w} : \nabla_F^{k+1} \underline{\boldsymbol{v}}_F = \int_F \nabla \boldsymbol{w} : \nabla_F^{k+1} \underline{\boldsymbol{v}}_F$ we have:

$$\begin{aligned}\tilde{\mathcal{E}}_{\boldsymbol{\Delta},h}(\boldsymbol{w}, \underline{\boldsymbol{v}}_h) = \sum_{F \in Fh} \Bigg(\int_F \nabla \boldsymbol{w} : (\nabla_F^{k+1} \underline{\boldsymbol{v}}_F - \nabla \gamma^{k+1}_{\nabla,F} \underline{\boldsymbol{v}}_F) + \mathrm{s}_{\boldsymbol{L}^2,F}\left(\underline{\boldsymbol{I}}^k_{\boldsymbol{L}^2(F)} \nabla \boldsymbol{w}, \underline{\nabla}_F^{k+1} \underline{\boldsymbol{v}}\right) \\ + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \gamma^{k+1}_{\nabla,F} \underline{\boldsymbol{v}}_F \nabla \boldsymbol{w} \boldsymbol{n}_E \Bigg)\end{aligned}$$

Since we assume $\nabla \boldsymbol{w} \cdot \boldsymbol{n}_\Omega = 0$ we have

$$\sum_{F \in \mathcal{F}_h} \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \boldsymbol{v}_E \nabla \boldsymbol{w} = 0 \tag{5.17}$$

so by 10 it holds $\forall \boldsymbol{w}_F \in \boldsymbol{\mathcal{P}}^{k+1}(F)$,

$$\int_F \boldsymbol{\Delta} \boldsymbol{w}_F \cdot \gamma^{k+1}_{\nabla,F} \underline{\boldsymbol{v}}_F + \int_F \nabla_F^{k+1} \underline{\boldsymbol{v}}_F : \nabla \boldsymbol{w}_F - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \boldsymbol{v}_E \nabla \boldsymbol{W} \boldsymbol{n}_E = 0,$$

so

$$\int_F \nabla \boldsymbol{w}_F : (\nabla_F^{k+1} \underline{\boldsymbol{v}}_F - \nabla \gamma^{k+1}_{\nabla,F} \underline{\boldsymbol{v}}_F) + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E (\gamma^{k+1}_{\nabla,F} \underline{\boldsymbol{v}}_F - \boldsymbol{v}_E) \nabla \boldsymbol{w}_F \boldsymbol{n}_E = 0.$$

This allows us to write for any $\underline{\boldsymbol{w}}_h = (\boldsymbol{w}_F)_{F \in Fh} \in \boldsymbol{\mathcal{P}}^{k+1}(F)$,

$$\begin{aligned}\tilde{\mathcal{E}}_{\boldsymbol{\Delta},h}(\boldsymbol{w}, \underline{\boldsymbol{v}}_h) = \sum_{F \in Fh} \Bigg(\int_F \nabla(\boldsymbol{w} - \boldsymbol{w}_F) : (\nabla_F^{k+1} \underline{\boldsymbol{v}}_F - \nabla \gamma^{k+1}_{\nabla,F} \underline{\boldsymbol{v}}_F) + \mathrm{s}_{\boldsymbol{L}^2,F}\left(\underline{\boldsymbol{I}}^k_{\boldsymbol{L}^2(F)} \nabla \boldsymbol{w}, \underline{\nabla}_F^{k+1} \underline{\boldsymbol{v}}\right) \\ + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E (\gamma^{k+1}_{\nabla,F} \underline{\boldsymbol{v}}_F - \boldsymbol{v}_E) \nabla(\boldsymbol{w} - \boldsymbol{w}_F) \boldsymbol{n}_E \Bigg),\end{aligned}$$

$$\begin{aligned}|\tilde{\mathcal{E}}_{\boldsymbol{\Delta},h}(\boldsymbol{w}, \underline{\boldsymbol{v}}_h)| \lesssim \sum_{F \in Fh} \Bigg(\| \nabla(\boldsymbol{w} - \boldsymbol{w}_F) \|_F \| \nabla_F^{k+1} \underline{\boldsymbol{v}}_F - \nabla \gamma^{k+1}_{\nabla,F} \underline{\boldsymbol{v}}_F \|_F \\ + \sum_{E \in FE} \| \gamma^{k+1}_{\nabla,F} \underline{\boldsymbol{v}}_F - \boldsymbol{v}_E \|_E \| \nabla(\boldsymbol{w} - \boldsymbol{w}_F) \|_E + | \mathrm{s}_{\boldsymbol{L}^2,F}\left(\underline{\boldsymbol{I}}^k_{\boldsymbol{L}^2(F)} \nabla \boldsymbol{w}, \underline{\nabla}_F^{k+1} \underline{\boldsymbol{v}}\right) | \Bigg).\end{aligned}$$

Applying 20 we get

$$\| \gamma^{k+1}_{\nabla,F} \underline{\boldsymbol{v}}_F - \boldsymbol{v}_E \|_E \| \nabla(\boldsymbol{w} - \boldsymbol{w}_F) \|_E \lesssim \left|\!\left|\!\left| \underline{\nabla}_F^{k+1} \underline{\boldsymbol{v}}_F \right|\!\right|\!\right|_{\boldsymbol{L}^2,F} h^{\frac{1}{2}} \| \nabla(\boldsymbol{w} - \boldsymbol{w}_F) \|_E.$$

By (5.4) and 15 we get

$$|\mathrm{s}_{\boldsymbol{L}^2,F}\left(\underline{\boldsymbol{I}}^k_{\boldsymbol{L}^2(F)}\nabla\boldsymbol{w},\underline{\nabla}_F^{k+1}\underline{\boldsymbol{v}}\right)|\lesssim h^{k+1}|\nabla\boldsymbol{w}|_{\boldsymbol{H}^{k+1}}\left|\!\left|\!\left|\underline{\nabla}_F^{k+1}\underline{\boldsymbol{v}}_F\right|\!\right|\!\right|_{\boldsymbol{L}^2,F}.$$

Hence, applying 21 we write:

$$\begin{aligned}|\tilde{\mathcal{E}}_{\boldsymbol{\Delta},h}(\boldsymbol{w},\underline{\boldsymbol{v}}_h)|\lesssim\sum_{F\in Fh}\left(\left|\!\left|\!\left|\underline{\nabla}_F^{k+1}\underline{\boldsymbol{v}}_F\right|\!\right|\!\right|_{\boldsymbol{L}^2,F}(\|\nabla(\boldsymbol{w}-\boldsymbol{w}_F)\|_F+h^{\frac12}\|\nabla(\boldsymbol{w}-\boldsymbol{w}_F)\|_E)\right)\\+h^{k+1}|\boldsymbol{w}|_{\boldsymbol{H}^{k+2}}\left|\!\left|\!\left|\underline{\nabla}_F^{k+1}\underline{\boldsymbol{v}}_h\right|\!\right|\!\right|_{\boldsymbol{L}^2,F}.\end{aligned}$$

We conclude by taking $\boldsymbol{w}_F=\boldsymbol{\pi}^{1,k+1}_{\boldsymbol{\mathcal{P}},F}\boldsymbol{w}$ the elliptic projection on F (see [[10] Definition 1.39]), then [[10] Theorem 1.48] gives:

$$\|\boldsymbol{w}-\boldsymbol{\pi}^{1,k+1}_{\boldsymbol{\mathcal{P}},F}\boldsymbol{w}\|_{\boldsymbol{H}^1(F)}\lesssim h^{k+1}|\boldsymbol{w}|_{\boldsymbol{H}^{k+2}},$$

$$h^{\frac12}\|\boldsymbol{w}-\boldsymbol{\pi}^{1,k+1}_{\boldsymbol{\mathcal{P}},F}\boldsymbol{w}\|_{\boldsymbol{H}^1(E)}\lesssim h^{k+1}|\boldsymbol{w}|_{\boldsymbol{H}^{k+2}}.$$

□

6 Stokes equations.

Finally, we illustrate this complex with the resolution of the Stokes equations. For the sake of simplicity we use Neumann boundary conditions over the whole boundary, that it to say with a free outlet condition. More general conditions are not difficult to enforce and are discussed in Section 7. The solution is therefore determined only up to a constant vector field. The leads to the introduction of a new space:

$$\underline{\boldsymbol{X}}^k_{\nabla,h,\star}:=\{\underline{\boldsymbol{v}}_h\in\underline{\boldsymbol{X}}^k_{\nabla,h}\,:\,\sum_{F\in\mathcal{F}_h}\int_F\gamma^{k+1}_{\nabla,F}\underline{\boldsymbol{v}}_F=0\}.\tag{6.1}$$

This is the discrete counterpart of $L^2_0(\Omega)$.

