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To

The Editor

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I am submitting the following paper entitled :

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Thanks,

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Study of Prey-Predator models with Allee effect for the Predator in presence of multiple interior equilibria

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Abstract

We consider prey-predator models with allee effect for the predator population. We first prove the existence of multiple equilibria of the models. In presence of multiple interior equilibria, we then study the dynamics of the models using the Conley connection matrix theory and establish analytically the results (namely the stability exchange bifurcation, Hopf bifurcations and homoclinic bifurcations) obtained by numerical simulations in previous studies. Some other results (including the global convergence of interior flows to the boundary) are also presented here.

keyword: prey-predator models, allee effect for the predator, Multiple interior equilibria, the Conley connection matrix, bifurcation analysis.

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1. Introduction

We consider the following prey predator models with allee effect for the predation population

$$\begin{aligned} x' &= xg(x) - yp(x) \\ y' &= y[\delta p(x)\left(\frac{y}{h+y}\right) - \epsilon_1 - \epsilon_2 y], \end{aligned} \tag{1}$$

with $g(x) = r\left(1 - \frac{x}{K}\right)$, and $p(x) = \frac{\alpha x}{1+\beta x}$ or $p(x) = \frac{\alpha x}{1+\beta x+\gamma y}$,

where $x(t)$ and $y(t)$ are the prey and predator population density at time t , respectively with $x(0), y(0) \geq 0$. $g(x)$ is the per capita growth rate of the prey in absence of the predator. $p(x)$ is the Holling type II functional response for the predator, or $p(x)$ is of Beddington De-Angelis form. $\epsilon_1 + \epsilon_2 y$ is the density dependent death rate of the predator. $\delta, h, K, \alpha, \beta, \gamma, \epsilon_1, \epsilon_2$ are positive constants. Out of a number of studies of prey-predator models in which the predator population is a subject to a component allee effect for reproduction (see the references in [7]), we mainly focus on the papers by Terry [7] and Verdy [8]. In [7] Terry extended the definitions of demographic and component allee effects for single species models to predators in prey-predator models while the prey population does not change. The models (1) can have multiple two species fixed points. By numerical simulation analysis Verdy [8] showed the dynamics of models with $\gamma = \epsilon_2 = 0$ in presence of two interior fixed points. By changing the values of the carrying capacity K the author [8] has shown the changes of dynamics of his models, namely, the existence of multiple equilibria, their stability and the occurrence of a stable limit cycle, and finally the convergence of the system to the extinction equilibrium for sufficiently large value of K . Terry considered the models (1) in [7] and studied numerically the dynamics of the models in presence of two or three interior equilibria. Through numerical analysis the author established the existence of fixed points, the existence of a stable limit cycle in one of the models and the coexistence of trapping regions.

In this paper we first prove the existence of interior fixed points. In presence of multiple fixed points we then study the dynamics of the models using the Conley connection matrix theory and obtained analytically the results shown by Verdy [8] and Terry [7]. Some other results (which are not possible by simulations) are also presented in this paper. Computing the connection matrices for models with $\gamma = \epsilon_2 = 0$, we prove Hopf bifurcations, homoclinic bifurcations and global convergence of the system to the extinction equilibrium. For models with $\gamma \neq 0, \epsilon_2 \neq 0$, we prove a result which shows the global convergence of the system to the extinction fixed point state. At the extinction equilibrium, only the prey population settles at the carrying capacity and the predator dies out. All of our results are given in section 2.

2. Assumptions and Results:

From Lemma 1 of Terry [7] we know that the solutions of the models (1) are uniformly bounded. From the prey and predator equations and the Jacobian matrix of the models (1) we get (i) the trivial equilibrium $M(0) = (0, 0)$, which always exists, is stable along the y axis and unstable along the x axis, and (ii) the axial equilibrium $M(K) = (K, 0)$ which always exists and is asymptotically stable (stable along the both x, y axes). Regarding the existence of interior fixed points we need two definitions. An equilibrium of a system of ecological equations $x' = xf(x)$ ($x \in R_+^n$) is called unsaturated (resp. saturated) if its eigenvalue along the orthogonal direction is positive (resp negative). A point $x \in U \subset R^n$, (where U is bounded open subset) is called regular if $\det D_x f \neq 0$. Next the result.

