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TYPED ANGULARLY DECORATED PLANAR ROOTED TREES AND GENERALIZED ROTA-BAXTER ALGEBRAS

LOÏC FOISSY AND XIAO-SONG PENG

ABSTRACT. We introduce a generalization of parametrized Rota-Baxter algebras, which includes family and matching Rota-Baxter algebras. We study the structure needed on the set Ω of parameters in order to obtain that free Rota-Baxter algebras are described in terms of typed and angularly decorated planar rooted trees: we obtain the notion of λ -extended diassociative semigroup, which includes sets (for matching Rota-Baxter algebras) and semigroups (for family Rota-Baxter algebras), and many other examples. We also describe free commutative Ω -Rota-Baxter algebras generated by a commutative algebra A in terms of typed words.

CONTENTS

1. Introduction	1
2. Ω -Rota Baxter algebras	3
2.1. Definitions	3
2.2. Ω -Rota Baxter algebras on typed angularly decorated planar rooted trees	7
2.3. Commutative Ω -Rota Baxter algebras on typed words	16
3. More results on λ -ETS and ETS	20
3.1. Description in terms of linear and bilinear maps	20
3.2. A description of all λ -ETS of cardinality two	21
3.3. A description of all ETS of cardinality two	23
References	25

1. INTRODUCTION

A Rota-Baxter algebra is an associative algebra A with a linear endomorphism $P : A \rightarrow A$, such that for any $a, b \in A$,

$$P(a)P(b) = P(aP(b)) + P(P(a)b) + \lambda P(ab),$$

where λ is a scalar called the weight of the Rota-Baxter operator P . Firstly introduced by Baxter [1] in a context of probability theory and popularized by Rota [7, 8, 9], they now appear in numerous fields of mathematics and physics, see for example [3] for examples and more details.

The first appearance of family Rota-Baxter algebras seems to be in [2], in the context of Renormalization in Quantum Field Theories. This terminology, due to Li Guo [6] refers to an associative algebra A with a family of linear endomorphisms $P_\alpha : A \rightarrow A$ indexed by the elements of a

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semigroup $(\Omega, *)$, such that for any $a, b \in A$, for any $\alpha, \beta \in \Omega$,

$$P_\alpha(a)P_\beta(b) = P_{\alpha*\beta}(P_\alpha(a)b + aP_\beta(b) + \lambda ab).$$

This notion of matching Rota-Baxter algebra is introduced in [10]. This time, the Rota-Baxter operators are indexed by the elements of a set Ω with no structure, and the weights are given by a family of scalars $(\lambda_\alpha)_{\alpha \in \Omega}$. For any $a, b \in A$, for any $\alpha, \beta \in \Omega$,

$$P_\alpha(a)P_\beta(b) = P_\beta(P_\alpha(a)b) + P_\alpha(aP_\beta(b)) + \lambda_\beta P_\alpha(ab).$$

These notions have been extended to other types of algebras (Lie, pre-Lie, dendriform. . .), see for example [10, 11, 12, 13].

Our aim here is a generalization of both family and matching Rota-Baxter algebras, in the spirit of what is made in [5] for dendriform algebras. We here consider that the set of parameters Ω is given five operations $\leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot$ and a family of scalars $\lambda = (\lambda_{\alpha,\beta})_{\alpha,\beta \in \Omega}$. An Ω -Rota-Baxter algebra of weight λ is an associative algebra A with a family of linear endomorphisms indexed by Ω such that for any $a, b \in A$, for any $\alpha, \beta \in \Omega$,

$$P_\alpha(a)P_\beta(b) = P_{\alpha \rightarrow \beta}(P_{\alpha \triangleright \beta}(a)b) + P_{\alpha \leftarrow \beta}(aP_{\alpha \triangleleft \beta}(b)) + \lambda_{\alpha,\beta} P_{\alpha,\beta}(ab).$$

Taking

$$\alpha \rightarrow \beta = \alpha \leftarrow \beta = \alpha \cdot \beta = \alpha * \beta, \quad \alpha \triangleright \beta = \alpha, \quad \alpha \triangleleft \beta = \beta,$$

and $\lambda_{\alpha,\beta}$ constant, we recover in this way family Rota-Baxter algebras. Taking

$$\alpha \rightarrow \beta = \beta, \quad \alpha \leftarrow \beta = \alpha, \quad \alpha \cdot \beta = \alpha, \quad \alpha \triangleright \beta = \alpha, \quad \alpha \triangleleft \beta = \beta,$$

and $\lambda_{\alpha,\beta}$ depending only on β , we recover matching Rota-Baxter algebras.

We impose that free Ω -Rota-Baxter algebras own a description in terms of angularly decorated (by the set of generators) and typed (by Ω) planar rooted trees as done in [13], that is to say in terms of planar rooted trees with angles decorated by the generators and internal edges decorated by elements of Ω , with an inductive description of the associative product and the Rota-Baxter operators being given by the grafting on a new root, the created internal edge begin of the required type. We show in Theorem 2.14 that this imposes strong constraints on Ω : we obtain that it should be a λ -ETS, as defined in Definition 2.3. In particular, $(\Omega, \leftarrow, \rightarrow)$ has to be a diassociative semigroup: for any $\alpha, \beta, \gamma \in \Omega$,

$$\begin{aligned} (\alpha \leftarrow \beta) \leftarrow \gamma &= \alpha \leftarrow (\beta \leftarrow \gamma) = \alpha \leftarrow (\beta \rightarrow \gamma), \\ (\alpha \rightarrow \beta) \leftarrow \gamma &= \alpha \rightarrow (\beta \leftarrow \gamma), \\ (\alpha \rightarrow \beta) \rightarrow \gamma &= (\alpha \leftarrow \beta) \rightarrow \gamma = \alpha \rightarrow (\beta \rightarrow \gamma). \end{aligned}$$

Moreover, $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright)$ is an extended semigroup (see Definition 2.2 below), a notion used in [5] for parametrization of dendriform algebras. Particular examples of λ -ETS attached to a set give matching Rota-Baxter algebras (see Example 2.4-(b), with $\psi.(\alpha \otimes \beta) = \lambda\alpha$) and particular examples of λ -ETS attached to a semigroup gives family Rota-Baxter algebras (see Example 2.4-(c)). In the case of weight 0, we obtain the generalization of the result [3] establishing that any Rota-Baxter of weight 0 is a dendriform algebra, see Proposition 2.11. Moreover, generalizing the construction of free commutative Rota-Baxter algebras, we obtain that free commutative Ω -Rota-Baxter algebras can be described in terms of Ω -typed words (Proposition 2.18 and Theorem 2.20).

This paper is organised as follows. The first section introduces the definitions of EDS, λ -ETS, ETS and of Ω -Rota-Baxter algebras. The main result on free Rota-Baxter algebras and λ -ETS

is then proved (Theorem 2.14), with a description of free Ω -Rota-Baxter algebras in terms of trees. The last subsection deals with commutative Ω -Rota-Baxter algebras and their description in terms of typed words (Theorem 2.20). The second section gives more examples of λ -ETS and ETS, and in particular a classification of these objects of cardinality 2.

Notation. Throughout this paper, let \mathbf{k} be a unitary commutative ring which will be the base ring of all modules, algebras, as well as linear maps, unless otherwise specified.

2. Ω -ROTA BAXTER ALGEBRAS

2.1. Definitions. We first recall the definition of diassociative semigroups and extended diassociative semigroups of [5], where these objects was used for parametrized versions of dendriform algebras.

Definition 2.1. [5, Definition 1] A **diassociative semigroup** is a family $(\Omega, \leftarrow, \rightarrow)$, where Ω is a set and $\leftarrow, \rightarrow: \Omega \times \Omega \rightarrow \Omega$ are maps such that

$$\begin{aligned} (\alpha \leftarrow \beta) \leftarrow \gamma &= \alpha \leftarrow (\beta \leftarrow \gamma) = \alpha \leftarrow (\beta \rightarrow \gamma), \\ (\alpha \rightarrow \beta) \leftarrow \gamma &= \alpha \rightarrow (\beta \leftarrow \gamma), \\ (\alpha \rightarrow \beta) \rightarrow \gamma &= (\alpha \leftarrow \beta) \rightarrow \gamma = \alpha \rightarrow (\beta \rightarrow \gamma), \end{aligned}$$

for all $\alpha, \beta, \gamma \in \Omega$.

Definition 2.2. [5, Definition 2] An **extended diassociative semigroup** (abbr. EDS) is a family $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright)$, where Ω is a set and $\leftarrow, \rightarrow, \triangleleft, \triangleright: \Omega \times \Omega \rightarrow \Omega$ such that $(\Omega, \leftarrow, \rightarrow)$ is a diassociative semigroup and

$$\begin{aligned} (1) \quad & \alpha \triangleright (\beta \leftarrow \gamma) = \alpha \triangleright \beta, \\ (2) \quad & (\alpha \rightarrow \beta) \triangleleft \gamma = \beta \triangleleft \gamma, \\ (3) \quad & (\alpha \triangleleft \beta) \leftarrow ((\alpha \leftarrow \beta) \triangleleft \gamma) = \alpha \triangleleft (\beta \leftarrow \gamma), \\ (4) \quad & (\alpha \triangleleft \beta) \triangleleft ((\alpha \leftarrow \beta) \triangleleft \gamma) = \beta \triangleleft \gamma, \\ (5) \quad & (\alpha \triangleleft \beta) \rightarrow ((\alpha \leftarrow \beta) \triangleleft \gamma) = \alpha \triangleleft (\beta \rightarrow \gamma), \\ (6) \quad & (\alpha \triangleleft \beta) \triangleright ((\alpha \leftarrow \beta) \triangleleft \gamma) = \beta \triangleright \gamma, \\ (7) \quad & (\alpha \triangleright (\beta \rightarrow \gamma)) \leftarrow (\beta \triangleright \gamma) = (\alpha \leftarrow \beta) \triangleright \gamma, \\ (8) \quad & (\alpha \triangleright (\beta \rightarrow \gamma)) \triangleleft (\beta \triangleright \gamma) = \alpha \triangleleft \beta, \\ (9) \quad & (\alpha \triangleright (\beta \rightarrow \gamma)) \rightarrow (\beta \triangleright \gamma) = (\alpha \rightarrow \beta) \triangleright \gamma, \\ (10) \quad & (\alpha \triangleright (\beta \rightarrow \gamma)) \triangleright (\beta \triangleright \gamma) = \alpha \triangleright \beta, \end{aligned}$$

for all $\alpha, \beta, \gamma \in \Omega$.

We shall use here the notion of λ -extended triassociative semigroup, where a family of scalars plays the role of weights.

Definition 2.3. An **λ -extended triassociative semigroup** (abbr. λ -ETS) is a family $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, *, \lambda)$, where $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright)$ is an EDS and $\lambda = (\lambda_{\alpha\beta})_{\alpha, \beta \in \Omega}$ is a family of elements in \mathbf{k} indexed by Ω^2 such that

$$(11) \quad \lambda_{\alpha \rightarrow \beta, \gamma} = \lambda_{\beta, \gamma}$$

$$(12) \quad \lambda_{\alpha \triangleleft \beta, (\alpha \leftarrow \beta) \triangleleft \gamma} = \lambda_{\beta, \gamma}$$

$$(13) \quad \lambda_{\alpha \leftarrow \beta, \gamma} = \lambda_{\alpha, \beta \rightarrow \gamma}$$

$$(14) \quad \lambda_{\alpha \triangleright (\beta \rightarrow \gamma), \beta \triangleright \gamma} = \lambda_{\alpha, \beta}$$

$$(15) \quad \lambda_{\alpha, \beta} = \lambda_{\alpha, \beta \leftarrow \gamma}$$

$$(16) \quad \lambda_{\alpha, \beta} \lambda_{\alpha, \beta, \gamma} = \lambda_{\beta, \gamma} \lambda_{\alpha, \beta \cdot \gamma}$$

and, for all $\alpha, \beta, \gamma \in \Omega$:

(a) If $\lambda_{\alpha \rightarrow \beta, \gamma} = \lambda_{\beta, \gamma} \neq 0$, then

$$(17) \quad \alpha \triangleright \beta = \alpha \triangleright (\beta \cdot \gamma),$$

$$(18) \quad (\alpha \rightarrow \beta) \cdot \gamma = \alpha \rightarrow (\beta \cdot \gamma).$$

(b) If $\lambda_{\alpha \triangleleft \beta, (\alpha \leftarrow \beta) \triangleleft \gamma} = \lambda_{\beta, \gamma} \neq 0$, then

$$(19) \quad (\alpha \triangleleft \beta) \cdot ((\alpha \leftarrow \beta) \triangleleft \gamma) = \alpha \triangleleft (\beta \cdot \gamma),$$

$$(20) \quad (\alpha \leftarrow \beta) \leftarrow \gamma = \alpha \leftarrow (\beta \cdot \gamma).$$

(c) If $\lambda_{\alpha \triangleright (\beta \rightarrow \gamma), \beta \triangleright \gamma} = \lambda_{\alpha, \beta} \neq 0$, then

$$(21) \quad \alpha \rightarrow (\beta \rightarrow \gamma) = (\alpha \cdot \beta) \rightarrow \gamma,$$

$$(22) \quad (\alpha \triangleright (\beta \rightarrow \gamma)) \cdot (\beta \triangleright \gamma) = (\alpha \cdot \beta) \triangleright \gamma.$$

(d) If $\lambda_{\alpha \leftarrow \beta, \gamma} = \lambda_{\alpha, \beta \rightarrow \gamma} \neq 0$, then

$$(23) \quad (\alpha \leftarrow \beta) \cdot \gamma = \alpha \cdot (\beta \rightarrow \gamma),$$

$$(24) \quad \alpha \triangleleft \beta = \beta \triangleright \gamma.$$

(e) If $\lambda_{\alpha, \beta} = \lambda_{\alpha, \beta \leftarrow \gamma} \neq 0$, then

$$(25) \quad (\alpha \cdot \beta) \triangleleft \gamma = \beta \triangleleft \gamma,$$

$$(26) \quad (\alpha \cdot \beta) \leftarrow \gamma = \alpha \cdot (\beta \leftarrow \gamma).$$

(f) If $\lambda_{\alpha, \beta} \lambda_{\alpha, \beta, \gamma} = \lambda_{\beta, \gamma} \lambda_{\alpha, \beta \cdot \gamma} \neq 0$, then

$$(27) \quad (\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma).$$

Example 2.4. (a) Let $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright)$ be an EDS. If we put $\lambda_{\alpha, \beta} = 0$ for any $\alpha, \beta \in \Omega$, then for any product \cdot , $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, \lambda)$ is a λ -ETS.

(b) If for any $\alpha, \beta \in \Omega$,

$$\alpha \leftarrow \beta = \beta \rightarrow \alpha = \beta \triangleleft \alpha = \alpha \triangleright \beta = \alpha,$$

then $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, \lambda)$ is a λ -ETS if, and only if, the following map defines an associative product:

$$\psi : \begin{cases} \mathbf{k}\Omega \otimes \mathbf{k}\Omega & \longrightarrow & \mathbf{k}\Omega \\ \alpha \otimes \beta & \longrightarrow & \lambda_{\alpha, \beta} \alpha \cdot \beta. \end{cases}$$

Indeed, for any $\alpha, \beta, \gamma \in \Omega$,

$$\psi \circ (\psi \otimes \text{id})(\alpha \otimes \beta \otimes \gamma) = \lambda_{\alpha, \beta} \lambda_{\alpha, \beta, \gamma} (\alpha \cdot \beta) \cdot \gamma,$$

$$\psi \circ (\text{id} \otimes \psi)(\alpha \otimes \beta \otimes \gamma) = \lambda_{\beta, \gamma} \lambda_{\alpha, \beta \cdot \gamma} \alpha \cdot (\beta \cdot \gamma),$$

which gives the missing condition (27).

(c) Let (Ω, \star) be a semigroup and $\lambda \in \mathbf{k}$. We put, for any $\alpha, \beta \in \Omega$:

$$\begin{aligned} \alpha \leftarrow \beta &= \alpha \star \beta, & \alpha \triangleleft \beta &= \beta, \\ \alpha \rightarrow \beta &= \alpha \star \beta, & \alpha \triangleright \beta &= \alpha, \\ \lambda_{\alpha, \beta} &= \lambda, & \alpha \cdot \beta &= \alpha \star \beta. \end{aligned}$$

Then $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, \lambda)$ is a λ -ETS.

(d) Let (Ω, \star) be an abelian group and let $\lambda \in \mathbf{k}$. For any $\alpha, \beta \in \Omega$, we put:

$$\begin{aligned} \alpha \leftarrow \beta &= \alpha, & \alpha \rightarrow \beta &= \beta, \\ \alpha \triangleleft \beta &= \alpha \star \beta^{\star^{-1}}, & \alpha \triangleright \beta &= \alpha^{\star^{-1}} \star \beta, \\ \lambda_{\alpha, \beta} &= \lambda \delta_{\alpha, \beta}, & \alpha \cdot \beta &= \alpha. \end{aligned}$$

Then $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, \lambda)$ is a λ -ETS.

(e) Let $\Omega = (\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, \lambda)$ be a λ -ETS. For any $\alpha, \beta \in \Omega$, we put

$$\begin{aligned} \alpha \leftarrow^{op} \beta &= \beta \rightarrow \alpha, & \alpha \triangleleft^{op} \alpha &= \beta \triangleright \alpha, \\ \alpha \rightarrow^{op} \beta &= \beta \leftarrow \alpha, & \alpha \triangleright^{op} \alpha &= \beta \triangleleft \alpha, \\ \alpha \cdot^{op} \beta &= \beta \cdot \alpha, & \lambda_{\alpha, \beta}^{op} &= \lambda_{\beta, \alpha}. \end{aligned}$$

Then $(\Omega, \leftarrow^{op}, \rightarrow^{op}, \triangleleft^{op}, \triangleright^{op}, \cdot^{op}, \lambda^{op})$ is also a λ -ETS, called the **opposite** of Ω and denoted by Ω^{op} . We shall say that Ω is commutative if it is equal to its opposite.