Let μ be a constant viscosity, we define the symmetric bilinear form $\mathrm{a}_h(\underline{\boldsymbol{v}}_h,\underline{\boldsymbol{w}}_h)\in\underline{\boldsymbol{X}}^k_{\nabla,h}\times\underline{\boldsymbol{X}}^k_{\nabla,h}\to\mathbb{R}$ on all $\underline{\boldsymbol{v}}_h,\underline{\boldsymbol{w}}_h\in\underline{\boldsymbol{X}}^k_{\nabla,h}$ by

$$\mathrm{a}_h(\underline{\boldsymbol{v}}_h,\underline{\boldsymbol{w}}_h):=\mu\left(\underline{\nabla}_h^{k+1}\,\underline{\boldsymbol{v}}_h,\underline{\nabla}_h^{k+1}\underline{\boldsymbol{w}}_h\right)_{\boldsymbol{L}^2,h}.\tag{6.2}$$

We also define the bilinear form $\mathrm{b}_h(\underline{\boldsymbol{v}}_h,\underline{q}_h)\in\underline{\boldsymbol{X}}^k_{\nabla,h}\times\underline{\boldsymbol{X}}^k_{L^2,h}\to\mathbb{R}$ on all $\underline{\boldsymbol{v}}_h\in\underline{\boldsymbol{X}}^k_{\nabla,h},\underline{q}_h\in\underline{\boldsymbol{X}}^k_{L^2,h}$ by

$$\mathrm{b}_h(\underline{\boldsymbol{v}}_h,\underline{q}_h):=\sum_{F\in\mathcal{F}_h}\int_F D^k_F\underline{\boldsymbol{v}}_Fq_F.\tag{6.3}$$

Then we define the bilinear form $\mathcal{A}_h((\underline{\boldsymbol{v}}_F,\underline{p}_h),(\underline{\boldsymbol{w}}_h,\underline{q}_h))\in(\underline{\boldsymbol{X}}^k_{\nabla,h,\star}\times\underline{\boldsymbol{X}}^k_{L^2,h})\times(\underline{\boldsymbol{X}}^k_{\nabla,h,\star}\times\underline{\boldsymbol{X}}^k_{L^2,h})\to\mathbb{R}$ by

$$\mathcal{A}_h((\underline{\boldsymbol{v}}_h,\underline{p}_h),(\underline{\boldsymbol{w}}_h,\underline{q}_h))=\mathrm{a}_h(\underline{\boldsymbol{v}}_h,\underline{\boldsymbol{w}}_h)-\mathrm{b}_h(\underline{\boldsymbol{w}}_h,\underline{p}_h)+\mathrm{b}_h(\underline{\boldsymbol{v}}_F,\underline{q}_h).\tag{6.4}$$

We define a suitable Sobolev-like norm on our discrete spaces such that $\forall\underline{\boldsymbol{v}}_h\in\underline{\boldsymbol{X}}^k_{\nabla,h}$,

$$\|\underline{\boldsymbol{v}}_h\|_{\mu,\nabla,1,h}:=\left(\|\underline{\boldsymbol{v}}_h\|^2_{\nabla,h}+\mathrm{a}_h(\underline{\boldsymbol{v}}_h,\underline{\boldsymbol{v}}_h)\right)^{1/2}.\tag{6.5}$$

And for $f \in \boldsymbol{L}^2(\Omega)$ we set $\mathcal{L}_h : \underline{\boldsymbol{X}}^k_{\nabla,h,\star} \to \mathbb{R}$ such that $\forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\nabla,h}$,

$$\mathcal{L}_h(\underline{\boldsymbol{v}}_h) := \sum_{F\in\mathcal{F}_h} \int_F \gamma^{k+1}_{\nabla,F}\underline{\boldsymbol{v}}_F \cdot f. \tag{6.6}$$

We define the discrete problem:
Find $(\underline{\boldsymbol{v}}_h, \underline{p}_h) \in \underline{\boldsymbol{X}}^k_{\nabla,h,\star} \times \underline{\boldsymbol{X}}^k_{L^2,h}$ such that for all $(\underline{\boldsymbol{w}}_h, \underline{q}_h) \in \underline{\boldsymbol{X}}^k_{\nabla,h,\star} \times \underline{\boldsymbol{X}}^k_{L^2,h}$

$$\mathcal{A}_h((\underline{\boldsymbol{v}}_F, \underline{p}_h), (\underline{\boldsymbol{w}}_F, \underline{q}_h)) = \mathcal{L}_h(\underline{\boldsymbol{v}}_h). \tag{6.7}$$

We show the well-posedness in lemma 30.

We consider the following Stokes problem:
Find $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega) \cap \boldsymbol{L}^2_0(\Omega)$, $p \in H^1_0(\Omega)$ such that

$$\begin{aligned} -\mu\,\boldsymbol{\Delta}\,\boldsymbol{u} + \mathbf{grad}\, p &= f, \text{ on } \Omega,\\ \operatorname{div} \boldsymbol{u} &= 0, \text{ on } \Omega,\\ \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}_\Omega} &= 0, \text{ on } \partial\Omega. \end{aligned} \tag{6.8}$$

Let $(\boldsymbol{u}, p)$ solves (6.8) and let $(\underline{\boldsymbol{v}}_h, \underline{p}_h)$ solves (6.7). We assume that the continuous solutions $\boldsymbol{u}$, p have the additional smoothness $\boldsymbol{u} \in \boldsymbol{H}^{k+2}(\mathcal{F}_h)$ and $p \in H^{k+2}(\mathcal{F}_h)$. We deduce the following error estimate.

Theorem 29 (Error estimate for Stokes)**.** *Under the smoothness assumption on $\boldsymbol{u}$ and p it holds that*

$$\|\underline{\boldsymbol{v}}_h - \underline{\boldsymbol{I}}^k_{\nabla,h}\boldsymbol{u}\|_{\mu,\nabla,1,h} + \|\underline{p}_h - \pi^k_{\mathcal{P},\mathcal{F}_h} p\|_{L^2,F} \lesssim h^{k+1}\left(|\boldsymbol{u}|_{\boldsymbol{H}^{k+2}(\mathcal{F}_h)} + |p|_{H^{k+1}(\mathcal{F}_h)} + |p|_{H^{k+2}(\mathcal{F}_h)}\right). \tag{6.9}$$

Proof. The proof is a direct application of the third Strang lemma (see [8]) to the estimates given by 30 and 31. □

Lemma 30 (Well-posedness.)**.** *For any $(\underline{\boldsymbol{v}}_h, \underline{p}_h) \in \underline{\boldsymbol{X}}^k_{\nabla,h,\star} \times \underline{\boldsymbol{X}}^k_{L^2,h}$ there is $(\underline{\boldsymbol{w}}_h, \underline{q}_h) \in \underline{\boldsymbol{X}}^k_{\nabla,h,\star} \times \underline{\boldsymbol{X}}^k_{L^2,h}$ such that $\|\underline{\boldsymbol{w}}_h\|_{\mu,\nabla,1,h} + \|\underline{q}_h\|_{L^2,F} \lesssim \|\underline{\boldsymbol{v}}_h\|_{\mu,\nabla,1,h} + \|\underline{p}_h\|_{L^2,F}$ and*

$$\mathcal{A}_h((\underline{\boldsymbol{v}}_F, \underline{p}_h), (\underline{\boldsymbol{w}}_F, \underline{q}_h)) \gtrsim \|\underline{\boldsymbol{v}}_h\|^2_{\mu,\nabla,1,h} + \|\underline{p}_h\|^2_{L^2,F}.$$

Proof. Let $(\underline{\boldsymbol{v}}_h, \underline{p}_h) \in \underline{\boldsymbol{X}}^k_{\nabla,h,\star} \times \underline{\boldsymbol{X}}^k_{L^2,h}$, we have

$$\mathcal{A}_h((\underline{\boldsymbol{v}}_F, \underline{p}_h), (\underline{\boldsymbol{v}}_F, \underline{p}_h)) = \mathrm{a}_h(\underline{\boldsymbol{v}}_h, \underline{\boldsymbol{v}}_h) \gtrsim \|\underline{\boldsymbol{v}}_h\|^2_{\mu,\nabla,1,h}. \tag{6.10}$$

