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ON SOME REACTION-DIFFUSION EQUATIONS GENERATED BY NON-DOMICILIATED TRIATOMINAE, VECTORS OF CHAGAS DISEASE

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Abstract. In this work, we study some reaction-diffusion equations set in two habitats which model the spatial dispersal of the triatomines, vectors of Chagas disease. We prove in particular that the dispersal operator generates an analytic semigroup in an adequate space and we prove the local existence of the solution for the corresponding Cauchy problem.

1. **Introduction.** Chagas disease or American trypanosomiasis is a life-threatening disease caused by the flagellated protozoan parasite *Trypanosoma cruzi (T.cruzi)*. It is mainly transmitted by blood-sucking bugs belonging to the subfamily of triatominae. Via these vectors, the parasite can infect humans as well as a large number of domestic or wild mammalians. If bugs live in the nests of non domesticated mammals or birds, they are said to be sylvatic. If they lives in shelters neighboring human habitations, they are said to be domestic.

The different process involved in *T.cruzi* transmission by non-domiciliated bugs are very complex and their understanding goes through knowledge of vector ecology. In particular, demography and spatial dispersal are important processes during the re-infestation of a domestic area. In most cases, they are not captured by means of laboratory studies. Mathematical modeling stand a good tool to gain insights into

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these processes [21]. These approach used, for example, ordinary differential equations [15], partial differential equations [24] and integro-difference equations [20], [18]. Other studies based on scientific calculus were also used as cellular automaton [23], [5] and agent-based model [6], [9].

In this study we consider the infestation of a village by the domestic household species *T. Dimidiata*. The village adjoins a forest representing the habitat of the vectors. The latter move to the village for food [22]. Food consists of a blood meal on humans or the mammals they raise. The transmission of *T. Cruzi* from the vector to the host takes place mainly during this phase. After the kissing bug, individuals move again to seek habitat either at the village or return to the forest to trigger their fecundity.

In this work, we consider a triatomine population structured in time and space. Demography and spatial dispersal processes are captured by the reaction-diffusion equations in a two-dimensional space. In adequate functional spaces, the partial differential equations system is transformed into an abstract differential equation. Our first aim is to show that the operator generates an analytic semigroup. We prove then the existence of a local solution to the corresponding Cauchy problem.

2. **The model.**

2.1. **Definition of the habitats.**

We represent the infected village by the rectangular domain $(-d, 0) \times (0, 1)$ denoted Ω_-. Between the village and the forest exists, as recommended by Shender and all [3], a buffer zone $(0, D) \times (0, 1)$ denoted Ω_+. The forest covers the half plane $x > D$ (Figure 1).

To simplify the study, we assume that the parts of the border of $\Omega_- \cup \Omega_+$ defined by $\Gamma_1 = [-d, D] \times \{1\}, \Gamma_2 = [-d, D] \times \{0\}, \Gamma_3 = \{-d\} \times [0, 1]$ are natural barriers delimiting lethal areas for the population (mountains for example).

The common border $\Gamma = \partial\Omega_- \cap \partial\Omega_+$, called interface (Figure 1), plays an important role in the spatio-temporal dynamics of the population. Indeed, the triatomines in the buffer zone Ω_+ neighboring $\partial\Omega_- \cap \partial\Omega_+$ are attracted towards the village to feed or find refuge. Their movement defines then a skew Brownian motion [2].

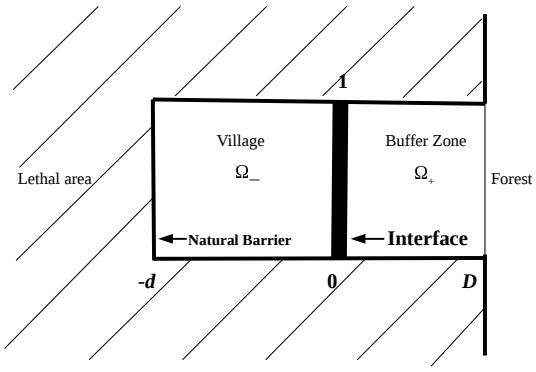

FIGURE 1. Population density in two habitats.

2.2. **Biological considerations.** Although the life cycle of triatomines consists of seven stages of development: an egg stage, five larval stages and an adult stage, the development of the egg in the fifth larval stage is considered to be a single stage which will be called the juvenile stage (Figure2). This assumption is realistic since the adults longevity is greater than the developmental time from the egg stage to the fifth larval stage [19].

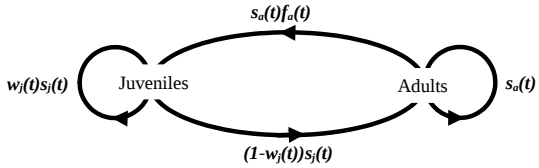

FIGURE 2. A schematic representation of the life cycle used in the triatomine's model.

Let us denote by $J(t,x,y)$ and $A(t,x,y)$ the respective densities of juveniles and adults classes at time $t>0$ and at a point (x,y). Assuming a balanced sex-ratio $\tau=1:1$ (one female for one male). We focus our modeling on female bugs density in the domain $\Omega_-\cup\Omega_+$. We assume that, inside this domain, the demographic parameters do not depend on the spatial position. During a time step dt between t and $t+dt$, juveniles having survived until t with a probability $s_j(t)$ will remain juveniles with a probability $w_j(t)$ or will transit to adult stage with a probability $(1-w_j(t))$. Adults who survived with a probability $s_a(t)$ will lay eggs with a rate $f_a(t)$. The entire life cycle is shown in(Figure 2).

Except in the neighborhood of the interface, the biological population spatial dispersal in the domain $\Omega_-\cup\Omega_+$ is modeled by a diffusion process for juveniles and adults with respective constants $d_j>0$ and $d_a>0$. These coefficients are non-negative as, whatever their developmental stage, bugs must move for their blood meal [16].

If during the demographic process we assumed identical demographic parameters in $\Omega_-\cup\Omega_+$, it is clear that the diffusion constants depend on the nature of each part of the domain. We will therefore denote, d_{j+} and d_{a+} the diffusion coefficients constants in Ω_+ and d_{j-} and d_{a-} the diffusion coefficients constants in Ω_-. The demographic and diffusion parameters of *T. Dimidiata* population are summarized in Table 1.

2.3. **Reaction-diffusion system.** By denoting J_- and A_-(respectively J_+ and A_+) the densities of the triatomines in Ω_-, (respectively in Ω_+), the biological system is modeled by the following system of reaction-diffusion equations:

$$\begin{cases} \dfrac{\partial J_-}{\partial t} = d_{j_-}\Delta J_- + \omega_j(t)s_j(t)J_- + f_a(t)s_a(t)A_- \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \Omega_- \\ \dfrac{\partial A_-}{\partial t} = d_{a_-}\Delta A_- + (1-\omega_j(t))s_j(t)J_- + s_a(t)A_- \\ \dfrac{\partial J_+}{\partial t} = d_{j_+}\Delta J_+ + \omega_j(t)s_j(t)J_+ + f_a(t)s_a(t)A_+ \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \Omega_+ \\ \dfrac{\partial A_+}{\partial t} = d_{a_+}\Delta A_+ + (1-\omega_j(t))s_j(t)J_+ + s_a(t)A_+ \ , \end{cases}$$

TABLE 1. The demographic and diffusion parameters of *T.Dimidiata* population.

Parameter	Definition	Properties
$s_j(t)$	Probability of survival of juveniles per unit of time	$0 \leq s_j(t) \leq 1$
$s_a(t)$	Probability of survival of adult per unit of time	$0 \leq s_a(t) \leq 1$
$w_j(t)$	Probability of transition from juvenile to adult	$0 \leq w_j(t) \leq 1$
$f_a(t)$	Female fertility per unit time	$f_a(t) \geq 0$
d_{j-}, d_{j+}	diffusion coefficient of juveniles	$d_{j-}, d_{j+} > 0$
d_{a-}, d_{a+}	diffusion coefficient of adults	$d_{a-}, d_{a+} > 0$

with initial conditions:

$$(IC) \begin{cases} J_-(0,x,y) = J_-^0(x,y) \\ A_-(0,x,y) = A_-^0(x,y) \end{cases} \quad \text{on } \Omega_-$$

$$\begin{cases} J_+(0,x,y) = J_+^0(x,y) \\ A_+(0,x,y) = A_+^0(x,y) \end{cases}, \quad \text{on } \Omega_+$$

and boundary conditions:

$$(EC_1) \begin{cases} J_-(.,x,0) = J_-(.,x,1) = 0 \\ A_-(.,x,0) = A_-(.,x,1) = 0 \end{cases} \quad x \in]-d, 0[$$

$$\begin{cases} J_+(.,x,0) = J_+(.,x,1) = 0 \\ A_+(.,x,0) = A_+(.,x,1) = 0 \end{cases}, \quad x \in]0, D[$$

$$(EC_2) \begin{cases} \begin{pmatrix} J_- \\ A_- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{on } \{-d\} \times]0,1[\\ \dfrac{\partial}{\partial \nu} \begin{pmatrix} J_+ \\ A_+ \end{pmatrix} = 0 & \text{on } \{D\} \times]0,1[\, . \end{cases}$$

In the last equation, $\partial/\partial\nu$ denotes the normal external derivative on $\{D\} \times]0,1[$. In our geometry, it coincides with $\partial/\partial x$. In (EC_2), the first condition means that the population dies on $\{-d\} \times]0,1[$ (as in (EC_1)) and that its flux vanishes on $\{D\} \times]0,1[$. Other boundary conditions may be considered instead of those in (EC_2).

At the interface Γ, following Cantrell and Cosner [3], we will also consider the skew brownian motion conditions and the continuity of the densities.

$$(Int.C)\quad \begin{cases} pd_{j_-}\dfrac{\partial J_-}{\partial \nu} = (1-p)d_{j_+}\dfrac{\partial J_+}{\partial \nu} \\ pd_{a_-}\dfrac{\partial A_-}{\partial \nu} = (1-p)d_{a_+}\dfrac{\partial A_+}{\partial \nu} \qquad \text{on} \quad \Gamma = \{0\}\times]0,1[\\ \begin{pmatrix} J_- \\ A_- \end{pmatrix} = \begin{pmatrix} J_+ \\ A_+ \end{pmatrix}, \end{cases}$$

where $p > 1/2$ is the probability to cross the interface from Ω_+ to Ω_-.

We put:

$$\begin{aligned} pd_{j_-} &= \mu_-, \quad (1-p)d_{j_+} = \mu_+, \\ pd_{a_-} &= \alpha_-, \quad (1-p)d_{a_+} = \alpha_+, \end{aligned}$$

$$V(t,x,y) = \begin{pmatrix} J(t,x,y) \\ A(t,x,y) \end{pmatrix} \text{ on } \Omega = \Omega_- \cup \Omega_+,\ t \geq 0,$$

$$V_- = V_{|\Omega_-} = \begin{pmatrix} J_- \\ A_- \end{pmatrix},\ V_+ = V_{|\Omega_+} = \begin{pmatrix} J_+ \\ A_+ \end{pmatrix},$$

$$\begin{aligned} \mathcal{P}_- + \mathcal{B}(t) &= \begin{pmatrix} d_{j_-}\Delta & 0 \\ 0 & d_{a_-}\Delta \end{pmatrix} + \begin{pmatrix} \omega_j(t)s_j(t)I & f_a(t)s_a(t)I \\ (1-\omega_j(t))s_j(t)I & s_a(t)I \end{pmatrix} \\ \mathcal{P}_+ + \mathcal{B}(t) &= \begin{pmatrix} d_{j_+}\Delta & 0 \\ 0 & d_{a_+}\Delta \end{pmatrix} + \begin{pmatrix} \omega_j(t)s_j(t)I & f_a(t)s_a(t)I \\ (1-\omega_j(t))s_j(t)I & s_a(t)I \end{pmatrix}. \end{aligned}$$

These two matrices are operator matrices.

