

# TYPED ANGULARLY DECORATED PLANAR ROOTED TREES AND GENERALIZED ROTA-BAXTER ALGEBRAS

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**ABSTRACT.** We introduce a generalization of parametrized Rota-Baxter algebras, named  $\Omega$ -Rota-Baxter algebra, which includes family and matching Rota-Baxter algebras. We study the structure needed on the set  $\Omega$  of parameters in order to obtain that free  $\Omega$ -Rota-Baxter algebras are described in terms of typed and angularly decorated planar rooted trees: we obtain the notion of  $\lambda$ -extended diassociative semigroup, which includes sets (for matching Rota-Baxter algebras) and semigroups (for family Rota-Baxter algebras), and many other examples. We also describe free commutative  $\Omega$ -Rota-Baxter algebras generated by a commutative algebra  $A$  in terms of typed words.

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## 1. INTRODUCTION

A Rota-Baxter algebra is an associative algebra  $A$  with a linear endomorphism  $P : A \rightarrow A$ , such that for any  $a, b \in A$ ,

$$P(a)P(b) = P(aP(b)) + P(P(a)b) + \lambda P(ab),$$

where  $\lambda$  is a scalar called the weight of the Rota-Baxter operator  $P$ . Firstly introduced by Baxter [1] in a context of probability theory and popularized by Rota [8, 9, 10], they now appear in numerous fields of mathematics and physics, see for example [3] for examples and more details.

The first appearance of family Rota-Baxter algebras seems to be in [2], in the context of Renormalization in Quantum Field Theories. This terminology, due to Li Guo [6] refers to an associative algebra  $A$  with a family of linear endomorphisms  $P_\alpha : A \rightarrow A$  indexed by the elements of a semigroup  $(\Omega, *)$ , such that for any  $a, b \in A$ , for any  $\alpha, \beta \in \Omega$ ,

$$P_\alpha(a)P_\beta(b) = P_{\alpha*\beta}(P_\alpha(a)b + aP_\beta(b) + \lambda ab).$$

This notion of matching Rota-Baxter algebra is introduced in [11]. This time, the Rota-Baxter operators are indexed by the elements of a set  $\Omega$  with no structure, and the weights are given by a family of scalars  $(\lambda_\alpha)_{\alpha \in \Omega}$ . For any  $a, b \in A$ , for any  $\alpha, \beta \in \Omega$ ,

$$P_\alpha(a)P_\beta(b) = P_\beta(P_\alpha(a)b) + P_\alpha(aP_\beta(b)) + \lambda_\beta P_\alpha(ab).$$

These notions have been extended to other types of algebras (Lie, pre-Lie, dendriform. . .), see for example [11, 12, 13, 14].

Our aim here is a generalization of both family and matching Rota-Baxter algebras, in the spirit of what is made in [5] for dendriform algebras. We here consider that the set of parameters  $\Omega$  is given five operations  $\leftarrow, \rightarrow, \triangleleft, \triangleright$  and  $\cdot$ , and a family of scalars  $\lambda = (\lambda_{\alpha,\beta})_{\alpha,\beta \in \Omega}$ . An  $\Omega$ -Rota-Baxter algebra of weight  $\lambda$  is an associative algebra  $A$  with a family of linear endomorphisms indexed by  $\Omega$  such that for any  $a, b \in A$ , for any  $\alpha, \beta \in \Omega$ ,

$$P_\alpha(a)P_\beta(b) = P_{\alpha \rightarrow \beta}(P_{\alpha \triangleright \beta}(a)b) + P_{\alpha \leftarrow \beta}(aP_{\alpha \triangleleft \beta}(b)) + \lambda_{\alpha,\beta} P_{\alpha \cdot \beta}(ab).$$

Taking

$$\alpha \rightarrow \beta = \alpha \leftarrow \beta = \alpha \cdot \beta = \alpha * \beta, \quad \alpha \triangleright \beta = \alpha, \quad \alpha \triangleleft \beta = \beta,$$

and  $\lambda_{\alpha,\beta}$  being constant, we recover in this way family Rota-Baxter algebras. Taking

$$\alpha \rightarrow \beta = \beta, \quad \alpha \leftarrow \beta = \alpha, \quad \alpha \cdot \beta = \alpha, \quad \alpha \triangleright \beta = \alpha, \quad \alpha \triangleleft \beta = \beta,$$

and  $\lambda_{\alpha,\beta}$  depending only on  $\beta$ , we recover matching Rota-Baxter algebras.

For any set  $\Omega$  with five operations and any family of scalars  $\lambda$ , we define an operad and a category of  $\Omega$ -Rota-Baxter algebras (Definition 2.8). This is far too general, and we impose the extra constraint that the combinatorics of Rota-Baxter algebras is somehow preserved. To be more precise, as free Rota-Baxter algebras are based on planar rooted trees [14], we impose that free  $\Omega$ -Rota-Baxter algebras own a description in terms of angularly decorated (by the set of generators) and typed (by  $\Omega$ ) planar rooted trees, that is to say in terms of planar rooted trees with angles decorated by the generators and internal edges decorated by elements of  $\Omega$ , with an inductive description of the associative product and the Rota-Baxter operators being given by the grafting on a new root, the created internal edge begin of the required type. We show in Theorem 2.14 that this imposes strong constraints on  $\Omega$ : we obtain that this combinatorial description holds if, and

only if  $\Omega$  is a  $\lambda$ -ETS, as defined in Definition 2.3. In particular,  $(\Omega, \leftarrow, \rightarrow)$  has to be a diassociative semigroup: for any  $\alpha, \beta, \gamma \in \Omega$ ,

$$\begin{aligned} (\alpha \leftarrow \beta) \leftarrow \gamma &= \alpha \leftarrow (\beta \leftarrow \gamma) = \alpha \leftarrow (\beta \rightarrow \gamma), \\ (\alpha \rightarrow \beta) \leftarrow \gamma &= \alpha \rightarrow (\beta \leftarrow \gamma), \\ (\alpha \rightarrow \beta) \rightarrow \gamma &= (\alpha \leftarrow \beta) \rightarrow \gamma = \alpha \rightarrow (\beta \rightarrow \gamma). \end{aligned}$$

This notion firstly appeared in Loday's work [7] under the name of (associative) dimonoid; the free dimonoid is also constructed in Loday's article. Moreover,  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright)$  is an extended semigroup (see Definition 2.2 below), a notion used in [5] for parametrization of dendriform algebras. Particular examples of  $\lambda$ -ETS attached to a set give matching Rota-Baxter algebras (see Example 2.4-(b), with  $\psi(\alpha \otimes \beta) = \lambda\alpha$ ) and particular examples of  $\lambda$ -ETS attached to a semigroup gives family Rota-Baxter algebras (see Example 2.4-(c)). In the case of weight 0, we obtain the generalization of the result [3] establishing that any Rota-Baxter of weight 0 is a dendriform algebra, see Proposition 2.11. Moreover, generalizing the construction of free commutative Rota-Baxter algebras, we obtain that free commutative  $\Omega$ -Rota-Baxter algebras can be described in terms of  $\Omega$ -typed words (Proposition 2.18 and Theorem 2.20).

This paper is organised as follows. The first section introduces the definitions of EDS,  $\lambda$ -ETS, ETS and of  $\Omega$ -Rota-Baxter algebras. The main result on free Rota-Baxter algebras and  $\lambda$ -ETS is then proved (Theorem 2.14), with a description of free  $\Omega$ -Rota-Baxter algebras in terms of trees. The last subsection deals with commutative  $\Omega$ -Rota-Baxter algebras and their description in terms of typed words (Theorem 2.20). The second section gives more examples of  $\lambda$ -ETS and ETS, and in particular a classification of these objects of cardinality 2.

**Notation.** Throughout this paper,  $\mathbf{k}$  is a unitary commutative ring which will be the base ring of all modules, algebras, as well as linear maps.

## 2. $\Omega$ -ROTA-BAXTER ALGEBRAS

**2.1. Definitions.** We first recall the definition of diassociative semigroups and extended diassociative semigroups of [5], where these objects were used for parametrized versions of dendriform algebras.

**Definition 2.1.** [5, 7] A **diassociative semigroup** is a family  $(\Omega, \leftarrow, \rightarrow)$ , where  $\Omega$  is a set and  $\leftarrow, \rightarrow: \Omega \times \Omega \rightarrow \Omega$  are maps such that

$$\begin{aligned} (\alpha \leftarrow \beta) \leftarrow \gamma &= \alpha \leftarrow (\beta \leftarrow \gamma) = \alpha \leftarrow (\beta \rightarrow \gamma), \\ (\alpha \rightarrow \beta) \leftarrow \gamma &= \alpha \rightarrow (\beta \leftarrow \gamma), \\ (\alpha \rightarrow \beta) \rightarrow \gamma &= (\alpha \leftarrow \beta) \rightarrow \gamma = \alpha \rightarrow (\beta \rightarrow \gamma), \end{aligned}$$

for all  $\alpha, \beta, \gamma \in \Omega$ .

**Definition 2.2.** [5, Definition 2] An **extended diassociative semigroup** (abbr. EDS) is a family  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright)$ , where  $\Omega$  is a set and  $\leftarrow, \rightarrow, \triangleleft, \triangleright: \Omega \times \Omega \rightarrow \Omega$  such that  $(\Omega, \leftarrow, \rightarrow)$  is a diassociative semigroup and

$$\begin{aligned} (1) \quad & \alpha \triangleright (\beta \leftarrow \gamma) = \alpha \triangleright \beta, \\ (2) \quad & (\alpha \rightarrow \beta) \triangleleft \gamma = \beta \triangleleft \gamma, \\ (3) \quad & (\alpha \triangleleft \beta) \leftarrow ((\alpha \leftarrow \beta) \triangleleft \gamma) = \alpha \triangleleft (\beta \leftarrow \gamma), \end{aligned}$$

- (4)  $(\alpha \triangleleft \beta) \triangleleft ((\alpha \leftarrow \beta) \triangleleft \gamma) = \beta \triangleleft \gamma,$
- (5)  $(\alpha \triangleleft \beta) \rightarrow ((\alpha \leftarrow \beta) \triangleleft \gamma) = \alpha \triangleleft (\beta \rightarrow \gamma),$
- (6)  $(\alpha \triangleleft \beta) \triangleright ((\alpha \leftarrow \beta) \triangleleft \gamma) = \beta \triangleright \gamma,$
- (7)  $(\alpha \triangleright (\beta \rightarrow \gamma)) \leftarrow (\beta \triangleright \gamma) = (\alpha \leftarrow \beta) \triangleright \gamma,$
- (8)  $(\alpha \triangleright (\beta \rightarrow \gamma)) \triangleleft (\beta \triangleright \gamma) = \alpha \triangleleft \beta,$
- (9)  $(\alpha \triangleright (\beta \rightarrow \gamma)) \rightarrow (\beta \triangleright \gamma) = (\alpha \rightarrow \beta) \triangleright \gamma,$
- (10)  $(\alpha \triangleright (\beta \rightarrow \gamma)) \triangleright (\beta \triangleright \gamma) = \alpha \triangleright \beta,$

for all  $\alpha, \beta, \gamma \in \Omega$ .

We shall use here the notion of  $\lambda$ -extended triassociative semigroup, where a family of scalars plays the role of weights.

**Definition 2.3.** An  $\lambda$ -extended triassociative semigroup (abbr.  $\lambda$ -ETS) is a family  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, *, \lambda)$ , where  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright)$  is an EDS and  $\lambda = (\lambda_{\alpha\beta})_{\alpha, \beta \in \Omega}$  is a family of elements in  $\mathbf{k}$  indexed by  $\Omega^2$  such that

- (11)  $\lambda_{\alpha \rightarrow \beta, \gamma} = \lambda_{\beta, \gamma}$
- (12)  $\lambda_{\alpha \triangleleft \beta, (\alpha \leftarrow \beta) \triangleleft \gamma} = \lambda_{\beta, \gamma}$
- (13)  $\lambda_{\alpha \leftarrow \beta, \gamma} = \lambda_{\alpha, \beta \rightarrow \gamma}$
- (14)  $\lambda_{\alpha \triangleright (\beta \rightarrow \gamma), \beta \triangleright \gamma} = \lambda_{\alpha, \beta}$
- (15)  $\lambda_{\alpha, \beta} = \lambda_{\alpha, \beta \leftarrow \gamma}$
- (16)  $\lambda_{\alpha, \beta} \lambda_{\alpha, \beta, \gamma} = \lambda_{\beta, \gamma} \lambda_{\alpha, \beta, \gamma}$

and, for all  $\alpha, \beta, \gamma \in \Omega$ :

(a) If  $\lambda_{\alpha \rightarrow \beta, \gamma} = \lambda_{\beta, \gamma} \neq 0$ , then

- (17)  $\alpha \triangleright \beta = \alpha \triangleright (\beta \cdot \gamma),$
- (18)  $(\alpha \rightarrow \beta) \cdot \gamma = \alpha \rightarrow (\beta \cdot \gamma).$

(b) If  $\lambda_{\alpha \triangleleft \beta, (\alpha \leftarrow \beta) \triangleleft \gamma} = \lambda_{\beta, \gamma} \neq 0$ , then

- (19)  $(\alpha \triangleleft \beta) \cdot ((\alpha \leftarrow \beta) \triangleleft \gamma) = \alpha \triangleleft (\beta \cdot \gamma),$
- (20)  $(\alpha \leftarrow \beta) \leftarrow \gamma = \alpha \leftarrow (\beta \cdot \gamma).$

(c) If  $\lambda_{\alpha \triangleright (\beta \rightarrow \gamma), \beta \triangleright \gamma} = \lambda_{\alpha, \beta} \neq 0$ , then

- (21)  $\alpha \rightarrow (\beta \rightarrow \gamma) = (\alpha \cdot \beta) \rightarrow \gamma,$
- (22)  $(\alpha \triangleright (\beta \rightarrow \gamma)) \cdot (\beta \triangleright \gamma) = (\alpha \cdot \beta) \triangleright \gamma.$

(d) If  $\lambda_{\alpha \leftarrow \beta, \gamma} = \lambda_{\alpha, \beta \rightarrow \gamma} \neq 0$ , then

- (23)  $(\alpha \leftarrow \beta) \cdot \gamma = \alpha \cdot (\beta \rightarrow \gamma),$
- (24)  $\alpha \triangleleft \beta = \beta \triangleright \gamma.$

(e) If  $\lambda_{\alpha, \beta} = \lambda_{\alpha, \beta \leftarrow \gamma} \neq 0$ , then

- (25)  $(\alpha \cdot \beta) \triangleleft \gamma = \beta \triangleleft \gamma,$
- (26)  $(\alpha \cdot \beta) \leftarrow \gamma = \alpha \cdot (\beta \leftarrow \gamma).$

(f) If  $\lambda_{\alpha,\beta}\lambda_{\alpha,\beta,\gamma} = \lambda_{\beta,\gamma}\lambda_{\alpha,\beta,\gamma} \neq 0$ , then

$$(27) \quad (\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma).$$

**Example 2.4.** (a) Let  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright)$  be an EDS. If we put  $\lambda_{\alpha,\beta} = 0$  for any  $\alpha, \beta \in \Omega$ , then for any product  $\cdot$ ,  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, \lambda)$  is a  $\lambda$ -ETS.

(b) If for any  $\alpha, \beta \in \Omega$ ,

$$\alpha \leftarrow \beta = \beta \rightarrow \alpha = \beta \triangleleft \alpha = \alpha \triangleright \beta = \alpha,$$

then  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, \lambda)$  is a  $\lambda$ -ETS if, and only if, the following map defines an associative product:

$$\psi : \begin{cases} \mathbf{k}\Omega \otimes \mathbf{k}\Omega & \longrightarrow \mathbf{k}\Omega \\ \alpha \otimes \beta & \longrightarrow \lambda_{\alpha,\beta} \alpha \cdot \beta. \end{cases}$$

Indeed, for any  $\alpha, \beta, \gamma \in \Omega$ ,

$$\psi \circ (\psi \otimes \text{id})(\alpha \otimes \beta \otimes \gamma) = \lambda_{\alpha,\beta} \lambda_{\alpha,\beta,\gamma} (\alpha \cdot \beta) \cdot \gamma,$$

$$\psi \circ (\text{id} \otimes \psi)(\alpha \otimes \beta \otimes \gamma) = \lambda_{\beta,\gamma} \lambda_{\alpha,\beta,\gamma} \alpha \cdot (\beta \cdot \gamma),$$

which gives the missing condition (27).

(c) Let  $(\Omega, \star)$  be a semigroup and  $\lambda \in \mathbf{k}$ . We put, for any  $\alpha, \beta \in \Omega$ :

$$\begin{aligned} \alpha \leftarrow \beta &= \alpha \star \beta, & \alpha \triangleleft \beta &= \beta, \\ \alpha \rightarrow \beta &= \alpha \star \beta, & \alpha \triangleright \beta &= \alpha, \\ \lambda_{\alpha,\beta} &= \lambda, & \alpha \cdot \beta &= \alpha \star \beta. \end{aligned}$$

Then  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, \lambda)$  is a  $\lambda$ -ETS.

(d) Let  $(\Omega, \star)$  be an abelian group and let  $\lambda \in \mathbf{k}$ . For any  $\alpha, \beta \in \Omega$ , we put:

$$\begin{aligned} \alpha \leftarrow \beta &= \alpha, & \alpha \rightarrow \beta &= \beta, \\ \alpha \triangleleft \beta &= \alpha \star \beta^{\star^{-1}}, & \alpha \triangleright \beta &= \alpha^{\star^{-1}} \star \beta, \\ \lambda_{\alpha,\beta} &= \lambda, & \alpha \cdot \beta &= \alpha. \end{aligned}$$

Then  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, \lambda)$  is a  $\lambda$ -ETS.