Where the last inequality comes from 22. Moreover by 25 there is $\underline{\boldsymbol{w}'}_h \in \underline{\boldsymbol{X}}^k_{\nabla,h,\star}$ such that $\underline{D}^k_h \underline{\boldsymbol{w}'}_h = -\underline{p}_h$ and $\|\underline{\boldsymbol{w}'}_h\|_{\mu,\nabla,1,h} \lesssim \|\underline{p}_h\|_{L^2,F}$. Hence

$$\begin{aligned} \mathcal{A}_h((\underline{\boldsymbol{v}}_F, \underline{p}_h), (\underline{\boldsymbol{w}'}_F, 0)) &= \mathrm{a}_h(\underline{\boldsymbol{v}}_h, \underline{\boldsymbol{w}'}_h) + \|\underline{p}_h\|^2_{L^2,F}\\ &\geq -\frac{1}{2}\|\underline{\boldsymbol{v}}_h\|^2_{\mu,\nabla,1,h} - \frac{1}{2}\|\underline{\boldsymbol{w}'}_h\|^2_{\mu,\nabla,1,h} + \|\underline{p}_h\|^2_{L^2,F}\\ &\gtrsim -\frac{1}{2}\|\underline{\boldsymbol{v}}_h\|^2_{\mu,\nabla,1,h} + \frac{1}{2}\|\underline{p}_h\|^2_{L^2,F}. \end{aligned} \tag{6.11}$$

And we conclude by summing (6.10) and (6.11). □

We define the consistency error $\mathcal{E}_h : \underline{\boldsymbol{X}}^k_{\nabla,h,\star} \times \underline{\boldsymbol{X}}^k_{L^2,h} \to \mathbb{R}$ by

$$\mathcal{E}_h((\underline{\boldsymbol{w}}_h, \underline{q}_h)) = \mathcal{L}_h(\underline{\boldsymbol{w}}_h) - \mathcal{A}_h((\underline{\boldsymbol{I}}^k_{\nabla,h}\boldsymbol{u}, \pi^k_{\mathcal{P},\mathcal{F}_h}p), (\underline{\boldsymbol{w}}_h, \underline{q}_h)). \tag{6.12}$$

Lemma 31. *For all* $\underline{\boldsymbol{w}}_h \in \underline{\boldsymbol{X}}^k_{\nabla,h,\star}$, $\underline{q}_h \in \underline{\boldsymbol{X}}^k_{L^2,h}$,

$$\begin{aligned}\mathcal{E}_h((\underline{\boldsymbol{w}}_h, \underline{q}_h)) \lesssim & h^{k+1}\left(|\boldsymbol{u}|_{\boldsymbol{H}^{k+2}(\mathcal{F}_h)} + |p|_{H^{k+1}(\mathcal{F}_h)} + |p|_{H^{k+2}(\mathcal{F}_h)}\right)\\ & \left(\|\underline{\boldsymbol{v}}_h\|_{\mu,\nabla,1,h} + \|\underline{p}_h\|_{L^2,F}\right).\end{aligned}$$

Proof. Let $\underline{\boldsymbol{w}}_h \in \underline{\boldsymbol{X}}^k_{\nabla,h,\star}$, $\underline{q}_h \in \underline{\boldsymbol{X}}^k_{L^2,h}$,

$$\begin{aligned}\mathcal{E}_h((\underline{\boldsymbol{w}}_h, \underline{q}_h)) =& \sum_{F\in\mathcal{F}_h}\int_F \gamma^{k+1}_{\nabla,F}\underline{\boldsymbol{w}}_F \cdot f - \mu\left(\underline{\nabla}^{k+1}_F \underline{\boldsymbol{I}}^k_{\nabla,F}\boldsymbol{u}, \underline{\nabla}^{k+1}_F\underline{\boldsymbol{w}}_F\right)_{\nabla,F}\\ &+ \int_F D^k_F\underline{\boldsymbol{w}}_F\pi^k_{\mathcal{P},F}p - \int_F D^k_F\underline{\boldsymbol{I}}^k_{\nabla,F}\boldsymbol{u}q_F\\ =& \sum_{F\in\mathcal{F}_h}\int_F \gamma^{k+1}_{\nabla,F}\underline{\boldsymbol{w}}_F\cdot\mathbf{grad}\, p + \int_F D^k_F\underline{\boldsymbol{w}}_F\pi^k_{\mathcal{P},F}p\\ &- \mu\left(\int_F \gamma^{k+1}_{\nabla,F}\underline{\boldsymbol{w}}_F\cdot\boldsymbol{\Delta}\,\boldsymbol{u} + \left(\underline{\nabla}^{k+1}_F\underline{\boldsymbol{I}}^k_{\nabla,F}\boldsymbol{u}, \underline{\nabla}^{k+1}_F\underline{\boldsymbol{w}}_F\right)_{\nabla,F}\right)\\ =& \tilde{\mathcal{E}}_{\nabla,h}(p\,\mathbf{I}_{2,2}, \underline{\boldsymbol{w}}_h) - \mathrm{s}_{\boldsymbol{L}^2,F}\left(\underline{\nabla}^{k+1}_F\underline{\boldsymbol{w}}_F, \underline{\boldsymbol{I}}^k_{\boldsymbol{L}^2(F)}(p\,\mathbf{I}_{2,2})\right) - \mu\tilde{\mathcal{E}}_{\boldsymbol{\Delta},h}(\boldsymbol{u}, \underline{\boldsymbol{w}}_h)\\ \le& |\tilde{\mathcal{E}}_{\nabla,h}(p\,\mathbf{I}_{2,2}, \underline{\boldsymbol{w}}_h)| + |\mathrm{s}_{\boldsymbol{L}^2,F}\left(\underline{\nabla}^{k+1}_F\underline{\boldsymbol{w}}_F, \underline{\nabla}^{k+1}_F\underline{\boldsymbol{w}}_F\right)|^{1/2}\\ &+ |\mathrm{s}_{\boldsymbol{L}^2,F}\left(\underline{\boldsymbol{I}}^k_{\boldsymbol{L}^2(F)}(p\,\mathbf{I}_{2,2}), \underline{\boldsymbol{I}}^k_{\boldsymbol{L}^2(F)}(p\,\mathbf{I}_{2,2})\right)|^{1/2} + \mu|\tilde{\mathcal{E}}_{\boldsymbol{\Delta},h}(\boldsymbol{u}, \underline{\boldsymbol{w}}_h)|.\end{aligned}$$

Where the second equality comes from we used $f = -\mu\ \boldsymbol{\Delta}\,\boldsymbol{u} + \mathbf{grad}\, p$, (4.2b) and $\operatorname{div}\boldsymbol{u} = 0$. And the third equality comes from (5.12), (5.15) as well as:

$$\int_F D^k_F\underline{\boldsymbol{w}}_F\pi^k_{\mathcal{P},F}p = \int_F \operatorname{Tr}\nabla^{k+1}_F\ \underline{\boldsymbol{w}}_F p = \int_F \nabla^{k+1}_F\,\underline{\boldsymbol{w}}_F : (p\,\mathbf{I}_{2,2}) = \int_F \nabla^{k+1}_F\,\underline{\boldsymbol{w}}_F : \boldsymbol{\pi}^{k+1}_{\overline{\boldsymbol{\mathcal{RT}}},F}(p\,\mathbf{I}_{2,2}).$$

We conclude inferring the estimates 26, 28 and the consistency (5.4). □

7 Alternative boundary conditions.

In this section we show how to extend the results of Section 6 when using Dirichlet boundary conditions on $\underline{\boldsymbol{X}}^k_{\nabla,h}$. This is useful for common condition such as the no slip condition or forced inlet condition and does not require much change.