Definition 2.5. A **extended triassociative semigroup** (abbr. ETS) is a family $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, *)$, where $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright)$ is an EDS and

- (28) $(\alpha \rightarrow \beta) * \gamma = \beta * \gamma,$
- (17) $(\alpha \rightarrow \beta) \cdot \gamma = \alpha \rightarrow (\beta \cdot \gamma),$
- (18) $\alpha \triangleright \beta = \alpha \triangleright (\beta \cdot \gamma),$
- (29) $(\alpha \triangleleft \beta) * ((\alpha \leftarrow \beta) \triangleleft \gamma) = \beta * \gamma,$
- (19) $(\alpha \triangleleft \beta) \cdot ((\alpha \leftarrow \beta) \triangleleft \gamma) = \alpha \triangleleft (\beta \cdot \gamma),$
- (20) $(\alpha \leftarrow \beta) \leftarrow \gamma = \alpha \leftarrow (\beta \cdot \gamma),$
- (30) $(\alpha \triangleright (\beta \rightarrow \gamma)) * (\beta \triangleright \gamma) = \alpha * \beta,$
- (21) $\alpha \rightarrow (\beta \rightarrow \gamma) = (\alpha \cdot \beta) \rightarrow \gamma,$
- (22) $(\alpha \triangleright (\beta \rightarrow \gamma)) \cdot (\beta \triangleright \gamma) = (\alpha \cdot \beta) \triangleright \gamma,$
- (31) $(\alpha \leftarrow \beta) * \gamma = \alpha * (\beta \rightarrow \gamma),$
- (23) $(\alpha \leftarrow \beta) \cdot \gamma = \alpha \cdot (\beta \rightarrow \gamma),$
- (24) $\alpha \triangleleft \beta = \beta \triangleright \gamma,$
- (32) $\alpha * \beta = \alpha * (\beta \leftarrow \gamma),$
- (25) $(\alpha \cdot \beta) \triangleleft \gamma = \beta \triangleleft \gamma,$
- (26) $(\alpha \cdot \beta) \leftarrow \gamma = \alpha \cdot (\beta \leftarrow \gamma),$
- (33) $\alpha * \beta = \alpha * (\beta \cdot \gamma),$
- (34) $(\alpha \cdot \beta) * \gamma = \beta * \gamma,$
- (27) $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma).$

Example 2.6. (a) Let $(\Omega, *, \cdot)$ be a set with two products such that for any $\alpha, \beta, \gamma \in \Omega$:

$$(35) \quad \alpha * \beta = \alpha * (\beta \cdot \gamma),$$

$$(36) \quad (\alpha \cdot \beta) * \gamma = \beta * \gamma,$$

$$(37) \quad (\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma).$$

We put, for any $\alpha, \beta \in \Omega$:

$$\alpha \leftarrow \beta = \alpha, \quad \alpha \triangleleft \beta = \beta,$$

$$\alpha \rightarrow \beta = \alpha, \quad \alpha \triangleright \beta = \beta.$$

Then $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, *)$ is a ETS.

(b) Let $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, *)$ be a ETS. For any $\alpha, \beta \in \Omega$, we put

$$\alpha \leftarrow^{op} \beta = \beta \rightarrow \alpha, \quad \alpha \triangleleft^{op} \alpha = \beta \triangleright \alpha,$$

$$\alpha \rightarrow^{op} \beta = \beta \leftarrow \alpha, \quad \alpha \triangleright^{op} \alpha = \beta \triangleleft \alpha,$$

$$\alpha *^{op} \beta = \beta * \alpha, \quad \alpha \cdot^{op} \beta = \beta \cdot \alpha.$$

Then $(\Omega, \leftarrow^{op}, \rightarrow^{op}, \triangleleft^{op}, \triangleright^{op}, *^{op}, \cdot^{op})$ is also a ETS, called the opposite of Ω . We shall say that Ω is commutative if it is equal to its opposite.

Actually, each ETS induces a λ -ETS, as the following result indicates:

Proposition 2.7. Let $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, *)$ be a ETS and let $(\mu_\alpha)_{\alpha \in \Omega}$ be a family of scalars. For any $\alpha, \beta \in \Omega$, we put:

$$\lambda_{\alpha, \beta} = \mu_{\alpha * \beta}.$$

Then $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, \lambda)$ is a λ -ETS.

Proof. Conditions (a)-(f) are obviously satisfied by (17)-(27). (11) is (28), (12) is (29), (13) is (31), (14) is (30), (15) is (32), and (16) comes from (33) and (34). \square

We now propose the concept of Ω -Rota-Baxter algebras as follows:

Definition 2.8. Let Ω be a set with five products $\leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot$ and $\lambda = (\lambda_{\alpha, \beta})_{\alpha, \beta \in \Omega}$ be a family of elements in \mathbf{k} indexed by Ω^2 . An Ω -Rota-Baxter algebra of weight λ is a family $(A, (P_\omega)_{\omega \in \Omega})$ where A is an associative algebra and $P_\omega : A \otimes A \rightarrow A$ is a linear map for each $\omega \in \Omega$, such that

$$P_\alpha(a)P_\beta(b) = P_{\alpha \rightarrow \beta}(P_{\alpha \triangleright \beta}(a)b) + P_{\alpha \leftarrow \beta}(aP_{\alpha \triangleleft \beta}(b)) + \lambda_{\alpha, \beta}P_{\alpha \cdot \beta}(ab),$$

for all $a, b \in A$ and $\alpha, \beta \in \Omega$. If, further, A is commutative, then $(A, (P_\omega)_{\omega \in \Omega})$ is a **commutative Ω -Rota-Baxter algebras**.

Taking all elements of λ to be 0, we get the concept of Ω -Rota-Baxter algebras of weight 0:

Definition 2.9. Let Ω be a set with four products $\leftarrow, \rightarrow, \triangleleft, \triangleright$. An Ω -Rota-Baxter algebra of weight 0 is a family $(A, (P_\omega)_{\omega \in \Omega})$ where A is an associative algebra and $P_\omega : A \otimes A \rightarrow A$ is a linear map for each $\omega \in \Omega$, such that

$$P_\alpha(a)P_\beta(b) = P_{\alpha \rightarrow \beta}(P_{\alpha \triangleright \beta}(a)b) + P_{\alpha \leftarrow \beta}(aP_{\alpha \triangleleft \beta}(b)),$$

for all $a, b \in A$ and $\alpha, \beta \in \Omega$.

Example 2.10. (a) If (Ω, \star) is a semigroup, we recover the definition of Rota-Baxter family algebras [6, 12] by defining

$$\alpha \leftarrow \beta = \alpha \rightarrow \beta = \alpha \cdot \beta = \alpha \star \beta, \quad \alpha \triangleright \beta = \alpha, \quad \alpha \triangleleft \beta = \beta,$$

and requiring all elements of λ to be equal. Note that this is the λ -ETS of Example 2.4 (c).

(b) For a set Ω , define

$$\alpha \rightarrow \beta = \alpha \triangleleft \beta = \beta, \quad \alpha \triangleright \beta = \alpha \leftarrow \beta = \alpha \cdot \beta = \alpha,$$

and $\lambda_{\alpha,\beta} = \lambda_\alpha$, for a family $(\lambda_\alpha)_{\alpha \in \Omega}$ of elements of \mathbf{k} . Then we get the concept of matching Rota-Baxter algebra [11], up to the change of the product of A into its opposite.

As we know, Rota-Baxter algebras of weight 0 induce dendriform algebras [3]. Similarly, we can show that each Ω -Rota-Baxter algebra of weight 0 has a structure of an Ω -dendriform algebra [5, definition 11]:

Proposition 2.11. *Let Ω be a set with four products $\leftarrow, \rightarrow, \triangleleft, \triangleright$ and $(A, (P_\omega)_{\omega \in \Omega})$ an Ω -Rota-Baxter algebra of weight 0. Then $(A, (<_\omega)_{\omega \in \Omega}, (>_\omega)_{\omega \in \Omega})$ is an Ω -dendriform algebra, where*

$$a <_\omega b := aP_\omega(b), \quad a >_\omega b := P_\omega(a)b,$$

for all $a, b \in A$ and $\omega \in \Omega$.

Proof. For $a, b, c \in A$ and $\alpha, \beta \in \Omega$,

$$\begin{aligned} (a <_\alpha b) <_\beta c &= (aP_\alpha(b))P_\beta(c) = a(P_\alpha(b)P_\beta(c)) = a(P_{\alpha \rightarrow \beta}(P_{\alpha \triangleright \beta}(b)c) + P_{\alpha \leftarrow \beta}(bP_{\alpha \triangleleft \beta}(c))) \\ &= aP_{\alpha \rightarrow \beta}(P_{\alpha \triangleright \beta}(b)c) + aP_{\alpha \leftarrow \beta}(bP_{\alpha \triangleleft \beta}(c)) = a <_{\alpha \rightarrow \beta} (b >_{\alpha \triangleright \beta} c) + a <_{\alpha \leftarrow \beta} (b <_{\alpha \triangleleft \beta} c), \\ a >_\alpha (b <_\beta c) &= P_\alpha(a)(bP_\beta(c)) = (P_\alpha(a)b)P_\beta(c) = (a >_\alpha b) <_\beta c, \\ a >_\alpha (b >_\beta c) &= P_\alpha(a)(P_\beta(b)c) = (P_\alpha(a)P_\beta(b))c = (P_{\alpha \rightarrow \beta}(P_{\alpha \triangleright \beta}(a)b) + P_{\alpha \leftarrow \beta}(aP_{\alpha \triangleleft \beta}(b)))c \\ &= P_{\alpha \rightarrow \beta}(P_{\alpha \triangleright \beta}(a)b)c + P_{\alpha \leftarrow \beta}(aP_{\alpha \triangleleft \beta}(b))c = (a >_{\alpha \triangleright \beta} b) >_{\alpha \rightarrow \beta} c + (a <_{\alpha \triangleleft \beta} b) >_{\alpha \leftarrow \beta} c. \quad \square \end{aligned}$$

2.2. Ω -Rota Baxter algebras on typed angularly decorated planar rooted trees. First, let us recall some notations on planar rooted trees (see [13] for more details). For a planar rooted tree T , we shall consider the root and the leaves of T as edges rather than vertices. Denote by $IE(T)$ the set of internal edges of T , i.e. edges which are neither leaves nor the root and denote by $V(T)$ the set of vertices of T . For each vertex v yields a (possibly empty) set of angles $A(v)$, an angle being a pair (e, e') of adjacent incoming edges for v . Let $A(T) = \bigsqcup_{v \in V(T)} A(v)$ be the set of angles of

T . Then:

Definition 2.12. [13, Definition 2.2] Let X and Ω be two sets. An X -angularly decorated Ω -typed (abbr. **typed angularly decorated**) planar rooted tree is a triple $T = (T, \text{dec}, \text{type})$, where T is a planar rooted tree, $\text{dec} : A(T) \rightarrow X$ and $\text{type} : IE(T) \rightarrow \Omega$ are maps.

For $n \geq 0$, let $\mathcal{T}_n(X, \Omega)$ denote the set of X -angularly decorated Ω -typed planar rooted trees with $n+1$ leaves and at least one internal vertex such that internal edges are decorated by elements of Ω . We put

$$\mathcal{T}(X, \Omega) := \bigsqcup_{n \geq 0} \mathcal{T}_n(X, \Omega) \quad \text{and} \quad \mathbf{k}\mathcal{T}(X, \Omega) := \bigoplus_{n \geq 0} \mathbf{k}\mathcal{T}_n(X, \Omega).$$

For example,

$$\mathcal{T}_0(X, \Omega) = \left\{ \begin{array}{c} | \\ | \\ | \end{array} \right\}, \left\{ \begin{array}{c} | \\ | \\ \alpha \end{array} \right\}, \left\{ \begin{array}{c} | \\ | \\ \beta \end{array} \right\}, \dots \left\{ \begin{array}{c} | \\ | \\ \alpha, \beta, \dots \in \Omega \end{array} \right\},$$

$$\mathcal{T}_1(X, \Omega) = \left\{ \begin{array}{c} \diagup \diagdown \\ x \quad \alpha \\ \downarrow \\ \alpha \end{array}, \begin{array}{c} \diagup \diagdown \\ x \quad \alpha \\ \downarrow \\ \alpha \end{array}, \begin{array}{c} \diagup \diagdown \\ \alpha \quad x \\ \downarrow \\ \beta \end{array}, \begin{array}{c} \diagup \diagdown \\ \alpha \quad x \\ \downarrow \\ \beta \end{array}, \begin{array}{c} \diagup \diagdown \\ \beta \quad x \\ \downarrow \\ \gamma \end{array}, \begin{array}{c} \diagup \diagdown \\ \beta \quad x \\ \downarrow \\ \gamma \end{array}, \dots \mid x \in X, \alpha, \beta, \gamma, \dots \in \Omega \right\},$$

$$\mathcal{T}_2(X, \Omega) = \left\{ \begin{array}{c} \diagup \diagdown \\ x \quad y \\ \downarrow \\ \beta \end{array}, \begin{array}{c} \diagup \diagdown \\ x \quad y \\ \downarrow \\ \alpha \end{array}, \begin{array}{c} \diagup \diagdown \\ y \quad x \\ \downarrow \\ \alpha \end{array}, \begin{array}{c} \diagup \diagdown \\ x \quad y \\ \downarrow \\ \alpha \end{array}, \begin{array}{c} \diagup \diagdown \\ x \quad y \\ \downarrow \\ \alpha \end{array}, \begin{array}{c} \diagup \diagdown \\ x \quad y \\ \downarrow \\ \alpha \end{array}, \dots \mid x, y \in X, \alpha, \beta, \gamma, \dots \in \Omega \right\},$$

Graphically, an element $T \in \mathcal{T}(X, \Omega)$ is of the form:

$$T = T_1 \circ_{\alpha_1} \begin{array}{c} \begin{array}{c} \circ \alpha_2 \\ \diagup \diagdown \\ x_1 \quad \dots \quad x_n \\ \downarrow \\ \alpha_{n+1} \end{array} \\ \diagup \diagdown \\ \alpha_1 \quad \alpha_{n+1} \end{array} \circ_{\alpha_{n+1}} T_{n+1}, \text{ with } n \geq 0, \text{ where } x_1, \dots, x_n \in X, \alpha_i \in \Omega \text{ if } T_i \neq | \text{ and otherwise}$$

α_i does not exist for $1 \leq i \leq n+1$.

For each $\omega \in \Omega$, there is a grafting operator $B_\omega^+ : \mathbf{k}\mathcal{T}(X, \Omega) \rightarrow \mathbf{k}\mathcal{T}(X, \Omega)$ which add a new root to a tree and an new internal edge typed by ω between the new root and the root of the tree.

For example,

$$B_\omega^+ \left(\begin{array}{c} \downarrow \\ \alpha \end{array} \right) = \begin{array}{c} \downarrow \\ \omega \\ \downarrow \\ \alpha \end{array}, \quad B_\omega^+ \left(\begin{array}{c} \diagup \diagdown \\ x \quad \alpha \\ \downarrow \\ \alpha \end{array} \right) = \begin{array}{c} \diagup \diagdown \\ x \quad \alpha \\ \downarrow \\ \omega \\ \downarrow \\ \alpha \end{array}.$$

The **depth** $\text{dep}(T)$ of a rooted tree T is the maximal length of linear chains from the root to the leaves of the tree. For example,

$$\text{dep} \left(\begin{array}{c} \downarrow \\ \alpha \end{array} \right) = \text{dep} \left(\begin{array}{c} \diagup \diagdown \\ x \quad \alpha \\ \downarrow \\ \alpha \end{array} \right) = 1 \quad \text{and} \quad \text{dep} \left(\begin{array}{c} \downarrow \\ \omega \end{array} \right) = \text{dep} \left(\begin{array}{c} \diagup \diagdown \\ x \quad y \\ \downarrow \\ \alpha \\ \downarrow \\ \alpha \end{array} \right) = 2.$$

We also consider the trivial tree $|$ and put by convention $\text{dep}(|) := 0$. For each typed angularly decorated planar rooted tree T , define the number of branches of T to be $\text{bra}(T) = 0$ if $T = |$. Otherwise, $\text{dep}(T) \geq 1$ and T is of the form

$$T = T_1 \circ_{\alpha_1} \begin{array}{c} \begin{array}{c} \circ \alpha_2 \\ \diagup \diagdown \\ x_1 \quad \dots \quad x_n \\ \downarrow \\ \alpha_{n+1} \end{array} \\ \diagup \diagdown \\ \alpha_1 \quad \alpha_{n+1} \end{array} \circ_{\alpha_{n+1}} T_{n+1} \text{ with } n \geq 0,$$

where $T_j \in \mathcal{T}(X, \Omega) \sqcup \{|\}$, $j = 1, \dots, n+1$. We define $\text{bra}(T) := n+1$. For example,

$$\text{bra} \left(\begin{array}{c} \downarrow \\ \alpha \end{array} \right) = 1, \quad \text{bra} \left(\begin{array}{c} \diagup \diagdown \\ x \quad y \\ \downarrow \\ \alpha \end{array} \right) = 2 \quad \text{and} \quad \text{bra} \left(\begin{array}{c} \diagup \diagdown \\ x \quad y \\ \downarrow \\ \alpha \end{array} \right) = 3.$$

Let X be a set, $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot)$ be a set with five products, and $\lambda = (\lambda_{\alpha, \beta})_{(\alpha, \beta) \in \Omega^2}$ be a family of elements in \mathbf{k} indexed by Ω^2 . We define a product \diamond on $\mathbf{k}\mathcal{T}(X, \Omega)$ as follows. For $T, T' \in \mathcal{T}(X, \Omega)$, we define $T \diamond T'$ by induction on $\text{dep}(T) + \text{dep}(T') \geq 2$. For the initial step $\text{dep}(T) + \text{dep}(T') = 2$, we have $\text{dep}(T) = \text{dep}(T') = 1$ and T, T' are of the form

$$T = \begin{array}{c} \diagup \dots \diagdown \\ x_1 \quad \dots \quad x_m \\ \downarrow \\ \alpha \end{array} \quad \text{and} \quad T' = \begin{array}{c} \diagup \dots \diagdown \\ y_1 \quad \dots \quad y_n \\ \downarrow \\ \alpha \end{array}, \quad \text{with } m, n \geq 0.$$

Define

$$(38) \quad T \diamond T' := \begin{array}{c} \dots \\ x_1 \dots x_m \\ \vdots \\ \diamond \\ y_1 \dots y_n \\ \vdots \\ x_1 \dots x_m y_1 \dots y_n \end{array}.$$

For the induction step $\text{dep}(T) + \text{dep}(T') \geq 3$, the trees T and T' are of the form

$$T = T_1 \circlearrowleft \begin{array}{c} T_2 \quad T_m \\ \alpha_2 \quad \alpha_m \\ \vdots \\ x_1 \dots x_m \\ \vdots \\ \alpha_{m+1} \end{array} \circlearrowright T_{m+1} \quad \text{and} \quad T' = T'_1 \circlearrowleft \begin{array}{c} T'_2 \quad T'_n \\ \beta_2 \quad \beta_n \\ \vdots \\ y_1 \dots y_n \\ \vdots \\ \beta_{n+1} \end{array} \circlearrowright T'_{n+1} \quad \text{with some } T_i \neq | \text{ or some } T'_j \neq |.$$

There are four cases to consider.