The following operator $\mathcal{L}$ (which acts with respect to the spatial variables (x,y)) is then defined by:

$$\begin{aligned} \mathcal{L}V(t,.,.) &= \mathcal{L}\begin{pmatrix} J(t,.,.) \\ A(t,.,.) \end{pmatrix} \\ &= \begin{cases} \begin{pmatrix} d_{j_-}\Delta & 0 \\ 0 & d_{a_-}\Delta \end{pmatrix}\begin{pmatrix} J_-(t,.,.) \\ A_-(t,.,.) \end{pmatrix} & \text{on } \Omega_-, \\ \begin{pmatrix} d_{j_+}\Delta & 0 \\ 0 & d_{a_+}\Delta \end{pmatrix}\begin{pmatrix} J_+(t,.,.) \\ A_+(t,.,.) \end{pmatrix} & \text{on } \Omega_+, \end{cases} \\ &= \begin{cases} \begin{pmatrix} d_{j_-}\Delta J_-(t,.,.) \\ d_{a_-}\Delta A_-(t,.,.) \end{pmatrix} & \text{on } \Omega_-, \\ \begin{pmatrix} d_{j_+}\Delta J_+(t,.,.) \\ d_{a_+}\Delta A_+(t,.,.) \end{pmatrix} & \text{on } \Omega_+, \end{cases} \end{aligned}$$

with

$$\begin{aligned} D(\mathcal{L}) = \Big\{ & W = (v,w) \in [L^q(\Omega)]^2 : W_- = W_{|\Omega_-} = (v_-,w_-) \in \left[W^{2,q}(\Omega_-)\right]^2 \\ & W_+ = W_{|\Omega_+} = (v_+,w_+) \in \left[W^{2,q}(\Omega_+)\right]^2 \text{ and} \\ & W_-, W_+ \text{ verifying } (Int.C),\ (EC_1),(EC_2) \Big\}, \end{aligned}$$

where $q \in]1,+\infty[$. The following classic operational vector notation will be adopted

$$V(t,x,y) := V(t)(x,y).$$

So the previous problem is written in the form of an abstract reaction-diffusion system:

$$\begin{cases} V'(t) = \mathcal{L}V(t) + \mathcal{B}(t)V(t) \quad t > 0 \\ \\ V(0) = V^0 = \begin{pmatrix} J^0(0,.,.) \\ A^0(0,.,.) \end{pmatrix} , \end{cases} \tag{1}$$

set in the Banach space $\mathcal{E} = [L^q(\Omega)]^2$ normed by:

$$\left\| \begin{pmatrix} v \\ w \end{pmatrix} \right\|_{\mathcal{E}} = \max \left(\|v\|_{L^q(\Omega)}, \|w\|_{L^q(\Omega)} \right) , \tag{2}$$

which is equivalent to

$$\max \left(\|v_-\|_{L^q(\Omega_-)} + \|w_-\|_{L^q(\Omega_-)}, \|v_+\|_{L^q(\Omega_+)} + \|w_+\|_{L^q(\Omega_+)} \right) .$$

3. **The spectral equation.** The study of problem (1) is based on the spectral equation

$$\mathcal{L}W - \lambda W = F \in [L^q(\Omega)]^2 , \tag{3}$$

for complex λ in a sector to specify and on the good behavior of:

$$\left\| (\mathcal{L} - \lambda I)^{-1} \right\|_{L\left([L^q(\Omega)]^2\right)} ,$$

in order to show that $\mathcal{L}$ generates an analytic semigroup**.** So, after the resolution of the spectral equation, we have to estimate:

$$\begin{aligned} \|W\|_{[L^q(\Omega)]^2} &= \max \left(\|v\|_{L^q(\Omega)}, \|w\|_{L^q(\Omega)} \right) \\ &= \max \left(\|v_-\|_{L^q(\Omega_-)} + \|v_+\|_{L^q(\Omega_+)}, \|w_-\|_{L^q(\Omega_-)} + \|w_+\|_{L^q(\Omega_+)} \right) , \end{aligned}$$

where:

$$W = \begin{pmatrix} v \\ w \end{pmatrix} , \quad F = \begin{pmatrix} f \\ g \end{pmatrix} ,$$

then (3) gives:

$$\begin{cases} \begin{pmatrix} d_{j_-}\Delta & 0 \\ 0 & d_{a_-}\Delta \end{pmatrix} \begin{pmatrix} v_- \\ w_- \end{pmatrix} - \lambda \begin{pmatrix} v_- \\ w_- \end{pmatrix} = \begin{pmatrix} f_- \\ g_- \end{pmatrix} \quad \text{on } \Omega_-, \\ \\ \begin{pmatrix} d_{j_+}\Delta & 0 \\ 0 & d_{a_+}\Delta \end{pmatrix} \begin{pmatrix} v_+ \\ w_+ \end{pmatrix} - \lambda \begin{pmatrix} v_+ \\ w_+ \end{pmatrix} = \begin{pmatrix} f_+ \\ g_+ \end{pmatrix} \quad \text{on } \Omega_+, \end{cases}$$

with the transmission conditions

$$(Int.C) \begin{cases} \begin{pmatrix} v_- \\ w_- \end{pmatrix} = \begin{pmatrix} v_+ \\ w_+ \end{pmatrix} \quad \text{on } \Gamma, \\ \\ \begin{pmatrix} \mu_- \dfrac{\partial v_-}{\partial x} \\ \alpha_- \dfrac{\partial w_-}{\partial x} \end{pmatrix} = \begin{pmatrix} \mu_+ \dfrac{\partial v_+}{\partial x} \\ \alpha_+ \dfrac{\partial w_+}{\partial x} \end{pmatrix} \quad \text{on } \Gamma, \end{cases}$$

boundary conditions

$$(EC_1)\begin{cases} v_-(x,0)=v_-(x,1)=0 \\ w_-(x,0)=w_-(x,1)=0\ , & x\in]-d,0[\\ \\ v_+(x,0)=v_+(x,1)=0 \\ w_+(x,0)=w_+(x,1)=0\ , & x\in]0,D[\end{cases}$$

and

$$(EC_2)\begin{cases} \begin{pmatrix} v_- \\ w_- \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ on } \{-d\}\times]0,1[\ , \\ \\ \dfrac{\partial}{\partial x}\begin{pmatrix} v_+ \\ w_+ \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ on } \{D\}\times]0,1[\ . \end{cases}$$

We explain the previous spectral system:

$$\begin{cases} d_{j_-}\Delta v_- - \lambda v_- = f_- \\ d_{a_-}\Delta w_- - \lambda w_- = g_-\ , & \text{on } \Omega_- \\ \\ d_{j_+}\Delta v_+ - \lambda v_+ = f_+ \\ d_{a_+}\Delta w_+ - \lambda w_+ = g_+\ , & \text{on } \Omega_+ \end{cases}$$

with

$$(Int.C)\begin{cases} v_-(0,y)=v_+(0,y) \quad y\in]0,1[\ , \\ w_-(0,y)=w_+(0,y) \quad y\in]0,1[\ , \\ \\ \mu_-\dfrac{\partial v_-}{\partial x}(0,y)=\mu_+\dfrac{\partial v_+}{\partial x}(0,y) \quad y\in]0,1[\ , \\ \\ \alpha_-\dfrac{\partial w_-}{\partial x}(0,y)=\alpha_+\dfrac{\partial w_+}{\partial x}(0,y) \quad y\in]0,1[\ , \end{cases}$$

and

$$(EC_1)\begin{cases} v_-(x,0)=v_-(x,1)=0 \\ w_-(x,0)=w_-(x,1)=0\ , & x\in]-d,0[\\ \\ v_+(x,0)=v_+(x,1)=0 \\ w_+(x,0)=w_+(x,1)=0\ , & x\in]0,D[\end{cases}$$

$$(EC_2)\begin{cases} v_-(-d,y)=w_-(-d,y)=0 \quad y\in]0,1[\ , \\ \\ \dfrac{\partial v_+}{\partial x}(D,y)=\dfrac{\partial w_+}{\partial x}(D,y)=0 \quad y\in]0,1[\ . \end{cases}$$

This system can be divided into two subsystems; one of which is governed by (v_-,v_+) (of the juveniles):

$$\begin{cases} d_{j_-}\Delta v_- - \lambda v_- = f_- \text{ on } \Omega_-\ , \\ d_{j_+}\Delta v_+ - \lambda v_+ = f_+ \text{ on } \Omega_+\ , \end{cases}$$

with

$$(Int.C_v)\begin{cases} v_-(0,y)=v_+(0,y) \quad y\in]0,1[\ , \\ \\ \mu_-\dfrac{\partial v_-}{\partial x}(0,y)=\mu_+\dfrac{\partial v_+}{\partial x}(0,y) \quad y\in]0,1[\ , \end{cases}$$

$$(EC_{1v})\begin{cases} v_-(x,0)=v_-(x,1)=0 \quad x\in]-d,0[\ , \\ v_+(x,0)=v_+(x,1)=0 \quad x\in]0,D[\ , \end{cases}$$

$$(EC_{2v})\begin{cases} v_-(-d,y)=0 \quad y\in]0,1[\ , \\ \\ \dfrac{\partial v_+}{\partial x}(D,y)=0 \quad y\in]0,1[\ ; \end{cases}$$

and the other by the couple (w_-,w_+) (of adults) checking a similar system. The difference between them lies in the diffusion coefficients, which are all different (here, we have assumed them to be constant). Therefore, it will be sufficient to analyze one system.

4. **The operational formulation.** Let's introduce, in Banach space $E=L^q(0,1)$, operator Λ defined by:

$$\begin{cases} D(\Lambda)=\left\{\varphi\in W^{2,q}(0,1):\varphi(0)=\varphi(1)=0\right\} \\ (\Lambda\varphi)(y)=\varphi''(y)\ . \end{cases}$$

For $\omega\in[0,\pi]$ we define the sector:

$$S_\omega:=\begin{cases} \{z\in\mathbb{C}\backslash\{0\}:|\arg z|<\omega\} & \text{if } \omega\in]0,\pi] \\ ]0,+\infty[& \text{if } \omega=0\ . \end{cases}$$

It is known that this operator is a closed linear operator with a dense domain and verifies:

$$\begin{cases} \overline{D(\Lambda)}=E, \text{ for all } \eta\in]0,\pi[\,,\rho(\Lambda)\supset S_{\pi-\eta}\cup\{0\} \text{ and} \\ \exists C>0:\forall z\in S_{\pi-\eta}\cup\{0\}\,,\left\|(zI-\Lambda)^{-1}\right\|_{L(E)}\leq\dfrac{C}{1+|z|}\ , \end{cases} \tag{4}$$

and there exists a ball $B(0,\delta)$, $\delta>0$, such that $\rho(\Lambda)\supset\overline{B(0,\delta)}$ and the above estimate is true in $S_{\pi-\eta}\cup\overline{B(0,\delta)}$. Here $\rho(\Lambda)$ denotes the resolvent set of Λ. The following usual operational notation of vector-valued functions:

$$v_\pm(x)(y):=v_\pm(x,y)\ ,$$

leads the previous system (3) for the couple (v_-,v_+), in space E, to be formulated by:

$$(S1)\begin{cases} v_-''(x)+\Lambda v_-(x)-\dfrac{\lambda}{d_{j_-}}v_-(x)=\dfrac{f_-(x)}{d_{j_-}} & \text{on }]-d,0[\ , \\ v_+''(x)+\Lambda v_+(x)-\dfrac{\lambda}{d_{j_+}}v_+(x)=\dfrac{f_+(x)}{d_{j_+}} & \text{on }]0,D[\ , \\ v_-(-d)=0\ , \\ v_+'(D)=0\ , \\ v_-(0)=v_+(0)\ , \\ \mu_-v_-'(0)=\mu_+v_+'(0)\ . \end{cases} \tag{5}$$

Boundary conditions (EC_{1v}) are implicit and expressed by the action of Λ.

