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## High-frequency leaky whispering-gallery modes in embedded elastic spheres

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The goal of this paper is to investigate the characteristics of high-frequency whispering-gallery modes in embedded elastic spheres, that is, surrounded by an infinite elastic matrix. Due to several modeling difficulties, the high-frequency regime of embedded spheres has remained unexplored in elasticity. Our approach consists of formulating a specific finite-element method in spherical coordinates. The basic idea is to discretize only the radial coordinate while describing analytically the angular distribution of elastodynamic fields. Then, we also introduce a radial perfectly matched layer to cope with the unbounded nature of the external medium. Our approach yields a linear matrix eigensystem, simple and costless to solve. In order to identify general trends, both stiff and soft configurations are considered, corresponding to a sphere stiffer and softer than the external medium, respectively. Including material loss, our results highlight the behavior of leaky elastic whispering-gallery modes in the high-frequency regime. This work is motivated by the well-known behavior of whispering-gallery modes in optical resonators, reaching high  $Q$  factors as the frequency increases. Identifying high- $Q$ -factor whispering-gallery modes in elastic spheres could find promising applications for sensing the mechanical properties of external media.

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### I. INTRODUCTION

In this paper, we address the computation of leaky resonances of embedded elastic spheres at high acoustical frequencies. Our aim is to investigate the characteristics of whispering-gallery modes (WGMs) with particular attention to their attenuation ( $Q$  factor), which has not been considered yet. This work is motivated by the well-known behavior of WGMs in optical resonators [1], reaching high  $Q$  factors as the frequency increases. High- $Q$ -factor optical WGMs have sustained the development of numerous optical sensors [2]. Identifying such modes in elastic spheres could therefore find many interesting applications, e.g., for sensing the mechanical properties of a media external to the sphere.

Resonances of elastic spheres have been largely studied *in vacuo*, first by Lamb in 1881 as a classical problem in mechanics [3]. This topic has then drawn the attention of geophysicists, using the normal modes of the Earth to analyze the internal structure of our planet [4,5]. More recently, elastic resonances of nanoparticles have been studied because of their significant role in Raman scattering [6,7], stimulated Brillouin scattering [8], or surface plasmonic resonances [9].

However, the case of embedded elastic spheres, i.e., buried in an external solid matrix, is more intricate than in a vacuum. As a consequence, this case has been far less considered in the literature, and the analyses are most often limited to quite low acoustical frequencies [10–14]. Indeed, the modeling of

an embedded elastic sphere can be described as an open resonator, which raises several difficulties.

First, the physics of open resonators strongly differ from closed systems: their spectrum involves a continuum of radiation modes, difficult to handle from a mathematical point of view. This continuum can be approximated by a discrete set, easier to manipulate, of so-called leaky modes (or quasinormal modes) [15]. Since energy leaks out of the sphere due to radiative loss, leaky resonances are damped in time: the eigenfrequencies are hence no longer real but necessarily complex. However, the behavior of leaky modes is somewhat unusual at infinity: while exponentially decreasing in time, these modes exponentially grow in the transverse direction [15,16] (that is, in the radial direction for our spherical problem).

Second, radiative loss dramatically increases in a low-frequency range, leading to a drastic drop of  $Q$  factors. This motivates the consideration of high-frequency regimes, i.e., normalized frequency greater than  $10^2$  in practice. However, high-frequency modes are difficult to solve by means of analytical methods because of the instabilities of secular equations [14,17,18].

To circumvent this problem, our approach consists of formulating a specific finite-element (FE) method in spherical coordinates. Full three-dimensional or two-dimensional FE models are prohibited due to their computational cost in the high-frequency regime [19]. Therefore, the basic idea is to consider a semianalytical approach, consisting of separating the angular and the radial variables. The solution along the angular variables is analytical, decomposed on the basis of vector spherical harmonics. The solution along the radial variable is discretized, approximated with one-dimensional FE (see Fig. 1). This semianalytical FE principle was applied

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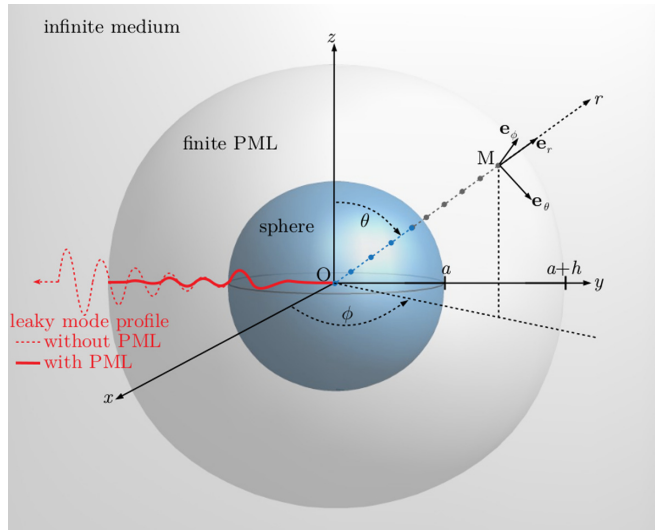


FIG. 1. Sphere of radius  $a$  embedded into an infinite medium truncated with a PML of thickness  $h$ . The PML cancels the natural growth of leaky modes in the radial direction  $r$ , discretized by one-dimensional FE (indicated by nodes). The spherical coordinates are the radius  $r$ , the polar angle  $\theta$  ( $0 \leq \theta < \pi$ ), and the azimuthal angle  $\phi$  ( $0 \leq \phi < 2\pi$ ).

to compute the resonances of *in vacuo* spheres in [20,21], but without taking full advantage of the analytical description leading to tedious integral calculus. Recently, a compact formulation has been proposed in [22]. The main difficulty to achieve such a compact form in elasticity is to uncouple the spherical harmonics, which requires orthogonality relations of tensor type (the wave equation is vectorial) [23].

The goal of this paper is twofold: extend the formulation of Ref. [22] to the embedded case and highlight the behavior of leaky WGMs in the high-frequency regime. Nevertheless, a major difficulty for numerical methods is raised with the unbounded nature of the problem. This difficulty is particularly severe due to the exponential growth of leaky modes. A powerful technique to bound the problem consists in using a perfectly matched layer (PML). A PML is based on the complex scaling of the unbounded coordinate [24], which cancels the growth of leaky modes and allows truncating the unbounded embedding medium to a user-defined finite thickness (see Fig. 1). Regardless of spherical problems, this technique has been used to compute resonances in fluid and optical open cavities (see Refs. [25–28], for instance).

In this paper, we propose to combine a semianalytical FE formulation in spherical coordinates with a radial PML in order to compute leaky modes in embedded elastic spheres (Sec. II). Our approach yields a linear eigenproblem, which is very simple and costless to solve, even in the high-frequency regime (the FE discretization is one-dimensional). Compared to analytical approaches, our formulation does not suffer from instabilities and is quite versatile (complex inhomogeneous spheres, e.g., made of several layers, can be readily considered).

Last but not least, the formulation remains applicable for viscoelastic materials (complex elastic constants are allowed). The intrinsic loss of materials is usually far greater in elas-

ticity than in optics. It is noteworthy that the effects of viscoelastic loss tend to increase with frequency. Therefore, in the high-frequency regime, including viscoelastic loss in addition to leakage loss appears to be essential for a proper understanding of the physics of leaky WGMs in elasticity. Numerical results will be presented in Sec. III.