(e) Let  $\Omega = (\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, \lambda)$  be a  $\lambda$ -ETS. For any  $\alpha, \beta \in \Omega$ , we put

$$\begin{aligned} \alpha \leftarrow^{op} \beta &= \beta \rightarrow \alpha, & \alpha \triangleleft^{op} \beta &= \beta \triangleright \alpha, \\ \alpha \rightarrow^{op} \beta &= \beta \leftarrow \alpha, & \alpha \triangleright^{op} \beta &= \beta \triangleleft \alpha, \\ \alpha \cdot^{op} \beta &= \beta \cdot \alpha, & \lambda_{\alpha,\beta}^{op} &= \lambda_{\beta,\alpha}. \end{aligned}$$

Then  $(\Omega, \leftarrow^{op}, \rightarrow^{op}, \triangleleft^{op}, \triangleright^{op}, \cdot^{op}, \lambda^{op})$  is also a  $\lambda$ -ETS, called the **opposite** of  $\Omega$  and denoted by  $\Omega^{op}$ . We shall say that  $\Omega$  is commutative if it is equal to its opposite.

**Definition 2.5.** A **extended triassociative semigroup** (abbr. ETS) is a family  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, \star)$ , where  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright)$  is an EDS and

$$(28) \quad (\alpha \rightarrow \beta) \star \gamma = \beta \star \gamma,$$

$$(17) \quad \alpha \triangleright \beta = \alpha \triangleright (\beta \cdot \gamma),$$

$$(18) \quad (\alpha \rightarrow \beta) \cdot \gamma = \alpha \rightarrow (\beta \cdot \gamma),$$

$$(29) \quad (\alpha \triangleleft \beta) \star ((\alpha \leftarrow \beta) \triangleleft \gamma) = \beta \star \gamma,$$

$$(19) \quad (\alpha \triangleleft \beta) \cdot ((\alpha \leftarrow \beta) \triangleleft \gamma) = \alpha \triangleleft (\beta \cdot \gamma),$$

- (20)  $(\alpha \leftarrow \beta) \leftarrow \gamma = \alpha \leftarrow (\beta \cdot \gamma),$   
(30)  $(\alpha \triangleright (\beta \rightarrow \gamma)) * (\beta \triangleright \gamma) = \alpha * \beta,$   
(21)  $\alpha \rightarrow (\beta \rightarrow \gamma) = (\alpha \cdot \beta) \rightarrow \gamma,$   
(22)  $(\alpha \triangleright (\beta \rightarrow \gamma)) \cdot (\beta \triangleright \gamma) = (\alpha \cdot \beta) \triangleright \gamma,$   
(31)  $(\alpha \leftarrow \beta) * \gamma = \alpha * (\beta \rightarrow \gamma),$   
(23)  $(\alpha \leftarrow \beta) \cdot \gamma = \alpha \cdot (\beta \rightarrow \gamma),$   
(24)  $\alpha \triangleleft \beta = \beta \triangleright \gamma,$   
(32)  $\alpha * \beta = \alpha * (\beta \leftarrow \gamma),$   
(25)  $(\alpha \cdot \beta) \triangleleft \gamma = \beta \triangleleft \gamma,$   
(26)  $(\alpha \cdot \beta) \leftarrow \gamma = \alpha \cdot (\beta \leftarrow \gamma),$   
(33)  $\alpha * \beta = \alpha * (\beta \cdot \gamma),$   
(34)  $(\alpha \cdot \beta) * \gamma = \beta * \gamma,$   
(27)  $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma).$

**Example 2.6.** (a) Let  $(\Omega, *, \cdot)$  be a set with two products such that for any  $\alpha, \beta, \gamma \in \Omega$ :

- (35)  $\alpha * \beta = \alpha * (\beta \cdot \gamma),$   
(36)  $(\alpha \cdot \beta) * \gamma = \beta * \gamma,$   
(37)  $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma).$

We put, for any  $\alpha, \beta \in \Omega$ :

$$\begin{aligned} \alpha \leftarrow \beta &= \alpha, & \alpha \triangleleft \beta &= \beta, \\ \alpha \rightarrow \beta &= \alpha, & \alpha \triangleright \beta &= \beta. \end{aligned}$$

Then  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, *)$  is an ETS.

(b) Let  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, *)$  be an ETS. For any  $\alpha, \beta \in \Omega$ , we put

$$\begin{aligned} \alpha \leftarrow^{op} \beta &= \beta \rightarrow \alpha, & \alpha \triangleleft^{op} \beta &= \beta \triangleright \alpha, \\ \alpha \rightarrow^{op} \beta &= \beta \leftarrow \alpha, & \alpha \triangleright^{op} \beta &= \beta \triangleleft \alpha, \\ \alpha *^{op} \beta &= \beta * \alpha, & \alpha \cdot^{op} \beta &= \beta \cdot \alpha. \end{aligned}$$

Then  $(\Omega, \leftarrow^{op}, \rightarrow^{op}, \triangleleft^{op}, \triangleright^{op}, *^{op}, \cdot^{op})$  is also an ETS, called the opposite of  $\Omega$ . We shall say that  $\Omega$  is commutative if it is equal to its opposite.

Actually, each ETS induces a  $\lambda$ -ETS, as the following result indicates:

**Proposition 2.7.** *Let  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, *)$  be an ETS and let  $(\mu_\alpha)_{\alpha \in \Omega}$  be a family of scalars. For any  $\alpha, \beta \in \Omega$ , we put:*

$$\lambda_{\alpha, \beta} = \mu_{\alpha * \beta}.$$

Then  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, \lambda)$  is a  $\lambda$ -ETS.

*Proof.* Conditions (a)-(f) of Definition 2.3 are obviously satisfied by (17)-(27). (11) is (28), (12) is (29), (13) is (31), (14) is (30), (15) is (32), and (16) comes from (33) and (34).  $\square$

We now propose the concept of  $\Omega$ -Rota-Baxter algebras as follows:

**Definition 2.8.** Let  $\Omega$  be a set with five products  $\leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot$  and  $\lambda = (\lambda_{\alpha,\beta})_{\alpha,\beta \in \Omega}$  be a family of elements in  $\mathbf{k}$  indexed by  $\Omega^2$ . An  $\Omega$ -Rota-Baxter algebra of weight  $\lambda$  is a family  $(A, (P_\omega)_{\omega \in \Omega})$  where  $A$  is an associative algebra and  $P_\omega : A \otimes A \rightarrow A$  is a linear map for each  $\omega \in \Omega$ , such that

$$P_\alpha(a)P_\beta(b) = P_{\alpha \rightarrow \beta}(P_{\alpha \triangleright \beta}(a)b) + P_{\alpha \leftarrow \beta}(aP_{\alpha \triangleleft \beta}(b)) + \lambda_{\alpha,\beta}P_{\alpha \cdot \beta}(ab),$$

for all  $a, b \in A$  and  $\alpha, \beta \in \Omega$ . If, further,  $A$  is commutative, then  $(A, (P_\omega)_{\omega \in \Omega})$  is a **commutative  $\Omega$ -Rota-Baxter algebra**.

Taking all elements of  $\lambda$  equal to 0, we get the concept of  $\Omega$ -Rota-Baxter algebras of weight 0:

**Definition 2.9.** Let  $\Omega$  be a set with four products  $\leftarrow, \rightarrow, \triangleleft, \triangleright$ . An  $\Omega$ -Rota-Baxter algebra of weight 0 is a family  $(A, (P_\omega)_{\omega \in \Omega})$  where  $A$  is an associative algebra and  $P_\omega : A \otimes A \rightarrow A$  is a linear map for each  $\omega \in \Omega$ , such that

$$P_\alpha(a)P_\beta(b) = P_{\alpha \rightarrow \beta}(P_{\alpha \triangleright \beta}(a)b) + P_{\alpha \leftarrow \beta}(aP_{\alpha \triangleleft \beta}(b)),$$

for all  $a, b \in A$  and  $\alpha, \beta \in \Omega$ .

**Example 2.10.** (a) If  $(\Omega, \star)$  is a semigroup, we recover the definition of Rota-Baxter family algebras [6, 13] by defining

$$\alpha \leftarrow \beta = \alpha \rightarrow \beta = \alpha \cdot \beta = \alpha \star \beta, \quad \alpha \triangleright \beta = \alpha, \quad \alpha \triangleleft \beta = \beta,$$

and requiring all elements of  $\lambda$  to be equal. Note that this is the  $\lambda$ -ETS of Example 2.4 (c).

(b) For a set  $\Omega$ , define

$$\alpha \rightarrow \beta = \alpha \triangleleft \beta = \beta, \quad \alpha \triangleright \beta = \alpha \leftarrow \beta = \alpha \cdot \beta = \alpha,$$

and  $\lambda_{\alpha,\beta} = \lambda_\alpha$ , for a family  $(\lambda_\alpha)_{\alpha \in \Omega}$  of elements of  $\mathbf{k}$ . Then we get the concept of matching Rota-Baxter algebra [12], up to the change of the product of  $A$  into its opposite.

As we know, Rota-Baxter algebras of weight 0 induce dendriform algebras [3]. Similarly, we can show that each  $\Omega$ -Rota-Baxter algebra of weight 0 has a structure of an  $\Omega$ -dendriform algebra [5, definition 11]:

**Proposition 2.11.** Let  $\Omega$  be a set with four products  $\leftarrow, \rightarrow, \triangleleft, \triangleright$  and  $(A, (P_\omega)_{\omega \in \Omega})$  an  $\Omega$ -Rota-Baxter algebra of weight 0. Then  $(A, (<_\omega)_{\omega \in \Omega}, (>_\omega)_{\omega \in \Omega})$  is an  $\Omega$ -dendriform algebra, where

$$a <_\omega b := aP_\omega(b), \quad a >_\omega b := P_\omega(a)b,$$

for all  $a, b \in A$  and  $\omega \in \Omega$ .

*Proof.* For  $a, b, c \in A$  and  $\alpha, \beta \in \Omega$ ,

$$\begin{aligned} (a <_\alpha b) <_\beta c &= (aP_\alpha(b))P_\beta(c) = a(P_\alpha(b)P_\beta(c)) = a(P_{\alpha \rightarrow \beta}(P_{\alpha \triangleright \beta}(b)c) + P_{\alpha \leftarrow \beta}(bP_{\alpha \triangleleft \beta}(c))) \\ &= aP_{\alpha \rightarrow \beta}(P_{\alpha \triangleright \beta}(b)c) + aP_{\alpha \leftarrow \beta}(bP_{\alpha \triangleleft \beta}(c)) = a <_{\alpha \rightarrow \beta} (b >_{\alpha \triangleright \beta} c) + a <_{\alpha \leftarrow \beta} (b <_{\alpha \triangleleft \beta} c), \\ a >_\alpha (b <_\beta c) &= P_\alpha(a)(bP_\beta(c)) = (P_\alpha(a)b)P_\beta(c) = (a >_\alpha b) <_\beta c, \\ a >_\alpha (b >_\beta c) &= P_\alpha(a)(P_\beta(b)c) = (P_\alpha(a)P_\beta(b))c = (P_{\alpha \rightarrow \beta}(P_{\alpha \triangleright \beta}(a)b) + P_{\alpha \leftarrow \beta}(aP_{\alpha \triangleleft \beta}(b)))c \\ &= P_{\alpha \rightarrow \beta}(P_{\alpha \triangleright \beta}(a)b)c + P_{\alpha \leftarrow \beta}(aP_{\alpha \triangleleft \beta}(b)c) = (a >_{\alpha \triangleright \beta} b) >_{\alpha \rightarrow \beta} c + (a <_{\alpha \triangleleft \beta} b) >_{\alpha \leftarrow \beta} c. \quad \square \end{aligned}$$

**2.2.  $\Omega$ -Rota-Baxter algebras on typed angularly decorated planar rooted trees.** First, let us recall some notations on planar rooted trees (see [14] for more details). For a planar rooted tree  $T$ , we shall consider the root and the leaves of  $T$  as edges rather than vertices. Denote by  $IE(T)$  the set of internal edges of  $T$ , i.e. edges which are neither leaves nor the root and denote by  $V(T)$  the set of vertices of  $T$ . For each vertex  $v$  yields a (possibly empty) set of angles  $A(v)$ , an angle being a pair  $(e, e')$  of adjacent incoming edges for  $v$ . Let  $A(T) = \bigsqcup_{v \in V(T)} A(v)$  be the set of angles of  $T$ . Then:

**Definition 2.12.** [14, Definition 2.2] Let  $X$  and  $\Omega$  be two sets. An  $X$ -angularly decorated  $\Omega$ -typed (abbr. **typed angularly decorated**) planar rooted tree is a triple  $T = (T, \text{dec}, \text{type})$ , where  $T$  is a planar rooted tree,  $\text{dec} : A(T) \rightarrow X$  and  $\text{type} : IE(T) \rightarrow \Omega$  are maps.

For  $n \geq 0$ , let  $\mathcal{T}_n(X, \Omega)$  denote the set of  $X$ -angularly decorated  $\Omega$ -typed planar rooted trees with  $n+1$  leaves and at least one internal vertex such that internal edges are decorated by elements of  $\Omega$ . We put

$$\mathcal{T}(X, \Omega) := \bigsqcup_{n \geq 0} \mathcal{T}_n(X, \Omega) \quad \text{and} \quad \mathbf{k}\mathcal{T}(X, \Omega) := \bigoplus_{n \geq 0} \mathbf{k}\mathcal{T}_n(X, \Omega).$$

For example,

$$\begin{aligned} \mathcal{T}_0(X, \Omega) &= \left\{ \begin{array}{c} | \\ \alpha \end{array}, \begin{array}{c} | \\ \beta \end{array}, \dots, \begin{array}{c} | \\ \alpha, \beta, \dots \in \Omega \end{array} \right\}, \\ \mathcal{T}_1(X, \Omega) &= \left\{ \begin{array}{c} x \\ \diagdown \quad \diagup \\ \alpha \end{array}, \begin{array}{c} x \\ \diagdown \quad \diagup \\ \beta \end{array}, \begin{array}{c} x \\ \diagdown \quad \diagup \\ \gamma \end{array}, \dots, \begin{array}{c} x \\ \diagdown \quad \diagup \\ \alpha \end{array} \mid x \in X, \alpha, \beta, \gamma, \dots \in \Omega \right\}, \\ \mathcal{T}_2(X, \Omega) &= \left\{ \begin{array}{c} x \quad y \\ \diagdown \quad \diagup \\ \beta \quad \alpha \end{array}, \begin{array}{c} x \quad y \\ \diagdown \quad \diagup \\ \alpha \end{array}, \begin{array}{c} x \quad y \\ \diagdown \quad \diagup \\ \beta \end{array}, \begin{array}{c} x \quad y \\ \diagdown \quad \diagup \\ \alpha \end{array}, \begin{array}{c} x \quad y \\ \diagdown \quad \diagup \\ \alpha \end{array}, \begin{array}{c} x \quad y \\ \diagdown \quad \diagup \\ \beta \end{array}, \dots \mid x, y \in X, \alpha, \beta, \gamma, \dots \in \Omega \right\}, \end{aligned}$$

Graphically, an element  $T \in \mathcal{T}(X, \Omega)$  is of the form:

$$T = T_1 \circ_{\alpha_1} \begin{array}{c} T_2 \quad T_n \\ \alpha_2 \quad \alpha_n \\ \diagdown \quad \diagup \\ \alpha_1 \quad \alpha_{n+1} \end{array} \circ_{\alpha_{n+1}} T_{n+1}, \quad \text{with } n \geq 0, \quad \text{where } x_1, \dots, x_n \in X, \alpha_i \in \Omega \text{ if } T_i \neq | \text{ and otherwise}$$

$\alpha_i$  does not exist for  $1 \leq i \leq n+1$ .

For each  $\omega \in \Omega$ , there is a grafting operator  $B_\omega^+ : \mathbf{k}\mathcal{T}(X, \Omega) \rightarrow \mathbf{k}\mathcal{T}(X, \Omega)$  which add a new root to a tree and an new internal edge typed by  $\omega$  between the new root and the root of the tree.