7.1 Dirichlet boundary conditions.

We introduce the space $\underline{\boldsymbol{X}}^k_{\nabla,h,0} := \{\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\nabla,h} : \forall E \in \mathcal{E}_h, E \subset \partial\Omega, \boldsymbol{v}_E \equiv 0\}$. The continuous and discrete problem are then pretty much the same: they take the same expression but on a different domain. Since the pressure is only defined up to a constant value, we introduce the natural space: $\underline{\boldsymbol{X}}^k_{L^2,h,\star} := \{\underline{q}_h \in \underline{\boldsymbol{X}}^k_{L^2,h} : \sum_{F\in\mathcal{F}_h}\int_F q_F = 0\}$. Then we define the bilinear form: $\mathcal{A}_h((\underline{\boldsymbol{v}}_F, \underline{p}_h), (\underline{\boldsymbol{w}}_h, \underline{q}_h)) \in (\underline{\boldsymbol{X}}^k_{\nabla,h,0} \times \underline{\boldsymbol{X}}^k_{L^2,h,\star}) \times (\underline{\boldsymbol{X}}^k_{\nabla,h,0} \times \underline{\boldsymbol{X}}^k_{L^2,h,\star}) \to \mathbb{R}$ by

$$\mathcal{A}_h((\underline{\boldsymbol{v}}_h, \underline{p}_h), (\underline{\boldsymbol{w}}_h, \underline{q}_h)) = \mathrm{a}_h(\underline{\boldsymbol{v}}_h, \underline{\boldsymbol{w}}_h) - \mathrm{b}_h(\underline{\boldsymbol{w}}_h, \underline{p}_h) + \mathrm{b}_h(\underline{\boldsymbol{v}}_F, \underline{q}_h). \tag{7.1}$$

With a_h and b_h defined by (6.2), (6.3), we also keep the same definition (6.6) of the source term $\mathcal{L}_h$. So the discrete problem is:
Find $(\underline{\boldsymbol{v}}_h, \underline{p}_h) \in \underline{\boldsymbol{X}}^k_{\nabla,h,0} \times \underline{\boldsymbol{X}}^k_{L^2,h,\star}$ such that for all $(\underline{\boldsymbol{w}}_h, \underline{q}_h) \in \underline{\boldsymbol{X}}^k_{\nabla,h,0} \times \underline{\boldsymbol{X}}^k_{L^2,h,\star}$

$$\mathcal{A}_h((\underline{\boldsymbol{v}}_F, \underline{p}_h), (\underline{\boldsymbol{w}}_F, \underline{q}_h)) = \mathcal{L}_h(\underline{\boldsymbol{v}}_h). \tag{7.2}$$

The Stokes problem becomes:
Find $\boldsymbol{u} \in \boldsymbol{H}^1_0(\Omega) \cap \boldsymbol{H}^2(\Omega)$, $p \in H^1(\Omega) \cap L^2_0(\Omega)$ such that

$$\begin{aligned} -\mu\, \boldsymbol{\Delta}\, \boldsymbol{u} + \mathbf{grad}\, p &= f, \text{ on } \Omega, \\ \operatorname{div} \boldsymbol{u} &= 0, \text{ on } \Omega. \end{aligned} \tag{7.3}$$

Theorem 32. *Under the same assumption as 29 we obtain the well-posedness of the problem* (7.2) *and a convergence toward the continuous solution of problem* (7.3) *with the same error estimate as* (29).

Proof. As stated before there is not much to adapt, namely: We need a suitable version of 24, and we can expect $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\nabla,h,0}$ if $\underline{p}_h \in \underline{\boldsymbol{X}}^k_{L^2,h,\star}$ by substituting the use of theorem 42 by theorem 43 in the proof of 24. The consistency errors 26 and 28 required respectively $\boldsymbol{W} \in \boldsymbol{H}^1_0(\Omega)$ and $\nabla \boldsymbol{w} \cdot \boldsymbol{n}_\Omega = 0$. However we can check that this is only used to get (5.14) and (5.17) both of which also hold if $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\nabla,h,0}$ instead, so that $\boldsymbol{v}_E \equiv 0$, $\forall E \subset \partial\Omega$. Finally, we relied on 22 to show that $\mathcal{A}_h$ is weakly coercive. This too can readily be adapted if we use [[10] Lemma 2.15] instead of [[10] Theorem 6.5] in the proof of 22. With these tree results we can proceed exactly in the same manner as we did for 29. □

7.2 Mixed boundary conditions.

We can also use Dirichlet conditions on a subset of the boundary and Neumann conditions elsewhere. Explicitly we write Γ_D a relatively open subset of $\partial\Omega$ with a non-zero measure and $\Gamma_N = \partial\Omega \setminus \Gamma_D$. We also assume that each boundary edge $\partial\Omega \supset E \in \mathcal{E}_h$ is either contained in Γ_N or in Γ_D but not in both (either $E \cap \Gamma_D = \emptyset$ or $E \cap \Gamma_N = \emptyset$) and that both contained at least one edge (else we degenerate to pure Neumann or pure Dirichlet with have already been deals with). The boundary defined by Γ_D will expectedly be where we use Dirichlet boundary conditions and Γ_N where we use Neumann boundary conditions. So that the Stokes problem is:
Find $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$, $p \in H^1(\Omega)$ such that

$$\begin{aligned} -\mu\, \boldsymbol{\Delta}\, \boldsymbol{u} + \mathbf{grad}\, p &= f, \text{ on } \Omega, \\ \operatorname{div} \boldsymbol{u} &= 0, \text{ on } \Omega, \\ \boldsymbol{u} &= 0, \text{ on } \Gamma_D, \\ \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}_\Omega} &= 0, \text{ on } \Gamma_N, \\ p &= 0, \text{ on } \Gamma_N. \end{aligned} \tag{7.4}$$

We introduce the discrete space $\underline{\boldsymbol{X}}^k_{\nabla,h,D} := \{\boldsymbol{v}_h \in \underline{\boldsymbol{X}}^k_{\nabla,h} : \forall E \in \mathcal{E}_h, E \subset \Gamma_D, \boldsymbol{v}_E \equiv 0\}$ and as before define: $\mathcal{A}_h((\underline{\boldsymbol{v}}_F, \underline{p}_h), (\underline{\boldsymbol{w}}_h, \underline{q}_h)) \in (\underline{\boldsymbol{X}}^k_{\nabla,h,D} \times \underline{\boldsymbol{X}}^k_{L^2,h}) \times (\underline{\boldsymbol{X}}^k_{\nabla,h,D} \times \underline{\boldsymbol{X}}^k_{L^2,h}) \to \mathbb{R}$ by

$$\mathcal{A}_h((\underline{\boldsymbol{v}}_h, \underline{p}_h), (\underline{\boldsymbol{w}}_h, \underline{q}_h)) = \mathrm{a}_h(\underline{\boldsymbol{v}}_h, \underline{\boldsymbol{w}}_h) - \mathrm{b}_h(\underline{\boldsymbol{w}}_h, \underline{p}_h) + \mathrm{b}_h(\underline{\boldsymbol{v}}_F, \underline{q}_h). \tag{7.5}$$

The discrete problem becomes once again:
Find $(\underline{\boldsymbol{v}}_h, \underline{p}_h) \in \underline{\boldsymbol{X}}^k_{\nabla,h,D} \times \underline{\boldsymbol{X}}^k_{L^2,h}$ such that for all $(\underline{\boldsymbol{w}}_h, \underline{q}_h) \in \underline{\boldsymbol{X}}^k_{\nabla,h,D} \times \underline{\boldsymbol{X}}^k_{L^2,h}$

$$\mathcal{A}_h((\underline{\boldsymbol{v}}_F, \underline{p}_h), (\underline{\boldsymbol{w}}_F, \underline{q}_h)) = \mathcal{L}_h(\underline{\boldsymbol{v}}_h). \tag{7.6}$$

Remark 33. In practical implementation we store continuous polynomials on edge by their lower order moment on each edge and their values on vertices. If a boundary edge is part of the Dirichlet boundary: $E \subset \Gamma_D$ we must set to zero all associated unknowns, including those on the vertices of this edge.

Remark 34. Although we took homogeneous boundary conditions for the sake of simplicity this is by no means a limitation. For inhomogeneous Dirichlet simply write $\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}_D$ with $\boldsymbol{u}_0 \in \underline{\boldsymbol{X}}^k_{\nabla,h,D}$ and $\boldsymbol{u}_D$ given by the value on the boundary and solve for $\boldsymbol{u}_0$.

