

Pareto-Optimality and Qualitative Aggregation Structures

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Among non-additive, ordinal methods for criteria aggregation and decision under uncertainty, some have their origin in an approach first proposed by Bellman and Zadeh in 1970. Instead of maximising sums of degrees of satisfaction pertaining to various criteria, they proposed to maximise the minimum of such degrees, thus leading to a calculus of fuzzy constraints, for instance [1]. Unfortunately, rankings of solutions using such qualitative techniques are usually rather coarse. This drawback seems to undermine the merits of qualitative techniques, whose appeal is to obviate the need for quantifying utility functions. Worse, some of the generally not unique maximin optimal solutions, may fail to be Pareto Optimal. Besides, other well-behaved aggregation operations on finite ordinal scales seem to be constant on significant subsets of their domains [4], which make these aggregations not so attractive in practice.

This work starts an investigation of some limitations of finitely-scaled methods for criteria aggregation, and a search for remedies to these limitations. Given a finite, totally ordered set (X, \geq) with top 1 and bottom 0, consider an aggregation function $f : X^n \rightarrow X$, which, by definition, is increasing in the wide sense and such that $f(1, 1 \dots 1) = 1$ and $f(0, \dots 0) = 0$. It can be shown that maximising f over a subset $S \subseteq X^n$ of n -tuples generally leads to a maximising set that contains non Pareto - optimal solutions. This fact is rather unsurprising since using f as a ranking function comes down to sorting $|X|^n$ elements into $|X|$ sets of equally ranked n -tuples.

Clearly it shows that the discriminating power of qualitative aggregation operations is bound to be unacceptably weak and intuitively debatable. One way out of this difficulty may be to use functions from X^n to a bigger finite scale Y . However, this idea is not satisfactory from a practical point of view since the combinatorics of functions from X^n to Y become rapidly prohibitive as Y is bigger, and are thus much higher than those of functions from X^n to X .

When $f = \min$, the natural way to tackle the problem has been to introduce relations that naturally refine the min-ordering, and restore the Pareto optimality of the selected maximal

solutions. Such relations are the discrimin ordering and the leximin ordering [2, 3]. The discrimination power of the latter is maximal, i.e. it is equal to that of the most discriminating symmetric aggregation operations.

In this work, we try to generalise this refinement technique to more general families of aggregation operation. We restrict to the case of symmetric functions. Consider a family $\{f^p\}_p$ of symmetric functions $X^p \rightarrow X$. For any positive integer p , f^p is supposed to be

-) *extensively preferentially consistent* with $f^{p-1} : \forall u \in X$,
 $f^{p-1}(x_1 \dots x_{p-1}) \geq f^{p-1}(y_1 \dots y_{p-1})$ imply $f^p(x_1 \dots x_{p-1}, u) \geq f^p(y_1 \dots y_{p-1}, u)$.
-) *globally strictly monotone* :
 if $x_i > y_i \ \forall i = 1, p$, then $f^p(x) > f^p(y)$ where x and $y \in X^n$.

These conditions look natural in the scope of applications. The first condition is a weak form of preferential independence. They are satisfied by the minimum, the maximum (but not other order-statistics). By convention f^1 is the identity function on X . Call $\{f^1 \dots f^p \dots\}$ a qualitative aggregation structure. We use the notation f when the number of arguments is not emphasized. The generalisation of discrimin and leximin to such aggregation structures is as follows:

Discri – f : Let $D(x, y) = \{i, x_i \neq y_i\}$ be the discriminating set of components for x and y . Then define $x \geq_{\text{discri-f}} y \Leftrightarrow f(\{x_i, i \in D(x, y)\}) \geq f(\{y_i, i \in D(x, y)\})$

Lexi – f : Let $x \in X^n$, and let $V(x) = \{x \in X, \exists i \in \{1 \dots n\}, x_i = x\}$ be the set of distinct values in the vector x . Let $k_x(x) =$ number of times the value x appears in x . Let $M(x)$ be the multi-set induced by $x : \forall x \in X$, the degree of “membership” of x to $M(x)$ is $k_x(x)$. Let $M(x) - M(y)$ be the multi-set with membership function $\max(0, k_x - k_y)$. Denoting $\sum_{x \in X} k_x(x)$ the cardinality of $M(x)$. It is obvious that $|M(x)| = n = |M(y)|, \forall x, y \in X^n$. Hence $|M(x) - M(y)| = |M(y) - M(x)|$. Then define : $x \geq_{\text{lexi-f}} y \Leftrightarrow f(M(x) - M(y)) \geq f(M(y) - M(x))$

It can be shown that under mild conditions such as global monotonicity and extensive preferential consistency, lexi-f and discri-f maximal solutions are indeed Pareto-optimal, and that the corresponding ordering of solutions is quite discriminant.