For example,

$$B_\omega^+ \left( \begin{array}{c} | \\ \alpha \end{array} \right) = \begin{array}{c} | \\ \omega \\ | \\ \alpha \end{array}, \quad B_\omega^+ \left( \begin{array}{c} x \\ \diagdown \quad \diagup \\ \alpha \end{array} \right) = \begin{array}{c} x \\ \diagdown \quad \diagup \\ \omega \\ | \\ \alpha \end{array}.$$

The **depth**  $\text{dep}(T)$  of a rooted tree  $T$  is the maximal length of linear chains from the root to the leaves of the tree. For example,

$$\text{dep}\left(\begin{array}{c} | \\ \vdots \\ | \end{array}\right) = \text{dep}\left(\begin{array}{c} \diagup \quad \diagdown \\ x \quad y \\ \diagdown \quad \diagup \\ \alpha \end{array}\right) = 1 \quad \text{and} \quad \text{dep}\left(\begin{array}{c} | \\ \vdots \\ | \end{array}\right) \omega = \text{dep}\left(\begin{array}{c} \diagup \quad \diagdown \\ x \quad y \\ \diagdown \quad \diagup \\ \alpha \end{array}\right) = 2.$$

We also consider the trivial tree  $|$  and put by convention  $\text{dep}(|) := 0$ . For each typed angularly decorated planar rooted tree  $T$ , define the number of branches of  $T$  to be  $\text{bra}(T) = 0$  if  $T = |$ . Otherwise,  $\text{dep}(T) \geq 1$  and  $T$  is of the form

$$T = T_1 \circlearrowleft \begin{array}{c} \alpha_1 \\ \vdots \\ \alpha_n \\ \alpha_{n+1} \end{array} \begin{array}{c} \alpha_2 \quad \alpha_n \\ \vdots \\ \alpha_n \end{array} \begin{array}{c} T_2 \quad T_n \\ \vdots \\ T_{n+1} \end{array} \quad \text{with } n \geq 0,$$

where  $T_j \in \mathcal{T}(X, \Omega) \sqcup \{| \}$ ,  $j = 1, \dots, n+1$ . We define  $\text{bra}(T) := n+1$ . For example,

$$\text{bra}\left(\begin{array}{c} | \\ \vdots \\ | \end{array}\right) = 1, \quad \text{bra}\left(\begin{array}{c} \diagup \quad \diagdown \\ x \quad y \\ \diagdown \quad \diagup \\ \alpha \end{array}\right) = 2 \quad \text{and} \quad \text{bra}\left(\begin{array}{c} \diagup \quad \diagdown \\ x \quad y \\ \diagdown \quad \diagup \\ \alpha \end{array}\right) = 3.$$

Let  $X$  be a set,  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot)$  be a set with five products, and  $\lambda = (\lambda_{\alpha, \beta})_{(\alpha, \beta) \in \Omega^2}$  be a family of elements in  $\mathbf{k}$  indexed by  $\Omega^2$ . By analogy with the construction of free Rota-Baxter algebras, we define a product  $\diamond$  on  $\mathbf{k}\mathcal{T}(X, \Omega)$  as follows. For  $T, T' \in \mathcal{T}(X, \Omega)$ , we define  $T \diamond T'$  by induction on  $\text{dep}(T) + \text{dep}(T') \geq 2$ . For the initial step  $\text{dep}(T) + \text{dep}(T') = 2$ , we have  $\text{dep}(T) = \text{dep}(T') = 1$  and  $T, T'$  are of the form

$$T = \begin{array}{c} \diagup \quad \vdots \quad \diagdown \\ x_1 \quad \quad \quad x_m \\ \diagdown \quad \quad \quad \diagup \\ \alpha \end{array} \quad \text{and} \quad T' = \begin{array}{c} \diagup \quad \vdots \quad \diagdown \\ y_1 \quad \quad \quad y_n \\ \diagdown \quad \quad \quad \diagup \\ \alpha \end{array}, \quad \text{with } m, n \geq 0.$$

Define

$$(38) \quad T \diamond T' := \begin{array}{c} \diagup \quad \vdots \quad \diagdown \\ x_1 \quad \quad \quad x_m \\ \diagdown \quad \quad \quad \diagup \\ \alpha \end{array} \diamond \begin{array}{c} \diagup \quad \vdots \quad \diagdown \\ y_1 \quad \quad \quad y_n \\ \diagdown \quad \quad \quad \diagup \\ \alpha \end{array} := \begin{array}{c} \diagup \quad \vdots \quad \diagdown \\ x_1 \quad \quad \quad x_m \quad | \quad y_1 \quad \quad \quad y_n \\ \diagdown \quad \quad \quad \diagup \\ \alpha \end{array}.$$

For the induction step  $\text{dep}(T) + \text{dep}(T') \geq 3$ , the trees  $T$  and  $T'$  are of the form

$$T = T_1 \circlearrowleft \begin{array}{c} \alpha_1 \\ \vdots \\ \alpha_m \\ \alpha_{m+1} \end{array} \begin{array}{c} \alpha_2 \quad \alpha_m \\ \vdots \\ \alpha_m \end{array} \begin{array}{c} T_2 \quad T_m \\ \vdots \\ T_{m+1} \end{array} \quad \text{and} \quad T' = T'_1 \circlearrowleft \begin{array}{c} \beta_1 \\ \vdots \\ \beta_n \\ \beta_{n+1} \end{array} \begin{array}{c} \beta_2 \quad \beta_n \\ \vdots \\ \beta_n \end{array} \begin{array}{c} T'_2 \quad T'_n \\ \vdots \\ T'_{n+1} \end{array} \quad \text{with some } T_i \neq | \text{ or some } T'_j \neq |.$$

There are four cases to consider.

**Case 1:**  $T_{m+1} = | = T'_1$ . Define

$$(39) \quad T \diamond T' := T_1 \circlearrowleft \begin{array}{c} \alpha_1 \\ \vdots \\ \alpha_m \\ \alpha_{m+1} \end{array} \begin{array}{c} \alpha_2 \quad \alpha_m \\ \vdots \\ \alpha_m \end{array} \begin{array}{c} T_2 \quad T_m \\ \vdots \\ T_{m+1} \end{array} \diamond \begin{array}{c} \beta_1 \\ \vdots \\ \beta_n \\ \beta_{n+1} \end{array} \begin{array}{c} \beta_2 \quad \beta_n \\ \vdots \\ \beta_n \end{array} \begin{array}{c} T'_2 \quad T'_n \\ \vdots \\ T'_{n+1} \end{array} := T_1 \circlearrowleft \begin{array}{c} \alpha_1 \\ \vdots \\ \alpha_m \quad x_m \quad y_1 \quad \beta_2 \quad \beta_n \\ \vdots \\ \alpha_2 \quad x_1 \quad y_n \quad \beta_{n+1} \end{array} \begin{array}{c} \alpha_2 \quad \alpha_m \\ \vdots \\ \alpha_m \end{array} \begin{array}{c} T_2 \quad T_m \\ \vdots \\ T_{m+1} \end{array}.$$

**Case 2:**  $T_{m+1} \neq | = T'_1$ . Define

$$(40) \quad T \diamond T' := T_1 \diamond \left( T_2 \begin{array}{c} \alpha_2 \dots \alpha_m \\ \circ \quad \circ \quad \circ \\ \vdots \\ \circ \quad \circ \quad \circ \\ \alpha_1 \quad x_1 \quad \dots \quad x_m \quad \alpha_{m+1} \end{array} T_{m+1} \right) \diamond \left( T'_2 \begin{array}{c} \beta_2 \dots \beta_n \\ \circ \quad \circ \quad \circ \\ \vdots \\ \circ \quad \circ \quad \circ \\ \beta_1 \quad y_1 \quad \dots \quad y_n \quad \beta_{n+1} \end{array} T'_{n+1} \right) := T_1 \diamond \left( T_2 \begin{array}{c} T_m \quad T_{m+1} \quad T'_2 \\ \alpha_m \quad x_m \quad y_1 \quad \beta_2 \\ \circ \quad \circ \quad \circ \quad \circ \\ \vdots \\ \circ \quad \circ \quad \circ \quad \circ \\ \alpha_1 \quad x_1 \quad \dots \quad y_n \quad \beta_{n+1} \end{array} T'_{n+1} \right).$$

**Case 3:**  $T_{m+1} = | \neq T'_1$ . Define

$$(41) \quad T \diamond T' := T_1 \diamond \left( T_2 \begin{array}{c} \alpha_2 \dots \alpha_m \\ \circ \quad \circ \quad \circ \\ \vdots \\ \circ \quad \circ \quad \circ \\ \alpha_1 \quad x_1 \quad \dots \quad x_m \end{array} T_m \right) \diamond T'_1 \diamond \left( T'_2 \begin{array}{c} \beta_2 \dots \beta_n \\ \circ \quad \circ \quad \circ \\ \vdots \\ \circ \quad \circ \quad \circ \\ \beta_1 \quad y_1 \quad \dots \quad y_n \quad \beta_{n+1} \end{array} T'_{n+1} \right) := T_1 \diamond \left( T_2 \begin{array}{c} T_m \quad T'_1 \quad T'_2 \\ \alpha_m \quad x_m \quad y_1 \quad \beta_2 \\ \circ \quad \circ \quad \circ \quad \circ \\ \vdots \\ \circ \quad \circ \quad \circ \quad \circ \\ \alpha_1 \quad x_1 \quad \dots \quad y_n \quad \beta_{n+1} \end{array} T'_{n+1} \right).$$

**Case 4:**  $T_{m+1} \neq | \neq T'_1$ . Define

$$(42) \quad \begin{aligned} T \diamond T' &:= T_1 \diamond \left( T_2 \begin{array}{c} \alpha_2 \dots \alpha_m \\ \circ \quad \circ \quad \circ \\ \vdots \\ \circ \quad \circ \quad \circ \\ \alpha_1 \quad x_1 \quad \dots \quad x_m \quad \alpha_{m+1} \end{array} T_{m+1} \right) \diamond T'_1 \diamond \left( T'_2 \begin{array}{c} \beta_2 \dots \beta_n \\ \circ \quad \circ \quad \circ \\ \vdots \\ \circ \quad \circ \quad \circ \\ \beta_1 \quad y_1 \quad \dots \quad y_n \quad \beta_{n+1} \end{array} T'_{n+1} \right) \\ &:= \left( T_1 \diamond \left( T_2 \begin{array}{c} \alpha_2 \dots \alpha_m \\ \circ \quad \circ \quad \circ \\ \vdots \\ \circ \quad \circ \quad \circ \\ \alpha_1 \quad x_1 \quad \dots \quad x_m \end{array} T_m \right) \diamond \left( B_{\alpha_{m+1}}^+(T_{m+1}) \diamond B_{\beta_1}^+(T'_1) \right) \right) \diamond \left( T'_2 \begin{array}{c} \beta_2 \dots \beta_n \\ \circ \quad \circ \quad \circ \\ \vdots \\ \circ \quad \circ \quad \circ \\ \beta_1 \quad y_1 \quad \dots \quad y_n \quad \beta_{n+1} \end{array} T'_{n+1} \right) \\ &:= \left( T_1 \diamond \left( T_2 \begin{array}{c} \alpha_2 \dots \alpha_m \\ \circ \quad \circ \quad \circ \\ \vdots \\ \circ \quad \circ \quad \circ \\ \alpha_1 \quad x_1 \quad \dots \quad x_m \end{array} T_m \right) \diamond \left( B_{\alpha_{m+1} \rightarrow \beta_1}^+(B_{\alpha_{m+1} > \beta_1}^+(T_{m+1}) \diamond T'_1) + B_{\alpha_{m+1} \leftarrow \beta_1}^+(T_{m+1} \diamond B_{\alpha_{m+1} < \beta_1}^+(T'_1)) \right. \right. \\ &\quad \left. \left. + \lambda_{\alpha_{m+1} \beta_1} B_{\alpha_{m+1} \beta_1}^+(T_{m+1} \diamond T'_1) \right) \right) \diamond \left( T'_2 \begin{array}{c} \beta_2 \dots \beta_n \\ \circ \quad \circ \quad \circ \\ \vdots \\ \circ \quad \circ \quad \circ \\ \beta_1 \quad y_1 \quad \dots \quad y_n \quad \beta_{n+1} \end{array} T'_{n+1} \right). \end{aligned}$$

Here the first  $\diamond$  is defined by Case 3, the second, third and fourth  $\diamond$  are defined by induction and the last  $\diamond$  is defined by Case 2. This inductively define the multiplication  $\diamond$  on  $\mathcal{T}(X, \Omega)$ . We then extend  $\diamond$  by linearity to  $\mathbf{k}\mathcal{T}(X, \Omega)$ . We then have the following result:

**Lemma 2.13.** *Let  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, \lambda)$  be a  $\lambda$ -ETS. Then  $(\mathbf{k}\mathcal{T}(X, \Omega), \diamond)$  is an associative algebra with identity  $\downarrow$ .*

*Proof.* By the construction of  $\diamond$ ,  $\mathbf{k}\mathcal{T}(X, \Omega)$  is closed under  $\diamond$  and  $\downarrow$  is the identity of  $\diamond$ .

Now we show the associativity of  $\diamond$ , i.e.

$$(43) \quad (T_1 \diamond T_2) \diamond T_3 = T_1 \diamond (T_2 \diamond T_3),$$

for all  $T_1, T_2, T_3 \in \mathcal{T}(X, \Omega)$ . We prove Eq. (43) by induction on the sum of depths  $p := \text{dep}(T_1) + \text{dep}(T_2) + \text{dep}(T_3)$ . If  $p = 3$ , then  $\text{dep}(T_1) = \text{dep}(T_2) = \text{dep}(T_3) = 1$  and  $T_1, T_2, T_3$  are of the form

$$T_1 = \begin{array}{c} x_1 \quad \dots \quad x_l \\ \circ \quad \circ \quad \circ \\ \vdots \\ \circ \quad \circ \quad \circ \\ \alpha_1 \end{array}, \quad T_2 = \begin{array}{c} y_1 \quad \dots \quad y_m \\ \circ \quad \circ \quad \circ \\ \vdots \\ \circ \quad \circ \quad \circ \\ \beta_1 \end{array}, \quad \text{and} \quad T_3 = \begin{array}{c} z_1 \quad \dots \quad z_n \\ \circ \quad \circ \quad \circ \\ \vdots \\ \circ \quad \circ \quad \circ \\ \gamma_1 \end{array} \quad \text{with } l, m, n \geq 0.$$

Then  $(T_1 \diamond T_2) \diamond T_3 = T_1 \diamond (T_2 \diamond T_3)$  by a direct calculation.

For the induction step  $p \geq 4$ , we use induction on the sum of branches  $q := \text{bra}(T_1) + \text{bra}(T_2) + \text{bra}(T_3)$ . If  $q = 3$  and one of  $T_1, T_2, T_3$  has depth 1, then this tree must be of the form  $\left\{ \begin{array}{l} \\ \\ \end{array} \right.$  and the associativity of  $\diamond$  follows directly. Assume

$$T_1 = B_\alpha^+(T'_1), T_2 = B_\beta^+(T'_2), T_3 = B_\gamma^+(T'_3) \text{ for some } \alpha, \beta, \gamma \in \Omega \text{ and } T'_1, T'_2, T'_3 \in \mathcal{T}(X, \Omega),$$