To estimate the resolvent operator, we must estimate:

$$\|v_-\|_{L^q(\Omega_-)}\,,\|v_+\|_{L^q(\Omega_+)}\,,\|w_-\|_{L^q(\Omega_-)} \text{ and } \|w_+\|_{L^q(\Omega_+)}\,.$$

Note that:

$$\|v_-\|_{L^q(\Omega_-)}=\|v_-\|_{L^q((-d,0);E)} \text{ and } \|v_+\|_{L^q(\Omega_+)}=\|v_+\|_{L^q((0,D);E)}\,.$$

5. **Some technical results.** We will use the following results.

Lemma 5.1. *Let $\eta \in]0, \pi/2[$. For any $z \in S_\eta$, we have*

1. $|\arg(1 - e^{-z}) - \arg(1 + e^{-z})| < \eta$
2. $|1 + e^{-z}| \geq C_\eta = 1 - e^{-\pi/(2\tan(\eta))} > 0$
3. $\dfrac{|z|\cos\eta}{1 + |z|\cos\eta} \leq |1 - e^{-z}| \leq \dfrac{2\,|z|}{1 + |z|\cos\eta}.$

□

See [7], Proposition 4.10, p.1880.

Lemma 5.2. *Let $w, z \in \mathbb{C}\setminus\{0\}$. We have*

$$|w + z| \geq (|w| + |z|)\left|\cos\frac{\arg w - \arg z}{2}\right|.$$

□

See Proposition 4.9, p.1879 in [7].

Now let us recall some results in [13].

Set $\omega \in [0, \pi[$, a linear operator K on a complex Banach space E is called sectorial of angle ω if

1. $\sigma(K) \subset \overline{S_\omega}$ and
2. $M(K, \omega') := \sup_{\lambda \in \mathbb{C}\setminus\overline{S_{\omega'}}} \left\|\lambda(K - \lambda I)^{-1}\right\| < \infty$ for all $\omega' \in]\omega, \pi[$.

We then write: $K \in Sect(\omega)$. The following angle

$$\omega_K := \min\{\omega \in [0, \pi[: K \in Sect(\omega)\},$$

is called the spectral angle of K. Statement2. implies necessarily that K is closed.

Proposition 1. *If $]-\infty, 0[\subset \rho(K)$ and*

$$M(K) := M(K, \pi) := \sup_{\mu > 0} \left\|\mu(K + \mu I)^{-1}\right\| < \infty,$$

then $M(K) \geq 1$ and

$$K \in Sect(\pi - \arcsin(1/M(K)).$$

See[13]*, proposition 2.1.1* □

Proposition 2. *Let K a sectorial operator and $\nu \in]0, 1/2]$. Then $K^\nu \in Sect(\nu\omega_K)$, and therefore $-K^\nu$ generates an analytic semigroup.*

See [13] *p. 80-81* □

Put

$$H^\infty(S_\omega) = \{f : f \text{ is analytic and bounded on } S_\omega\},$$

with $\omega \in]0, \pi[$; we recall that if $f \in H^\infty(S_\omega)$ is such that $1/f \in H^\infty(S_\omega)$ and

$$(1/f)(K) \in \mathcal{L}(X),$$

then $f(K)$ is boundedly invertible and

$$[f(K)]^{-1} = (1/f)(K), \tag{6}$$

see, for example [4].

6. **Resolution of the first system (5).** For the resolution of the previous system we apply an analogous method to the one used in paper [11].

Here we will assume that λ is such that

$$|\arg(\lambda)| < \pi - \varepsilon_0, \tag{7}$$

where ε_0 is a small fixed number.

Consider the two following operators:

$$\Lambda_\lambda^- = \Lambda - \frac{\lambda}{d_{j_-}} I, \qquad \Lambda_\lambda^+ = \Lambda - \frac{\lambda}{d_{j_+}} I,$$

which have the same domain

$$D(\Lambda_\lambda^-) = D(\Lambda_\lambda^+) = D(\Lambda).$$

As for operator $-\Lambda$, we verify that

$$-\Lambda_\lambda^- = -\Lambda + \frac{\lambda}{d_{j_-}} I,$$

is sectorial in $E = L^q(0,1)$. Indeed, we have: $]-\infty, 0] \subset \rho(-\Lambda_\lambda^-)$ and if we put

$$M(-\Lambda_\lambda^-) := M(-\Lambda_\lambda^-, \pi) := \sup_{\mu>0} \left\| \mu(-\Lambda_\lambda^- + \mu I)^{-1} \right\|,$$

then for λ such that

$$|\arg(\lambda)| < \pi - \varepsilon_0,$$

and by an explicit calculus we obtain:

$$M(-\Lambda_\lambda^-) \leq \sup_{\mu>0} \left(\frac{\mu}{\cos\left[\frac{1}{2}\arg\left(\frac{\lambda}{d_{j_-}} + \mu\right)\right]} \frac{1}{\left|\frac{\lambda}{d_{j_-}} + \mu\right|} \right).$$

We have two possible cases:

1. if $|\arg(\lambda)| < \pi/2$ then for all $\mu > 0$:
$$\left|\frac{\lambda}{d_{j_-}} + \mu\right| \geq \mu,$$
2. if $\pi/2 \leq |\arg(\lambda)| < \pi - \varepsilon_0$, then
$$\left|\frac{\lambda}{d_{j_-}} + \mu\right| \geq \mu \sin\alpha,$$
where $\alpha \in]\varepsilon_0, \pi/2]$; therefore:
$$\left|\frac{\lambda}{d_{j_-}} + \mu\right| \geq \mu \sin\varepsilon_0,$$

we deduce that there exists a constant C independent of λ such that

$$\begin{aligned} M(-\Lambda_\lambda^-) &\leq \sup_{\mu>0} \left(\frac{\mu}{\cos\left[\frac{1}{2}\arg\left(\frac{\lambda}{d_{j_-}} + \mu\right)\right]} \frac{1}{\left|\frac{\lambda}{d_{j_-}} + \mu\right|} \right) \\ &\leq \frac{C}{\cos\frac{\pi-\varepsilon_0}{2}} = \frac{C}{\sin(\varepsilon_0/2)} < \infty. \end{aligned}$$

Hence

$$-\Lambda_\lambda^- \in Sect\left[\pi - \arcsin\left(1/M(-\Lambda_\lambda^-)\right)\right],$$

where $M(-\Lambda_\lambda^-)$ is independent of λ, see Proposition 2.1.1 in [13].

In a similar way, we obtain

$$-\Lambda_\lambda^+ \in Sect\left(\pi - \arcsin\left[1/M(-\Lambda_\lambda^+)\right]\right),$$

with $M(-\Lambda_\lambda^+)$ is independent of λ.

We also deduce that the two following operators:

$$Q_\lambda^- = -\left[-\left(\Lambda - \frac{\lambda}{d_{j_-}}I\right)\right]^{1/2}, \quad Q_\lambda^+ = -\left[-\left(\Lambda - \frac{\lambda}{d_{j_+}}I\right)\right]^{1/2},$$

of the same domain

$$D(Q_\lambda^-) = D(Q_\lambda^+) = D\left([-\Lambda]^{1/2}\right) := D(Q),$$

(here $Q := -[-\Lambda]^{1/2}$), are well defined and generate analytic semigroups on E, see [13] p.81 and also [1].

Using Lemma 4.2, and estimates (28)-(29) in [10], there exist $\epsilon_- > 0$, $\epsilon_+ > 0$, $C_- > 0$ and $C_+ > 0$ independent of λ such that:

$$\begin{cases} \text{for all } z \in \left\{z \in \mathbb{C}\backslash\{0\} : |\arg(z)| \le \frac{\pi}{2} + \epsilon_-\right\}, \left\|\left(Q_\lambda^- - zI\right)^{-1}\right\| \le \dfrac{C_-}{\sqrt{1+|\lambda|}+|z|}, \\ \text{for all } z \in \left\{z \in \mathbb{C}\backslash\{0\} : |\arg(z)| \le \frac{\pi}{2} + \epsilon_+\right\}, \left\|\left(Q_\lambda^+ - zI\right)^{-1}\right\| \le \dfrac{C_+}{\sqrt{1+|\lambda|}+|z|}. \end{cases} \tag{8}$$

We will focus oureselves to solve the system

$$(S1)\begin{cases} v_-''(x) + \Lambda v_-(x) - \dfrac{\lambda}{d_{j_-}}v_-(x) = \dfrac{f_-(x)}{d_{j_-}} := N_-(x) \quad \text{on }]-d, 0[\\ v_+''(x) + \Lambda v_+(x) - \dfrac{\lambda}{d_{j_+}}v_+(x) = \dfrac{f_+(x)}{d_{j_+}} := N_+(x) \quad \text{on }]0, D[\\ v_-(-d) = 0 \\ v_+'(D) = 0 \\ v_-(0) = v_+(0) \\ \mu_- v_-'(0) = \mu_+ v_+'(0), \end{cases}$$

which is written in the form

$$\begin{cases} v_-''(x) - (Q_\lambda^-)^2 v_-(x) = N_-(x) \quad \text{on }]-d, 0[\\ \\ v_+''(x) - (Q_\lambda^+)^2 v_+(x) = N_+(x) \quad \text{on }]0, D[\\ v_-(-d) = 0 \\ v_+'(D) = 0 \\ v_-(0) = v_+(0) \\ \mu_- v_-'(0) = \mu_+ v_+'(0). \end{cases}$$

Then

$$v_\pm(x) = e^{(x-c_\pm)Q_\lambda^\pm}\gamma_\pm + e^{(b_\pm - x)Q_\lambda^\pm}\delta_\pm + w_\pm(N)(x),$$

where $\gamma_\pm$, $\delta_\pm \in E$; $c_- = -d, b_- = 0 : c_+ = 0, b_+ = D$ and

$$w_\pm(N_\pm)(x) = \frac{1}{2}\int_{c_\pm}^{x} e^{(x-t)Q_\lambda^\pm}(Q_\lambda^\pm)^{-1}N_\pm(t)dt + \frac{1}{2}\int_{x}^{b_\pm} e^{(t-x)Q_\lambda^\pm}(Q_\lambda^\pm)^{-1}N_\pm(t)dt.$$

