Local Limit Theorem for Complex Valued Sequences

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Abstract

In this article, we study the pointwise asymptotic behavior of iterated convolutions on $\mathbb{Z}$. We generalize the so-called local limit theorem in probability theory to complex valued sequences. A sharp rate of convergence is proved together with a generalized Gaussian bound for the first corrector.

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For $1 \leq q < +\infty$, we let $\ell^q(\mathbb{Z})$ denote the Banach space of complex valued sequences indexed by $\mathbb{Z}$ and such that the norm:

$$\|u\|_{\ell^q} := \left(\sum_{j \in \mathbb{Z}} |u_j|^q \right)^{\frac{1}{q}}$$

is finite. We also let $\ell^\infty(\mathbb{Z})$ denote the Banach space of bounded complex valued sequences indexed by $\mathbb{Z}$ equipped with the norm

$$\|u\|_{\ell^\infty} := \sup_{j \in \mathbb{Z}} |u_j|.$$

Throughout this article, we define the following sets:

$$\mathcal{U} := \{z \in \mathbb{C}, |z| > 1\}, \quad \mathbb{D} := \{z \in \mathbb{C}, |z| < 1\}, \quad \mathbb{S}^1 := \{z \in \mathbb{C}, |z| = 1\},$$

$$\overline{\mathcal{U}} := \mathbb{S}^1 \cup \mathcal{U}, \quad \overline{\mathbb{D}} := \mathbb{S}^1 \cup \mathbb{D}.$$

For $z \in \mathbb{C}$ and $r > 0$, we let $B_r(z)$ denote the open ball in $\mathbb{C}$ centered at z with radius r.

For E a Banach space, we denote $\mathcal{L}(E)$ the space of bounded operators acting on E and $\|\cdot\|_{\mathcal{L}(E)}$ the operator norm. For T in $\mathcal{L}(E)$, the notation $\sigma(T)$ stands for the spectrum of the operator T and $\rho(T)$ denotes the spectral radius of T.

Lastly, we let $\mathcal{M}_n(\mathbb{C})$ denote the space of complex valued square matrices of size n and for an element M of $\mathcal{M}_n(\mathbb{C})$, the notation M^T stands for the transpose of M.

We use the notation $\lesssim$ to express an inequality up to a multiplicative constant. Eventually, we let C (resp. c) denote some large (resp. small) positive constants that may vary throughout the text (sometimes within the same line).

1 Introduction and main result

1.1 Context

We define the convolution $a * b$ of two elements a and b of $\ell^1(\mathbb{Z})$ by

$$\forall j \in \mathbb{Z}, \quad (a * b)_j := \sum_{l \in \mathbb{Z}} a_l b_{j-l}.$$

When equipped with this product, $\ell^1(\mathbb{Z})$ is a Banach algebra. For $a \in \ell^1(\mathbb{Z})$, we define the Laurent operator L_a associated with a which acts on $\ell^p(\mathbb{Z})$ as

$$\forall u \in \ell^p(\mathbb{Z}), \quad L_a u := a * u \in \ell^p(\mathbb{Z}).$$

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Young's inequality implies that those operators are well defined and are bounded for all $p \in [1, +\infty]$. Furthermore, we have that $L_{a*b} = L_a \circ L_b$ for $a, b \in \ell^1(\mathbb{Z})$. Finally, Wiener's theorem [10] characterizes the invertible elements of $\ell^1(\mathbb{Z})$ and thus allows us to describe the spectrum of L_a via the Fourier series F associated with a:

$$\sigma(L_a) = \left\{ F(t) := \sum_{k \in \mathbb{Z}} a_k e^{itk}, t \in \mathbb{R} \right\}.$$

We observe that the spectrum is independent of p and that F is continuous since a belongs to $\ell^1(\mathbb{Z})$.

If we suppose that the sequence a has real non negative coefficients and $\sum_{k\in\mathbb{Z}} a_k = 1$, then the sequence $a^n = a * \ldots * a$ is the probability distribution of the sum of n independent random variables supported on $\mathbb{Z}$ each with the probability distribution[2] a. A lot is known on the pointwise asymptotic behavior of the sequence a^n in this case. In particular, the local limit theorem states, under suitable hypotheses on a, that

$$a_j^n - \frac{1}{\sqrt{2\pi\sigma^2 n}} \exp\left(-\frac{|j-n\alpha|^2}{2n\sigma^2}\right) \underset{n\to+\infty}{=} o\left(\frac{1}{\sqrt{n}}\right), \tag{1}$$

where $\alpha = \sum_{k\in\mathbb{Z}} k a_k$ and $\sigma^2 = \sum_{k\in\mathbb{Z}} k^2 a_k - \alpha^2$ are respectively the mean and the variance of a random variable with probability distribution a and where the error term is uniform on $\mathbb{Z}$ (see [11, Chapter VII] for more details). This gives a precise description of the asymptotic behavior of a_j^n in the range $|j - n\alpha| \lesssim \sqrt{n}$ and implies that the convolution powers of a are attracted towards the heat kernel.

Following, among other works, [6, 12, 3], we are interested in generalizing the local limit theorem to the case where a is complex valued. Extending the works of Schoenberg [15], Greville [8] and Diaconis and Saloff-Coste [6, Theorem 2.6], the article [12] of Randles and Saloff-Coste already provides a generalization of the local limit theorem for a large class of complex valued finitely supported elements of $\ell^1(\mathbb{Z})$. By doing so, the authors of [12] describe an asymptotic expansion similar to (1) with an explicit expression of the attractors. Our goal in this paper is to obtain a more precise description of the asymptotic result proved in [12] by proving a sharp rate of convergence together with a generalized Gaussian bound for the remainder (see Theorem 1 below). In the case where a is the probability distribution of a random variable, as above, our theorem would translate in saying that, under suitable assumptions on a (namely that a is finitely supported with at least two nonzero elements), there exist two constants $C, c > 0$ such that

$$\forall n \in \mathbb{N}^*, \forall j \in \mathbb{Z}, \quad \left| a_j^n - \frac{1}{\sqrt{2\pi\sigma^2 n}} \exp\left(-\frac{|j-n\alpha|^2}{2n\sigma^2}\right) \right| \leq \frac{C}{n} \exp\left(-c\frac{|j-n\alpha|^2}{n}\right).$$

Using this result, we are able to prove the well-known Berry-Essen inequality (see [1, 7]) which states that there exists a constant $C > 0$ such that

$$\forall n \in \mathbb{N}^*, \forall J \in \mathbb{Z}, \quad \left| \sum_{j\leq J} a_j^n - \sum_{j\leq J} \frac{1}{\sqrt{2\pi\sigma^2 n}} \exp\left(-\frac{|j-n\alpha|^2}{2n\sigma^2}\right) \right| \leq \frac{C}{\sqrt{n}}.$$

However, we will need stronger hypotheses on the elements of $\ell^1(\mathbb{Z})$ than the conditions imposed in [12]. We will consider here elements a of $\ell^1(\mathbb{Z})$ which are finitely supported and such that the sequence $(a^n)_{n\in\mathbb{N}}$ is bounded in $\ell^1(\mathbb{Z})$. The fundamental contribution [18] by Thomée completely characterizes such elements and is an important starting point for our work.

In the articles [6] and [12], the proofs mainly rely on the use of Fourier analysis to express the elements a_j^n via the Fourier series associated with a. In this paper, we will rather follow an approach usually referred to in partial differential equations as "spatial dynamics". It aims at using the functional calculus (see [2, Chapter VII]) to express the temporal Green's function (here the coefficients a_j^n) with the resolvent of the operator L_a via the spatial Green's function which is the unique solution of

$$(zId - L_a)u = \delta, \quad z \in \mathbb{C}\backslash\sigma(L_a),$$

where δ is the discrete Dirac mass $\delta = (\delta_{j,0})_{j\in\mathbb{Z}}$. This approach has already been used in [3] to extend the result of [6, Theorem 1.1] and obtain a uniform generalized Gaussian bound for the elements a_j^n. It has also been used in [4] to prove similar results on finite rank perturbations of Toeplitz operators (convolution operators on $\ell^p(\mathbb{N})$ rather than on $\ell^p(\mathbb{Z})$). The present paper is very much inspired by [3, 4] and we will use notations and proofs similar to those articles. We will now present in more details the hypotheses we need on the elements $a \in \ell^1(\mathbb{Z})$ that we shall consider and we shall then our main theorem.

[2] We say that a sequence a is the probability distribution of a random variable X with values in $\mathbb{Z}$ when $\mathbb{P}(X = j) = a_j$ for all $j \in \mathbb{Z}$.

1.2 Hypotheses and theorem

We consider a given sequence $a \in \ell^1(\mathbb{Z})$. We let $\mathscr{L}_a$ be the bounded operator acting on $\ell^p(\mathbb{Z})$ defined as

$$\forall u \in \ell^p(\mathbb{Z}), \quad \mathscr{L}_a u := \left(\sum_{l \in \mathbb{Z}} a_l u_{j+l} \right)_{j \in \mathbb{Z}}.$$

This operator is obviously linked to Laurent operators and could be written as one of them ($\mathscr{L}_a = L_b$ for $b = (a_{-j})_{j \in \mathbb{Z}}$). Our goal will be to study the powers $\mathscr{L}_a^n$ for n large. This problem arises as the large time behavior of finite difference approximations of partial differential equations. We also define the symbol F associated with a as

$$\forall \kappa \in \mathbb{S}^1, \quad F(\kappa) := \sum_{j \in \mathbb{Z}} a_j \kappa^j. \tag{2}$$

The Wiener theorem [10] implies that

$$\sigma(\mathscr{L}_a) = F(\mathbb{S}^1).$$

We are now going to introduce some hypotheses that are necessary for the rest of the paper.

Hypothesis 1. *The sequence a is finitely supported and has at least two nonzero coefficients.*

This hypothesis implies that we can extend the definition (2) of F on $\mathbb{C}\setminus\{0\}$ and F becomes a holomorphic function on this domain. We introduce the two following elements

$$k_m := \min\{k \in \mathbb{Z}, \quad a_k \neq 0\}, \quad k_M := \max\{k \in \mathbb{Z}, \quad a_k \neq 0\}.$$

Observing that Hypothesis 1 implies $k_m < k_M$, we then distinguish three different possibilities:

- We have $k_M \leq -1$. Then, we define $r := -k_m$ and $p := 0$.
- We have $k_m \leq 0 \leq k_M$. Then, we define $r := -k_m$ and $p := k_M$.
- We have $1 \leq k_m$. Then, we define $r := 0$ and $p := k_M$.

In every case, we have $r, p \in \mathbb{N}$ and $-r > p$. Also, we have that

$$\forall u \in \ell^p(\mathbb{Z}), \forall j \in \mathbb{Z}, \quad (\mathscr{L}_a u)_j = \sum_{k=-r}^{p} a_k u_{j+k}.$$

The natural integers r and p we just introduced define the common stencil of the operators $\mathcal{L}$ and Id and they will be useful to study the so-called resolvent equation (13). We now introduce an assumption on F which is based on [18]. Just like in [6, 12, 3], we normalize a so that the maximum of F on $\mathbb{S}^1$ is 1.

Hypothesis 2. *There exists a finite set of distinct points $\{\underline{\kappa}_1, \dots, \underline{\kappa}_K\}$, $K \geq 1$, in $\mathbb{S}^1$ such that for all $k \in \{1, \dots, K\}$, $\underline{z}_k := F(\underline{\kappa}_k)$ belongs to $\mathbb{S}^1$ and*

$$\forall z \in \mathbb{S}^1 \setminus \{\underline{\kappa}_1, \dots, \underline{\kappa}_K\}, \quad |F(z)| < 1.$$

Moreover, we suppose that for each $k \in \{1, \dots, K\}$, there exist a nonzero real number α_k, an integer $\mu_k \geq 1$ and a complex number β_k with positive real part such that

$$F(\underline{\kappa}_k e^{i\xi}) \underset{\xi \to 0}{=} \underline{z}_k \exp(-i\alpha_k \xi - \beta_k \xi^{2\mu_k} + O(|\xi|^{2\mu_k+1})). \tag{3}$$

Graphically, this means that the spectrum $\sigma(\mathscr{L}_a)$ is contained in the disk $\overline{\mathbb{D}}$ and intersects $\mathbb{S}^1$ at a finite number of points (see Figure 1 for an example with $K = 2$, $\underline{z}_1 = 1$, $\underline{z}_2 = -1$) and that F has a specific asymptotic expansion at those intersection points. The condition (3) has been studied closely because of its link with the stability properties of finite difference approximations for PDEs (see [18]). We can observe that, under hypotheses 1 and 2, there holds

$$\forall n \in \mathbb{N}^*, \quad \|\mathscr{L}_a^n\|_{\mathcal{L}(\ell^2(\mathbb{Z}))} = \|F^n\|_{L^\infty(\mathbb{S}^1)} = 1.$$

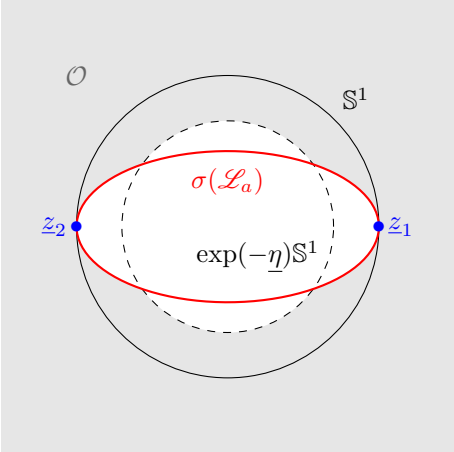

Figure 1: An example of spectrum $\sigma(\mathscr{L}_a)$. The spectrum $\sigma(\mathscr{L}_a)$ (in red) is inside the closed disk $\bar{\mathbb{D}}$ and touches the boundary $\mathbb{S}^1$ in finitely many points. In gray, we have $\mathcal{O}$ the intersection of the unbounded connected component of $\mathbb{C}\backslash\sigma(\mathscr{L}_a)$ and $\left\{z \in \mathbb{C}, |z| > \exp(-\underline{\eta})\right\}$.

It assures the ℓ^2-stability, or strong stability (see [16], [17]), of the numerical scheme defined as

$$\begin{cases} u^{n+1} = \mathscr{L}_a u^n, \quad n \geq 0, \\ u^0 \in \ell^2(\mathbb{Z}). \end{cases} \tag{4}$$

However, it has much further consequences, as the asymptotic expansion (3) assures the ℓ^p-stability of the scheme (4) for every p in $[1, +\infty]$ (see [18, Theorem 1] which focuses on the ℓ^∞-stability but also studies the ℓ^p-stability as a consequence). From a more general point of view, it is proved in [18, Theorem 1] that Hypothesis 2 is one of two conditions to characterize the elements a of $\ell^1(\mathbb{Z})$ such that the family $(a^n)_{n\in\mathbb{N}}$ is bounded in $\ell^1(\mathbb{Z})$.

For the rest of the paper, it will be useful to observe that the asymptotic expansion (3) implies that

$$\forall k \in \{1, \dots, K\}, \quad \alpha_k = -F'(\underline{\kappa}_k)\frac{\underline{\kappa}_k}{\underline{z}_k} \in \mathbb{R}^*. \tag{5}$$

Finally, we have the following lemma.

Lemma 1. *For $a \in \ell^1(\mathbb{Z})$ which verifies hypotheses 1 and 2, we have that a_{-r} and a_p belong to $\mathbb{D}$.*

Proof We introduce the polynomial function g defined by

$$\forall \kappa \in \mathbb{C}, \quad g(\kappa) := \sum_{l=-r}^{p} a_l \kappa^{l+r}.$$

For all $\kappa \in \mathbb{S}^1$, Hypothesis 2 implies that

$$|g(\kappa)| = |\kappa^r F(\kappa)| = |F(\kappa)| \leq 1.$$

Observing that g is not a constant function, the maximum principle for holomorphic functions [14] allows us to conclude that

$$|a_{-r}| = |g(0)| < 1.$$

The same kind of argument allows us to conclude for the coefficient a_p. □

We now introduce yet another hypothesis.

Hypothesis 3. *For all $k \in \{1, \dots, K\}$, we have that the set*

$$\mathcal{I}_k := \{\nu \in \{1, \dots, K\}, \quad \underline{z}_\nu = \underline{z}_k\}$$

has either one or two elements. Moreover, if there are two distinct elements $\nu_{k,1}$ and $\nu_{k,2}$ in it, then $\alpha_{\nu_{k,1}}\alpha_{\nu_{k,2}} < 0$.

Hypothesis 3 combined with the fact that the α_k are nonzero real numbers (see Hypothesis 2) implies that, for $k \in \{1, \dots, K\}$, we have three different possibilities:

- **Case I:** $\mathcal{I}_k$ is the singleton $\{k\}$ and $\alpha_k > 0$,
- **Case II:** $\mathcal{I}_k$ is the singleton $\{k\}$ and $\alpha_k < 0$,
- **Case III:** $\mathcal{I}_k$ has two distinct elements $\nu_{k,1}$ and $\nu_{k,2}$ such that $\alpha_{\nu_{k,1}} > 0$ and $\alpha_{\nu_{k,2}} < 0$.

Distinguishing those three cases will be useful later on. We define for $z \in \mathbb{C}$ and $j \in \{-r, \dots, p\}$

$$\mathbb{A}_j(z) := z\delta_{j,0} - a_j. \tag{6}$$

The definition of r and p implies that $\mathbb{A}_{-r}$ and $\mathbb{A}_p$ can vanish at most on one point which are respectively a_{-r} and a_p. Lemma 1 allows us to find $\underline{\eta} > 0$ such that $\mathbb{A}_{-r}$ and $\mathbb{A}_p$ do not vanish on $\left\{z \in \mathbb{C}, |z| > \exp(-\underline{\eta})\right\}$. We can therefore define for all $z \in \mathbb{C}$ such that $|z| > \exp(-\underline{\eta})$ the matrix

$$\mathbb{M}(z) := \begin{pmatrix} -\frac{\mathbb{A}_{p-1}(z)}{\mathbb{A}_p(z)} & \dots & \dots & -\frac{\mathbb{A}_{-r}(z)}{\mathbb{A}_p(z)} \\ 1 & 0 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \mathcal{M}_{p+r}(\mathbb{C}).$$

The application which associates z with $\mathbb{M}(z)$ is holomorphic. Moreover, since $\mathbb{A}_{-r}(z) \neq 0$, the upper right coefficient of $\mathbb{M}(z)$ is always nonzero and $\mathbb{M}(z)$ is invertible for all $z \in \mathbb{C}$ such that $|z| > \exp(-\underline{\eta})$.

The three hypotheses we presented above will be crucial in the rest of the paper. Some of them could maybe be less restrictive, but this would be consideration for future works.

Finally, by defining the discrete Dirac mass $\delta = (\delta_{j,0})_{j\in\mathbb{Z}}$, we introduce the so-called temporal Green's function defined by

$$\forall n \in \mathbb{N}^*, \forall j \in \mathbb{Z}, \quad \mathscr{G}_j^n := \left(\mathscr{L}_a^n \delta\right)_j.$$

Our main goal is to determine the asymptotic behavior of $\mathscr{G}_j^n$ when n grows and to determine a speed of convergence with a uniform estimate of the remainder for $j \in \mathbb{Z}$. As discussed in the previous section, the asymptotic behavior is already known and justified in [12]. For $\mu \in \mathbb{N}^*$ and $\beta \in \mathbb{C}$ with positive real part, we let $H_{2\mu}^\beta : \mathbb{R} \to \mathbb{C}$ be the function defined as

$$\forall x \in \mathbb{R}, \quad H_{2\mu}^\beta(x) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ixu} e^{-\beta u^{2\mu}} du.$$

The following lemma gives sharp estimates on the function $H_{2\mu}^\beta$.

Lemma 2. *For $\mu \in \mathbb{N}^*$ and $\beta \in \mathbb{C}$ with positive real part, there exist two constants $C, c > 0$ such that*

$$\forall x \in \mathbb{R}, \quad \begin{cases} \left|H_{2\mu}^\beta(x)\right| \leq C \exp\left(-c|x|^{\frac{2\mu}{2\mu-1}}\right), \\ \left|\left(H_{2\mu}^\beta\right)'(x)\right| \leq C \exp\left(-c|x|^{\frac{2\mu}{2\mu-1}}\right). \end{cases}$$

This lemma is proved in [13, Proposition 5.3]. For the sake of completeness, we give a complete proof of it in the appendix (Section 5). We can now define the so-called "attractors":

$$\forall k \in \{1, \dots, K\}, \forall n \in \mathbb{N}^*, \forall j \in \mathbb{Z}, \quad \mathscr{H}_{k,j}^n := \frac{{z_k}^n {\kappa_k}^j}{n^{\frac{1}{2\mu_k}}} H_{2\mu_k}^{\beta_k}\left(\frac{j - n\alpha_k}{n^{\frac{1}{2\mu_k}}}\right).$$

In [12, Theorem 1.2], it is proved that if we introduce $\mu = \max_{k\in\{1,\dots,K\}} \mu_k$ and $\mathcal{K}_\mu = \{k \in \{1, \dots, K\}, \mu_k = \mu\}$, then

$$\mathscr{G}_j^n - \sum_{k\in\mathcal{K}_\mu} \mathscr{H}_{k,j}^n \underset{n\to+\infty}{=} o\left(\frac{1}{n^{\frac{1}{2\mu}}}\right) \tag{7}$$

where the error term is uniform on $\mathbb{Z}$. This result is obviously a generalization of (1) to complex valued sequences. Using Lemma 2 for the estimate on $\mathscr{H}_{k,j}^n$ and [3, Theorem 1] for the estimate on $\mathscr{G}_j^n$, we can prove that there exist two constants $C, c > 0$ such that

$$\forall n \in \mathbb{N}^*, \forall j \in \mathbb{Z}, \quad \left|\mathscr{G}_j^n\right| \leq C \sum_{k=1}^K \frac{1}{n^{\frac{1}{2\mu_k}}} \exp\left(-c\left(\frac{|j - n\alpha_k|}{n^{\frac{1}{2\mu_k}}}\right)^{\frac{2\mu_k}{2\mu_k-1}}\right),$$

and

$$\forall k \in \{1,\dots,K\}, \forall n \in \mathbb{N}^*, \forall j \in \mathbb{Z}, , \quad \left|\mathscr{H}_{k,j}^n\right| \leq \frac{C}{n^{\frac{1}{2\mu_k}}} \exp\left(-c\left(\frac{|j-n\alpha_k|}{n^{\frac{1}{2\mu_k}}}\right)^{\frac{2\mu_k}{2\mu_k-1}}\right).$$

Thus, the result of Randles and Saloff-Coste gives a precise description of the behavior of $\mathscr{G}_j^n$ for j such that

$$|j-n\alpha_k| \lesssim n^{\frac{1}{2\mu}}, \tag{8}$$

where $k \in \mathcal{K}_\mu$. Our main result is this next theorem.