## II. SEMIANALYTICAL FE METHOD IN COMPLEX SPHERICAL COORDINATES

### A. Elastodynamic weak form with radial PML

Let us consider an elastic sphere of radius  $a$  embedded into an infinite elastic matrix. With time-harmonic dependence  $e^{-i\omega t}$ , the displacement field  $\tilde{\mathbf{u}}(r, \theta, \phi) = [\tilde{u}_r(r, \theta, \phi), \tilde{u}_\theta(r, \theta, \phi), \tilde{u}_\phi(r, \theta, \phi)]^T$  (T denotes matrix transpose) is the solution of the weak form of elastodynamics [29]:

$$\int_{\tilde{V}} \delta \tilde{\boldsymbol{\epsilon}}^T \tilde{\boldsymbol{\sigma}} d\tilde{V} - \omega^2 \int_{\tilde{V}} \tilde{\rho} \delta \tilde{\mathbf{u}}^T \tilde{\mathbf{u}} d\tilde{V} = 0, \quad (1)$$

with  $d\tilde{V} = \tilde{r}^2 d\tilde{r} \sin \theta d\theta d\phi$  in the spherical frame depicted in Fig. 1. Using Voigt notation, the stress and strain vectors are  $\tilde{\boldsymbol{\sigma}} = [\tilde{\sigma}_{rr}, \tilde{\sigma}_{\theta\theta}, \tilde{\sigma}_{\phi\phi}, \tilde{\sigma}_{\theta\phi}, \tilde{\sigma}_{r\phi}, \tilde{\sigma}_{r\theta}]^T$  and  $\tilde{\boldsymbol{\epsilon}} = [\tilde{\epsilon}_{rr}, \tilde{\epsilon}_{\theta\theta}, \tilde{\epsilon}_{\phi\phi}, 2\tilde{\epsilon}_{\theta\phi}, 2\tilde{\epsilon}_{r\phi}, 2\tilde{\epsilon}_{r\theta}]^T$ . The stress-strain relationship is  $\tilde{\boldsymbol{\sigma}} = \tilde{\mathbf{C}}\tilde{\boldsymbol{\epsilon}}$ , where  $\tilde{\mathbf{C}}$  is the matrix of material properties. The external medium is assumed as homogeneous and isotropic. The sphere can be transversely isotropic and inhomogeneous (e.g., constituted by several layers). For transversely isotropic materials, the matrix depends on five independent coefficients and can be written as

$$\tilde{\mathbf{C}} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{23} + 2C_{44} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{23} + 2C_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{55} \end{bmatrix}. \quad (2)$$

The strain-displacement relationship is  $\tilde{\boldsymbol{\epsilon}} = \tilde{\mathbf{L}}\tilde{\mathbf{u}}$  with

$$\tilde{\mathbf{L}} = \mathbf{L}_r \frac{\partial}{\partial \tilde{r}} + \mathbf{L}_\theta \frac{\partial}{\tilde{r} \partial \theta} + \mathbf{L}_\phi \frac{\partial}{\tilde{r} \sin \theta \partial \phi} + \frac{1}{\tilde{r}} \mathbf{L}_1 + \frac{\cot \theta}{\tilde{r}} \mathbf{L}_2, \quad (3)$$

where

$$\mathbf{L}_r = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{L}_\theta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{L}_\phi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{L}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \mathbf{L}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (4)$$

Along the radius, we introduce a radial PML thanks to an analytic continuation [24] of the weak form (1) into the

143 complex transverse coordinate  $\tilde{r}$ :

$$\tilde{r}(r) = \int_0^r \gamma(\xi) d\xi. \quad (5)$$

144 The function  $\gamma(r)$  is a user-defined function, with  $\text{Im } \gamma(r) >$   
 145  $0$  in the PML region, which enables the absorption of outgoing  
 146 waves in the embedding medium. The PML is truncated to a  
 147 thickness  $h$  to obtain a reflectionless bounded problem. In this  
 148 paper, we choose a parabolic attenuation profile:

$$\gamma(r) = \begin{cases} 1, & \text{if } r \leq a, \\ 1 + 3(\hat{\gamma} - 1)\left(\frac{r-d}{h}\right)^2, & \text{if } r > a. \end{cases} \quad (6)$$

149 The PML interface is set at  $r = a$ ; that is, stick to the physical  
 150 interface between the sphere and the embedding medium, to  
 151 avoid the computation of spurious eigenvalues [26]. For con-  
 152 venience, we define the parameter  $\hat{\gamma} = \frac{1}{h} \int_d^{d+h} \gamma(\xi) d\xi$ , the  
 153 user-defined averaged value of  $\gamma(r)$  inside the PML. At the  
 154 end of the PML ( $r = a + h$ ), a Dirichlet boundary condition  
 155 is applied.

156 Finally, the weak form (1) can be transformed to go back  
 157 to the real radial direction  $r$  thanks to the following change of  
 158 variable  $\tilde{r} \mapsto r$ , for any function  $\tilde{g}(r)$ :

$$\begin{aligned} \tilde{g}(\tilde{r}) &= g(r), \quad d\tilde{r} = \gamma(r) dr, \\ \frac{\partial \tilde{g}}{\partial \tilde{r}} &= \frac{1}{\gamma(r)} \frac{\partial g}{\partial r}. \end{aligned} \quad (7)$$

## 159 B. Semianalytical FE formulation

160 Since the problem has been bounded by a finite PML, we  
 161 can now apply the same procedure as in Ref. [22] to obtain a  
 162 semianalytical FE formulation. For brevity, we recall the main  
 163 steps of the procedure in the following (readers are invited to  
 164 see Ref. [22] for further details).

165 The basic idea is to treat analytically the angular behavior  
 166 of the solution and to use numerical discretization along the  
 167 radial coordinate. Based on Refs. [30,31], the displacement  
 168 field can be decomposed into vector spherical harmonics:

$$\mathbf{u}(r, \theta, \phi) = \sum_{l \geq 0} \sum_{|m| \leq l} \mathbf{S}_l^m(\theta, \phi) \hat{\mathbf{u}}_l^m(r). \quad (8)$$

169 The matrix  $\mathbf{S}_l^m(\theta, \phi)$  concatenates the vector spherical har-  
 170 monics which describe the angular distribution of the three  
 171 components of the displacement field. This matrix is explicitly  
 172 given by

$$\mathbf{S}_l^m(\theta, \phi) = \begin{bmatrix} \mathbf{Y}_l^m(\theta, \phi) & 0 & 0 \\ 0 & \frac{\partial \mathbf{Y}_l^m(\theta, \phi)}{\partial \theta} & -\frac{\partial \mathbf{Y}_l^m(\theta, \phi)}{\sin \theta \partial \phi} \\ 0 & \frac{\partial \mathbf{Y}_l^m(\theta, \phi)}{\sin \theta \partial \phi} & \frac{\partial \mathbf{Y}_l^m(\theta, \phi)}{\partial \theta} \end{bmatrix}, \quad (9)$$

173 where  $\mathbf{Y}_l^m(\theta, \phi)$  corresponds to normalized spherical harmon-  
 174 ics of integer degree  $l$  and order  $m$  ( $|m| \leq l$ ) [32].  $l$  and  $m$  are  
 175 also called polar and azimuthal wave numbers, respectively.  
 176 The vector  $\hat{\mathbf{u}}_l^m(r) = [\hat{u}_l^m(r), \hat{v}_l^m(r), \hat{w}_l^m(r)]^T$  is the  $(l, m)$  co-  
 177 efficient of the vector spherical harmonic transform of the  
 178 physical field  $\mathbf{u}(r, \theta, \phi)$ . With a full analytical method, these  
 179 coefficients could be expressed as spherical Bessel functions  
 180 (for the interior problem with  $r \leq a$ ) and spherical Hankel  
 181 functions (for the exterior problem with  $r > a$ ) [17,31]. In-  
 182 stead, a one-dimensional FE approximation is used in this