Case 1: $T_{m+1} = | = T'_1$. Define

$$(39) \quad T \diamond T' := T_1 \circlearrowleft \begin{array}{c} T_2 \quad T_m \\ \alpha_2 \quad \alpha_m \\ \vdots \\ x_1 \dots x_m \\ \vdots \\ \alpha_{m+1} \end{array} \diamond \begin{array}{c} T'_2 \quad T'_n \\ \beta_2 \quad \beta_n \\ \vdots \\ y_1 \dots y_n \\ \vdots \\ \beta_{n+1} \end{array} \circlearrowright T'_{n+1} := T_1 \circlearrowleft \begin{array}{c} T_m \quad T'_2 \\ \alpha_m \quad \alpha_{m+1} \quad \beta_2 \\ \vdots \\ x_1 \dots x_m y_1 \dots y_n \\ \vdots \\ \alpha_1 \quad \beta_{n+1} \end{array} \circlearrowright T'_{n+1}.$$

Case 2: $T_{m+1} \neq | = T'_1$. Define

$$(40) \quad T \diamond T' := T_1 \circlearrowleft \begin{array}{c} T_2 \quad T_m \\ \alpha_2 \quad \alpha_m \\ \vdots \\ x_1 \dots x_m \\ \vdots \\ \alpha_{m+1} \end{array} \diamond \begin{array}{c} T'_2 \quad T'_n \\ \beta_2 \quad \beta_n \\ \vdots \\ y_1 \dots y_n \\ \vdots \\ \beta_{n+1} \end{array} \circlearrowright T'_{n+1} := T_1 \circlearrowleft \begin{array}{c} T_m \quad T_{m+1} \quad T'_2 \\ \alpha_m \quad \alpha_{m+1} \quad \beta_2 \\ \vdots \\ x_1 \dots x_m y_1 \dots y_n \\ \vdots \\ \alpha_1 \quad \beta_{n+1} \end{array} \circlearrowright T'_{n+1}.$$

Case 3: $T_{m+1} = | \neq T'_1$. Define

$$(41) \quad T \diamond T' := T_1 \circlearrowleft \begin{array}{c} T_2 \quad T_m \\ \alpha_2 \quad \alpha_m \\ \vdots \\ x_1 \dots x_m \\ \vdots \\ \alpha_{m+1} \end{array} \diamond T'_1 \circlearrowleft \begin{array}{c} T'_2 \quad T'_n \\ \beta_2 \quad \beta_n \\ \vdots \\ y_1 \dots y_n \\ \vdots \\ \beta_{n+1} \end{array} \circlearrowright T'_{n+1} := T_1 \circlearrowleft \begin{array}{c} T_m \quad T'_1 \quad T'_2 \\ \alpha_m \quad \beta_1 \quad \beta_2 \\ \vdots \\ x_1 \dots x_m y_1 \dots y_n \\ \vdots \\ \alpha_1 \quad \beta_{n+1} \end{array} \circlearrowright T'_{n+1}.$$

Case 4: $T_{m+1} \neq | \neq T'_1$. Define

$$(42) \quad \begin{aligned} T \diamond T' &:= T_1 \circlearrowleft \begin{array}{c} T_2 \quad T_m \\ \alpha_2 \quad \alpha_m \\ \vdots \\ x_1 \dots x_m \\ \vdots \\ \alpha_{m+1} \end{array} \diamond T'_1 \circlearrowleft \begin{array}{c} T'_2 \quad T'_n \\ \beta_2 \quad \beta_n \\ \vdots \\ y_1 \dots y_n \\ \vdots \\ \beta_{n+1} \end{array} \circlearrowright T'_{n+1} \\ &:= \left(T_1 \circlearrowleft \begin{array}{c} T_2 \quad T_m \\ \alpha_2 \quad \alpha_m \\ \vdots \\ x_1 \dots x_m \\ \vdots \\ \alpha_{m+1} \end{array} \diamond \left(B_{\alpha_{m+1}}^+ (T_{m+1}) \diamond B_{\beta_1}^+ (T'_1) \right) \right) \diamond \begin{array}{c} T'_2 \quad T'_n \\ \beta_2 \quad \beta_n \\ \vdots \\ y_1 \dots y_n \\ \vdots \\ \beta_{n+1} \end{array} \circlearrowright T'_{n+1} \\ &:= \left(T_1 \circlearrowleft \begin{array}{c} T_2 \quad T_m \\ \alpha_2 \quad \alpha_m \\ \vdots \\ x_1 \dots x_m \\ \vdots \\ \alpha_{m+1} \end{array} \diamond \left(B_{\alpha_{m+1} \rightarrow \beta_1}^+ (B_{\alpha_{m+1} > \beta_1}^+ (T_{m+1}) \diamond T'_1) + B_{\alpha_{m+1} \leftarrow \beta_1}^+ (T_{m+1} \diamond B_{\alpha_{m+1} < \beta_1}^+ (T'_1)) \right) \right) \diamond \begin{array}{c} T'_2 \quad T'_n \\ \beta_2 \quad \beta_n \\ \vdots \\ y_1 \dots y_n \\ \vdots \\ \beta_{n+1} \end{array} \circlearrowright T'_{n+1} \end{aligned}$$

$$+ \lambda_{\alpha_{m+1}, \beta_1} B_{\alpha_{m+1}, \beta_1}^+(T_{m+1} \diamond T'_1) \Big) \diamond \begin{array}{c} T'_2 \quad T'_n \\ \circ \beta_2 \quad \beta_n \\ \swarrow \quad \searrow \\ y_1 \quad y_n \\ \searrow \quad \swarrow \\ \beta_{n+1} \\ \circ T'_{n+1} \end{array} \cdot$$

Here the first \diamond is defined by Case 3, the second, third and fourth \diamond are defined by induction and the last \diamond is defined by Case 2. This inductively define the multiplication \diamond on $\mathcal{T}(X, \Omega)$. We then extend \diamond by linearity to $\mathbf{k}\mathcal{T}(X, \Omega)$. We then have the following result:

Lemma 2.13. *Let $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, \lambda)$ be a λ -ETS. Then $(\mathbf{k}\mathcal{T}(X, \Omega), \diamond)$ is an associative algebra with identity \downarrow .*

Proof. By the construction of \diamond , $\mathbf{k}\mathcal{T}(X, \Omega)$ is closed under \diamond and \downarrow is the identity of \diamond .

Now we show the associativity of \diamond , i.e.

$$(43) \quad (T_1 \diamond T_2) \diamond T_3 = T_1 \diamond (T_2 \diamond T_3),$$

for all $T_1, T_2, T_3 \in \mathcal{T}(X, \Omega)$. We prove Eq. (43) by induction on the sum of depths $p := \text{dep}(T_1) + \text{dep}(T_2) + \text{dep}(T_3)$. If $p = 3$, then $\text{dep}(T_1) = \text{dep}(T_2) = \text{dep}(T_3) = 1$ and T_1, T_2, T_3 are of the form

$$T_1 = \begin{array}{c} \dots \\ \swarrow \quad \searrow \\ x_1 \quad x_l \\ \searrow \\ \downarrow \end{array}, \quad T_2 = \begin{array}{c} \dots \\ \swarrow \quad \searrow \\ y_1 \quad y_m \\ \searrow \\ \downarrow \end{array}, \quad \text{and} \quad T_3 = \begin{array}{c} \dots \\ \swarrow \quad \searrow \\ z_1 \quad z_n \\ \searrow \\ \downarrow \end{array} \quad \text{with } l, m, n \geq 0.$$

Then $(T_1 \diamond T_2) \diamond T_3 = T_1 \diamond (T_2 \diamond T_3)$ by a direct calculation.

For the induction step $p \geq 4$, we use induction on the sum of branches $q := \text{bra}(T_1) + \text{bra}(T_2) + \text{bra}(T_3)$. If $q = 3$ and one of T_1, T_2, T_3 has depth 1, then this tree must be of the form \downarrow and the associativity of \diamond follows directly. Assume

$$T_1 = B_{\alpha}^+(T'_1), \quad T_2 = B_{\beta}^+(T'_2), \quad T_3 = B_{\gamma}^+(T'_3) \quad \text{for some } \alpha, \beta, \gamma \in \Omega \text{ and } T'_1, T'_2, T'_3 \in \mathcal{T}(X, \Omega),$$

then

$$\begin{aligned} & (T_1 \diamond T_2) \diamond T_3 = (B_{\alpha}^+(T'_1) \diamond B_{\beta}^+(T'_2)) \diamond B_{\gamma}^+(T'_3) \\ &= B_{\alpha \triangleright \beta}^+(B_{\alpha \triangleright \beta}^+(T'_1) \diamond T'_2) \diamond B_{\gamma}^+(T'_3) + B_{\alpha \leftarrow \beta}^+(T'_1 \diamond B_{\alpha \leftarrow \beta}^+(T'_2)) \diamond B_{\gamma}^+(T'_3) + \lambda_{\alpha, \beta} B_{\alpha, \beta}^+(T'_1 \diamond T'_2) \diamond B_{\gamma}^+(T'_3) \\ &= B_{(\alpha \rightarrow \beta) \rightarrow \gamma}^+(B_{(\alpha \rightarrow \beta) \triangleright \gamma}^+(B_{\alpha \triangleright \beta}^+(T'_1) \diamond T'_2) \diamond T'_3) + B_{(\alpha \rightarrow \beta) \leftarrow \gamma}^+(B_{\alpha \triangleright \beta}^+(T'_1) \diamond T'_2) \diamond B_{(\alpha \rightarrow \beta) \leftarrow \gamma}^+(T'_3)) \\ &\quad + \lambda_{\alpha \rightarrow \beta, \gamma} B_{(\alpha \rightarrow \beta) \gamma}^+(B_{\alpha \triangleright \beta}^+(T'_1) \diamond T'_2) \diamond T'_3) + B_{(\alpha \leftarrow \beta) \rightarrow \gamma}^+(B_{\alpha \leftarrow \beta}^+(T'_1 \diamond B_{\alpha \leftarrow \beta}^+(T'_2)) \diamond T'_3) \\ &\quad + B_{(\alpha \leftarrow \beta) \leftarrow \gamma}^+(T'_1 \diamond B_{\alpha \leftarrow \beta}^+(T'_2)) \diamond B_{(\alpha \leftarrow \beta) \leftarrow \gamma}^+(T'_3)) + \lambda_{\alpha \leftarrow \beta, \gamma} B_{(\alpha \leftarrow \beta) \gamma}^+(T'_1 \diamond B_{\alpha \leftarrow \beta}^+(T'_2)) \diamond T'_3) \\ &\quad + \lambda_{\alpha, \beta} B_{(\alpha, \beta) \rightarrow \gamma}^+(B_{(\alpha, \beta) \triangleright \gamma}^+(T'_1 \diamond T'_2) \diamond T'_3) + \lambda_{\alpha, \beta} B_{(\alpha, \beta) \leftarrow \gamma}^+(T'_1 \diamond T'_2) \diamond B_{(\alpha, \beta) \leftarrow \gamma}^+(T'_3) \\ &\quad + \lambda_{\alpha, \beta} \lambda_{\alpha, \beta, \gamma} B_{(\alpha, \beta) \gamma}^+(T'_1 \diamond T'_2) \diamond T'_3) \\ &= B_{(\alpha \rightarrow \beta) \rightarrow \gamma}^+(B_{(\alpha \rightarrow \beta) \triangleright \gamma}^+(B_{\alpha \triangleright \beta}^+(T'_1) \diamond T'_2) \diamond T'_3) + B_{(\alpha \rightarrow \beta) \leftarrow \gamma}^+(B_{\alpha \triangleright \beta}^+(T'_1) \diamond T'_2) \diamond B_{(\alpha \rightarrow \beta) \leftarrow \gamma}^+(T'_3)) \\ &\quad + \lambda_{\alpha \rightarrow \beta, \gamma} B_{(\alpha \rightarrow \beta) \gamma}^+(B_{\alpha \triangleright \beta}^+(T'_1) \diamond T'_2) \diamond T'_3) + B_{(\alpha \leftarrow \beta) \rightarrow \gamma}^+(B_{\alpha \leftarrow \beta}^+(T'_1 \diamond B_{\alpha \leftarrow \beta}^+(T'_2)) \diamond T'_3) \\ &\quad + B_{(\alpha \leftarrow \beta) \leftarrow \gamma}^+(T'_1 \diamond B_{\alpha \leftarrow \beta}^+(T'_2) \diamond B_{(\alpha \leftarrow \beta) \leftarrow \gamma}^+(T'_3))) + \lambda_{\alpha \leftarrow \beta, \gamma} B_{(\alpha \leftarrow \beta) \gamma}^+(T'_1 \diamond B_{\alpha \leftarrow \beta}^+(T'_2)) \diamond T'_3) \\ &\quad + \lambda_{\alpha, \beta} B_{(\alpha, \beta) \rightarrow \gamma}^+(B_{(\alpha, \beta) \triangleright \gamma}^+(T'_1 \diamond T'_2) \diamond T'_3) + \lambda_{\alpha, \beta} B_{(\alpha, \beta) \leftarrow \gamma}^+(T'_1 \diamond T'_2) \diamond B_{(\alpha, \beta) \leftarrow \gamma}^+(T'_3) \\ &\quad + \lambda_{\alpha, \beta} \lambda_{\alpha, \beta, \gamma} B_{(\alpha, \beta) \gamma}^+(T'_1 \diamond T'_2) \diamond T'_3) \quad (\text{by induction hypothesis}) \\ &= B_{(\alpha \rightarrow \beta) \rightarrow \gamma}^+(B_{(\alpha \rightarrow \beta) \triangleright \gamma}^+(B_{\alpha \triangleright \beta}^+(T'_1) \diamond T'_2) \diamond T'_3) + B_{(\alpha \rightarrow \beta) \leftarrow \gamma}^+(B_{\alpha \triangleright \beta}^+(T'_1) \diamond T'_2) \diamond B_{(\alpha \rightarrow \beta) \leftarrow \gamma}^+(T'_3)) \\ &\quad + \lambda_{\alpha \rightarrow \beta, \gamma} B_{(\alpha \rightarrow \beta) \gamma}^+(B_{\alpha \triangleright \beta}^+(T'_1) \diamond T'_2) \diamond T'_3) + B_{(\alpha \leftarrow \beta) \rightarrow \gamma}^+(B_{\alpha \leftarrow \beta}^+(T'_1 \diamond B_{\alpha \leftarrow \beta}^+(T'_2)) \diamond T'_3) \\ &\quad + B_{(\alpha \leftarrow \beta) \leftarrow \gamma}^+(T'_1 \diamond B_{(\alpha \leftarrow \beta) \rightarrow ((\alpha \leftarrow \beta) \leftarrow \gamma)}^+(B_{\alpha \leftarrow \beta}^+(T'_2) \diamond T'_3))) \\ &\quad + B_{(\alpha \leftarrow \beta) \leftarrow \gamma}^+(T'_1 \diamond B_{(\alpha \leftarrow \beta) \leftarrow ((\alpha \leftarrow \beta) \leftarrow \gamma)}^+(T'_2 \diamond B_{(\alpha \leftarrow \beta) \leftarrow ((\alpha \leftarrow \beta) \leftarrow \gamma)}^+(T'_3)))) \end{aligned}$$

$$\begin{aligned}
 & + \lambda_{\alpha \triangleleft \beta, (\alpha \leftarrow \beta) \triangleleft \gamma} B_{(\alpha \leftarrow \beta) \leftarrow \gamma}^+(T_1' \diamond B_{(\alpha \triangleleft \beta) \cdot ((\alpha \leftarrow \beta) \triangleleft \gamma)}^+(T_2' \diamond T_3')) + \lambda_{\alpha \leftarrow \beta, \gamma} B_{(\alpha \leftarrow \beta) \cdot \gamma}^+((T_1' \diamond B_{\alpha \triangleleft \beta}^+(T_2')) \diamond T_3') \\
 & + \lambda_{\alpha, \beta} B_{(\alpha \rightarrow \beta) \triangleright \gamma}^+(B_{(\alpha \rightarrow \beta) \triangleright \gamma}^+(T_1' \diamond T_2') \diamond T_3') + \lambda_{\alpha, \beta} B_{(\alpha \rightarrow \beta) \leftarrow \gamma}^+((T_1' \diamond T_2') \diamond B_{(\alpha \rightarrow \beta) \triangleleft \gamma}^+(T_3')) \\
 & + \lambda_{\alpha, \beta} \lambda_{\alpha, \beta, \gamma} B_{(\alpha \rightarrow \beta) \cdot \gamma}^+((T_1' \diamond T_2') \diamond T_3'),
 \end{aligned}$$