Theorem 1. *If we consider $a \in \ell^1(\mathbb{Z})$ which verifies hypotheses 1, 2 and 3, we can find two constants $C, c > 0$ such that*

$$\forall n \in \mathbb{N}^*, \forall j \in \mathbb{Z}, \quad \left|\mathscr{G}_j^n - \sum_{k=1}^K \mathscr{H}_{k,j}^n\right| \leq C \sum_{k=1}^K \frac{1}{n^{\frac{1}{\mu_k}}} \exp\left(-c\left(\frac{|j-n\alpha_k|}{n^{\frac{1}{2\mu_k}}}\right)^{\frac{2\mu_k}{2\mu_k-1}}\right).$$

In particular, this implies that there exists a constant $\tilde{C} > 0$ such that if we define $\mu = \max_{k\in\{1,\dots,K\}} \mu_k$, then

$$\forall n \in \mathbb{N}^*, \forall J \in \mathbb{Z}, \quad \left|\sum_{j\leq J} \mathscr{G}_j^n - \sum_{k=1}^K \sum_{j\leq J} \mathscr{H}_{k,j}^n\right| \leq \frac{\tilde{C}}{n^{\frac{1}{2\mu}}}.$$

Our result gives a sharp estimate on the remainder for any $j \in \mathbb{Z}$ and allows to extend the result of Randles and Saloff-Coste by describing the asymptotic behavior $\mathscr{G}_j^n$ for every j in $\mathbb{Z}$ with a more precise speed of convergence, in particular for j that verifies (8) where $k \in \{1,\dots,K\}$.

However, Theorem 1 has some limits. A complete generalization of the local limit theorem would treat the case were a and F satisfy wider hypotheses. For example, [12, Theorem 1.2] also treats the case where the asymptotic expansion (3) has the form

$$F(\underline{\kappa}_k e^{i\xi}) \underset{\xi\to 0}{=} \underline{z}_k \exp(-i\alpha_k\xi + i\gamma_k\xi^{2\mu_k+1} + O(|\xi|^{2\mu_k+2})),$$

where γ_k is a real number. A generalization of Theorem 1 in this difficult case has not yet been found.

Theorem 1 is also constrained by Hypothesis 2 which imposes that α_k is nonzero. The result [12, Theorem 1.2] of Randles and Saloff-Coste does not have this kind of restriction. The hypothesis $\alpha_k \neq 0$ is essential in the proof of Theorem 1 below but it seems like the theorem should also be proved when α_k can be equal to zero. We therefore introduce a relaxed version of Hypothesis 2.

Hypothesis 4 (Hypothesis 2 bis)**.** *The sequence a verifies Hypothesis 2 but with the possibility that some α_k are equal to 0.*

We now consider a finitely supported sequence $a \in \ell^1(\mathbb{Z})$ which verifies Hypothesis 4 and let $J \in \mathbb{Z}$. Then, if we define the sequence $b = (a_{j+J})_{j\in\mathbb{Z}}$ and $\widetilde{F}$ the symbol associated with b, we have that b satisfies Hypothesis 4 since

$$\forall \kappa \in \mathbb{S}^1, \quad \widetilde{F}(\kappa) = \kappa^{-J} F(\kappa),$$

and therefore

$$\forall \kappa \in \mathbb{S}^1, \quad \left|\widetilde{F}(\kappa)\right| = |F(\kappa)|.$$

Also, we have for $k \in \{1,\dots,K\}$

$$\widetilde{F}(\underline{\kappa}_k e^{i\xi}) \underset{\xi\to 0}{=} \underline{\kappa}_k^{-J} \underline{z}_k \exp(-i(\alpha_k+J)\xi - \beta_k\xi^{2\mu_k} + o(|\xi|^{2\mu_k})). \tag{9}$$

Considering this new sequence b allows us to "shift" the elements α_k. In particular, if we choose J large enough, then b satisfies Hypothesis 2. We can also define the attractors associated to the sequence b thanks to the asymptotic expansion (9). We can then prove the following corollary of Theorem 1 which generalizes Theorem 1 in the case where α_k can be equal to 0.

Corollary 1. *We consider $a \in \ell^1(\mathbb{Z})$ which verifies hypotheses 1 and 4. If there exists some integer $J \in \mathbb{Z}$ such that the sequence $(a_{j+J})_{j\in\mathbb{Z}}$ verifies hypotheses 2 and 3, then we can find two constants $C, c > 0$ such that*

$$\forall n \in \mathbb{N}^*, \forall j \in \mathbb{Z}, \quad \left|\mathscr{G}_j^n - \sum_{k=1}^K \mathscr{H}_{k,j}^n\right| \leq C \sum_{k=1}^K \frac{1}{n^{\frac{1}{\mu_k}}} \exp\left(-c\left(\frac{|j-n\alpha_k|}{n^{\frac{1}{2\mu_k}}}\right)^{\frac{2\mu_k}{2\mu_k-1}}\right).$$

Before we start the proofs, we will see in the next subsection that the estimate of Theorem 1 is sharp with some examples and numerical applications. Then, we start with the proof of Theorem 1. To do so, in Section 2, we will introduce the spatial Green's function on which Coulombel and Faye proved holomorphic extension properties and sharp bounds in [3, Section 2]. Our goal in Section 2 is to improve the analysis of [3] and to obtain the precise behavior of the spatial Green's function for z close to $\underline{z}_k$ and to prove sharp bounds on the remainder. In Section 3, we will express the elements $\mathscr{G}_j^n$ with the spatial Green's function using the functional calculus. The results of Section 2 will then allow us to conclude the proof of Theorem 1. The last Section 4 is dedicated to the proof of Corollary 1.

1.3 Examples and numerical results

We consider some examples of elements $a \in \ell^1(\mathbb{Z})$ which satisfy the conditions of Theorem 1 and see how sharp the estimations we found are.

1.3.1 Probability distribution : real non negative sequences

First, we consider the case where a has real non negative coefficients. If we introduce the sequence $b = (a_{-j})_{j\in\mathbb{Z}}$, then b is the probability distribution of some random variable X supported on $\mathbb{Z}$. We observe that $L_b = \mathscr{L}_a$, so, recalling that $b^n = b * \ldots * b$, we have

$$\forall n \in \mathbb{N}^*, \forall j \in \mathbb{Z}, \quad b_j^n = \mathscr{G}_j^n.$$

We will settle on $a \in \ell^1(\mathbb{Z})$ such that $a_j = 0$ for $j \neq -1, 0, 1$ and

$$a_{-1} = 2/3, a_0 = 1/6, a_1 = 1/6.$$

This sequence verifies Hypothesis 1. In this case, we have $r = p = 1$. Also, $F(1) = 1$ and

$$\forall \kappa \in \mathbb{S}^1 \setminus \{1\}, \quad |F(\kappa)| < 1.$$

The function F satisfies that

$$F(e^{i\xi}) \underset{\xi\to 0}{=} \exp(-i\alpha\xi - \beta\xi^2 + o(\xi^2))$$

where $\alpha = \mathbb{E}(X) = 0.5$ and $\beta = \frac{V(X)}{2} = \frac{7}{24}$. It also directly satisfies Hypothesis 3 since $K = 1$. We can then define the attractors

$$\forall n \in \mathbb{N}^*, \forall j \in \mathbb{Z}, \quad \mathscr{H}_j^n = \frac{1}{\sqrt{n}} H_2^\beta\left(\frac{j - n\alpha}{\sqrt{n}}\right) = \frac{1}{\sqrt{4\pi\beta n}} \exp\left(-\frac{|j - n\alpha|^2}{4\beta n}\right).$$

Theorem 1 states that there exists two constants $C, c > 0$ such that

$$\forall n \in \mathbb{N}^*, \forall j \in \mathbb{Z}, \quad \left|b_j^n - \mathscr{H}_j^n\right| = \left|\mathscr{G}_j^n - \mathscr{H}_j^n\right| \leq \frac{C}{n} \exp\left(-c\left(\frac{|j - n\alpha|^2}{n}\right)\right). \tag{10}$$

This behavior is represented on Figure 2. The local limit theorem[3] actually states for sequences b with finite third moment (see [11, Section VII, Theorem 13]) that

$$\mathscr{G}_j^n - \mathscr{H}_j^n = b_j^n - \mathscr{H}_j^n \underset{n\to+\infty}{=} \frac{1}{n\sqrt{2\beta}} q_1\left(\frac{j - n\alpha}{\sqrt{2\beta n}}\right) + o\left(\frac{1}{n}\right), \tag{11}$$

where the error term is uniform on $\mathbb{Z}$ and

$$\forall x \in \mathbb{R}, \quad q_1(x) := \frac{\mathbb{E}((X - \mathbb{E}(X))^3)}{\sqrt{2\pi}(V(X))^{\frac{3}{2}}}(x^3 - 3x)e^{-\frac{x^2}{2}}.$$

Knowing that there exist two constants $C, c > 0$ such that the function q_1 verifies

$$\forall x \in \mathbb{R}, \quad |q_1(x)| \leq C \exp\left(-cx^2\right),$$

we conclude that the inequality (10) is sharp, meaning that the factor $\frac{1}{n}$ cannot be improved.

[3] We refer to [11, Section VII] for a complete asymptotic expansion of b_j^n involving Hermite polynomials.

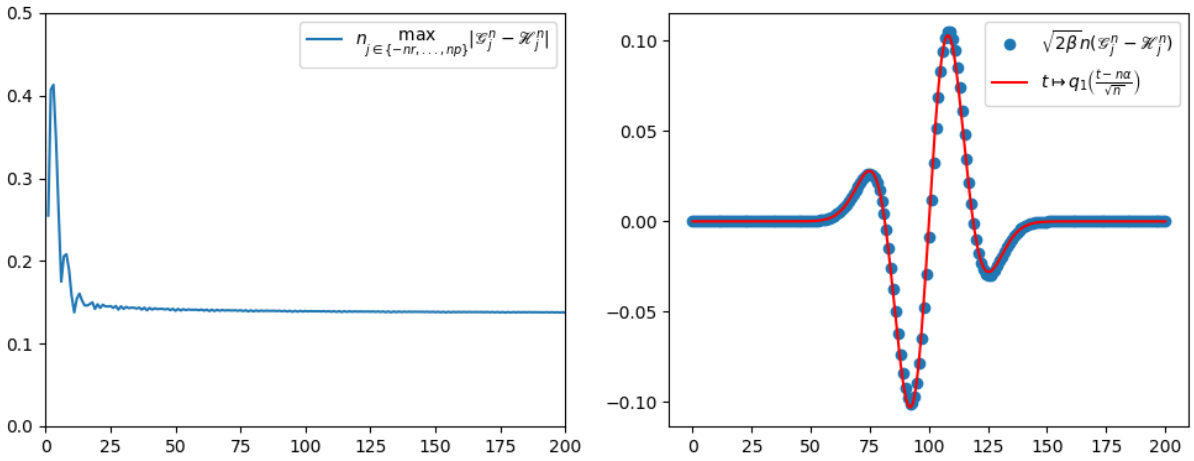

Figure 2: On the left : A representation of $n \max_{j\in\{-nr,\dots,np\}} |\mathscr{G}_j^n - \mathscr{H}_j^n|$ depending on n. As expected knowing that $-r < \alpha < p$, we see that the function is bounded and even seems to converge. On the right : We fixed $n = 200$ and represented $t \mapsto q_1\left(\frac{t-n\alpha}{\sqrt{n}}\right)$ and $j \in \mathbb{Z} \mapsto \sqrt{2\beta n}\left(\mathscr{G}_j^n - \mathscr{H}_j^n\right)$. We find the behavior we expected from equation (11).

1.3.2 The O3 scheme for the transport equation

We will now consider an example linked to finite difference schemes. We consider the transport equation

$$\partial_t u + a\partial_x u = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}$$

with Cauchy data at $t = 0$. The O3 scheme is an explicit third order accurate finite difference approximation of the previous transport equation. We refer to [5] for a detailed analysis of this scheme. It corresponds to the numerical scheme (4) for $a \in \ell^1(\mathbb{Z})$ such that $a_j = 0$ for $j \notin \{-2,-1,0,1\}$ and

$$a_{-2} = -\frac{\lambda a(1-(\lambda a)^2)}{6}, \quad a_{-1} = \frac{\lambda a(1+\lambda a)(2-\lambda a)}{2}, \quad a_0 = \frac{(1-(\lambda a)^2)(2-\lambda a)}{2}, \quad a_1 = -\frac{\lambda a(1-\lambda a)(2-\lambda a)}{6},$$

with $\lambda = \frac{\Delta t}{\Delta x} > 0$. The parameter λa is the Courant number. We have in this case that $r = 2$ and $p = 1$. For $\lambda a \in]-1,1[\setminus\{0\}$, we have that $F(1) = 1$ and

$$\forall \kappa \in \mathbb{S}^1 \setminus \{1\}, \quad |F(\kappa)| < 1.$$

Also, there exists $\beta \in \mathbb{R}_+^*$ such that

$$F(e^{i\xi}) \underset{\xi\to 0}{=} \exp(-i\lambda a\xi - \beta\xi^4 + o(\xi^4)).$$

We have $\mu = 2$ in this case and Hypothesis 2 is satisfied with $K = 1$, $\underline{\kappa}_1 = 1$ and $\underline{z}_1 = 1$. The sequence a verifies hypotheses 1, 2 and 3, so we can apply Theorem 1. We introduce the attractors

$$\forall n \in \mathbb{N}^*, \forall j \in \mathbb{Z}, \quad \mathscr{H}_j^n = \frac{1}{n^{\frac{1}{4}}} H_4^\beta\left(\frac{j-n\alpha}{n^{\frac{1}{4}}}\right).$$

Theorem 1 then states that there exists two constants $C, c > 0$ such that

$$\forall n \in \mathbb{N}^*, \forall j \in \mathbb{Z}, \quad \left|\mathscr{G}_j^n - \mathscr{H}_j^n\right| \leq \frac{C}{\sqrt{n}} \exp\left(-c\left(\frac{|j-n\alpha|}{n^{\frac{1}{4}}}\right)^{\frac{4}{3}}\right). \tag{12}$$

This behavior is represented on Figure 3 where we even see that the remainder $\sqrt{n}(\mathscr{G}_j^n - \mathscr{H}_j^n)$ seems to scale like $f\left(\frac{j-n\alpha}{\sqrt{n}}\right)$ like in (11) for the case of probability distribution. Hence, the $\frac{1}{\sqrt{n}}$ pre-factor in (12) seems to be sharp. We now turn to the proof of Theorem 1.

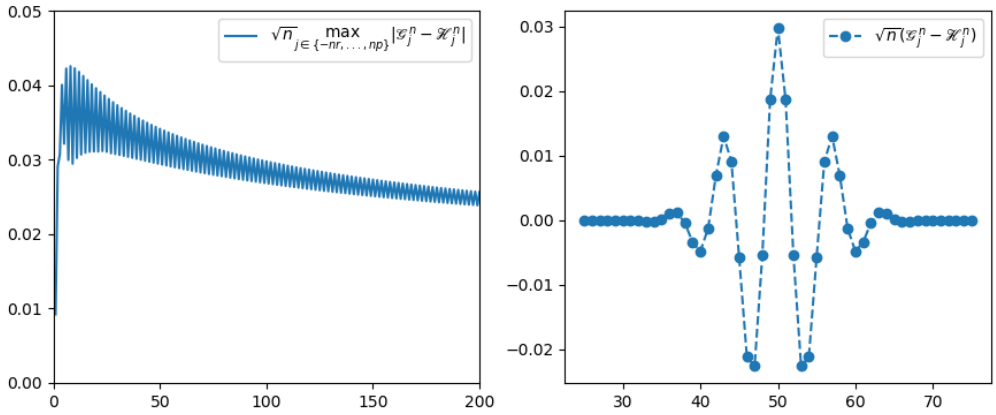

Figure 3: For these figures, we chose $\lambda a = 1/2$. On the left : A representation of $\sqrt{n}\max_{j\in\{-nr,\dots,np\}}|\mathscr{G}_j^n-\mathscr{H}_j^n|$ depending on n. As expected, the function seems to be bounded. On the right : We fixed $n = 100$ and represented $j \in \mathbb{Z} \mapsto \sqrt{n}\left(\mathscr{G}_j^n - \mathscr{H}_j^n\right)$. We observe the exponential decay in j. Also, we can see a particular shape of curve that arises. It might be possible to go even further in the asymptotic expansion of $\mathscr{G}_j^n$ and find a result similar to (11), but this would be consideration for other papers.

2 Spatial Green's function

From now on, we consider a sequence a that satisfies the hypotheses 1, 2 and 3. In this section, we are going to introduce the spatial Green's function and prove some estimates. We start by defining the open set $\mathcal{O}$ which corresponds to the intersection of the unbounded connected component of $\mathbb{C}\backslash F(\mathbb{S}^1)$ and $\left\{z \in \mathbb{C}, |z| > \exp(-\underline{\eta})\right\}$ (see Figure 1). The Hypothesis 2 implies that $\overline{\mathcal{U}}\backslash\{\underline{z}_1,\dots,\underline{z}_K\}$ is contained within $\mathcal{O}$. By recalling that $\sigma(\mathscr{L}_a) = F(\mathbb{S}^1)$, when we consider that $\mathscr{L}_a$ acts on $\ell^2(\mathbb{Z})$, we have for every $z \in \mathcal{O}$ of a unique $G(z) \in \ell^2(\mathbb{Z})$ such that

$$(zI - \mathscr{L}_a)G(z) = \delta, \tag{13}$$

where δ still denotes the discrete Dirac mass. The sequence $G(z)$ is the so-called spatial Green's function. In [3, Lemma 2, 3 and 4], there are already sharp bounds on this sequence $G(z)$ that have been proved. Our goal is to get a more precise description of the behavior of the sequence $G(z)$, at least for z close to $\underline{z}_k$. The description of $G(z)$ in [3] far from the points $\underline{z}_k$ will be sufficient for our purpose. This section will thus on many aspects closely look like [3, Section 2] and we will specify where our study of the sequence $G(z)$ will differ.

Using the functions $\mathbb{A}_l$ which are defined by (6), the equation (13) can be rewritten as

$$\forall j \in \mathbb{Z}, \quad \sum_{l=-r}^{p} \mathbb{A}_l(z)G_{j+l}(z) = \delta_{j,0}.$$

We introduce the vectors

$$\forall j \in \mathbb{Z}, \quad W_j(z) := \begin{pmatrix} G_{j+p-1}(z) \\ \vdots \\ G_{j-r}(z) \end{pmatrix} \in \mathbb{C}^{p+r}, \quad e := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{C}^{p+r}.$$

We then have

$$\forall z \in \mathcal{O}, \forall j \in \mathbb{Z}, \quad W_{j+1}(z) - \mathbb{M}(z)W_j(z) = -\frac{\delta_{j,0}}{\mathbb{A}_p(z)}e. \tag{14}$$

The study of the recurrence relation (14) relies on the following lemma introduced in [9] that studies the eigenvalues of $\mathbb{M}(z)$ for $z \in \mathcal{O}$ and $z \in \{\underline{z}_k, 1 \le k \le K\}$. We recall that we defined cases I, II and III according to the cardinality of $\mathcal{I}_k$ and the sign of α_k right after Hypothesis 3. We also recall that we consider that a verifies Hypotheses 1, 2 and 3.

Lemma 3 (Spectral Splitting). *Let $z \in \mathcal{O}$. Then the matrix $\mathbb{M}(z)$ has*

- *no eigenvalue on $\mathbb{S}^1$,*
- *r eigenvalues in $\mathbb{D}\setminus\{0\}$ (that we call stable eigenvalues),*
- *p eigenvalues in $\mathcal{U}$ (that we call unstable eigenvalues).*

We now consider $k \in \{1,\dots,K\}$. The eigenvalues of the matrix $\mathbb{M}(\underline{z}_k)$ are described by the following possibilities depending on k.