then

$$\begin{aligned} (T_1 \diamond T_2) \diamond T_3 &= (B_\alpha^+(T'_1) \diamond B_\beta^+(T'_2)) \diamond B_\gamma^+(T'_3) \\ &= B_{\alpha \rightarrow \beta}^+(B_{\alpha \triangleright \beta}^+(T'_1) \diamond T'_2) \diamond B_\gamma^+(T'_3) + B_{\alpha \leftarrow \beta}^+(T'_1 \diamond B_{\alpha \triangleleft \beta}^+(T'_2)) \diamond B_\gamma^+(T'_3) + \lambda_{\alpha, \beta} B_{\alpha, \beta}^+(T'_1 \diamond T'_2) \diamond B_\gamma^+(T'_3) \\ &= B_{(\alpha \rightarrow \beta) \rightarrow \gamma}^+(B_{(\alpha \rightarrow \beta) \triangleright \gamma}^+(B_{\alpha \triangleright \beta}^+(T'_1) \diamond T'_2) \diamond T'_3) + B_{(\alpha \rightarrow \beta) \leftarrow \gamma}^+((B_{\alpha \triangleright \beta}^+(T'_1) \diamond T'_2) \diamond B_{(\alpha \rightarrow \beta) \triangleleft \gamma}^+(T'_3)) \\ &\quad + \lambda_{\alpha \rightarrow \beta, \gamma} B_{(\alpha \rightarrow \beta) \rightarrow \gamma}^+((B_{\alpha \triangleright \beta}^+(T'_1) \diamond T'_2) \diamond T'_3) + B_{(\alpha \leftarrow \beta) \rightarrow \gamma}^+(B_{(\alpha \leftarrow \beta) \triangleright \gamma}^+(T'_1 \diamond B_{\alpha \triangleleft \beta}^+(T'_2)) \diamond T'_3) \\ &\quad + B_{(\alpha \leftarrow \beta) \leftarrow \gamma}^+((T'_1 \diamond B_{\alpha \triangleleft \beta}^+(T'_2)) \diamond B_{(\alpha \leftarrow \beta) \triangleleft \gamma}^+(T'_3)) + \lambda_{\alpha \leftarrow \beta, \gamma} B_{(\alpha \leftarrow \beta) \leftarrow \gamma}^+((T'_1 \diamond B_{\alpha \triangleleft \beta}^+(T'_2)) \diamond T'_3) \\ &\quad + \lambda_{\alpha, \beta} B_{(\alpha, \beta) \rightarrow \gamma}^+(B_{(\alpha, \beta) \triangleright \gamma}^+(T'_1 \diamond T'_2) \diamond T'_3) + \lambda_{\alpha, \beta} B_{(\alpha, \beta) \leftarrow \gamma}^+((T'_1 \diamond T'_2) \diamond B_{(\alpha, \beta) \triangleleft \gamma}^+(T'_3)) \\ &\quad + \lambda_{\alpha, \beta} \lambda_{\alpha, \beta, \gamma} B_{(\alpha, \beta) \rightarrow \gamma}^+((T'_1 \diamond T'_2) \diamond T'_3) \\ &= B_{(\alpha \rightarrow \beta) \rightarrow \gamma}^+(B_{(\alpha \rightarrow \beta) \triangleright \gamma}^+(B_{\alpha \triangleright \beta}^+(T'_1) \diamond T'_2) \diamond T'_3) + B_{(\alpha \rightarrow \beta) \leftarrow \gamma}^+((B_{\alpha \triangleright \beta}^+(T'_1) \diamond T'_2) \diamond B_{(\alpha \rightarrow \beta) \triangleleft \gamma}^+(T'_3)) \\ &\quad + \lambda_{\alpha \rightarrow \beta, \gamma} B_{(\alpha \rightarrow \beta) \rightarrow \gamma}^+((B_{\alpha \triangleright \beta}^+(T'_1) \diamond T'_2) \diamond T'_3) + B_{(\alpha \leftarrow \beta) \rightarrow \gamma}^+(B_{(\alpha \leftarrow \beta) \triangleright \gamma}^+(T'_1 \diamond B_{\alpha \triangleleft \beta}^+(T'_2)) \diamond T'_3) \\ &\quad + B_{(\alpha \leftarrow \beta) \leftarrow \gamma}^+(T'_1 \diamond (B_{\alpha \triangleleft \beta}^+(T'_2) \diamond B_{(\alpha \leftarrow \beta) \triangleleft \gamma}^+(T'_3))) + \lambda_{\alpha \leftarrow \beta, \gamma} B_{(\alpha \leftarrow \beta) \leftarrow \gamma}^+((T'_1 \diamond B_{\alpha \triangleleft \beta}^+(T'_2)) \diamond T'_3) \\ &\quad + \lambda_{\alpha, \beta} B_{(\alpha, \beta) \rightarrow \gamma}^+(B_{(\alpha, \beta) \triangleright \gamma}^+(T'_1 \diamond T'_2) \diamond T'_3) + \lambda_{\alpha, \beta} B_{(\alpha, \beta) \leftarrow \gamma}^+((T'_1 \diamond T'_2) \diamond B_{(\alpha, \beta) \triangleleft \gamma}^+(T'_3)) \\ &\quad + \lambda_{\alpha, \beta} \lambda_{\alpha, \beta, \gamma} B_{(\alpha, \beta) \rightarrow \gamma}^+((T'_1 \diamond T'_2) \diamond T'_3) \quad (\text{by the induction hypothesis}) \\ &= B_{(\alpha \rightarrow \beta) \rightarrow \gamma}^+(B_{(\alpha \rightarrow \beta) \triangleright \gamma}^+(B_{\alpha \triangleright \beta}^+(T'_1) \diamond T'_2) \diamond T'_3) + B_{(\alpha \rightarrow \beta) \leftarrow \gamma}^+((B_{\alpha \triangleright \beta}^+(T'_1) \diamond T'_2) \diamond B_{(\alpha \rightarrow \beta) \triangleleft \gamma}^+(T'_3)) \\ &\quad + \lambda_{\alpha \rightarrow \beta, \gamma} B_{(\alpha \rightarrow \beta) \rightarrow \gamma}^+((B_{\alpha \triangleright \beta}^+(T'_1) \diamond T'_2) \diamond T'_3) + B_{(\alpha \leftarrow \beta) \rightarrow \gamma}^+(B_{(\alpha \leftarrow \beta) \triangleright \gamma}^+(T'_1 \diamond B_{\alpha \triangleleft \beta}^+(T'_2)) \diamond T'_3) \\ &\quad + B_{(\alpha \leftarrow \beta) \leftarrow \gamma}^+(T'_1 \diamond B_{(\alpha \triangleleft \beta) \rightarrow ((\alpha \leftarrow \beta) \triangleleft \gamma)}^+(B_{(\alpha \triangleleft \beta) \triangleright ((\alpha \leftarrow \beta) \triangleleft \gamma)}^+(T'_2) \diamond T'_3)) \\ &\quad + B_{(\alpha \leftarrow \beta) \leftarrow \gamma}^+(T'_1 \diamond B_{(\alpha \triangleleft \beta) \leftarrow ((\alpha \leftarrow \beta) \triangleleft \gamma)}^+(T'_2 \diamond B_{(\alpha \triangleleft \beta) \triangleleft ((\alpha \leftarrow \beta) \triangleleft \gamma)}^+(T'_3))) \\ &\quad + \lambda_{\alpha \triangleleft \beta, (\alpha \leftarrow \beta) \triangleleft \gamma} B_{(\alpha \leftarrow \beta) \leftarrow \gamma}^+(T'_1 \diamond B_{(\alpha \triangleleft \beta) \leftarrow ((\alpha \leftarrow \beta) \triangleleft \gamma)}^+(T'_2 \diamond T'_3)) + \lambda_{\alpha \leftarrow \beta, \gamma} B_{(\alpha \leftarrow \beta) \leftarrow \gamma}^+((T'_1 \diamond B_{\alpha \triangleleft \beta}^+(T'_2)) \diamond T'_3) \\ &\quad + \lambda_{\alpha, \beta} B_{(\alpha, \beta) \rightarrow \gamma}^+(B_{(\alpha, \beta) \triangleright \gamma}^+(T'_1 \diamond T'_2) \diamond T'_3) + \lambda_{\alpha, \beta} B_{(\alpha, \beta) \leftarrow \gamma}^+((T'_1 \diamond T'_2) \diamond B_{(\alpha, \beta) \triangleleft \gamma}^+(T'_3)) \\ &\quad + \lambda_{\alpha, \beta} \lambda_{\alpha, \beta, \gamma} B_{(\alpha, \beta) \rightarrow \gamma}^+((T'_1 \diamond T'_2) \diamond T'_3), \end{aligned}$$

and

$$\begin{aligned} T_1 \diamond (T_2 \diamond T_3) &= B_\alpha^+(T'_1) \diamond (B_\beta^+(T'_2) \diamond B_\gamma^+(T'_3)) \\ &= B_\alpha^+(T'_1) \diamond B_{\beta \rightarrow \gamma}^+(B_{\beta \triangleright \gamma}^+(T'_2) \diamond T'_3) + B_\alpha^+(T'_1) \diamond B_{\beta \leftarrow \gamma}^+(T'_2 \diamond B_{\beta \triangleleft \gamma}^+(T'_3)) + \lambda_{\beta, \gamma} B_\alpha^+(T'_1) \diamond B_{\beta, \gamma}^+(T'_2 \diamond T'_3) \\ &= B_{\alpha \rightarrow (\beta \rightarrow \gamma)}^+(B_{\alpha \triangleright (\beta \rightarrow \gamma)}^+(T'_1) \diamond (B_{\beta \triangleright \gamma}^+(T'_2) \diamond T'_3)) + B_{\alpha \leftarrow (\beta \rightarrow \gamma)}^+(T'_1 \diamond B_{\alpha \triangleleft (\beta \rightarrow \gamma)}^+(B_{\beta \triangleright \gamma}^+(T'_2) \diamond T'_3)) \\ &\quad + \lambda_{\alpha \rightarrow \beta, \gamma} B_{\alpha \rightarrow (\beta \rightarrow \gamma)}^+(T'_1 \diamond (B_{\beta \triangleright \gamma}^+(T'_2) \diamond T'_3)) + B_{\alpha \rightarrow (\beta \leftarrow \gamma)}^+(B_{\alpha \triangleright (\beta \leftarrow \gamma)}^+(T'_1) \diamond (T'_2 \diamond B_{\beta \triangleleft \gamma}^+(T'_3))) \\ &\quad + B_{\alpha \leftarrow (\beta \leftarrow \gamma)}^+(T'_1 \diamond B_{\alpha \triangleleft (\beta \leftarrow \gamma)}^+(T'_2 \diamond B_{\beta \triangleleft \gamma}^+(T'_3))) + \lambda_{\alpha \leftarrow \beta, \gamma} B_{\alpha \leftarrow (\beta \leftarrow \gamma)}^+(T'_1 \diamond (T'_2 \diamond B_{\beta \triangleleft \gamma}^+(T'_3))) \\ &\quad + \lambda_{\beta, \gamma} B_{\alpha \rightarrow (\beta, \gamma)}^+(B_{\alpha \triangleright (\beta, \gamma)}^+(T'_1) \diamond (T'_2 \diamond T'_3)) + \lambda_{\beta, \gamma} B_{\alpha \leftarrow (\beta, \gamma)}^+(T'_1 \diamond B_{\alpha \triangleleft (\beta, \gamma)}^+(T'_2 \diamond T'_3)) \\ &\quad + \lambda_{\beta, \gamma} \lambda_{\alpha, \beta, \gamma} B_{\alpha \rightarrow (\beta, \gamma)}^+(T'_1 \diamond (T'_2 \diamond T'_3)) \\ &= B_{\alpha \rightarrow (\beta \rightarrow \gamma)}^+((B_{\alpha \triangleright (\beta \rightarrow \gamma)}^+(T'_1) \diamond B_{\beta \triangleright \gamma}^+(T'_2)) \diamond T'_3) + B_{\alpha \leftarrow (\beta \rightarrow \gamma)}^+(T'_1 \diamond B_{\alpha \triangleleft (\beta \rightarrow \gamma)}^+(B_{\beta \triangleright \gamma}^+(T'_2) \diamond T'_3)) \\ &\quad + \lambda_{\alpha \rightarrow \beta, \gamma} B_{\alpha \rightarrow (\beta \rightarrow \gamma)}^+(T'_1 \diamond (B_{\beta \triangleright \gamma}^+(T'_2) \diamond T'_3)) + B_{\alpha \rightarrow (\beta \leftarrow \gamma)}^+(B_{\alpha \triangleright (\beta \leftarrow \gamma)}^+(T'_1) \diamond (T'_2 \diamond B_{\beta \triangleleft \gamma}^+(T'_3))) \end{aligned}$$

$$\begin{aligned}
& + B_{\alpha \leftarrow (\beta \leftarrow \gamma)}^+(T'_1 \diamond B_{\alpha \triangleleft (\beta \leftarrow \gamma)}^+(T'_2 \diamond B_{\beta \triangleleft \gamma}^+(T'_3))) + \lambda_{\alpha, \beta \leftarrow \gamma} B_{\alpha \cdot (\beta \leftarrow \gamma)}^+(T'_1 \diamond (T'_2 \diamond B_{\beta \triangleleft \gamma}^+(T'_3))) \\
& + \lambda_{\beta, \gamma} B_{\alpha \rightarrow (\beta \cdot \gamma)}^+(B_{\alpha \triangleright (\beta \cdot \gamma)}^+(T'_1) \diamond (T'_2 \diamond T'_3)) + \lambda_{\beta, \gamma} B_{\alpha \leftarrow (\beta \cdot \gamma)}^+(T'_1 \diamond B_{\alpha \leftarrow (\beta \cdot \gamma)}^+(T'_2 \diamond T'_3)) \\
& + \lambda_{\beta, \gamma} \lambda_{\alpha, \beta \cdot \gamma} B_{\alpha \cdot (\beta \cdot \gamma)}^+(T'_1 \diamond (T'_2 \diamond T'_3)) \quad (\text{by the induction hypothesis}) \\
= & B_{\alpha \rightarrow (\beta \rightarrow \gamma)}^+(B_{(\alpha \triangleright (\beta \rightarrow \gamma)) \rightarrow (\beta \triangleright \gamma)}^+(B_{(\alpha \triangleright (\beta \rightarrow \gamma)) \triangleright (\beta \triangleright \gamma)}^+(T'_1) \diamond T'_2) \diamond T'_3) \\
& + B_{\alpha \rightarrow (\beta \rightarrow \gamma)}^+(B_{\alpha \triangleright (\beta \rightarrow \gamma) \leftarrow (\beta \triangleright \gamma)}^+(T'_1 \diamond B_{(\alpha \triangleright (\beta \rightarrow \gamma)) \triangleleft (\beta \triangleright \gamma)}^+(T'_2)) \diamond T'_3) \\
& + \lambda_{\alpha \triangleright (\beta \rightarrow \gamma), \beta \triangleright \gamma} B_{\alpha \rightarrow (\beta \rightarrow \gamma)}^+(B_{(\alpha \triangleright (\beta \rightarrow \gamma)) \cdot (\beta \triangleright \gamma)}^+(T'_1 \diamond T'_2) \diamond T'_3) + B_{\alpha \leftarrow (\beta \rightarrow \gamma)}^+(T'_1 \diamond B_{\alpha \triangleleft (\beta \rightarrow \gamma)}^+(B_{\beta \triangleright \gamma}^+(T'_2) \diamond T'_3)) \\
& + \lambda_{\alpha, \beta \rightarrow \gamma} B_{\alpha \cdot (\beta \rightarrow \gamma)}^+(T'_1 \diamond (B_{\beta \triangleright \gamma}^+(T'_2) \diamond T'_3)) + B_{\alpha \rightarrow (\beta \leftarrow \gamma)}^+(B_{\alpha \triangleright (\beta \leftarrow \gamma)}^+(T'_1) \diamond (T'_2 \diamond B_{\beta \triangleleft \gamma}^+(T'_3))) \\
& + B_{\alpha \leftarrow (\beta \leftarrow \gamma)}^+(T'_1 \diamond B_{\alpha \triangleleft (\beta \leftarrow \gamma)}^+(T'_2 \diamond B_{\beta \triangleleft \gamma}^+(T'_3))) + \lambda_{\alpha, \beta \leftarrow \gamma} B_{\alpha \cdot (\beta \leftarrow \gamma)}^+(T'_1 \diamond (T'_2 \diamond B_{\beta \triangleleft \gamma}^+(T'_3))) \\
& + \lambda_{\beta, \gamma} B_{\alpha \rightarrow (\beta \cdot \gamma)}^+(B_{\alpha \triangleright (\beta \cdot \gamma)}^+(T'_1) \diamond (T'_2 \diamond T'_3)) + \lambda_{\beta, \gamma} B_{\alpha \leftarrow (\beta \cdot \gamma)}^+(T'_1 \diamond B_{\alpha \leftarrow (\beta \cdot \gamma)}^+(T'_2 \diamond T'_3)) \\
& + \lambda_{\beta, \gamma} \lambda_{\alpha, \beta \cdot \gamma} B_{\alpha \cdot (\beta \cdot \gamma)}^+(T'_1 \diamond (T'_2 \diamond T'_3)).
\end{aligned}$$

By the induction hypothesis and  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, \lambda)$  being a  $\lambda$ -ETS, we get

$$(T_1 \diamond T_2) \diamond T_3 = T_1 \diamond (T_2 \diamond T_3).$$

If  $q > 3$ , then at least one of  $T_1, T_2, T_3$  have branches greater than or equal to 2. If  $\text{bra}(T_1) \geq 2$ , then there exist  $T'_1, T''_1$  of the form

$$T'_1 = \begin{array}{c} \bar{T}'_2 \quad \bar{T}'_m \\ \circ \quad \circ \\ \alpha_2 \quad \alpha_m \\ \circ \quad \circ \\ x_1 \quad x_m \\ \circ \\ \alpha_1 \end{array} \quad \text{and} \quad T''_1 = \begin{array}{c} \bar{T}''_2 \quad \bar{T}''_n \\ \circ \quad \circ \\ \beta_2 \quad \beta_n \\ \circ \quad \circ \\ y_1 \quad y_n \\ \circ \\ \beta_{n+1} \end{array} \bar{T}''_{n+1}$$

such that  $T_1 = T'_1 \diamond T''_1$ . Hence

$$\begin{aligned}
(T_1 \diamond T_2) \diamond T_3 & = ((T'_1 \diamond T''_1) \diamond T_2) \diamond T_3 \\
& = (T'_1 \diamond (T''_1 \diamond T_2)) \diamond T_3 && (\text{by the induction hypothesis}) \\
& = T'_1 \diamond ((T''_1 \diamond T_2) \diamond T_3) && (\text{by the form of } T'_1 \text{ and the definition of } \diamond) \\
& = T'_1 \diamond (T''_1 \diamond (T_2 \diamond T_3)) && (\text{by the induction hypothesis}) \\
& = T'_1 \diamond T''_1 \diamond (T_2 \diamond T_3) && (\text{by the form of } T'_1 \text{ and the definition of } \diamond) \\
& = T_1 \diamond (T_2 \diamond T_3).
\end{aligned}$$

If  $\text{bra}(T_2) \geq 2$  or  $\text{bra}(T_3) \geq 2$ , the associativity can be proved similarly.  $\square$

Let  $i : X \rightarrow \mathbf{k}\mathcal{T}(X, \Omega)$ ,  $x \mapsto \begin{array}{c} \diagup \\ x \\ \diagdown \end{array}$  be the natural inclusion. Then

**Theorem 2.14.** *Let  $\Omega$  be a set with five products  $\leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot$  and  $\lambda$  a family of elements in  $\mathbf{k}$  indexed by  $\Omega^2$ . Then the following conditions are equivalent:*

- $(\mathbf{k}\mathcal{T}(X, \Omega), \diamond, (B_{\omega}^+)_{\omega \in \Omega})$  together with the map  $i$  is the free  $\Omega$ -Rota-Baxter algebra generated by  $X$ .
- $(\mathbf{k}\mathcal{T}(X, \Omega), \diamond, (B_{\omega}^+)_{\omega \in \Omega})$  is an  $\Omega$ -Rota-Baxter algebra.
- $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, \lambda)$  is a  $\lambda$ -ETS.

*Proof.* (a)  $\implies$  (b) It is obvious.