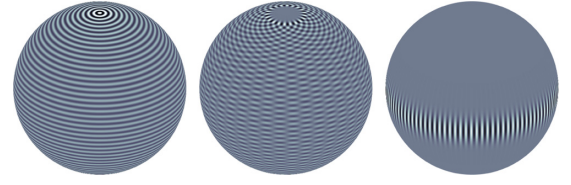


FIG. 2. Normal displacement distribution for the following spherical harmonics (from left to right):  $(l, m) = (100, 0)$ ,  $(l, m) = (100, 20)$ ,  $(l, m) = (100, 100)$ , corresponding to zonal, tesseral, and sectoral patterns, respectively (WGMs are of sectoral type).

paper, so that

$$\hat{\mathbf{u}}_l^{m,e}(r) = \mathbf{N}^e(r) \hat{\mathbf{U}}_l^{m,e}, \quad (10)$$

183 where  $\mathbf{N}^e(r)$  is the matrix of interpolating functions and  $\hat{\mathbf{U}}_l^{m,e}$   
 184 is the vector of degrees of freedom (dofs) at the element level  
 185 ( $e$ ).  
 186

187 To achieve the separation of angular and radial variables,  
 188 the key point is to properly choose the arbitrary test fields. We  
 189 choose the virtual displacements as

$$\delta \mathbf{U}^{eT}(r, \theta, \phi) = \delta \hat{\mathbf{U}}^{eT} \mathbf{N}^{eT}(r) \mathbf{S}_k^{p*}(\theta, \phi), \quad (11)$$

190 where the operation  $*$  stands for transpose conjugate. Thanks  
 191 to this choice, it can be shown that the orthogonality relations  
 192 of both vector [31] and tensor [23] spherical harmonics appear  
 193 in Eq. (1) when integrating over the angular coordinates (see  
 194 Ref. [22] for details). This choice is therefore fundamental to  
 195 get uncoupled equations for every spherical harmonics  $(l, m)$   
 196 of the displacement field. After lengthy calculations, the fol-  
 197 lowing compact matrix system is obtained for a given pair  
 198  $(l, m)$ :

$$[\mathbf{K}(l) - \omega^2 \mathbf{M}(l)] \hat{\mathbf{U}}_l^m = \mathbf{0}. \quad (12)$$

199 The detailed expressions of the elementary stiffness and mass  
 200 matrices are given in Appendix. Equation (12) is a linear  
 201 eigenproblem with respect to  $\omega^2$ . For a given value of  $l$ , it  
 202 yields  $n = 1, \dots, N$  eigenfrequencies  $\omega_l^{(n)}$  and eigenvectors  
 203  $\hat{\mathbf{U}}_l^{(n)}$ . The latter corresponds to radial mode shapes.

204 Regarding analytical approaches [17,30], two main fea-  
 205 tures are recovered by the matrix eigensystem (12). First, the  
 206 eigensystem can be subdivided into two independent linear  
 207 eigenproblems corresponding on the one hand to spheroidal  
 208 modes (with nonzero displacements in each direction), and  
 209 on the other hand to torsional modes (with zero displacement  
 210 in the radial direction). This can be easily deduced from the  
 211 structure of matrices in Eqs. (A1)–(A5).

212 Second, it can be noticed that the eigensystem is degenerate  
 213 with respect to the azimuthal wave number  $m$ . This means that  
 214  $2l + 1$  modes have the same eigenfrequency and the same  
 215 radial mode shape [33]. These multiple modes only differ  
 216 from their angular distribution.

217 Figure 2 shows the angular distribution of the normal  
 218 displacement  $\hat{u}_l^m$ , given by  $Y_l^m(\theta, \phi)$  [see Eq. (9)], for three  
 219 spherical harmonics:  $(l, m) = (100, 0)$ ,  $(l, m) = (100, 20)$ ,  
 220 and  $(l, m) = (100, 100)$ . The distributions corresponding to  
 221  $m = 0$ ,  $m \neq l$ , and  $m = l$  are the zonal, tesseral, and sectoral  
 222 patterns of spherical harmonics, respectively [31,32]. As ob-  
 223 served in Fig. 2, sectoral modes are confined at the equator



224 ( $\theta = \pi/2$ ) and are the so-called WGMs, which have found  
 225 many applications in optics [1,2,34]. Note that the distribu-  
 226 tion of WGMs gets narrower near the equator for increasing  
 227 wave numbers  $l = m$ . This means that narrow WGMs can be  
 228 achieved only at a high-frequency regime in practice.

### 229 C. Improved conditioning for large wave number

230 With Eq. (12), ill conditioning may result for large value  
 231 of  $l$ . This can be easily deduced from the expression of the  
 232 mass matrix, given by Eq. (A1), where terms of order 1 and  
 233  $l^2$  occur together in the diagonal inner matrix. To circumvent  
 234 this problem, we introduce the following transformation for  
 235  $l \neq 0$ :

$$\hat{\mathbf{u}}_l^m = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{l}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{l}} \end{bmatrix} \hat{\boldsymbol{\phi}}_l^m, \quad (13)$$

236 which merely consists in dividing by  $\sqrt{l}$  the angular compo-  
 237 nents of the transformed displacement field, denoted as  $\hat{\boldsymbol{\phi}}_l^m$ .

238 At the global FE level, we denote this transformation as  
 239 follows:

$$\hat{\mathbf{U}}_l^m = \mathbf{T}(l) \hat{\boldsymbol{\Phi}}_l^m, \quad (14)$$

240 where  $\mathbf{T}(l)$  is the global transformation matrix. Then, the  
 241 initial eigensystem (12) is transformed into

$$(\mathbf{T}^T \mathbf{K} \mathbf{T} - \omega^2 \mathbf{T}^T \mathbf{M} \mathbf{T}) \hat{\boldsymbol{\Phi}}_l^m = \mathbf{0}, \quad (15)$$

242 where the dependence on  $l$  of matrices has been dropped for  
 243 conciseness of notations.

244 With this transformation, it can be checked that the inner  
 245 diagonal terms of the element mass matrix (A1) are now all  
 246 transformed to unity. The eigensystem (15) hence remains  
 247 well conditioned for large  $l$ . Note also that the symmetry of  
 248 the eigensystem has been preserved.

### 249 D. Spectrum characteristics

250 As outlined in the introduction, the physics of open reso-  
 251 nators is significantly different from *in vacuo* systems.  
 252 Understanding their spectral behavior is essential to properly  
 253 exploit the numerical results.

254 We define  $k_{l,s_\infty}^2 = \omega^2/c_{l,s_\infty}^2$ , the shear and longitudinal  
 255 wave numbers in the infinite medium. Let us first briefly  
 256 recall some theoretical results by considering the unbounded  
 257 problem without introducing any PML. Theoretically, the  
 258 spectrum of an unbounded problem is constituted by a con-  
 259 tinuum of radiation modes and proper discrete poles [15,35–  
 260 37].