- *In case I, $\mathbb{M}(\underline{z}_k)$ has $\underline{\kappa}_k \in \mathbb{S}^1$ as a simple eigenvalue, $r-1$ eigenvalues in $\mathbb{D}$ and p eigenvalues in $\mathcal{U}$.*
- *In case II, $\mathbb{M}(\underline{z}_k)$ has $\underline{\kappa}_k \in \mathbb{S}^1$ as a simple eigenvalue, r eigenvalues in $\mathbb{D}$ and $p-1$ eigenvalues in $\mathcal{U}$.*
- *In case III, if we denote $\nu_{k,1}$ and $\nu_{k,2}$ the two distinct elements of $\mathcal{I}_k$, then $\mathbb{M}(\underline{z}_k)$ has $\underline{\kappa}_{\nu_{k,1}} \in \mathbb{S}^1$ and $\underline{\kappa}_{\nu_{k,2}} \in \mathbb{S}^1$ as simple eigenvalues, $r-1$ eigenvalues in $\mathbb{D}$ and $p-1$ eigenvalues in $\mathcal{U}$.*

Lemma 3 is proved in [3, Lemma 1] and is the key to study the recurrence relation (14). We now want to prove some estimates on the spatial Green's function $G(z)$. We begin with the following lemma.

Lemma 4 (Bounds far from the tangency points [3]). *For all $\underline{z} \in \mathcal{O}$, there exist a radius $\delta > 0$ and constants $C, c > 0$ such that for all $j \in \mathbb{Z}$, $z \mapsto G_j(z)$ is holomorphic on $B_\delta(\underline{z})$ and satisfies*

$$\forall z \in B_\delta(\underline{z}), \forall j \in \mathbb{Z}, \quad |G_j(z)| \leq C\exp(-c|j|).$$

Lemma 4 is proved in [3, Lemma 2] and allows us to study the spatial Green's function far from the points $\underline{z}_k$, where the spectrum of $\mathscr{L}_a$ intersects the unit circle $\mathbb{S}^1$. We will now have to study the spatial Green's function $G(z)$ near those points $\underline{z}_k$ while still remembering that $G_j(z)$ and $W_j(z)$ are only defined on $\mathcal{O}$ in the neighborhood of $\underline{z}_k$. We are going to extend holomorphically $G_j(z)$ in a whole neighborhood of $\underline{z}_k$, and thus pass through the spectrum $\sigma(\mathscr{L}_a)$.

Lemma 5 (Bounds close to the tangency points : cases I and II). *Let $k \in \{1,\dots,K\}$ so that we are either in case I or II. Then, there exist a radius $\varepsilon > 0$, some constants $C, c > 0$ and some holomorphic functions $\kappa_k, f_k : B_\varepsilon(\underline{z}_k) \to \mathbb{C}$ such that for all $z \in B_\varepsilon(\underline{z}_k)$, $\kappa_k(z)$ is a simple eigenvalue of $\mathbb{M}(z)$ with $\kappa_k(\underline{z}_k) = \underline{\kappa}_k$, for all $j \in \mathbb{Z}$, the function $z \in B_\varepsilon(\underline{z}_k) \cap \mathcal{O} \mapsto G_j(z)$ can be holomorphically extended on $B_\varepsilon(\underline{z}_k)$ and*

Case I: *($\alpha_k > 0$)*

$$\forall z \in B_\varepsilon(\underline{z}_k), \forall j \geq 1, \quad |G_j(z) - f_k(z)\kappa_k(z)^j| \leq C\exp(-cj). \tag{15}$$
$$\forall z \in B_\varepsilon(\underline{z}_k), \forall j \leq 0, \quad |G_j(z)| \leq C\exp(-c|j|). \tag{16}$$

Case II: *($\alpha_k < 0$)*

$$\forall z \in B_\varepsilon(\underline{z}_k), \forall j \geq 1, \quad |G_j(z)| \leq C\exp(-cj). \tag{17}$$
$$\forall z \in B_\varepsilon(\underline{z}_k), \forall j \leq 0, \quad |G_j(z) - f_k(z)\kappa_k(z)^j| \leq C\exp(-c|j|). \tag{18}$$

Furthermore, we have in both cases that

$$f_k(\underline{z}_k) = \frac{1}{|\alpha_k|\underline{z}_k}. \tag{19}$$

Lemma 6 (Bounds close to the tangency points : case III). *Let $k \in \{1,\dots,K\}$ so that we are in case III. The set $\mathcal{I}_k$ has two elements $\nu_{k,1}$ and $\nu_{k,2}$ so that $\alpha_{\nu_{k,1}} > 0$ and $\alpha_{\nu_{k,2}} < 0$. Then, there exist a radius $\varepsilon > 0$, some constants $C, c > 0$ and some holomorphic functions $\kappa_{\nu_{k,1}}, \kappa_{\nu_{k,2}}, f_{\nu_{k,1}}, f_{\nu_{k,2}} : B_\varepsilon(\underline{z}_k) \to \mathbb{C}$ such that for all $z \in B_\varepsilon(\underline{z}_k)$, $\kappa_{\nu_{k,1}}(z)$ and $\kappa_{\nu_{k,2}}(z)$ are simple eigenvalues of $\mathbb{M}(z)$ with $\kappa_{\nu_{k,1}}(\underline{z}_k) = \underline{\kappa}_{\nu_{k,1}}$ and $\kappa_{\nu_{k,2}}(\underline{z}_k) = \underline{\kappa}_{\nu_{k,1}}$, for all $j \in \mathbb{Z}$, the function $z \in B_\varepsilon(\underline{z}_k) \cap \mathcal{O} \mapsto G_j(z)$ can be holomorphically extended on $B_\varepsilon(\underline{z}_k)$ and*

$$\forall z \in B_\varepsilon(\underline{z}_k), \forall j \geq 1, \quad |G_j(z) - f_{\nu_{k,1}}(z)\kappa_{\nu_{k,1}}(z)^j| \leq C\exp(-cj). \tag{20}$$
$$\forall z \in B_\varepsilon(\underline{z}_k), \forall j \leq 0, \quad |G_j(z) - f_{\nu_{k,2}}(z)\kappa_{\nu_{k,2}}(z)^j| \leq C\exp(-c|j|). \tag{21}$$

Furthermore, knowing that $\underline{z}_k = \underline{z}_{\nu_{k,1}} = \underline{z}_{\nu_{k,2}}$, we have that

$$f_{\nu_{k,1}}(\underline{z}_k) = \frac{1}{|\alpha_{\nu_{k,1}}|\underline{z}_k}, \quad f_{\nu_{k,2}}(\underline{z}_k) = \frac{1}{|\alpha_{\nu_{k,2}}|\underline{z}_k}. \tag{22}$$

Lemmas 5 and 6 are similar to [3, Lemmas 3 and 4] but instead of proving sharp bounds on the spatial Green's function, we express its precise behavior near the points $\underline{z}_k$. This is the crucial point that will allow us to not just bound the elements $\mathscr{G}_j^n$ but find their asymptotic behavior and prove a bound for the remainder.

Proof of Lemma 5 Our proof will be mostly similar to the proof of [3, Lemmas 3, 4]. First, we observe that case II would be dealt similarly as case I and that case III is a mixture of both cases I and II. Therefore, we will only detail the proof in case I. We therefore consider $k \in \{1, \dots, K\}$ so that we are in case I. Lemma 3 implies that $\underline{\kappa}_k$ is a simple eigenvalue of $\mathbb{M}(\underline{z}_k)$. Thus, we can find a holomorphic function κ_k defined on a neighborhood $B_\varepsilon(\underline{z}_k)$ of $\underline{z}_k$ such that for all $z \in B_\varepsilon(\underline{z}_k)$, $\kappa_k(z)$ is a simple eigenvalue of $\mathbb{M}(z)$ and $\kappa_k(\underline{z}_k) = \underline{\kappa}_k$. We also know that for all $z \in B_\varepsilon(\underline{z}_k)$, the vector

$$R_k(z) := \begin{pmatrix} \kappa_k(z)^{p+r-1} \\ \vdots \\ \kappa_k(z) \\ 1 \end{pmatrix}$$

is an eigenvector of $\mathbb{M}(z)$ associated with $\kappa_k(z)$. Because of Lemma 3, even if we have to take a smaller radius ε, we can assume that for all $z \in B_\varepsilon(\underline{z}_k)$, $\mathbb{M}(z)$ has $\kappa_k(z)$ as a simple eigenvalue, $r-1$ eigenvalues different from $\kappa_k(z)$ in $\mathbb{D}$ and p eigenvalues different from $\kappa_k(z)$ in $\mathcal{U}$. We define $E^s(z)$ (resp. $E^u(z)$) the strictly stable (resp. strictly unstable) subspace of $\mathbb{M}(z)$ which corresponds to the subspace spanned by the generalized eigenvectors of $\mathbb{M}(z)$ associated to eigenvalues different from $\kappa_k(z)$ in $\mathbb{D}$ (resp. $\mathcal{U}$). We therefore know that $E^s(z)$ (resp. $E^u(z)$) has dimension $r-1$ (resp. p) thanks to Lemma 3 and we have the decomposition

$$\mathbb{C}^{p+r} = E^s(z) \oplus E^u(z) \oplus \text{ Span } R_k(z).$$

The associated projectors are denoted $\pi^s(z)$, $\pi^u(z)$ and $\pi^k(z)$. Those linear maps commute with $\mathbb{M}(z)$ and depend holomorphically on $z \in B_\varepsilon(\underline{z}_k)$.

For all $z \in B_\varepsilon(\underline{z}_k) \cap \mathcal{O}$ and $j \in \mathbb{Z}$, $G_j(z)$ and $W_j(z)$ are well defined. Also, by Lemma 3, we have that $|\kappa_k(z)| < 1$ for all $z \in B_\varepsilon(\underline{z}_k) \cap \mathcal{O}$. By reasoning in the same manner as in the proof of [3, Lemma 3], we have for all $z \in B_\varepsilon(\underline{z}_k) \cap \mathcal{O}$ and $j \in \mathbb{Z}$

$$\pi^u(z)W_j(z) = \frac{\mathbb{1}_{j\in]-\infty,0]}}{\mathbb{A}_p(z)}\mathbb{M}(z)^{j-1}\pi^u(z)e, \tag{23}$$

$$\pi^s(z)W_j(z) = -\frac{\mathbb{1}_{j\in[1,+\infty[}}{\mathbb{A}_p(z)}\mathbb{M}(z)^{j-1}\pi^s(z)e, \tag{24}$$

$$\pi^k(z)W_j(z) = -\frac{\mathbb{1}_{j\in[1,+\infty[}}{\mathbb{A}_p(z)}\mathbb{M}(z)^{j-1}\pi^k(z)e = -\frac{\mathbb{1}_{j\in[1,+\infty[}}{\mathbb{A}_p(z)}\kappa_k(z)^{j-1}\pi^k(z)e. \tag{25}$$

We observe that the right hand side in the equations (23), (24) and (25) can be holomorphically extended on $B_\varepsilon(\underline{z}_k)$. Therefore, we can extend holomorphically the applications which associates z to $\pi^s(z)W_j(z)$, $\pi^u(z)W_j(z)$ and $\pi^k(z)W_j(z)$ on the whole open ball $B_\varepsilon(\underline{z}_k)$ and this allows us to extend $W_j(z)$ on $B_\varepsilon(\underline{z}_k)$. Since $G_j(z)$ is a coordinate of the vector $W_j(z)$, the holomorphic extension property is proved.

By reasoning in the same manner as in the proof of the inequality [3, (23)], we prove that there exist two constants $C, c > 0$ such that

$$\forall z \in B_\varepsilon(\underline{z}_k), \forall j \in \mathbb{Z}, \quad \|\pi^s(z)W_j(z) + \pi^u(z)W_j(z)\| \leq C\exp(-c|j|).$$

This implies that

$$\forall z \in B_\varepsilon(\underline{z}_k), \forall j \in \mathbb{Z}, \quad \|W_j(z) - \pi^k(z)W_j(z)\| \leq C\exp(-c|j|).$$

This is were our proof will now differ from the proof of [3, Lemmas 3, 4]. In [3], the authors find bounds on $\pi^k(z)W_j(z)$ to be able to find estimates on $G_j(z)$. In our case, we have a stronger hypothesis (Hypothesis 1) that allows us to have a much simpler expression (25) of $\pi^k(z)W_j(z)$ and this will enable us to find the precise behavior of $G_j(z)$.

For $j \leq 0$, we observe that $\pi^k(z)W_j(z) = 0$ and that $G_j(z)$ is a component of $W_j(z)$. We therefore get the inequality (16).

We now consider the case $j \geq 1$. We have that $G_j(z) = (W_j(z))_p$ for all $z \in B_\varepsilon(\underline{z}_k)$ where $(X)_p$ refers to the p-th coordinate of a vector $X \in \mathbb{C}^{p+r}$. Then,

$$\forall z \in B_\varepsilon(\underline{z}_k), \quad |G_j(z) - (\pi^k(z)W_j(z))_p| \leq C\exp(-c|j|).$$

We then define the holomorphic function

$$\begin{array}{rccc} f_k: & B_\varepsilon(\underline{z}_k) & \to & \mathbb{C} \\ & z & \mapsto & -\frac{1}{\mathbb{A}_p(z)\kappa_k(z)}(\pi^k(z)e)_p \end{array}.$$

By observing that $(\pi^k(z)W_j(z))_p = f_k(z)\kappa_k(z)^j$, we get the inequality (15) and it now remains to obtain the expression (19). We have that

$$f_k(\underline{z}_k) = -\frac{1}{\mathbb{A}_p(\underline{z}_k)\underline{\kappa}_k}(\pi^k(\underline{z}_k)e)_p, \quad e = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

We first need to determine the spectral projector $\pi^k(\underline{z}_k)$. We recall that $\underline{\kappa}_k \in \mathbb{S}^1$ is a simple eigenvalue of $\mathbb{M}(\underline{z}_k)$ and the vector

$$\underline{R}_k = \begin{pmatrix} \underline{\kappa}_k^{p+r-1} \\ \vdots \\ \underline{\kappa}_k \\ 1 \end{pmatrix} \in \mathbb{C}^{p+r}$$

is an eigenvector of $\mathbb{M}(\underline{z}_k)$ associated with $\underline{\kappa}_k$. We also know that there exists a unique eigenvector $\underline{L}_k = (l_j)_{j\in\{1,\dots,p+r\}} \in \mathbb{C}^{p+r}$ of $\mathbb{M}(\underline{z}_k)^T$ associated with the eigenvalue $\underline{\kappa}_k$ such that

$$\underline{L}_k \cdot \underline{R}_k = 1$$

where the symmetric bilinear form $\cdot$ on $\mathbb{C}^{p+r}$ is defined as

$$\forall X, Y \in \mathbb{C}^{p+r}, \quad X \cdot Y := \sum_{l=1}^{p+r} X_i Y_i.$$

Then, we have that

$$\forall Y \in \mathbb{C}^{p+r}, \quad \pi^k(\underline{z}_k)Y = (\underline{L}_k \cdot Y)\underline{R}_k.$$

Thus, applying it for $Y = e$ implies that

$$f_k(\underline{z}_k) = -\frac{l_1 \underline{\kappa}_k^{r-1}}{\mathbb{A}_p(\underline{z}_k)}.$$

We thus need to find the value of the coefficient l_1. Since $\underline{L}_k$ is an eigenvalue of $\mathbb{M}(\underline{z}_k)^T$ for the eigenvalue $\underline{\kappa}_k$, we get

$$\forall j \in \{1, \dots, p+r\}, \quad l_j = \left(\sum_{l=-r}^{p-j} \frac{\mathbb{A}_l(\underline{z}_k)}{\underline{\kappa}_k^{p-j+1-l}} \right) \frac{l_1}{\mathbb{A}_p(\underline{z}_k)}.$$

We now have an expression of each l_j depending on l_1. To determine the value of l_1, we have to use the normalization choice that we have made between $\underline{L}_k$ and $\underline{R}_k$. We have

$$1 = \underline{L}_k \cdot \underline{R}_k = \sum_{j=1}^{p+r} \underline{\kappa}_k^{p+r-j} l_j = \left(\sum_{j=1}^{p+r} \sum_{l=-r}^{p-j} \mathbb{A}_l(\underline{z}_k) \underline{\kappa}_k^{l+r-1} \right) \frac{l_1}{\mathbb{A}_p(\underline{z}_k)}.$$

By the expression of $\mathbb{A}_l(\underline{z}_k)$, this implies that

$$\begin{aligned} 1 = \left(\sum_{l=-r}^{p} (p-l)\mathbb{A}_l(\underline{z}_k)\underline{\kappa}_k^{l+r-1} \right) \frac{l_1}{\mathbb{A}_p(\underline{z}_k)} &= \left(p\underline{\kappa}_k^{r-1}\underline{z}_k - \sum_{l=-r}^{p} (p-l) a_l \underline{\kappa}_k^{l+r-1} \right) \frac{l_1}{\mathbb{A}_p(\underline{z}_k)} \\ &= \left(p\underline{\kappa}_k^{r-1}(\underline{z}_k - F(\underline{\kappa}_k)) + \underline{\kappa}_k^r F'(\underline{\kappa}_k) \right) \frac{l_1}{\mathbb{A}_p(\underline{z}_k)} \\ &= \underline{\kappa}_k^r F'(\underline{\kappa}_k) \frac{l_1}{\mathbb{A}_p(\underline{z}_k)}. \end{aligned}$$

Using the relation (5), we finally get the equation (19). □

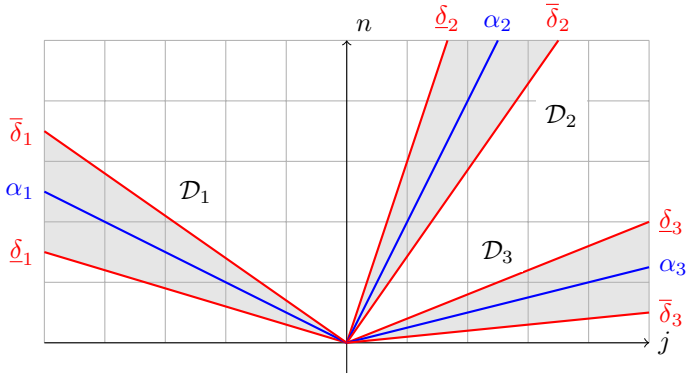

Figure 4: An illustration of the sectors $\mathcal{D}_k$. Here, we have $\alpha_1 = -2$, $\alpha_2 = 0.5$ and $\alpha_3 = 4$. The rays labeled α_k (resp. $\underline{\delta}_k$, $\overline{\delta}_k$) correspond to the ray $j = n\alpha_k$ (resp. $j = n\underline{\delta}_k$, $j = n\overline{\delta}_k$). We observe that, because $\underline{\delta}_k$, α_k and $\overline{\delta}_k$ have the same sign, j and α_k have the same sign for $(n, j) \in \mathcal{D}_k$. Also, the sectors $\mathcal{D}_k$ do not intersect each other.

3 Temporal Green's function

We are now ready to start proving Theorem 1. In a first step, we will prove the results of the Theorem far from the axes $j = n\alpha_k$. In this regime, the proof will not rely on the previous section. Then, we will see the link between the spatial Green's function $G(z)$ and the temporal Green's function (here the elements $\mathscr{G}_j^n$) using functional calculus (see [2, Chapter VII]) and use the estimates we proved in the Section 2 to determine the estimates on $\mathscr{G}_j^n$ near the axes $j = n\alpha_k$.

Before we start, we are going to make two hypotheses to simplify the proof. The first one is that $-1 \notin \{\underline{z}_1, \dots, \underline{z}_K\}$. This hypothesis is actually not restrictive. If it were not verified, we would just have to multiply the sequence a by some well chosen element of $\mathbb{S}^1$ to find a new sequence b that will verify this hypothesis and prove the theorem for this new sequence. The theorem for our previous sequence a would directly follow.

The second hypothesis we make is that all α_k are distinct from one another. This hypothesis has a real impact on the proof, symplifying greatly some parts of the calculations. We will come back at the end (Section 3.5) on the case where the elements α_k are not distinct and explain what elements of the proof would change.

3.1 Estimates far from the axes $j = n\alpha_k$

As we said in the previous section, we suppose that all α_k are distinct from one another. Without loss of generality, we suppose that

$$\alpha_1 < \ldots < \alpha_k < \ldots < \alpha_K.$$

For all $k \in \{1, \dots, K\}$, we define two elements $\underline{\delta}_k, \overline{\delta}_k \in \mathbb{R}^*$ such that $\underline{\delta}_k, \overline{\delta}_k$ and α_k have the same sign and

$$\underline{\delta}_1 < \alpha_1 < \overline{\delta}_1 < \ldots < \underline{\delta}_k < \alpha_k < \overline{\delta}_k < \ldots < \underline{\delta}_K < \alpha_K < \overline{\delta}_K.$$

We are now defining for every $k \in \{1, \dots, K\}$ the sectors

$$\mathcal{D}_k := \left\{(n, j) \in \mathbb{N}^* \times \mathbb{Z}, \quad n\underline{\delta}_k \leq j \leq n\overline{\delta}_k\right\}.$$

We observe that the sectors $\mathcal{D}_k$ do not intersect each other. We also introduce

$$\mathcal{D} := \bigcup_{k=1}^{K} \mathcal{D}_k.$$

We represent the sectors $\mathcal{D}_k$ on the Figure 4. In this section, we are going to prove the following two lemmas, which gives estimates on the elements $\mathscr{G}_j^n$ and $\mathscr{H}_{k,j}^n$ outside the sectors $\mathcal{D}_k$.