Lemma 1 *For models (1), there exist either no, two or four interior fixed points in the positive quadrant*

Proof : In model (1) the boundary equilibrium $M(k)$ is always saturated with index (+1) and $M(0)$ is nonsaturated with index (-1). Solutions are bounded. The sum of indices of the saturated regular equilibria of model (1) in $(-1)^2 = 1$, by Theorem 1 in [2] (INDEX theorem for ecological equations). Thus there exist either no equilibrium (if the prey and predator isoclines do not intersect) giving the sum +1 or two or four equilibria $M(x_i) = (\bar{x}_i, \bar{y}_i)$ for $i = 1, 2$ or $1, 2, 3, 4$ (if the isoclines intersect at (\bar{x}_i, \bar{y}_i)) giving the sum +1. If $i = 1, 2$, then one of the two must be a saddle point with index -1 and the rest is either source or sink with index +1. If $i = 1, 2, 3, 4$, then two of them are saddle points with total indices -2 and the rest two are sinks or sources with total indices +2. \square

We study models (1) by considering $\gamma = \epsilon_2 = 0$, (MODEL 1), $\epsilon_2 = 0$ (MODEL 2) and $\gamma \neq 0, \epsilon_2 \neq 0$ (MODEL 3).

MODEL 1 with $\gamma = \epsilon_2 = 0$,

The isoclines $x' = y' = 0$ intersect at two equilibria $M(x_i), i = 1, 2$. We consider that $M(x_2)$ is saddle and $M(x_1)$ is either sink or source, whenever $x_1 < x_2 < K$. Moreover, for our analysis purpose we assume

(H) If $K < x_1 < x_2$, then $M(x_1)$ saddle and $M(x_2)$ is sink and the connection orbit $M(x_1) \rightarrow M(K)$ is unique and also $M(x_1) \rightarrow M(x_2)$ is unique. These bounded orbits lie outside the closure of the positive quadrant and they lie above the line $y = L$ for some positive L .

We define (i) $S = \{\text{bounded solutions of Model 1}\}$ and (ii) π be the minimal isolated invariant set of the positive quadrant containing all periodic solutions of S . Then we have the followings.

Proposition 1

a. $H_*(h(S), \mathbf{Z}_2) \approx (0, 0, 0, \dots)$; $h(S)$ is the Conley index of S and $H_*(h(S), \mathbf{Z}_2)$ is the homologies of the index over \mathbf{Z}_2 .

b. $H_*(M(0)) \approx (0, \mathbf{Z}_2, 0, 0, \dots)$ for all $K > 0$.

c. $H_*(M(K)) \approx (\mathbf{Z}_2, 0, 0, 0, \dots)$

d. $H_*(M(x_1)) \approx \begin{cases} (0, \mathbf{Z}_2, 0, 0, \dots) & \text{if } K < x_1 \\ (\mathbf{Z}_2, 0, 0, 0, \dots) & \text{if } x_1 < K. \end{cases}$

e. $H_*(M(x_2)) \approx \begin{cases} (\mathbf{Z}_2, 0, 0, 0, \dots) & \text{if } K < x_1 < x_2 \\ (0, \mathbf{Z}_2, 0, 0, \dots) & \text{if } x_1 < x_2 < K. \end{cases}$

f. If $H_*(M(x_1)) \approx (\mathbf{Z}_2, 0, 0, 0, \dots)$, then $H_*(\pi) \approx (0, 0, \dots)$ or $(0, \mathbf{Z}_2, \mathbf{Z}_2, 0, 0, \dots)$

g If $H_*(M(x_1)) \approx (0, 0, \mathbf{Z}_2, \dots)$, then $H_*(\pi) \approx (0, 0, \dots)$ or $(\mathbf{Z}_2, \mathbf{Z}_2, 0, 0, \dots)$

h If (x_1, π) is an attracting interval, then $H_*(x_1\pi) \approx (\mathbf{Z}_2, 0, 0, 0, \dots)$. Here we denote $M(x_i)$ by only x_i .

Proof: (a) See [4]

(b) - (e) These follows from the linear analysis of the critical points and assumption (H).

