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| Guillaume Theyssier. Cold Dynamics in Cellular Automata: a Tutorial. 2021. hal-03462040v1

**HAL Id: hal-03462040**

**<https://hal.science/hal-03462040v1>**

Preprint submitted on 1 Dec 2021 (v1), last revised 14 Jun 2022 (v2)

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# Cold Dynamics in Cellular Automata: a Tutorial

Guillaume Theyssier\*

December 1, 2021

## Abstract

This tutorial is about cellular automata that exhibit 'cold dynamics'. By this we mean zero entropy, stabilization of all orbits, trivial asymptotic dynamics, etc. These are purely transient irreversible dynamics, but they capture many examples from the literature. A rich zoo of properties is presented and discussed: nilpotency and asymptotic, generic or mu-variants, unique ergodicity, convergence, bounded-changeness, freezingness. They all correspond to the 'cold dynamics' paradigm in some way, and we study their links and differences by various examples and results from the literature. Besides dynamical considerations, we also focus on computational aspects: we show how such 'cold cellular automata' can still compute under their dynamical constraint, and what are their computational limitation.

## 1 Introduction

This tutorial is about cellular automata (CA for short), a well-known class of discrete dynamical systems, useful for modeling natural phenomena, and also a model of computation. Informally, a CA is a lattice of finite-state cells that evolve according to a uniform local rule. CA have been extensively studied as chaotic dynamical systems [31], as models of physical phenomena with positive entropy [12, 51], as a model of reversible computing [37, 19] or as groups when considering only reversible CA [7, 43].

None of these kinds of CA are considered here. On the contrary, we are interested in examples with zero entropy, that have a strong convergence property, whose asymptotic dynamics is essentially trivial, and that are strongly irreversible. These examples are essentially transient dynamically, they erase information from the initial configuration, and display a strong effect of the 'arrow of time'.

We will refer to this class as *cold dynamics*, without trying to precisely define it for now. It turns out that literature on CA abounds in results and examples that fit into this class. The purpose of this tutorial is to present many of them and, as much as possible, sketch a coherent global picture around CA cold dynamics.

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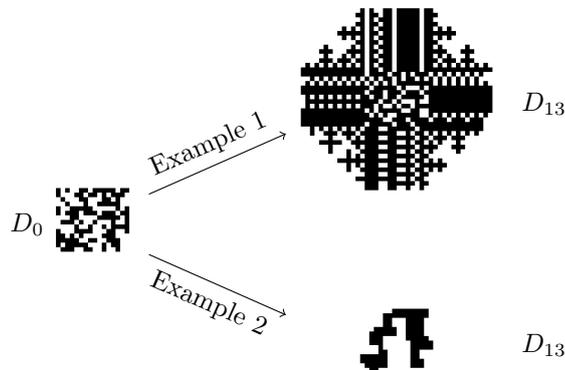


Figure 1: Evolution starting from the same initial configuration for the two examples of cold dynamics presented in this section. Positions belonging to sets  $D_i$  are represented in black.

**Two examples.** Let us consider the discrete plane  $\mathbb{Z}^2$  and say that two elements are neighbors if one is at distance 1 to the north, east, south or west of the other. Now consider any initial set  $D_0 \subseteq \mathbb{Z}^2$  and define  $D_{i+1}$  for any  $i \in \mathbb{N}$  as the union of  $D_i$  and all elements of  $\mathbb{Z}^2$  that have exactly one neighbor in  $D_i$ . This actually defines a CA with a cold dynamics. Indeed, the sets  $D_i$  are only increasing and therefore each position of  $\mathbb{Z}^2$  can only have two destiny: either it will never belong to any  $D_i$ , or it will appear in some  $D_i$  and then stay in all subsequent ones. This first example was introduced in [47] and is actually one of the very first CA ever defined.

As our second example, consider again any initial set  $D_0 \subseteq \mathbb{Z}^2$  and define  $D_{i+1}$  as set of positions from  $\mathbb{Z}^2$  such that, among itself and its 4 neighbors, a majority of positions is in  $D_i$  (*i.e.* at least 3). In this second example, the destiny of a position is less clear as the local majority could change several times. It turns out that if we consider one step every two, this example also has a strong convergence property (we will establish this for a large class of examples in Theorem 10). Note that majority dynamics like this one might well be the single most studied class of CA.