261 Proper discrete poles are referred to as trapped modes,  
 262 having a pure real resonance frequency if material loss is  
 263 neglected (infinite  $Q$  factor). They are located on the Riemann  
 264 sheet  $\text{Im}(k_{l,s_\infty}) \geq 0$ , where fields spatially decay at infinity.  
 265 The occurrence of trapped modes depends on the configu-  
 266 ration of the problem. Actually, such modes never exist in  
 267 spherical problems. This can be shown using Rellich's lemma  
 268 [38], stating that if the energy is zero far from the sphere then  
 269 the displacement is zero everywhere (that is, only the zero  
 270 eigenvalue satisfies the decaying wave condition at infinity).

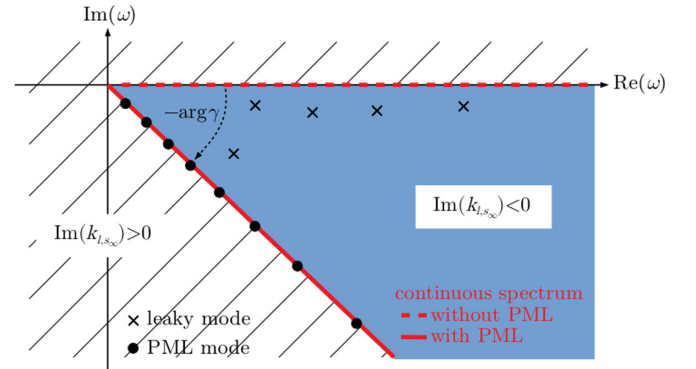


FIG. 3. Representation of the eigenspectrum of an open resonator in the complex frequency plane. Red dashed line: Continuum of radiation modes without PML [branch cuts  $\text{Im}(k_{l,s_\infty}) = 0$ ]. Red solid line: Continuum rotated by the infinite PML [branch cuts  $\text{Im}(\gamma k_{l,s_\infty}) = 0$ ]. Blue region: Portion of the initial improper Riemann sheets,  $\text{Im}(k_{l,s_\infty}) < 0$ , accessible thanks to the PML rotation and containing leaky modes (crosses). Black bullets indicate PML modes (discretization of the continuum caused by truncation of PML to a finite thickness).

This a major difference with flat problems, for which trapped waves can occur (such as Stoneley waves [39]).

As a consequence, only the continuum of radiation modes theoretically takes part in the solution of our problem. For elastic waves, this continuum corresponds to the contribution of two branch cuts given by  $\text{Im}(k_{l,s_\infty}) = 0$ . These two branch cuts define a fourfolded Riemann surface for the square roots of  $k_{l,s_\infty}^2$  (multivalued functions), according to the sign of  $\text{Im}(k_{l_\infty})$  and  $\text{Im}(k_{s_\infty})$ . Figure 3 depicts the branch cuts in the complex frequency plane. Both branch cuts coincide with each other for pure real bulk wave speeds  $c_{l,s_\infty}$  (in the case of viscoelastic materials, the imaginary part of wave speeds is yet relatively small compared to the real part so that both branch cuts remain almost coincident in practice).

Actually, a second type of discrete mode occurs, corresponding to complex-valued poles located in the improper Riemann sheets [i.e.,  $\text{Im}(k_{l_\infty}) < 0$  and/or  $\text{Im}(k_{s_\infty}) < 0$ ]. These improper eigenvalues are the so-called leaky modes (or quasinormal modes). They do not satisfy the spatially decaying wave condition at infinity, and hence, do not theoretically contribute to the exact solution. However, leaky modes can be conveniently used to approximate the continuum of radiation modes as a discrete sum. Besides, they reveal key information, hidden inside the continuum, about wave properties (group velocity, attenuation, etc.) [40]. Note that, because trapped modes do not exist in spherical problems, the WGMs of embedded spheres are indeed necessarily leaky modes.

Now, let us introduce an infinite PML. As demonstrated in Refs. [26,37], this further modifies the eigenspectrum. For simplicity, we assume a constant attenuation function  $\gamma$  inside the PML. Introducing an infinite PML changes the branch cuts to  $\text{Im}(\gamma k_{l,s_\infty}) = 0$ . This corresponds to a branch-cut rotation in the complex-frequency plane by the angle of rotation  $-\arg \gamma$  (see Fig. 3). With a PML, the proper Riemann surface is now given by  $\text{Im}(\gamma k_{l,s_\infty}) \geq 0$ . This surface contains leaky modes, revealed by the branch-cut rotation (blue region in

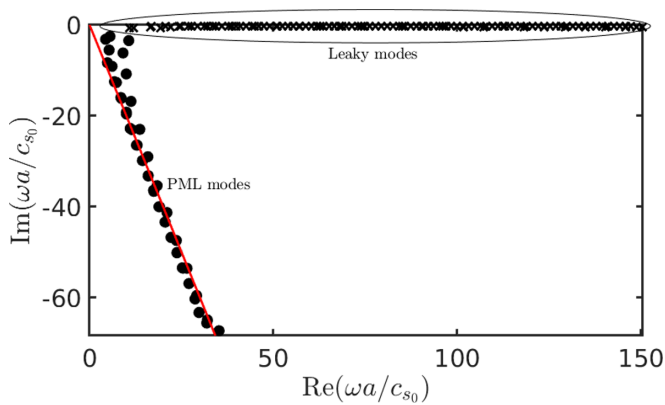


FIG. 4. Spectrum computed for a steel sphere into concrete ( $l = 10$ ,  $\hat{\gamma} = 1 + 2i$ ,  $h = 0.25a$ ). Black bullets: PML modes; black crosses: leaky modes identified from the criterion given by Eq. (16). Red line: Theoretical branch cut (rotated by  $-\arg \hat{\gamma}$ ).

continuous sets are discretized by the finite PML, yielding PML modes which are close to the theoretical branch cuts (rotated by  $-\arg \hat{\gamma}$ ). Leaky modes have a small imaginary part (small leakage) and are clearly distinguished from PML modes. The criterion given by Eq. (16) allows an efficient and straightforward separation of leaky and PML modes.

### F. Remarks on mode normalization

From a theoretical point of view, the exponential growth of leaky modes raises normalization issues. As proved in optics (see, e.g., [15]), a PML solves this intricate problem by transforming the divergent field of leaky modes into a decaying field.

Actually, a PML provides normalization and orthogonality for both leaky resonances and PML modes. Considering Eq. (12) and owing to the symmetry of the complex-valued stiffness and mass matrices, the following orthogonality relationships holds:

$$\hat{\mathbf{U}}_l^{(j)\text{T}} \mathbf{K}(l) \hat{\mathbf{U}}_l^{(i)} = \omega_l^{(i)2} \delta_{ij}, \quad (17)$$

$$\hat{\mathbf{U}}_l^{(j)\text{T}} \mathbf{M}(l) \hat{\mathbf{U}}_l^{(i)} = \delta_{ij}, \quad (18)$$

where the superscripts  $m$  have been dropped for conciseness of notations (and since the eigensystem is degenerate with respect to  $m$ ). Based on modal expansions, these orthogonality relationships can be used to compute the forced response of spheres to optimize the generation of WGMs. Details about this type of calculation can be found in Ref. [22] for *in vacuo* spheres (i.e., without PML). With a PML, the result remains unchanged because the relationships (18) still hold in both cases. For paper conciseness, the presentation of the computation process of the forced response is not repeated here.

## III. RESULTS

This section presents numerical results computed with the semianalytical FE formulation. In particular, we investigate the high-frequency behavior of leaky WGMs in embedded spheres.