(b)  $\implies$  (c) For  $\alpha, \beta, \gamma \in \Omega$  and  $\begin{array}{c} \diagup \\ x \\ \diagdown \end{array}, \begin{array}{c} \diagup \\ y \\ \diagdown \end{array}, \begin{array}{c} \diagup \\ z \\ \diagdown \end{array} \in \mathcal{T}(X, \Omega)$ , we have

$$(B_{\alpha}^+(\begin{array}{c} \diagup \\ x \\ \diagdown \end{array}) \diamond B_{\beta}^+(\begin{array}{c} \diagup \\ y \\ \diagdown \end{array})) \diamond B_{\gamma}^+(\begin{array}{c} \diagup \\ z \\ \diagdown \end{array})$$





$$\begin{aligned}
 & + \lambda_{\beta,\gamma} B_{\alpha \leftarrow (\beta \cdot \gamma)}^+ \left( \begin{array}{c} x \\ \swarrow \quad \searrow \\ \diamond \end{array} B_{\alpha \leftarrow (\beta \cdot \gamma)}^+ \left( \begin{array}{c} y \\ \swarrow \quad \searrow \\ \diamond \end{array} \begin{array}{c} z \\ \swarrow \quad \searrow \\ \diamond \end{array} \right) \right) \\
 & + \lambda_{\beta,\gamma} \lambda_{\alpha,\beta \cdot \gamma} B_{\alpha \leftarrow (\beta \cdot \gamma)}^+ \left( \begin{array}{c} x \\ \swarrow \quad \searrow \\ \diamond \end{array} \left( \begin{array}{c} y \\ \swarrow \quad \searrow \\ \diamond \end{array} \begin{array}{c} z \\ \swarrow \quad \searrow \\ \diamond \end{array} \right) \right) \\
 = & \begin{array}{c} (\alpha \triangleright (\beta \rightarrow \gamma)) \triangleright (\beta \triangleright \gamma) \\ (\alpha \triangleright (\beta \rightarrow \gamma)) \rightarrow (\beta \triangleright \gamma) \\ \alpha \rightarrow (\beta \rightarrow \gamma) \end{array} \begin{array}{c} x \\ \swarrow \quad \searrow \\ \diamond \end{array} \begin{array}{c} y \\ \swarrow \quad \searrow \\ \diamond \end{array} \begin{array}{c} z \\ \swarrow \quad \searrow \\ \diamond \end{array} \\
 & + \begin{array}{c} (\alpha \triangleright (\beta \rightarrow \gamma)) \leftarrow (\beta \triangleright \gamma) \\ \alpha \rightarrow (\beta \rightarrow \gamma) \end{array} \begin{array}{c} x \\ \swarrow \quad \searrow \\ \diamond \end{array} \begin{array}{c} y \\ \swarrow \quad \searrow \\ \diamond \end{array} \begin{array}{c} z \\ \swarrow \quad \searrow \\ \diamond \end{array} \\
 & + \lambda_{\alpha \triangleright (\beta \rightarrow \gamma), \beta \triangleright \gamma} (\alpha \triangleright (\beta \rightarrow \gamma)) \cdot (\beta \triangleright \gamma) \begin{array}{c} x \\ \swarrow \quad \searrow \\ \diamond \end{array} \begin{array}{c} y \\ \swarrow \quad \searrow \\ \diamond \end{array} \begin{array}{c} z \\ \swarrow \quad \searrow \\ \diamond \end{array} \\
 & + \begin{array}{c} \beta \triangleright \gamma \\ \alpha \leftarrow (\beta \rightarrow \gamma) \end{array} \begin{array}{c} y \\ \swarrow \quad \searrow \\ \diamond \end{array} \begin{array}{c} z \\ \swarrow \quad \searrow \\ \diamond \end{array} + \lambda_{\alpha,\beta \rightarrow \gamma} \begin{array}{c} z \\ \swarrow \quad \searrow \\ \diamond \end{array} \begin{array}{c} x \\ \swarrow \quad \searrow \\ \diamond \end{array} \begin{array}{c} y \\ \swarrow \quad \searrow \\ \diamond \end{array} \\
 & + \begin{array}{c} \alpha \triangleright (\beta \leftarrow \gamma) \\ \alpha \rightarrow (\beta \leftarrow \gamma) \end{array} \begin{array}{c} x \\ \swarrow \quad \searrow \\ \diamond \end{array} \begin{array}{c} y \\ \swarrow \quad \searrow \\ \diamond \end{array} \begin{array}{c} z \\ \swarrow \quad \searrow \\ \diamond \end{array} + \begin{array}{c} \beta \triangleleft \gamma \\ \alpha \leftarrow (\beta \leftarrow \gamma) \end{array} \begin{array}{c} x \\ \swarrow \quad \searrow \\ \diamond \end{array} \begin{array}{c} y \\ \swarrow \quad \searrow \\ \diamond \end{array} \begin{array}{c} z \\ \swarrow \quad \searrow \\ \diamond \end{array} \\
 & + \lambda_{\alpha,\beta \leftarrow \gamma} \begin{array}{c} x \\ \swarrow \quad \searrow \\ \diamond \end{array} \begin{array}{c} y \\ \swarrow \quad \searrow \\ \diamond \end{array} \begin{array}{c} z \\ \swarrow \quad \searrow \\ \diamond \end{array} + \lambda_{\beta,\gamma} \begin{array}{c} \alpha \triangleright (\beta \cdot \gamma) \\ \alpha \rightarrow (\beta \cdot \gamma) \end{array} \begin{array}{c} x \\ \swarrow \quad \searrow \\ \diamond \end{array} \begin{array}{c} y \\ \swarrow \quad \searrow \\ \diamond \end{array} \begin{array}{c} z \\ \swarrow \quad \searrow \\ \diamond \end{array} \\
 & + \lambda_{\beta,\gamma} \begin{array}{c} \alpha \triangleleft (\beta \cdot \gamma) \\ \alpha \leftarrow (\beta \cdot \gamma) \end{array} \begin{array}{c} x \\ \swarrow \quad \searrow \\ \diamond \end{array} \begin{array}{c} y \\ \swarrow \quad \searrow \\ \diamond \end{array} \begin{array}{c} z \\ \swarrow \quad \searrow \\ \diamond \end{array} + \lambda_{\beta,\gamma} \lambda_{\alpha,\beta \cdot \gamma} \begin{array}{c} x \\ \swarrow \quad \searrow \\ \diamond \end{array} \begin{array}{c} y \\ \swarrow \quad \searrow \\ \diamond \end{array} \begin{array}{c} z \\ \swarrow \quad \searrow \\ \diamond \end{array} \begin{array}{c} \alpha \cdot (\beta \cdot \gamma) \end{array}
 \end{aligned}$$

By Lemma 2.13 and identifying the types of the planar rooted trees, we get that  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, \lambda)$  is a  $\lambda$ -ETS.

(c)  $\implies$  (a) By Lemma 2.13 and the definition of  $\diamond$ ,  $(\mathbf{kT}(X, \Omega), \diamond, (B_{\omega}^+)_{\omega \in \Omega})$  is an  $\Omega$ -Rota-Baxter algebra. Now we show the freeness of  $\mathbf{kT}(X, \Omega)$ .

Let  $(R, \cdot, (P_{\omega})_{\omega \in \Omega})$  be an  $\Omega$ -Rota-Baxter algebra of weight  $\lambda_{\Omega}$  and  $f : X \rightarrow R$  a set map. We extend  $f$  to an  $\Omega$ -Rota-Baxter algebra morphism  $\bar{f} : \mathbf{kT}(X, \Omega) \rightarrow R$  such that  $\bar{f} \circ i = f$ .

For  $T \in \mathcal{T}(X, \Omega)$ , we define  $\bar{f}(T)$  by induction on  $\text{dep}(T)$ . If  $\text{dep}(T) = 1$ , then  $T$  is of the form

$$T = \begin{array}{c} \cdots \\ \swarrow \quad \searrow \\ x_1 \quad \cdots \quad x_m \\ \swarrow \quad \searrow \\ \diamond \end{array}$$

Define

$$\bar{f}(T) := f(x_1) \cdot f(x_2) \cdots f(x_m).$$

For the induction step of  $\text{dep}(T) \geq 2$ , we define  $\bar{f}(T)$  by induction on the branches of  $T$ . If  $\text{bra}(T) = 1$ , then  $T$  is of the form

$$T = \begin{array}{c} T_2 \quad T_m \\ \alpha_2 \quad \alpha_m \\ \swarrow \quad \searrow \\ x_1 \quad \cdots \quad x_m \\ \swarrow \quad \searrow \\ \alpha_1 \quad \alpha_{m+1} \\ \swarrow \quad \searrow \\ \omega \end{array}$$

Define

$$\bar{f}(T) := P_{\omega} \left( P_{\alpha_1}(\bar{f}(T_1)) \cdot f(x_1) \cdot P_{\alpha_2}(\bar{f}(T_2)) \cdots P_{\alpha_m}(\bar{f}(T_m)) \cdot f(x_m) \cdot P_{\alpha}(\bar{f}(T_{m+1})) \right).$$

If  $\text{bra}(T) > 1$ , then  $T$  is of the form

$$T = \begin{array}{c} T_2 \quad T_m \\ \alpha_2 \quad \alpha_m \\ \swarrow \quad \searrow \\ x_1 \quad \cdots \quad x_m \\ \swarrow \quad \searrow \\ \alpha_1 \quad \alpha_{m+1} \\ \swarrow \quad \searrow \\ \diamond \end{array}$$

Define

$$\bar{f}(T) := P_{\alpha_1}(\bar{f}(T_1)) \cdot f(x_1) \cdot P_{\alpha_2}(\bar{f}(T_2)) \cdots P_{\alpha_m}(\bar{f}(T_m)) \cdot f(x_m) \cdot P_{\alpha}(\bar{f}(T_{m+1})).$$

By construction of  $\bar{f}$ ,  $\bar{f} \circ i = f$  and  $P_\omega \bar{f} = \bar{f} B_\omega^+$  for all  $\omega \in \Omega$ . Next we show that  $\bar{f}$  is an algebra homomorphism, i.e.

$$(44) \quad \bar{f}(T \diamond T') = \bar{f}(T) \cdot \bar{f}(T') \quad \text{for all } T, T' \in \mathcal{T}(X, \Omega).$$

We prove Eq. (44) by induction on  $\text{dep}(T) + \text{dep}(T')$ . If  $\text{dep}(T) + \text{dep}(T') = 2$ , then  $\text{dep}(T) = \text{dep}(T') = 1$  and

$$T = \begin{array}{c} \cdots \\ x_1 \quad \cdots \quad x_m \\ \diagdown \quad \diagup \\ \text{---} \\ \text{---} \end{array} \quad \text{and} \quad T' = \begin{array}{c} \cdots \\ y_1 \quad \cdots \quad y_n \\ \diagdown \quad \diagup \\ \text{---} \\ \text{---} \end{array}, \quad \text{with } m, n \geq 0,$$

and

$$\bar{f}(T \diamond T') = f(x_1) \cdots f(x_m) \cdot f(y_1) \cdots f(y_n) = (f(x_1) \cdots f(x_m)) \cdot (f(y_1) \cdots f(y_n)) = \bar{f}(T) \diamond \bar{f}(T').$$

For the induction step of  $\text{dep}(T) + \text{dep}(T') \geq 3$ . If  $T \diamond T'$  belongs to the first three cases, then  $\bar{f}(T \diamond T') = \bar{f}(T) \cdot \bar{f}(T')$  by the definition of  $\diamond$  and the construction of  $\bar{f}$ . So we only need to consider the fourth case. Then

$$\begin{aligned} \bar{f}(T \diamond T') &= \bar{f} \left( \begin{array}{c} T_2 \quad T_m \\ \alpha_2 \quad \alpha_m \\ \circ \quad \circ \\ \diagdown \quad \diagup \\ T_1 \quad \alpha_1 \quad x_1 \quad \cdots \quad x_m \\ \diagdown \quad \diagup \\ \text{---} \\ \text{---} \end{array} \diamond \left( B_{\alpha_{m+1} \rightarrow \beta_1}^+ (B_{\alpha_{m+1} \triangleright \beta_1}^+ (T_{m+1}) \diamond T'_1) + B_{\alpha_{m+1} \leftarrow \beta_1}^+ (T_{m+1} \diamond B_{\alpha_{m+1} \triangleleft \beta_1}^+ (T'_1)) \right. \right. \\ &\quad \left. \left. + \lambda_{\alpha_{m+1} \beta_1} B_{\alpha_{m+1} \beta_1}^+ (T_{m+1} \diamond T'_1) \right) \right) \diamond \begin{array}{c} T'_2 \quad T'_n \\ \beta_2 \quad \beta_n \\ \circ \quad \circ \\ \diagdown \quad \diagup \\ y_1 \quad \cdots \quad y_n \quad \beta_{n+1} \\ \diagdown \quad \diagup \\ \text{---} \\ \text{---} \end{array} \\ &= \left( P_{\alpha_1}(\bar{f}(T_1)) \cdot f(x_1) \cdot P_{\alpha_2}(\bar{f}(T_2)) \cdots P_{\alpha_m}(\bar{f}(T_m)) \cdot f(x_m) \cdot \bar{f} \left( B_{\alpha_{m+1} \rightarrow \beta_1}^+ (B_{\alpha_{m+1} \triangleright \beta_1}^+ (T_{m+1}) \diamond T'_1) \right. \right. \\ &\quad \left. \left. + B_{\alpha_{m+1} \leftarrow \beta_1}^+ (T_{m+1} \diamond B_{\alpha_{m+1} \triangleleft \beta_1}^+ (T'_1)) + \lambda_{\alpha_{m+1} \beta_1} B_{\alpha_{m+1} \beta_1}^+ (T_{m+1} \diamond T'_1) \right) \right) \\ &\quad \cdot f(y_1) \cdot P_{\beta_2}(\bar{f}(T'_2)) \cdots P_{\beta_{n+1}}(\bar{f}(T'_{n+1})) \\ &= \left( P_{\alpha_1}(\bar{f}(T_1)) \cdot f(x_1) \cdot P_{\alpha_2}(\bar{f}(T_2)) \cdots P_{\alpha_m}(\bar{f}(T_m)) \cdot f(x_m) \cdot P_{\alpha_{m+1}}(\bar{f}(T_{m+1})) \right) \\ &\quad \cdot \left( P_{\beta_1}(\bar{f}(T'_1)) \cdot f(y_1) \cdot P_{\beta_2}(\bar{f}(T'_2)) \cdots P_{\beta_{n+1}}(\bar{f}(T'_{n+1})) \right) \\ &= \bar{f}(T) \diamond \bar{f}(T'). \end{aligned}$$

Moreover, by the construction of  $\bar{f}$ , it is the unique way to extend  $f$  as an  $\Omega$ -Rota-Baxter algebra morphism. Hence  $(\mathbf{k}\mathcal{T}(X, \Omega), \diamond, (B_\omega^+)_{\omega \in \Omega})$  together with the map  $i$  is the free  $\Omega$ -Rota-Baxter algebra generated by  $X$ .  $\square$

**Remark 2.15.** (a) In Definition 2.8,  $\Omega$  is required to be a set with five products  $\leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot$  and  $\lambda$  is required to be a family of elements in  $\mathbf{k}$  indexed by  $\Omega^2$ . This defines a category of  $\Omega$ -Rota-Baxter algebras for any such  $\Omega$ . Generally, free  $\Omega$ -Rota-Baxter algebras are not based on  $\Omega$ -angularly decorated planar trees. However, by Theorem 2.14, the condition of a free  $\Omega$ -Rota-Baxter algebra based on the combinatorics of  $\Omega$ -angularly decorated planar trees, similar to the one of (classical) Rota-Baxter algebras, is equivalent to  $(\Omega, \leftarrow, \rightarrow, \cdot, \triangleleft, \triangleright, \cdot, \lambda)$  being a  $\lambda$ -ETS.

- (b) As a particular case, we recover the description of free family Rota-Baxter algebras of [14]. An alternative description of free Rota-Baxter algebras (with rooted forests) is done in [4].

Taking all elements in  $\lambda$  to be 0, we get the following result:

**Corollary 2.16.** *Let  $\Omega$  be a set with four products  $\leftarrow, \rightarrow, \triangleleft, \triangleright$ . Then the following conditions are equivalent:*

- (a)  $(\mathbf{k}\mathcal{T}(X, \Omega), \diamond, (B_\omega^+)_{\omega \in \Omega})$  together with the map  $i$  is the free  $\Omega$ -Rota-Baxter algebra of weight 0 generated by  $X$ .  
 (b)  $(\mathbf{k}\mathcal{T}(X, \Omega), \diamond, (B_\omega^+)_{\omega \in \Omega})$  is an  $\Omega$ -Rota-Baxter algebra of weight 0.  
 (c)  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright)$  is an EDS.

**2.3. Commutative  $\Omega$ -Rota-Baxter algebras on typed words.** Let  $\Omega$  be a set and  $V$  a vector space. Recall from [5] that the space of  $\Omega$ -typed words in  $V$  is

$$\text{Sh}_\Omega^+(V) = \bigoplus_{n \geq 1} (\mathbf{k}\Omega)^{\otimes(n-1)} \otimes V^{\otimes n}.$$

For the ease of statement, we redefine the space of  $\Omega$ -typed words in  $V$  as

$$\text{Sh}_\Omega^+(V) = \bigoplus_{n \geq 0} \underbrace{V \otimes (\mathbf{k}\Omega) \otimes \cdots \otimes (\mathbf{k}\Omega) \otimes V}_{(n+1)\text{'s } V \text{ and } n\text{'s } (\mathbf{k}\Omega)}$$

and write each pure tensor  $\mathbf{v} = v_0 \otimes \omega_1 \otimes \cdots \otimes \omega_n \otimes v_n \in \Omega$  under the form

$$\mathbf{v} = v_0 \otimes_{\omega_1} v_1 \otimes_{\omega_2} \cdots \otimes_{\omega_n} v_n,$$

where  $n \geq 0$ ,  $\omega_1, \dots, \omega_n \in \Omega$  and  $v_0, \dots, v_n \in V$  with the convention  $\mathbf{v} = v_0$  if  $n = 0$ . We call  $\mathbf{v}$  an  $\Omega$ -typed word in  $V$  and define its **length**  $\ell(\mathbf{v}) := n + 1$ .