An example of evolution of both CA is show in Figure 1. One could start to analyze these specific examples in depth, but we will rather try to understand what are the properties that make them special, and work on entire classes of examples.

**Main questions around cold dynamics.** This tutorial is organized into five parts: we will first precise our formal settings in Section 2 (using the language of orbits, topology and probability measures) and introduce in Section 3 the pivotal notion for cold dynamics: convergence. Then, we will address three general questions as starting points to present many results and examples:

1. *What kind of behaviors can be obtained if we push the convergence property*

*to an extreme and ask that the system collapse all initial configurations into a single one? What if we only specify that the asymptotic dynamics be trivial (a singleton) and how does it depend on the language used (configurations versus probability measures)?*

We will tackle these questions in Section 4 by studying the classical notion of nilpotency and its variants.

2. *How to establish that a given CA is convergent?*

In Section 5, we will both present proof techniques and introduce useful sub-classes of convergent CA (bounded-change and freezing CA) that put stronger constraints on the dynamics but already capture many natural examples, like the two presented above.

3. *How the ability of general CA to make arbitrary computations survive under the constraint of convergence, bounded-change or freezingness?*

In Section 6, we will see that even under the strongest constraints, computational universality is preserved, but the effectiveness of this computational power greatly varies with the constraint considered.

**A tutorial.** This tutorial is an invitation to discover what we believe are nice examples, concepts and proof technics. It is not a survey, and we try to find a good compromise between accessibility and generality. We do include some proofs (or sketch of) as they give much insight in the topic. However, many results are stated without proof. This is mostly due to space constraint and not an indication that the result is less important: the reader is warmly invited to consult the corresponding references.

## 2 Definitions and Notations

In this section we present standard definitions about CA and establish a precise formal framework as well as some notations. The reader unfamiliar with these definitions or expecting a more detailed presentation is invited to consult [10, 39, 35, 31, 50]. Most of the results presented in this tutorial do not require in depth knowledge of the notions defined below.

A dynamical system is a pair  $(X, F)$  where  $X$  is a compact metric space and  $F : X \rightarrow X$  a continuous map. We will consider only two kind of dynamical systems in this tutorial: deterministic CA acting on configurations, and deterministic CA acting on probability measures. A CA is defined by some uniform spatial structure of cells, an alphabet and a local evolution rule.

**Spatial structure of cells.** A natural and fairly general settings for the spatial cell structure is to consider a finitely generated group  $\mathbb{G}$  (without mentioning it each time, all groups  $\mathbb{G}$  below will be finitely generated). We will use additive notation for groups and denote by  $0_{\mathbb{G}}$  the identity of  $\mathbb{G}$ . For  $z \in \mathbb{G}$ , we denote by  $\|z\|_{\mathbb{G}}$  the length  $n$  of the smallest sequence of generators  $(g_i)_{1 \leq i \leq n}$  such that  $g_1 + \dots + g_n = z$ , and we let  $\|0_{\mathbb{G}}\|_{\mathbb{G}} = 0$ . We denote by  $B_n$  the set of elements  $z \in \mathbb{G}$  such that  $\|z\|_{\mathbb{G}} \leq n$ . Most of the time we will consider  $\mathbb{G} = \mathbb{Z}^d$  and discuss about cases  $d = 1$  or  $d \geq 2$  (called 1D CA, 2D CA, etc). We will

only scratch the surface of the deep and largely open question of how CA theory depends on  $\mathbb{G}$ . We will sometimes use the notion of amenability which is an emblematic example of this dependence through the so-called 'Garden of Eden' theorem (see [10]). A finitely additive probability measure on  $\mathbb{G}$  is a map  $\mu$  from subsets of  $\mathbb{G}$  to  $[0, 1]$  such that  $\mu(\mathbb{G}) = 1$  and  $\mu(A \cup B) = \mu(A) + \mu(B)$  whenever  $A \cap B = \emptyset$ . A finitely additive probability measure  $\mu$  is said left-invariant if  $\mu(A) = \mu(g + A)$  for any  $g \in \mathbb{G}$  and any  $A \subseteq \mathbb{G}$ . Then we say that  $\mathbb{G}$  is *amenable* if admits a left-invariant finitely additive probability measure. Groups  $\mathbb{Z}^d$  are amenable and we will sometimes consider free groups with 2 or more generators as an example of non-amenable group.