(f) - (h) See [4].□

A finite set P with a strict partial order $>$ satisfying i) $i > i$ never holds, ii) if $i > j$ and $j > k$, then $i > k$ for all $i, j, k \in P$, is a partially ordered set, $(P, >)$. An interval is a subset $I \subset P$, for which $i, j \in I$ and $i < k < j$ imply $k \in I$. The set of intervals is denoted by $I(>)$. $I \in I(>)$ is called an attracting interval if $i \in I$ and $i > j$, then $j \in I$. (Also see [1, 3, 4, 5, 6]) A Morse decomposition $\mathcal{M}(S) = \{M(i) \mid i \in (P, >)\}$ of S is a collection of mutually disjoint isolated invariant subsets of S , indexed by $(P, >)$ and given $x \in S$ either $x \in M(i)$ for some i , or there exists $i, j \in P$, such that $i > j$ with $\omega(x) \in M_j$ and $\alpha(x) \in M_i$. $\omega(x)$ and $\alpha(x)$ denote the omega limit set and alpha limit set of x . For our MODEL 1, $P = \{O, x_1, x_2, K, \pi\}$ is the index set for a Morse decomposition (if not $M(\pi) = \emptyset$). The possible partial orders on P are given below

Proposition 2

(i) If $K < x_1 < x_2$, then $>_1$ is $0 >_1 p, x_1 >_1 K, x_1 >_1 x_2$.

(ii) If $x_1 < x_2 < K$, then $>_2$ is $0 >_2 p, x_2 >_2 K, x_2 >_2 \pi, \pi >_2 x_1$.

(iii) If $x_1 < x_2 < K$, then $>_3$ is $0 >_3 p, x_2 >_3 K, \pi >_3 x_2, \pi >_3 x_1$.

(iii) If $x_1 < x_2 < K$, then $>_4$ is $0 >_4 p, x_2 >_4 K, x_2 >_4 \pi, x_1 >_4 \pi$.

(iii) If $x_1 < x_2 < K$, then $>_5$ is $0 >_5 p, x_2 >_5 K, \pi >_5 x_2, x_1 >_5 \pi$.

where $p \in P$ and $\{0, K\}, \{x_1, \pi\} \in I(>_j)$ for all j .

Proof: The origin $M(0)$ is stable along the y axis, hence there are no connecting orbits from 0 to any other Morse sets of P . So $0 > p$ for $p \in P$. $M(0)$ is stable along the x axis, thus $\{0, K\} \in I(>)$. $M(K)$ is stable and there are no connecting orbit from $M(K)$ to any other set, thus $p > K$, for all values of $p \in P \setminus \{M(K)\}$. If γ be the inner most periodic orbit (or the outer most periodic orbit) in $M(\pi)$ (if exists), then only $M(x_1)$ belongs to the compact region Γ bounded by γ . Thus $\{x_1, \pi\} \in I(>)$. As a saddle point $M(x_2)$ does not belongs to Γ . \square

Every Morse decomposition of S has at least one connection matrix. (See [1]). We denote A_i be the connection matrix in respect to the Morse decomposition $\mathcal{M}(S) = \{0, K, x_1, x_2, \pi \mid p \in (P, >_i)\}$. The connection matrix is a degree (-1) maps and upper triangular matrix.

The Connection Matrix A_1 for order $>_1$ from Proposition 2

It is of the form

$$A_1 = \begin{matrix} & H_0(K) & H_0(x_2) & H_1(x_1) & H_1(0) \\ \begin{matrix} H_0(K) \\ H_0(x_2) \\ H_1(x_1) \\ H_1(0) \end{matrix} & \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & \star \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

By assumption (H) the entries $\Delta(x_1, K) = 1$ and $\Delta(x_1, x_2) = 1$. The condition that $A_1^2 = 0$ and $\text{rank} A_1 = 2$ gives the value of the unknown entry \star is 0.

The Connection Matrix A_2 for order $>_2$ from Proposition 2

It is of the form

$$A_1 = \begin{matrix} & H_0(K) & H_0(x_1\pi) & H_1(x_2) & H_1(0) \\ \begin{matrix} H_0(K) \\ H_0(x_1\pi) \\ H_1(x_2) \\ H_1(0) \end{matrix} & \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & \star \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

By Proposition 1h, $H_*(x_1\pi) = (\mathbf{Z}_2, 0, 0, \dots)$. Similar to A_1 the value of the unknown entry \star is zero.