and

$$\begin{aligned}
 T_1 \diamond (T_2 \diamond T_3) & = B_{\alpha}^+(T_1') \diamond (B_{\beta}^+(T_2') \diamond B_{\gamma}^+(T_3')) \\
 & = B_{\alpha}^+(T_1') \diamond B_{\beta \triangleright \gamma}^+(B_{\beta \triangleright \gamma}^+(T_2') \diamond T_3') + B_{\alpha}^+(T_1') \diamond B_{\beta \leftarrow \gamma}^+(T_2' \diamond B_{\beta \triangleleft \gamma}^+(T_3')) + \lambda_{\beta, \gamma} B_{\alpha}^+(T_1') \diamond B_{\beta \cdot \gamma}^+(T_2' \diamond T_3') \\
 & = B_{\alpha \rightarrow (\beta \rightarrow \gamma)}^+(B_{\alpha \triangleright (\beta \rightarrow \gamma)}^+(T_1') \diamond (B_{\beta \triangleright \gamma}^+(T_2') \diamond T_3')) + B_{\alpha \leftarrow (\beta \rightarrow \gamma)}^+(T_1' \diamond B_{\alpha \triangleleft (\beta \rightarrow \gamma)}^+(B_{\beta \triangleright \gamma}^+(T_2') \diamond T_3')) \\
 & \quad + \lambda_{\alpha, \beta \rightarrow \gamma} B_{\alpha \cdot (\beta \rightarrow \gamma)}^+(T_1' \diamond (B_{\beta \triangleright \gamma}^+(T_2') \diamond T_3')) + B_{\alpha \rightarrow (\beta \leftarrow \gamma)}^+(B_{\alpha \triangleright (\beta \leftarrow \gamma)}^+(T_1') \diamond (T_2' \diamond B_{\beta \triangleleft \gamma}^+(T_3')))) \\
 & \quad + B_{\alpha \leftarrow (\beta \leftarrow \gamma)}^+(T_1' \diamond B_{\alpha \triangleleft (\beta \leftarrow \gamma)}^+(T_2' \diamond B_{\beta \triangleleft \gamma}^+(T_3')))) + \lambda_{\alpha, \beta \leftarrow \gamma} B_{\alpha \cdot (\beta \leftarrow \gamma)}^+(T_1' \diamond (T_2' \diamond B_{\beta \triangleleft \gamma}^+(T_3')))) \\
 & \quad + \lambda_{\beta, \gamma} B_{\alpha \rightarrow (\beta \cdot \gamma)}^+(B_{\alpha \triangleright (\beta \cdot \gamma)}^+(T_1') \diamond (T_2' \diamond T_3')) + \lambda_{\beta, \gamma} B_{\alpha \leftarrow (\beta \cdot \gamma)}^+(T_1' \diamond B_{\alpha \triangleleft (\beta \cdot \gamma)}^+(T_2' \diamond T_3')) \\
 & \quad + \lambda_{\beta, \gamma} \lambda_{\alpha, \beta, \gamma} B_{\alpha \cdot (\beta \cdot \gamma)}^+(T_1' \diamond (T_2' \diamond T_3')) \\
 & = B_{\alpha \rightarrow (\beta \rightarrow \gamma)}^+((B_{\alpha \triangleright (\beta \rightarrow \gamma)}^+(T_1') \diamond B_{\beta \triangleright \gamma}^+(T_2')) \diamond T_3') + B_{\alpha \leftarrow (\beta \rightarrow \gamma)}^+(T_1' \diamond B_{\alpha \triangleleft (\beta \rightarrow \gamma)}^+(B_{\beta \triangleright \gamma}^+(T_2') \diamond T_3')) \\
 & \quad + \lambda_{\alpha, \beta \rightarrow \gamma} B_{\alpha \cdot (\beta \rightarrow \gamma)}^+(T_1' \diamond (B_{\beta \triangleright \gamma}^+(T_2') \diamond T_3')) + B_{\alpha \rightarrow (\beta \leftarrow \gamma)}^+(B_{\alpha \triangleright (\beta \leftarrow \gamma)}^+(T_1') \diamond (T_2' \diamond B_{\beta \triangleleft \gamma}^+(T_3')))) \\
 & \quad + B_{\alpha \leftarrow (\beta \leftarrow \gamma)}^+(T_1' \diamond B_{\alpha \triangleleft (\beta \leftarrow \gamma)}^+(T_2' \diamond B_{\beta \triangleleft \gamma}^+(T_3')))) + \lambda_{\alpha, \beta \leftarrow \gamma} B_{\alpha \cdot (\beta \leftarrow \gamma)}^+(T_1' \diamond (T_2' \diamond B_{\beta \triangleleft \gamma}^+(T_3')))) \\
 & \quad + \lambda_{\beta, \gamma} B_{\alpha \rightarrow (\beta \cdot \gamma)}^+(B_{\alpha \triangleright (\beta \cdot \gamma)}^+(T_1') \diamond (T_2' \diamond T_3')) + \lambda_{\beta, \gamma} B_{\alpha \leftarrow (\beta \cdot \gamma)}^+(T_1' \diamond B_{\alpha \triangleleft (\beta \cdot \gamma)}^+(T_2' \diamond T_3')) \\
 & \quad + \lambda_{\beta, \gamma} \lambda_{\alpha, \beta, \gamma} B_{\alpha \cdot (\beta \cdot \gamma)}^+(T_1' \diamond (T_2' \diamond T_3')) \quad (\text{by induction hypothesis}) \\
 & = B_{\alpha \rightarrow (\beta \rightarrow \gamma)}^+(B_{(\alpha \triangleright (\beta \rightarrow \gamma)) \rightarrow (\beta \triangleright \gamma)}^+(B_{(\alpha \triangleright (\beta \rightarrow \gamma)) \triangleright (\beta \triangleright \gamma)}^+(T_1') \diamond T_2') \diamond T_3') \\
 & \quad + B_{\alpha \rightarrow (\beta \rightarrow \gamma)}^+(B_{\alpha \triangleright (\beta \rightarrow \gamma) \leftarrow (\beta \triangleright \gamma)}^+(T_1' \diamond B_{(\alpha \triangleright (\beta \rightarrow \gamma)) \triangleleft (\beta \triangleright \gamma)}^+(T_2')) \diamond T_3') \\
 & \quad + \lambda_{\alpha \triangleright (\beta \rightarrow \gamma), \beta \triangleright \gamma} B_{\alpha \rightarrow (\beta \rightarrow \gamma)}^+(B_{(\alpha \triangleright (\beta \rightarrow \gamma)) \cdot (\beta \triangleright \gamma)}^+(T_1' \diamond T_2') \diamond T_3') + B_{\alpha \leftarrow (\beta \rightarrow \gamma)}^+(T_1' \diamond B_{\alpha \triangleleft (\beta \rightarrow \gamma)}^+(B_{\beta \triangleright \gamma}^+(T_2') \diamond T_3')) \\
 & \quad + \lambda_{\alpha, \beta \rightarrow \gamma} B_{\alpha \cdot (\beta \rightarrow \gamma)}^+(T_1' \diamond (B_{\beta \triangleright \gamma}^+(T_2') \diamond T_3')) + B_{\alpha \rightarrow (\beta \leftarrow \gamma)}^+(B_{\alpha \triangleright (\beta \leftarrow \gamma)}^+(T_1') \diamond (T_2' \diamond B_{\beta \triangleleft \gamma}^+(T_3')))) \\
 & \quad + B_{\alpha \leftarrow (\beta \leftarrow \gamma)}^+(T_1' \diamond B_{\alpha \triangleleft (\beta \leftarrow \gamma)}^+(T_2' \diamond B_{\beta \triangleleft \gamma}^+(T_3')))) + \lambda_{\alpha, \beta \leftarrow \gamma} B_{\alpha \cdot (\beta \leftarrow \gamma)}^+(T_1' \diamond (T_2' \diamond B_{\beta \triangleleft \gamma}^+(T_3')))) \\
 & \quad + \lambda_{\beta, \gamma} B_{\alpha \rightarrow (\beta \cdot \gamma)}^+(B_{\alpha \triangleright (\beta \cdot \gamma)}^+(T_1') \diamond (T_2' \diamond T_3')) + \lambda_{\beta, \gamma} B_{\alpha \leftarrow (\beta \cdot \gamma)}^+(T_1' \diamond B_{\alpha \triangleleft (\beta \cdot \gamma)}^+(T_2' \diamond T_3')) \\
 & \quad + \lambda_{\beta, \gamma} \lambda_{\alpha, \beta, \gamma} B_{\alpha \cdot (\beta \cdot \gamma)}^+(T_1' \diamond (T_2' \diamond T_3')).
 \end{aligned}$$

By induction hypothesis and $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, \lambda)$ being a λ -ETS, we get

$$(T_1 \diamond T_2) \diamond T_3 = T_1 \diamond (T_2 \diamond T_3).$$

If $q > 3$, then at least one of T_1, T_2, T_3 have branches greater than or equal to 2. If $\text{bra}(T_1) \geq 2$, then there exist T_1', T_1'' of the form

$$\begin{array}{ccc}
 \begin{array}{c} \bar{T}_2' \quad \bar{T}_m' \\ \circ \quad \circ \\ \alpha_2 \quad \alpha_m \\ \circ \quad \circ \\ \dots \\ \alpha_1 \quad \alpha_m \\ \circ \quad \circ \\ x_1 \quad x_m \\ \circ \\ T_1' \end{array} & \text{and} & \begin{array}{c} \bar{T}_2'' \quad \bar{T}_n'' \\ \circ \quad \circ \\ \beta_2 \quad \beta_n \\ \circ \quad \circ \\ \dots \\ \beta_{n+1} \\ \circ \\ T_1'' \end{array}
 \end{array}$$

such that $T_1 = T_1' \diamond T_1''$. Hence

$$\begin{aligned}
 (T_1 \diamond T_2) \diamond T_3 & = ((T_1' \diamond T_1'') \diamond T_2) \diamond T_3 \\
 & = (T_1' \diamond (T_1'' \diamond T_2)) \diamond T_3 && (\text{by induction hypothesis}) \\
 & = T_1' \diamond ((T_1'' \diamond T_2) \diamond T_3) && (\text{by the form of } T_1' \text{ and the definition of } \diamond) \\
 & = T_1' \diamond (T_1' \diamond (T_2 \diamond T_3)) && (\text{by induction hypothesis}) \\
 & = T_1' \diamond T_1' \diamond (T_2 \diamond T_3) && (\text{by the form of } T_1' \text{ and the definition of } \diamond)
 \end{aligned}$$

$$= T_1 \diamond (T_2 \diamond T_3).$$

If $\text{bra}(T_2) \geq 2$ or $\text{bra}(T_3) \geq 2$, the associativity can be proved similarly. \square

Let $i : X \rightarrow \mathbf{k}\mathcal{T}(X, \Omega)$, $x \mapsto \begin{array}{c} \diagup x \diagdown \\ | \\ \text{Y} \end{array}$ be the natural inclusion. Then

Theorem 2.14. *Let Ω be a set with five products $\leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot$ and λ a family of elements in \mathbf{k} indexed by Ω^2 . Then the following conditions are equivalent:*

- (a) $(\mathbf{k}\mathcal{T}(X, \Omega), \diamond, (B_\omega^+)_{\omega \in \Omega})$ together with the map i is the free Ω -Rota-Baxter algebra generated by X .
- (b) $(\mathbf{k}\mathcal{T}(X, \Omega), \diamond, (B_\omega^+)_{\omega \in \Omega})$ is an Ω -Rota-Baxter algebra.
- (c) $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, \lambda)$ is a λ -ETS.

Proof. (a) \implies (b) It is obvious.

(b) \implies (c) For $\alpha, \beta, \gamma \in \Omega$ and $\begin{array}{c} \diagup x \diagdown \\ | \\ \text{Y} \end{array}, \begin{array}{c} \diagup y \diagdown \\ | \\ \text{Y} \end{array}, \begin{array}{c} \diagup z \diagdown \\ | \\ \text{Y} \end{array} \in \mathcal{T}(X, \Omega)$, we have

$$\begin{aligned}
& (B_\alpha^+(\begin{array}{c} \diagup x \diagdown \\ | \\ \text{Y} \end{array}) \diamond B_\beta^+(\begin{array}{c} \diagup y \diagdown \\ | \\ \text{Y} \end{array})) \diamond B_\gamma^+(\begin{array}{c} \diagup z \diagdown \\ | \\ \text{Y} \end{array}) \\
= & B_{\alpha \rightarrow \beta}^+(B_{\alpha \triangleright \beta}^+(\begin{array}{c} \diagup x \diagdown \\ | \\ \text{Y} \end{array}) \diamond \begin{array}{c} \diagup y \diagdown \\ | \\ \text{Y} \end{array}) \diamond B_\gamma^+(\begin{array}{c} \diagup z \diagdown \\ | \\ \text{Y} \end{array}) + B_{\alpha \leftarrow \beta}^+(\begin{array}{c} \diagup x \diagdown \\ | \\ \text{Y} \end{array}) \diamond B_{\alpha \triangleleft \beta}^+(\begin{array}{c} \diagup y \diagdown \\ | \\ \text{Y} \end{array}) \diamond B_\gamma^+(\begin{array}{c} \diagup z \diagdown \\ | \\ \text{Y} \end{array}) \\
& + \lambda_{\alpha, \beta} B_{\alpha \cdot \beta}^+(\begin{array}{c} \diagup x \diagdown \\ | \\ \text{Y} \end{array}) \diamond \begin{array}{c} \diagup y \diagdown \\ | \\ \text{Y} \end{array}) \diamond B_\gamma^+(\begin{array}{c} \diagup z \diagdown \\ | \\ \text{Y} \end{array}) \\
= & B_{(\alpha \rightarrow \beta) \rightarrow \gamma}^+(B_{(\alpha \rightarrow \beta) \triangleright \gamma}^+(B_{\alpha \triangleright \beta}^+(\begin{array}{c} \diagup x \diagdown \\ | \\ \text{Y} \end{array}) \diamond \begin{array}{c} \diagup y \diagdown \\ | \\ \text{Y} \end{array}) \diamond \begin{array}{c} \diagup z \diagdown \\ | \\ \text{Y} \end{array}) \\
& + B_{(\alpha \rightarrow \beta) \leftarrow \gamma}^+((B_{\alpha \triangleright \beta}^+(\begin{array}{c} \diagup x \diagdown \\ | \\ \text{Y} \end{array}) \diamond \begin{array}{c} \diagup y \diagdown \\ | \\ \text{Y} \end{array}) \diamond B_{(\alpha \rightarrow \beta) \triangleleft \gamma}^+(\begin{array}{c} \diagup z \diagdown \\ | \\ \text{Y} \end{array})) \\
& + \lambda_{\alpha \rightarrow \beta, \gamma} B_{(\alpha \rightarrow \beta) \cdot \gamma}^+((B_{\alpha \triangleright \beta}^+(\begin{array}{c} \diagup x \diagdown \\ | \\ \text{Y} \end{array}) \diamond \begin{array}{c} \diagup y \diagdown \\ | \\ \text{Y} \end{array}) \diamond \begin{array}{c} \diagup z \diagdown \\ | \\ \text{Y} \end{array}) \\
& + B_{(\alpha \leftarrow \beta) \rightarrow \gamma}^+(B_{(\alpha \leftarrow \beta) \triangleright \gamma}^+(\begin{array}{c} \diagup x \diagdown \\ | \\ \text{Y} \end{array}) \diamond B_{\alpha \triangleleft \beta}^+(\begin{array}{c} \diagup y \diagdown \\ | \\ \text{Y} \end{array})) \diamond \begin{array}{c} \diagup z \diagdown \\ | \\ \text{Y} \end{array}) \\
& + B_{(\alpha \leftarrow \beta) \leftarrow \gamma}^+((\begin{array}{c} \diagup x \diagdown \\ | \\ \text{Y} \end{array}) \diamond B_{\alpha \triangleleft \beta}^+(\begin{array}{c} \diagup y \diagdown \\ | \\ \text{Y} \end{array})) \diamond B_{(\alpha \leftarrow \beta) \triangleleft \gamma}^+(\begin{array}{c} \diagup z \diagdown \\ | \\ \text{Y} \end{array})) \\
& + \lambda_{\alpha \leftarrow \beta, \gamma} B_{(\alpha \leftarrow \beta) \cdot \gamma}^+((\begin{array}{c} \diagup x \diagdown \\ | \\ \text{Y} \end{array}) \diamond B_{\alpha \triangleleft \beta}^+(\begin{array}{c} \diagup y \diagdown \\ | \\ \text{Y} \end{array})) \diamond \begin{array}{c} \diagup z \diagdown \\ | \\ \text{Y} \end{array}) \\
& + \lambda_{\alpha, \beta} B_{(\alpha \cdot \beta) \rightarrow \gamma}^+(B_{(\alpha \cdot \beta) \triangleright \gamma}^+(\begin{array}{c} \diagup x \diagdown \\ | \\ \text{Y} \end{array}) \diamond \begin{array}{c} \diagup y \diagdown \\ | \\ \text{Y} \end{array}) \diamond \begin{array}{c} \diagup z \diagdown \\ | \\ \text{Y} \end{array}) \\
& + \lambda_{\alpha, \beta} B_{(\alpha \cdot \beta) \leftarrow \gamma}^+((\begin{array}{c} \diagup x \diagdown \\ | \\ \text{Y} \end{array}) \diamond \begin{array}{c} \diagup y \diagdown \\ | \\ \text{Y} \end{array}) \diamond B_{(\alpha \cdot \beta) \triangleleft \gamma}^+(\begin{array}{c} \diagup z \diagdown \\ | \\ \text{Y} \end{array})) \\
& + \lambda_{\alpha, \beta} \lambda_{\alpha \cdot \beta, \gamma} B_{(\alpha \cdot \beta) \cdot \gamma}^+((\begin{array}{c} \diagup x \diagdown \\ | \\ \text{Y} \end{array}) \diamond \begin{array}{c} \diagup y \diagdown \\ | \\ \text{Y} \end{array}) \diamond \begin{array}{c} \diagup z \diagdown \\ | \\ \text{Y} \end{array}) \\
= & B_{(\alpha \rightarrow \beta) \rightarrow \gamma}^+(B_{(\alpha \rightarrow \beta) \triangleright \gamma}^+(B_{\alpha \triangleright \beta}^+(\begin{array}{c} \diagup x \diagdown \\ | \\ \text{Y} \end{array}) \diamond \begin{array}{c} \diagup y \diagdown \\ | \\ \text{Y} \end{array}) \diamond \begin{array}{c} \diagup z \diagdown \\ | \\ \text{Y} \end{array}) \\
& + B_{(\alpha \rightarrow \beta) \leftarrow \gamma}^+((B_{\alpha \triangleright \beta}^+(\begin{array}{c} \diagup x \diagdown \\ | \\ \text{Y} \end{array}) \diamond \begin{array}{c} \diagup y \diagdown \\ | \\ \text{Y} \end{array}) \diamond B_{(\alpha \rightarrow \beta) \triangleleft \gamma}^+(\begin{array}{c} \diagup z \diagdown \\ | \\ \text{Y} \end{array})) \\
& + \lambda_{\alpha \rightarrow \beta, \gamma} B_{(\alpha \rightarrow \beta) \cdot \gamma}^+((B_{\alpha \triangleright \beta}^+(\begin{array}{c} \diagup x \diagdown \\ | \\ \text{Y} \end{array}) \diamond \begin{array}{c} \diagup y \diagdown \\ | \\ \text{Y} \end{array}) \diamond \begin{array}{c} \diagup z \diagdown \\ | \\ \text{Y} \end{array}) \\
& + B_{(\alpha \leftarrow \beta) \rightarrow \gamma}^+(B_{(\alpha \leftarrow \beta) \triangleright \gamma}^+(\begin{array}{c} \diagup x \diagdown \\ | \\ \text{Y} \end{array}) \diamond B_{\alpha \triangleleft \beta}^+(\begin{array}{c} \diagup y \diagdown \\ | \\ \text{Y} \end{array})) \diamond \begin{array}{c} \diagup z \diagdown \\ | \\ \text{Y} \end{array}) \\
& + B_{(\alpha \leftarrow \beta) \leftarrow \gamma}^+(\begin{array}{c} \diagup x \diagdown \\ | \\ \text{Y} \end{array}) \diamond B_{(\alpha \triangleleft \beta) \rightarrow ((\alpha \leftarrow \beta) \triangleleft \gamma)}^+(B_{(\alpha \triangleleft \beta) \triangleright ((\alpha \leftarrow \beta) \triangleleft \gamma)}^+(\begin{array}{c} \diagup y \diagdown \\ | \\ \text{Y} \end{array}) \diamond \begin{array}{c} \diagup z \diagdown \\ | \\ \text{Y} \end{array})) \\
& + B_{(\alpha \leftarrow \beta) \leftarrow \gamma}^+(\begin{array}{c} \diagup x \diagdown \\ | \\ \text{Y} \end{array}) \diamond B_{(\alpha \triangleleft \beta) \leftarrow ((\alpha \leftarrow \beta) \triangleleft \gamma)}^+(\begin{array}{c} \diagup y \diagdown \\ | \\ \text{Y} \end{array}) \diamond B_{(\alpha \triangleleft \beta) \triangleleft ((\alpha \leftarrow \beta) \triangleleft \gamma)}^+(\begin{array}{c} \diagup z \diagdown \\ | \\ \text{Y} \end{array})))
\end{aligned}$$