As outlined in the introduction, it is of importance to take into account the material loss. For simplicity, we will consider a viscoelastic frequency independent hysteretic model. With this model, the bulk wave velocities  $\tilde{c}_l$  and  $\tilde{c}_s$  of the materials are complex and given by

$$\tilde{c}_{l,s} = c_{l,s} \left( 1 + i \frac{\kappa_{l,s}}{2\pi} \right)^{-1}, \quad (19)$$

where  $\kappa_l$  and  $\kappa_s$  denote the bulk wave attenuations in nepers per wavelength.

Two test cases are considered. The first case is that of a stiff sphere, that is, a sphere material with bulk wave velocities greater than the external medium. The second test case is that of a soft sphere (bulk wave velocities lower than the external medium). Results computed for embedded spheres are compared with those for *in vacuo* spheres. A thorough investigation of these configurations allows the identification of general trends about the behavior of leaky WGMs.

Fig. 3). The number of revealed leaky resonances depends on the argument of  $\gamma$ .

Finally, let us truncate the infinite PML to a finite thickness  $h$ . Truncation induces discretization of branch cuts: the continuum of radiation modes is transformed into a discrete set of radiation modes [26,41], with finer discretization as the PML thickness increases [15,27,40]. These modes are often called PML modes (see Fig. 3). They resonate mainly inside the PML and are hence not intrinsic to the physics (they strongly depend on the user-defined PML parametrization). Although a solution of the eigenvalue problem, PML modes are therefore of little interest for the analysis of this paper.

### E. Mode filtering

As a postprocessing step, it is necessary to filter out PML modes from the solution in order to properly visualize only the leaky WGMs. Following Refs. [42,43], leaky modes can be identified efficiently from the following energy-based criterion:

$$1 - \frac{\text{Im}(E_l^m)}{|E_l^m|} > \eta_{\min}, \quad (16)$$

where  $\eta_{\min}$  is a user-defined threshold and  $E_l^m = \frac{1}{2} |\omega_l^2| \hat{\mathbf{U}}_l^{m*} \mathbf{M} \hat{\mathbf{U}}_l^m$ .  $E_l^m$  can be interpreted as the kinetic energy integrated over the radius, including the PML. The imaginary part of this energy is expected to be much greater inside the PML for PML modes than for leaky resonances (which are expected to resonate mostly inside the sphere).

As an example, Fig. 4 shows a typical spectrum computed with the semianalytical FE formulation. The frequency is normalized as  $\omega a/c_{s0}$ , where  $c_{s0}$  denote the shear bulk wave velocity of the sphere material. The test case, a steel sphere buried into concrete, is described later in the next section. As explained earlier at the end of Sec. II B, the whole spectrum consists of two separate sets of modes, spheroidal and torsional modes (both types of modes have been computed together and are not distinguished in the figure). The latter are related to shear waves only [44] and give only one continuous set. The former are a mixture of longitudinal and shear waves [44] and give two continuous sets. As observed in Fig. 4, these

TABLE I. Material properties.

Material	$\rho$ (kgm <sup>-3</sup> )	$c_l$ (ms <sup>-1</sup> )	$c_s$ (ms <sup>-1</sup> )	$\kappa_l$ (Np $\lambda^{-1}$ )	$\kappa_s$ (Np $\lambda^{-1}$ )
Steel <sup>a</sup>	7932	5500.7	3175.8	0.003	0.008
Concrete <sup>b</sup>	2152	3758	2090	0.186	0.229
Epoxy <sup>c</sup>	1600	2960	1450	0.029	0.043

<sup>a</sup>Properties taken from Ref. [45].

<sup>b</sup>Properties taken from Ref. [46].

<sup>c</sup>Properties taken from Ref. [47].

### A. Model parameters

A concrete medium is chosen for the embedment. The sphere is made of steel in the stiff case and epoxy in the soft case. Material parameters are given in Table I. A perfectly bonded interface is assumed between the sphere and the external matrix (i.e., continuity of stress and displacement in the three directions).

The PML parameters are set to  $h = 0.25a$ ,  $\hat{\gamma} = 1 + 2i$ . This choice follows from a convergence study (not shown for conciseness), by varying both  $h$  and  $\hat{\gamma}$ . The radial coordinate is meshed with one-dimensional quadratic FE of length  $0.005a$ , yielding 1500 degrees of freedom (dofs) (reduced to 1200 dofs in the *in vacuo* case). The eigenproblem (12) is solved repeatedly for a wide range of wave numbers,  $l = 0, \dots, 150$ . In order to compute mainly high-frequency leaky modes, the normalized eigenvalues  $\omega a/c_{s_0}$  are solved around a user-defined shift equal to  $1.2l$  in the eigenvalue solver (see ARPACK library [48]).

Dispersion curves will be given as a function of  $l$  for the phase velocity  $v_p$ , the group velocity  $v_g$  [30], and the  $Q$  factor, defined by

$$v_p = \frac{\text{Re}(\omega_l^{(n)})a}{l + 1/2}, \quad v_g = \text{Re}\left(\frac{\partial \omega_l^{(n)}}{\partial l}\right)a, \quad Q = -\frac{\text{Re}(\omega_l^{(n)})}{2\text{Im}(\omega_l^{(n)})}. \quad (20)$$

The group velocity can be postprocessed as follows (see Ref. [22] for proof):

$$v_g = \text{Re}\left\{ \frac{\hat{\mathbf{U}}_l^{(n)\text{T}} \left( \frac{\partial \mathbf{K}(l)}{\partial l} - \omega_l^{(n)2} \frac{\partial \mathbf{M}(l)}{\partial l} \right) \hat{\mathbf{U}}_l^{(n)}}{2\omega_l^{(n)} \hat{\mathbf{U}}_l^{(n)\text{T}} \mathbf{M}(l) \hat{\mathbf{U}}_l^{(n)}} \right\} a. \quad (21)$$

Depending on the configuration, the parameter  $\eta_{\min}$  for filtering PML modes has been set between 0.8 and 0.9.

### B. Case 1: Stiff-sphere configuration

Figure 5 shows the dispersion curves of spheroidal modes for the stiff configuration (steel into concrete), as well as for the *in vacuo* steel sphere. The dispersion curves for torsional modes are shown in Fig. 6.

As a general trend, the phase velocity tends toward  $c_{s_0}$ , the shear wave bulk velocity of the sphere, as  $l$  increases (i.e., in the high-frequency regime), for all modes except the fundamental spheroidal modes  $n = 0$  (this particular mode will be discussed later). This velocity limit can be more clearly observed for the group velocity. WGMs have hence a nondispersive behavior as the frequency increases. Besides, both

phase and group velocities with the embedment remain close to their *in vacuo* counterparts.

Compared to torsional modes [Fig. 6(b)], the nonmonotonic changes observed in the group velocities of spheroidal modes [Fig. 5(b)] can be explained by the complex body wave combination of P waves and SV waves traveling through the sphere, reflected and converted several times at the surface, and fulfilling constructive interference conditions [5,44]. As the polar wave number  $l$  increases, the spheroidal modes become mainly formed from a combination of multiple reflected SV waves (the polar displacement  $u_\theta$  nearly goes to zero), which explains why they tend toward a shear wave behavior. These modes are the so-called transverse spheroidal modes, observed for a sphere in vacuum [8], for which the SV-wave content prevails. Compared to spheroidal modes, torsional modes rather tend monotonically toward the shear wave behavior because torsional modes are formed from multiple reflected horizontal shear waves (SH waves) only (no conversion) [5,44].