**Space of configuration and its topology.** Given a finite alphabet  $Q$  and a group  $\mathbb{G}$ , a *configuration* is a map  $c : \mathbb{G} \rightarrow Q$ . In order to remove some parenthesis in expressions, we will often denote  $c(z)$  by  $c_z$ . For any  $q \in Q$ , we denote by  $\bar{q}$  the uniform configuration  $c : z \mapsto q$ . The set of configurations  $Q^{\mathbb{G}}$  can be endowed with the prodiscrete topology (product of the discrete topology on  $Q$ ) to form a compact space by Tychonoff theorem. Given a finite set  $D \subseteq \mathbb{G}$  and a partial configuration  $u : D \rightarrow Q$  we denote by  $[u]$  the *cylinder set* of configurations that coincide with  $u$  on domain  $D$ :

$$[u] = \{c \in Q^{\mathbb{G}} : c|_D = u\}.$$

By a slight abuse of notation, we denote by  $[q]$  for  $q \in Q$  the cylinder set  $[u]$  of domain  $\{0_{\mathbb{G}}\}$  where  $u = 0_{\mathbb{G}} \mapsto q$ . Cylinder sets are clopen sets and form a base of the prodiscrete topology. This topology is also metrizable as follows. For any pair of configuration  $x, y \in Q^{\mathbb{G}}$  define their distance by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 2^{-n} & \text{where } n = \max\{i : x|_{B_i} = y|_{B_i}\}. \end{cases}$$

$\mathbb{G}$  naturally acts on  $Q^{\mathbb{G}}$  by *translation* as follows: for any  $z \in \mathbb{G}$  we define the shift map  $\sigma_z : Q^{\mathbb{G}} \rightarrow Q^{\mathbb{G}}$  by

$$\sigma_z(c)_{z'} = c_{z+z'}.$$

Shift maps are continuous and one-to-one. For  $c \in Q^{\mathbb{G}}$ , we denote by  $\mathcal{O}_{\mathbb{G}}(c)$  its orbit under translations, that is:  $\mathcal{O}_{\mathbb{G}}(c) = \{\sigma_z(c) : z \in \mathbb{G}\}$ . A natural kind of subset of configurations are closed translation invariant subsets of  $Q^{\mathbb{G}}$ . Such a subset  $X$  is called a *subshift* and it is characterized by its *forbidden language*, *i.e.* the set of cylinder set it avoids:  $\{[u] : [u] \cap X = \emptyset\}$  (see [35]).

**Evolution rule.** A CA on  $Q^{\mathbb{G}}$  is defined by a local evolution rule  $\lambda : Q^N \rightarrow Q$  where  $N$  is a finite subset of  $\mathbb{G}$  called *neighborhood*, and whose associated global evolution rule  $F : Q^{\mathbb{G}} \rightarrow Q^{\mathbb{G}}$  is defined as follows:

$$F(c)_z = \lambda(\sigma_z(c)|_N)$$

for all  $z \in \mathbb{G}$ . We say that  $F$  has *radius*  $r$  if  $N \subseteq B_r$  (note that the same map  $F$  could be defined by different local maps  $\lambda$  with different neighborhoods  $N$ , but finding the minimal such  $N$  and hence the minimal radius is