Let  $A$  be an algebra with identity  $1_A$ ,  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot)$  be a set with five products and  $\lambda = (\lambda_{\alpha, \beta})_{(\alpha, \beta) \in \Omega^2}$  be a family of elements in  $\mathbf{k}$  indexed by  $\Omega^2$ . For any pure tensors  $\mathbf{a} = a_0 \otimes_{\alpha_1} \mathbf{a}'$ ,  $\mathbf{b} = b_0 \otimes_{\beta_1} \mathbf{b}' \in \text{Sh}_\Omega^+(A)$  with  $\ell(\mathbf{a}) = m$  and  $\ell(\mathbf{b}) = n$ , define  $\mathbf{a} \diamond \mathbf{b}$  inductively as follows:

$$(45) \quad \mathbf{a} \diamond \mathbf{b} := \begin{cases} a_0 b_0, & \text{if } m = n = 0, \\ a_0 b_0 \otimes_{\alpha_1} \mathbf{a}', & \text{if } m > 0, n = 0, \\ a_0 b_0 \otimes_{\beta_1} \mathbf{b}', & \text{if } m = 0, n > 0, \\ a_0 b_0 \otimes_{\alpha_1 \rightarrow \beta_1} ((1_A \otimes_{\alpha_1 \triangleright \beta_1} \mathbf{a}') \diamond \mathbf{b}') + a_0 b_0 \otimes_{\alpha \leftarrow \beta_1} (\mathbf{a}' \diamond (1_A \otimes_{\alpha_1 \triangleleft \beta_1} \mathbf{b}')) \\ \quad + \lambda_{\alpha_1, \beta_1} a_0 b_0 \otimes_{\alpha_1, \beta_1} (\mathbf{a}' \diamond \mathbf{b}'), & \text{if } m > 0, n > 0. \end{cases}$$

Extending bilinearly, we construct a product  $\diamond$  on  $\text{Sh}_\Omega^+(A)$ .

**Lemma 2.17.** *Let  $A$  be an algebra with identity  $1_A$ ,  $\Omega$  a set with five products  $\leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot$  and  $\lambda$  a family of elements in  $\mathbf{k}$  indexed by  $\Omega^2$ . If  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, \lambda)$  is a  $\lambda$ -ESD, then  $(\text{Sh}_\Omega^+(A), \diamond)$  is an associative algebra with identity  $1_A$ .*

*Proof.* By Eq. (45),  $\text{Sh}_\Omega^+(A)$  is closed under  $\diamond$  and  $1_A$  is the identity of  $\diamond$ .

For pure tensors  $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \text{Sh}_\Omega^+(A)$ , we prove

$$(46) \quad (\mathbf{a} \diamond \mathbf{b}) \diamond \mathbf{c} = \mathbf{a} \diamond (\mathbf{b} \diamond \mathbf{c})$$

by induction on  $\ell(\mathbf{a}) + \ell(\mathbf{b}) + \ell(\mathbf{c})$ . If  $\ell(\mathbf{a}) + \ell(\mathbf{b}) + \ell(\mathbf{c}) = 3$ , then  $\ell(\mathbf{a}) = \ell(\mathbf{b}) = \ell(\mathbf{c}) = 1$  and  $\mathbf{a} = a_0$ ,  $\mathbf{b} = b_0$ ,  $\mathbf{c} = c_0$ . Hence

$$(\mathbf{a} \diamond \mathbf{b}) \diamond \mathbf{c} = a_0 b_0 c_0 = \mathbf{a} \diamond (\mathbf{b} \diamond \mathbf{c}).$$

Suppose Eq. (46) holds for  $\ell(\mathbf{a}) + \ell(\mathbf{b}) + \ell(\mathbf{c}) \leq p$ , where  $p \geq 3$  is a fixed integer. Consider the case of  $\ell(\mathbf{a}) + \ell(\mathbf{b}) + \ell(\mathbf{c}) = p + 1$ . If one of  $\ell(\mathbf{a}), \ell(\mathbf{b}), \ell(\mathbf{c})$  is equal to 1, then Eq. (46) holds by direct calculation. Hence we assume  $\ell(\mathbf{a}) > 1, \ell(\mathbf{b}) > 1, \ell(\mathbf{c}) > 1$  and

$$\mathbf{a} = a_0 \otimes_{\alpha_1} \mathbf{a}', \quad \mathbf{b} = b_0 \otimes_{\beta_1} \mathbf{b}', \quad \mathbf{c} = c_0 \otimes_{\gamma_1} \mathbf{c}'.$$

Then

$$\begin{aligned} & (\mathbf{a} \diamond \mathbf{b}) \diamond \mathbf{c} \\ = & (a_0 b_0) c_0 \otimes_{(\alpha_1 \rightarrow \beta_1) \rightarrow \gamma_1} ((1_A \otimes_{(\alpha_1 \rightarrow \beta_1) \triangleright \gamma_1} ((1_A \otimes_{\alpha_1 \otimes \beta_1} \mathbf{a}') \diamond \mathbf{b}')) \diamond \mathbf{c}') \\ & + (a_0 b_0) c_0 \otimes_{(\alpha_1 \rightarrow \beta_1) \leftarrow \gamma_1} (((1_A \otimes_{\alpha_1 \triangleright \beta_1} \mathbf{a}') \diamond \mathbf{b}') \diamond (1_A \otimes_{(\alpha_1 \rightarrow \beta_1) \triangleleft \gamma_1} \mathbf{c}')) \\ & + \lambda_{(\alpha_1 \rightarrow \beta_1), \gamma_1} (a_0 b_0) c_0 \otimes_{(\alpha_1 \rightarrow \beta_1) \cdot \gamma_1} (((1_A \otimes_{\alpha_1 \triangleright \beta_1} \mathbf{a}') \diamond \mathbf{b}') \diamond \mathbf{c}') \\ & + (a_0 b_0) c_0 \otimes_{(\alpha_1 \leftarrow \beta_1) \rightarrow \gamma_1} ((1_A \otimes_{(\alpha_1 \leftarrow \beta_1) \triangleright \gamma_1} (\mathbf{a}' \diamond (1_A \otimes_{\alpha_1 \triangleleft \beta_1} \mathbf{b}')))) \diamond \mathbf{c}') \\ & + (a_0 b_0) c_0 \otimes_{(\alpha_1 \leftarrow \beta_1) \leftarrow \gamma_1} (\mathbf{a}' \diamond (1_A \otimes_{(\alpha_1 \triangleleft \beta_1) \rightarrow ((\alpha_1 \leftarrow \beta_1) \triangleleft \gamma_1)} ((1_A \otimes_{(\alpha_1 \triangleleft \beta_1) \triangleright ((\alpha_1 \leftarrow \beta_1) \triangleleft \gamma_1)} \mathbf{b}') \diamond \mathbf{c}')))) \\ & + (a_0 b_0) c_0 \otimes_{(\alpha_1 \leftarrow \beta_1) \leftarrow \gamma_1} (\mathbf{a}' \diamond (1_A \otimes_{(\alpha_1 \triangleleft \beta_1) \leftarrow ((\alpha_1 \leftarrow \beta_1) \triangleleft \gamma_1)} (\mathbf{b}' \diamond (1_A \otimes_{(\alpha_1 \triangleleft \beta_1) \triangleleft ((\alpha_1 \leftarrow \beta_1) \triangleleft \gamma_1)} \mathbf{c}'))))) \\ & + \lambda_{(\alpha_1 \triangleleft \beta_1), ((\alpha_1 \leftarrow \beta_1) \triangleleft \gamma_1)} (a_0 b_0) c_0 \otimes_{(\alpha_1 \leftarrow \beta_1) \leftarrow \gamma_1} (\mathbf{a}' \diamond (1_A \otimes_{(\alpha_1 \triangleleft \beta_1), ((\alpha_1 \leftarrow \beta_1) \triangleleft \gamma_1)} (\mathbf{b}' \diamond \mathbf{c}')))) \\ & + \lambda_{(\alpha_1 \leftarrow \beta_1), \gamma_1} (a_0 b_0) c_0 \otimes_{(\alpha_1 \leftarrow \beta_1) \cdot \gamma_1} ((\mathbf{a}' \diamond (1_A \otimes_{\alpha_1 \triangleleft \beta_1} \mathbf{b}')) \diamond \mathbf{c}') \\ & + \lambda_{\alpha_1, \beta_1} (a_0 b_0) c_0 \otimes_{(\alpha_1, \beta_1) \rightarrow \gamma_1} ((1_A \otimes_{(\alpha_1, \beta_1) \triangleright \gamma_1} (\mathbf{a}' \diamond \mathbf{b}')) \diamond \mathbf{c}') \\ & + \lambda_{\alpha_1, \beta_1} (a_0 b_0) c_0 \otimes_{(\alpha_1, \beta_1) \leftarrow \gamma_1} ((\mathbf{a}' \diamond \mathbf{b}') \diamond (1_A \otimes_{(\alpha_1, \beta_1) \triangleleft \gamma_1} \mathbf{c}')) \\ & + \lambda_{\alpha_1, \beta_1} \lambda_{(\alpha_1, \beta_1), \gamma_1} (a_0 b_0) c_0 \otimes_{(\alpha_1, \beta_1) \cdot \gamma_1} ((\mathbf{a}' \diamond \mathbf{b}') \diamond \mathbf{c}') \end{aligned}$$

and

$$\begin{aligned} & \mathbf{a} \diamond (\mathbf{b} \diamond \mathbf{c}) \\ = & a_0 (b_0 c_0) \otimes_{\alpha_1 \rightarrow (\beta_1 \rightarrow \gamma_1)} ((1_A \otimes_{(\alpha_1 \triangleright (\beta_1 \rightarrow \gamma_1)) \rightarrow (\beta_1 \triangleright \gamma_1)} ((1_A \otimes_{(\alpha_1 \triangleright (\beta_1 \rightarrow \gamma_1)) \triangleright (\beta_1 \triangleright \gamma_1)} \mathbf{a}') \diamond \mathbf{b}')) \diamond \mathbf{c}') \\ & + a_0 (b_0 c_0) \otimes_{\alpha_1 \rightarrow (\beta_1 \rightarrow \gamma_1)} ((1_A \otimes_{(\alpha_1 \triangleright (\beta_1 \rightarrow \gamma_1)) \leftarrow (\beta_1 \triangleright \gamma_1)} (\mathbf{a}' \diamond (1_A \otimes_{(\alpha_1 \triangleright (\beta_1 \rightarrow \gamma_1)) \triangleleft (\beta_1 \triangleright \gamma_1)} \mathbf{b}')))) \diamond \mathbf{c}') \\ & + \lambda_{(\alpha_1 \triangleright (\beta_1 \rightarrow \gamma_1)), (\beta_1 \triangleright \gamma_1)} a_0 (b_0 c_0) \otimes_{\alpha_1 \rightarrow (\beta_1 \rightarrow \gamma_1)} ((1_A \otimes_{(\alpha_1 \triangleright (\beta_1 \rightarrow \gamma_1)), (\beta_1 \triangleright \gamma_1)} (\mathbf{a}' \diamond \mathbf{b}')) \diamond \mathbf{c}') \\ & + a_0 (b_0 c_0) \otimes_{\alpha_1 \leftarrow (\beta_1 \rightarrow \gamma_1)} (\mathbf{a}' \diamond (1_A \otimes_{\alpha_1 \triangleleft (\beta_1 \rightarrow \gamma_1)} ((1_A \otimes_{\beta_1 \triangleright \gamma_1} \mathbf{b}') \diamond \mathbf{c}')))) \\ & + \lambda_{\alpha_1, (\beta_1 \rightarrow \gamma_1)} a_0 (b_0 c_0) \otimes_{\alpha_1 \cdot (\beta_1 \rightarrow \gamma_1)} (\mathbf{a}' \diamond ((1_A \otimes_{\beta_1 \triangleright \gamma_1} \mathbf{b}') \diamond \mathbf{c}')) \\ & + a_0 (b_0 c_0) \otimes_{\alpha_1 \rightarrow (\beta_1 \leftarrow \gamma_1)} ((1_A \otimes_{\alpha_1 \triangleright (\beta_1 \leftarrow \gamma_1)} \mathbf{a}') \diamond (\mathbf{b}' \diamond (1_A \otimes_{\beta_1 \triangleleft \gamma_1} \mathbf{c}')))) \\ & + a_0 (b_0 c_0) \otimes_{\alpha_1 \leftarrow (\beta_1 \leftarrow \gamma_1)} (\mathbf{a}' \diamond (1_A \otimes_{\alpha_1 \triangleleft (\beta_1 \leftarrow \gamma_1)} (\mathbf{b}' \diamond (1_A \otimes_{\beta_1 \triangleleft \gamma_1} \mathbf{c}'))))) \\ & + \lambda_{\alpha_1, (\beta_1 \leftarrow \gamma_1)} a_0 (b_0 c_0) \otimes_{\alpha_1 \cdot (\beta_1 \leftarrow \gamma_1)} (\mathbf{a}' \diamond (\mathbf{b}' \diamond (1_A \otimes_{\beta_1 \triangleleft \gamma_1} \mathbf{c}')))) \\ & + \lambda_{\beta_1, \gamma_1} a_0 (b_0 c_0) \otimes_{\alpha_1 \rightarrow (\beta_1 \cdot \gamma_1)} ((1_A \otimes_{\alpha_1 \triangleright (\beta_1 \cdot \gamma_1)} \mathbf{a}') \diamond (\mathbf{b}' \diamond \mathbf{c}')) \\ & + \lambda_{\beta_1, \gamma_1} a_0 (b_0 c_0) \otimes_{\alpha_1 \leftarrow (\beta_1 \cdot \gamma_1)} (\mathbf{a}' \diamond (1_A \otimes_{\alpha_1 \triangleleft (\beta_1 \cdot \gamma_1)} (\mathbf{b}' \diamond \mathbf{c}')))) \\ & + \lambda_{\beta_1, \gamma_1} \lambda_{\alpha_1, (\beta_1 \cdot \gamma_1)} a_0 (b_0 c_0) \otimes_{\alpha_1 \cdot (\beta_1 \cdot \gamma_1)} (\mathbf{a}' \diamond (\mathbf{b}' \diamond \mathbf{c}')). \end{aligned}$$

By induction hypothesis and  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, \lambda)$  being a  $\lambda$ -ETS,  $(\mathbf{a} \diamond \mathbf{b}) \diamond \mathbf{c} = \mathbf{a} \diamond (\mathbf{b} \diamond \mathbf{c})$ . Hence  $(\text{Sh}_\Omega^+(A), \diamond)$  is an associative algebra with identity  $1_A$ .  $\square$

For each  $\omega \in \Omega$ , define a linear map  $P_\omega : \text{Sh}_\Omega^+(A) \rightarrow \text{Sh}_\Omega^+(A)$ ,  $\mathbf{a} \mapsto 1_A \otimes_\omega \mathbf{a}$ . If further  $A$  is a commutative algebra and  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, \lambda)$  is a commutative  $\lambda$ -ETS, we get the following result:

**Proposition 2.18.** *If  $A$  is a commutative algebra with identity  $1_A$  and  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, \lambda)$  is a commutative  $\lambda$ -ETS, then  $(\text{Sh}_\Omega^+(A), \diamond, (P_\omega)_{\omega \in \Omega})$  is the free commutative  $\Omega$ -Rota-Baxter algebra generated by  $A$ .*

*Proof.* For  $\mathbf{a}, \mathbf{b} \in \text{Sh}_\Omega^+(A)$  and  $\alpha, \beta \in \Omega$ ,

$$\begin{aligned} P_\alpha(\mathbf{a}) \diamond P_\beta(\mathbf{b}) &= (1_A \otimes_\alpha \mathbf{a}) \diamond (1_A \otimes_\beta \mathbf{b}) \\ &= 1_A \otimes_{\alpha \rightarrow \beta} ((1 \otimes_{\alpha \triangleright \beta} \mathbf{a}) \diamond \mathbf{b}) + 1 \otimes_{\alpha \leftarrow \beta} (\mathbf{a} \diamond (1_A \otimes_{\alpha \triangleleft \beta} \mathbf{b})) \\ &\quad + \lambda_{\alpha, \beta} 1_A \otimes_{\alpha \cdot \beta} (\mathbf{a} \diamond \mathbf{b}) \\ &= P_{\alpha \rightarrow \beta}(P_{\alpha \triangleright \beta}(\mathbf{a}) \diamond \mathbf{b}) + P_{\alpha \leftarrow \beta}(\mathbf{a} \diamond P_{\alpha \triangleleft \beta}(\mathbf{b})) + \lambda_{\alpha, \beta} P_{\alpha \cdot \beta}(\mathbf{a} \diamond \mathbf{b}), \end{aligned}$$

hence  $\text{Sh}_\Omega(A)$  is an  $\Omega$ -Rota-Baxter algebra. Next we show

$$(47) \quad \mathbf{a} \diamond \mathbf{b} = \mathbf{b} \diamond \mathbf{a}$$

by induction on  $\ell(\mathbf{a}) + \ell(\mathbf{b})$ . If  $\ell(\mathbf{a}) + \ell(\mathbf{b}) = 2$ , then  $\ell(\mathbf{a}) = \ell(\mathbf{b}) = 1$  and

$$\mathbf{a} \diamond \mathbf{b} = a_0 \diamond b_0 = a_0 b_0 = b_0 a_0 = b_0 \diamond a_0 = \mathbf{b} \diamond \mathbf{a}.$$