Lemma 7. *There exist two constants $C, c > 0$ such that*

$$\forall (n, j) \in (\mathbb{N}^* \times \mathbb{Z}) \backslash \mathcal{D}, \quad \left|\mathscr{G}_j^n\right| \leq C \sum_{k=1}^{K} \frac{1}{n^{\frac{1}{\mu_k}}} \exp\left(-c\left(\frac{|j - n\alpha_k|}{n^{\frac{1}{2\mu_k}}}\right)^{\frac{2\mu_k}{2\mu_k - 1}}\right).$$

Lemma 8. *For all $k \in \{1, \dots, K\}$, there exist two constants $C, c > 0$ such that*

$$\forall (n,j) \in (\mathbb{N}^* \times \mathbb{Z}) \backslash \mathcal{D}_k, \quad \left| \mathscr{H}^n_{k,j} \right| \leq C \frac{1}{n^{\frac{1}{\mu_k}}} \exp\left(-c \left(\frac{|j - n\alpha_k|}{n^{\frac{1}{2\mu_k}}} \right)^{\frac{2\mu_k}{2\mu_k - 1}} \right).$$

Both lemmas are proved in the same way. We will detail the proof of the first lemma and give some indications for the second afterwards.

Proof of Lemma 7 The result [3, Theorem 1] gives us the existence of two constants $C, c > 0$ such that

$$\forall n \in \mathbb{N}^*, \forall j \in \mathbb{Z}, \quad |\mathscr{G}^n_j| \leq C \sum_{k=1}^{K} \frac{1}{n^{\frac{1}{2\mu_k}}} \exp\left(-c \left(\frac{|j - n\alpha_k|}{n^{\frac{1}{2\mu_k}}} \right)^{\frac{2\mu_k}{2\mu_k - 1}} \right).$$

For a sufficiently small $\tilde{c} > 0$, we have that

$$\forall k \in \{1, \dots, K\}, \forall (n,j) \in (\mathbb{N}^* \times \mathbb{Z}) \backslash \mathcal{D}, \quad \frac{c}{2} \left(\frac{|j - n\alpha_k|}{n^{\frac{1}{2\mu_k}}} \right)^{\frac{2\mu_k}{2\mu_k - 1}} \geq \tilde{c} n.$$

Also, we can find a constant $\tilde{C} > 0$ such that

$$\forall k \in \{1, \dots, K\}, \forall n \in \mathbb{N}^*, \quad \exp(-\tilde{c} n) \leq \frac{\tilde{C}}{n^{\frac{1}{2\mu_k}}}.$$

Therefore, we get the expected result. □

We could even have proved for those j and n a stronger estimate of the form

$$\left| \mathscr{G}^n_j \right| \leq C e^{-c(n + |j|)}$$

but the limiting estimate will occur in $\mathcal{D}$ so we can avoid being sharp here.

For the proof of Lemma 8, we use the first inequality of Lemma 2 (estimate on $H^{\beta}_{2\mu}$), which implies the existence of two constants $C, c > 0$ such that

$$\forall k \in \{1, \dots, K\}, \forall n \in \mathbb{N}^*, \forall j \in \mathbb{Z}, \quad |\mathscr{H}^n_{k,j}| \leq C \frac{1}{n^{\frac{1}{2\mu_k}}} \exp\left(-c \left(\frac{|j - n\alpha_k|}{n^{\frac{1}{2\mu_k}}} \right)^{\frac{2\mu_k}{2\mu_k - 1}} \right).$$

The rest of the proof is close to what we did for Lemma 7. The two Lemmas 7 and 8 imply that we have two constants $C, c > 0$ such that

$$\forall (n,j) \in (\mathbb{N}^* \times \mathbb{Z}) \backslash \mathcal{D}, \quad \left| \mathscr{G}^n_j - \sum_{k=1}^{K} \mathscr{H}^n_{k,j} \right| \leq C \sum_{k=1}^{K} \frac{1}{n^{\frac{1}{\mu_k}}} \exp\left(-c \left(\frac{|j - n\alpha_k|}{n^{\frac{1}{2\mu_k}}} \right)^{\frac{2\mu_k}{2\mu_k - 1}} \right) \tag{26}$$

and, since the sectors $\mathcal{D}_k$ do not intersect each other, there also exists two constants $C, c > 0$ such that

$$\forall k_0 \in \{1, \dots, K\}, \forall (n,j) \in \mathcal{D}_{k_0}, \quad \left| \mathscr{G}^n_j - \sum_{k=1}^{K} \mathscr{H}^n_{k,j} \right| \leq |\mathscr{G}^n_j - \mathscr{H}^n_{k_0,j}| + C \sum_{\substack{k=1 \\ k \neq k_0}}^{K} \frac{1}{n^{\frac{1}{\mu_k}}} \exp\left(-c \left(\frac{|j - n\alpha_k|}{n^{\frac{1}{2\mu_k}}} \right)^{\frac{2\mu_k}{2\mu_k - 1}} \right). \tag{27}$$

3.2 Link between the spatial and temporal Green's functions

Our goal will now be to prove the following proposition.

Proposition 1. *For any $k \in \{1, \dots, K\}$, there exist two constants $C, c > 0$ such that*

$$\forall (n,j) \in \mathcal{D}_k, \quad \left| \mathscr{G}^n_j - \mathscr{H}^n_{k,j} \right| \leq \frac{C}{n^{\frac{1}{\mu_k}}} \exp\left(-c \left(\frac{|j - n\alpha_k|}{n^{\frac{1}{2\mu_k}}} \right)^{\frac{2\mu_k}{2\mu_k - 1}} \right).$$

Because of the inequalities (26) and (27), if we prove Proposition 1, we will have proved Theorem 1. The estimates on the spatial Green's function that we introduced in the previous Section 2 will become one of the main elements of the proof of Proposition 1. We will prove this proposition in Section 3.4. But first, we will need to study the elements $\mathscr{G}_j^n$ more closely.

To prove the Proposition 1, the first step will be to express the elements $\mathscr{G}_j^n$ via the spatial Green's function $G_j(z)$ using functional calculus. The equation (13) implies by using the inverse Laplace transform that if we define a path which surrounds $\sigma(\mathscr{L}_a) = F(\mathbb{S}^1)$, like for example $\widetilde{\Gamma}_\rho = \exp(\rho)\mathbb{S}^1$ for $0 < \rho \leq \pi$, then

$$\forall n \in \mathbb{N}^*, \forall j \in \mathbb{Z}, \quad \mathscr{G}_j^n = \frac{1}{2i\pi} \int_{\widetilde{\Gamma}_\rho} z^n G_j(z) dz.$$

We fix this choice of path for now but we are going to modify it in what follows. The idea will be to deform the path on which we integrate so that we can best use the estimates on $G_j(z)$ proved in Section 2. We start with a change of variable $z = \exp(\tau)$ in the previous equality. Therefore, if we define $\Gamma_\rho := \{\rho + il, l \in [-\pi, \pi]\}$ and $\mathbf{G}_j(\tau) = e^\tau G_j(e^\tau)$, then

$$\forall n \in \mathbb{N}^*, \forall j \in \mathbb{Z}, \quad \mathscr{G}_j^n = \frac{1}{2i\pi} \int_{\Gamma_\rho} e^{n\tau} \mathbf{G}_j(\tau) d\tau. \tag{28}$$

We will therefore need a lemma that allows us to get from estimates on $G_j(z)$ to estimates on $\mathbf{G}_j(\tau)$. First, recalling that $\underline{z}_k \neq -1$, we define for all $k \in \{1, \dots, K\}$ the unique element $\underline{\tau}_k := i\theta_k$ of $i]-\pi, \pi[$ such that

$$\underline{z}_k = \exp(\underline{\tau}_k) = \exp(i\theta_k).$$

We also introduce for all $k \in \{1, \dots, K\}$ the unique $\tilde{\theta}_k \in]-\pi, \pi]$ such that

$$\underline{\kappa}_k = e^{i\tilde{\theta}_k}.$$

We now introduce a lemma to pass from estimates on $G_j(z)$ to estimates on $\mathbf{G}_j(\tau)$.

Lemma 9. *There exists a radius $\varepsilon_\star > 0$ and for all $k \in \{1, \dots, K\}$ two holomorphic functions $\varpi_k : B_{\varepsilon_\star}(\underline{\tau}_k) \to \mathbb{C}$ and $g_k : B_{\varepsilon_\star}(\underline{\tau}_k) \to \mathbb{C}$ such that for all $\varepsilon \in]0, \varepsilon_\star[$, there exist a width $\eta_\varepsilon > 0$ and two constants $C, c > 0$ such that if we define*

$$U_\varepsilon := \{\tau \in \mathbb{C}, \Re(\tau) \in]-\eta_\varepsilon, \pi], \Im(\tau) \in [-\pi, \pi]\} \quad \text{and} \quad \Omega_\varepsilon := U_\varepsilon \setminus \bigcup_{k=1}^K B_\varepsilon(\underline{\tau}_k),$$

then for all $j \in \mathbb{Z}$, the application $\tau \mapsto \boldsymbol{G}_j(\tau)$ can be holomorphically extended on $U_\varepsilon \cup \bigcup_{k=1}^K B_\varepsilon(\underline{\tau}_k)$ and we have that

$$\forall \tau \in \Omega_\varepsilon, \forall j \in \mathbb{Z}, \quad |\boldsymbol{G}_j(\tau)| \leq Ce^{-c|j|}. \tag{29}$$

Also, for all $k \in \{1, \dots, K\}$, depending on the case, we have that

Case I:

$$\forall \tau \in B_\varepsilon(\underline{\tau}_k), \forall j \geq 1, \quad |\boldsymbol{G}_j(\tau) - e^\tau g_k(\tau) e^{j\varpi_k(\tau)}| \leq Ce^{-c|j|}, \tag{30}$$

$$\forall \tau \in B_\varepsilon(\underline{\tau}_k), \forall j \leq 0, \quad |\boldsymbol{G}_j(\tau)| \leq Ce^{-c|j|}, \tag{31}$$

Case II:

$$\forall \tau \in B_\varepsilon(\underline{\tau}_k), \forall j \geq 1, \quad |\boldsymbol{G}_j(\tau)| \leq Ce^{-c|j|}, \tag{32}$$

$$\forall \tau \in B_\varepsilon(\underline{\tau}_k), \forall j \leq 0, \quad |\boldsymbol{G}_j(\tau) - e^\tau g_k(\tau) e^{j\varpi_k(\tau)}| \leq Ce^{-c|j|}, \tag{33}$$

Case III:

$$\forall \tau \in B_\varepsilon(\underline{\tau}_k), \forall j \geq 1, \quad |\boldsymbol{G}_j(\tau) - e^\tau g_{\nu_{k,1}}(\tau) e^{j\varpi_{\nu_{k,1}}(\tau)}| \leq Ce^{-c|j|}, \tag{34}$$

$$\forall \tau \in B_\varepsilon(\underline{\tau}_k), \forall j \leq 0, \quad |\boldsymbol{G}_j(\tau) - e^\tau g_{\nu_{k,2}}(\tau) e^{j\varpi_{\nu_{k,2}}(\tau)}| \leq Ce^{-c|j|}, \tag{35}$$

where we have $\mathcal{I}_k = \{\nu_{k,1}, \nu_{k,2}\}$, $\alpha_{\nu_{k,1}} > 0$ and $\alpha_{\nu_{k,2}} < 0$.

For all $k \in \{1, \dots, K\}$, we define $\varphi_k(\tau) := -\frac{(\tau - \underline{\tau}_k)}{\alpha_k} + (-1)^{\mu_k+1} \frac{\beta_k}{{\alpha_k}^{2\mu_k+1}} (\tau - \underline{\tau}_k)^{2\mu_k}$ for $\tau \in \mathbb{C}$. Then, there exists a holomorphic function $\xi_k : B_{\varepsilon_\star}(\underline{\tau}_k) \to \mathbb{C}$ such that

$$\forall \tau \in B_{\varepsilon_\star}(\underline{\tau}_k), \quad \varpi_k(\tau) = i\tilde{\theta}_k + \varphi_k(\tau) + (\tau - \underline{\tau}_k)^{2\mu_k+1} \xi_k(\tau). \tag{36}$$

Finally, there exist two constants $A_R, A_I > 0$ such that for all $k \in \{1, \dots, K\}$,

$$\forall \tau \in B_{\varepsilon_\star}(\underline{\tau}_k), \quad \alpha_k \left(\Re(\varpi_k(\tau)) + |\Re(\xi_k(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+1})| \right) \\ \leq -\Re(\tau - \underline{\tau}_k) + A_R \Re(\tau - \underline{\tau}_k)^{2\mu_k} - A_I \Im(\tau - \underline{\tau}_k)^{2\mu_k}, \tag{37}$$

$$\forall \tau \in \mathbb{C}, \quad \alpha_k \Re(\varphi_k(\tau)) \leq -\Re(\tau - \underline{\tau}_k) + A_R \Re(\tau - \underline{\tau}_k)^{2\mu_k} - A_I \Im(\tau - \underline{\tau}_k)^{2\mu_k}. \tag{38}$$

Proof Using the Lemmas 5 and 6 and writing $\kappa_k(z) = \exp(\omega_k(z))$ for z near $\underline{z}_k$ with $\omega_k(\underline{z}_k) = i\tilde{\theta}_k$, we can define for a choice of $\varepsilon_\star$ small enough two holomorphic functions ϖ_k and g_k such that

$$\forall \tau \in B_{\varepsilon_\star}(\underline{\tau}_k), \quad \varpi_k(\tau) = \omega_k(e^\tau), g_k(\tau) = f_k(e^\tau).$$

Lemmas 5 and 6 directly imply the inequalities (30), (31), (32), (33), (34) and (35) on the open balls $B_{\varepsilon_\star}(\underline{\tau}_k)$ and the fact that the functions $\tau \mapsto \mathbf{G}_j(\tau)$ are holomorphic on $B_{\varepsilon_\star}(\underline{\tau}_k)$. We now consider $\varepsilon \in]0, \varepsilon_\star[$. The inequalities we just proved remain true on $B_\varepsilon(\underline{\tau}_k)$. Using a compactness argument and Lemma 4, we also get the existence of η_ε and the inequality (29).

We observe that the asymptotic expansion (3) implies that

$$\tau - \underline{\tau}_k \underset{\tau \to \underline{\tau}_k}{=} -\alpha_k(\varpi_k(\tau) - i\tilde{\theta}_k) + (-1)^{\mu_k+1} \beta_k (\varpi_k(\tau) - i\tilde{\theta}_k)^{2\mu_k} + O\left(\left| \varpi_k(\tau) - i\tilde{\theta}_k \right|^{2\mu_k+1} \right).$$

We then deduce the existence of ξ_k and the equation (36).

There only remains to prove the existence of A_R and A_I to verify the equations (37) and (38).

We are going to prove (38) first. Because of Young's inequality, we have that for $l \in \{1, \dots, 2\mu_k - 1\}$, for all $\delta > 0$, there exists $C_\delta > 0$ such that for all $\tau \in \mathbb{C}$

$$|\Re(\tau)|^l |\Im(\tau)|^{2\mu_k - l} \leq \delta \Im(\tau)^{2\mu_k} + C_\delta \Re(\tau)^{2\mu_k}.$$

Furthermore, we have that

$$\alpha_k \Re(\varphi_k(\tau)) = -\Re(\tau - \underline{\tau}_k) + (-1)^{\mu_k+1} \left(\frac{\Re(\beta_k)}{\alpha_k^{2\mu_k}} \Re((\tau - \underline{\tau}_k)^{2\mu_k}) - \frac{\Im(\beta_k)}{\alpha_k^{2\mu_k}} \Im((\tau - \underline{\tau}_k)^{2\mu_k}) \right).$$

Then, for $\delta > 0$, there exists $C_\delta > 0$ such that

$$\alpha_k \Re(\varphi_k(\tau)) \leq -\Re(\tau - \underline{\tau}_k) + \Re(\tau - \underline{\tau}_k)^{2\mu_k} \left(\frac{\Re(\beta_k)}{\alpha_k^{2\mu_k}} + C_\delta \right) + \Im(\tau - \underline{\tau}_k)^{2\mu_k} \left(-\frac{\Re(\beta_k)}{\alpha_k^{2\mu_k}} + \delta \right).$$

Therefore, by taking δ small enough, we can end the proof of inequality (38). The proof of inequality (37) is similar. We have for $\tau \in B_{\varepsilon_\star}(\underline{\tau}_k)$

$$\alpha_k \left(\Re(\varpi_k(\tau)) + |\Re(\xi_k(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+1})| \right) \leq -\Re(\tau - \underline{\tau}_k) + 2|\alpha_k||\xi_k(\tau)||\tau - \underline{\tau}_k|^{2\mu_k+1} \\ + (-1)^{\mu_k+1} \left(\frac{\Re(\beta_k)}{\alpha_k^{2\mu_k}} \Re((\tau - \underline{\tau}_k)^{2\mu_k}) - \frac{\Im(\beta_k)}{\alpha_k^{2\mu_k}} \Im((\tau - \underline{\tau}_k)^{2\mu_k}) \right).$$

We know there exists $c_1, c_2 > 0$ such that

$$\forall k \in \{1, \dots, K\}, \forall \tau \in \mathbb{C}, \quad |\tau|^{2\mu_k} \leq c_1 \Re(\tau)^{2\mu_k} + c_2 \Im(\tau)^{2\mu_k}.$$

Since ξ_k can be bounded by some constant $\tilde{C} > 0$ on $B_{\varepsilon_\star}(\underline{\tau}_k)$, using the same reasoning as previously gives us

$$\alpha_k \left(\Re(\varpi_k(\tau)) + |\Re(\xi_k(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+1})| \right) \leq -\Re(\tau - \underline{\tau}_k) + 2\tilde{C}|\alpha_k|\varepsilon_\star(c_1 \Re(\tau - \underline{\tau}_k)^{2\mu_k} + c_2 \Im(\tau - \underline{\tau}_k)^{2\mu_k}) \\ + \Re(\tau - \underline{\tau}_k)^{2\mu_k} \left(\frac{\Re(\beta_k)}{\alpha_k^{2\mu_k}} + C_\delta \right) + \Im(\tau - \underline{\tau}_k)^{2\mu_k} \left(-\frac{\Re(\beta_k)}{\alpha_k^{2\mu_k}} + \delta \right).$$

Taking δ and $\varepsilon_\star$ small enough allows us to prove (37).

□

Remark 1. We observe that the constants in the inequalities (30), (31), (32), (33), (34) and (35) could be chosen uniformly for all $\varepsilon \in]0, \varepsilon_\star[$. However, it is not the case for the constants in inequality (29).

Lemma 9 leads us to introduce the elements

$$\forall k \in \{1, \dots, K\}, \forall n \in \mathbb{N}^*, \forall j \in \mathbb{Z}, \quad \mathscr{F}_{k,j}^n := \frac{\underline{z}_k g_k(\underline{\tau}_k) {\underline{\kappa}_k}^j}{2i\pi} \int_{\Gamma_0} \exp(n\tau + j\varphi_k(\tau)) d\tau$$

where $\Gamma_0 := \{it, t \in [-\pi, \pi]\}$. Because of the equalities (19) and (22) on $f_k(\underline{z}_k) = g_k(\underline{\tau}_k)$, we get with an affine change of variables that

$$\mathscr{F}_{k,j}^n = \frac{{\underline{z}_k}^n {\underline{\kappa}_k}^j}{2\pi} sgn(\alpha_k) \int_{\frac{-\pi-\theta_k}{\alpha_k}}^{\frac{\pi-\theta_k}{\alpha_k}} \exp\left(is(n\alpha_k - j) - \frac{j}{\alpha_k} \beta_k s^{2\mu_k} \right) ds.$$

We aim at proving the following proposition.

Proposition 2. *For any $k \in \{1, \dots, K\}$, there exist two constants $C, c > 0$ such that*

$$\forall (n, j) \in \mathcal{D}_k, \quad \left| \mathscr{G}_j^n - \mathscr{F}_{k,j}^n \right| \leq \frac{C}{n^{\frac{1}{\mu_k}}} \exp\left(-c \left(\frac{|j - n\alpha_k|}{n^{\frac{1}{2\mu_k}}} \right)^{\frac{2\mu_k}{2\mu_k - 1}} \right).$$

Proving this proposition is an important step in proving Theorem 1. In Section 3.4, we will just have to estimate the elements $\mathscr{F}_{k,j}^n - \mathscr{H}_{k,j}^n$ to prove Proposition 1 and thus Theorem 1.

3.3 Proof of Proposition 2

From now on, we fix a $k \in \{1, \dots, K\}$ and our goal is to prove the claim of Proposition 2 for this k. We will suppose that $\alpha_k > 0$. The major consequence is that for $(n, j) \in \mathcal{D}_k$, we have $j \geq 1$. This implies that we will use the inequalities (30), (32) and (34). The case where $\alpha_k < 0$ would need some little modifications, in particular we will have that $j \leq 0$ for $(n, j) \in \mathcal{D}_k$ and we would use the inequalities (31), (33) and (35).

Before we begin with the proof, we will need to introduce some lemmas and define some elements. First, we can easily prove the following lemma which allows us to pass from bounds that are exponentially decaying in n to the generalized gaussian bounds expected in Proposition 2.