$$\begin{aligned}
& + \lambda_{\alpha \triangleright (\beta \rightarrow \gamma), \beta \triangleright \gamma} B_{\alpha \rightarrow (\beta \rightarrow \gamma)}^+ (B_{\alpha \triangleright (\beta \rightarrow \gamma), (\beta \triangleright \gamma)}^+ (\begin{array}{c} x \\ \diagdown \quad \diagup \\ \diamond \end{array} \begin{array}{c} y \\ \diagdown \quad \diagup \\ \diamond \end{array}) \diamond \begin{array}{c} z \\ \diagdown \quad \diagup \\ \diamond \end{array}) \\
& + B_{\alpha \leftarrow (\beta \rightarrow \gamma)}^+ (\begin{array}{c} x \\ \diagdown \quad \diagup \\ \diamond \end{array} \diamond B_{\alpha \triangleleft (\beta \rightarrow \gamma)}^+ (B_{\beta \triangleright \gamma}^+ (\begin{array}{c} y \\ \diagdown \quad \diagup \\ \diamond \end{array}) \diamond \begin{array}{c} z \\ \diagdown \quad \diagup \\ \diamond \end{array})) \\
& + \lambda_{\alpha, \beta \rightarrow \gamma} B_{\alpha \cdot (\beta \rightarrow \gamma)}^+ (\begin{array}{c} x \\ \diagdown \quad \diagup \\ \diamond \end{array} \diamond (B_{\beta \triangleright \gamma}^+ (\begin{array}{c} y \\ \diagdown \quad \diagup \\ \diamond \end{array}) \diamond \begin{array}{c} z \\ \diagdown \quad \diagup \\ \diamond \end{array})) \\
& + B_{\alpha \rightarrow (\beta \leftarrow \gamma)}^+ (B_{\alpha \triangleright (\beta \leftarrow \gamma)}^+ (\begin{array}{c} x \\ \diagdown \quad \diagup \\ \diamond \end{array}) \diamond (\begin{array}{c} y \\ \diagdown \quad \diagup \\ \diamond \end{array} \diamond B_{\beta \triangleleft \gamma}^+ (\begin{array}{c} z \\ \diagdown \quad \diagup \\ \diamond \end{array}))) \\
& + B_{\alpha \leftarrow (\beta \leftarrow \gamma)}^+ (\begin{array}{c} x \\ \diagdown \quad \diagup \\ \diamond \end{array} \diamond B_{\alpha \triangleleft (\beta \leftarrow \gamma)}^+ (\begin{array}{c} y \\ \diagdown \quad \diagup \\ \diamond \end{array} \diamond B_{\beta \triangleleft \gamma}^+ (\begin{array}{c} z \\ \diagdown \quad \diagup \\ \diamond \end{array}))) \\
& + \lambda_{\alpha, \beta \leftarrow \gamma} B_{\alpha \cdot (\beta \leftarrow \gamma)}^+ (\begin{array}{c} x \\ \diagdown \quad \diagup \\ \diamond \end{array} \diamond (\begin{array}{c} y \\ \diagdown \quad \diagup \\ \diamond \end{array} \diamond B_{\beta \triangleleft \gamma}^+ (\begin{array}{c} z \\ \diagdown \quad \diagup \\ \diamond \end{array}))) \\
& + \lambda_{\beta, \gamma} B_{\alpha \rightarrow (\beta \cdot \gamma)}^+ (B_{\alpha \triangleright (\beta \cdot \gamma)}^+ (\begin{array}{c} x \\ \diagdown \quad \diagup \\ \diamond \end{array}) \diamond (\begin{array}{c} y \\ \diagdown \quad \diagup \\ \diamond \end{array} \diamond \begin{array}{c} z \\ \diagdown \quad \diagup \\ \diamond \end{array})) \\
& + \lambda_{\beta, \gamma} B_{\alpha \leftarrow (\beta \cdot \gamma)}^+ (\begin{array}{c} x \\ \diagdown \quad \diagup \\ \diamond \end{array} \diamond B_{\alpha \leftarrow (\beta \cdot \gamma)}^+ (\begin{array}{c} y \\ \diagdown \quad \diagup \\ \diamond \end{array} \diamond \begin{array}{c} z \\ \diagdown \quad \diagup \\ \diamond \end{array})) \\
& + \lambda_{\beta, \gamma} \lambda_{\alpha, \beta \cdot \gamma} B_{\alpha \cdot (\beta \cdot \gamma)}^+ (\begin{array}{c} x \\ \diagdown \quad \diagup \\ \diamond \end{array} \diamond (\begin{array}{c} y \\ \diagdown \quad \diagup \\ \diamond \end{array} \diamond \begin{array}{c} z \\ \diagdown \quad \diagup \\ \diamond \end{array})) \\
& = \begin{array}{c} \begin{array}{c} x \\ \diagdown \quad \diagup \\ \beta \triangleright \gamma \\ \alpha \rightarrow (\beta \rightarrow \gamma) \end{array} \begin{array}{c} y \\ \diagdown \quad \diagup \\ \beta \triangleright \gamma \\ \alpha \rightarrow (\beta \rightarrow \gamma) \end{array} \begin{array}{c} z \\ \diagdown \quad \diagup \\ \beta \triangleright \gamma \\ \alpha \rightarrow (\beta \rightarrow \gamma) \end{array} \\ + \begin{array}{c} \begin{array}{c} x \\ \diagdown \quad \diagup \\ \alpha \triangleright (\beta \rightarrow \gamma) \end{array} \begin{array}{c} y \\ \diagdown \quad \diagup \\ \alpha \triangleright (\beta \rightarrow \gamma) \end{array} \begin{array}{c} z \\ \diagdown \quad \diagup \\ \alpha \triangleright (\beta \rightarrow \gamma) \end{array} \\ + \lambda_{\alpha \triangleright (\beta \rightarrow \gamma), \beta \triangleright \gamma} (\alpha \triangleright (\beta \rightarrow \gamma)) \cdot (\beta \triangleright \gamma) \\ \alpha \rightarrow (\beta \rightarrow \gamma) \end{array} \\
& + \begin{array}{c} \begin{array}{c} y \\ \diagdown \quad \diagup \\ \beta \triangleright \gamma \\ \alpha \leftarrow (\beta \rightarrow \gamma) \end{array} \begin{array}{c} z \\ \diagdown \quad \diagup \\ \beta \triangleright \gamma \\ \alpha \leftarrow (\beta \rightarrow \gamma) \end{array} \\ + \lambda_{\alpha, \beta \rightarrow \gamma} \begin{array}{c} \begin{array}{c} z \\ \diagdown \quad \diagup \\ \alpha \cdot (\beta \rightarrow \gamma) \end{array} \\ + \alpha \triangleright (\beta \leftarrow \gamma) \begin{array}{c} \begin{array}{c} x \\ \diagdown \quad \diagup \\ \beta \triangleleft \gamma \\ \alpha \rightarrow (\beta \leftarrow \gamma) \end{array} \begin{array}{c} y \\ \diagdown \quad \diagup \\ \beta \triangleleft \gamma \\ \alpha \rightarrow (\beta \leftarrow \gamma) \end{array} \\ + \begin{array}{c} \begin{array}{c} x \\ \diagdown \quad \diagup \\ \alpha \triangleleft (\beta \leftarrow \gamma) \end{array} \begin{array}{c} y \\ \diagdown \quad \diagup \\ \alpha \triangleleft (\beta \leftarrow \gamma) \end{array} \begin{array}{c} z \\ \diagdown \quad \diagup \\ \alpha \triangleleft (\beta \leftarrow \gamma) \end{array} \\ + \lambda_{\alpha, \beta \leftarrow \gamma} \begin{array}{c} \begin{array}{c} x \\ \diagdown \quad \diagup \\ \alpha \cdot (\beta \leftarrow \gamma) \end{array} \begin{array}{c} y \\ \diagdown \quad \diagup \\ \alpha \cdot (\beta \leftarrow \gamma) \end{array} \begin{array}{c} z \\ \diagdown \quad \diagup \\ \alpha \cdot (\beta \leftarrow \gamma) \end{array} \\ + \lambda_{\beta, \gamma} \begin{array}{c} \begin{array}{c} x \\ \diagdown \quad \diagup \\ \alpha \triangleright (\beta \cdot \gamma) \end{array} \begin{array}{c} y \\ \diagdown \quad \diagup \\ \alpha \triangleright (\beta \cdot \gamma) \end{array} \begin{array}{c} z \\ \diagdown \quad \diagup \\ \alpha \triangleright (\beta \cdot \gamma) \end{array} \\ + \lambda_{\beta, \gamma} \begin{array}{c} \begin{array}{c} x \\ \diagdown \quad \diagup \\ \alpha \triangleleft (\beta \cdot \gamma) \end{array} \begin{array}{c} y \\ \diagdown \quad \diagup \\ \alpha \triangleleft (\beta \cdot \gamma) \end{array} \begin{array}{c} z \\ \diagdown \quad \diagup \\ \alpha \triangleleft (\beta \cdot \gamma) \end{array} \\ + \lambda_{\beta, \gamma} \lambda_{\alpha, \beta \cdot \gamma} \begin{array}{c} \begin{array}{c} x \\ \diagdown \quad \diagup \\ \alpha \cdot (\beta \cdot \gamma) \end{array} \begin{array}{c} y \\ \diagdown \quad \diagup \\ \alpha \cdot (\beta \cdot \gamma) \end{array} \begin{array}{c} z \\ \diagdown \quad \diagup \\ \alpha \cdot (\beta \cdot \gamma) \end{array} \end{array}
\end{aligned}$$

By Lemma 2.13 and identifying the types of the planar rooted trees, we get that $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, \lambda)$ is a λ -ETS.

(c) \implies (a) By Lemma 2.13 and the definition of \diamond , $(\mathbf{kT}(X, \Omega), \diamond, (B_{\omega}^+)_{\omega \in \Omega})$ is an Ω -Rota-Baxter algebra. Now we show the freeness of $\mathbf{kT}(X, \Omega)$.

Let $(R, \cdot, (P_{\omega})_{\omega \in \Omega})$ be an Ω -Rota-Baxter algebra of weight λ_{Ω} and $f : X \rightarrow R$ a set map. We extend f to an Ω -Rota-Baxter algebra morphism $\bar{f} : \mathbf{kT}(X, \Omega) \rightarrow R$ such that $\bar{f} \circ i = f$.

For $T \in \mathcal{T}(X, \Omega)$, we define $\bar{f}(T)$ by induction on $\text{dep}(T)$. If $\text{dep}(T) = 1$, then T is of the form

$$T = \begin{array}{c} \cdots \\ x_1 \quad \cdots \quad x_m \\ \diagdown \quad \diagup \\ \omega \end{array}$$

Define

$$\bar{f}(T) := f(x_1) \cdot f(x_2) \cdots f(x_m).$$

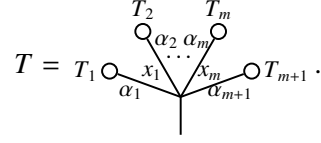
For the induction step of $\text{dep}(T) \geq 2$, we define $\bar{f}(T)$ by induction on the branches of T . If $\text{bra}(T) = 1$, then T is of the form

$$T = \begin{array}{c} T_2 \quad T_m \\ \alpha_2 \quad \alpha_m \\ \diagdown \quad \diagup \\ \alpha_1 \quad \alpha_{m+1} \\ \omega \end{array} \begin{array}{c} x_1 \quad \cdots \quad x_m \\ \diagdown \quad \diagup \\ \omega \end{array} \begin{array}{c} T_{m+1} \end{array}$$

Define

$$\bar{f}(T) := P_{\omega} (P_{\alpha_1} (\bar{f}(T_1)) \cdot f(x_1) \cdot P_{\alpha_2} (\bar{f}(T_2)) \cdots P_{\alpha_m} (\bar{f}(T_m)) \cdot f(x_m) \cdot P_{\alpha} (\bar{f}(T_{m+1}))).$$

If $\text{bra}(T) > 1$, then T is of the form



Define

$$\bar{f}(T) := P_{\alpha_1}(\bar{f}(T_1)) \cdot f(x_1) \cdot P_{\alpha_2}(\bar{f}(T_2)) \cdots P_{\alpha_m}(\bar{f}(T_m)) \cdot f(x_m) \cdot P_{\alpha_{m+1}}(\bar{f}(T_{m+1})).$$

By construction of \bar{f} , $\bar{f} \circ i = f$ and $P_{\omega} \bar{f} = \bar{f} B_{\omega}^+$ for all $\omega \in \Omega$. Next we show that \bar{f} is an algebra homomorphism, i.e.

$$(44) \quad \bar{f}(T \diamond T') = \bar{f}(T) \cdot \bar{f}(T') \quad \text{for all } T, T' \in \mathcal{T}(X, \Omega).$$

We prove Eq. (44) by induction on $\text{dep}(T) + \text{dep}(T')$. If $\text{dep}(T) + \text{dep}(T') = 2$, then $\text{dep}(T) = \text{dep}(T') = 1$ and

$$T = \begin{array}{c} x_1 \quad \dots \quad x_m \\ \diagdown \quad \quad \diagup \\ \text{---} \\ \diagup \quad \quad \diagdown \end{array} \quad \text{and} \quad T' = \begin{array}{c} y_1 \quad \dots \quad y_n \\ \diagdown \quad \quad \diagup \\ \text{---} \\ \diagup \quad \quad \diagdown \end{array}, \quad \text{with } m, n \geq 0,$$

and

$$\bar{f}(T \diamond T') = f(x_1) \cdots f(x_m) \cdot f(y_1) \cdots f(y_n) = (f(x_1) \cdots f(x_m)) \cdot (f(y_1) \cdots f(y_n)) = \bar{f}(T) \diamond \bar{f}(T').$$

For the induction step of $\text{dep}(T) + \text{dep}(T') \geq 3$. If $T \diamond T'$ belongs to the first three cases, then $\bar{f}(T \diamond T') = \bar{f}(T) \cdot \bar{f}(T')$ by the definition of \diamond and the construction of \bar{f} . So we only need to consider the fourth case. Then

$$\begin{aligned} \bar{f}(T \diamond T') &= \bar{f} \left(\left(T_1 \circ_{\alpha_1} \begin{array}{c} T_2 \quad T_m \\ \alpha_2 \quad \alpha_m \\ \dots \\ x_1 \quad x_m \\ \alpha_{m+1} \end{array} \diamond \left(B_{\alpha_{m+1} \rightarrow \beta_1}^+ (B_{\alpha_{m+1} > \beta_1}^+ (T_{m+1}) \diamond T'_1) + B_{\alpha_{m+1} \leftarrow \beta_1}^+ (T_{m+1}) \diamond B_{\alpha_{m+1} < \beta_1}^+ (T'_1) \right) \right. \right. \\ &\quad \left. \left. + \lambda_{\alpha_{m+1}, \beta_1} B_{\alpha_{m+1}, \beta_1}^+ (T_{m+1} \diamond T'_1) \right) \diamond \begin{array}{c} T'_2 \quad T'_n \\ \beta_2 \quad \beta_n \\ \dots \\ y_1 \quad y_n \\ \beta_{n+1} \end{array} \circ_{\beta_{n+1}} T'_{n+1} \right) \\ &= \left(P_{\alpha_1}(\bar{f}(T_1)) \cdot f(x_1) \cdot P_{\alpha_2}(\bar{f}(T_2)) \cdots P_{\alpha_m}(\bar{f}(T_m)) \cdot f(x_m) \cdot \bar{f} \left(B_{\alpha_{m+1} \rightarrow \beta_1}^+ (B_{\alpha_{m+1} > \beta_1}^+ (T_{m+1}) \diamond T'_1) \right) \right. \\ &\quad \left. + B_{\alpha_{m+1} \leftarrow \beta_1}^+ (T_{m+1}) \diamond B_{\alpha_{m+1} < \beta_1}^+ (T'_1) + \lambda_{\alpha_{m+1}, \beta_1} B_{\alpha_{m+1}, \beta_1}^+ (T_{m+1} \diamond T'_1) \right) \\ &\quad \cdot f(y_1) \cdot P_{\beta_2}(\bar{f}(T'_2)) \cdots P_{\beta_{n+1}}(\bar{f}(T'_{n+1})) \\ &= \left(P_{\alpha_1}(\bar{f}(T_1)) \cdot f(x_1) \cdot P_{\alpha_2}(\bar{f}(T_2)) \cdots P_{\alpha_m}(\bar{f}(T_m)) \cdot f(x_m) \cdot P_{\alpha_{m+1}}(\bar{f}(T_{m+1})) \right) \\ &\quad \cdot \left(P_{\beta_1}(\bar{f}(T'_1)) \cdot f(y_1) \cdot P_{\beta_2}(\bar{f}(T'_2)) \cdots P_{\beta_{n+1}}(\bar{f}(T'_{n+1})) \right) \\ &= \bar{f}(T) \diamond \bar{f}(T'). \end{aligned}$$

Moreover, by the construction of \bar{f} , it is the unique way to extend f as an Ω -Rota-Baxter algebra morphism. Hence $(\mathbf{k}\mathcal{T}(X, \Omega), \diamond, (B_{\omega}^+)_{\omega \in \Omega})$ together with the map i is the free Ω -Rota-Baxter algebra generated by X . \square

Remark 2.15. As a particular case, we recover the description of free family Rota-Baxter algebras of [13]. An alternative description of free Rota-Baxter algebras (with rooted forests) is done in [4].