Suppose Eq. (47) holds for  $\ell(\mathbf{a}) + \ell(\mathbf{b}) < p$ , where  $p \geq 2$  is a fixed integer. We consider the case of  $\ell(\mathbf{a}) + \ell(\mathbf{b}) = p + 1$ . If one of  $\ell(\mathbf{a}), \ell(\mathbf{b})$  is equal to 1, then Eq. (47) holds directly. We assume that  $\mathbf{a} = a_0 \otimes_{\alpha_1} \mathbf{a}'$ ,  $\mathbf{b} = b_0 \otimes_{\beta_1} \mathbf{b}'$ , then

$$\begin{aligned} \mathbf{a} \diamond \mathbf{b} &= (a_0 \otimes_{\alpha_1} \mathbf{a}') \diamond (b_0 \otimes_{\beta_1} \mathbf{b}') \\ &= a_0 b_0 \otimes_{\alpha_1 \rightarrow \beta_1} ((1_A \otimes_{\alpha_1 \triangleright \beta_1} \mathbf{a}') \diamond \mathbf{b}') + a_0 b_0 \otimes_{\alpha_1 \leftarrow \beta_1} (\mathbf{a}' \diamond (1_A \otimes_{\alpha_1 \triangleleft \beta_1} \mathbf{b}')) + \lambda_{\alpha_1, \beta_1} a_0 b_0 \otimes_{\alpha_1 \cdot \beta_1} (\mathbf{a}' \diamond \mathbf{b}') \\ &= b_0 a_0 \otimes_{\alpha_1 \rightarrow \beta_1} ((1_A \otimes_{\alpha_1 \triangleright \beta_1} \mathbf{a}') \diamond \mathbf{b}') + b_0 a_0 \otimes_{\alpha_1 \leftarrow \beta_1} (\mathbf{a}' \diamond (1_A \otimes_{\alpha_1 \triangleleft \beta_1} \mathbf{b}')) + \lambda_{\alpha_1, \beta_1} b_0 a_0 \otimes_{\alpha_1 \cdot \beta_1} (\mathbf{a}' \diamond \mathbf{b}') \\ &\quad \text{(by } A \text{ being a commutative algebra)} \\ &= b_0 a_0 \otimes_{\alpha_1 \rightarrow \beta_1} (\mathbf{b}' \diamond (1_A \otimes_{\alpha_1 \triangleright \beta_1} \mathbf{a}')) + b_0 a_0 \otimes_{\alpha_1 \leftarrow \beta_1} ((1_A \otimes_{\alpha_1 \triangleleft \beta_1} \mathbf{b}') \diamond \mathbf{a}') + \lambda_{\alpha_1, \beta_1} b_0 a_0 \otimes_{\alpha_1 \cdot \beta_1} (\mathbf{b}' \diamond \mathbf{a}') \\ &\quad \text{(by the induction hypothesis)} \\ &= b_0 a_0 \otimes_{\beta_1 \leftarrow \alpha_1} (\mathbf{b}' \diamond (1_A \otimes_{\beta_1 \triangleleft \alpha_1} \mathbf{a}')) + b_0 a_0 \otimes_{\beta_1 \rightarrow \alpha_1} ((1_A \otimes_{\beta_1 \triangleright \alpha_1} \mathbf{b}') \diamond \mathbf{a}') + \lambda_{\beta_1, \alpha_1} b_0 a_0 \otimes_{\beta_1 \cdot \alpha_1} (\mathbf{b}' \diamond \mathbf{a}') \\ &\quad \text{(by } \Omega \text{ being commutative)} \\ &= (b_0 \otimes_{\beta_1} \mathbf{b}') \diamond (a_0 \otimes_{\alpha_1} \mathbf{a}') = \mathbf{b} \diamond \mathbf{a}. \end{aligned}$$

Hence  $(\text{Sh}_\Omega^+(A), \diamond)$  is a commutative algebra.

Let  $(R, \cdot, (P_\omega)_{\omega \in \Omega})$  be a commutative  $\Omega$ -Rota-Baxter algebra and  $f : A \rightarrow R$  a commutative algebra homomorphism. We extend  $f$  to an  $\Omega$ -Rota-Baxter algebra morphism  $\bar{f} : \text{Sh}_\Omega^+(A) \rightarrow R$  as follows: for  $\mathbf{a} \in \text{Sh}_\Omega^+(A)$ , we define  $\bar{f}(\mathbf{a})$  by induction on  $\ell(\mathbf{a})$ . If  $\ell(\mathbf{a}) = 1$ , then define  $\bar{f}(\mathbf{a}) = f(\mathbf{a})$ . Suppose  $\bar{f}(\mathbf{a})$  has been defined for all  $\mathbf{a}$  with  $\ell(\mathbf{a}) \leq p$ , where  $p \geq 1$  is a fixed integer. Consider the case of  $\ell(\mathbf{a}) = p + 1$ . We suppose that  $\mathbf{a} = a_0 \otimes_{\alpha_1} \mathbf{a}'$ , and we then put:

$$\bar{f}(\mathbf{a}) := f(a_0) \cdot P_{\alpha_1}(\bar{f}(\mathbf{a}')).$$

We can get that it is the unique way to extend  $f$  as an  $\Omega$ -Rota-Baxter algebra morphism. Hence  $(\text{Sh}_\Omega^+(A), \diamond)$  is the free commutative  $\Omega$ -Rota-Baxter algebra generated by  $A$ .  $\square$

Let us assume that  $A$  is unitary. We denote its unit by  $1_A$ . For each  $\omega \in \Omega$ , define a linear map  $P_\omega : \text{Sh}_\Omega(A) \rightarrow \text{Sh}_\Omega(A)$ ,  $\mathbf{a} \mapsto 1_A \otimes_\omega \mathbf{a}$ .

**Proposition 2.19.** *If  $A$  is a unitary commutative algebra and  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, \lambda)$  is a commutative  $\lambda$ -ETS, then  $(\text{Sh}_\Omega(A), \diamond, (P_\omega)_{\omega \in \Omega})$  is a commutative  $\Omega$ -Rota-Baxter algebra.*

*Proof.* For  $\mathbf{a}, \mathbf{b} \in \text{Sh}_\Omega(A)$  and  $\alpha, \beta \in \Omega$ ,

$$\begin{aligned} P_\alpha(\mathbf{a}) \diamond P_\beta(\mathbf{b}) &= (1_A \otimes_\alpha \mathbf{a}) \diamond (1_A \otimes_\beta \mathbf{b}) \\ &= 1_A \otimes_{\alpha \rightarrow \beta} ((1_A \otimes_{\alpha \triangleright \beta} \mathbf{a}) \diamond \mathbf{b}) + 1_A \otimes_{\alpha \leftarrow \beta} (\mathbf{a} \diamond (1_A \otimes_{\alpha \triangleleft \beta} \mathbf{b})) \\ &\quad + \lambda_{\alpha, \beta} 1_A \otimes_{\alpha \cdot \beta} (\mathbf{a} \diamond \mathbf{b}) \\ &= P_{\alpha \rightarrow \beta}(P_{\alpha \triangleright \beta}(\mathbf{a}) \diamond \mathbf{b}) + P_{\alpha \leftarrow \beta}(\mathbf{a} \diamond P_{\alpha \triangleleft \beta}(\mathbf{b})) + \lambda_{\alpha, \beta} P_{\alpha \cdot \beta}(\mathbf{a} \diamond \mathbf{b}), \end{aligned}$$

hence  $\text{Sh}_\Omega(A)$  is an  $\Omega$ -Rota-Baxter algebra. Next we show

$$(48) \quad \mathbf{a} \diamond \mathbf{b} = \mathbf{b} \diamond \mathbf{a}$$

by induction on  $\ell(\mathbf{a}) + \ell(\mathbf{b})$ . If  $\ell(\mathbf{a}) + \ell(\mathbf{b}) = 2$ , then  $\ell(\mathbf{a}) = \ell(\mathbf{b}) = 1$  and

$$\mathbf{a} \diamond \mathbf{b} = a_0 \diamond b_0 = a_0 b_0 = b_0 a_0 = b_0 \diamond a_0 = \mathbf{b} \diamond \mathbf{a}.$$

Suppose Eq. (48) holds for  $\ell(\mathbf{a}) + \ell(\mathbf{b}) < p$ , where  $p \geq 2$  is a fixed integer. We consider the case of  $\ell(\mathbf{a}) + \ell(\mathbf{b}) = p + 1$ . If one of  $\ell(\mathbf{a}), \ell(\mathbf{b})$  is equal to 1, then Eq. (48) holds directly. So assume  $\mathbf{a} = a_0 \otimes_{\alpha_1} \mathbf{a}', \mathbf{b} = b_0 \otimes_{\beta_1} \mathbf{b}'$ , then

$$\begin{aligned} \mathbf{a} \diamond \mathbf{b} &= (a_0 \otimes_{\alpha_1} \mathbf{a}') \diamond (b_0 \otimes_{\beta_1} \mathbf{b}') \\ &= a_0 b_0 \otimes_{\alpha_1 \rightarrow \beta_1} ((1_A \otimes_{\alpha_1 \triangleright \beta_1} \mathbf{a}') \diamond \mathbf{b}') + a_0 b_0 \otimes_{\alpha_1 \leftarrow \beta_1} (\mathbf{a}' \diamond (1_A \otimes_{\alpha_1 \triangleleft \beta_1} \mathbf{b}')) + \lambda_{\alpha_1, \beta_1} a_0 b_0 \otimes_{\alpha_1 \cdot \beta_1} (\mathbf{a}' \diamond \mathbf{b}') \\ &= b_0 a_0 \otimes_{\alpha_1 \rightarrow \beta_1} ((1_A \otimes_{\alpha_1 \triangleright \beta_1} \mathbf{a}') \diamond \mathbf{b}') + b_0 a_0 \otimes_{\alpha_1 \leftarrow \beta_1} (\mathbf{a}' \diamond (1_A \otimes_{\alpha_1 \triangleleft \beta_1} \mathbf{b}')) + \lambda_{\alpha_1, \beta_1} b_0 a_0 \otimes_{\alpha_1 \cdot \beta_1} (\mathbf{a}' \diamond \mathbf{b}') \\ &\quad \text{(by } A \text{ being a commutative algebra)} \\ &= b_0 a_0 \otimes_{\alpha_1 \rightarrow \beta_1} (\mathbf{b}' \diamond (1_A \otimes_{\alpha_1 \triangleright \beta_1} \mathbf{a}')) + b_0 a_0 \otimes_{\alpha_1 \leftarrow \beta_1} ((1_A \otimes_{\alpha_1 \triangleleft \beta_1} \mathbf{b}') \diamond \mathbf{a}') + \lambda_{\alpha_1, \beta_1} b_0 a_0 \otimes_{\alpha_1 \cdot \beta_1} (\mathbf{b}' \diamond \mathbf{a}') \\ &\quad \text{(by the induction hypothesis)} \\ &= b_0 a_0 \otimes_{\beta_1 \leftarrow \alpha_1} (\mathbf{b}' \diamond (1_A \otimes_{\beta_1 \triangleleft \alpha_1} \mathbf{a}')) + b_0 a_0 \otimes_{\beta_1 \rightarrow \alpha_1} ((1_A \otimes_{\beta_1 \triangleright \alpha_1} \mathbf{b}') \diamond \mathbf{a}') + \lambda_{\beta_1, \alpha_1} b_0 a_0 \otimes_{\beta_1 \cdot \alpha_1} (\mathbf{b}' \diamond \mathbf{a}') \\ &\quad \text{(by } \Omega \text{ being commutative)} \\ &= (b_0 \otimes_{\beta_1} \mathbf{b}') \diamond (a_0 \otimes_{\alpha_1} \mathbf{a}') = \mathbf{b} \otimes \mathbf{a}. \end{aligned}$$

Hence  $(\text{Sh}_\Omega(A), \diamond)$  is a commutative algebra.  $\square$

Let  $A$  be a commutative algebra. We put  $uA = \mathbf{k} \oplus A$  and give it a product defined by

$$(\lambda + a)(\mu + b) = \lambda\mu + (\lambda b + \mu a + ab).$$

Then  $uA$  is a commutative unitary algebra and its unit  $1_A$  is the unit 1 of  $\mathbf{k}$ .

**Theorem 2.20.** *We put*

$$\text{Sh}'_\Omega(A) = A \oplus \bigoplus_{n \geq 2} \underbrace{uA \otimes (\mathbf{k}\Omega) \otimes \cdots \otimes (\mathbf{k}\Omega) \otimes uA}_{n \text{ 's } V \text{ and } (n-1) \text{ 's } (\mathbf{k}\Omega)}.$$

*Then  $\text{Sh}'_\Omega(A)$  is the free commutative  $\Omega$ -Rota-Baxter algebra generated by the algebra  $A$ .*

*Proof.* Let  $(R, \cdot, (P_\omega)_{\omega \in \Omega})$  be a commutative  $\Omega$ -Rota-Baxter algebra and  $f : A \rightarrow R$  a (nonunitary) algebra homomorphism. We extend  $f$ , first from  $uA$  to  $R$  as a unitary algebra morphism by sending  $1_{uA}$  to  $1_R$ , then as an  $\Omega$ -Rota-Baxter algebra morphism  $\bar{f} : \text{Sh}'_\Omega(A) \rightarrow R$  as follows: for  $\mathbf{a} \in \text{Sh}_\Omega(A)$ , we define  $\bar{f}(\mathbf{a})$  by induction on  $\ell(\mathbf{a})$ . If  $\ell(\mathbf{a}) = 1$ , then define  $\bar{f}(\mathbf{a}) = f(\mathbf{a})$ . Suppose  $\bar{f}(\mathbf{a})$  has been defined for all  $\mathbf{a}$  with  $\ell(\mathbf{a}) \leq p$ , where  $p \geq 1$  is a fixed integer. Consider the case of  $\ell(\mathbf{a}) = p + 1$ . Suppose  $\mathbf{a} = a_0 \otimes_{\alpha_1} \mathbf{a}'$ , then define

$$\bar{f}(\mathbf{a}) := f(a_0) \cdot P_{\alpha_1}(\bar{f}(\mathbf{a}')).$$

For any  $\mathbf{a} \in \text{Sh}'_{\Omega}(A)$  and for any  $\alpha \in \Omega$ :

$$\bar{f} \circ P_{\alpha}(\mathbf{a}) = \bar{f}(1_A \otimes_{\alpha} \mathbf{a}) = 1_B \cdot P_{\alpha}(\bar{f}(\mathbf{a})) = P_{\alpha} \circ \bar{f}(\mathbf{a}).$$

Let us prove that this is an algebra morphism. Let  $\mathbf{a}, \mathbf{b} \in \text{Sh}'_{\Omega}(A)$ , let us prove that  $\bar{f}(\mathbf{a} \diamond \mathbf{b}) = \bar{f}(\mathbf{a})\bar{f}(\mathbf{b})$  by induction on  $n = \ell(\mathbf{a}) + \ell(\mathbf{b})$ . If  $\ell(\mathbf{a}) = \ell(\mathbf{b}) = 1$ , then

$$\bar{f}(\mathbf{a} \diamond \mathbf{b}) = \bar{f}(a_0 b_0) = f(a_0 b_0) = f(a_0) \cdot f(b_0) = \bar{f}(\mathbf{a}) \cdot \bar{f}(\mathbf{b}).$$

If  $\ell(\bar{a}) = 1$  and  $\ell(\bar{b}) > 1$ , then

$$\begin{aligned} \bar{f}(\mathbf{a} \diamond \mathbf{b}) &= \bar{f}(a_0 b_0 \otimes_{\alpha_1} \mathbf{a}') \\ &= f(a_0 b_0) \cdot P_{\alpha_1} \circ \bar{f}(\mathbf{a}') \\ &= f(a_0) \cdot f(b_0) \cdot P_{\alpha_1} \circ \bar{f}(\mathbf{a}') \\ &= \bar{f}(\mathbf{a}) \cdot \bar{f}(\mathbf{b}). \end{aligned}$$