Therefore

$$\begin{aligned} v_-(x) &= e^{(x+d)Q_\lambda^-}\gamma_- + e^{-xQ_\lambda^-}\delta_- + w_-(N_-)(x), \ \ x \in]-d, 0[\\ v_+(x) &= e^{xQ_\lambda^+}\gamma_+ + e^{(D-x)Q_\lambda^+}\delta_+ + w_+(N_+)(x), \ \ x \in]0, D[, \end{aligned}$$

and

$$\begin{aligned} w_-(N_-)(x) &= \frac{1}{2}\int_{-d}^{x} e^{(x-t)Q_\lambda^-}(Q_\lambda^-)^{-1}N_-(t)dt + \frac{1}{2}\int_{x}^{0} e^{(t-x)Q_\lambda^-}(Q_\lambda^-)^{-1}N_-(t)dt, \\ w_+(N_+)(x) &= \frac{1}{2}\int_{0}^{x} e^{(x-t)Q_\lambda^+}(Q_\lambda^+)^{-1}N_+(t)dt + \frac{1}{2}\int_{x}^{D} e^{(t-x)Q_\lambda^+}(Q_\lambda^+)^{-1}N_+(t)dt, \end{aligned}$$

hence

$$\begin{aligned} v_-'(x) &= Q_\lambda^- e^{(x+d)Q_\lambda^-}\gamma_- - Q_\lambda^- e^{-xQ_\lambda^-}\delta_- + w_-'(N_-)(x), \ \ x\in\,]-d,0[\\ v_+'(x) &= Q_\lambda^+ e^{xQ_\lambda^+}\gamma_+ - Q_\lambda^+ e^{(D-x)Q_\lambda^+}\delta_+ + w_+'(N_+)(x), \ \ x\in\,]0,D[\,, \end{aligned}$$

$$\begin{aligned} v_-'(0) &= Q_\lambda^- e^{dQ_\lambda^-}\gamma_- - Q_\lambda^-\delta_- + w_-'(N_-)(0), \\ v_+'(0) &= Q_\lambda^+\gamma_+ - Q_\lambda^+ e^{DQ_\lambda^+}\delta_+ + w_+'(N_+)(0). \end{aligned}$$

Boundary conditions

$$\begin{cases} v_-(-d) = \gamma_- + e^{dQ_\lambda^-}\delta_- + w_-(N_-)(-d) = 0 \\ v_+'(D) = Q_\lambda^+ e^{DQ_\lambda^+}\gamma_+ - Q_\lambda^+\delta_+ + w_+'(N_+)(D) = 0, \end{cases}$$

imply

$$\begin{cases} \gamma_- = -e^{dQ_\lambda^-}\delta_- - w_-(N_-)(-d) \\ \delta_+ = e^{DQ_\lambda^+}\gamma_+ + (Q_\lambda^+)^{-1}w_+'(N_+)(D). \end{cases}$$

The transmission conditions give

$$\begin{cases} e^{dQ_\lambda^-}\gamma_- + \delta_- + w_-(N_-)(0) = \gamma_+ + e^{DQ_\lambda^+}\delta_+ + w_+(N_+)(0) \\ \mu_-\left[R_- e^{dQ_\lambda^-}\gamma_- - Q_\lambda^-\delta_- + w_-'(N_-)(0)\right] = \mu_+\left[Q_\lambda^+\gamma_+ - Q_\lambda^+ e^{DQ_\lambda^+}\delta_+ + w_+'(N_+)(0)\right]. \end{cases}$$

We then obtain:

$$\begin{cases} \gamma_- = -e^{dQ_\lambda^-}\delta_- - w_-(N_-)(-d), \\ \delta_+ = e^{DQ_\lambda^+}\gamma_+ + (Q_\lambda^+)^{-1}w_+'(N_+)(D), \\ \mu_-\left(Q_\lambda^- e^{dQ_\lambda^-}\gamma_- - Q_\lambda^-\delta_-\right) - \mu_+\left(Q_\lambda^+\gamma_+ - Q_\lambda^+ e^{DQ_\lambda^+}\delta_+\right) = \\ \qquad\qquad -\mu_- w_-'(N_-)(0) + \mu_+ w_+'(N_+)(0), \\ e^{dQ_\lambda^-}\gamma_- + \delta_- + w_-(N_-)(0) = \gamma_+ + e^{DQ_\lambda^+}\delta_+ + w_+(N_+)(0), \end{cases}$$

or

$$\begin{cases} \gamma_- = -e^{dQ_\lambda^-}\delta_- - w_-(N_-)(-d), \\ \delta_+ = e^{DQ_\lambda^+}\gamma_+ + (Q_\lambda^+)^{-1}w_+'(N_+)(D), \\ \mu_-\left(e^{dQ_\lambda^-}\gamma_- - \delta_-\right) - \mu_+\left((Q_\lambda^-)^{-1}Q_\lambda^+\gamma_+ - (Q_\lambda^-)^{-1}Q_\lambda^+ e^{DQ_\lambda^+}\delta_+\right) = (I) \\ \left[e^{dQ_\lambda^-}\gamma_- + \delta_-\right] - \left[\gamma_+ + e^{DQ_\lambda^+}\delta_+\right] = (II), \end{cases}$$

where

$$\begin{cases} (I) := (Q_\lambda^-)^{-1}\left[-\mu_- w_-'(N_-)(0) + \mu_+ w_+'(N_+)(0)\right] \\ (II) := w_+(N_+)(0) - w_-(N_-)(0). \end{cases}$$

Using the first two equations, we obtain

$$\begin{aligned}
& \mu_-\left(e^{dQ_\lambda^-}\gamma_- - \delta_-\right) - \mu_+\left((Q_\lambda^-)^{-1}Q_\lambda^+\gamma_+ - (Q_\lambda)^{--1}Q_\lambda^+ e^{DQ_\lambda^+}\delta_+\right)\\
= \;& \mu_-\left(e^{dQ_\lambda^-}\left[-e^{dQ_\lambda^-}\delta_- - w_-(N_-)(-d)\right] - \delta_-\right)\\
& -\mu_+\left(R_-^{-1}R_+\gamma_+ - R_-^{-1}Q_\lambda^+ e^{DQ_\lambda^+}\left[e^{DQ_\lambda^+}\gamma_+ + (Q_\lambda^+)^{-1}w_+'(N_+)(D)\right]\right)\\
= \;& -\mu_-\left[e^{2dQ_\lambda^-}\delta_- + \delta_-\right] - \mu_- e^{dQ_\lambda^-}w_-(N_-)(-d) - \mu_+(Q_\lambda^-)^{-1}Q_\lambda^+\gamma_+\\
& +\mu_+(Q_\lambda^-)^{-1}Q_\lambda^+ e^{2DQ_\lambda^+}\gamma_+ + \mu_+(Q_\lambda^-)^{-1}Q_\lambda^+ e^{DQ_\lambda^+}(Q_\lambda^+)^{-1}w_+'(N_+)(D)\\
= \;& -\mu_-\left[I + e^{2dQ_\lambda^-}\right]\delta_- - \mu_+(Q_\lambda^-)^{-1}Q_\lambda^+\left[I - e^{2DQ_\lambda^+}\right]\gamma_+\\
& +\mu_+(Q_\lambda^-)^{-1}e^{DQ_\lambda^+}w_+'(N_+)(D) - \mu_- e^{dQ_\lambda^-}w_-(N_-)(-d),
\end{aligned}$$

and

$$\begin{aligned}
& \left[e^{dQ_\lambda^-}\gamma_- + \delta_-\right] - \left[\gamma_+ + e^{DQ_\lambda^+}\delta_+\right]\\
= \;& e^{dQ_\lambda^-}\left[-e^{dQ_\lambda^-}\delta_- - w_-(N_-)(-d)\right] + \delta_-\\
& -\left[\gamma_+ + e^{DQ_\lambda^+}\left[e^{DQ_\lambda^+}\gamma_+ + (Q_\lambda^+)^{-1}w_+'(N_+)(D)\right]\right]\\
= \;& \left(I - e^{2dQ_\lambda^-}\right)\gamma_- - \left(I + e^{2DQ_\lambda^+}\right)\gamma_+\\
& -e^{dQ_\lambda^-}w_-(N_-)(-d) - e^{DQ_\lambda^+}(Q_\lambda^+)^{-1}w_+'(N_+)(D).
\end{aligned}$$

The system becomes

$$\begin{cases}
-\mu_-\left[I + e^{2dQ_\lambda^-}\right]\delta_- - \mu_+(Q_\lambda^-)^{-1}Q_\lambda^+\left[I - e^{2DQ_\lambda^+}\right]\gamma_+\\
= (I) - \mu_+(Q_\lambda^-)^{-1}e^{DQ_\lambda^+}w_+'(N_+)(D) + \mu_- e^{dQ_\lambda^-}w_-(N_-)(-d) := (I')\\
\left(I - e^{2dQ_\lambda^-}\right)\delta_- - \left(I + e^{2DQ_\lambda^+}\right)\gamma_+\\
= (II) + e^{dQ_\lambda^-}w_-(N_-)(-d) + e^{DQ_\lambda^+}(Q_\lambda^+)^{-1}w_+'(N_+)(D) := (II'),
\end{cases}$$

or

$$\begin{cases}
-\mu_-\left[I + e^{2dQ_\lambda^-}\right]\delta_- - \mu_+(Q_\lambda^-)^{-1}Q_\lambda^+\left[I - e^{2DQ_\lambda^+}\right]\gamma_+ = (I')\\
\left(I - e^{2dQ_\lambda^-}\right)\delta_- - \left(I + e^{2DQ_\lambda^+}\right)\gamma_+ = (II').
\end{cases}$$

The abstract determinant of this system is:

$$\begin{aligned}
& \Delta_{\lambda,\mu_-,\mu_+}\\
= \;& \mu_-\left(I + e^{2dQ_\lambda^-}\right)\left(I + e^{2DQ_\lambda^+}\right) + \mu_+(Q_\lambda^-)^{-1}Q_\lambda^+\left(I - e^{2DQ_\lambda^+}\right)\left(I - e^{2dQ_\lambda^-}\right).
\end{aligned}$$

For its invertibility, we use the H^∞- calculus for sectorial operators.

It is known that operators $\left(I + e^{2dQ_\lambda^-}\right), \left(I + e^{2DQ_\lambda^+}\right), \left(I - e^{2DQ_\lambda^+}\right)$ and $\left(I - e^{2dQ_\lambda^-}\right)$ are boundedly invertible, see Proposition 2.3.6, page 60 in [17].