8 Numerical tests.

We display the numerical results for the Stokes problem with Neumann boundary conditions (6.7), with Dirichlet boundary conditions (7.2) and with mixed boundary conditions (7.6). This was implemented with the HArDCore C++ framework (see `https://github.com/jdroniou/HArDCore`), using the linear algebra facilities from the Eigen3 library (see `https://eigen.tuxfamily.org`). An implementation of the spaces and operators defined in this paper as well as a Stokes solver can be found at `https://github.com/mlhanot/HArDCore2D-Stokes`.

We used a constant viscosity $\mu = 1$ and measure the rate of convergence for various polynomial degrees $k \in \{0, 1, 2, 3\}$. We compute the error by

$$\|\underline{\boldsymbol{v}}_h - \underline{\boldsymbol{I}}^k_{\nabla,h}\boldsymbol{u}\|_{\mu,\nabla,1,h} + \|\underline{p}_h - \pi^k_{\mathcal{P},\mathcal{F}_h} p\|_{L^2,h}.$$

We expect the error to decrease at a rate $\mathcal{O}(h^{k+1})$ thanks to theorem 29 and 32. These tests are done on various mesh sequences which can be seen in Figure 2 showing the flexibility of the method. We show our results in Figure 3. We always obtain results consistent with the theory and the various features of the meshes do not deteriorate the convergence toward the exact solution.

A Results on polynomial spaces.

We begin by showing a few results to complete the introduction of spaces (2.12) and (2.14).

Lemma 35. *For any $F \in \mathcal{F}_h$, it holds $\overline{\boldsymbol{\mathcal{R}}}^{c,k}(F) \cap \overline{\boldsymbol{\mathcal{R}}}^{k}(F) = \{0\}$.*

Proof. If $w \in \mathcal{P}^{0,k}(F)$ and $W \in (\boldsymbol{\mathcal{R}}^{c,k}(F))^2$ are such that $\operatorname{Tr} W = 0$ and $\nabla\cdot W = \mathbf{grad}\, w$ then $W = 0$ and $w = 0$. Indeed assuming $\boldsymbol{x}_F = \mathbf{0}$ without loss of generality any element $W \in \overline{\boldsymbol{\mathcal{R}}}^{c,k}(F)$ is written $W = \begin{pmatrix} xyP_{i,j}x^iy^j & y^2P_{i,j}x^iy^j \\ -x^2P_{i,j}x^iy^j & -xyP_{i,j}x^iy^j \end{pmatrix}, P_{i,j}x^jy^j \in \mathcal{P}^{k-2}(F)$. So that

$$\nabla\cdot W = \begin{pmatrix} (2+i+j)P_{i,j-1}x^iy^j \\ -(2+i+j)P_{i-1,j}x^iy^j \end{pmatrix}.$$

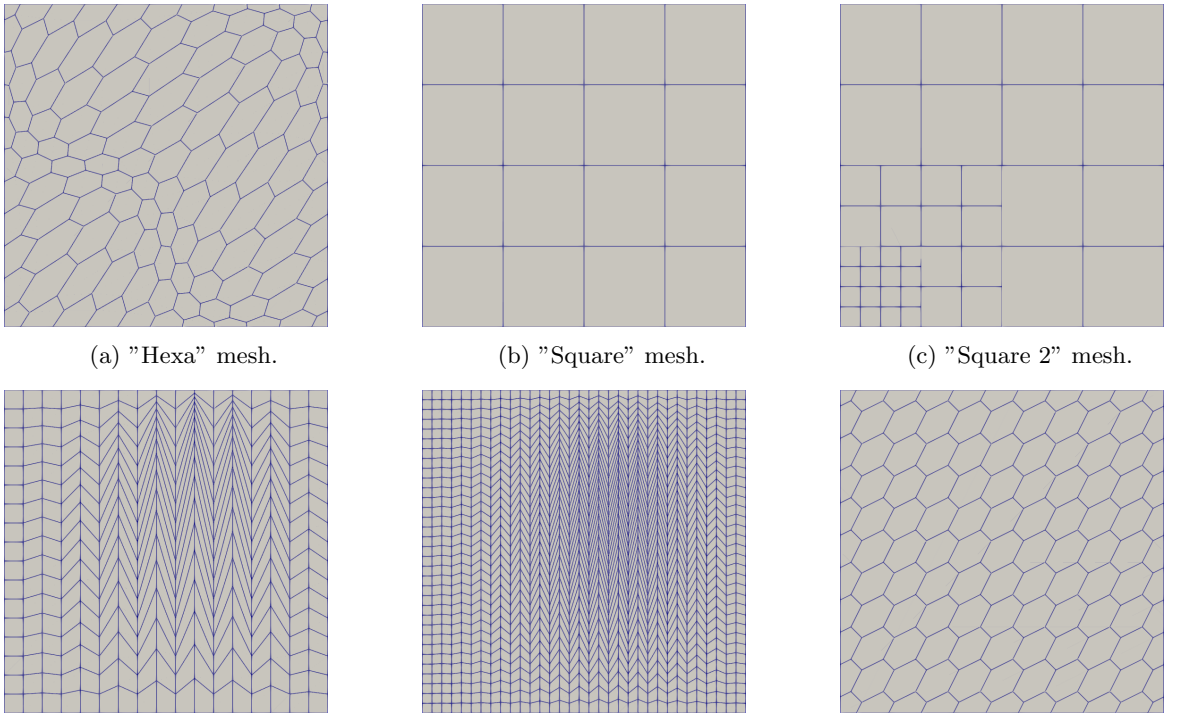
On the other side for $w = w_{i,j}x^iy^j \in \mathcal{P}^{0,k}(F)$,

$$\mathbf{grad}\, w = \begin{pmatrix} (i+1)w_{i+1,j}x^iy^j \\ (j+1)w_{i,j+1}x^iy^j \end{pmatrix}.$$

Hence, $\nabla\cdot W = \mathbf{grad}\, w$ if and only if for all $i, j \geq 0$,

$$\begin{pmatrix} (1+i+j)P_{i-1,j-1} \\ -(1+i+j)P_{i-1,j-1} \end{pmatrix} = \begin{pmatrix} iw_{i,j} \\ jw_{i,j} \end{pmatrix}.$$

This is only satisfied for $w \equiv 0$. □



(a) "Hexa" mesh. (b) "Square" mesh. (c) "Square 2" mesh.

(d) "Tilted" mesh. (e) "Tilted 2" mesh. (f) "Hexa anisotropic" mesh.

Figure 2: Families of mesh used.

Lemma 36. *For any $F \in \mathcal{F}_h$, it holds $(\boldsymbol{\mathcal{R}}^{c,k}(F))^2 = \overline{\boldsymbol{\mathcal{R}}}^{c,k}(F) \oplus \overline{\boldsymbol{\mathcal{R}}}^{k}(F)$.*

Proof. Lemma 35 already shows that $\overline{\boldsymbol{\mathcal{R}}}^{c,k}(F) \cap \overline{\boldsymbol{\mathcal{R}}}^{k}(F) = \{0\}$. It is enough to compare the dimension of these spaces:

$$\dim \overline{\boldsymbol{\mathcal{R}}}^{c,k}(F) = \dim \mathcal{P}^{k-2}(F) = \frac{k!}{2(k-2)!} = \frac{k(k-1)}{2},$$

$$\dim \overline{\boldsymbol{\mathcal{R}}}^{k}(F) = \dim \mathcal{P}^{0,k}(F) = \frac{(k+2)(k+1)}{2} - 1 = \frac{k^2+3k+2-2}{2}.$$

The sum of both is $\frac{k^2+3k+k^2-k}{2} = k(k+1)$, which is the same as

$$\dim (\boldsymbol{\mathcal{R}}^{c,k}(F))^2 = 2 \dim \mathcal{P}^{k-1}(F) = 2\frac{(k+1)k}{2} = k(k+1).$$