The Connection Matrix A_3 for order $>_3$ from Proposition 2

It is of the form

$$A_3 = \begin{array}{c} H_0(K) \\ H_0(x_1) \\ H_1(0) \\ H_1(x_2) \\ H_1(\pi) \\ H_2(\pi) \end{array} \begin{array}{c} H_0(K) \\ H_0(x_1) \\ H_1(0) \\ H_1(x_2) \\ H_1(\pi) \\ H_2(\pi) \end{array} \begin{bmatrix} 0 & 0 & 1 & a & b & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

That the $\text{rank}A_3 = 3$ gives $a = 0$. $A_3^2 = 0$ forces two connection matrices from A_3 , as A_3' with $b = 1$, and A_3'' with $b = 0$. That is in this case connection matrix is not unique. Two or more possible connection matrices will not make any problem to our latter analysis. We need only the existence of a connection matrix for a Morse decomposition with a particular partial ordering.

The Connection Matrix A_4 for order $>_4$ from Proposition 2

It is of the form

$$A_4 = \begin{array}{c} H_0(K) \\ H_0(\pi) \\ H_1(\pi) \\ H_1(0) \\ H_1(x_2) \\ H_2(x_1) \end{array} \begin{array}{c} H_0(K) \\ H_0(\pi) \\ H_1(\pi) \\ H_1(0) \\ H_1(x_2) \\ H_2(x_1) \end{array} \begin{bmatrix} 0 & 0 & 0 & 1 & a & 0 \\ 0 & 0 & 0 & b & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Using $A_4^2 = 0$ and $\text{rank}A_4 = 3$ we get four connection matrices from A_4 ; A_4' with $a = 0, b = c = 1$, A_4'' with $a = 1, b = 1, c = 0$, A_4''' with $a = 0, b = 0, c = 1$, A_4'''' with $a = 1, b = 0, c = 1$,

The Connection Matrix A_5 for order $>_5$ from Proposition 2

It is of the form

$$\begin{array}{c}
H_0(K) \quad H_1(0) \quad H_1(x_2) \quad H_2(x_1) \\
\begin{array}{c}
H_0(K) \\
H_1(0) \\
H_1(x_2) \\
H_2(x_1)
\end{array}
\begin{bmatrix}
0 & 1 & * & 0 \\
0 & 0 & 0 & * \\
0 & 0 & 0 & * \\
0 & 0 & 0 & 0
\end{bmatrix}
\end{array}$$

$M(x_1)$ is source, $\pi >_5 x_2$, thus $H_*(\pi) \approx 0$. Using square matrix zero and rank of the matrix two we get the following connection matrix

$$\begin{array}{c}
H_0(K) \quad H_1(0) \quad H_1(x_2) \quad H_2(x_1) \\
A_5 = \begin{array}{c}
H_0(K) \\
H_1(0) \\
H_1(x_2) \\
H_2(x_1)
\end{array}
\begin{bmatrix}
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\end{array}$$

After establishing the possible connection matrices for the MODEL 1 we get the following result with respect to the values of the carrying capacity K for the prey.

Theorem 1

(a) *There exists a unique value of K for which exchange of stability bifurcation of two species fixed points occur through the existence of the connection matrix from A_1 to A_2 .*

(b) *There exist unique values of K for which Hopf bifurcations occur through the change of connection matrices from A_2 to A_4 and from A_3 to A_5 .*

(c) *There exist unique values of K for which homoclinic bifurcations occur through the changes from A_2 to A_3 and from A_4 to A_5 .*

(d) *(After getting the connection matrix A_5) system admits a 2-disc of connecting orbits from $M(x_1)$ to $M(K)$.*

Proof: (a) For small values of K , i.e, for $K < x_1 < x_2$, from connection matrix A_1 we note that all interior solutions of the model converges to the $M(K)$ (except the orbit from $M(x_1)$ to $M(x_2)$ but this orbit does not lie in the positive quadrant). Now the parameter K is increased i.e, for $x_1 < x_2 < K$, we obtain the connection matrix from A_1 to A_2 and this change indicates that as K is increased stability exchange bifurcation of euilibria happens and $M(x_1)$ and $M(x_2)$ appear in the positive quadrant. $M(x_1)$ becomes the attractor and

$M(x_2)$ is a saddle point.

(b) By the existence of the connection matrices A_2 and A_4 it shows that there exist a Hopf bifurcation for some unique value of K due to the change from A_2 to A_4 . Similarly Hopf bifurcation results in a change of A_3 to A_5 .