6.1. **Invertibility of $\Delta_{\lambda,\mu_-,\mu_+}$.** Put

$$\frac{\lambda}{d_{j_-}} = \lambda_-, \quad \frac{\lambda}{d_{j_+}} = \lambda_+,$$

where, as mentioned above, λ is fixed such that

$$|\arg(\lambda)| < \pi - \varepsilon_0\ ;$$

recall that, for $\omega \in]0,\pi[$:

$$S_\omega = \{z \in \mathbb{C}\backslash\{0\} : |\arg z| < \omega\}.$$

Consider the function

$$l_{\lambda,\mu_-,\mu_+} : S_\omega \ni z \longmapsto l_{\lambda,\mu_-,\mu_+}(z),$$

defined by

$$\begin{aligned} l_{\lambda,\mu_-,\mu_+}(z) &= \mu_- \left(1+e^{-2d(z+\lambda_-)^{1/2}}\right)\left(1+e^{-2D(z+\lambda_+)^{1/2}}\right) \\ &\quad +\mu_+ \frac{(z+\lambda_+)^{1/2}}{(z+\lambda_-)^{1/2}}\left(1-e^{-2D(z+\lambda_+)^{1/2}}\right)\left(1-e^{-2d(z+\lambda_-)^{1/2}}\right); \end{aligned}$$

this function is analytic and bounded on S_ω since

$$Re\,(z+\lambda_-)^{1/2} > 0 \quad \text{and} \quad Re\,(z+\lambda_+)^{1/2} > 0,$$

and thus $l_{\lambda,\mu_-,\mu_+} \in H^\infty(S_\omega)$. On the other hand, we know that $-\Lambda$ has a bounded H^∞- calculus on S_ω ; therefore by the functional calculus, we have

$$\Delta_{\lambda,\mu_-,\mu_+} = l_{\lambda,\mu_-,\mu_+}(-\Lambda).$$

Now, in virtue of our Lemmas; we have, for all $z \in S_\omega$:

$$\begin{aligned} \left|l_{\lambda,\mu_-,\mu_+}(z)\right| &\geq \left(\mu_+ \left|\frac{(z+\lambda_+)^{1/2}}{(z+\lambda_-)^{1/2}}\right| \left|1-e^{-2D(z+\lambda_+)^{1/2}}\right| \left|1-e^{-2d(z+\lambda_-)^{1/2}}\right| \right. \\ &\quad \left. +\,\mu_- \left|1+e^{-2d(z+\lambda_-)^{1/2}}\right| \left|1+e^{-2D(z+\lambda_+)^{1/2}}\right|\right) \cdot \left|\cos\left(\Pi/2\right)\right|, \end{aligned}$$

where

$$\begin{aligned} \Pi &= \arg\left(\frac{(z+\lambda_+)^{1/2}}{(z+\lambda_-)^{1/2}}\right) + \arg\left(1-e^{-2D(z+\lambda_+)^{1/2}}\right) \\ &\quad +\arg\left(1-e^{-2d(z+\lambda_-)^{1/2}}\right) - \arg\left(1+e^{-2d(z+\lambda_-)^{1/2}}\right) \\ &\quad -\arg\left(1+e^{-2D(z+\lambda_+)^{1/2}}\right), \end{aligned}$$

then

$$\begin{aligned} |\Pi| &\leq \left|\arg(z+\lambda_+)^{1/2} - \arg(z+\lambda_-)^{1/2}\right| \\ &\quad +\left|\arg\left(1-e^{-2d(z+\lambda_-)^{1/2}}\right) - \arg\left(1+e^{-2d(z+\lambda_-)^{1/2}}\right)\right| \\ &\quad +\left|\arg\left(1-e^{-2D(z+\lambda_+)^{1/2}}\right) - \arg\left(1+e^{-2D(z+\lambda_+)^{1/2}}\right)\right|. \end{aligned}$$

And it is easy to see that there are $\omega_{\lambda_+} \in]0,\omega/2[$ and $\omega_{\lambda_-} \in]0,\omega/2[$ (with $\omega_{\lambda_+} < \omega_{\lambda_-}$ for example) such that:

$$\begin{cases} 2d\,(z+\lambda_-)^{1/2} \in S_{\omega_{\lambda_-}} \\ 2D\,(z+\lambda_+)^{1/2} \in S_{\omega_{\lambda_+}} \end{cases}$$

and therefore, since

$$\begin{cases} \left|\arg\left(1-e^{-2D(z+\lambda_+)^{1/2}}\right) - \arg\left(1+e^{-2D(z+\lambda_+)^{1/2}}\right)\right| < \omega_{\lambda_+} \\ \left|\arg\left(1-e^{-2d(z+\lambda_-)^{1/2}}\right) - \arg\left(1+e^{-2d(z+\lambda_-)^{1/2}}\right)\right| < \omega_{\lambda_-}, \end{cases}$$

we deduce that

$$|\Pi| \leq \omega_{\lambda_+} - \omega_{\lambda_-} + \omega_{\lambda_-} + \omega_{\lambda_+} = 2\omega_{\lambda_+} < \omega,$$

thus

$$\begin{aligned} \left|l_{\lambda,\mu_-,\mu_+}(z)\right| \quad \geq \quad & \left(\mu_+ \left|\frac{(z+\lambda_+)^{1/2}}{(z+\lambda_-)^{1/2}}\right| \left|1 - e^{-2D(z+\lambda_+)^{1/2}}\right| \left|1 - e^{-2d(z+\lambda_-)^{1/2}}\right| \right. \\ & \left. + \mu_- \left|1 + e^{-2d(z+\lambda_-)^{1/2}}\right| \left|1 + e^{-2D(z+\lambda_+)^{1/2}}\right|\right) \cdot \cos\left(\frac{\omega}{2}\right), \end{aligned}$$

or

$$\begin{aligned} \left|l_{\lambda,\mu_-,\mu_+}(z)\right| \quad & \geq \quad \mu_- \left|1 + e^{-2d(z+\lambda_-)^{1/2}}\right| \left|1 + e^{-2D(z+\lambda_+)^{1/2}}\right| \cos\left(\frac{\omega}{2}\right) \\ & \geq \quad \mu_- \left[1 - e^{-\pi/(2\tan(\omega_{\lambda_-})}\right] \left[1 - e^{-\pi/(2\tan(\omega_{\lambda_+})}\right] \cos\left(\frac{\omega}{2}\right) \\ & \geq \quad \mu_- \left(1 - e^{-\pi/(2\tan(\omega/2)}\right)^2 \cos\left(\frac{\omega}{2}\right) > 0. \end{aligned}$$

So, the function $l_{\lambda,\mu_-,\mu_+}(.)$ does not vanish on S_ω and the function $1/l_{\lambda,\mu_-,\mu_+}(.)$ is bounded, hence it belongs to $H^\infty(S_\omega)$.

Finally, $\Delta_{\lambda,\mu_-,\mu_+}$ is invertible with a bounded inverse and

$$\begin{cases} \Delta^{-1}_{\lambda,\mu_-,\mu_+} = \left(1/l_{\lambda,\mu_-,\mu_+}\right)(-\Lambda) \\ \left\|\Delta^{-1}_{\lambda,\mu_-,\mu_+}\right\|_{\mathcal{L}(X)} \leq \dfrac{C}{\mu_-}. \end{cases}$$

The constant C is independent of the parameter λ.

Now, from equality

$$\Delta_{\lambda,\mu_-,\mu_+}\Lambda^{-1} = \Lambda^{-1}\Delta_{\lambda,\mu_-,\mu_+},$$

it follows that

$$\Delta^{-1}_{\lambda,\mu_-,\mu_+}\Lambda = \Lambda\Delta^{-1}_{\lambda,\mu_-,\mu_+},$$

on $D(\Lambda)$, hence $\Delta^{-1}_{\lambda,\mu_-,\mu_+}$ is a bounded operator from $D(\Lambda)$ into itself. Therefore, by interpolation $\Delta^{-1}_{\lambda,\mu_-,\mu_+}$ is bounded from any interpolation space $(D(\Lambda), E)_{\alpha,s}$ (see the definition in [12]) into itself and clearly we have also the same estimate

$$\left\|\Delta^{-1}_{\lambda,\mu_-,\mu_+}\right\|_{\mathcal{L}((D(\Lambda),E)_{\alpha,s})} \leq \frac{C}{\mu_-}.$$

6.2. **The resolution of (S1).** Recall that:

$$\begin{cases} (I) := (Q^-_\lambda)^{-1}\left[-\mu_- w'_-(N_-)(0) + \mu_+ w'_+(N_+)(0)\right] \\ (II) := w_+(N_+)(0) - w_-(N_-)(0), \end{cases}$$

and

$$\begin{cases} (I') := (I) - \mu_+(Q^-_\lambda)^{-1}e^{DQ^+_\lambda}w'_+(N_+)(D) + \mu_- e^{dQ^-_\lambda}w_-(N_-)(-d) \\ (II') := (II) + e^{dQ^-_\lambda}w_-(N_-)(-d) + e^{DQ^+_\lambda}(Q^+_\lambda)^{-1}w'_+(N_+)(D). \end{cases}$$

Then

$$\begin{aligned} (I') \quad &= \quad (I) - \mu_+(Q^-_\lambda)^{-1}e^{DQ^+_\lambda}w'_+(N_+)(D) + \mu_- e^{dQ^-_\lambda}w_-(N_-)(-d) \\ &= \quad (Q^-_\lambda)^{-1}\left[-\mu_- w'_-(N_-)(0) + \mu_+ w'_+(N_+)(0)\right] \\ &\quad -\mu_+(Q^-_\lambda)^{-1}e^{DQ^+_\lambda}w'_+(N_+)(D) + \mu_- e^{dQ^-_\lambda}w_-(N_-)(-d), \end{aligned}$$

$$\begin{aligned}
& (II') \\
= \;& (II) + e^{dQ_\lambda^-} w_-(N_-)(-d) + e^{DQ_\lambda^+} (Q_\lambda^+)^{-1} w'_+(N_+)(D) \\
= \;& w_+(N_+)(0) - w_-(N_-)(0) + e^{dQ_\lambda^-} w_-(N_-)(-d) + e^{DQ_\lambda^+} {Q_\lambda^+}^{-1} w'_+(N_+)(D).
\end{aligned}$$

From the system

$$\begin{cases} -\mu_- \left(I + e^{2dQ_\lambda^-}\right) \delta_- - \mu_+ (Q_\lambda^-)^{-1} Q_\lambda^+ \left(I - e^{2DQ_\lambda^+}\right) \gamma_+ = (I') \\ \left(I - e^{2dQ_\lambda^-}\right) \delta_- - \left(I + e^{2DQ_\lambda^+}\right) \gamma_+ = (II'), \end{cases}$$

we get

$$\begin{cases} \delta_- = \Delta^{-1}_{\lambda,\mu_-,\mu_+} \left[\mu_+ (Q_\lambda^-)^{-1} Q_\lambda^+ \left(I - e^{2DQ_\lambda^+}\right) (II') - \left(I + e^{2DQ_\lambda^+}\right) (I')\right] \\ \\ \gamma_+ = \Delta^{-1}_{\lambda,\mu_-,\mu_+} \left[-\mu_- \left(I + e^{2dQ_\lambda^-}\right) (II') - \left(I - e^{2dQ_\lambda^-}\right) (I')\right], \end{cases}$$

and

$$\begin{aligned}
\gamma_- \;=\;& -e^{dQ_\lambda^-} \delta_- - w_-(N_-)(-d) \\
=\;& -\Delta^{-1}_{\lambda,\mu_-,\mu_+} e^{dQ_\lambda^-} \left(\left[\mu_+ (Q_\lambda^-)^{-1} Q_\lambda^+ \left(I - e^{2DQ_\lambda^+}\right) (II') - \left(I + e^{2DQ_\lambda^+}\right) (I')\right]\right) \\
& -w_-(N_-)(-d),
\end{aligned}$$

from which we obtain for $x \in]-d, 0[$

$$\begin{aligned}
& v_-(x) \\
=\;& e^{(x+d)Q_\lambda^-} \gamma_- + e^{-xQ_\lambda^-} \delta_- + w_-(N_-)(x) \\
=\;& -\Delta^{-1}_{\lambda,\mu_-,\mu_+} e^{dQ_\lambda^-} e^{(x+d)Q_\lambda^-} \\
& * \left[\mu_+ (Q_\lambda^-)^{-1} Q_\lambda^+ \left(I - e^{2DQ_\lambda^+}\right) (II') - \left(I + e^{2DQ_\lambda^+}\right) (I')\right] \\
& -e^{(x+d)Q_\lambda^-} \left[w_-(N_-)(-d)\right] \\
& +\Delta^{-1}_{\lambda,\mu_-,\mu_+} e^{-xQ_\lambda^-} \left[\mu_+ (Q_\lambda^-)^{-1} Q_\lambda^+ \left(I - e^{2DQ_\lambda^+}\right) (II') - \left(I + e^{2DQ_\lambda^+}\right) (I')\right] \\
& +w_-(N_-)(x);
\end{aligned}$$