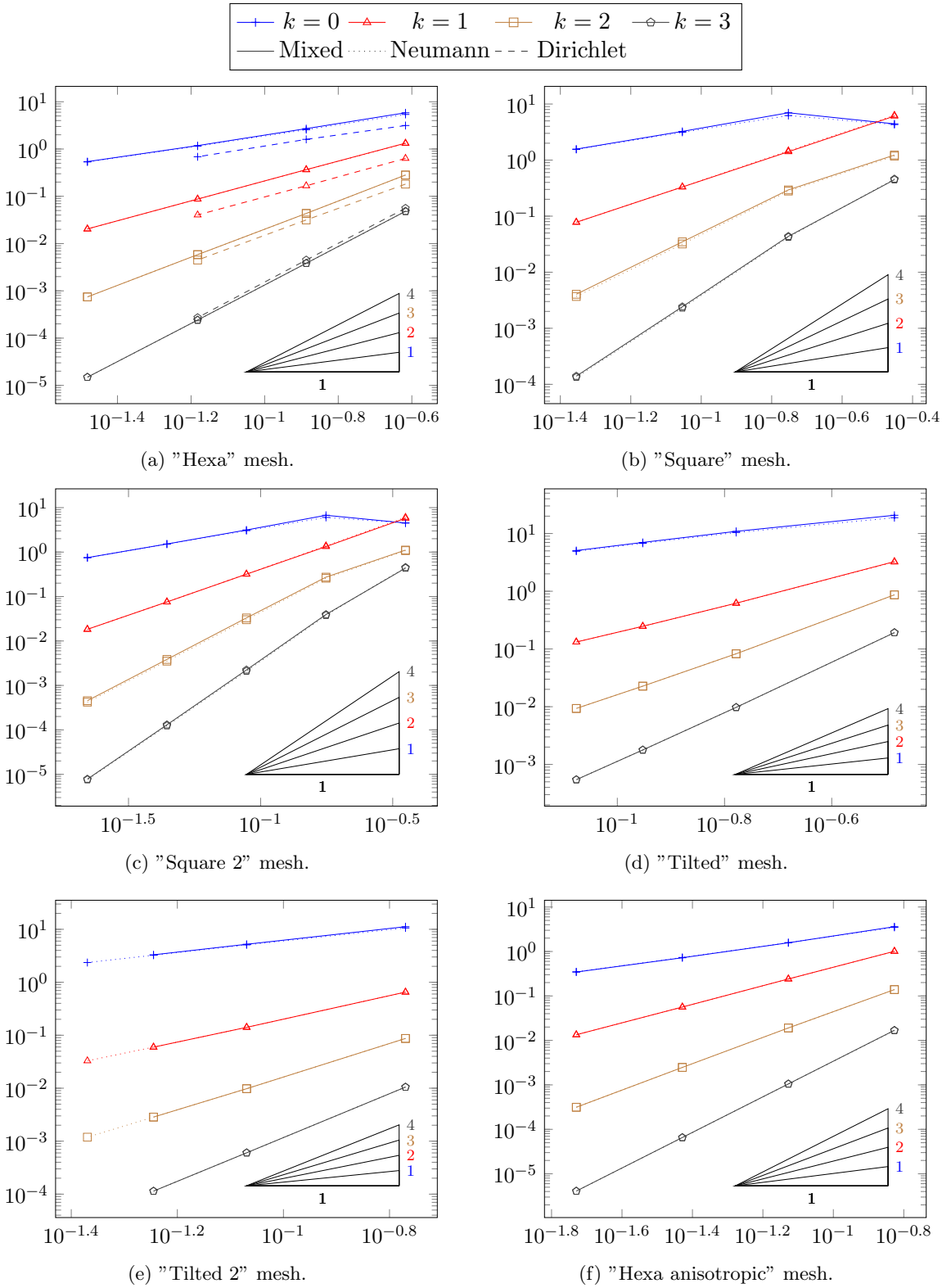
□

Next we show some lemmas on convex polytopes.

Lemma 37. *Let $F \in \mathcal{F}_h$, $\boldsymbol{x}_F$ defined as in (2.5), if $B = B(\boldsymbol{x}_F, h_B) \subset F$ with $h_B \gtrsim h_F$ and $Q \in \mathcal{P}^k(F)$ then $\|Q\|_{L^\infty(B)} \approx \|Q\|_\infty$.*

Proof. Let $h_o \in \mathbb{R}_+^*$ such that $F \subset B(\boldsymbol{x}_F, h_o)$ and $h_o \lesssim h_F$ (h_o exists by the regularity assumption on the mesh sequence $\mathcal{M}_h$). Let $\boldsymbol{v}$ by any vector such that $\|\boldsymbol{v}\| = 1$, then $\forall \alpha > 0$ such that $\boldsymbol{x}_F + \alpha \boldsymbol{v} \in F$,

$$Q(\boldsymbol{x}_F + \alpha \boldsymbol{v}) = \sum_{i=0}^{k} \frac{\partial_{\boldsymbol{v}}^i Q(\boldsymbol{x}_F)}{i!} \alpha^i,$$



(a) "Hexa" mesh.

(b) "Square" mesh.

(c) "Square 2" mesh.

(d) "Tilted" mesh.

(e) "Tilted 2" mesh.

(f) "Hexa anisotropic" mesh.

Figure 3: Absolute error estimate in discrete norm $\|\cdot\|_{\mu,\nabla,1,h} + \|\cdot\|_{L^2,h}$ vs. mesh size h.

so

$$|Q(\boldsymbol{x}_F + \alpha\boldsymbol{v})| \leq \sum_{i=0}^{k} \frac{\|\partial_{\boldsymbol{v}}^i Q\|_{L^\infty(B)}}{i!}\alpha^i.$$

By the discrete Poincare inequality 2 we have $\forall i$, $\|\partial_{\boldsymbol{v}}^i Q\|_{L^\infty(B)} \lesssim h_B^{-i}\|Q\|_{L^\infty(B)}$. Lastly $\boldsymbol{x}_F + \alpha\boldsymbol{v} \in F$ so $\alpha < h_o$ and $|Q(\boldsymbol{x}_F + \alpha\boldsymbol{v})| \lesssim \|Q\|_{L^\infty(B)} h_o^i h_B^{-i}$. Since $h_o \lesssim h_B$ we can conclude $\|Q\|_\infty \lesssim \|Q\|_{L^\infty(B)}$. □

Any $F \in \mathcal{F}_h$ is a convex, open polygon. Let $(E_i)_{i\leq|\mathcal{E}_F|}$ be the set of its edges, each of normal vector $\boldsymbol{n}_E$. For all E_i there exists $P_i \in P^1(\mathbb{R}^2)$ such that $E_i \subset \mathrm{Ker}(P_i)$, moreover we can normalize P_i such that $\boldsymbol{x} \in F \implies P_i(\boldsymbol{x}) > 0$ (since F is convex) and $|\partial_{\boldsymbol{n}_E} P_i| = 1$.

Lemma 38. *Set $P = \prod_{i\leq|\mathcal{E}_F|} P_i$, $\boldsymbol{x}_0 \in F$ such that $B_0 = B(\boldsymbol{x}_0, h_F/2) \subset F$ and $B = B(\boldsymbol{x}_0, h_F/4)$ then $\inf_{\boldsymbol{x}\in B} P(\boldsymbol{x}) > \left(\frac{h_F}{4}\right)^{|\mathcal{E}_F|}$ and $\|P\|_{L^\infty(F)} \lesssim h_F^{|\mathcal{E}_F|}$.*

Proof. For any $i \leq |\mathcal{E}_F|$, the value $P_i(\boldsymbol{x})$ at any point $\boldsymbol{x}$ is the distance between $\boldsymbol{x}$ and the straight line defined by E_i and is positive on F. For any $\boldsymbol{x} \in B$, $\boldsymbol{x}$ is a least at a distance $h_F/4$ of any edge since $B_0 \subset F$. We obtain the lower bound by taking the product over all edges. Conversely, using the mesh regularity we can find $h_o > 0$, $h_o \lesssim h_F$ such that F is inscribed in a circle of diameter h_o. Then $\forall i \leq |\mathcal{E}_F|$, $\forall \boldsymbol{x} \in F$, it holds $0 < P_i(\boldsymbol{x}) < h_o \lesssim h_F$. Again we conclude by taking the product over all edges. □

Lemma 39. *For any $F \in \mathcal{F}_h$ and $q \in \mathcal{P}^k(F)$ there is $P \in \mathcal{P}^{k+|\mathcal{E}_F|}(\overline{F})$ such that $P_{|\partial F} = 0$, $\pi^k_{\mathcal{P},F} P = q$ and $\|P\| \approx \|q\|$.*