Taking all elements in λ to be 0, we get the following result:

Corollary 2.16. *Let Ω be a set with four products $\leftarrow, \rightarrow, \triangleleft, \triangleright$. Then the following conditions are equivalent:*

- (a) $(\mathbf{k}\mathcal{J}(X, \Omega), \diamond, (B_\omega^+)_{\omega \in \Omega})$ together with the map i is the free Ω -Rota-Baxter algebra of weight 0 generated by X .
- (b) $(\mathbf{k}\mathcal{J}(X, \Omega), \diamond, (B_\omega^+)_{\omega \in \Omega})$ is an Ω -Rota-Baxter algebra of weight 0.
- (c) $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright)$ is an EDS.

2.3. Commutative Ω -Rota Baxter algebras on typed words. Let Ω be a set and V a vector space. Recall from [5] that the space of Ω -typed words in V is

$$\text{Sh}_\Omega^+(V) = \bigoplus_{n \geq 1} (\mathbf{k}\Omega)^{\otimes(n-1)} \otimes V^{\otimes n}.$$

For the ease of statement, we redefine the space of Ω -typed words in V as

$$\text{Sh}_\Omega^+(V) = \bigoplus_{n \geq 0} \underbrace{V \otimes (\mathbf{k}\Omega) \otimes \cdots \otimes (\mathbf{k}\Omega) \otimes V}_{(n+1)\text{'s } V \text{ and } n\text{'s } (\mathbf{k}\Omega)}$$

and write each pure tensor $\mathbf{v} = v_0 \otimes \omega_1 \otimes \cdots \otimes \omega_n \otimes v_n \in \Omega$ under the form

$$\mathbf{v} = v_0 \otimes_{\omega_1} v_1 \otimes_{\omega_2} \cdots \otimes_{\omega_n} v_n,$$

where $n \geq 0$, $\omega_1, \dots, \omega_n \in \Omega$ and $v_0, \dots, v_n \in V$ with the convention $\mathbf{v} = v_0$ if $n = 0$. We call \mathbf{v} an **Ω -typed word** in V and define its **length** $\ell(\mathbf{v}) := n + 1$.

Let A be an algebra with identity 1_A , $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot)$ be a set with five products and $\lambda = (\lambda_{\alpha, \beta})_{(\alpha, \beta) \in \Omega^2}$ be a family of elements in \mathbf{k} indexed by Ω^2 . For any pure tensors $\mathbf{a} = a_0 \otimes_{\alpha_1} \mathbf{a}'$, $\mathbf{b} = b_0 \otimes_{\beta_1} \mathbf{b}' \in \text{Sh}_\Omega^+(A)$ with $\ell(\mathbf{a}) = m$ and $\ell(\mathbf{b}) = n$, define $\mathbf{a} \diamond \mathbf{b}$ inductively as follows:

$$(45) \quad \mathbf{a} \diamond \mathbf{b} := \begin{cases} a_0 b_0, & \text{if } m = n = 0, \\ a_0 b_0 \otimes_{\alpha_1} \mathbf{a}', & \text{if } m > 0, n = 0, \\ a_0 b_0 \otimes_{\beta_1} \mathbf{b}', & \text{if } m = 0, n > 0, \\ a_0 b_0 \otimes_{\alpha_1 \rightarrow \beta_1} ((1_A \otimes_{\alpha_1 \triangleright \beta_1} \mathbf{a}') \diamond \mathbf{b}') + a_0 b_0 \otimes_{\alpha \leftarrow \beta_1} (\mathbf{a}' \diamond (1_A \otimes_{\alpha_1 \triangleleft \beta_1} \mathbf{b}')) \\ \quad + \lambda_{\alpha_1, \beta_1} a_0 b_0 \otimes_{\alpha_1, \beta_1} (\mathbf{a}' \diamond \mathbf{b}'), & \text{if } m > 0, n > 0. \end{cases}$$

Extending bilinearly, we construct a product \diamond on $\text{Sh}_\Omega^+(A)$.

Lemma 2.17. *Let A be an algebra with identity 1_A , Ω a set with five products $\leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot$ and λ a family of elements in \mathbf{k} indexed by Ω^2 . If $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, \lambda)$ is a λ -ESD, then $(\text{Sh}_\Omega^+(A), \diamond)$ is an associative algebra with identity 1_A .*

Proof. By Eq. (45), $\text{Sh}_\Omega^+(A)$ is closed under \diamond and 1_A is the identity of \diamond .

For pure tensor $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \text{Sh}_\Omega^+(A)$, we prove

$$(46) \quad (\mathbf{a} \diamond \mathbf{b}) \diamond \mathbf{c} = \mathbf{a} \diamond (\mathbf{b} \diamond \mathbf{c})$$

by induction on $\ell(\mathbf{a}) + \ell(\mathbf{b}) + \ell(\mathbf{c})$. If $\ell(\mathbf{a}) + \ell(\mathbf{b}) + \ell(\mathbf{c}) = 3$, then $\ell(\mathbf{a}) = \ell(\mathbf{b}) = \ell(\mathbf{c}) = 1$ and $\mathbf{a} = a_0$, $\mathbf{b} = b_0$, $\mathbf{c} = c_0$. Hence

$$(\mathbf{a} \diamond \mathbf{b}) \diamond \mathbf{c} = a_0 b_0 c_0 = \mathbf{a} \diamond (\mathbf{b} \diamond \mathbf{c}).$$

Suppose Eq. (46) holds for $\ell(\mathbf{a}) + \ell(\mathbf{b}) + \ell(\mathbf{c}) \leq p$, where $p \geq 3$ is a fixed integer. Consider the case of $\ell(\mathbf{a}) + \ell(\mathbf{b}) + \ell(\mathbf{c}) = p + 1$. If one of $\ell(\mathbf{a}), \ell(\mathbf{b}), \ell(\mathbf{c})$ is equal to 1, then Eq. (46) holds by direct calculation. Hence we assume $\ell(\mathbf{a}) > 1, \ell(\mathbf{b}) > 1, \ell(\mathbf{c}) > 1$ and

$$\mathbf{a} = a_0 \otimes_{\alpha_1} \mathbf{a}', \quad \mathbf{b} = b_0 \otimes_{\beta_1} \mathbf{b}', \quad \mathbf{c} = c_0 \otimes_{\gamma_1} \mathbf{c}'.$$

Then

$$\begin{aligned} & (\mathbf{a} \diamond \mathbf{b}) \diamond \mathbf{c} \\ &= (a_0 b_0) c_0 \otimes_{(\alpha_1 \rightarrow \beta_1) \rightarrow \gamma_1} ((1_A \otimes_{(\alpha_1 \rightarrow \beta_1) \triangleright \gamma_1} ((1_A \otimes_{\alpha_1 \triangleright \beta_1} \mathbf{a}') \diamond \mathbf{b}')) \diamond \mathbf{c}') \end{aligned}$$

$$\begin{aligned}
 & + (a_0 b_0) c_0 \otimes_{(\alpha_1 \rightarrow \beta_1) \leftarrow \gamma_1} (((1_A \otimes_{\alpha_1 \triangleright \beta_1} \mathbf{a}') \diamond \mathbf{b}') \diamond (1_A \otimes_{(\alpha_1 \rightarrow \beta_1) \triangleleft \gamma_1} \mathbf{c}')) \\
 & + \lambda_{(\alpha_1 \rightarrow \beta_1), \gamma_1} (a_0 b_0) c_0 \otimes_{(\alpha_1 \rightarrow \beta_1) \cdot \gamma_1} (((1_A \otimes_{\alpha_1 \triangleright \beta_1} \mathbf{a}') \diamond \mathbf{b}') \diamond \mathbf{c}') \\
 & + (a_0 b_0) c_0 \otimes_{(\alpha_1 \leftarrow \beta_1) \rightarrow \gamma_1} ((1_A \otimes_{(\alpha_1 \leftarrow \beta_1) \triangleright \gamma_1} (\mathbf{a}' \diamond (1_A \otimes_{\alpha_1 \triangleleft \beta_1} \mathbf{b}')))) \diamond \mathbf{c}') \\
 & + (a_0 b_0) c_0 \otimes_{(\alpha_1 \leftarrow \beta_1) \leftarrow \gamma_1} (\mathbf{a}' \diamond (1_A \otimes_{(\alpha_1 \triangleleft \beta_1) \rightarrow ((\alpha_1 \leftarrow \beta_1) \triangleleft \gamma_1)} ((1_A \otimes_{(\alpha_1 \triangleleft \beta_1) \triangleright ((\alpha_1 \leftarrow \beta_1) \triangleleft \gamma_1)} \mathbf{b}') \diamond \mathbf{c}')))) \\
 & + (a_0 b_0) c_0 \otimes_{(\alpha_1 \leftarrow \beta_1) \leftarrow \gamma_1} (\mathbf{a}' \diamond (1_A \otimes_{(\alpha_1 \triangleleft \beta_1) \leftarrow ((\alpha_1 \leftarrow \beta_1) \triangleleft \gamma_1)} (\mathbf{b}' \diamond (1_A \otimes_{(\alpha_1 \triangleleft \beta_1) \triangleleft ((\alpha_1 \leftarrow \beta_1) \triangleleft \gamma_1)} \mathbf{c}')))) \\
 & + \lambda_{(\alpha_1 \triangleleft \beta_1), ((\alpha_1 \leftarrow \beta_1) \triangleleft \gamma_1)} (a_0 b_0) c_0 \otimes_{(\alpha_1 \leftarrow \beta_1) \leftarrow \gamma_1} (\mathbf{a}' \diamond (1_A \otimes_{(\alpha_1 \triangleleft \beta_1), ((\alpha_1 \leftarrow \beta_1) \triangleleft \gamma_1)} (\mathbf{b}' \diamond \mathbf{c}')))) \\
 & + \lambda_{(\alpha_1 \leftarrow \beta_1), \gamma_1} (a_0 b_0) c_0 \otimes_{(\alpha_1 \leftarrow \beta_1) \cdot \gamma_1} ((\mathbf{a}' \diamond (1_A \otimes_{\alpha_1 \triangleleft \beta_1} \mathbf{b}')) \diamond \mathbf{c}') \\
 & + \lambda_{\alpha_1, \beta_1} (a_0 b_0) c_0 \otimes_{(\alpha_1 \cdot \beta_1) \rightarrow \gamma_1} ((1_A \otimes_{(\alpha_1 \cdot \beta_1) \triangleright \gamma_1} (\mathbf{a}' \diamond \mathbf{b}')) \diamond \mathbf{c}') \\
 & + \lambda_{\alpha_1, \beta_1} (a_0 b_0) c_0 \otimes_{(\alpha_1 \cdot \beta_1) \leftarrow \gamma_1} ((\mathbf{a}' \diamond \mathbf{b}') \diamond (1_A \otimes_{(\alpha_1 \cdot \beta_1) \triangleleft \gamma_1} \mathbf{c}')) \\
 & + \lambda_{\alpha_1, \beta_1} \lambda_{(\alpha_1 \cdot \beta_1), \gamma_1} (a_0 b_0) c_0 \otimes_{(\alpha_1 \cdot \beta_1) \cdot \gamma_1} ((\mathbf{a}' \diamond \mathbf{b}') \diamond \mathbf{c}')
 \end{aligned}$$

and

$$\begin{aligned}
 & \mathbf{a} \diamond (\mathbf{b} \diamond \mathbf{c}) \\
 = & a_0 (b_0 c_0) \otimes_{\alpha_1 \rightarrow (\beta_1 \rightarrow \gamma_1)} ((1_A \otimes_{(\alpha_1 \triangleright (\beta_1 \rightarrow \gamma_1)) \rightarrow (\beta_1 \triangleright \gamma_1)} ((1_A \otimes_{(\alpha_1 \triangleright (\beta_1 \rightarrow \gamma_1)) \triangleright (\beta_1 \triangleright \gamma_1)} \mathbf{a}') \diamond \mathbf{b}')) \diamond \mathbf{c}') \\
 & + a_0 (b_0 c_0) \otimes_{\alpha_1 \rightarrow (\beta_1 \rightarrow \gamma_1)} ((1_A \otimes_{(\alpha_1 \triangleright (\beta_1 \rightarrow \gamma_1)) \leftarrow (\beta_1 \triangleright \gamma_1)} (\mathbf{a}' \diamond (1_A \otimes_{(\alpha_1 \triangleright (\beta_1 \rightarrow \gamma_1)) \triangleleft (\beta_1 \triangleright \gamma_1)} \mathbf{b}')))) \diamond \mathbf{c}') \\
 & + \lambda_{(\alpha_1 \triangleright (\beta_1 \rightarrow \gamma_1)), (\beta_1 \triangleright \gamma_1)} a_0 (b_0 c_0) \otimes_{\alpha_1 \rightarrow (\beta_1 \rightarrow \gamma_1)} ((1_A \otimes_{(\alpha_1 \triangleright (\beta_1 \rightarrow \gamma_1)) \cdot (\beta_1 \triangleright \gamma_1)} (\mathbf{a}' \diamond \mathbf{b}')) \diamond \mathbf{c}') \\
 & + a_0 (b_0 c_0) \otimes_{\alpha_1 \leftarrow (\beta_1 \rightarrow \gamma_1)} (\mathbf{a}' \diamond (1_A \otimes_{\alpha_1 \triangleleft (\beta_1 \rightarrow \gamma_1)} ((1_A \otimes_{\beta_1 \triangleright \gamma_1} \mathbf{b}') \diamond \mathbf{c}')))) \\
 & + \lambda_{\alpha_1, (\beta_1 \rightarrow \gamma_1)} a_0 (b_0 c_0) \otimes_{\alpha_1 \cdot (\beta_1 \rightarrow \gamma_1)} (\mathbf{a}' \diamond ((1_A \otimes_{\beta_1 \triangleright \gamma_1} \mathbf{b}') \diamond \mathbf{c}')) \\
 & + a_0 (b_0 c_0) \otimes_{\alpha_1 \rightarrow (\beta_1 \leftarrow \gamma_1)} ((1_A \otimes_{\alpha_1 \triangleright (\beta_1 \leftarrow \gamma_1)} \mathbf{a}') \diamond (\mathbf{b}' \diamond (1_A \otimes_{\beta_1 \triangleleft \gamma_1} \mathbf{c}')))) \\
 & + a_0 (b_0 c_0) \otimes_{\alpha_1 \leftarrow (\beta_1 \leftarrow \gamma_1)} (\mathbf{a}' \diamond (1_A \otimes_{\alpha_1 \triangleleft (\beta_1 \leftarrow \gamma_1)} (\mathbf{b}' \diamond (1_A \otimes_{\beta_1 \triangleleft \gamma_1} \mathbf{c}')))) \\
 & + \lambda_{\alpha_1, (\beta_1 \leftarrow \gamma_1)} a_0 (b_0 c_0) \otimes_{\alpha_1 \cdot (\beta_1 \leftarrow \gamma_1)} (\mathbf{a}' \diamond (\mathbf{b}' \diamond (1_A \otimes_{\beta_1 \triangleleft \gamma_1} \mathbf{c}')))) \\
 & + \lambda_{\beta_1, \gamma_1} a_0 (b_0 c_0) \otimes_{\alpha_1 \rightarrow (\beta_1 \cdot \gamma_1)} ((1_A \otimes_{\alpha_1 \triangleright (\beta_1 \cdot \gamma_1)} \mathbf{a}') \diamond (\mathbf{b}' \diamond \mathbf{c}')) \\
 & + \lambda_{\beta_1, \gamma_1} a_0 (b_0 c_0) \otimes_{\alpha_1 \leftarrow (\beta_1 \cdot \gamma_1)} (\mathbf{a}' \diamond (1_A \otimes_{\alpha_1 \triangleleft (\beta_1 \cdot \gamma_1)} (\mathbf{b}' \diamond \mathbf{c}')))) \\
 & + \lambda_{\beta_1, \gamma_1} \lambda_{\alpha_1, (\beta_1 \cdot \gamma_1)} a_0 (b_0 c_0) \otimes_{\alpha_1 \cdot (\beta_1 \cdot \gamma_1)} (\mathbf{a}' \diamond (\mathbf{b}' \diamond \mathbf{c}')).
 \end{aligned}$$

By induction hypothesis and $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, \lambda)$ being a λ -ETS, $(\mathbf{a} \diamond \mathbf{b}) \diamond \mathbf{c} = \mathbf{a} \diamond (\mathbf{b} \diamond \mathbf{c})$. Hence $(\text{Sh}_\Omega^+(A), \diamond)$ is an associative algebra with identity 1_A . \square