where:

$$\begin{aligned}
& \mu_+ (Q_\lambda^-)^{-1} Q_\lambda^+ \left(I - e^{2DQ_\lambda^+}\right) (II') - \left(I + e^{2DQ_\lambda^+}\right) (I') \\
=\;& \mu_+ (Q_\lambda^-)^{-1} Q_\lambda^+ \left(I - e^{2DQ_\lambda^+}\right) \left[w_+(N_+)(0) - w_-(N_-)(0)\right] \\
& +\mu_+ (Q_\lambda^-)^{-1} Q_\lambda^+ \left(I - e^{2DQ_\lambda^+}\right) \left[e^{dQ_\lambda^-} w_-(N_-)(-d) + e^{DQ_+} (Q_\lambda^+)^{-1} w'_+(N_+)(D)\right] \\
& -(Q_\lambda^-)^{-1} \left(I + e^{2DQ_\lambda^+}\right) \left[-\mu_- w'_-(N_-)(0) + \mu_+ w'_+(N_+)(0)\right] \\
& - \left(I + e^{2DQ_\lambda^+}\right) \left[\mu_- e^{dQ_\lambda^-} w_-(N_-)(-d) - \mu_+ (Q_\lambda^-)^{-1} e^{DQ_\lambda^+} w'_+(N_+)(D)\right];
\end{aligned}$$

we can write for all $x \in]-d, 0[$:

$$\begin{aligned}
v_-(x) \;=\;& \Delta^{-1}_{\lambda,\mu_-,\mu_+} \left(e^{-xQ_\lambda^-} - e^{(x+2d)Q_\lambda^-}\right) \left[\mu_+ (Q_\lambda^-)^{-1} Q_\lambda^+ \left(I - e^{2DQ_\lambda^+}\right) (II')\right] \\
& -\Delta^{-1}_{\lambda,\mu_-,\mu_+} \left(e^{-xQ_\lambda^-} - e^{(x+2d)Q_\lambda^-}\right) \left[\left(I + e^{2DQ_\lambda^+}\right) (I')\right] \\
& -e^{(x+d)Q_\lambda^-} \left[w_-(N_-)(-d)\right] \\
& +w_-(N_-)(x).
\end{aligned}$$

In the same way, we obtain

$$\begin{aligned}\delta_+ &= e^{DQ_\lambda^+}\gamma_+ + (Q_\lambda^+)^{-1}w'_+(N_+)(D)\\ &= \Delta^{-1}_{\lambda,\mu_-,\mu_+}e^{DQ_\lambda^+}\left(\left[-\mu_-\left(I+e^{2dQ_\lambda^-}\right)(II') - \left(I-e^{2dQ_\lambda^-}\right)(I')\right]\right)\\ &\quad +(Q_\lambda^+)^{-1}w'_+(N_+)(D),\end{aligned}$$

and for all $x\in\,]0,D[\,$,

$$\begin{aligned}&v_+(x)\\ =\;&\Delta^{-1}_{\lambda,\mu_-,\mu_+}\left(e^{xQ_\lambda^+}+e^{(2D-x)Q_\lambda^+}\right)\left[-\mu_-\left(I+e^{2dQ_\lambda^-}\right)(II') - \left(I-e^{2dQ_\lambda^-}\right)(I')\right]\\ &+(Q_\lambda^+)^{-1}e^{(D-x)Q_\lambda^+}w'_+(N_+)(D)+w_+(N_+)(x).\end{aligned}$$

6.3. **Optimal regularity of v_- and v_+ and estimates.** In this section, we will use the following lemma (see the proof in [10]), where the authors used the lemma 2.6 of [8].

Lemma 6.1. *Let $-\infty<a<b<+\infty$. Then:*

1. *For $\lambda\in S_\omega$, $Q_\lambda^- = -\left[-\left(\Lambda_\lambda^-\right)\right]^{1/2}$ which generates a semigroup $\left(e^{tQ_\lambda^-}\right)_{t\geq 0}$ bounded, analytic for $t>0$ and strongly continuous for $t\geq 0$ satisfies moreover*
$$\begin{cases}\exists K_0>0,\exists c_0>0,\forall t\geq 1/2,\forall\lambda\in S_\omega:\\ \max\left\{\|\ e^{tQ_\lambda^-}\ \|_{L(E)},\|\ G_\lambda e^{tQ_\lambda^-}\ \|_{L(E)}\right\}\leq K_0e^{-tc_0|\lambda|^{1/2}}.\end{cases}$$
2. *For $x\in[a,b]$, $\lambda\in S_\omega$ and $f\in L^q(a,b;E), 1<q<+\infty$, we set,*
$$\begin{cases}U_{\lambda,f}(x)=\int_a^x e^{(x-t)Q_\lambda^-}f(t)dt, & a\leq x\\ \\ V_{\lambda,f}(x)=\int_x^b e^{(t-x)Q_\lambda^-}f(t)dt, & x\leq b.\end{cases}\tag{9}$$
There exists $M>0$ such that for any $f\in L^q(a,b;E)$ and any $\lambda\in S_\omega$
$$\begin{cases}\|U_{\lambda,f}\|_{L^p(a,b;E)}\leq\dfrac{M}{\sqrt{|\lambda|+1}}\|f\|_{L^p(a,b;E)}\\ \|V_{\lambda,f}\|_{L^p(a,b;E)}\leq\dfrac{M}{\sqrt{|\lambda|+1}}\|f\|_{L^p(a,b;E)}.\end{cases}$$

□

We must show that:

$$\begin{cases}v_-\in W^{2,q}(-d,0;E)\cap L^q(-d,0;D(\Lambda_\lambda^-))\\ v_+\in W^{2,q}(0,D;E)\cap L^q(0,D;D(\Lambda_\lambda^+))\end{cases}$$

and all boundary and transmission conditions are verified.

For $x\in]-d,0[$, we have:

$$\begin{aligned}&\Lambda_\lambda^- v_-(x)\\ =\;&(Q_\lambda^-)^2v_-(x)\\ =\;&\Delta^{-1}_{\lambda,\mu_-,\mu_+}(Q_\lambda^-)^2\left(e^{-xQ_\lambda^-}-e^{(x+2d)Q_\lambda^-}\right)\left[\mu_+(Q_\lambda^-)^{-1}Q_\lambda^+\left(I-e^{2DQ_\lambda^+}\right)(II')\right]\\ &-\Delta^{-1}_{\lambda,\mu_-,\mu_+}(Q_\lambda^-)^2\left(e^{-xQ_\lambda^-}-e^{(x+2d)Q_\lambda^-}\right)\left(I+e^{2DQ_\lambda^+}\right)(I')\\ &-(Q_\lambda^-)^2e^{(x+d)Q_\lambda^-}\left[w_-(N_-)(-d)\right]+(Q_\lambda^-)^2w_-(N_-)(x)\\ =\;&(I)(x)+(II)(x)+(III)(x)+(IV)(x).\end{aligned}$$

Let us show that

$$x \mapsto (Q_\lambda^-)^2 v_-(x) \in L^q(-d,0;E).$$

The term

$$\begin{aligned} & v_-''(x) \\ = \; & \Delta^{-1}_{\lambda,\mu_-,\mu_+}(Q_\lambda^-)^2\left(e^{-xQ_\lambda^-}-e^{(x+2d)Q_\lambda^-}\right)\left[\mu_+(Q_\lambda^-)^{-1}Q_\lambda^+\left(I-e^{2DQ_\lambda^+}\right)(II')\right] \\ & -\Delta^{-1}_{\lambda,\mu_-,\mu_+}(Q_\lambda^-)^2\left(e^{-xQ_\lambda^-}-e^{(x+2d)Q_\lambda^-}\right)\left(I+e^{2DQ_\lambda^+}\right)(I') \\ & -(Q_\lambda^-)^2 e^{(x+d)Q_\lambda^-}\left[w_-(N_-)(-d)\right]+w_-''(N_-)(x), \end{aligned}$$

can be treated similarly.

We have, for $x \in]-d,0[$:

$$\begin{aligned} & (I)(x) \\ = \; & \Delta^{-1}_{\lambda,\mu_-,\mu_+}(Q_\lambda^-)^2\left(e^{-xQ_\lambda^-}-e^{(x+2d)Q_\lambda^-}\right)\left[\mu_+(Q_\lambda^-)^{-1}Q_\lambda^+\left(I-e^{2DQ_\lambda^+}\right)(II')\right] \\ = \; & \Delta^{-1}_{\lambda,\mu_-,\mu_+}Q_\lambda^-\left(e^{-xQ_\lambda^-}-e^{(x+2d)Q_\lambda^-}\right)\left[\mu_+Q_\lambda^+\left(I-e^{2DQ_\lambda^+}\right)(II')\right], \end{aligned}$$

where

$$\begin{aligned} & (II') \\ = \; & (II)+e^{dQ_\lambda^-}w_-(N_-)(-d)+e^{DQ_\lambda^+}(Q_\lambda^+)^{-1}w_+'(N_+)(D) \\ = \; & w_+(N_+)(0)-w_-(N_-)(0)+e^{dQ_\lambda^-}w_-(N_-)(-d)+e^{DQ_\lambda^+}(Q_\lambda^+)^{-1}w_+'(N_+)(D), \end{aligned}$$

from which we get

$$\begin{aligned} & \left[\mu_+Q_\lambda^+\left(I-e^{2DQ_\lambda^+}\right)(II')\right] \\ = \; & \mu_+Q_\lambda^+\left(I-e^{2DQ_\lambda^+}\right)\left[w_+(N_+)(0)-w_-(N_-)(0)\right] \\ & +\mu_+Q_\lambda^+\left(I-e^{2DQ_\lambda^+}\right)\left[e^{dQ_\lambda^-}w_-(N_-)(-d)+e^{DQ_\lambda^+}(Q_\lambda^+)^{-1}w_+'(N_+)(D)\right], \end{aligned}$$

so, the first term belongs to an interpolation space and the second is very regular, therefore

$$x \mapsto (I)(x) \in L^q(-d,0;E);$$

the same is true for the two terms $(II)(x)$ and $(III)(x)$.

For the fourth term we have:

$$x \mapsto (IV)(x) \in L^q(-d,0;E),$$

due to the Dore-Venni theorem.