usually not important in the sequel). Any CA map  $F$  is continuous and commutes with translations:  $\sigma_z \circ F = F \circ \sigma_z$ . Then  $(Q^{\mathbb{G}}, F)$  is a dynamical system. Conversely, by Curtis-Hedlund-Lyndon theorem [27, 10], any dynamical system  $(Q^{\mathbb{G}}, \phi)$  where  $\phi$  commutes with translations is actually a CA. Given a CA  $F : Q^{\mathbb{G}} \rightarrow Q^{\mathbb{G}}$ , we are naturally interested in orbits, *i.e.* sequences of the form  $c, F(c), \dots, F^t(c), \dots$  for some configuration  $c$ . A partial information on orbits called *canonical trace* will play an important role: it is the sequence  $\mathcal{T}_c^F : \mathbb{N} \rightarrow Q$  defined as  $c_{0_{\mathbb{G}}}, F(c)_{0_{\mathbb{G}}}, \dots, F^t(c)_{0_{\mathbb{G}}}, \dots$  for a given  $c \in Q^{\mathbb{G}}$ . The map  $c \mapsto \mathcal{T}_c^F$  is a factor map from  $Q^{\mathbb{G}} \rightarrow Q^{\mathbb{N}}$ . It is generally neither surjective, nor injective.

**Space of probability measures and action of a CA on it.** The set of Borel probability measures on  $Q^{\mathbb{G}}$  will be denoted by  $\mathcal{M}(Q^{\mathbb{G}})$ . A measure  $\mu \in \mathcal{M}(Q^{\mathbb{G}})$  is a countably additive function from Borel sets to  $[0, 1]$  such that  $\mu(Q^{\mathbb{G}}) = 1$ . The *support* of a measure is the smallest closed set  $X$  such that  $\mu(X) = 1$ . We say that a measure is *full-support* if its support is  $Q^{\mathbb{G}}$ . Concretely, a measure is characterized by its value on cylinders (by the Carathéodory-Fréchet extension theorem since cylinders form a semi-ring that generates the  $\sigma$ -algebra of Borel sets). It is also convenient to define a measure this way. For instance, a natural and important class of measures are the *Bernoulli measures* which are product measures defined by coefficients  $\beta_q$  for each  $q \in Q$  such that  $\sum_{q \in Q} \beta_q = 1$  as follows: for any cylinder  $[u]$  with  $u : D \rightarrow Q$ ,

$$\mu([u]) = \prod_{z \in D} \beta_{u_z}.$$

The uniform Bernoulli measure is the one where all coefficients  $\beta_q$  are equal to  $1/\#Q$  where notation  $\#X$  represents the cardinal of set  $X$ . A Bernoulli measure is full-support if and only if  $\beta_q > 0$  for all  $q \in Q$ . A fundamental property of Bernoulli measure is that they are *translation-ergodic*, meaning that for any Borel set  $X$  which is invariant under translation,  $\mu(X)$  is either 1 or 0 [50]. The somewhat opposite example of measures are dirac measures. The dirac measure associated to configuration  $c \in Q^{\mathbb{G}}$ , denoted  $\delta_c$ , is defined by

$$\delta_c(A) = \begin{cases} 1 & \text{if } c \in A, \\ 0 & \text{if } c \notin A \end{cases}$$

for any Borel set  $A$ .

The set  $\mathcal{M}(Q^{\mathbb{G}})$  is naturally endowed with the weak-\* topology which is the coarsest topology such that the map  $\mu \mapsto \mu([u])$  is continuous for each cylinder set  $[u]$ . This topology is compact and metrizable by the distance

$$d(\mu, \nu) = \sum_{n \in \mathbb{N}} 2^{-n} \max\{|\mu([u]) - \nu([u])| : D \subseteq B_n, u : D \rightarrow Q\}.$$

In this topology, a sequence of measures  $(\mu_n)_n$  converges to  $\mu$  if and only if, for each cylinder set  $[u]$ ,  $\mu_n([u])$  converges to  $\mu([u])$ .