The  $Q$  factor behaves completely differently for the embedded sphere [see Figs. 5(c) and 6(c)]. *In vacuo*, the  $Q$  factor of spheroidal modes quickly decreases toward  $Q_{s_0} = \pi/\kappa_{s_0}$ , the  $Q$  factor of shear waves inside the sphere. The  $Q$  factor of torsional modes remains constant and equal to  $Q_{s_0}$ . Conversely, with embedment, the  $Q$  factor is weak in the low-frequency regime and then slowly increases in the high-frequency regime, up to  $Q_{s_0}$  according to the shear wave limit.

These results are consistent with the competing effects of viscoelasticity and leakage found in embedded elastic waveguides [45]: as the frequency increases, the attenuation due to viscoelasticity tends to increase while the attenuation due to leakage tends to decrease. Note that if viscoelasticity was neglected ( $\kappa_s \rightarrow 0$ ),  $Q$  would tend to infinity in the high-frequency regime, which would be unrealistic (hence the importance of taking into account material loss).

In a stiff configuration, the  $Q$  factor tends to improve as the mode order  $n$  grows. This trend is particularly visible for torsional modes, although less significant for spheroidal modes. It can be explained by the energy distribution, concentrated near the interface for low-order modes, hence increasing their leakage in the external medium.

As far as the fundamental spheroidal mode  $n = 0$  is concerned (see Fig. 5), its *in vacuo* characteristics tend toward those of the Rayleigh wave in the high-frequency regime. In a half space, the Rayleigh wave speed can be approximated by  $\tilde{c}_{r_0} \simeq \tilde{c}_{s_0}(0.862 + 1.14\tilde{\nu}_0)/(1 + \tilde{\nu}_0)$  [where  $\tilde{\nu}_0 = 0.5(\tilde{c}_{l_0}^2 - 2\tilde{c}_{s_0}^2)/(\tilde{c}_{l_0}^2 - \tilde{c}_{s_0}^2)$  is the complex Poisson ratio], yielding a quality factor roughly equal to  $Q_{r_0} \simeq -0.5\text{Im}(\tilde{c}_{r_0})/\text{Re}(\tilde{c}_{r_0})$ .



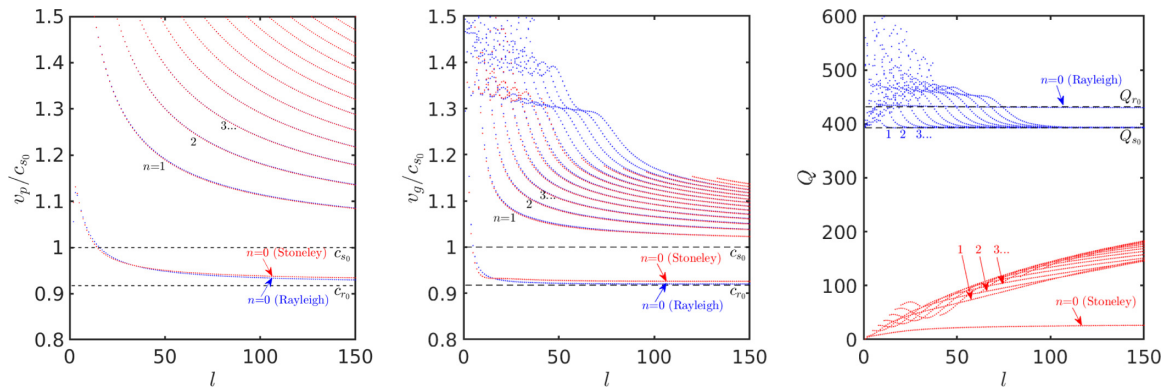


FIG. 5. Stiff-sphere configuration, spheroidal modes. From left to right: Phase velocity, group velocity, and  $Q$  factor as a function of the polar wave number  $l$  ( $\eta_{\min} = 0.8$ ). Red: Steel into concrete; blue: steel in vacuum.

478 Embedded in concrete, the fundamental spheroidal mode  
 479 corresponds to a solid-solid interface wave, which hence tends  
 480 toward the behavior of a Stoneley wave between two elastic  
 481 half spaces. Note that its phase velocity remains higher than  
 482 the shear bulk wave velocities so that this Stoneley wave is of  
 483 leaky type (attenuating). The  $Q$  factor of the  $n = 0$  Stoneley-  
 484 like wave is very low compared to the other modes. Such a  
 485 wave hence appears of less interest for generating WGMs.

486 The main conclusion is that the  $Q$  factor of leaky WGMs  
 487 cannot exceed  $\pi/\kappa_{s0}$ , the quality factor of shear waves inside  
 488 the material constituting the sphere ( $\simeq 400$  for steel). The  $Q$   
 489 factor reached by elastic WGMs is hence far weaker than in  
 490 optics ( $Q \approx 10^9$  for  $l \approx 1000$  in practice).

491 Additional numerical tests have been conducted to further  
 492 explore the behavior of leaky WGMs (results not shown for  
 493 conciseness). By increasing the contrast between materials,  
 494 the convergence of the  $Q$  factor toward  $Q_{s0}$  for large  $l$  becomes  
 495 faster (this has been observed by artificially multiplying the  
 496 density of steel by 10). As far as Stoneley waves are con-  
 497 cerned, their behavior is more complex than Rayleigh waves.  
 498 Their existence depends on material combination. This has  
 499 been checked by artificially dividing the density of steel by  
 500 10. In this case, no Stoneley-like mode has been found. This  
 501 is consistent with the theoretical findings of Ref. [49] (the  
 502 Stoneley wave disappears if the material of greater density has  
 503 a velocity much lower).

C. Case 2: Soft-sphere configuration

504 A soft-sphere configuration offers a stronger analogy with  
 505 WGMs in optics. In optics, total internal reflection can be ob-  
 506 tained when light propagates faster in the embedding medium  
 507 than in the sphere. In elasticity, total internal reflection is  
 508 more complicated to obtain because of multiple reflection,  
 509 transmission, and conversion of three kinds of waves (P, SV,  
 510 and SH waves) [50]. Yet, we can expect a behavior similar to  
 511 light when both shear wave and longitudinal wave velocities  
 512 are smaller in the sphere than in the embedding medium, that  
 513 is, in the case of a soft inclusion into a stiffer matrix.

514 For a steel sphere (stiff configuration), there was a critical  
 515 angle only for the conversion of SV waves into P waves, there-  
 516 fore leading to significant leakage loss and a slow increase of  
 517 the  $Q$  factor. Let us replace steel with epoxy (soft configura-  
 518 tion). The shear and longitudinal bulk waves of epoxy are  
 519 slower than in the embedding medium, so that both waves in-  
 520 deed admit critical angles in reflection [50] (although P-wave  
 521 reflection into a reflected SV wave can still occur).

522 Figure 7 shows the dispersion curves of spheroidal modes  
 523 for the soft configuration, as well as for the *in vacuo* epoxy  
 524 sphere. The dispersion curves for torsional modes are shown  
 525 in Fig. 8.