Lemma 10. *We consider $C, c > 0$. Then, there exist $\tilde{C}, \tilde{c} > 0$ such that*

$$\forall (n, j) \in \mathcal{D}_k, \quad C \exp(-cn) \leq \frac{\tilde{C}}{n^{\frac{1}{\mu_k}}} \exp\left(-\tilde{c} \left(\frac{|j - \alpha_k n|}{n^{\frac{1}{2\mu_k}}} \right)^{\frac{2\mu_k}{2\mu_k - 1}} \right).$$

We now apply Lemma 9 and consider $\varepsilon \in]0, \varepsilon_\star[$ small enough so that

$$\forall i, j \in \{1, \dots, K\}, \quad \underline{z}_i \neq \underline{z}_j \Rightarrow B_\varepsilon(\underline{\tau}_i) \cap B_\varepsilon(\underline{\tau}_j) = \emptyset$$

and

$$\forall l \in \{1, \dots, K\}, \quad B_\varepsilon(\underline{\tau}_l) \subset \{\tau \in \mathbb{C}, \quad \Im(\tau) \in [-\pi, \pi]\}.$$

This can be done because we supposed that $\underline{\tau}_l \notin \{-i\pi, i\pi\}$ for all l. We also define a constant $\eta \in]0, \eta_\varepsilon[$.

We will now follow a strategy developed in [19], which has also been used in [3] and [4], and introduce a family of parameterized curves. For $\tau_p \in \mathbb{R}$, we introduce

$$\Psi_k(\tau_p) = \tau_p - A_R {\tau_p}^{2\mu_k}.$$

The function Ψ_k is continuous and strictly increasing on $\left] -\infty, \left(\frac{1}{2\mu_k A_R} \right)^{\frac{1}{2\mu_k - 1}} \right[$. We choose ε small enough so that it is strictly increasing on $]-\infty, \varepsilon]$. We can therefore introduce for $\tau_p \in [-\eta, \varepsilon]$

$$\Gamma_{k,p} = \left\{ \tau \in \mathbb{C}, -\eta \leq \Re(\tau) \leq \tau_p, \quad \Re(\tau - \underline{\tau}_k) - A_R \Re(\tau - \underline{\tau}_k)^{2\mu_k} + A_I \Im(\tau - \underline{\tau}_k)^{2\mu_k} = \Psi_k(\tau_p) \right\}.$$

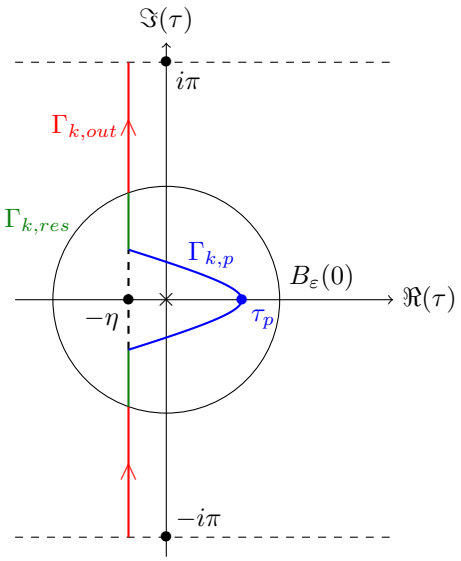

Figure 5: A representation of the path Γ_k for $\underline{\tau}_k = 0$. It is composed of $\Gamma_{k,out}$ (in red), $\Gamma_{k,res}$ (in green) and $\Gamma_{k,p}$ (in blue). The section of Γ_k which lies inside the ball $B_\varepsilon(\underline{\tau}_k)$ (i.e. the reunion of $\Gamma_{k,res}$ and $\Gamma_{k,p}$) is notated $\Gamma_{k,in}$.

It is a symmetric curve with respect to the axis $\mathbb{R} + \underline{\tau}_k$ which intersects this axis on the point $\tau_p + \underline{\tau}_k$. If we introduce $\ell_{k,p} = \left(\frac{\Psi_k(\tau_p) - \Psi_k(-\eta)}{A_I}\right)^{\frac{1}{2\mu_k}}$, then $-\eta + i(\theta_k + \ell_{k,p})$ and $-\eta + i(\theta_k - \ell_{k,p})$ are the end points of $\Gamma_{k,p}$. We can also introduce a parametrization of this curve by defining $\gamma_{k,p} : [-\ell_{k,p}, \ell_{k,p}] \to \mathbb{C}$ such that

$$\forall \tau_p \in [-\eta, \varepsilon], \forall t \in [-\ell_{k,p}, \ell_{k,p}], \quad \Im(\gamma_{k,p}(t)) = t + \theta_k, \quad \Re(\gamma_{k,p}(t)) = h_{k,p}(t) := \Psi_k^{-1}\left(\Psi_k(\tau_p) - A_I t^{2\mu_k}\right). \tag{39}$$

The above parametrization immediately yields that there exists a constant $M > 0$ such that

$$\forall \tau_p \in [-\eta, \varepsilon], \forall t \in [-\ell_{k,p}, \ell_{k,p}], \quad |h'_{k,p}(t)| \leq M. \tag{40}$$

Also, there exists a constant $c_\star > 0$ such that

$$\forall \tau_p \in [-\eta, \varepsilon], \forall \tau \in \Gamma_{k,p}, \quad \Re(\tau - \underline{\tau}_k) - \tau_p \leq -c_\star \Im(\tau - \underline{\tau}_k)^{2\mu_k}. \tag{41}$$

Even if we have to consider a smaller η, we can define a $0 < \varepsilon_{k,0} < \varepsilon$ such that the curve $\Gamma_{k,p}$ associated to $\tau_p = \varepsilon_{k,0}$ intersects the axis $-\eta + i\mathbb{R}$ within $B_\varepsilon(\underline{\tau}_k)$.

For $\tau_p \in \left[-\frac{\eta}{2}, \varepsilon_{k,0}\right]$, we want to define the path Γ_k defined on the Figure 5. As we can see, it follows the ray $-\eta + i[-\pi, \pi]$ and is deformed inside $B_\varepsilon(\underline{\tau}_k)$ into the path $\Gamma_{k,p}$. We define

$$\begin{aligned}
\Gamma_{k,res} &:= \{-\eta + it, t \in [-\pi, \pi] \backslash [\theta_k - \ell_{k,p}, \theta_k + \ell_{k,p}]\} \cap B_\varepsilon(\underline{\tau}_k), \\
\Gamma_{k,out} &:= \{-\eta + it, t \in [-\pi, \pi]\} \cap B_\varepsilon(\underline{\tau}_k)^c, \\
\Gamma_{k,in} &:= \Gamma_{k,p} \cup \Gamma_{k,res}, \\
\Gamma_k &:= \Gamma_{k,in} \cup \Gamma_{k,out}.
\end{aligned}$$

We still have to define the choice of τ_p depending on n, j and k. We let

$$\zeta_k = \frac{j - n\alpha_k}{2\mu_k n}, \quad \gamma_k = \frac{A_R j}{n}, \quad \rho_k\left(\frac{\zeta_k}{\gamma_k}\right) = sgn(\zeta_k)\left(\frac{|\zeta_k|}{\gamma_k}\right)^{\frac{1}{2\mu_k - 1}}.$$

Our limiting estimates will come from the case where ζ_k is close to 0. We observe that the condition $(n, j) \in \mathcal{D}_k$ implies

$$A_R \underline{\delta}_k \leq \gamma_k \leq A_R \overline{\delta}_k. \tag{42}$$

Moreover, we have that $\rho_k\left(\frac{\zeta_k}{\gamma_k}\right)$ is the unique real root of the polynomial

$$\gamma_k x^{2\mu_k-1} = \zeta_k.$$

Then, we take

$$\tau_p := \begin{cases} \varepsilon_{k,0}, & \text{if } \rho_k\left(\frac{\zeta_k}{\gamma_k}\right) > \varepsilon_{k,0}, \quad \text{(Case A)} \\ \rho_k\left(\frac{\zeta_k}{\gamma_k}\right), & \text{if } \rho_k\left(\frac{\zeta_k}{\gamma_k}\right) \in [-\frac{\eta}{2}, \varepsilon_{k,0}], \quad \text{(Case B)} \\ -\frac{\eta}{2}, & \text{if } \rho_k\left(\frac{\zeta_k}{\gamma_k}\right) < -\frac{\eta}{2}. \quad \text{(Case C)} \end{cases}$$

We define for $(n,j) \in \mathcal{D}_k$

$$\widetilde{\mathscr{F}}^n_{k,j} := \frac{\underline{z}_k g_k(\underline{\tau}_k) \underline{\kappa}_k{}^j}{2i\pi} \int_{\Gamma_k} \exp(n\tau + j\varphi_k(\tau))d\tau.$$

We have the following lemma.

Lemma 11. *There exist constants $C, c > 0$ such that*

$$\forall (n,j) \in \mathcal{D}_k, \quad \left|\mathscr{F}^n_{k,j} - \widetilde{\mathscr{F}}^n_{k,j}\right| \leq \frac{C}{n^{\frac{1}{\mu_k}}} \exp\left(-c\left(\frac{|j - n\alpha_k|}{n^{\frac{1}{2\mu_k}}}\right)^{\frac{2\mu_k}{2\mu_k-1}}\right).$$

Proof We define the paths $\Gamma_{comp,+} := \{t + i\pi, t \in [-\eta, 0]\}$ and $\Gamma_{comp,-} := \{t - i\pi, t \in [-\eta, 0]\}$. The triangle inequality and Cauchy's formula gives us

$$\frac{2\pi}{|g_k(\underline{\tau}_k)|}|\widetilde{\mathscr{F}}^n_{k,j} - \mathscr{F}^n_{k,j}| \leq \left|\int_{\Gamma_{comp,+}} \exp(n\tau + j\varphi_k(\tau))d\tau\right| + \left|\int_{\Gamma_{comp,-}} \exp(n\tau + j\varphi_k(\tau))d\tau\right|.$$

We need estimations on both terms. We allow ourselves to only present the first estimation. We have

$$\left|\int_{\Gamma_{comp,+}} \exp(n\tau + j\varphi_k(\tau))d\tau\right| \leq \int_{-\eta}^{0} \exp(nt + j\Re(\varphi_k(t + i\pi)))dt.$$

Because of the inequality (38) and knowing that $j \geq n\underline{\delta}_k$ with $\underline{\delta}_k > 0$, we have for $t \in [-\eta, 0]$

$$nt + j\Re(\varphi_k(t + i\pi)) \leq -\frac{j}{\alpha_k}\left(-\eta - A_R\eta^{2\mu_k} + A_I|\pi - \theta_k|^{2\mu_k}\right).$$

Therefore, because $\theta_k \notin \{-\pi, \pi\}$, if we take η small enough, we have a constant $c > 0$ independent from n and j such that

$$\left|\int_{\Gamma_{comp,+}} \exp(n\tau + j\varphi_k(\tau))d\tau\right| \leq \eta e^{-cn}.$$

Because of the Lemma 10, there exist $C, c > 0$ independent from j and n such that

$$\left|\int_{\Gamma_{comp,+}} \exp(n\tau + j\varphi_k(\tau))d\tau\right| \leq \frac{C}{n^{\frac{1}{\mu_k}}} \exp\left(-c\left(\frac{|j - \alpha_k n|}{n^{\frac{1}{2\mu_k}}}\right)^{\frac{2\mu_k}{2\mu_k-1}}\right).$$

□

Using Cauchy's formula and taking into account the "$2i\pi$-periodicity" of $\mathbf{G}_j(\tau)$, we also have that for all $n \in \mathbb{N}^*$ and $j \in \mathbb{Z}$

$$\mathscr{G}^n_j = \frac{1}{2i\pi}\int_{\Gamma_\rho} e^{n\tau}\mathbf{G}_j(\tau)d\tau = \frac{1}{2i\pi}\int_{\Gamma_k} e^{n\tau}\mathbf{G}_j(\tau)d\tau. \tag{43}$$

To prove Proposition 2, Lemma 11 implies that we need to prove the following lemma.

Lemma 12. *For every $k \in \{1, \dots, K\}$, there exist two constants $C, c > 0$ such that*

$$\forall (n,j) \in \mathcal{D}_k, \quad \left| \mathscr{G}_j^n - \widetilde{\mathscr{F}}_{k,j}^n \right| \leq \frac{C}{n^{\frac{1}{\mu_k}}} \exp\left(-c \left(\frac{|j - n\alpha_k|}{n^{\frac{1}{2\mu_k}}} \right)^{\frac{2\mu_k}{2\mu_k - 1}} \right).$$

The main part of this paper will now be to prove this lemma. Both elements can be expressed has integrals along the path Γ_k. By using the triangle inequality, we have that

$$\begin{aligned} 2\pi \left| \mathscr{G}_j^n - \widetilde{\mathscr{F}}_{k,j}^n \right| \leq & \left| \int_{\Gamma_{k,in}} e^{n\tau} \mathbf{G}_j(\tau) - \underline{z}_k \underline{\kappa}_k^j g_k(\underline{\tau}_k) \exp(n\tau + j\varphi_k(\tau)) d\tau \right| + \left| \int_{\Gamma_{k,out}} e^{n\tau} \mathbf{G}_j(\tau) d\tau \right| \\ & + \left| \int_{\Gamma_{k,out}} \underline{z}_k \underline{\kappa}_k^j g_k(\underline{\tau}_k) \exp(n\tau + j\varphi_k(\tau)) d\tau \right|. \end{aligned}$$

Furthermore, we have using the triangle inequality

$$\begin{aligned} & \left| \int_{\Gamma_{k,in}} e^{n\tau} \mathbf{G}_j(\tau) - \underline{z}_k \underline{\kappa}_k^j g_k(\underline{\tau}_k) \exp(n\tau + j\varphi_k(\tau)) d\tau \right| \\ \leq & \left| \int_{\Gamma_{k,in}} e^{n\tau} \left(\mathbf{G}_j(\tau) - e^{\tau} g_k(\tau) \exp(j\varpi_k(\tau)) \right) d\tau \right| + \left| \int_{\Gamma_{k,in}} e^{n\tau + j\varpi_k(\tau)} \left(e^{\tau} g_k(\tau) - \underline{z}_k g_k(\underline{\tau}_k) \right) d\tau \right| \\ & + \left| \int_{\Gamma_{k,in}} e^{n\tau} \underline{z}_k g_k(\underline{\tau}_k) \left(e^{j\varpi_k(\tau)} - \underline{\kappa}_k^j e^{j\varphi_k(\tau)} \right) d\tau \right|. \end{aligned}$$

Thus, using the triangle inequality and decomposing some of the integrals along $\Gamma_{k,in}$ in integrals along $\Gamma_{k,p}$ and $\Gamma_{k,res}$, we have

$$\left| \mathscr{G}_j^n - \widetilde{\mathscr{F}}_{k,j}^n \right| \leq \frac{1}{2\pi} \left(E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + E_7 \right), \tag{44}$$

where

$$\begin{aligned} E_1 &= \left| \int_{\Gamma_{k,out}} \underline{z}_k \underline{\kappa}_k^j g_k(\underline{\tau}_k) \exp(n\tau + j\varphi_k(\tau)) d\tau \right|, & E_2 &= \left| \int_{\Gamma_{k,in}} e^{n\tau} \left(\mathbf{G}_j(\tau) - e^{\tau} g_k(\tau) \exp(j\varpi_k(\tau)) \right) d\tau \right|, \\ E_3 &= \left| \int_{\Gamma_{k,p}} e^{n\tau + j\varpi_k(\tau)} \left(e^{\tau} g_k(\tau) - \underline{z}_k g_k(\underline{\tau}_k) \right) d\tau \right|, & E_4 &= \left| \int_{\Gamma_{k,res}} e^{n\tau + j\varpi_k(\tau)} \left(e^{\tau} g_k(\tau) - \underline{z}_k g_k(\underline{\tau}_k) \right) d\tau \right|, \\ E_5 &= \left| \int_{\Gamma_{k,p}} e^{n\tau} \underline{z}_k g_k(\underline{\tau}_k) \left(e^{j\varpi_k(\tau)} - \underline{\kappa}_k^j e^{j\varphi_k(\tau)} \right) d\tau \right|, & E_6 &= \left| \int_{\Gamma_{k,res}} e^{n\tau} \underline{z}_k g_k(\underline{\tau}_k) \left(e^{j\varpi_k(\tau)} - \underline{\kappa}_k^j e^{j\varphi_k(\tau)} \right) d\tau \right|, \\ E_7 &= \left| \int_{\Gamma_{k,out}} e^{n\tau} \mathbf{G}_j(\tau) d\tau \right|. \end{aligned}$$

We will now have to determine estimates on all these terms depending on k (case I, II and III) and also on τ_p and $\Gamma_{k,p}$:

- **Case A:** $\rho_k \left(\frac{\zeta_k}{\gamma_k} \right) \in \left[-\frac{\eta}{2}, \varepsilon_{k,0} \right]$,
- **Case B:** $\rho_k \left(\frac{\zeta_k}{\gamma_k} \right) > \varepsilon_{k,0}$,
- **Case C:** $\rho_k \left(\frac{\zeta_k}{\gamma_k} \right) < -\frac{\eta}{2}$.

The main contribution will come from the terms E_3 and E_5. We will prove much sharper estimates for the other terms.

3.3.1 Preliminary lemmas

Before we start to determine the estimates on the different terms, we are going to introduce some lemmas to simplify the redaction. Those lemmas assemble inequalities in the different cases (A, B and C) for which the proofs are similar with variations depending on the case we are in. They mainly rely on the inequalities (37) and (38). The proofs of those lemmas can be found in the appendix.

We start with a lemma which will be useful to study the terms E_5 and E_6.

Lemma 13 (An inequality in $B_\varepsilon(\underline{\tau}_k)$)**.** *There exists $C > 0$ such that for all $\tau \in B_\varepsilon(\underline{\tau}_k)$ and $(n, j) \in \mathcal{D}_k$, we have*

$$\left|e^{n\tau}\left(e^{j\varpi_k(\tau)} - \underline{\kappa}_k^j e^{j\varphi_k(\tau)}\right)\right| \leq Cn|\tau - \underline{\tau}_k|^{2\mu_k+1} \exp(n\Re(\tau - \underline{\tau}_k) + j(\Re(\varpi_k(\tau)) + |\Re(\xi_k(\tau)(\tau - \underline{\tau}_k)^{2\mu+1})|)).$$

This next lemma will be useful for terms where the integral is defined along the path $\Gamma_{k,p}$ (terms E_3 and E_5).

Lemma 14 (Inequalities on $\Gamma_{k,p}$)**.** *For $(n, j) \in \mathbb{N}^* \times \mathbb{N}^*$ and $\tau \in \Gamma_{k,p}$, we have*

- *Case A:* $\rho_k\left(\frac{\zeta_k}{\gamma_k}\right) \in \left[-\frac{\eta}{2}, \varepsilon_{k,0}\right]$

$$n\Re(\tau - \underline{\tau}_k) + j(\Re(\varpi_k(\tau)) + \left|\Re\left(\xi_k(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+1}\right)\right|) \leq -nc_\star\Im(\tau - \underline{\tau}_k)^{2\mu_k} - \frac{n}{\alpha_k}(2\mu_k - 1)\gamma_k\left(\frac{|\zeta_k|}{\gamma_k}\right)^{\frac{2\mu_k}{2\mu_k-1}}, \tag{45}$$

$$n\Re(\tau - \underline{\tau}_k) + j\Re(\varpi_k(\tau)) \leq -nc_\star\Im(\tau - \underline{\tau}_k)^{2\mu_k} - \frac{n}{\alpha_k}(2\mu_k - 1)\gamma_k\left(\frac{|\zeta_k|}{\gamma_k}\right)^{\frac{2\mu_k}{2\mu_k-1}}. \tag{46}$$

- *Case B:* $\rho_k\left(\frac{\zeta_k}{\gamma_k}\right) > \varepsilon_{k,0}$

$$n\Re(\tau - \underline{\tau}_k) + j(\Re(\varpi_k(\tau)) + \left|\Re\left(\xi_k(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+1}\right)\right|) \leq -\frac{n}{\alpha_k}(2\mu_k - 1)A_R\underline{\delta}_k\varepsilon_{k,0}^{2\mu_k}, \tag{47}$$

$$n\Re(\tau - \underline{\tau}_k) + j\Re(\varpi_k(\tau)) \leq -\frac{n}{\alpha_k}(2\mu_k - 1)A_R\underline{\delta}_k\varepsilon_{k,0}^{2\mu_k}. \tag{48}$$

- *Case C:* $\rho_k\left(\frac{\zeta_k}{\gamma_k}\right) < -\frac{\eta}{2}$

$$n\Re(\tau - \underline{\tau}_k) + j(\Re(\varpi_k(\tau)) + \left|\Re\left(\xi_k(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+1}\right)\right|) \leq -\frac{n}{\alpha_k}(2\mu_k - 1)A_R\underline{\delta}_k\left(\frac{\eta}{2}\right)^{2\mu_k}, \tag{49}$$

$$n\Re(\tau - \underline{\tau}_k) + j\Re(\varpi_k(\tau)) \leq -\frac{n}{\alpha_k}(2\mu_k - 1)A_R\underline{\delta}_k\left(\frac{\eta}{2}\right)^{2\mu_k}. \tag{50}$$

We introduce in the next lemma some inequalities that will help us for the terms with integrals defined on $\Gamma_{k,res}$ (terms E_4 and E_6).