(c) Since the connection matrices change from A_2 (or A_3) to A_4 (or A_5) there exists a transition matrix T satisfying the equation (see Theorem 3.8 [4])

$$A_2T + TA_4 = 0 \quad [\text{ or } A_3T + TA_5 = 0]$$

and T is a degree 0 map and it takes the form

$$T = \begin{matrix} & H_0(K) & H_0(x_1) & H_1(0) & H_1(x_2) & H_1(\pi) & H_2(\pi) \\ \begin{matrix} H_0(K) \\ H_0(x_1) \\ H_1(x_2) \\ H_1(0) \end{matrix} & \left[\begin{array}{cccccc} a & b & 0 & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 & 0 \\ 0 & 0 & e & f & g & 0 \\ 0 & 0 & h & i & j & 0 \end{array} \right] \end{matrix}$$

The above equations (taking matrix A_4''') will be satisfied if the following set of equations are satisfied

$$a + f + i = 0, b + g + j = 0, c + f = 0, g + d = 0$$

Then it is possible to get the value of f is equal to zero, i.e, $\Delta(x_2, x_2) = 0$. By Corollary 4.4 of [5], it follows that there exists a homoclinic orbit to $M(x_2)$. The other result follows by similiar argument.

(c) From A_5 we get a 2-disc of connecting orbits from $M(x_1)$ to $M(K)$ and this follows from Theorem 3.2 [3]. Moreover the boundary of this disc are the connecting orbits from $M(x_2)$ to $M(K)$ and from $M(x_1)$ to $M(0)$. That is the extinction equilibrium $M(K)$ becomes globally stable, though there are two coexistence equilibria $M(x_2)$ and $M(x_1)$ in the positive quadrant. \square

Remark 1 *The region for the 2-disc of connecting orbits along with the boundary orbits from $M(x_2)$ to $M(K)$ and from $M(x_1)$ to $M(0)$ may be the trapping region of interior flows and from this region then interior orbit can move to the extinction state of the system*

Remark 2 *For parameter values K_4 and K_5 (of K) for which there are A_4 and A_5 connection matrices and $K_4 < K_5$.*

MODEL 2 with $\epsilon_2 = 0$,

The prey and predator isoclines $x' = y' = 0$, in this model intersect at two interior fixed points (see equation (39) of Terry[7]) by Lemma 1. One of two

fixed points is a saddle and the rest is either sink or source and there exists periodic solutions also in the positive quadrant, hence we will get the same result as Theorem 1. In addition, there may exist a homoclinic orbit to the saddle point and after increasing K sufficiently large one may have global convergence of interior solutions to the extinction state $M(K)$ (not reported in [7] but this may happen as it is forced by the connection matrix in Theorem 1).

MODEL 3 with $\gamma \neq 0, \epsilon_2 \neq 0$,

From $x' = 0$ and $y' = 0$ (see equation 37 in [7]) we have four two species fixed points $M(x_i), i = 1, 2, 3, 4$. We consider that $M(x_1)$ and $M(x_3)$ are sink, and $M(x_2)$ and $M(x_4)$ are saddle points. We form a Morse decomposition with $M(x_i), i = 1, 2, 3, 4, M(0), M(K)$, and the partial ordering $x_1 < x_2 < x_3 < x_4 < K$ (thus excluding the periodic orbits in the positive quadrant). Then the connection matrix will be of the form

$$C = \begin{array}{c} H_0(K) \quad H_0(x_1) \quad H_0(x_3) \quad H_1(0) \quad H_1(x_2) \quad H_1(x_4) \\ \begin{array}{c} H_0(K) \\ H_0(x_1) \\ H_0(x_3) \\ H_1(0) \\ H_1(x_2) \\ H_1(x_4) \end{array} \left[\begin{array}{cccccc} 0 & 0 & 0 & 1 & a & b \\ 0 & 0 & 0 & c & d & e \\ 0 & 0 & 0 & f & g & h \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \end{array}$$

From C we get three possible connection matrices using $C^2 = 0$ and $\text{rank}C = 3$, namely, $C_i, i = 1, 2, 3$;

- (i) C_1 with $a = e = f = 0$ and $b = c = d = g = h = 1$.
- (ii) C_2 with $a = c = e = f = 0$ and $b = d = g = h = 1$.
- (iii) C_3 with $a = c = e = f = g = 0$ and $b = d = h = 1$.