526 Several differences can be observed compared to the stiff  
 527 configuration. First, no Stoneley-like wave is found for this  
 528 material combination (as shown in Fig. 7, a Rayleigh wave is  
 529

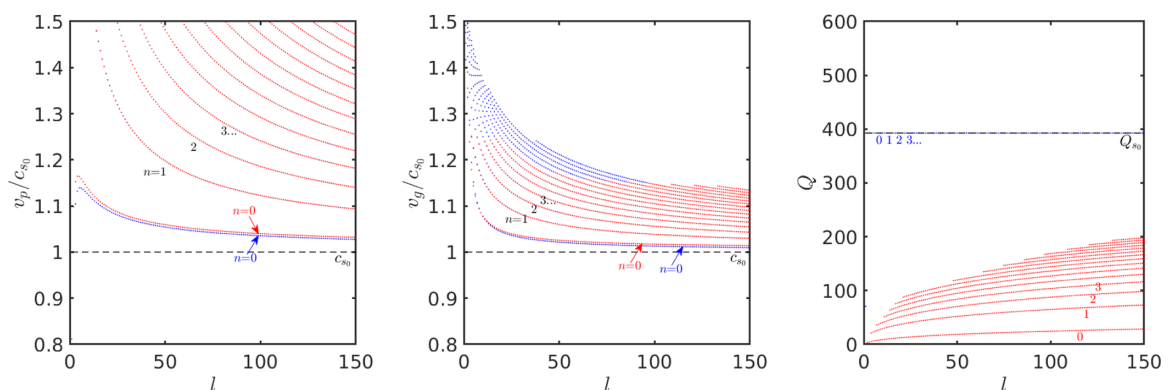


FIG. 6. Same caption as Fig. 5 but for torsional modes.

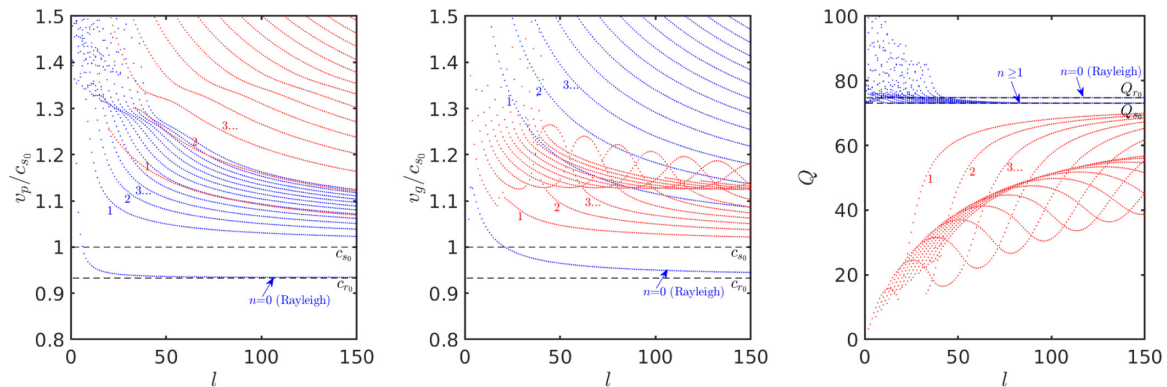


FIG. 7. Soft-sphere configuration, spheroidal modes. From left to right: Phase velocity, group velocity, and  $Q$  factor as a function of the polar wave number  $l$  ( $\eta_{\min} = 0.8$ ). Red: Epoxy into concrete; blue: epoxy in vacuum.

530 found *in vacuo* but no Stoneley wave is found in the embedded  
 531 sphere). Second, the velocities (both phase and group) are  
 532 significantly different from their *in vacuo* counterparts. Third,  
 533 the  $Q$  factor tends toward  $Q_{s_0}$  far more rapidly than in the  
 534 stiff configuration. Fourth, for a given  $l$ , the  $Q$  factor tends  
 535 to decrease as the mode order  $n$  increases (as opposed to the  
 536 stiff configuration). This behavior is quite similar to those of  
 537 WGMs in optics [1,51].

538 As a side remark, it can be observed that the group velocity  
 539 of spheroidal modes [Fig. 7(b)] changes in a strong nonmono-  
 540 tonic manner, with maxima of group velocities coinciding  
 541 with minima of  $Q$  factors (this phenomenon also occurs in  
 542 Fig. 5 but is less visible). These maxima correspond to a polar-  
 543 ization of spheroidal modes in the azimuthal direction. Such  
 544 modes correspond to the so-called pseudolongitudinal modes  
 545 of a sphere in vacuum [8], for which the P-wave contribution  
 546 is significant. The  $Q$  factor of pseudolongitudinal modes and  
 547 torsional modes strongly depends on the bonding condition of  
 548 the interface, as will be briefly discussed in Sec. III D.

549 The main result with this test case is that the  $Q$  factor  
 550 can quickly tend toward  $Q_{s_0}$  in a soft-sphere configuration.  
 551 However, this advantage is counterbalanced by the fact that  
 552 quality factors are usually significantly weaker in soft than in  
 553 stiff materials in practice ( $Q_{s_0} < 100$  for epoxy).

#### D. Perfectly sliding interface

554 To further illustrate the influence of the polarization of  
 555 WGMs, we consider the case of a perfectly sliding interface.  
 556 Only the continuity of normal stress and normal displacement  
 557 is therefore allowed at the interface between the sphere and  
 558 the external medium. The tangential components of the dis-  
 559 placement,  $u_\theta$  and  $u_\phi$ , are free.

560 With a sliding interface, note that the torsional modes are  
 561 no longer coupled to the external medium. Their character-  
 562 istics are exactly the same as *in vacuo* (in particular, the  $Q$   
 563 factor of torsional modes remains constant and equal to  $Q_{s_0}$ );  
 564 see blue curves in Figs. 6 and 8.

565 Conversely, the behavior of spheroidal modes is strongly  
 566 affected. Figures 9 and 10 show the  $Q$  factor computed with  
 567 a sliding interface in the stiff and soft configurations, respec-  
 568 tively. As opposed to a perfectly bonded interface, the  
 569  $Q$  factor can reach high values in a low-frequency range.  
 570 These values appear to be close to the Rayleigh  $Q$  factor.  
 571 This striking behavior can be explained by the polarization of  
 572 spheroidal modes in the low-frequency regime. As mentioned  
 573 previously, spheroidal modes in this frequency regime corre-  
 574 spond to pseudolongitudinal modes, mainly oriented along the  
 575 azimuthal direction, which minimizes their sensitivity to the  
 576 external medium.  
 577

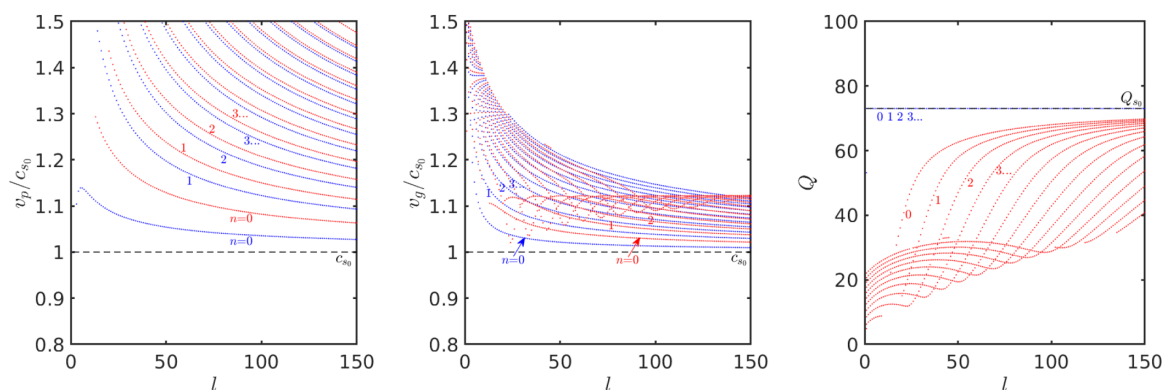


FIG. 8. Same caption as Fig. 7 but for torsional modes ( $\eta_{\min} = 0.9$ ).