Lemma 15 (Inequalities on $\Gamma_{k,res}$)**.** *For $(n, j) \in \mathbb{N}^* \times \mathbb{N}^*$ and $\tau \in \Gamma_{k,res}$, we have in all cases*

$$n\Re(\tau - \underline{\tau}_k) + j(\Re(\varpi_k(\tau)) + \left|\Re\left(\xi_k(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+1}\right)\right|) \leq -n\frac{\eta}{2}, \tag{51}$$

$$n\Re(\tau - \underline{\tau}_k) + j(\Re(\varpi_k(\tau)) \leq -n\frac{\eta}{2}. \tag{52}$$

Finally, we need a lemma for the term where the integral is defined along the path $\Gamma_{k,out}$ (term E_1).

Lemma 16 (An inequality on $\Gamma_{k,out}$)**.** *For $(n, j) \in \mathbb{N}^* \times \mathbb{N}^*$ and $\tau \in \Gamma_{k,out}$, we have in all cases*

$$n\Re(\tau - \underline{\tau}_k) + j\Re(\varphi_k(\tau)) \leq -n\frac{\eta}{2}. \tag{53}$$

Remark 2. Those lemmas could be generalized in the case where $\alpha_k < 0$ but there would be some changes. For example, we would have to prove the Lemmas 14, 15 and 16 for $(n, j) \in \mathbb{N} \times \mathbb{Z}$ with $j \leq 0$. The main element to use for those generalization would be to see that $\frac{j}{\alpha_k} \geq 0$.

3.3.2 Estimates of a part of the terms

We are going to first prove estimates for the terms where the proof will not depend on the case A, B or C in which we are.

• Estimate for E_1:
If we use the inequality (53), we have

$$\left|\int_{\Gamma_{k,out}} \underline{z}_k \underline{\kappa}_k^j g_k(\underline{\tau}_k) \exp(n\tau + j\varphi_k(\tau)) d\tau\right| \lesssim \int_{\Gamma_{k,out}} \exp(n\Re(\tau) + j\Re(\varphi_k(\tau)))|d\tau| \lesssim e^{-n\frac{\eta}{2}}.$$

• Estimate for E_2:
We introduce the path $\Gamma_{\eta,k}$ defined as

$$\Gamma_{\eta,k} := \{-\eta + it, t \in [-\pi, \pi]\} \cap B_\varepsilon(\underline{\tau}_k).$$

Using Cauchy's formula, we have that

$$\int_{\Gamma_{k,in}} e^{n\tau} \left(\mathbf{G}_j(\tau) - e^\tau g_k(\tau) \exp(j\varpi_k(\tau))\right) d\tau = \int_{\Gamma_{\eta,k}} e^{n\tau} \left(\mathbf{G}_j(\tau) - e^\tau g_k(\tau) \exp(j\varpi_k(\tau))\right) d\tau.$$

Because $\alpha_k > 0$, depending on whether we are in case I or III, the previous equality and the inequalities (30) and (34) imply

$$\left|\int_{\Gamma_{k,in}} e^{n\tau} \left(\mathbf{G}_j(\tau) - e^\tau g_k(\tau) \exp(j\varpi_k(\tau))\right) d\tau\right| \lesssim e^{-n\eta - cj}.$$

• Estimate for E_4:
We can suppose that the functions $\tau \in B_\varepsilon(\underline{\tau}_k) \mapsto e^\tau g_k(\tau)$ have bounded derivatives on $B_\varepsilon(\underline{\tau}_k)$. Therefore, the mean value inequality and the inequality (52) imply

$$\left|\int_{\Gamma_{k,res}} e^{n\tau + j\varpi_k(\tau)} \left(e^\tau g_k(\tau) - \underline{z}_k g_k(\underline{\tau}_k)\right) d\tau\right| \lesssim \int_{\Gamma_{k,res}} |\tau - \underline{\tau}_k| \exp\left(n\Re(\tau) + j\Re(\varpi_k(\tau))\right) |d\tau| \lesssim e^{-n\frac{\eta}{2}}.$$

• Estimate for E_6:
If we use the Lemma 13, we have

$$\left|\int_{\Gamma_{k,res}} e^{n\tau} \underline{z}_k g_k(\underline{\tau}_k) \left(e^{j\varpi_k(\tau)} - \underline{\kappa}_k^j e^{j\varphi_k(\tau)}\right) d\tau\right| \lesssim \int_{\Gamma_{k,res}} \exp(n\Re(\tau - \underline{\tau}_k) + j(\Re(\varpi_k(\tau)) + |\Re(\xi_k(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+1})|)) \\ n|\tau - \underline{\tau}_k|^{2\mu_k+1} |d\tau|.$$

Therefore, the inequality (51) implies

$$\left|\int_{\Gamma_{k,res}} e^{n\tau} \underline{z}_k g_k(\underline{\tau}_k) \left(e^{j\varpi_k(\tau)} - \underline{\kappa}_k^j e^{j\varphi_k(\tau)}\right) d\tau\right| \lesssim n e^{-n\frac{\eta}{2}} \lesssim e^{-n\frac{\eta}{4}}.$$

It remains to study the terms E_3, E_5 and E_7.

3.3.3 The terms E_3 and E_5, Case A : $\rho_k\left(\frac{\zeta_k}{\gamma_k}\right) \in \left[-\frac{\eta}{2}, \varepsilon_{k,0}\right]$

This part of the proof is the most important because those terms will create the limiting estimates.

• Estimate for E_3:
Because of the mean value inequality, we have

$$E_3 = \left|\int_{\Gamma_{k,p}} \left(e^\tau g_k(\tau) - \underline{z}_k g_k(\underline{\tau}_k)\right) e^{n\tau + j\varpi_k(\tau)} d\tau\right| \lesssim \int_{\Gamma_{k,p}} |\tau - \underline{\tau}_k| \exp\left(n\Re(\tau) + j\Re(\varpi_k(\tau))\right) |d\tau|.$$

The inequality (46) implies

$$E_3 \lesssim \int_{\Gamma_{k,p}} |\tau - \underline{\tau}_k| e^{-nc_\star \Im(\tau - \underline{\tau}_k)^{2\mu_k}} |d\tau| \exp\left(-\frac{n}{\alpha_k}(2\mu_k - 1)\gamma_k \left(\frac{|\zeta_k|}{\gamma_k}\right)^{\frac{2\mu_k}{2\mu_k - 1}}\right).$$

But, the inequality (42) and the fact that $\rho_k\left(\frac{\zeta_k}{\gamma_k}\right)=\tau_p$ imply

$$\frac{n}{\alpha_k}(2\mu_k-1)\gamma_k\left(\frac{|\zeta_k|}{\gamma_k}\right)^{\frac{2\mu_k}{2\mu_k-1}}\geq\frac{2\mu_k-1}{\alpha_k}A_R\underline{\delta}_k n|\tau_p|^{2\mu_k}.$$

If we introduce $c>0$ small enough, then

$$E_3\lesssim\int_{\Gamma_{k,p}}|\tau-\underline{\tau}_k|e^{-nc_\star\Im(\tau-\underline{\tau}_k)^{2\mu_k}}|d\tau|\exp\left(-cn|\tau_p|^{2\mu_k}\right).$$

Using the parametrization (39) and the inequality (40), we have

$$\int_{\Gamma_{k,p}}|\tau-\underline{\tau}_k|e^{-nc_\star\Im(\tau-\underline{\tau}_k)^{2\mu_k}}|d\tau|\lesssim\int_{-\ell_{k,p}}^{\ell_{k,p}}(|\tau_p|+|t|)e^{-nc_*t^{2\mu_k}}dt.$$

The change of variables $s=n^{\frac{1}{2\mu_k}}t$ and the fact that the function $x\geq0\mapsto x\exp\left(-\frac{c}{2}x^{2\mu_k}\right)$ is bounded imply

$$\begin{cases}\displaystyle\int_{-\ell_{k,p}}^{\ell_{k,p}}|t|e^{-nc_*t^{2\mu_k}}dt\lesssim\frac{1}{n^{\frac{1}{\mu_k}}},\\ \displaystyle\int_{-\ell_{k,p}}^{\ell_{k,p}}|\tau_p|e^{-nc_*t^{2\mu_k}}dt\lesssim\frac{1}{n^{\frac{1}{\mu_k}}}\exp\left(\frac{c}{2}n|\tau_p|^{2\mu_k}\right).\end{cases}$$

Thus,

$$E_3\lesssim\frac{1}{n^{\frac{1}{\mu_k}}}\exp\left(-\frac{c}{2}n|\tau_p|^{2\mu_k}\right).$$

Lastly, the inequality (42) implies that we have a constant $\tilde{c}>0$ independent from j and n such that

$$\frac{c}{2}n|\tau_p|^{2\mu_k}\geq\tilde{c}\left(\frac{|j-n\alpha_k|}{n^{\frac{1}{2\mu_k}}}\right)^{\frac{2\mu_k}{2\mu_k-1}}$$

so,

$$E_3\lesssim\frac{1}{n^{\frac{1}{\mu_k}}}\exp\left(-\tilde{c}\left(\frac{|j-n\alpha_k|}{n^{\frac{1}{2\mu_k}}}\right)^{\frac{2\mu_k}{2\mu_k-1}}\right).$$

• Estimate for E_5:
Using the Lemma 13 and the inequality (45), we have

$$\begin{aligned}E_5&=\left|\int_{\Gamma_{k,p}}e^{n\tau}\underline{z}_k g_k(\underline{\tau}_k)\left(e^{j\varpi_k(\tau)}-\underline{\kappa}_k^j e^{j\varphi_k(\tau)}\right)d\tau\right|\\&\lesssim\int_{\Gamma_{k,p}}n|\tau-\underline{\tau}_k|^{2\mu_k+1}\exp(n\Re(\tau-\underline{\tau}_k)+j(\Re(\varpi_k(\tau))+|\Re(\xi_k(\tau)(\tau-\underline{\tau}_k)^{2\mu_k+1})|))|d\tau|\\&\lesssim\exp\left(-\frac{n}{\alpha_k}(2\mu_k-1)\gamma_k\left(\frac{|\zeta_k|}{\gamma_k}\right)^{\frac{2\mu_k}{2\mu_k-1}}\right)n\int_{\Gamma_{k,p}}|\tau-\underline{\tau}_k|^{2\mu_k+1}\exp(-nc_\star\Im(\tau-\underline{\tau}_k)^{2\mu_k})|d\tau|.\end{aligned}$$

Just like in the estimation of the previous term, because of the inequality (42), if we introduce $c>0$ small enough, we have

$$E_5\lesssim n\exp\left(-cn|\tau_p|^{2\mu_k}\right)\int_{\Gamma_{k,p}}|\tau-\underline{\tau}_k|^{2\mu_k+1}\exp(-nc_\star\Im(\tau-\underline{\tau}_k)^{2\mu_k})|d\tau|.$$

The same reasoning as for the estimate of E_3 implies that

$$n\int_{\Gamma_{k,p}}|\tau-\underline{\tau}_k|^{2\mu_k+1}\exp(-nc_\star\Im(\tau-\underline{\tau}_k)^{2\mu_k})|d\tau|\lesssim n\int_{-\ell_{k,p}}^{\ell_{k,p}}|t|^{2\mu_k+1}e^{-nc_*t^{2\mu_k}}dt+n\int_{-\ell_{k,p}}^{\ell_{k,p}}|\tau_p|^{2\mu_k+1}e^{-nc_*t^{2\mu_k}}dt.$$

The change of variables $s = n^{\frac{1}{2\mu_k}} t$ and the fact that the function $x \geq 0 \mapsto x^{2\mu_k+1} \exp\left(-\frac{c}{2}x^{2\mu_k}\right)$ is bounded imply

$$\begin{cases} n\displaystyle\int_{-\ell_{k,p}}^{\ell_{k,p}} |t|^{2\mu_k+1} e^{-nc_* t^{2\mu_k}} dt \lesssim \frac{1}{n^{\frac{1}{\mu_k}}}, \\ n\displaystyle\int_{-\ell_{k,p}}^{\ell_{k,p}} |\tau_p|^{2\mu_k+1} e^{-nc_* t^{2\mu_k}} dt \lesssim \frac{1}{n^{\frac{1}{\mu_k}}} \exp\left(\frac{c}{2} n|\tau_p|^{2\mu_k}\right). \end{cases}$$

Thus,

$$E_5 \lesssim \frac{1}{n^{\frac{1}{\mu_k}}} \exp\left(-\frac{c}{2} n|\tau_p|^{2\mu_k}\right).$$

Lastly, the inequality on γ_k (42) implies that we have a constant $\tilde{c} > 0$ independent from j and n such that

$$\frac{c}{2} n|\tau_p|^{2\mu_k} \geq \tilde{c} \left(\frac{|j - n\alpha_k|}{n^{\frac{1}{2\mu_k}}}\right)^{\frac{2\mu_k}{2\mu_k - 1}}$$

so,

$$E_5 \lesssim \frac{1}{n^{\frac{1}{\mu_k}}} \exp\left(-\tilde{c} \left(\frac{|j - n\alpha_k|}{n^{\frac{1}{2\mu_k}}}\right)^{\frac{2\mu_k}{2\mu_k - 1}}\right).$$

3.3.4 The terms E_3 and E_5, Case B and C:

We now consider that we are either in case B or case C (i.e. $\rho_k(\frac{\zeta_k}{\gamma_k}) \notin \left[-\frac{\eta}{2}, \varepsilon_{k,0}\right]$).

• Estimate for E_3:

Because of the mean value inequality, we have

$$E_3 = \left| \int_{\Gamma_{k,p}} e^{n\tau + j\varpi_k(\tau)} \left(e^{\tau} g_k(\tau) - \underline{z}_k g_k(\underline{\tau}_k)\right) d\tau \right| \lesssim \int_{\Gamma_{k,p}} |\tau - \underline{\tau}_k| \exp\left(n\Re(\tau) + j\Re(\varpi_k(\tau))\right) |d\tau|.$$

Using the inequality (48) or (50) whether we are in case B or C, they imply that there exists $c > 0$ independent from j and n such that

$$E_3 \lesssim e^{-cn}.$$

• Estimate for E_5:

Using the Lemma 13, we have

$$\begin{aligned} E_5 &= \left| \int_{\Gamma_{k,p}} e^{n\tau} \underline{z}_k g_k(\underline{\tau}_k) \left(e^{j\varpi_k(\tau)} - \underline{\kappa}_k^j e^{j\varphi_k(\tau)}\right) d\tau \right| \\ &\lesssim \int_{\Gamma_{k,p}} n|\tau - \underline{\tau}_k|^{2\mu_k+1} \exp(n\Re(\tau - \underline{\tau}_k) + j(\Re(\varpi_k(\tau)) + |\Re(\xi_k(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+1})|)) |d\tau|. \end{aligned}$$

Using the inequality (47) or (49) whether we are in case B or C, they imply that there exists $c > 0$ independent from j and n such that

$$E_5 \lesssim ne^{-cn} \lesssim e^{-\frac{c}{2}n}.$$

3.3.5 Estimate of the term E_7

We recall that

$$E_7 = \left| \int_{\Gamma_{k,out}} e^{n\tau} \mathbf{G}_j(\tau) d\tau \right|.$$

For $\tau \in \Gamma_{k,out}$, we have different estimates depending on whether we are inside a $B_\varepsilon(\underline{\tau}_l)$ or not. Therefore, we introduce the set of distinct points

$$\{\hat{\tau}_1, \ldots, \hat{\tau}_R\} = \{\underline{\tau}_l, \quad l \in \{1, \ldots, K\}\} \setminus \{\underline{\tau}_k\}.$$

It allows us to decompose the path $\Gamma_{k,out}$ as

$$\Gamma_{k,out} := \bigcup_{l=0}^{R} \widehat{\Gamma}_l,$$

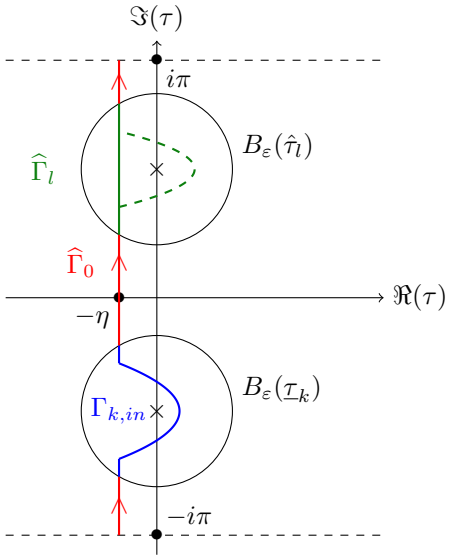

Figure 6: This is a representation of Γ_k were we decompose $\Gamma_{k,out}$. The red path corresponds to $\widehat{\Gamma}_0$ the part of $\Gamma_{k,out}$ which lies outside the balls $B_\varepsilon(\hat{\tau}_l)$. The green path corresponds to $\widehat{\Gamma}_l$ the part of $\Gamma_{k,out}$ which lies inside the ball $B_\varepsilon(\hat{\tau}_l)$. The dashed green path corresponds to the deformation we use in the proof of the estimate for E_7.

where for all $l \in \{1, \dots, R\}$

$$\widehat{\Gamma}_l := \Gamma_{k,out} \cap B_\varepsilon(\hat{\tau}_l)$$

and

$$\widehat{\Gamma}_0 := \Gamma_{k,out} \setminus \bigcup_{l=1}^{R} \widehat{\Gamma}_l.$$

This decomposition of $\Gamma_{k,out}$ is represented on the Figure 6. The inequality (29) gives us that

$$\left| \int_{\widehat{\Gamma}_0} e^{n\tau} \mathbf{G}_j(\tau) d\tau \right| \lesssim e^{-n\eta - c|j|}.$$

We now consider $l \in \{1, \dots, R\}$. There are two possibilities because of Hypothesis 3:

• The set $\{i \in \{1, \dots, R\}, \quad \underline{\tau}_i = \hat{\tau}_l\}$ is the singleton $\{i\}$ with $\alpha_i < 0$ (i.e. we are in case II). Then, knowing that for $(n, j) \in \mathcal{D}_k$ we have $j \geq 1$, because of the inequality (32), we have

$$\left| \int_{\widehat{\Gamma}_l} e^{n\tau} \mathbf{G}_j(\tau) d\tau \right| \lesssim e^{-n\eta - c|j|}.$$

• The set $\{i \in \{1, \dots, R\}, \quad \underline{\tau}_i = \hat{\tau}_l\}$ is the singleton $\{i\}$ with $\alpha_i > 0$ (i.e. we are in case I) or it has two distinct elements $\{i, j\}$ with $\alpha_i > 0$ and $\alpha_j < 0$ (i.e. we are in case III). Either way, the inequalities (30) and (34) imply that

$$\left| \int_{\widehat{\Gamma}_l} e^{n\tau} \mathbf{G}_j(\tau) d\tau \right| \leq 2\pi C e^{-n\eta - c|j|} + \left| \int_{\widehat{\Gamma}_l} \exp(n\tau + j\varpi_i(\tau)) e^{\tau} g_i(\tau) d\tau \right|.$$

Just like we defined the path $\Gamma_{k,p}$, $\Gamma_{k,res}$ and $\Gamma_{k,in} := \Gamma_{k,p} \sqcup \Gamma_{k,res}$, we can define a path $\Gamma_{i,p}$, $\Gamma_{i,res}$ and $\Gamma_{i,in} := \Gamma_{i,p} \sqcup \Gamma_{i,res}$. The path $\Gamma_{i,in}$ is represented with a dashed green line on the Figure 6. Using Cauchy's formula, we then have

$$\int_{\widehat{\Gamma}_l} \exp(n\tau + j\varpi_i(\tau)) e^{\tau} g_i(\tau) d\tau = \int_{\Gamma_{i,in}} \exp(n\tau + j\varpi_i(\tau)) e^{\tau} g_i(\tau) d\tau$$

The function $\tau \mapsto e^{\tau} g_i(\tau)$ can be bounded so we just have to bound $\int_{\Gamma_{i,in}} \exp(n\Re(\tau - \underline{\tau}_i) + j\Re(\varpi_i(\tau))) d|\tau|$. We observe that the proofs of the Lemmas 14 and 15 are also true for $\Gamma_{i,p}$ and $\Gamma_{i,res}$ because α_k and α_i have

the same sign. Using the inequality (52) for the integral along the path $\Gamma_{i,res}$, we prove that there exists a constant $c>0$ independent from n and j so that

$$\int_{\Gamma_{i,res}} \exp(n\Re(\tau-\underline{\tau}_i)+j\Re(\varpi_i(\tau)))d|\tau| \lesssim e^{-cn}.$$

It remains to bound the integral along the path $\Gamma_{i,p}$. In the case A (i.e. $\rho_i(\frac{\zeta_i}{\gamma_i})\in[-\frac{\eta}{2},\varepsilon_{i,0}]$), we observe that for $(n,j)\in\mathcal{D}_k$, γ_i is bounded between two positive constants and

$$|\zeta_i| \geq \frac{1}{2\mu_i}\min(|\alpha_i-\underline{\delta}_k|,|\alpha_i-\overline{\delta}_k|).$$

Therefore, using the inequality (46) and the previous observation in case A and using the inequalities (48) and (50) in cases B and C, we prove that there exists a constant $c>0$ independent from n and j so that

$$\int_{\Gamma_{i,p}} \exp(n\Re(\tau-\underline{\tau}_i)+j\Re(\varpi_i(\tau)))d|\tau| \lesssim e^{-cn}.$$

Therefore, there exists a constant $c>0$ such that

$$\forall(n,j)\in\mathcal{D}_k, \quad \left|\int_{\widehat{\Gamma}_l} e^{n\tau}\mathbf{G}_j(\tau)d\tau\right| \lesssim e^{-cn}.$$

This gives a sharp estimate of E_7.