This is similar if  $\ell(\bar{a}) > 1$  and  $\ell(\bar{b}) = 1$ . If  $\ell(\bar{a}) > 1$  and  $\ell(\bar{b}) > 1$ , then

$$\begin{aligned} \bar{f}(\mathbf{a} \diamond \mathbf{b}) &= \bar{f}(a_0 b_0 \otimes_{\alpha_1 \rightarrow \beta_1} ((1 \otimes_{\alpha_1 \triangleright \beta_1} \mathbf{a}') \diamond \mathbf{b}')) + \bar{f}(a_0 b_0 \otimes_{\alpha_1 \leftarrow \beta_1} (\mathbf{a}' \diamond (1 \otimes_{\alpha_1 \triangleleft \beta_1} \mathbf{b}'))) \\ &\quad + \bar{f}(\lambda_{\alpha_1, \beta_1} a_0 b_0 \otimes_{\alpha_1, \beta_1} (\mathbf{a}' \diamond \mathbf{b}')) \\ &= f(a_0 b_0) \cdot P_{\alpha_1 \rightarrow \beta_1} \circ \bar{f}((1 \otimes_{\alpha_1 \triangleright \beta_1} \mathbf{a}') \diamond \mathbf{b}') + f(a_0 b_0) \cdot P_{\alpha_1 \leftarrow \beta_1} \circ \bar{f}(\mathbf{a}' \diamond (1 \otimes_{\alpha_1 \triangleleft \beta_1} \mathbf{b}')) \\ &\quad + \lambda_{\alpha_1, \beta_1} f(a_0 b_0) \cdot P_{\alpha_1, \beta_1} \circ \bar{f}(\mathbf{a}' \diamond \mathbf{b}') \\ &= f(a_0) \cdot f(b_0) \cdot \bar{f}(P_{\alpha_1 \rightarrow \beta_1}(P_{\alpha_1 \triangleright \beta_1}(\mathbf{a}') \diamond \mathbf{b}')) + f(a_0) \cdot f(b_0) \cdot \bar{f}(P_{\alpha_1 \leftarrow \beta_1}(\mathbf{a}' \diamond P_{\alpha_1 \triangleleft \beta_1}(\mathbf{b}'))) \\ &\quad + \lambda_{\alpha_1, \beta_1} f(a_0) \cdot f(b_0) \cdot \bar{f}(P_{\alpha_1, \beta_1}(\mathbf{a}' \diamond \mathbf{b}')) \\ &= f(a_0) \cdot f(b_0) \cdot \bar{f}(P_{\alpha_1}(\mathbf{a}') P_{\beta_1}(\mathbf{b}')) \\ &= f(a_0) \cdot f(b_0) \cdot \bar{f}(P_{\alpha_1}(\mathbf{a}')) \cdot \bar{f}(P_{\beta_1}(\mathbf{b}')) \quad (\text{by the induction hypothesis}) \\ &= f(a_0) \cdot f(b_0) \cdot P_{\alpha_1} \circ \bar{f}(\mathbf{a}') \cdot P_{\beta_1} \circ \bar{f}(\mathbf{b}') \\ &= f(a_0) \cdot P_{\alpha_1} \circ \bar{f}(\mathbf{a}') \cdot f(b_0) \cdot P_{\beta_1} \circ \bar{f}(\mathbf{b}') \quad (\text{as } B \text{ is commutative}) \\ &= \bar{f}(\mathbf{a}) \cdot \bar{f}(\mathbf{b}). \end{aligned}$$

We get that it is the unique way to extend  $f$  as an  $\Omega$ -Rota-Baxter algebra morphism. Hence  $\text{Sh}'_{\Omega}(A)$  is the free commutative  $\Omega$ -Rota-Baxter algebra generated by  $A$ .  $\square$

### 3. MORE RESULTS ON $\lambda$ -ETS AND ETS

**3.1. Description in terms of linear and bilinear maps.** As in Lemma 5 of [5], we obtain:

**Lemma 3.1.** *Let  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot)$  be a set with five operations and  $\lambda = (\lambda_{\alpha, \beta})_{\alpha, \beta \in \Omega}$  be a family of elements in  $\mathbf{k}$  indexed by  $\Omega^2$ . We denote by  $\mathbf{k}\Omega$  the vector space generated by  $\Omega$ . We put:*

$$\begin{aligned} \varphi_{\leftarrow} : & \begin{cases} \mathbf{k}\Omega^{\otimes 2} & \longrightarrow & \mathbf{k}\Omega^{\otimes 2} \\ \alpha \otimes \beta & \longrightarrow & \alpha \leftarrow \beta \otimes \alpha \triangleleft \beta, \end{cases} \\ \varphi_{\rightarrow} : & \begin{cases} \mathbf{k}\Omega^{\otimes 2} & \longrightarrow & \mathbf{k}\Omega^{\otimes 2} \\ \alpha \otimes \beta & \longrightarrow & \alpha \rightarrow \beta \otimes \alpha \triangleright \beta, \end{cases} \\ \psi : & \begin{cases} \mathbf{k}\Omega^{\otimes 2} & \longrightarrow & \mathbf{k}\Omega \\ \alpha \otimes \beta & \longrightarrow & \lambda_{\alpha, \beta} \alpha \cdot \beta. \end{cases} \end{aligned}$$

Then  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, \lambda)$  is a  $\lambda$ -ETS if, and only if:

$$\begin{aligned}
(49) \quad & (\tau \otimes \text{id}) \circ (\text{id} \otimes \varphi_{\leftarrow}) \circ (\tau \otimes \text{id}) \circ (\varphi_{\rightarrow} \otimes \text{id}) = (\varphi_{\rightarrow} \otimes \text{id}) \circ (\text{id} \otimes \varphi_{\leftarrow}), \\
(50) \quad & (\text{id} \otimes \varphi_{\leftarrow}) \circ (\tau \otimes \text{id}) \circ (\text{id} \otimes \varphi_{\leftarrow}) \circ (\tau \otimes \text{id}) \circ (\varphi_{\leftarrow} \otimes \text{id}) = (\varphi_{\leftarrow} \otimes \text{id}) \circ (\text{id} \otimes \varphi_{\leftarrow}), \\
(51) \quad & (\text{id} \otimes \varphi_{\rightarrow}) \circ (\tau \otimes \text{id}) \circ (\text{id} \otimes \varphi_{\leftarrow}) \circ (\tau \otimes \text{id}) \circ (\varphi_{\leftarrow} \otimes \text{id}) = (\varphi_{\leftarrow} \otimes \text{id}) \circ (\text{id} \otimes \varphi_{\rightarrow}), \\
(52) \quad & (\text{id} \otimes \varphi_{\leftarrow}) \circ (\varphi_{\rightarrow} \otimes \text{id}) \circ (\text{id} \otimes \varphi_{\rightarrow}) = (\varphi_{\rightarrow} \otimes \text{id}) \circ (\text{id} \otimes \tau) \circ (\varphi_{\leftarrow} \otimes \text{id}), \\
(53) \quad & (\text{id} \otimes \varphi_{\rightarrow}) \circ (\varphi_{\rightarrow} \otimes \text{id}) \circ (\text{id} \otimes \varphi_{\rightarrow}) = (\varphi_{\rightarrow} \otimes \text{id}) \circ (\text{id} \otimes \tau) \circ (\varphi_{\rightarrow} \otimes \text{id}), \\
(54) \quad & \varphi_{\rightarrow} \circ (\text{id} \otimes \psi) = (\psi \otimes \text{id}) \circ (\text{id} \otimes \tau) \circ (\varphi_{\rightarrow} \otimes \text{id}), \\
(55) \quad & (\psi \otimes \text{id}) \circ (\text{id} \otimes \tau) \circ (\text{id} \otimes \varphi_{\leftarrow}) \circ (\tau \otimes \text{id}) \circ (\varphi_{\leftarrow} \otimes \text{id}) = \tau \circ \varphi_{\leftarrow} \circ (\text{id} \otimes \psi), \\
(56) \quad & (\text{id} \otimes \psi) \circ (\varphi_{\rightarrow} \otimes \text{id}) \circ (\text{id} \otimes \varphi_{\rightarrow}) = \varphi_{\rightarrow} \circ (\psi \otimes \text{id}), \\
(57) \quad & (\psi \otimes \text{id}) \circ (\text{id} \otimes \tau) \circ (\varphi_{\leftarrow} \otimes \text{id}) = (\psi \otimes \text{id}) \circ (\text{id} \otimes \varphi_{\rightarrow}), \\
(58) \quad & (\psi \otimes \text{id}) \circ (\text{id} \otimes \varphi_{\leftarrow}) = \varphi_{\leftarrow} \circ (\psi \otimes \text{id}), \\
(59) \quad & \psi \circ (\psi \otimes \text{id}) = \psi \circ (\text{id} \otimes \psi).
\end{aligned}$$

In particular,  $\psi$  is an associative product.

*Proof.* By Lemma 5 in [5], Eqs. (49)-(53) are equivalent to  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright)$  being an EDS. Moreover, direct computations prove that Eq. (54) is equivalent to Eq. (11) and condition (a); Eq. (55) is equivalent to Eq. (12) and condition (b); Eq. (56) is equivalent to Eq. (13) and condition (c); Eq. (57) is equivalent to Eq. (14) and condition (d); Eq. (58) is equivalent to Eq. (15) and condition (e); Eq. (59) is equivalent to Eq. (16) and condition (f) in Definition 2.3.  $\square$

Similarly, we obtain for ETS:

**Lemma 3.2.** Let  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, *, \cdot)$  be a set with six operations. We put:

$$\begin{aligned}
\varphi_{\leftarrow} &: \begin{cases} \Omega^2 & \longrightarrow \Omega^2 \\ (\alpha, \beta) & \longrightarrow (\alpha \leftarrow \beta, \alpha \triangleleft \beta), \end{cases} \\
\varphi_{\rightarrow} &: \begin{cases} \Omega^2 & \longrightarrow \Omega^2 \\ (\alpha, \beta) & \longrightarrow (\alpha \rightarrow \beta, \alpha \triangleright \beta), \end{cases} \\
\varphi_* &: \begin{cases} \Omega^2 & \longrightarrow \Omega^2 \\ (\alpha, \beta) & \longrightarrow (\alpha \cdot \beta, \alpha * \beta). \end{cases}
\end{aligned}$$

Then  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, *, \cdot)$  is an ETS if, and only if, (34)-(38) of [5] are satisfied and:

$$\begin{aligned}
(60) \quad & (\varphi_{\rightarrow} \otimes \text{id}) \circ (\text{id} \otimes \varphi_*) = (\tau \otimes \text{id}) \circ (\text{id} \otimes \varphi_*) \circ (\tau \otimes \text{id}) \circ (\varphi_{\rightarrow} \otimes \text{id}), \\
(61) \quad & (\varphi_{\leftarrow} \otimes \text{id}) \circ (\text{id} \otimes \varphi_*) = (\text{id} \otimes \varphi_*) \circ (\tau \otimes \text{id}) \circ (\text{id} \otimes \varphi_{\leftarrow}) \circ (\tau \otimes \text{id}) \circ (\varphi_{\leftarrow} \otimes \text{id}), \\
(62) \quad & (\text{id} \otimes \varphi_*) \circ (\varphi_{\rightarrow} \otimes \text{id}) \circ (\text{id} \otimes \varphi_{\rightarrow}) = (\varphi_{\rightarrow} \otimes \text{id}) \circ (\text{id} \otimes \tau) \circ (\text{id} \otimes \varphi_*), \\
(63) \quad & (\text{id} \otimes \varphi_*) \circ (\tau \otimes \text{id}) \circ (\varphi_{\leftarrow} \otimes \text{id}) = (\text{id} \otimes \varphi_*) \circ (\tau \otimes \text{id}) \circ (\text{id} \otimes \tau) \circ (\text{id} \otimes \varphi_{\rightarrow}), \\
(64) \quad & (\varphi_* \otimes \text{id}) \circ (\text{id} \otimes \varphi_{\leftarrow}) = (\tau \otimes \text{id}) \circ (\text{id} \otimes \varphi_{\leftarrow}) \circ (\tau \otimes \text{id}) \circ (\varphi_* \otimes \text{id}), \\
(65) \quad & (\varphi_* \otimes \text{id}) \circ (\text{id} \otimes \tau) \circ (\varphi_* \otimes \text{id}) = (\text{id} \otimes \tau) \circ (\varphi_* \otimes \text{id}) \circ (\text{id} \otimes \varphi_*).
\end{aligned}$$

*Proof.* By Lemma 5 in [5], Eqs. (34)-(38) are equivalent to  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright)$  being an EDS. Moreover, direct computations prove that Eq. (60) is equivalent to Eqs. (17), (18) and (28); Eq. (61) is equivalent to Eqs. (19), (20) and (29); Eq. (62) is equivalent to Eqs. (21), (22) and (30); Eq. (63) is equivalent to Eqs. (23), (24) and (31); Eq. (64) is equivalent to Eqs. (25), (26) and (32); Eq. (65) is equivalent to Eqs. (27), (33) and (34).  $\square$

3.2. **A description of all  $\lambda$ -ETS of cardinality two.** The following table gives all  $\lambda$ -ETS. We slightly generalize our definition, by accepting more general maps  $\varphi. : \mathbf{k}\Omega^{\otimes 2} \rightarrow \mathbf{k}\Omega$ . The underlying set is  $\{a, b\}$  and all the products are given by a  $2 \times 2$  table. Here,  $\lambda, \mu$  are elements of the base field  $\mathbf{k}$ .

Type	$\leftarrow$	$\rightarrow$	$\triangleleft$	$\triangleright$	$\varphi_*$	Name
A	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} (\lambda + \mu)a & (\lambda + \mu)a \\ (\lambda + \mu)a & \lambda a + \mu b \end{pmatrix}$	$A_1(\lambda, \mu)$
			$\begin{pmatrix} a & b \\ a & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ b & b \end{pmatrix}$	$\begin{pmatrix} (\lambda + \mu)a & (\lambda + \mu)a \\ (\lambda + \mu)a & \lambda a + \mu b \end{pmatrix}$	$A_2(\lambda, \mu)$
B	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} a & b \\ a & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} \lambda a & \lambda a \\ \lambda a & \lambda a \end{pmatrix}, \begin{pmatrix} \lambda a & \lambda b \\ \lambda a & \lambda b \end{pmatrix}$	$B'_1(\lambda), B''_1(\lambda)$
			$\begin{pmatrix} a & b \\ a & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ b & b \end{pmatrix}$		$B'_2(\lambda), B''_2(\lambda)$
C	$\begin{pmatrix} a & a \\ a & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} \lambda a & \lambda a \\ \lambda a & \lambda b \end{pmatrix}$	$C_1(\lambda)$
			$\begin{pmatrix} a & b \\ a & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ b & b \end{pmatrix}$		$C_3(\lambda)$
			$\begin{pmatrix} b & b \\ b & b \end{pmatrix}$	$\begin{pmatrix} b & b \\ b & b \end{pmatrix}$		$C_5(\lambda)$
			$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} b & b \\ b & b \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$C_2$
			$\begin{pmatrix} b & b \\ b & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$		$C_4$

Type	$\leftarrow$	$\rightarrow$	$\triangleleft$	$\triangleright$	$\varphi_*$	Name
$D$	$\begin{pmatrix} a & a \\ b & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} \lambda a & \lambda a \\ \lambda a & \lambda a \end{pmatrix}, \begin{pmatrix} \lambda a & \lambda a \\ \lambda b & \lambda b \end{pmatrix}$	$D'_1(\lambda), D''_1(\lambda)$
			$\begin{pmatrix} a & b \\ a & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ b & b \end{pmatrix}$		$D'_2(\lambda), D''_2(\lambda)$
$E$	$\begin{pmatrix} a & a \\ b & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ b & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} \lambda a & \lambda a \\ \lambda b & \lambda b \end{pmatrix}$	$E_1(\lambda)$
			$\begin{pmatrix} a & b \\ a & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ b & b \end{pmatrix}$		$E_3(\lambda)$
			$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} b & b \\ b & b \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$E_2$
$F$	$\begin{pmatrix} a & a \\ b & b \end{pmatrix}$	$\begin{pmatrix} a & b \\ a & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} (\lambda + \mu)a & (\lambda + \mu)a \\ (\lambda + \mu)a & \lambda a + \mu b \end{pmatrix}, \begin{pmatrix} (\lambda + \mu)a & (\lambda + \mu)b \\ (\lambda + \mu)b & \lambda a + \mu b \end{pmatrix},$ $\begin{pmatrix} \lambda a & \lambda b \\ \lambda a & \lambda b \end{pmatrix}, \begin{pmatrix} \lambda a & \lambda a \\ \lambda b & \lambda b \end{pmatrix}$	$F'_1(\lambda, \mu), F''_1(\lambda, \mu)$ $F'_1(\lambda), F''_1(\lambda)$
			$\begin{pmatrix} a & b \\ a & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ b & b \end{pmatrix}$		any associative product *
			$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$	$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$	$\begin{pmatrix} \lambda a & 0 \\ 0 & \lambda b \end{pmatrix}$	$F_4(\lambda)$
			$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} b & b \\ b & b \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$F_2$
			$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$	$\begin{pmatrix} b & a \\ a & b \end{pmatrix}$		$F_5$
$G$	$\begin{pmatrix} a & b \\ a & b \end{pmatrix}$	$\begin{pmatrix} a & b \\ a & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} \lambda a & \lambda b \\ \lambda a & \lambda b \end{pmatrix}$	$G_1(\lambda)$
			$\begin{pmatrix} a & b \\ a & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ b & b \end{pmatrix}$		$G_3(\lambda)$
			$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} b & b \\ b & b \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$G_2$
$H$	$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$	$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} \lambda a & \lambda b \\ \lambda b & \lambda a \end{pmatrix}$	$H_1(\lambda)$
			$\begin{pmatrix} a & b \\ a & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ b & b \end{pmatrix}$		$H_2(\lambda)$

The commutative  $\lambda$ -ETS are the ones of type  $A$  and  $H$ ,  $C_1(\lambda)$ ,  $C_3(\lambda)$ ,  $C_5(\lambda)$ ,  $F'_1(\lambda, \mu)$ ,  $F''_1(\lambda, \mu)$  and  $F_4(\lambda)$ . The opposite of  $B'_1(\lambda)$ ,  $B''_1(\lambda)$ ,  $B'_2(\lambda)$  and  $B''_2(\lambda)$  are respectively  $D'_1(\lambda)$ ,  $D''_1(\lambda)$ ,  $D'_2(\lambda)$  and  $D''_2(\lambda)$ . The opposite of  $C_2$  is  $C_4$ . The opposite of  $E_1(\lambda)$ ,  $E_2$  and  $E_3(\lambda)$  are respectively  $G_1(\lambda)$ ,  $G_2$  and  $G_3(\lambda)$ . The opposite of  $F'_1(\lambda)$  is  $F''_1(\lambda)$ . The  $\lambda$ -ETS  $F_2$  and  $F_5$  are not commutative but are isomorphic to their opposite in a non trivial way. Finally, if  $*$  is an associative product, the opposite of  $F_3(*)$  is  $F_3(*^{op})$ .

**3.3. A description of all ETS of cardinality two.** The following table gives all the ETS of cardinality 2.



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