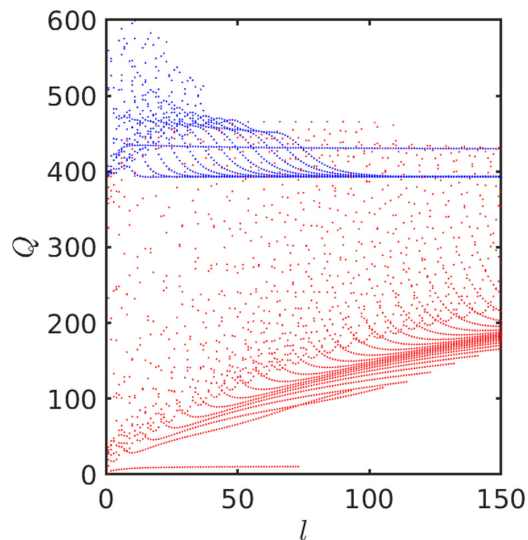


FIG. 9. Same caption as Fig. 5 but for a sliding interface (for conciseness, only the  $Q$  factor of spheroidal modes is shown).

#### IV. CONCLUSION

578

579 In this paper, we have investigated the characteristics of  
 580 high-frequency whispering-gallery modes in embedded elas-  
 581 tic spheres. For this purpose, a specific FE method in spherical  
 582 coordinates has been proposed. With this method, only the  
 583 radial coordinate is discretized while the angular distribution  
 584 of elastodynamic fields is described analytically. Besides, a  
 585 radial PML is introduced to cope with the unbounded nature  
 586 of the problem. Our approach leads to a linear matrix eigen-  
 587 system, simple and fast to solve in the high-frequency regime.  
 588 The solution of the eigenproblem delivers leaky and PML  
 589 modes. The leaky modes are revealed thanks to the PML, by  
 590 the rotation of the continuum in the complex frequency plane,  
 591 allowing access to the improper Riemann sheets. The PML  
 592 modes, related to the continuum of radiation modes, are not

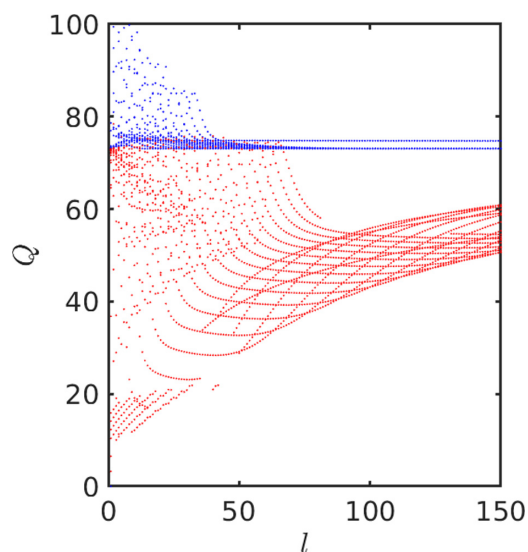


FIG. 10. Same caption as Fig. 7 but for a sliding interface (for conciseness, only the  $Q$  factor of spheroidal modes is shown).

593 intrinsic to the physics but can be efficiently filtered out, in a  
 594 postprocessing step, thanks to an energy-based criterion. Be-  
 595 sides, the PML can provide normalization and orthogonality  
 596 for the leaky modes, which allows the computation of forced  
 597 responses based on modal expansions.

598 Two configurations have been considered, corresponding  
 599 to a stiff sphere and a soft sphere. Material loss has been  
 600 included by considering a viscoelastic hysteretic model. Our  
 601 results highlight the behavior of leaky elastic WGMs in the  
 602 high-frequency regime, not yet considered in the literature.  
 603 Except for the Stoneley-like fundamental spheroidal mode  
 604 (peculiar to elasticity but of low  $Q$  factor), the modal char-  
 605 acteristics of leaky WGMs tends, as the frequency increases,  
 606 toward the shear wave properties of the material constituting  
 607 the sphere. As opposed to leakage, the material viscoelasticity  
 608 tends to increase the attenuation of waves with frequency,  
 609 leading to relatively low  $Q$  factor limits.

610 In particular, it has been shown that the  $Q$  factor of leaky  
 611 WGMs tends to slowly increase up to the quality factor of the  
 612 shear bulk waves of the sphere. Owing to internal reflection  
 613 mechanisms and by analogy with optics, this slow increase  
 614 can be enhanced in the case of a soft-sphere configuration.  
 615 However, this advantage is counterbalanced by the fact that  
 616 quality factors are usually weaker in soft than in stiff materi-  
 617 als.

618 Elastic WGMs in spheres could be of prime interest for  
 619 sensing the mechanical properties of an external medium,  
 620 e.g., by measuring a shift of their eigenfrequency [52]. More  
 621 generally, we expect that elastic WGMs could be optimized  
 622 in a more complex configuration, considering multilayered  
 623 spheres for instance. The formulation proposed in this paper is  
 624 versatile and allows modeling such complex spheres without  
 625 difficulties.

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 634 manuscript.  
 635

#### APPENDIX: FE MATRICES

637 In this Appendix, we give the detailed expression of the FE  
 638 matrices involved in Eq. (12).

639 The elementary mass matrix is given by

$$\mathbf{M}^e(l) = \int \rho \mathbf{N}^{eT} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \bar{l} & 0 \\ 0 & 0 & \bar{l} \end{bmatrix} \mathbf{N}^e r^2 \gamma dr. \quad (\text{A1})$$

640 The stiffness matrix can be written as

$$\mathbf{K}(l) = \mathbf{K}_1(l) + \mathbf{K}_2(l) + \mathbf{K}_2^T(l) + \mathbf{K}_3(l), \quad (\text{A2})$$

641 with

$$\mathbf{K}_1^e(l) = \int \frac{d\mathbf{N}^{eT}}{dr} \begin{bmatrix} C_{11} & 0 & 0 \\ 0 & \bar{l}C_{55} & 0 \\ 0 & 0 & \bar{l}C_{55} \end{bmatrix} \frac{d\mathbf{N}^e}{dr} \frac{\tilde{r}^2}{\gamma} dr, \quad (\text{A3})$$

$$\mathbf{K}_2^e(l) = \int \frac{d\mathbf{N}^{eT}}{dr} \begin{bmatrix} 2C_{12} & -\bar{l}C_{12} & 0 \\ \bar{l}C_{55} & -\bar{l}C_{55} & 0 \\ 0 & 0 & -\bar{l}C_{55} \end{bmatrix} \mathbf{N}^e \tilde{r} dr, \quad (\text{A4})$$

$$\mathbf{K}_3^e(l) = \int \mathbf{N}^{eT} \begin{bmatrix} \bar{l}C_{55} + 4(C_{23} + C_{44}) & -\bar{l}[C_{55} + 2(C_{23} + C_{44})] & 0 \\ -\bar{l}[C_{55} + 2(C_{23} + C_{44})] & \bar{l}^2 C_{23} + \bar{l}C_{55} + 2\bar{l}(\bar{l} - 1)C_{44} & 0 \\ 0 & 0 & \bar{l}C_{55} + \bar{l}(\bar{l} - 2)C_{44} \end{bmatrix} \mathbf{N}^e \gamma dr. \quad (\text{A5})$$

642 The above expressions differ from those of Ref. [22] due to  
643 the introduction of the PML, which yields complex-valued

matrices. Without PML ( $\gamma = 1$ ,  $\tilde{r} = r$ ), the expressions well  
degenerate to the *in vacuo* case treated in [22].

644  
645

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