If we recapitulate the estimates we found, we can define two constants $C,c>0$ such that

$$\forall(n,j)\in\mathcal{D}_k,\forall l\in\{1,2,4,6,7\}, \quad E_l \leq Ce^{-cn},$$

and

$$\forall(n,j)\in\mathcal{D}_k,\forall l\in\{3,5\}, \quad E_l \leq \frac{C}{n^{\frac{1}{\mu_k}}}\exp\left(-c\left(\frac{|j-n\alpha_k|}{n^{\frac{1}{2\mu_k}}}\right)^{\frac{2\mu_k}{2\mu_k-1}}\right).$$

The estimates we proved on all the terms and the Lemma 10 allow us to conclude the proof of Lemma 12 and therefore of Proposition 2.

3.4 End of the proofs of Proposition 1 and Theorem 1

We recall that the inequalities (26) and (27) implied that to prove Theorem 1 we only had to prove Proposition 1 which announces generalized Gaussian bounds on $\mathscr{G}_j^n-\mathscr{H}_{k,j}^n$ for $(n,j)\in\mathcal{D}_k$. Proving Proposition 1 will be the goal of this section. Proposition 2 implies that there exist two constants $C,c>0$ such that

$$\forall k\in\{1,\dots,K\},\forall(n,j)\in\mathcal{D}_k, \quad |\mathscr{G}_j^n-\mathscr{F}_{k,j}^n| \leq \frac{C}{n^{\frac{1}{\mu_k}}}\exp\left(-c\left(\frac{|j-\alpha_k n|}{n^{\frac{1}{2\mu_k}}}\right)^{\frac{2\mu_k}{2\mu_k-1}}\right). \tag{54}$$

To prove Proposition 1, we understand that we just need to prove similar generalized Gaussian bounds for $\mathscr{F}_{k,j}^n-\mathscr{H}_{k,j}^n$. To do so, we introduce the following elements

$$\forall k\in\{1,\dots,K\},\forall(n,j)\in\mathcal{D}_k, \quad \widetilde{\mathscr{H}_{k,j}^n} := \frac{{\underline{z}_k}^n{\underline{\kappa}_k}^j}{\left(\frac{j}{\alpha_k}\right)^{\frac{1}{2\mu_k}}}H_{2\mu_k}^{\beta_k}\left(\frac{j-n\alpha_k}{\left(\frac{j}{\alpha_k}\right)^{\frac{1}{2\mu_k}}}\right).$$

For $(n,j)\in\mathcal{D}_k$, $\frac{j}{\alpha_k}$ is positive so those elements are well defined. Also, $\frac{j}{\alpha_k}$ is close to n so it would seem obvious that $\widetilde{\mathscr{H}_{k,j}^n}$ is close to $\mathscr{H}_{k,j}^n$. Furthermore, with an affine change of variables, we have that

$$\forall k\in\{1,\dots,K\},\forall(n,j)\in\mathcal{D}_k, \quad \widetilde{\mathscr{H}_j^n} := \frac{{\underline{z}_k}^n{\underline{\kappa}_k}^j}{2\pi}\int_{-\infty}^{+\infty}\exp\left(it(n\alpha_k-j)-\frac{j}{\alpha_k}\beta_k t^{2\mu_k}\right)dt.$$

Therefore, $\widetilde{\mathscr{H}_{k,j}^n}$ should also be close to $\mathscr{F}_{k,j}^n$. We introduce the two following lemmas.

Lemma 17. *There exist $C, c > 0$ such that*

$$\forall k \in \{1, \dots, K\}, \forall (n,j) \in \mathcal{D}_k, \quad \left| \mathscr{F}_{k,j}^n - \widetilde{\mathscr{H}}_{k,j}^n \right| \leq \frac{C}{n^{\frac{1}{\mu_k}}} \exp\left(-c \left(\frac{|j - \alpha_k n|}{n^{\frac{1}{2\mu_k}}} \right)^{\frac{2\mu_k}{2\mu_k - 1}} \right).$$

Lemma 18. *There exist $C, c > 0$ such that*

$$\forall k \in \{1, \dots, K\}, \forall (n,j) \in \mathcal{D}_k, \quad \left| \mathscr{H}_{k,j}^n - \widetilde{\mathscr{H}}_{k,j}^n \right| \leq \frac{C}{n^{\frac{1}{\mu_k}}} \exp\left(-c \left(\frac{|j - \alpha_k n|}{n^{\frac{1}{2\mu_k}}} \right)^{\frac{2\mu_k}{2\mu_k - 1}} \right).$$

If we prove those two lemmas, then we will have proved that there exists two constants $C, c > 0$ such that

$$\forall k \in \{1, \dots, K\}, \forall (n,j) \in \mathcal{D}_k, \quad \left| \mathscr{F}_{k,j}^n - \mathscr{H}_{k,j}^n \right| \leq \frac{C}{n^{\frac{1}{\mu_k}}} \exp\left(-c \left(\frac{|j - \alpha_k n|}{n^{\frac{1}{2\mu_k}}} \right)^{\frac{2\mu_k}{2\mu_k - 1}} \right).$$

This inequality and the inequality (54) will end the proof of Proposition 1 and therefore the proof of Theorem 1.

We consider $k \in \{1, \dots, K\}$ and we will now prove Lemmas 17 and 18 for this k. We will suppose that $\alpha_k > 0$ and observe that the proof in the case where $\alpha_k < 0$ would be done in a similar way.

3.4.1 Proof of Lemma 17

We start with the proof of Lemma 17. We have

$$\forall (n,j) \in \mathcal{D}_k, \quad \begin{cases} \mathscr{F}_{k,j}^n := & \frac{z_k{}^n \underline{\kappa}_k{}^j}{2\pi} \int_{\frac{-\pi - \theta_k}{\alpha_k}}^{\frac{\pi - \theta_k}{\alpha_k}} \exp\left(it(n\alpha_k - j) - \frac{j}{\alpha_k} \beta_k t^{2\mu_k} \right) dt, \\ \widetilde{\mathscr{H}}_{k,j}^n := & \frac{z_k{}^n \underline{\kappa}_k{}^j}{2\pi} \int_{-\infty}^{+\infty} \exp\left(it(n\alpha_k - j) - \frac{j}{\alpha_k} \beta_k t^{2\mu_k} \right) dt. \end{cases}$$

We then need estimates on $\int_{-\infty}^{-\frac{\pi + \theta_k}{\alpha_k}} e^{it(n\alpha_k - j) - \frac{j}{\alpha_k}\beta_k t^{2\mu_k}} dt$ and $\int_{\frac{\pi - \theta_k}{\alpha_k}}^{+\infty} e^{it(n\alpha_k - j) - \frac{j}{\alpha_k}\beta_k t^{2\mu_k}} dt$. We will present the proof for the second term. Integrating by parts and recalling that $\theta_k \in]-\pi, \pi[$, we have

$$\begin{aligned} \left| \int_{\frac{\pi - \theta_k}{\alpha_k}}^{+\infty} e^{it(n\alpha_k - j) - \frac{j}{\alpha_k}\beta_k t^{2\mu_k}} dt \right| &\leq \int_{\frac{\pi - \theta_k}{\alpha_k}}^{+\infty} e^{-\frac{j}{\alpha_k}\Re(\beta_k) t^{2\mu_k}} dt \\ &\leq \frac{1}{\Re(\beta_k) 2\mu_k \frac{j}{\alpha_k} \left(\frac{\pi - \theta_k}{\alpha_k} \right)^{2\mu_k - 1}} e^{-\left(\frac{\pi - \theta_k}{\alpha_k} \right)^{2\mu_k} \Re(\beta_k) \frac{j}{\alpha_k}} - \int_{\frac{\pi - \theta_k}{\alpha_k}}^{+\infty} \frac{(2\mu_k - 1)}{2\mu_k \Re(\beta_k) \frac{j}{\alpha_k} t^{2\mu_k}} e^{-\Re(\beta_k)\frac{j}{\alpha_k} t^{2\mu_k}} dt. \end{aligned}$$

The second term is non positive. Knowing that $j \geq n\underline{\delta}_k \geq \underline{\delta}_k$, we have a constant $c > 0$ independent from j and n such that

$$\left| \int_{\frac{\pi - \theta_k}{\alpha_k}}^{+\infty} e^{it(n\alpha_k - j) - \frac{j}{\alpha_k}\beta_k t^{2\mu_k}} dt \right| \lesssim e^{-cn}.$$

Lemma 10 allows us to conclude.

3.4.2 Proof of Lemma 18

We now prove Lemma 18. We have for $(n,j) \in \mathcal{D}_k$

$$2\pi |\widetilde{\mathscr{H}}_{k,j}^n - \mathscr{H}_{k,j}^n| \leq \left(\frac{\alpha_k}{j} \right)^{\frac{1}{2\mu_k}} \left| H_{2\mu_k}^{\beta_k} \left(\frac{n\alpha_k - j}{n^{\frac{1}{2\mu_k}}} \right) - H_{2\mu_k}^{\beta_k} \left(\frac{n\alpha_k - j}{\left(\frac{j}{\alpha_k} \right)^{\frac{1}{2\mu_k}}} \right) \right| + \left| H_{2\mu_k}^{\beta_k} \left(\frac{n\alpha_k - j}{n^{\frac{1}{2\mu_k}}} \right) \right| \left| \frac{1}{n^{\frac{1}{2\mu_k}}} - \frac{1}{\left(\frac{j}{\alpha_k} \right)^{\frac{1}{2\mu_k}}} \right|.$$

We will have to use the Lemma 2 that we introduced earlier. It directly implies that there exist two constant $C, c > 0$ such that

$$\forall x \in \mathbb{R}, \quad \begin{cases} |H_{2\mu_k}^{\beta_k}(x)| \leq C \exp\left(-c|x|^{\frac{2\mu_k}{2\mu_k-1}}\right), \\ |H_{2\mu_k}^{\beta_k}{}'(x)| \leq C \exp\left(-c|x|^{\frac{2\mu_k}{2\mu_k-1}}\right). \end{cases}$$

Those inequalities will allow us to determine estimates for both terms of the sum. We start with the second term. The mean value inequality implies that

$$\begin{aligned} \left|H_{2\mu_k}^{\beta_k}\left(\frac{n\alpha_k - j}{n^{\frac{1}{2\mu_k}}}\right)\right| \left|\frac{1}{n^{\frac{1}{2\mu_k}}} - \frac{1}{\left(\frac{j}{\alpha_k}\right)^{\frac{1}{2\mu_k}}}\right| &\lesssim \exp\left(-c\left(\frac{|n\alpha_k - j|}{n^{\frac{1}{2\mu_k}}}\right)^{\frac{2\mu_k}{2\mu_k-1}}\right) |n\alpha_k - j| \sup_{t\in[n,\frac{j}{\alpha_k}]} \left|\frac{1}{t^{1+\frac{1}{2\mu_k}}}\right| \\ &\lesssim \frac{|n\alpha_k - j|}{n^{1+\frac{1}{2\mu_k}}} \exp\left(-c\left(\frac{|n\alpha_k - j|}{n^{\frac{1}{2\mu_k}}}\right)^{\frac{2\mu_k}{2\mu_k-1}}\right). \end{aligned}$$

The function $x \in \mathbb{R} \to x \exp\left(-\frac{c}{2}|x|^{\frac{2\mu_k}{2\mu_k-1}}\right)$ is bounded so

$$\left|H_{2\mu_k}^{\beta_k}\left(\frac{n\alpha_k - j}{n^{\frac{1}{2\mu_k}}}\right)\right| \left|\frac{1}{n^{\frac{1}{2\mu_k}}} - \frac{1}{\left(\frac{j}{\alpha_k}\right)^{\frac{1}{2\mu_k}}}\right| \lesssim \frac{1}{n} \exp\left(-\frac{c}{2}\left(\frac{|n\alpha_k - j|}{n^{\frac{1}{2\mu_k}}}\right)^{\frac{2\mu_k}{2\mu_k-1}}\right).$$

There only remains to prove the estimation for the first term of the sum. Because of the mean value inequality and knowing that $(n, j) \in \mathcal{D}_k$, we have

$$\begin{aligned} &\left(\frac{\alpha_k}{j}\right)^{\frac{1}{2\mu_k}} \left|H_{2\mu_k}^{\beta_k}\left(\frac{n\alpha_k - j}{n^{\frac{1}{2\mu_k}}}\right) - H_{2\mu_k}^{\beta_k}\left(\frac{n\alpha_k - j}{\left(\frac{j}{\alpha_k}\right)^{\frac{1}{2\mu_k}}}\right)\right| \\ &\leq \left(\frac{\alpha_k}{\delta_k}\right)^{\frac{1}{2\mu_k}} \frac{|n\alpha_k - j|}{n^{\frac{1}{2\mu_k}}} \left|\frac{1}{n^{\frac{1}{2\mu_k}}} - \frac{1}{\left(\frac{j}{\alpha_k}\right)^{\frac{1}{2\mu_k}}}\right| \sup_{t\in\left[\frac{n\alpha_k - j}{n^{\frac{1}{2\mu_k}}}, \frac{n\alpha_k - j}{\left(\frac{j}{\alpha_k}\right)^{\frac{1}{2\mu_k}}}\right]} \left|H_{2\mu_k}^{\beta_k}{}'(t)\right|. \end{aligned}$$

Using a similar proof as previously to estimate $\left|\frac{1}{n^{\frac{1}{2\mu_k}}} - \frac{1}{\left(\frac{j}{\alpha_k}\right)^{\frac{1}{2\mu_k}}}\right|$ and using the inequality on $H_{2\mu_k}^{\beta_k}{}'$, we have

$$\left(\frac{\alpha_k}{j}\right)^{\frac{1}{2\mu_k}} \left|H_{2\mu_k}^{\beta_k}\left(\frac{n\alpha_k - j}{n^{\frac{1}{2\mu_k}}}\right) - H_{2\mu_k}^{\beta_k}\left(\frac{n\alpha_k - j}{\left(\frac{j}{\alpha_k}\right)^{\frac{1}{2\mu_k}}}\right)\right| \lesssim \frac{1}{n}\left(\frac{|n\alpha_k - j|}{n^{\frac{1}{2\mu_k}}}\right)^2 \exp\left(-c\left(\frac{|n\alpha_k - j|}{n^{\frac{1}{2\mu_k}}}\right)^{\frac{2\mu_k}{2\mu_k-1}}\right).$$

The function $x \in \mathbb{R} \to x^2 \exp\left(-\frac{c}{2}|x|^{\frac{2\mu_k}{2\mu_k-1}}\right)$ is bounded so

$$\left(\frac{\alpha_k}{j}\right)^{\frac{1}{2\mu_k}} \left|H_{2\mu_k}^{\beta_k}\left(\frac{n\alpha_k - j}{n^{\frac{1}{2\mu_k}}}\right) - H_{2\mu_k}^{\beta_k}\left(\frac{n\alpha_k - j}{\left(\frac{j}{\alpha_k}\right)^{\frac{1}{2\mu_k}}}\right)\right| \lesssim \frac{1}{n} \exp\left(-\frac{c}{2}\left(\frac{|n\alpha_k - j|}{n^{\frac{1}{2\mu_k}}}\right)^{\frac{2\mu_k}{2\mu_k-1}}\right).$$

This concludes the proof of Proposition 1 and Theorem 1.

3.5 Coming back to the hypothesis on the coefficients α_k

As we said in the beginning on Section 3, we supposed in the proof that the elements α_k were distinct from one another. In the case where the α_k can be equal, there are some changes that need to be done but the calculations remain similar.

First, just as in Section 3.1, we would define $\overline{\delta}_k$, $\underline{\delta}_k$ and $\mathcal{D}_k$ in the same manner but with the added condition that if $\alpha_k = \alpha_l$, then $\overline{\delta}_k = \overline{\delta}_l$ and $\underline{\delta}_k = \underline{\delta}_l$.

If we consider $k_0 \in \{1, \dots, K\}$, we define

$$\mathcal{J}_{k_0} := \{k \in \{1, \dots, K\}, \quad \alpha_k = \alpha_{k_0}\}.$$

We observe that for $k \in \mathcal{J}_{k_0}$, we have $\mathcal{D}_k = \mathcal{D}_{k_0}$ because of our new condition.

The estimate we found on $\mathscr{G}_j^n$ and $\mathscr{H}_{k,j}^n$ far from the axis $j = n\alpha_k$ (Lemmas 7 and 8) will remain true. Thus, the inequality (26) would also remain true but the inequality (27) would become

$$\forall k_0 \in \{1, \dots, K\}, \forall (n,j) \in \mathcal{D}_{k_0}, \quad \left| \mathscr{G}_j^n - \sum_{k=1}^{K} \mathscr{H}_{k,j}^n \right| \leq C \sum_{k \in \{1,\dots,K\} \setminus \mathcal{J}_{k_0}} \frac{1}{n^{\frac{1}{\mu_k}}} \exp\left(-c \left(\frac{|j - n\alpha_k|}{n^{\frac{1}{2\mu_k}}} \right)^{\frac{2\mu_k}{2\mu_k - 1}} \right) + \left| \mathscr{G}_j^n - \sum_{k \in \mathcal{J}_{k_0}} \mathscr{H}_{k,j}^n \right|.$$

Therefore, we now need estimates on $\left| \mathscr{G}_j^n - \sum_{k \in \mathcal{J}_{k_0}} \mathscr{H}_{k,j}^n \right|$ for $(n,j) \in \mathcal{D}_{k_0}$. We would still introduce for $k \in \mathcal{J}_{k_0}$ the paths Γ_k and the elements $\mathscr{F}_{k,j}^n$, $\widetilde{\mathscr{F}}_{k,j}^n$ and $\widetilde{\mathscr{H}}_{k,j}^n$ with the same definition as in the previous proof. Knowing that the Lemmas 11, 17 and 18 would still be true without any changes to the proofs, we would just need to prove the following lemma, which is a modification of Lemma 12, to prove the Theorem 1.

Lemma 19. *For every $k_0 \in \{1, \dots, K\}$, there exist two constants $C, c > 0$ such that*

$$\forall (n,j) \in \mathcal{D}_{k_0}, \quad \left| \mathscr{G}_j^n - \sum_{k \in \mathcal{J}_{k_0}} \widetilde{\mathscr{F}}_{k,j}^n \right| \leq \sum_{k \in \mathcal{J}_{k_0}} \frac{C}{n^{\frac{1}{\mu_k}}} \exp\left(-c \left(\frac{|j - n\alpha_k|}{n^{\frac{1}{2\mu_k}}} \right)^{\frac{2\mu_k}{2\mu_k - 1}} \right).$$

We recall that in the case where the elements α_k were distinct from one another, we found an expression of the elements $\mathscr{G}_j^n$ as an integral along the path Γ_k and used the triangular inequality to find the inequality (44). We then bounded all the terms C_i to find an estimate on $\mathscr{G}_j^n - \widetilde{\mathscr{F}}_{k,j}^n$. We will do the same thing in the case where the elements α_k can be equal but with a better suited choice of path to express the elements $\mathscr{G}_j^n$. We fix $k_0 \in \{1, \dots, K\}$ and introduce the path $\widetilde{\Gamma}_{k_0}$ which is the ray $\{-\eta + it, t \in [-\pi, \pi]\}$ deformed into the path $\Gamma_{k,in}$ inside the balls $B_\varepsilon(\underline{\tau}_k)$ for $k \in \mathcal{J}_{k_0}$ (see Figure 7). Using Cauchy's formula and taking into account the "$2i\pi$-periodicity" of $\mathbf{G}_j(\tau)$, we have that

$$\forall n \in \mathbb{N}^*, \forall j \in \mathbb{Z}, \quad \mathscr{G}_j^n = \frac{1}{2i\pi} \int_{\widetilde{\Gamma}_{k_0}} e^{n\tau} \mathbf{G}_j(\tau) d\tau.$$

Using a similar proof as in the Section 3.3, we end up with an inequality similar to (44)

$$\left| \mathscr{G}_j^n - \sum_{k \in \mathcal{J}_{k_0}} \widetilde{\mathscr{F}}_{k,j}^n \right| \leq \frac{1}{2\pi} \left(E_{out} + \sum_{k \in \mathcal{J}_{k_0}} (E_{1,k} + E_{2,k} + E_{3,k} + E_{4,k} + E_{5,k} + E_{6,k}) \right),$$

where $E_{i,k}$ has the same definition as E_i in (44) but depends on the $k \in \mathcal{J}_{k_0}$ we consider. The term E_{out} is similar to E_7 in (44) and is equal to

$$E_{out} = \left| \int_{\widetilde{\Gamma}_{k_0,out}} e^{n\tau} \mathbf{G}_j(\tau) d\tau \right|,$$

where $\widetilde{\Gamma}_{k_0,out}$ corresponds to the part of $\widetilde{\Gamma}_{k_0}$ outside the balls $B_\varepsilon(\underline{\tau}_k)$ for $k \in \mathcal{J}_{k_0}$ (see the red path on Figure 7). Reasoning in the same manner as in the case where the elements α_k are different from one another, we get estimates on the different terms. The minor modifications are left to the reader. This concludes the proof of Theorem 1. There only remains to prove Corollary 1.