For each $\omega \in \Omega$, define a linear map $P_\omega : \text{Sh}_\Omega^+(A) \rightarrow \text{Sh}_\Omega^+(A)$, $\mathbf{a} \mapsto 1_A \otimes_\omega \mathbf{a}$. If further A is a commutative algebra and $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, \lambda)$ is a commutative λ -ETS, we get the following result:

Proposition 2.18. *If A is a commutative algebra with identity 1_A and $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, \lambda)$ is a commutative λ -ETS, then $(\text{Sh}_\Omega^+(A), \diamond, (P_\omega)_{\omega \in \Omega})$ is the free commutative Ω -Rota-Baxter algebra generated by A .*

Proof. For $\mathbf{a}, \mathbf{b} \in \text{Sh}_\Omega^+(A)$ and $\alpha, \beta \in \Omega$,

$$\begin{aligned}
 P_\alpha(\mathbf{a}) \diamond P_\beta(\mathbf{b}) & = (1_A \otimes_\alpha \mathbf{a}) \diamond (1_A \otimes_\beta \mathbf{b}) \\
 & = 1_A \otimes_{\alpha \rightarrow \beta} ((1 \otimes_{\alpha \triangleright \beta} \mathbf{a}) \diamond \mathbf{b}) + 1 \otimes_{\alpha \leftarrow \beta} (\mathbf{a} \diamond (1_A \otimes_{\alpha \triangleleft \beta} \mathbf{b})) \\
 & \quad + \lambda_{\alpha, \beta} 1_A \otimes_{\alpha \cdot \beta} (\mathbf{a} \diamond \mathbf{b}) \\
 & = P_{\alpha \rightarrow \beta}(P_{\alpha \triangleright \beta}(\mathbf{a}) \diamond \mathbf{b}) + P_{\alpha \leftarrow \beta}(\mathbf{a} \diamond P_{\alpha \triangleleft \beta}(\mathbf{b})) + \lambda_{\alpha, \beta} P_{\alpha \cdot \beta}(\mathbf{a} \diamond \mathbf{b}),
 \end{aligned}$$

hence $\text{Sh}_\Omega(A)$ is an Ω -Rota-Baxter algebra. Next we show

$$(47) \quad \mathbf{a} \diamond \mathbf{b} = \mathbf{b} \diamond \mathbf{a}$$

by induction on $\ell(\mathbf{a}) + \ell(\mathbf{b})$. If $\ell(\mathbf{a}) + \ell(\mathbf{b}) = 2$, then $\ell(\mathbf{a}) = \ell(\mathbf{b}) = 1$ and

$$\mathbf{a} \diamond \mathbf{b} = a_0 \diamond b_0 = a_0 b_0 = b_0 a_0 = b_0 \diamond a_0 = \mathbf{b} \diamond \mathbf{a}.$$

Suppose Eq. (47) holds for $\ell(\mathbf{a}) + \ell(\mathbf{b}) < p$, where $p \geq 2$ is a fixed integer. We consider the case of $\ell(\mathbf{a}) + \ell(\mathbf{b}) = p + 1$. If one of $\ell(\mathbf{a}), \ell(\mathbf{b})$ is equal to 1, then Eq. (47) holds directly. So assume $\mathbf{a} = a_0 \otimes_{\alpha_1} \mathbf{a}', \mathbf{b} = b_0 \otimes_{\beta_1} \mathbf{b}'$, then

$$\begin{aligned}
\mathbf{a} \diamond \mathbf{b} &= (a_0 \otimes_{\alpha_1} \mathbf{a}') \diamond (b_0 \otimes_{\beta_1} \mathbf{b}') \\
&= a_0 b_0 \otimes_{\alpha_1 \rightarrow \beta_1} ((1_A \otimes_{\alpha_1 \triangleright \beta_1} \mathbf{a}') \diamond \mathbf{b}') + a_0 b_0 \otimes_{\alpha_1 \leftarrow \beta_1} (\mathbf{a}' \diamond (1_A \otimes_{\alpha_1 \triangleleft \beta_1} \mathbf{b}')) + \lambda_{\alpha_1, \beta_1} a_0 b_0 \otimes_{\alpha_1 \cdot \beta_1} (\mathbf{a}' \diamond \mathbf{b}') \\
&= b_0 a_0 \otimes_{\alpha_1 \rightarrow \beta_1} ((1_A \otimes_{\alpha_1 \triangleright \beta_1} \mathbf{a}') \diamond \mathbf{b}') + b_0 a_0 \otimes_{\alpha_1 \leftarrow \beta_1} (\mathbf{a}' \diamond (1_A \otimes_{\alpha_1 \triangleleft \beta_1} \mathbf{b}')) + \lambda_{\alpha_1, \beta_1} b_0 a_0 \otimes_{\alpha_1 \cdot \beta_1} (\mathbf{a}' \diamond \mathbf{b}') \\
&\quad \text{(by } A \text{ being a commutative algebra)} \\
&= b_0 a_0 \otimes_{\alpha_1 \rightarrow \beta_1} (\mathbf{b}' \diamond (1_A \otimes_{\alpha_1 \triangleright \beta_1} \mathbf{a}')) + b_0 a_0 \otimes_{\alpha_1 \leftarrow \beta_1} ((1_A \otimes_{\alpha_1 \triangleleft \beta_1} \mathbf{b}') \diamond \mathbf{a}') + \lambda_{\alpha_1, \beta_1} b_0 a_0 \otimes_{\alpha_1 \cdot \beta_1} (\mathbf{b}' \diamond \mathbf{a}') \\
&\quad \text{(by induction hypothesis)} \\
&= b_0 a_0 \otimes_{\beta_1 \leftarrow \alpha_1} (\mathbf{b}' \diamond (1_A \otimes_{\beta_1 \triangleleft \alpha_1} \mathbf{a}')) + b_0 a_0 \otimes_{\beta_1 \rightarrow \alpha_1} ((1_A \otimes_{\beta_1 \triangleright \alpha_1} \mathbf{b}') \diamond \mathbf{a}') + \lambda_{\beta_1, \alpha_1} b_0 a_0 \otimes_{\beta_1 \cdot \alpha_1} (\mathbf{b}' \diamond \mathbf{a}') \\
&\quad \text{(by } \Omega \text{ being commutative)} \\
&= (b_0 \otimes_{\beta_1} \mathbf{b}') \diamond (a_0 \otimes_{\alpha_1} \mathbf{a}') = \mathbf{b} \otimes \mathbf{a}.
\end{aligned}$$

Hence $(\text{Sh}_\Omega^+(A), \diamond)$ is a commutative algebra.

Let $(R, \cdot, (P_\omega)_{\omega \in \Omega})$ be a commutative Ω -Rota-Baxter algebra and $f : A \rightarrow R$ a commutative algebra homomorphism. We extend f to an Ω -Rota-Baxter algebra morphism $\bar{f} : \text{Sh}_\Omega^+(A) \rightarrow R$ as follows: for $\mathbf{a} \in \text{Sh}_\Omega^+(A)$, we define $\bar{f}(\mathbf{a})$ by induction on $\ell(\mathbf{a})$. If $\ell(\mathbf{a}) = 1$, then define $\bar{f}(\mathbf{a}) = f(\mathbf{a})$. Suppose $\bar{f}(\mathbf{a})$ has been defined for all \mathbf{a} with $\ell(\mathbf{a}) \leq p$, where $p \geq 1$ is a fixed integer. Consider the case of $\ell(\mathbf{a}) = p + 1$. Suppose $\mathbf{a} = a_0 \otimes_{\alpha_1} \mathbf{a}'$, then define

$$\bar{f}(\mathbf{a}) := f(a_0) \cdot P_{\alpha_1}(\bar{f}(\mathbf{a}')).$$

We can get that it is the unique way to extend f as an Ω -Rota-Baxter algebra morphism. Hence $(\text{Sh}_\Omega^+(A), \diamond)$ is the free commutative Ω -Rota-Baxter algebra generated by A . \square

Let us assume that A is unitary. We denote its unit by 1_A . For each $\omega \in \Omega$, define a linear map $P_\omega : \text{Sh}_\Omega(A) \rightarrow \text{Sh}_\Omega(A), \mathbf{a} \mapsto 1_A \otimes_\omega \mathbf{a}$.

Proposition 2.19. *If A is a unitary commutative algebra and $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, \lambda)$ is a commutative λ -ETS, then $(\text{Sh}_\Omega(A), \diamond, (P_\omega)_{\omega \in \Omega})$ is a commutative Ω -Rota-Baxter algebra.*

Proof. For $\mathbf{a}, \mathbf{b} \in \text{Sh}_\Omega(A)$ and $\alpha, \beta \in \Omega$,

$$\begin{aligned}
P_\alpha(\mathbf{a}) \diamond P_\beta(\mathbf{b}) &= (1_A \otimes_\alpha \mathbf{a}) \diamond (1_A \otimes_\beta \mathbf{b}) \\
&= 1_A \otimes_{\alpha \rightarrow \beta} ((1_A \otimes_{\alpha \triangleright \beta} \mathbf{a}) \diamond \mathbf{b}) + 1_A \otimes_{\alpha \leftarrow \beta} (\mathbf{a} \diamond (1_A \otimes_{\alpha \triangleleft \beta} \mathbf{b})) \\
&\quad + \lambda_{\alpha, \beta} 1_A \otimes_{\alpha \cdot \beta} (\mathbf{a} \diamond \mathbf{b}) \\
&= P_{\alpha \rightarrow \beta}(P_{\alpha \triangleright \beta}(\mathbf{a}) \diamond \mathbf{b}) + P_{\alpha \leftarrow \beta}(\mathbf{a} \diamond P_{\alpha \triangleleft \beta}(\mathbf{b})) + \lambda_{\alpha, \beta} P_{\alpha \cdot \beta}(\mathbf{a} \diamond \mathbf{b}),
\end{aligned}$$

hence $\text{Sh}_\Omega(A)$ is an Ω -Rota-Baxter algebra. Next we show

$$(48) \quad \mathbf{a} \diamond \mathbf{b} = \mathbf{b} \diamond \mathbf{a}$$

by induction on $\ell(\mathbf{a}) + \ell(\mathbf{b})$. If $\ell(\mathbf{a}) + \ell(\mathbf{b}) = 2$, then $\ell(\mathbf{a}) = \ell(\mathbf{b}) = 1$ and

$$\mathbf{a} \diamond \mathbf{b} = a_0 \diamond b_0 = a_0 b_0 = b_0 a_0 = b_0 \diamond a_0 = \mathbf{b} \diamond \mathbf{a}.$$

Suppose Eq. (48) holds for $\ell(\mathbf{a}) + \ell(\mathbf{b}) < p$, where $p \geq 2$ is a fixed integer. We consider the case of $\ell(\mathbf{a}) + \ell(\mathbf{b}) = p + 1$. If one of $\ell(\mathbf{a}), \ell(\mathbf{b})$ is equal to 1, then Eq. (48) holds directly. So assume $\mathbf{a} = a_0 \otimes_{\alpha_1} \mathbf{a}', \mathbf{b} = b_0 \otimes_{\beta_1} \mathbf{b}'$, then

$$\begin{aligned}
\mathbf{a} \diamond \mathbf{b} &= (a_0 \otimes_{\alpha_1} \mathbf{a}') \diamond (b_0 \otimes_{\beta_1} \mathbf{b}') \\
&= a_0 b_0 \otimes_{\alpha_1 \rightarrow \beta_1} ((1_A \otimes_{\alpha_1 \triangleright \beta_1} \mathbf{a}') \diamond \mathbf{b}') + a_0 b_0 \otimes_{\alpha_1 \leftarrow \beta_1} (\mathbf{a}' \diamond (1_A \otimes_{\alpha_1 \triangleleft \beta_1} \mathbf{b}')) + \lambda_{\alpha_1, \beta_1} a_0 b_0 \otimes_{\alpha_1 \cdot \beta_1} (\mathbf{a}' \diamond \mathbf{b}')
\end{aligned}$$

$$\begin{aligned}
 &= b_0 a_0 \otimes_{\alpha_1 \rightarrow \beta_1} ((1_A \otimes_{\alpha_1 \triangleright \beta_1} \mathbf{a}') \diamond \mathbf{b}') + b_0 a_0 \otimes_{\alpha_1 \leftarrow \beta_1} (\mathbf{a}' \diamond (1_A \otimes_{\alpha_1 \triangleleft \beta_1} \mathbf{b}')) + \lambda_{\alpha_1, \beta_1} b_0 a_0 \otimes_{\alpha_1, \beta_1} (\mathbf{a}' \diamond \mathbf{b}') \\
 &\quad \text{(by } A \text{ being a commutative algebra)} \\
 &= b_0 a_0 \otimes_{\alpha_1 \rightarrow \beta_1} (\mathbf{b}' \diamond (1_A \otimes_{\alpha_1 \triangleright \beta_1} \mathbf{a}')) + b_0 a_0 \otimes_{\alpha_1 \leftarrow \beta_1} ((1_A \otimes_{\alpha_1 \triangleleft \beta_1} \mathbf{b}') \diamond \mathbf{a}') + \lambda_{\alpha_1, \beta_1} b_0 a_0 \otimes_{\alpha_1, \beta_1} (\mathbf{b}' \diamond \mathbf{a}') \\
 &\quad \text{(by induction hypothesis)} \\
 &= b_0 a_0 \otimes_{\beta_1 \leftarrow \alpha_1} (\mathbf{b}' \diamond (1_A \otimes_{\beta_1 \triangleleft \alpha_1} \mathbf{a}')) + b_0 a_0 \otimes_{\beta_1 \rightarrow \alpha_1} ((1_A \otimes_{\beta_1 \triangleright \alpha_1} \mathbf{b}') \diamond \mathbf{a}') + \lambda_{\beta_1, \alpha_1} b_0 a_0 \otimes_{\beta_1, \alpha_1} (\mathbf{b}' \diamond \mathbf{a}') \\
 &\quad \text{(by } \Omega \text{ being commutative)} \\
 &= (b_0 \otimes_{\beta_1} \mathbf{b}') \diamond (a_0 \otimes_{\alpha_1} \mathbf{a}') = \mathbf{b} \otimes \mathbf{a}.
 \end{aligned}$$

Hence $(\text{Sh}_\Omega(A), \diamond)$ is a commutative algebra. \square

Let A be a commutative algebra. We put $uA = \mathbf{k} \oplus A$ and give it a product defined by

$$(\lambda + a)(\mu + b) = \lambda\mu + (\lambda b + \mu a + ab).$$

Then uA is a commutative unitary algebra and its unit 1_A is the unit 1 of \mathbf{k} .

Theorem 2.20. *We put*

$$\text{Sh}'_\Omega(A) = A \oplus \bigoplus_{n \geq 2} \underbrace{uA \otimes (\mathbf{k}\Omega) \otimes \cdots \otimes (\mathbf{k}\Omega) \otimes uA}_{n \text{'s } V \text{ and } (n-1) \text{'s } (\mathbf{k}\Omega)}.$$

Then $\text{Sh}'_\Omega(A)$ is the free commutative Ω -Rota-Baxter algebra generated by the algebra A .

Proof. Let $(R, \cdot, (P_\omega)_{\omega \in \Omega})$ be a commutative Ω -Rota-Baxter algebra and $f : A \rightarrow R$ a (nonunitary) algebra homomorphism. We extend f , first from uA to R as a unitary algebra morphism by sending 1_{uA} to 1_R , then as an Ω -Rota-Baxter algebra morphism $\bar{f} : \text{Sh}'_\Omega(A) \rightarrow R$ as follows: for $\mathbf{a} \in \text{Sh}_\Omega(A)$, we define $\bar{f}(\mathbf{a})$ by induction on $\ell(\mathbf{a})$. If $\ell(\mathbf{a}) = 1$, then define $\bar{f}(\mathbf{a}) = f(\mathbf{a})$. Suppose $\bar{f}(\mathbf{a})$ has been defined for all \mathbf{a} with $\ell(\mathbf{a}) \leq p$, where $p \geq 1$ is a fixed integer. Consider the case of $\ell(\mathbf{a}) = p + 1$. Suppose $\mathbf{a} = a_0 \otimes_{\alpha_1} \mathbf{a}'$, then define

$$\bar{f}(\mathbf{a}) := f(a_0) \cdot P_{\alpha_1}(\bar{f}(\mathbf{a}')).$$

For any $\mathbf{a} \in \text{Sh}'_\Omega(A)$ and for any $\alpha \in \Omega$:

$$\bar{f} \circ P_\alpha(\mathbf{a}) = \bar{f}(1_A \otimes_\alpha \mathbf{a}) = 1_B \cdot P_\alpha(\bar{f}(\mathbf{a})) = P_\alpha \circ \bar{f}(\mathbf{a}).$$

Let us prove that this is an algebra morphism. Let $\mathbf{a}, \mathbf{b} \in \text{Sh}'_\Omega(A)$, let us prove that $\bar{f}(\mathbf{a} \diamond \mathbf{b}) = \bar{f}(\mathbf{a})\bar{f}(\mathbf{b})$ by induction on $n = \ell(\mathbf{a}) + \ell(\mathbf{b})$. If $\ell(\mathbf{a}) = \ell(\mathbf{b}) = 1$, then

$$\bar{f}(\mathbf{a} \diamond \mathbf{b}) = \bar{f}(a_0 b_0) = f(a_0 b_0) = f(a_0) \cdot f(b_0) = \bar{f}(\mathbf{a}) \cdot \bar{f}(\mathbf{b}).$$

If $\ell(\bar{a}) = 1$ and $\ell(\bar{b}) > 1$, then

$$\begin{aligned}
 \bar{f}(\mathbf{a} \diamond \mathbf{b}) &= \bar{f}(a_0 b_0 \otimes_{\alpha_1} \mathbf{a}') \\
 &= f(a_0 b_0) \cdot P_{\alpha_1} \circ \bar{f}(\mathbf{a}') \\
 &= f(a_0) \cdot f(b_0) \cdot P_{\alpha_1} \circ \bar{f}(\mathbf{a}') \\
 &= \bar{f}(\mathbf{a}) \cdot \bar{f}(\mathbf{b}).
 \end{aligned}$$