Globally strictly monotone aggregation functions on X are easily proved to be idempotent on finite scales, since if $0 = x_1 < x_2 < \dots < x_m = 1$ it follows that $f(x_1 \dots x_i) = x_i$. It rules out the Archimedean t-norms and conorm-like operations on finite sets [4]. Moreover, the only associative idempotent aggregation operations different from min and max, the α -medians ($f(x, y) = \text{median}(\alpha, x, y)$) are generally not globally strictly monotone, since they are constant on large

subsets of X^n . The above extensions of leximin and discrim orderings to these operations thus do not possess enough discrimination power; they can only be refined by directly adopting Pareto-ordering on the ranges where these aggregation functions are constant .

The simplest non trivial example of aggregation structure is for $X = \{1, 2, 3\}$. Adopting the lexi- f^2 ordering does not leave many degrees of freedom: one must indicate the relative position of $f^2(1, 3)$ and $f^2(2, 2)$. If $f^2(1, 3) > f^2(2, 2)$, this is the leximax ordering. If $f^2(1, 3) < f^2(2, 2)$, this is the leximin ordering. If $f^2(1, 3) = f^2(2, 2)$, this is a kind of ordinal average (which is less discriminant). With three arguments, the ordering of 3-tuples is fixed by further positioning $(2, 2, 2)$ with respect to $(1, 1, 3)$ and $(1, 3, 3)$ (note that $(1, 3, 3) >_{\text{lexi-f}} (1, 1, 3)$ in any case). The lexi-f positions of $(1, 2, 2)$ with respect to $(1, 1, 3)$, $(1, 2, 3)$ with respect to $(2, 2, 2)$, and $(2, 2, 3)$ with respect to $(1, 3, 3)$, are enforced by the position of $(2, 2)$ with respect to $(1, 3)$. If $f^2(1, 3) > f^2(2, 2)$, then $f^3(1, 3, 3) \geq f^3(2, 2, 2)$ and only the relative position of $(2, 2, 2)$ and $(1, 1, 3)$ is left open. If $f^2(2, 2) > f^2(1, 3)$, then $f^3(2, 2, 2) \geq f^3(1, 1, 3)$ and only the relative position of $(2, 2, 2)$ and $(1, 3, 3)$ is left open.

The generation of complete preorderings of tuples of elements from a finite ordered scale in agreement with Pareto-ordering and symmetry has been considered in Moura-Pires and Prade[5] in the scope of fuzzy constraint satisfaction problems. A natural question is whether any such complete preorderings of X^n can be obtained as a lexi-f ordering for some qualitative aggregation structure $\{f^1 \dots f^n\}$, and more generally, can be generated by a small number of extra constraints on the relative positioning of a few tuples. Unfortunately the answer for lexi-f orderings is negative. A counterexample is obtained using a 4-element scale $X = \{0, 1, 2, 3\}$. Then f^2 is characterized by the relative positionings of $(0, 2)$ w.r.t. $(1, 1)$, $(1, 3)$ w.r.t. $(2, 2)$, and $(0, 3)$ w.r.t. $(1, 1)$ and $(2, 2)$. However, using $f : X^2 \rightarrow X$, $f^2(0, 3) = \{f^2(0, 0), f^2(1, 1), f^2(2, 2), f^2(3, 3)\}$. Neither the discri-f nor the lexi-f extension, nor even the adding of Pareto-ordering itself can generate the complete preorderings such that $(1, 1) < (0, 3) < (2, 2)$. There are 12 total orderings which are Pareto-compatible and respect symmetry in this example, and only 8 of them can be generated as a lexi-f ordering via an aggregation structure $X^n \rightarrow X$. Generating the other total orderings requires a function $X^2 \rightarrow Y$ where Y has 5 levels.

Another property which may simplify the study of qualitative aggregation structures is the following regularity : let $(i, j) \in X^2 = \{0, 1, 2, \dots, n\}$;

$$\text{if } f^2(i, j) > f^2(i+1, j-1), \text{ then } f^2(i+1, j+1) > f^2(i+2, j).$$

The combinatorics of such regular aggregations functions look moderate and deserve further exploration. However the above study has exhibited some intrinsic limitations of the

otherwise appealing finite setting for criteria aggregation using a single finite scale, whereby concise representations and functions having good algebraic properties turn out to lack expressivity, even under natural lexicographic-like extensions .

References

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