Now we will estimate

$$\|v_-\|_{L^q(-d,0;E)}.$$

Recall that

$$\begin{aligned} v_-(x) \; = \; & \Delta^{-1}_{\lambda,\mu_-,\mu_+}\left(e^{-xQ_\lambda^-}-e^{(x+2d)Q_\lambda^-}\right)\left[\mu_+(Q_\lambda^-)^{-1}Q_\lambda^+\left(I-e^{2DQ_\lambda^+}\right)(II')\right] \\ & -\Delta^{-1}_{\lambda,\mu_-,\mu_+}\left(e^{-xQ_\lambda^-}-e^{(x+2d)Q_\lambda^-}\right)\left[\left(I+e^{2DQ_\lambda^+}\right)(I')\right] \\ & -e^{(x+d)Q_\lambda^-}\left[w_-(N_-)(-d)\right]+w_-(N_-)(x), \end{aligned}$$

where

$$w_-(N_-)(x) \; = \; \frac{1}{2}\int_{-d}^{x} e^{(x-t)Q_\lambda^-}(Q_\lambda^-)^{-1}N_-(t)dt+\frac{1}{2}\int_{x}^{0} e^{(t-x)Q_\lambda^-}(Q_\lambda^-)^{-1}N_-(t)dt,$$

and λ is such

$$|\arg(\lambda)| < \pi - \varepsilon_0.$$

Now, we use the lemma 6.1:

We have

$$\begin{aligned} w_-(N_-)(x) &= \frac{1}{2}\int_{-d}^{x} e^{(x-t)Q_\lambda^-}(Q_\lambda^-)^{-1}N_-(t)dt + \frac{1}{2}\int_{x}^{0} e^{(t-x)Q_\lambda^-}(Q_\lambda^-)^{-1}N_-(t)dt \\ &= \frac{1}{2}(Q_\lambda^-)^{-1}\int_{-d}^{x} e^{(x-t)Q_\lambda^-}N_-(t)dt + \frac{1}{2}(Q_\lambda^-)^{-1}\int_{x}^{0} e^{(t-x)Q_\lambda^-}N_-(t)dt, \end{aligned}$$

thanks to lemma 6.1 and (8); there exists two constants $M_1 > 0$ and $M_2 > 0$ (independent of λ) such that:

$$\|w_-(N_-)\|_{L^q(-d,0;E)} \leq \frac{M_1}{|\lambda|^{\frac{1}{2}}}\left[\frac{M_2}{\sqrt{|\lambda|+1}}\right]\|N_-\|_{L^q(-d,0;E)},$$

hence the existence of a constant $C > 0$ (independent of λ) such that:

$$\|w_-\|_{L(L^q(-d,0;E))} \leq \frac{C}{|\lambda|}.$$

A similar estimate is obtained for the other terms.

Summing up, we obtain

$$\begin{cases} \|v_-\|_{L(L^q(-d,0;E))} = \|v_-\|_{L(L^q(\Omega_-))} \leq \frac{C}{|\lambda|} \\ \|v_+\|_{L(L^q(\Omega_+))} \leq \frac{C}{|\lambda|}\ . \end{cases}$$

By the same methods, we obtain:

$$\begin{cases} \|w_-\|_{L(L^q(-d,0;E))} = \|w_-\|_{L(L^q(\Omega_-))} \leq \dfrac{C}{|\lambda|} \\ \|w_+\|_{L(L^q(\Omega_+))} \leq \dfrac{C}{|\lambda|}\ . \end{cases}$$

Then in the space $\mathcal{E} = [L^q(\Omega)]^2$, we get

$$\begin{aligned} \left\|\begin{pmatrix} v \\ w \end{pmatrix}\right\|_{\mathcal{E}} &= \|W\|_{\mathcal{E}} = \max\left(\|v\|_{L^q(\Omega)}, \|w\|_{L^q(\Omega)}\right) \\ &\leq \frac{C}{|\lambda|}\left\|\begin{pmatrix} f \\ g \end{pmatrix}\right\|_{\mathcal{E}} = \frac{C}{|\lambda|}\ \|F\|_{\mathcal{E}}\,; \end{aligned}$$

we conclude by the theorem:

Theorem 6.2. *Operator $\mathcal{L}$ generates an analytic semigroup in the space $\mathcal{E} = [L^q(\Omega)]^2$.* □

7. **Return to the evolution equation.** We go back to our abstract evolution equation:

$$\begin{cases} V'(t) = \mathcal{L}V(t) + \mathcal{B}(t)V(t)\ , \quad t > 0 \\ V(0) = V^0. \end{cases}$$

Since $\mathcal{L}$ generates an analytic semigroup in $\mathcal{E} = [L^q(\Omega)]^2$ (which is exponentially decreasing), then necessarily the solution (if it exists) is written as:

$$V(t) = e^{t\mathcal{L}}V^0 + \int_0^t e^{(t-s)\mathcal{L}}\left[\mathcal{B}(s)V(s)\right]ds\ , \tag{10}$$

with

$$\begin{aligned}\mathcal{B}(t)V(t) &= \begin{pmatrix} \omega_j(t)s_j(t)I & f_a(t)s_a(t)I \\ (1-\omega_j(t))s_j(t)I & s_a(t)I \end{pmatrix}\begin{pmatrix} J(t) \\ A(t) \end{pmatrix} \\ &= \begin{pmatrix} \omega_j(t)s_j(t)J(t) + f_a(t)s_a(t)A(t) \\ (1-\omega_j(t))s_j(t)J(t) + s_a(t)A(t) \end{pmatrix} \\ &:= G(t,V(t))\ ,\end{aligned}$$

where we have used the operational notation:

$$\begin{pmatrix} J(t,.,.) \\ A(t,.,.) \end{pmatrix} = \begin{pmatrix} J(t) \\ A(t) \end{pmatrix}\ ,$$

with, for example, for $t>0$:

$$J(t)(x,y) = J(t,x,y) = \begin{cases} J_-(t,x,y) & \text{for } (x,y)\in\Omega_- \\ J_+(t,x,y) & \text{for } (x,y)\in\Omega_+. \end{cases}$$

7.1. **Study in $L^q(0,T;\mathcal{E})$.** We note that the Banach space $L^q(0,T;\mathcal{E})$ where $q\in\,]1,+\infty[$ is UMD.

Here, the representation (10) is well defined, if for all

$$t\mapsto \begin{pmatrix} J(t) \\ A(t) \end{pmatrix} \in L^q(0,T;\mathcal{E}),$$

we have

$$t\mapsto \begin{pmatrix} \omega_j(t)s_j(t)J(t) + f_a(t)s_a(t)A(t) \\ (1-\omega_j(t))s_j(t)J(t) + s_a(t)A(t) \end{pmatrix} \in L^q(0,T;\mathcal{E}).$$

This is checked if, for example, the rates ω_j, s_j, f_a and s_a are bounded. Therefore

$$t\mapsto G(t,V(t))\in L^q(0,T;\mathcal{E}),$$

and thus the integral

$$\int_0^t e^{(t-s)\mathcal{L}}G(s,V(s))ds$$

is well defined and due to the well known Dore-Venni Theorem we have:

$$t\mapsto \mathcal{L}\int_0^t e^{(t-s)\mathcal{L}}G(s,V(s))ds \in L^q(0,T;\mathcal{E}).$$

7.2. **Application of the fixed point theorem.** We will apply the fixed point theorem to the equation:

$$\begin{cases} V'(t) = \mathcal{L}V(t) + \mathcal{B}(t)V(t),\ t\in(0,T) \\ V(0) = V^0. \end{cases}$$

Note that we can also consider this equation as a non autonomous linear equation since we know that $\mathcal{L}+\mathcal{B}(t)$ for every $t\geq 0$ generates an analytic semigroup ($\mathcal{B}(t)$ is bounded).

It can be written as

$$\begin{cases} V'(t) = \mathcal{L}V(t) + G(t,V(t)),\ t\in(0,T) \\ V(0) = V^0. \end{cases}$$

It is well known that the equation:

$$\begin{cases} \Phi'(t) = \mathcal{L}\Phi(t) + g(t),\ t\in(0,T) \\ \Phi(0) = 0, \end{cases} \tag{11}$$

whith $g \in L^q(0,T;\mathcal{E})$ has the Lq-maximal regularity. So for all $g \in L^q(0,T;\mathcal{E})$, there exists a unique solution

$$\Phi \in W_0^{1,q}(0,T;\mathcal{E}) \cap L^q(0,T;D(\mathcal{L})),$$

and there exists a constant $C_0 > 0$ such that

$$\|\Phi\|_1 := \|\Phi\|_{L^q(0,T;D(\mathcal{L}))} + \|\Phi'\|_{L^q(0,T;\mathcal{E})} \leq C_0 \|g\|_{L^q(0,T;\mathcal{E})}.$$

On the other hand we know that

$$W_0^{1,q}(0,T;\mathcal{E}) \cap L^q(0,T;D(\mathcal{L})) \subset C\left([0,T];\left(D\left(\mathcal{L}\right);\mathcal{E}\right)_{1/q,q}\right)$$

for the interpolation space, see [12] with continuous injection and therefore there exists a constant $C_1 > 0$ such that

$$\sup_{t\in[0,T]} \|\Phi(t)\|_{(D(\mathcal{L});\mathcal{E})_{1/q,q}} \leq C_1 \|\Phi\|_1.$$

These constants do not depend on T here as $\mathcal{L}$ is invertible, the semigroup is exponentially decreasing, from which we obtain the maximal Lq-regularity on $(0,+\infty)$.

Now let us consider the problem:

$$\begin{cases} \Phi'(t) = \mathcal{L}\Phi(t) + G(t,0) = \mathcal{L}\Phi(t), \ t \in (0,T) \\ \Phi(0) = V^0, \end{cases}$$

which has a unique solution given by:

$$\Phi^*(t) = e^{t\mathcal{L}}V^0,$$

belonging to $W^{1,q}(0,T;\mathcal{E}) \cap L^q(0,T;D(\mathcal{L}))$ for all

$$V^0 \in \left(D\left(\mathcal{L}\right);\mathcal{E}\right)_{1/q,q}.$$

Let us introduce the following closed ball of $L^q(0,T;\mathcal{E})$ of center Φ^* and radius $r \in]0,1[$:

$$\begin{aligned} \mathbb{B}_r &= \Big\{W \in L^q(0,T;\mathcal{E}) : W - \Phi^* \in W_0^{1,q}(0,T;\mathcal{E}) \cap L^q(0,T;D(\mathcal{L})) \quad \text{and} \\ &\quad \|W - \Phi^*\|_1 \leq r\Big\}; \end{aligned}$$

We are going to apply the fixed point theorem for contraction mappings.

1. For every $W \in \mathbb{B}_r$, the problem

$$\begin{cases} V'(t) = \mathcal{L}V(t) + G(t,W(t)), \ t \in (0,T) \\ V(0) = V^0, \end{cases}$$

has a unique solution $V \in W^{1,q}(0,T;\mathcal{E}) \cap L^q(0,T;D(\mathcal{L}))$.

Let us define the following application:

$$\begin{array}{rccc} \Psi: & \mathbb{B}_r & \longrightarrow & W^{1,q}(0,T;\mathcal{E}) \cap L^q(0,T;D(\mathcal{L})) \\ & W & \longmapsto & \Psi(W) = V. \end{array}$$

2. Let us prove thatΨ is a strict contraction on $\mathbb{B}_r$.

We have

$$\begin{aligned} \|\Psi(W) - \Phi^*\|_1 &= \|V - \Phi^*\|_1 \\ &\leq C_0 \|G(.,W)\|_{L^q(0,T;\mathcal{E})} \\ &\leq C_0 \|G(.,W) - G(.,\Phi^*)\|_{L^q(0,T;\mathcal{E})} \\ &\quad + C_0 \|G(.,\Phi^*) - G(.,0)\|_{L^q(0,T;\mathcal{E})} + C_0 \|G(.,0)\|_{L^q(0,T;\mathcal{E})}; \end{aligned}$$

where obviously $G(.,0) = 0$.