Proof. Let $P = \Pi Q$ with Π given by 38 and $Q \in \mathcal{P}^k(F)$. The application

$$\mathcal{P}^k(F) \ni Q \to \left(\lambda \to \int_F P\lambda\right) \in \mathcal{P}^k(F)' \tag{A.1}$$

is linear and between two spaces of same dimension, thus it is enough to check that it is injective. Let $Q \in \mathcal{P}^k(F)$ such that $\forall \lambda \in \mathcal{P}^k(F)$, $\int_F \Pi Q\lambda = 0$ hence $\int_F \Pi Q^2 = 0$. However since on F, $\Pi > 0$ we can define the function $\sqrt{\Pi} \in L^2(F)$ and have $\int_F \left(\sqrt{\Pi}Q\right)^2 = \|\sqrt{\Pi}Q\|^2 = 0$. So $\sqrt{\Pi}Q \equiv 0$ and $Q \equiv 0$ which prove that (A.1) is injective thus prove the existence of a polynomial $P \in \mathcal{P}^{k+|\mathcal{E}_F|}(\overline{F})$ such that $P_{|\partial F} = 0$ and $\pi^k_{\mathcal{P},F} P = q$. Let us show that P also satisfy the norm equivalence: In particular we must have

$$\int_F (P - q)Q = 0$$
$$\int_F \Pi Q^2 = \int_F qQ$$
$$\|\sqrt{\Pi}Q\|^2 \leq \|q\|\|Q\| \lesssim \|q\| h_F \|Q\|_\infty.$$

Therefore the discrete Sobolev inequality [[10] Lemma 1.25] gives

$$\|\sqrt{\Pi}Q\|_\infty^2 \approx h_F^{-2}\|\sqrt{\Pi}Q\|^2 \lesssim h_F^{-1}\|Q\|_\infty\|q\|.$$

On the other consider $B = B(\boldsymbol{x}_0, h_F/4)$ given in 38, it holds $\inf_{\boldsymbol{x}\in B} \sqrt{\Pi}(\boldsymbol{x}) \gtrsim h_F^{|\mathcal{E}_F|/2}$. Thus by 37:

$$\|\sqrt{\Pi}Q\|_\infty \geq \|\sqrt{\Pi}Q\|_{L^\infty(B)} \gtrsim h_F^{|\mathcal{E}_F|/2}\|Q\|_{L^\infty(B)} \approx h_F^{|\mathcal{E}_F|/2}\|Q\|_\infty.$$

Hence $h_F^{|\mathcal{E}_F|}\|Q\|_\infty^2 \lesssim h_F^{-1}\|Q\|_\infty\|q\|$, $\|Q\|_\infty \lesssim h_F^{-1-|\mathcal{E}_F|}\|q\|$ and

$$\|\Pi Q\| \approx h_F\|\Pi Q\|_\infty \leq h_F\|\Pi\|_\infty\|Q\|_\infty \lesssim h_F^{|\mathcal{E}_F|+1}\|Q\|_\infty \lesssim \|q\|.$$

□

B Trace lifting.

In order to prove consistency results we often need functions of Sobolev spaces with suitable properties. We construct them in this section.

Theorem 40. *For all $\underline{\boldsymbol{v}}_F \in \underline{\boldsymbol{X}}_{\nabla,F}^k$ there is $\boldsymbol{R}_{\underline{\boldsymbol{v}}_F} \in \boldsymbol{H}^1(F)$ such that*

$$\begin{aligned} \boldsymbol{R}_{\underline{\boldsymbol{v}}_F} =& \boldsymbol{v}_E \text{ on } \partial F, \\ \|\boldsymbol{R}_{\underline{\boldsymbol{v}}_F}\|_F + \|\nabla \boldsymbol{R}_{\underline{\boldsymbol{v}}_F}\|_F \lesssim& |||\underline{\boldsymbol{v}}_F|||_{\nabla,F} + \left|\left|\left|\underline{\nabla}_F^{k+1}\underline{\boldsymbol{v}}_F\right|\right|\right|_{\boldsymbol{L}^2,F}. \end{aligned} \tag{B.1}$$

This lift is built upon [[2] Theorem 18.40]: Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be an open set whose boundary $\partial\Omega$ is uniformly Lipschitz continuous of parameters ϵ, L and M (see [[2] Definition 13.11]). Then for all $g \in \boldsymbol{B}^{1/2,2}(\partial\Omega)$, there is $c \in \mathbb{R}$ depending only on N and a function $u \in H^1(\Omega)$ such that $\mathrm{Tr}(u) = g$,

$$\|u\|_{L^2(\Omega)} \leq M^{1/2}\epsilon^{1/2}\|g\|_{L^2(\partial\Omega)} \tag{B.2}$$

and

$$\|\mathbf{grad}\, u\|_{\boldsymbol{L}^2(\Omega)} \leq cM(1+L)^{3+N/2}\epsilon^{-1/2}\|g\|_{L^2(\partial\Omega)} + cM(1+L)^{2+(N+1)/2}|g|^\diamondsuit_{\boldsymbol{B}^{1/2,2}(\partial\Omega)}. \tag{B.3}$$

With the Besov seminorm defined by (see [[2] Definition 18.36]):

$$|g|^\diamondsuit_{\boldsymbol{B}^{1/2,2}(\partial\Omega)} := \left(\int_{\partial\Omega}\int_{\partial\Omega\cap B(x,\epsilon)} \frac{|g(x)-g(y)|^2}{\|x-y\|^N} dydx\right)^{1/2}. \tag{B.4}$$

Proof of theorem 40. We apply the above-mentioned theorem [[2] Theorem 18.40] to $\Omega = F$ and g a component of $\boldsymbol{v}_{\mathcal{E}_F}$. Here $N = 2$ and for the open cover of ∂F (see [[2] Definition 13.11]) we take a ball centered at each vertex of radius half the length of the shortest adjacent edge as well as a ball centered on middle of each edge of radius half the length of the edge. This way we have $L = 1$, $M = 2$ and $\epsilon \approx h_F$. We conclude with the estimate on the Besov seminorm 41.

Indeed, let $\boldsymbol{R}_{\underline{\boldsymbol{v}}_F}$ be such that $\mathrm{Tr}(\boldsymbol{R}_{\underline{\boldsymbol{v}}_F}) = \boldsymbol{v}_{\mathcal{E}_F}$ and that $\boldsymbol{R}_{\underline{\boldsymbol{v}}_F}$ satisfy (B.2) and (B.3). Let g be a component of $\boldsymbol{v}_{\mathcal{E}_h}$ and u be given by (B.2) and (B.3). Without loss of generality we assume that $\int_{\partial F} g = 0$: Else we take instead $\overline{g} = \int_{\partial F} g$ and $u = \overline{g}$ so $\mathbf{grad}\, u = 0$ and $\|u\|_{L^2(F)} \approx h_F^2|\overline{g}|$, $\|\overline{g}\|_{L^2(\partial F)} \approx h_F|\overline{g}|$ and u, $\overline{g}$ satisfy (B.1). We reduce to the case $\int_{\partial F} g' = 0$ for $g' = g - \overline{g}$.

Equation (B.2) gives $\|\boldsymbol{R}_{\underline{\boldsymbol{v}}_F}\|_F \lesssim |||\underline{\boldsymbol{v}}_F|||_{\nabla,F}$ since $\epsilon \approx h_F$. Applying the Poincare-Wirtinger inequality to ∂F (since g is continuous and assumed to have zero average) we get $\|g\|_{L^2(\partial F)} \lesssim h_F\|\dot{g}\|_{L^2(\partial F)}$ hence

$$\begin{aligned} \|\mathbf{grad}\, u\|_{L^2(\Omega)} \lesssim& h_F^{-\frac{1}{2}}\|g\|_{L^2(\partial F)} + |g|^\diamondsuit_{\boldsymbol{B}^{1/2,2}(\partial\Omega)} \\ \lesssim& h_F^{-\frac{1}{2}}h_F\|\dot{g}\|_{L^2(\partial F)} + h_F^{\frac{1}{2}}\|\dot{g}\|_{L^2(\partial F)} \\ \lesssim& h_F^{\frac{1}{2}}\|\dot{g}\|_{L^2(\partial F)}. \end{aligned}$$

Recalling definitions (3.28) and (3.14) we get the expected results. □

Lemma 41. *Keeping the notations of the proof of 40, it holds*

$$|g|^{\diamondsuit}_{\boldsymbol{B}^{1/2,2}(\partial\Omega)} \lesssim h_F^{\frac{1}{2}} \|\dot{g}\|_{L^2(\partial\Omega)}. \tag{B.5}$$