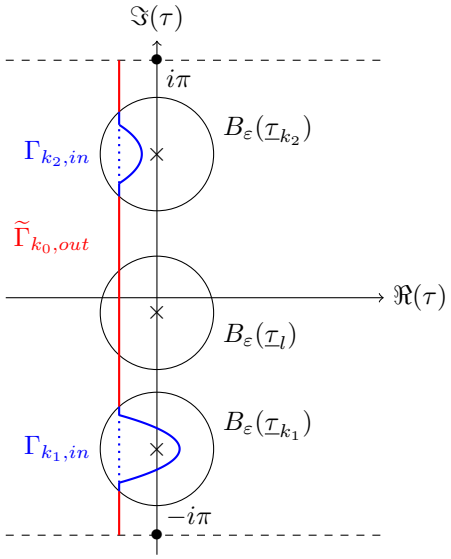

Figure 7: A representation of the path $\widetilde{\Gamma}_{k_0}$. Inside the balls $B_\varepsilon(\underline{\tau}_k)$ where k belongs to $\mathcal{J}_{k_0}$, it follows the path $\Gamma_{k,in}$ composed of $\Gamma_{k,res}$ and $\Gamma_{k,p}$. For $l \in \{1,\dots,K\}$, if there is no $k \in \mathcal{J}_{k_0}$ such that $\underline{\tau}_k = \underline{\tau}_l$, then the path $\widetilde{\Gamma}_{k_0}$ inside $B_\varepsilon(\underline{\tau}_l)$ just corresponds to the ray $\{-\eta + it, t \in [-\pi,\pi]\}$.

4 Proof of Corollary 1

We are now going to prove Corollary 1. We consider that a satisfies the hypotheses of Corollary 1, i.e. a satisfies hypotheses 1 and 4 and we have an integer $J \in \mathbb{Z}$ such that the sequence $b = (a_{j+J})_{j\in\mathbb{Z}}$ satisfies hypotheses 2 and 3. As we said just before we introduced the corollary, we observe that if we define $\widetilde{F}$ the symbol associated with b, then we have that

$$\forall \kappa \in \mathbb{S}^1, \quad \widetilde{F}(\kappa) = \kappa^{-J} F(\kappa).$$

and we have for $k \in \{1,\dots,K\}$

$$\widetilde{F}(\underline{\kappa}_k e^{i\xi}) \underset{\xi\to 0}{=} \underline{\kappa}_k^{-J} \underline{z}_k \exp(-i(\alpha_k + J)\xi - \beta_k \xi^{2\mu_k} + o(|\xi|^{2\mu_k})). \tag{55}$$

We can also define the attractors associated to the sequence thanks to the asymptotic expansion (55). Applying Theorem 1 for the sequence b, there exists two constants $C, c > 0$ such that

$$\forall n \in \mathbb{N}^*, \forall j \in \mathbb{Z}, \quad \left| (\mathscr{L}_b^n \delta)_j - \sum_{k=1}^K \frac{(\underline{\kappa}_k^{-J} \underline{z}_k)^n \underline{\kappa}_k{}^j}{n^{\frac{1}{2\mu_k}}} H_{2\mu_k}^{\beta_k}\left(\frac{j - n(\alpha_k + J)}{n^{\frac{1}{2\mu_k}}}\right)\right|$$
$$\leq C \sum_{k=1}^K \frac{1}{n^{\frac{1}{\mu_k}}} \exp\left(-c\left(\frac{|j - n(\alpha_k + J)|}{n^{\frac{1}{2\mu_k}}}\right)^{\frac{2\mu_k}{2\mu_k - 1}}\right).$$

By observing that

$$\forall n \in \mathbb{N}^*, \forall j \in \mathbb{Z}, \quad (\mathscr{L}_b^n \delta)_j = (\mathscr{L}_a^n \delta)_{j-nJ} = \mathscr{G}_{j-nJ}^n,$$

we conclude the proof of Corollary 1.

5 Appendix

Proof of the Lemma 2

We recall here the statement of Lemma 2.

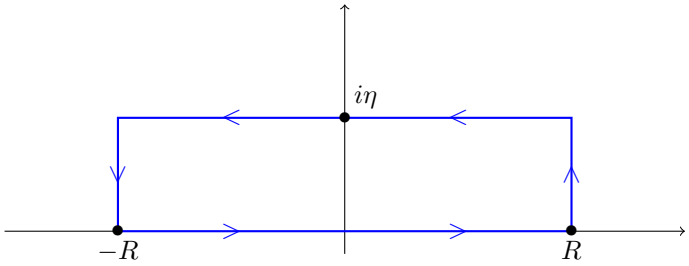

Figure 8: Integrating path for the proof of Lemma 2.

Lemma 20. *For all $\mu \in \mathbb{N}^*$ and $\beta \in \mathbb{C}$ with a positive real part, there exist two constants $C, c > 0$ such that*

$$\forall x \in \mathbb{R}, \quad \begin{cases} |H_{2\mu}^{\beta}(x)| \leq C \exp\left(-c|x|^{\frac{2\mu}{2\mu-1}}\right), \\ |H_{2\mu}^{\beta\,\prime}(x)| \leq C \exp\left(-c|x|^{\frac{2\mu}{2\mu-1}}\right). \end{cases}$$

Proof We will prove here the first inequality, the second one being proved in a similar way. We fix $\eta \in \mathbb{R}$ that we will choose more precisely later. Integrating the function $z \mapsto \exp(-izx - \beta z^{2\mu})$ on the rectangle depicted in the Figure 8 using the Cauchy formula and passing to the limit $R \to +\infty$, we obtain

$$\forall \eta \in \mathbb{R}, \quad H_{2\mu}^{\beta}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i(t+i\eta)x} e^{-\beta(t+i\eta)^{2\mu}} dt.$$

Thus,

$$\left|H_{2\mu}^{\beta}(x)\right| \leq \frac{e^{\eta x}}{2\pi} \int_{\mathbb{R}} \exp\left(-\Re\left(\beta(t+i\eta)^{2\mu}\right)\right) dt.$$

Using Young's inequality, we can show that there exists a constant $c > 0$ such that

$$\forall t \in \mathbb{R}, \quad \Re\left(\beta(t+i\eta)^{2\mu}\right) \geq \frac{\Re(\beta)}{2} t^{2\mu} - c\eta^{2\mu}.$$

and thus there exists $C > 0$ such that

$$\left|H_{2\mu}^{\beta}(x)\right| \leq C e^{\eta x + c\eta^{2\mu}}.$$

Optimizing with respect to η yields the desired result.

□

Proof of the Lemma 13

We recall here the statement of the Lemma 13.

Lemma 21. *There exists $C > 0$ such that $\tau \in B_{\varepsilon}(\underline{\tau}_k)$ and $(n, j) \in \mathcal{D}_k$, we have*

$$\left|e^{n\tau}\left(e^{j\varpi_k(\tau)} - \underline{\kappa}_k^j e^{j\varphi_k(\tau)}\right)\right| \leq Cn|\tau - \underline{\tau}_k|^{2\mu_k+1} \exp(n\Re(\tau - \underline{\tau}_k) + j(\Re(\varpi_k(\tau)) + |\Re(\xi_k(\tau)(\tau - \underline{\tau}_k)^{2\mu+1})|)).$$

Proof We define the holomorphic function $\mathcal{S}$ such that

$$\forall z \in \mathbb{C}, \quad \mathcal{S}(z) = \begin{cases} 1 & \text{if } z = 0, \\ \frac{\sinh(z)}{z} & \text{else.} \end{cases}$$

We consider $(n, j) \in \mathcal{D}_k$ and $\tau \in B_{\varepsilon}(\underline{\tau}_k)$. We have because $j \geq n\underline{\delta}_k \geq 0$,

$$\left|e^{n\tau}\left(e^{j\varpi_k(\tau)} - \underline{\kappa}_k^j e^{j\varphi_k(\tau)}\right)\right| = j|\xi_k(\tau)||\tau - \underline{\tau}_k|^{2\mu_k+1} \left|\mathcal{S}\left(j\frac{\xi_k(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+1}}{2}\right)\right| \exp\left(n\Re(\tau) + j\left(\Re(\varpi_k(\tau)) - \Re\left(\frac{\xi_k(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+1}}{2}\right)\right)\right).$$

We observe that the function $z \in \mathbb{C} \mapsto |\mathcal{S}(z)| \exp(-|\Re(z)|)$ is bounded. Therefore, because the function ξ_k can be bounded on $B_\varepsilon(\underline{\tau}_k)$, $j \leq n\bar{\delta}_k$ and $\Re(\tau) = \Re(\tau - \underline{\tau}_k)$,

$$\left| e^{n\tau} \left(e^{j\varpi_k(\tau)} - \underline{\kappa}_k^j e^{j\varphi_k(\tau)} \right) \right| \lesssim n|\tau - \underline{\tau}_k|^{2\mu_k+1} \exp\left(n\Re(\tau - \underline{\tau}_k) + j\left(\Re(\varpi_k(\tau)) + \left| \Re\left(\xi_k(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+1} \right) \right| \right) \right).$$

□

Proof of the Lemma 14

We recall here the statement of the Lemma 14.

Lemma 22. *For $(n, j) \in \mathbb{N}^* \times \mathbb{N}^*$ and $\tau \in \Gamma_{k,p}$, we have*

- *Case A: $\rho_k\left(\frac{\zeta_k}{\gamma_k}\right) \in \left[-\frac{\eta}{2}, \varepsilon_{k,0}\right]$*

$$n\Re(\tau - \underline{\tau}_k) + j(\Re(\varpi_k(\tau)) + \left| \Re\left(\xi_k(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+1} \right) \right|) \leq -nc_\star \Im(\tau - \underline{\tau}_k)^{2\mu_k} - \frac{n}{\alpha_k}(2\mu_k - 1)\gamma_k \left(\frac{|\zeta_k|}{\gamma_k} \right)^{\frac{2\mu_k}{2\mu_k - 1}},$$
$$n\Re(\tau - \underline{\tau}_k) + j\Re(\varpi_k(\tau)) \leq -nc_\star \Im(\tau - \underline{\tau}_k)^{2\mu_k} - \frac{n}{\alpha_k}(2\mu_k - 1)\gamma_k \left(\frac{|\zeta_k|}{\gamma_k} \right)^{\frac{2\mu_k}{2\mu_k - 1}}.$$

- *Case B: $\rho_k\left(\frac{\zeta_k}{\gamma_k}\right) > \varepsilon_{k,0}$*

$$n\Re(\tau - \underline{\tau}_k) + j(\Re(\varpi_k(\tau)) + \left| \Re\left(\xi_k(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+1} \right) \right|) \leq -\frac{n}{\alpha_k}(2\mu_k - 1)A_R\underline{\delta}_k\varepsilon_{k,0}^{2\mu_k},$$
$$n\Re(\tau - \underline{\tau}_k) + j\Re(\varpi_k(\tau)) \leq -\frac{n}{\alpha_k}(2\mu_k - 1)A_R\underline{\delta}_k\varepsilon_{k,0}^{2\mu_k}.$$

- *Case C: $\rho_k\left(\frac{\zeta_k}{\gamma_k}\right) < -\frac{\eta}{2}$*

$$n\Re(\tau - \underline{\tau}_k) + j(\Re(\varpi_k(\tau)) + \left| \Re\left(\xi_k(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+1} \right) \right|) \leq -\frac{n}{\alpha_k}(2\mu_k - 1)A_R\underline{\delta}_k \left(\frac{\eta}{2} \right)^{2\mu_k},$$
$$n\Re(\tau - \underline{\tau}_k) + j\Re(\varpi_k(\tau)) \leq -\frac{n}{\alpha_k}(2\mu_k - 1)A_R\underline{\delta}_k \left(\frac{\eta}{2} \right)^{2\mu_k}.$$

Proof We consider $(n, j) \in \mathbb{N}^* \times \mathbb{N}^*$ and $\tau \in \Gamma_{k,p}$. Using first the inequality (37), the fact that $\tau \in \Gamma_{k,p}$ and finally the inequality (41), we have

$$\begin{aligned} n\Re(\tau - \underline{\tau}_k) + j(\Re(\varpi_k(\tau)) + |\Re(\xi_k(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+1})|) &\leq n\Re(\tau - \underline{\tau}_k) - \frac{j}{\alpha_k}\Psi_k(\tau_p) \\ &\leq -nc_\star \Im(\tau - \underline{\tau}_k)^{2\mu_k} + \frac{n}{\alpha_k}\left(\gamma_k \tau_p^{2\mu_k} - 2\mu_k\zeta_k\tau_p \right). \end{aligned}$$

- First, we consider the case A. Then, we have $\tau_p = \rho_k\left(\frac{\zeta_k}{\gamma_k}\right)$. Therefore,

$$\gamma_k \tau_p^{2\mu_k} - 2\mu_k\zeta_k\tau_p = -(2\mu_k - 1)\gamma_k \left(\frac{|\zeta_k|}{\gamma_k} \right)^{\frac{2\mu_k}{2\mu_k - 1}} \leq 0. \tag{56}$$

We deduce (45). The inequality (46) is a direct consequence of (45).

- We consider the case B. Because $\tau_p = \varepsilon_{k,0}$, we have

$$n\Re(\tau - \underline{\tau}_k) + j(\Re(\varpi_k(\tau)) + |\Re(\xi_k(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+1})|) \leq \frac{n}{\alpha_k}\left(\gamma_k\varepsilon_{k,0}^{2\mu_k} - 2\mu_k\zeta_k\varepsilon_{k,0} \right).$$

We recall that $\rho_k\left(\frac{\zeta_k}{\gamma_k}\right) > \varepsilon_{k,0}$ and that $\rho_k\left(\frac{\zeta_k}{\gamma_k}\right)$ is the only real root of $-\zeta_k + \gamma_k x^{2\mu_k - 1} = 0$. Therefore, $-\zeta_k \leq -\gamma_k\varepsilon_{k,0}^{2\mu_k - 1}$ and

$$\gamma_k \tau_p^{2\mu_k} - 2\mu_k\zeta_k\tau_p \leq -(2\mu_k - 1)\gamma_k\varepsilon_{k,0}^{2\mu_k} \leq 0. \tag{57}$$

Using (42) to bound γ_k, we deduce the inequality (47). The inequality (48) is a direct consequence of (47).

• Finally, we place ourselves in case C. We have that $\tau_p = -\frac{\eta}{2}$, so

$$n\Re(\tau - \underline{\tau}_k) + j(\Re(\varpi_k(\tau)) + |\Re(\xi_k(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+1})|) \leq \frac{n}{\alpha_k}\left(\gamma_k\left(\frac{\eta}{2}\right)^{2\mu_k} + 2\mu_k\zeta_k\frac{\eta}{2}\right).$$

We recall that $\rho_k\left(\frac{\zeta_k}{\gamma_k}\right) < -\frac{\eta}{2}$ and that $\rho_k\left(\frac{\zeta_k}{\gamma_k}\right)$ is the only real root of $-\zeta_k + \gamma_k x^{2\mu_k-1} = 0$. Then, $\zeta_k \leq -\gamma_k\left(\frac{\eta}{2}\right)^{2\mu_k-1}$ and

$$\gamma_k\tau_p^{2\mu_k} - 2\mu_k\zeta_k\tau_p \leq -(2\mu_k - 1)\gamma_k\left(\frac{\eta}{2}\right)^{2\mu_k} \leq 0. \tag{58}$$

Using (42) to bound γ_k, we deduce the inequality (49). The inequality (50) is a direct consequence of (49). □

Proof of the Lemma 15

We recall here the statement of the Lemma 15.

Lemma 23. *For $(n, j) \in \mathbb{N}^* \times \mathbb{N}^*$ and $\tau \in \Gamma_{k,res}$, we have in all cases*

$$n\Re(\tau - \underline{\tau}_k) + j(\Re(\varpi_k(\tau)) + \left|\Re\left(\xi_k(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+1}\right)\right|) \leq -n\frac{\eta}{2},$$
$$n\Re(\tau - \underline{\tau}_k) + j(\Re(\varpi_k(\tau)) \leq -n\frac{\eta}{2}.$$

Proof We consider $(n, j) \in \mathbb{N}^* \times \mathbb{N}^*$ and $\tau \in \Gamma_{k,res}$. Using the inequality (37) and the facts that $\Im(\tau - \underline{\tau}_k)^{2\mu_k} \geq \ell_{k,p}^{2\mu_k}$ and $-\eta + i\ell_{k,p} + \underline{\tau}_k \in \Gamma_{k,p}$, we have

$$\begin{aligned} n\Re(\tau - \underline{\tau}_k) + j(\Re(\varpi_k(\tau)) + \left|\Re\left(\xi_k(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+1}\right)\right|) &\leq -n\eta - \frac{j}{\alpha_k}\left(-\eta - A_R\eta^{2\mu_k} + A_I\Im(\tau - \underline{\tau}_k)^{2\mu_k}\right) \\ &\leq -n\eta - \frac{j}{\alpha_k}\Psi_k(\tau_p). \end{aligned}$$

We know that $\eta + \tau_p \geq \frac{\eta}{2}$, so

$$-n\eta - \frac{j}{\alpha_k}(\tau_p - A_R\tau_p^{2\mu_k}) = -n(\eta + \tau_p) + \frac{n}{\alpha_k}\left(\gamma_k\tau_p^{2\mu_k} - 2\mu_k\zeta_k\tau_p\right) \leq -n\frac{\eta}{2} + \frac{n}{\alpha_k}\left(\gamma_k\tau_p^{2\mu_k} - 2\mu_k\zeta_k\tau_p\right).$$

We proved at the end of the proof of Lemma 14 that, in the three cases A, B and C, $\gamma_k\tau_p^{2\mu_k} - 2\mu_k\zeta_k\tau_p$ are non positive (see (56), (57) and (58)). This concludes the proof. □

Proof of the Lemma 16

We recall here the statement of the Lemma 16.

Lemma 24. *For $(n, j) \in \mathbb{N}^* \times \mathbb{N}^*$ and $\tau \in \Gamma_{k,out}$, we have in all cases*

$$n\Re(\tau - \underline{\tau}_k) + j\Re(\varphi_k(\tau)) \leq -n\frac{\eta}{2}.$$

Proof We consider $(n, j) \in \mathbb{N}^* \times \mathbb{N}^*$ and $\tau \in \Gamma_{k,out}$. Just as in the proof of Lemma 15, using the inequality (38) and using the fact that $\Im(\tau - \underline{\tau}_k)^{2\mu_k} \geq \ell_{k,p}^{2\mu_k}$ and $-\eta + i\ell_{k,p} + \underline{\tau}_k \in \Gamma_{k,p}$, we have

$$n\Re(\tau - \underline{\tau}_k) + j\Re(\varphi_k(\tau)) \leq -n\eta - \frac{j}{\alpha_k}\Psi_k(\tau_p).$$

We end the proof in the same way as for Lemma 15. □

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References

[1] A. C. Berry. The accuracy of the Gaussian approximation to the sum of independent variates. *Trans. Amer. Math. Soc.*, 49:122–136, 1941.

[2] J. B. Conway. *A course in functional analysis*, volume 96 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1990.

[3] J.-F. Coulombel and G. Faye. Generalized gaussian bounds for discrete convolution powers. preprint, 2021.

[4] J.-F. Coulombel and G. Faye. Sharp stability for finite difference approximations of hyperbolic equations with boundary conditions. *IMA Journal of Numerical Analysis*, 2021.

[5] B. Després. Finite volume transport schemes. *Numer. Math.*, 108(4):529–556, 2008.

[6] P. Diaconis and L. Saloff-Coste. Convolution powers of complex functions on $\mathbb{Z}$. *Math. Nachr.*, 287(10):1106–1130, 2014.

[7] C.-G. Esseen. On the Liapounoff limit of error in the theory of probability. *Ark. Mat. Astr. Fys.*, 28A(9):19, 1942.

[8] T. N. E. Greville. On stability of linear smoothing formulas. *SIAM J. Numer. Anal.*, 3(1):157–170, 1966.

[9] H.-O. Kreiss. Stability theory for difference approximations of mixed initial boundary value problems. I. *Math. Comp.*, 22:703–714, 1968.

[10] D. J. Newman. A simple proof of Wiener's $1/f$ theorem. *Proc. Amer. Math. Soc.*, 48:264–265, 1975.

[11] V. V. Petrov. *Sums of independent random variables*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 82. Springer-Verlag, New York-Heidelberg, 1975.

[12] E. Randles and L. Saloff-Coste. On the convolution powers of complex functions on $\mathbb{Z}$. *J. Fourier Anal. Appl.*, 21(4):754–798, 2015.

[13] D. W. Robinson. *Elliptic operators and Lie groups*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1991. Oxford Science Publications.

[14] W. Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987.

[15] I. J. Schoenberg. On smoothing operations and their generating functions. *Bull. Amer. Math. Soc.*, 59:199–230, 1953.

[16] G. Strang. On the construction and comparison of difference schemes. *SIAM J. Numer. Anal.*, 5:506–517, 1968.

[17] E. Tadmor. Complex symmetric matrices with strongly stable iterates. *Linear Algebra Appl.*, 78:65–77, 1986.

[18] V. Thomée. Stability of difference schemes in the maximum-norm. *J. Differential Equations*, 1:273–292, 1965.

[19] K. Zumbrun and P. Howard. Pointwise semigroup methods and stability of viscous shock waves. *Indiana Univ. Math. J.*, 47(3):741–871, 1998.