From C_1 (this is actually with the given partial ordering), one can get the same result what Terry [7] obtained by numerical simulation (see Fig. 4 [7]). Thus the existence of C_1 confirms the result of Terry [7] for MODEL 3. Moreover, there are also other possible connection matrices, namely C_2 and C_3 , through which other pattern of the interior flows can be obtained (these are not reported in [7]).

Remark 3 *We attempt to get a saddle-saddle connection from $M(0)$ to $M(x_2)$ through forming a transition matrix from connection matrices C_2 to C_1 (or C_3 to C_1) for the product flow (see [6]), but this does not occur. This is because that there are periodic orbits around the sink $M(x_1)$ and a homoclinic orbit to $M(x_2)$ which does not allow the formation of such saddle-saddle connection.*

We consider another Morse decomposition containing $M(x_1)$ (source), $M(x_3)$ (sink), $M(x_2)$ (saddle), $M(x_4)$ (saddle), $M(0)$, and $M(K)$. and form the connection matrix of the form

$$D = \begin{array}{c} H_0(K) \quad H_0(x_3) \quad H_0(0) \quad H_1(x_2) \quad H_1(x_4) \quad H_2(x_1) \\ \begin{array}{c} H_0(K) \\ H_0(x_3) \\ H_0(0) \\ H_1(x_2) \\ H_1(x_4) \\ H_2(x_1) \end{array} \left[\begin{array}{cccccc} 0 & 0 & 1 & * & 1 & 0 \\ 0 & 0 & * & * & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \end{array}$$

Rank $D = 3$ forces $\Delta(x_2, K) = 0$ and $\Delta(x_2, x_3) = 0$. $D^2 = 0$ implies $\Delta(x_1, 0) = 0$, $\Delta(x_4, K) = 0$, $\Delta(x_1, x_2) = 1$ and $\Delta(x_1, x_4) = 0$. Then one possible connection matrix is given below (there may be many others).

$$D_1 = \begin{array}{c} H_0(K) \quad H_0(x_3) \quad H_0(0) \quad H_1(x_2) \quad H_1(x_4) \quad H_2(x_1) \\ \begin{array}{c} H_0(K) \\ H_0(x_3) \\ H_0(0) \\ H_1(x_2) \\ H_1(x_4) \\ H_2(x_1) \end{array} \left[\begin{array}{cccccc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \end{array}$$

In D there is a connection $M(x_1) \rightarrow M(0)$ but the zero in D_1 indicates there is a double $M(x_1) \rightarrow M(0)$ connections. Thus if we consider the subset $\{M(K), M(0), M(x_1)\}$, by Theorems 3.1 and 3.2 in [3] we get a 2-discs of connecting orbits connecting $M(x_1)$ to $M(K)$. That is, interior solutions converge to $M(K)$ in presence of other three interior fixed points (see Fig. 1). From the connection matrices C and D , we get our next partial result

Proposition 3

(a) For MODEL 3 the connection matrix C indicates the existence of four states where the system can coexist and

(b) The connection matrix D_1 shows a 2-disc of connecting orbits in the interior of the positive quadrant so that interior orbits can converge to the boundary.

In addition to the above two cases, one may consider periodic orbits in the positive quadrant for MODEL 3, then other many types of connection matrices will exist and accordingly other types of dynamics of MODEL 3 will emerge (that may differ from the results of this study).

Conclusion

In our analysis we need only that (i) solutions of the models are uniformly bounded and (ii) the existence of equilibria (or periodic orbits) and their local stability structure. From this knowledge we are able to obtain the results shown by Verdy [8] and Terry [7] through numerical analysis. By computing connection matrices we show the occurrence of stability exchange bifurcation of equilibria, the occurrence of Hopf bifurcations and homoclinic bifurcations for MODEL 1. Further, for all models we also show the existence of a 2-disc of connecting orbits which allows the global convergence of interior flows to the boundary. Thus our study support the views of ecologists that the mathematical modeling of allee effects (what the Lotka-Volterra models ignore) in prey-predator systems greatly increases the likelihood of local and global extinction.

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Figure legend:

Fig. 1 This figure shows the existence of a 2-disc of connecting orbits from $M(x_1)$ to $M(K)$. From region N interior solutions can converge to $M(K)$.