Proof. We know that g is a continuous piecewise polynomial. Let $E \in \mathcal{E}_F$. Far away from the vertices, i.e. for $x \in E$ such that $B(x,\epsilon) \cap \mathcal{V}_F = \emptyset$ it holds $\forall y \in B(x,\epsilon)$, $\exists c \in B(x,\epsilon)$ such that

$$g(x) = g(y) + \dot{g}(y)(x-y) + \ddot{g}(c)\frac{(x-y)^2}{2}.$$

Hence

$$\begin{aligned}\frac{|g(x)-g(y)|^2}{\|x-y\|^2} =& \frac{|\dot{g}(y)(x-y) + \ddot{g}(c)\frac{(x-y)^2}{2}|^2}{\|x-y\|^2} \\ =& |\dot{g}(y) + \ddot{g}(c)\frac{(x-y)}{2}|^2 \\ \lesssim& |\dot{g}(y)|^2 + \|\ddot{g}\|_\infty |(x-y)|^2.\end{aligned}$$

If x_V is the curvilinear coordinate of a vertex of E, the formula still holds for $g(x) - g(x_V)$ since g is continuous on $[x, x_V]$ and C^∞ on $]x, x_V[$. Thus the formula holds for all x, y using the triangular inequality if x and y are not on the same edge. Moreover, since $\epsilon \approx h_F$ and $\int_{\partial F} 1 \approx h_F$ it holds:

$$\int_{x\in\partial F}\int_{y\in\partial F\cap B(x,\epsilon)} \|\ddot{g}\|_\infty |(x-y)|^2 \leq \|\ddot{g}\|_\infty \int_{x\in\partial F} \frac{4}{3}\epsilon^3 \lesssim \|\ddot{g}\|_\infty^2 h_F^4.$$

We have $\|\ddot{g}\|_\infty \lesssim h_F^{-\frac{1}{2}} \|\ddot{g}\|_{L^2(\partial F)}$ by [[10] Lemma 1.25], $\|\ddot{g}\|_{L^2(\partial F)} \lesssim h_F^{-1}\|\dot{g}\|_{L^2(\partial F)}$ by 2 so $\|\ddot{g}\|_\infty^2 \lesssim h_F^{-3}\|\dot{g}\|^2_{L^2(\partial F)}$ and

$$\int_{x\in\partial F}\int_{y\in\partial F\cap B(x,\epsilon)} \|\ddot{g}\|_\infty |(x-y)|^2 \lesssim h_F \|\dot{g}\|^2_{L^2(\partial F)}.$$

On the other hand by Fubini-Tonelli it holds

$$\begin{aligned}\int_{x\in\partial F}\int_{y\in\partial F\cap B(x,\epsilon)} |\dot{g}(y)|^2 &= \int_{x\in\partial F}\int_{y\in\partial F} |\dot{g}(y)|^2 \mathbf{1}_{B(x,\epsilon)}(y) \\ &= \int_{y\in\partial F} |\dot{g}(y)|^2 \int_{x\in\partial F} \mathbf{1}_{B(x,\epsilon)}(y).\end{aligned}$$

However $\int_{x\in\partial F} \mathbf{1}_{B(x,\epsilon)}(y) \lesssim h_F$ thus

$$\int_{x\in\partial F}\int_{y\in\partial F\cap B(x,\epsilon)} |\dot{g}(y)|^2 \lesssim h_F \|\dot{g}\|^2_{L^2(\partial F)}.$$

□

Theorem 42. *If $p \in H^1_0(\Omega)$ then there is $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$ such that* $\operatorname{div} \boldsymbol{u} = p$, $\|\boldsymbol{u}\|_{\boldsymbol{H}^2} \lesssim \|p\|_{H^1}$ *and* $\|\boldsymbol{u}\|_{\boldsymbol{H}^1} \lesssim \|p\|_{L^2}$.

Proof. Consider a smooth bounded extension (at least $C^{2,1}$) B of Ω. For all function $g \in H^{-1}(B)$, following [[4] Theorem III.4.1] there is a unique solution $f \in H^1_0(B)$ to the equation $\Delta f = g$ in B. Moreover this solution satisfy $\|f\|_{H^1} \lesssim \|g\|_{H^{-1}}$ and [[4] Theorem III.4.2] shows that if B is $C^{k+1,1}$, $k \geq 0$ and $g \in H^k(B)$ then $\|f\|_{H^{k+2}} \lesssim \|g\|_{H^k}$. Since $p \in H^1_0(\Omega)$ we can

extend p by zero and define $\tilde{p} \in H^1_0(B)$ with $\|\tilde{p}\|_{H^k(B)} = \|p\|_{H^k(\Omega)}$. Hence if we take $f \in H^1_0(B)$ such that $\Delta f = \operatorname{div} \mathbf{grad}\, f = \tilde{p}$ we have $f \in H^3(B)$ since $\tilde{p} \in H^1(B)$ with $\|f\|_{H^2(B)} \lesssim \|p\|_{L^2}$ and $\|f\|_{H^3(B)} \lesssim \|p\|_{H^1}$. Let $\boldsymbol{u} = \mathbf{grad}\, f_{|\Omega}$ then we have $\operatorname{div} \boldsymbol{u} = p$ in Ω and the expected bounds. □

We can adapt the theorem to cover other boundary conditions.

Theorem 43. *If $p \in H^1_0(\Omega)$ such that $\int_\Omega p = 0$ then there is $\boldsymbol{u} \in \boldsymbol{H}^1_0(\Omega) \cap \boldsymbol{H}^2(\Omega)$ such that $\operatorname{div} \boldsymbol{u} = p$, $\|\boldsymbol{u}\|_{\boldsymbol{H}^2} \lesssim \|p\|_{H^1}$ and $\|\boldsymbol{u}\|_{\boldsymbol{H}^1} \lesssim \|p\|_{L^2}$.*

Proof. Consider B an open bounded extension $C^{2,1}$ of Ω such that $\overline{\Omega} \subset B$ and such that any point of $\partial\Omega$ can be connected by a path staying in $B \setminus \Omega$ to ∂B. This is a mild assumption that prevent B from filling holes of Ω. We construct $\boldsymbol{u}$ in the same manner as in 42, using [[4] Theorem III.4.3] which is possible since the extension by zero of p on B, $\tilde{p}$ satisfy $\int_B \tilde{p} = \int_\Omega p = 0$. This time we have $f \in H^1(B)$ such that $\Delta f = \tilde{p}$ on B and $\partial_n f = 0$ on ∂B with the increased smoothness $f \in H^3(B)$, $\|f\|_{H^{k+2}} \lesssim \|p\|_{H^k}$, $k \in \{0, 1\}$. Let $\boldsymbol{u} = \mathbf{grad}\, f_{|\Omega}$, $\boldsymbol{u} \in H^2(\Omega)$, $\operatorname{div} \boldsymbol{u} = p$ on Ω and satisfy the bounds on norms. It only remains to show that $\boldsymbol{u} \in H^1_0(\Omega)$. Since $\mathbf{grad}\, f \in H^2(B)$ the Sobolev imbedding theorems give $\mathbf{grad}\, f \in C(\overline{B})$. Moreover since $\tilde{p} = 0$ outside Ω we have $\operatorname{div} \mathbf{grad}\, f = 0$ and in general $\operatorname{rot} \mathbf{grad}\, f = 0$ thus on each connected component of $B \setminus \Omega$ there is $\boldsymbol{C} \in \mathbb{R}^2$ such that $\mathbf{grad}\, f = \boldsymbol{C}$ on this component. The boundary condition of f on ∂B requires $\mathbf{grad}\, f \cdot \boldsymbol{n} = 0$ with $\boldsymbol{n}$ the normal vector. Hence, $\boldsymbol{C} \cdot \boldsymbol{n} = 0$ and since the boundary is not reduced to a single straight line there must be $\boldsymbol{n_1}$ and $\boldsymbol{n_2}$ two normal vectors of ∂B taken at two different locations that span a basis of $\mathbb{R}^2$. Thus $\boldsymbol{C} = 0$, $\mathbf{grad}\, f = 0$ on $B \setminus \Omega$ and by continuity we must have $\boldsymbol{u} = \mathbf{grad}\, f = 0$ on $\partial\Omega$. □

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