3. Recall that

$$G(t,V(t)) = G\left(t, \begin{pmatrix} J(t) \\ A(t) \end{pmatrix}\right) = \begin{pmatrix} \omega_j(t)s_j(t)J(t) + f_a(t)s_a(t)A(t) \\ (1-\omega_j(t))s_j(t)J(t) + s_a(t)A(t) \end{pmatrix},$$

set

$$W(.) = \begin{pmatrix} W_1(.) \\ W_2(.) \end{pmatrix}, \quad \Phi^*(.) = \begin{pmatrix} \Phi_1^*(.) \\ \Phi_2^*(.) \end{pmatrix};$$

We know that

$$W - \Phi^* \in W_0^{1,q}(0,T;\mathcal{E}) \cap L^q(0,T;D(\mathcal{L}))$$

and by definition of $\mathbb{B}_r$, we have

$$W - \Phi^* \in W_0^{1,q}(0,T;\mathcal{E}) \cap L^q(0,T;D(\mathcal{L})) \subset C\left([0,T];\left(D\left(\mathcal{L}\right);\mathcal{E}\right)_{1/q,q}\right)$$

so for all $t \in [0,T]$:

$$W(t) - \Phi^*(t) \in \left(D\left(\mathcal{L}\right);\mathcal{E}\right)_{1/q,q}.$$

We have

$$\begin{array}{l} G(t,W(t)) - G(t,\Phi^*(t)) \\ = \begin{pmatrix} \omega_j(t)s_j(t)\left[W_1(t) - \Phi_1^*(t)\right] + f_a(t)s_a(t)\left[W_2(t) - \Phi_2^*(t)\right] \\ (1-\omega_j(t))s_j(t)\left[W_1(t) - \Phi_1^*(t)\right] + s_a(t)\left[W_2(t) - \Phi_2^*(t)\right] \end{pmatrix} \end{array}$$

and

$$\begin{array}{rl} & \left\|G(t,W(t)) - G(t,\Phi^*(t))\right\|_{\mathcal{E}} \\ = & \max\Big(\left\|\omega_j(t)s_j(t)\left[W_1(t) - \Phi_1^*(t)\right] + f_a(t)s_a(t)\left[W_2(t) - \Phi_2^*(t)\right]\right\|_{L^q(\Omega)}, \\ & \left\|(1-\omega_j(t))s_j(t)\left[W_1(t) - \Phi_1^*(t)\right] + s_a(t)\left[W_2(t) - \Phi_2^*(t)\right]\right\|_{L^q(\Omega)}\Big) \\ \leq & \phi(t)\left(\left\|\left[W_1(t) - \Phi_1^*(t)\right]\right\|_{L^q(\Omega)} + \left\|\left[W_2(t) - \Phi_2^*(t)\right]\right\|_{L^q(\Omega)}\right) \end{array}$$

where

$$\phi(t) = \sup\left(\omega_j(t)s_j(t), f_a(t)s_a(t), \left|(1-\omega_j(t))\right| s_j(t), s_a(t)\right)$$

4. Note that all these functions in the sup are positive and are not vector-valued only coefficients. Then in the space $\mathcal{E} = \left[L^q(\Omega)\right]^2$, we have

$$\begin{array}{rcl} \left\|G(t,W(t)) - G(t,\Phi^*(t))\right\|_{\mathcal{E}} & \leq & \phi(t)\left\|W(t) - \Phi^*(t)\right\|_{\mathcal{E}} \\ & \leq & \phi(t)\left\|W(t) - \Phi^*(t)\right\|_{\left(D(\mathcal{L});\mathcal{E}\right)_{1/q,q}} \\ & \leq & \phi(t)\sup\limits_{t\in[0,T]}\left\|W(t) - \Phi^*(t)\right\|_{\left(D(\mathcal{L});\mathcal{E}\right)_{1/q,q}}, \end{array}$$

hence

$$\left\|G(.,W) - G(.,\Phi^*)\right\|_{L^q(0,T;\mathcal{E})} \leq \left\|\phi\right\|_{L^q(0,T)}\sup_{t\in[0,T]}\left\|W(t) - \Phi^*(t)\right\|_{\left(D(\mathcal{L});\mathcal{E}\right)_{1/q,q}}.$$

We have similary:

$$\begin{array}{rcl} \left\|G(.,\Phi^*) - G(.,0)\right\|_{L^q(0,T;\mathcal{E})} & = & \left\|G(.,\Phi^*)\right\|_{L^q(0,T;\mathcal{E})} \\ & \leq & \left\|\phi\right\|_{L^q(0,T)}\sup\limits_{t\in[0,T]}\left\|\Phi^*(t)\right\|_{\left(D(\mathcal{L});\mathcal{E}\right)_{1/q,q}}. \end{array}$$

5. Finally:

$$\begin{aligned}
&\|\Psi(W)-\Phi^*\|_1\\
=\;&\|V-\Phi^*\|_1\\
\le\;&C_0\|G(.,W)\|_{L^q(0,T;\mathcal{E})}\\
\le\;&C_0\|G(.,W)-G(.,\Phi^*)\|_{L^q(0,T;\mathcal{E})}+C_0\|G(.,\Phi^*)\|_{L^q(0,T;\mathcal{E})}\\
\le\;&C_0\|\phi\|_{L^q(0,T)}\left[\sup_{t\in[0,T]}\|W(t)-\Phi^*(t)\|_{(D(\mathcal{L});\mathcal{E})_{1/q,q}}+\sup_{t\in[0,T]}\|\Phi^*(t)\|_{(D(\mathcal{L});\mathcal{E})_{1/q,q}}\right]\\
\le\;&C_0\|\phi\|_{L^q(0,T)}\left[C_1\|W-\Phi^*\|_1+\sup_{t\in[0,T]}\|\Phi^*(t)\|_{(D(\mathcal{L});\mathcal{E})_{1/q,q}}\right]\\
\le\;&C_0\|\phi\|_{L^q(0,T)}\left[C_1r+\sup_{t\in[0,T]}\|\Phi^*(t)\|_{(D(\mathcal{L});\mathcal{E})_{1/q,q}}\right].
\end{aligned}$$

6. In the same way we have, for any $W,\overline{W}$ in $\mathbb{B}_r$

$$\begin{aligned}
\left\|\Psi(W)-\Psi(\overline{W})\right\|_1&\le C_0\left\|G(.,W)-G(.,\overline{W})\right\|_{L^q(0,T;\mathcal{E})}\\
&\le C_0\|\phi\|_{L^q(0,T)}\sup_{t\in[0,T]}\left\|(W-\overline{W})(t)\right\|_{(D(\mathcal{L});\mathcal{E})_{1/q,q}}\\
&\le C_0C_1\|\phi\|_{L^q(0,T)}\left\|W-\overline{W}\right\|_1.
\end{aligned}$$

The function

$$\mu(\tau)=\|\phi\|_{L^q(0,\tau)}=\left(\int_0^\tau|\phi(s)|^q\,ds\right)^{1/q},$$

is continuous (with respect to τ) positive, strictly increasing on $[0,+\infty[$ taking its values in the intervall $[0,\mu(T)]$; so there exists $T^*\le T$ such that

$$C_0C_1\|\phi\|_{L^q(0,T^*)}\le 1/2, \tag{12}$$

from wich we obtain that the function:

$$\begin{array}{rccc}\Psi: & \mathbb{B}_r^* & \longrightarrow & W^{1,q}(0,T;\mathcal{E})\cap L^q(0,T;D(\mathcal{L}))\\ & W & \longmapsto & \Psi(W)=V.\end{array}$$

is a strict contraction, where

$$\mathbb{B}_r^*=\left\{W\in L^q(0,T^*;\mathcal{E}):(W-\Phi^*)\in\Xi\ \text{ and }\ \|W-\Phi^*\|_1\le r\right\}. \tag{13}$$

with $\Xi=W_0^{1,q}(0,T^*;\mathcal{E})\cap L^q(0,T^*;D(\mathcal{L}))$

7. It remains to prove that Ψ applies $\mathbb{B}_r^*$ into $\mathbb{B}_r^*$. We have seen that

$$\begin{aligned}
\|\Psi(W)-\Phi^*\|_1&\le C_0\|\phi\|_{L^q(0,T)}\left[C_1r+\sup_{t\in[0,T]}\|\Phi^*(t)\|_{(D(\mathcal{L});\mathcal{E})_{1/q,q}}\right] &(14)\\
&\le C_0C_1\|\phi\|_{L^q(0,T)}r+C_0\|\phi\|_{L^q(0,T)}\sup_{t\in[0,T]}\|\Phi^*(t)\|_{(D(\mathcal{L});\mathcal{E})_{1/q,q}} &(15)
\end{aligned}$$

Recall that

$$\Phi^*(t)=e^{t\mathcal{L}}V^0,$$

where

$$V^0\in\left(D\left(\mathcal{L}\right);\mathcal{E}\right)_{1/q,q};$$

Φ^* belongs to $W^{1,q}(0,+\infty;\mathcal{E}) \cap L^q(0,+\infty;D(\mathcal{L}))$ for all data

$$V^0 \in (D(\mathcal{L});\mathcal{E})_{1/q,q}.$$

Since $\mathcal{L}$ is exponentially decreasing, there exists $\delta > 0$ and $M \geq 1$ such that

$$\left\|e^{t\mathcal{L}}\right\| \leq Me^{-\delta t},$$

so

$$\|\Phi^*(t)\|_{(D(\mathcal{L});\mathcal{E})_{1/q,q}} \leq M \left\|V^0\right\|_{(D(\mathcal{L});\mathcal{E})_{1/q,q}},$$

then there exists $T^{**} \leq T^*$ such that

$$\begin{aligned} C_0 \|\phi\|_{L^q(0,T^{**})} \sup_{t\in[0,T]} \|\Phi^*(t)\|_{(D(\mathcal{L});\mathcal{E})_{1/q,q}} &\leq MC_0 \|\phi\|_{L^q(0,T^{**})} \left\|V^0\right\|_{(D(\mathcal{L});\mathcal{E})_{1/q,q}} \quad (16)\\ &\leq r/2, \quad (17)\end{aligned}$$

from (12) and (17), we get:

$$\begin{aligned} \|\Psi(W) - \Phi^*\|_1 &\leq r\, C_0C_1 \|\phi\|_{L^q(0,T)} + C_0 \|\phi\|_{L^q(0,T)} \sup_{t\in[0,T]} \|\Phi^*(t)\|_{(D(\mathcal{L});\mathcal{E})_{1/q,q}}\\ &\leq r/2 + r/2\\ &\leq r \end{aligned}$$

hence $\Psi(W)$ belongs to $\mathbb{B}_r^*$, see (13).

Then Ψ applies $\mathbb{B}_r^*$ into $\mathbb{B}_r^*$ which is a complete metric space.

Finally we have:

i) Ψ is defined from $\mathbb{B}_r^*$ into $\mathbb{B}_r^*$,

ii) Ψ is a strict contraction from $\mathbb{B}_r^*$ to $\mathbb{B}_r^*$,

iii) $\mathbb{B}_r^*$ is a closed set in a Banach space, then it is a complete metric space, therefore the fixed point theorem applies to Ψ in $\mathbb{B}_r^*$.

We then obtain our main result:

Theorem 7.1. *Assume that* $V^0 \in (D(\mathcal{L});\mathcal{E})_{1/q,q}$, *then there exists* $0 < T^{**} \leq T$ *such that the problem*

$$\begin{cases} V'(t) = \mathcal{L}V(t) + G(t,W) \\ V(0) = V^0. \end{cases}$$

has a unique local solution in $[0,T^{**}]$ *verifying the maximal* L^q*-regularity*

$$V \in W^{1,q}(0,T^{**};\mathcal{E}) \cap L^q(0,T^{**};D(\mathcal{L})).$$

□

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