This is similar if $\ell(\bar{a}) > 1$ and $\ell(\bar{b}) = 1$. If $\ell(\bar{a}) > 1$ and $\ell(\bar{b}) > 1$, then

$$\begin{aligned}
 \bar{f}(\mathbf{a} \diamond \mathbf{b}) &= \bar{f}(a_0 b_0 \otimes_{\alpha_1 \rightarrow \beta_1} ((1 \otimes_{\alpha_1 \triangleright \beta_1} \mathbf{a}') \diamond \mathbf{b}')) + \bar{f}(a_0 b_0 \otimes_{\alpha_1 \leftarrow \beta_1} (\mathbf{a}' \diamond (1 \otimes_{\alpha_1 \triangleleft \beta_1} \mathbf{b}')))) \\
 &\quad + \bar{f}(\lambda_{\alpha_1, \beta_1} a_0 b_0 \otimes_{\alpha_1, \beta_1} (\mathbf{a}' \diamond \mathbf{b}')) \\
 &= f(a_0 b_0) \cdot P_{\alpha_1 \rightarrow \beta_1} \circ \bar{f}((1 \otimes_{\alpha_1 \triangleright \beta_1} \mathbf{a}') \diamond \mathbf{b}') + f(a_0 b_0) \cdot P_{\alpha_1 \leftarrow \beta_1} \circ \bar{f}(\mathbf{a}' \diamond (1 \otimes_{\alpha_1 \triangleleft \beta_1} \mathbf{b}')) \\
 &\quad + \lambda_{\alpha_1, \beta_1} f(a_0 b_0) \cdot P_{\alpha_1, \beta_1} \circ \bar{f}(\mathbf{a}' \diamond \mathbf{b}') \\
 &= f(a_0) \cdot f(b_0) \cdot \bar{f}(P_{\alpha_1 \rightarrow \beta_1}(P_{\alpha_1 \triangleright \beta_1}(\mathbf{a}') \diamond \mathbf{b}')) + f(a_0) \cdot f(b_0) \cdot \bar{f}(P_{\alpha_1 \leftarrow \beta_1}(\mathbf{a}' \diamond P_{\alpha_1 \triangleleft \beta_1}(\mathbf{b}'))))
 \end{aligned}$$

$$\begin{aligned}
& + \lambda_{\alpha_1, \beta_1} f(a_0) \cdot f(b_0) \cdot \overline{f}(P_{\alpha_1, \beta_1}(\mathbf{a}' \diamond \mathbf{b}')) \\
& = f(a_0) \cdot f(b_0) \cdot \overline{f}(P_{\alpha_1}(\mathbf{a}') P_{\beta_1}(\mathbf{b}')) \\
& = f(a_0) \cdot f(b_0) \cdot \overline{f}(P_{\alpha_1}(\mathbf{a}')) \cdot \overline{f}(P_{\beta_1}(\mathbf{b}')) && \text{(by the induction hypothesis)} \\
& = f(a_0) \cdot f(b_0) \cdot P_{\alpha_1} \circ \overline{f}(\mathbf{a}') \cdot P_{\beta_1} \circ \overline{f}(\mathbf{b}') \\
& = f(a_0) \cdot P_{\alpha_1} \circ \overline{f}(\mathbf{a}') \cdot f(b_0) \cdot P_{\beta_1} \circ \overline{f}(\mathbf{b}') && \text{(as } B \text{ is commutative)} \\
& = \overline{f}(\mathbf{a}) \cdot \overline{f}(\mathbf{b}).
\end{aligned}$$

We get that it is the unique way to extend f as an Ω -Rota-Baxter algebra morphism. Hence $\text{Sh}'_{\Omega}(A)$ is the free commutative Ω -Rota-Baxter algebra generated by A . \square

3. MORE RESULTS ON λ -ETS AND ETS

3.1. Description in terms of linear and bilinear maps. As in Lemma 5 of [5], we obtain:

Lemma 3.1. *Let $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot)$ be a set with five operations and $\lambda = (\lambda_{\alpha, \beta})_{\alpha, \beta \in \Omega}$ be a family of elements in \mathbf{k} indexed by Ω^2 . We denote by $\mathbf{k}\Omega$ the vector space generated by Ω . We put:*

$$\begin{aligned}
\varphi_{\leftarrow} & : \begin{cases} \mathbf{k}\Omega^{\otimes 2} & \longrightarrow & \mathbf{k}\Omega^{\otimes 2} \\ \alpha \otimes \beta & \longrightarrow & \alpha \leftarrow \beta \otimes \alpha \triangleleft \beta \end{cases} \\
\varphi_{\rightarrow} & : \begin{cases} \mathbf{k}\Omega^{\otimes 2} & \longrightarrow & \mathbf{k}\Omega^{\otimes 2} \\ \alpha \otimes \beta & \longrightarrow & \alpha \rightarrow \beta \otimes \alpha \triangleright \beta \end{cases} \\
\psi & : \begin{cases} \mathbf{k}\Omega^{\otimes 2} & \longrightarrow & \mathbf{k}\Omega \\ \alpha \otimes \beta & \longrightarrow & \lambda_{\alpha, \beta} \alpha \cdot \beta \end{cases}
\end{aligned}$$

Then $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, \lambda)$ is a λ -ETS if, and only if:

$$\begin{aligned}
(49) \quad & (\tau \otimes Id) \circ (Id \otimes \varphi_{\leftarrow}) \circ (\tau \otimes Id) \circ (\varphi_{\rightarrow} \otimes Id) = (\varphi_{\rightarrow} \otimes Id) \circ (Id \otimes \varphi_{\leftarrow}), \\
(50) \quad & (Id \otimes \varphi_{\leftarrow}) \circ (\tau \otimes Id) \circ (Id \otimes \varphi_{\leftarrow}) \otimes (\tau \otimes Id) \circ (\varphi_{\leftarrow} \otimes Id) = (\varphi_{\leftarrow} \otimes Id) \circ (Id \otimes \varphi_{\leftarrow}), \\
(51) \quad & (Id \otimes \varphi_{\rightarrow}) \circ (\tau \otimes Id) \circ (Id \otimes \varphi_{\leftarrow}) \circ (\tau \otimes Id) \circ (\varphi_{\leftarrow} \otimes Id) = (\varphi_{\leftarrow} \otimes Id) \circ (Id \otimes \varphi_{\rightarrow}), \\
(52) \quad & (Id \otimes \varphi_{\leftarrow}) \circ (\varphi_{\rightarrow} \otimes Id) \circ (Id \otimes \varphi_{\rightarrow}) = (\varphi_{\rightarrow} \otimes Id) \circ (Id \otimes \tau) \circ (\varphi_{\leftarrow} \otimes Id), \\
(53) \quad & (Id \otimes \varphi_{\rightarrow}) \circ (\varphi_{\rightarrow} \otimes Id) \circ (Id \otimes \varphi_{\rightarrow}) = (\varphi_{\rightarrow} \otimes Id) \circ (Id \otimes \tau) \circ (\varphi_{\rightarrow} \otimes Id), \\
(54) \quad & \varphi_{\rightarrow} \circ (id \otimes \psi) = (\psi \otimes id) \circ (id \otimes \tau) \circ (\varphi_{\rightarrow} \otimes id) \\
(55) \quad & (\psi \otimes id) \circ (id \otimes \tau) \circ (id \otimes \varphi_{\leftarrow}) \circ (\tau \otimes id) \circ (\varphi_{\leftarrow} \otimes id) = \tau \circ \varphi_{\leftarrow} \circ (id \otimes \psi) \\
(56) \quad & (id \otimes \psi) \circ (\varphi_{\rightarrow} \otimes id) \circ (id \otimes \varphi_{\rightarrow}) = \varphi_{\rightarrow} \circ (\psi \otimes id) \\
(57) \quad & (\psi \otimes id) \circ (id \otimes \tau) \circ (\varphi_{\leftarrow} \otimes id) = (\psi \otimes id) \circ (id \otimes \varphi_{\rightarrow}) \\
(58) \quad & (\psi \otimes id) \circ (id \otimes \varphi_{\leftarrow}) = \varphi_{\leftarrow} \circ (\psi \otimes id) \\
(59) \quad & \psi \circ (\psi \otimes id) = \psi \circ (id \otimes \psi).
\end{aligned}$$

In particular, ψ is an associative product.

Proof. By Lemma 5 in [5], Eqs. (49)-(53) are equivalent to $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright)$ being an EDS. Moreover, direct computations prove that Eq. (54) is equivalent to Eq. (11) and condition (a); Eq. (55) is equivalent to Eq. (12) and condition (b); Eq. (56) is equivalent to Eq. (13) and condition (c); Eq. (57) is equivalent to Eq. (14) and condition (d); Eq. (58) is equivalent to Eq. (15) and condition (e); Eq. (59) is equivalent to Eq. (16) and condition (f) in Definition 2.3. \square

Similarly, we obtain for ETS:

Lemma 3.2. Let $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, *, \cdot)$ be a set with six operations. We put:

$$\begin{aligned} \varphi_{\leftarrow} &: \begin{cases} \Omega^2 & \longrightarrow & \Omega^2 \\ (\alpha, \beta) & \longrightarrow & (\alpha \leftarrow \beta, \alpha \triangleleft \beta) \end{cases} \\ \varphi_{\rightarrow} &: \begin{cases} \Omega^2 & \longrightarrow & \Omega^2 \\ (\alpha, \beta) & \longrightarrow & (\alpha \rightarrow \beta, \alpha \triangleright \beta) \end{cases} \\ \varphi_* &: \begin{cases} \Omega^2 & \longrightarrow & \Omega^2 \\ (\alpha, \beta) & \longrightarrow & (\alpha \cdot \beta, \alpha * \beta) \end{cases} \end{aligned}$$

Then $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, *, \cdot)$ is a ETS if, and only if, (34)-(38) of [5] are satisfied and:

$$(60) \quad (\tau \otimes \text{id}) \circ (\text{id} \otimes \varphi_*) \circ (\tau \otimes \text{id}) \circ (\varphi_{\rightarrow} \otimes \text{id}) = (\varphi_{\rightarrow} \otimes \text{id}) \circ (\text{id} \otimes \varphi_*),$$

$$(61) \quad (\text{id} \otimes \varphi_*) \circ (\tau \otimes \text{id}) \circ (\text{id} \otimes \varphi_{\leftarrow}) \circ (\tau \otimes \text{id}) \circ (\varphi_{\leftarrow} \otimes \text{id}) = (\varphi_{\leftarrow} \otimes \text{id}) \circ (\text{id} \otimes \varphi_*),$$

$$(62) \quad (\text{id} \otimes \varphi_*) \circ (\varphi_{\rightarrow} \otimes \text{id}) \circ (\text{id} \otimes \varphi_{\rightarrow}) = (\varphi_{\rightarrow} \otimes \text{id}) \circ (\text{id} \otimes \tau) \circ (\text{id} \otimes \varphi_*),$$

$$(63) \quad (\text{id} \otimes \varphi_*) \circ (\tau \otimes \text{id}) \circ (\varphi_{\leftarrow} \otimes \text{id}) = (\text{id} \otimes \varphi_*) \circ (\tau \otimes \text{id}) \circ (\text{id} \otimes \tau) \circ (\text{id} \otimes \varphi_{\rightarrow}),$$

$$(64) \quad (\varphi_* \otimes \text{id}) \circ (\text{id} \otimes \varphi_{\leftarrow}) = (\tau \otimes \text{id}) \circ (\text{id} \otimes \varphi_{\leftarrow}) \circ (\tau \otimes \text{id}) \circ (\varphi_* \otimes \text{id}),$$

$$(65) \quad (\varphi_* \otimes \text{id}) \circ (\text{id} \otimes \tau) \circ (\varphi_* \otimes \text{id}) = (\text{id} \otimes \tau) \circ (\varphi_* \otimes \text{id}) \circ (\text{id} \otimes \varphi_*).$$

Proof. By Lemma 5 in [5], Eqs. (34)-(38) are equivalent to $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright)$ being an EDS. Moreover, direct computations prove that Eq. (60) is equivalent to Eqs. (17),(18) and (28); Eq. (61) is equivalent to Eqs. (19), (20) and (29); Eq. (62) is equivalent to Eqs. (21), (22) and (30); Eq. (63) is equivalent to Eqs. (23), (24) and (31); Eq. (64) is equivalent to Eqs. (25), (26) and (32); Eq. (65) is equivalent to Eqs. (27), (33) and (34). \square

3.2. A description of all λ -ETS of cardinality two. The following table gives all λ -ETS. We slightly generalize our definition, by accepting more general maps $\varphi : \mathbf{k}\Omega^{\otimes 2} \longrightarrow \mathbf{k}\Omega$. The underlying set is $\{a, b\}$ and all the products are given by a 2×2 table. Here, λ, μ are elements of the base field \mathbf{k} .

Type	\leftarrow	\rightarrow	\triangleleft	\triangleright	φ_*	Name
A	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} (\lambda + \mu)a & (\lambda + \mu)a \\ (\lambda + \mu)a & \lambda a + \mu b \end{pmatrix}$	$A_1(\lambda, \mu)$
			$\begin{pmatrix} a & b \\ a & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ b & b \end{pmatrix}$	$\begin{pmatrix} (\lambda + \mu)a & (\lambda + \mu)a \\ (\lambda + \mu)a & \lambda a + \mu b \end{pmatrix}$	$A_2(\lambda, \mu)$
B	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} a & b \\ a & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} \lambda a & \lambda a \\ \lambda a & \lambda a \end{pmatrix}, \begin{pmatrix} \lambda a & \lambda b \\ \lambda a & \lambda b \end{pmatrix}$	$B'_1(\lambda), B''_1(\lambda)$
			$\begin{pmatrix} a & b \\ a & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ b & b \end{pmatrix}$		$B'_2(\lambda), B''_2(\lambda)$
C	$\begin{pmatrix} a & a \\ a & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} \lambda a & \lambda a \\ \lambda a & \lambda b \end{pmatrix}$	$C_1(\lambda)$
			$\begin{pmatrix} a & b \\ a & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ b & b \end{pmatrix}$		$C_3(\lambda)$
			$\begin{pmatrix} b & b \\ b & b \end{pmatrix}$	$\begin{pmatrix} b & b \\ b & b \end{pmatrix}$		$C_5(\lambda)$
			$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} b & b \\ b & b \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	C_2
			$\begin{pmatrix} b & b \\ b & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$		C_4

Type	\leftarrow	\rightarrow	\triangleleft	\triangleright	φ_*	Name
D	$\begin{pmatrix} a & a \\ b & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} \lambda a & \lambda a \\ \lambda a & \lambda a \end{pmatrix}, \begin{pmatrix} \lambda a & \lambda a \\ \lambda b & \lambda b \end{pmatrix}$	$D'_1(\lambda), D''_1(\lambda)$
			$\begin{pmatrix} a & b \\ a & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ b & b \end{pmatrix}$		$D'_2(\lambda), D''_2(\lambda)$
E	$\begin{pmatrix} a & a \\ b & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ b & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} \lambda a & \lambda a \\ \lambda b & \lambda b \end{pmatrix}$	$E_1(\lambda)$
			$\begin{pmatrix} a & b \\ a & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ b & b \end{pmatrix}$		$E_3(\lambda)$
			$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} b & b \\ b & b \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	E_2
F	$\begin{pmatrix} a & a \\ b & b \end{pmatrix}$	$\begin{pmatrix} a & b \\ a & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} (\lambda + \mu)a & (\lambda + \mu)a \\ (\lambda + \mu)a & \lambda a + \mu b \end{pmatrix}, \begin{pmatrix} (\lambda + \mu)a & (\lambda + \mu)b \\ (\lambda + \mu)b & \lambda a + \mu b \end{pmatrix},$ $\begin{pmatrix} \lambda a & \lambda b \\ \lambda a & \lambda b \end{pmatrix}, \begin{pmatrix} \lambda a & \lambda a \\ \lambda b & \lambda b \end{pmatrix}$	$F'_1(\lambda, \mu), F''_1(\lambda, \mu)$ $F'_1(\lambda), F''_1(\lambda)$
			$\begin{pmatrix} a & b \\ a & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ b & b \end{pmatrix}$		any associative product
			$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$	$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$	$\begin{pmatrix} \lambda a & 0 \\ 0 & \lambda b \end{pmatrix}$	$F_4(\lambda)$
			$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} b & b \\ b & b \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	F_2
			$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$	$\begin{pmatrix} b & a \\ a & b \end{pmatrix}$		F_5
G	$\begin{pmatrix} a & b \\ a & b \end{pmatrix}$	$\begin{pmatrix} a & b \\ a & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} \lambda a & \lambda b \\ \lambda a & \lambda b \end{pmatrix}$	$G_1(\lambda)$
			$\begin{pmatrix} a & b \\ a & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ b & b \end{pmatrix}$		$G_3(\lambda)$
			$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} b & b \\ b & b \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	G_2
H	$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$	$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} \lambda a & \lambda b \\ \lambda b & \lambda a \end{pmatrix}$	$H_1(\lambda)$
			$\begin{pmatrix} a & b \\ a & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ b & b \end{pmatrix}$		$H_2(\lambda)$

The commutative λ -ETS are the ones of type A and H , $C_1(\lambda)$, $C_3(\lambda)$, $C_5(\lambda)$, $F'_1(\lambda, \mu)$, $F''_1(\lambda, \mu)$ and $F_4(\lambda)$. The opposite of $B'_1(\lambda)$, $B''_1(\lambda)$, $B'_2(\lambda)$ and $B''_2(\lambda)$ are respectively $D'_1(\lambda)$, $D''_1(\lambda)$, $D'_2(\lambda)$ and $D''_2(\lambda)$. The opposite of C_2 is C_4 . The opposite of $E_1(\lambda)$, E_2 and $E_3(\lambda)$ are respectively $G_1(\lambda)$, G_2 and $G_3(\lambda)$. The opposite of $F'_1(\lambda)$ is $F''_1(\lambda)$. The λ -ETS F_2 and F_5 are not commutative but are isomorphic to their opposite in a non trivial way. Finally, if \cdot is a product, the opposite of $F_3(\cdot)$ is $F_3(\cdot^{op})$.

3.3. A description of all ETS of cardinality two. The following table gives all the ETS of cardinality 2.

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