Any CA  $F : Q^{\mathbb{G}} \rightarrow Q^{\mathbb{G}}$  acts naturally on the set of measures  $\mathcal{M}(Q^{\mathbb{G}})$  as follows:  $F(\mu) = \mu \circ F^{-1}$  or, said differently,  $F(\mu)$  is the measure such that for each Borel set one has

$$F(\mu)(X) = \mu(F^{-1}(X)).$$

This way  $(\mathcal{M}(Q^{\mathbb{G}}), F)$  is also a dynamical system. Note that for any  $c \in Q^{\mathbb{G}}$  one has  $F(\delta_c) = \delta_{F(c)}$  because  $F(c) \in A \iff c \in F^{-1}(A)$ . Hence the dynamical system  $(Q^{\mathbb{G}}, F)$  actually embeds into the dynamical system  $(\mathcal{M}(Q^{\mathbb{G}}), F)$ .

**Decision problems and undecidability.** To conclude this definition section, let us briefly clarify the settings behind the decision problems that we will consider. Our problems deal with CA and/or configurations as input. A CA is always given by its finite description (alphabet and local evolution rule). A configuration in input will be given as a map  $c : \mathbb{G} \rightarrow Q$ , which is actually a map from  $A^*$  to  $Q$  where  $A$  is a fixed set of generators of  $\mathbb{G}$ . Depending on the context (computable, finite, periodic configuration) there are various way to represent this map as an input. In the general case, it will be a Turing machine that computes the map.

Our undecidability results will sometimes refer to the arithmetical hierarchy, we refer to [40] for an in depth presentation.

### 3 Convergent CA and their basic properties

The central definition considered in this tutorial is convergence.

**Definition 1.** A dynamical system  $(X, T)$  is convergent if for any  $x \in X$  the orbit of  $x$  under  $T$  converges towards some limit denoted  $T^\omega(x)$ :

$$\forall x \in X : \lim_{n \rightarrow \infty} T^n(x) = T^\omega(x).$$

The convergence of a CA  $(Q^{\mathbb{G}}, F)$  is equivalent to the property that for any  $c \in Q^{\mathbb{G}}$ ,  $\mathcal{T}_c^F$  is eventually constant. When it is the case, we denote by  $\zeta_F(c, z)$  the *freezing time* of cell  $z$  in the orbit of  $c$ , *i.e.* the first time after which cell  $z$  becomes constant in this orbit:

$$\zeta_F(c, z) = \min\{t : \forall t' \geq t, F^{t'+1}(c)_z = F^{t'}(c)_z\}.$$

We will see various examples of convergent CA in the next sections, but let's start by one of the simplest to fix ideas.

**Example 1.** Let  $Q = \{0, 1\}$  and define the CA  $F$  on  $Q^{\mathbb{G}}$  by

$$F(x)_z = \max_{a \in A} x_{z+a}$$

where  $A$  is a fixed finite neighborhood containing the identity of  $\mathbb{G}$ . Clearly,  $F$  can only turn 0s into 1s in any configuration, so it is a convergent CA. It turns out that this simple dynamical systems actually give much information on the group  $\mathbb{G}$  and its generating sets (see [16]).

A first observation is that convergence property on configurations extends to the action on measures.

**Lemma 1.** A CA  $(Q^{\mathbb{G}}, F)$  is convergent if and only if  $(\mathcal{M}(Q^{\mathbb{G}}), F)$  is convergent.

*Proof.* Suppose first that  $(\mathcal{M}(Q^{\mathbb{G}}), F)$  is convergent and consider any  $c \in Q^{\mathbb{G}}$ . Since  $F^t(\delta_c) = \delta_{F^t(c)}$  and since  $(F^t(\delta_c))_t$  converges to some measure  $F^\omega(\delta_c)$  by hypothesis, we have that for any  $q \in Q$ ,  $\delta_{F^t(c)}([q])$  must converge to  $F^\omega(\delta_c)([q])$ . But  $\delta_{F^t(c)}([q])$  is either 1 if  $\mathcal{T}_c^F(t) = q$  or 0 otherwise. This shows that  $\mathcal{T}_c^F$  is eventually constant.

Suppose now for the other direction that  $(Q^{\mathbb{G}}, F)$  is convergent. Consider any measure  $\mu$  and any pattern  $u \in Q^D$  with  $D \subseteq B_n$  for some  $n \in \mathbb{N}$ . We want to show that  $(F^t(\mu)([u]))_t$  converges. Consider the set  $X_t$  of initial configurations whose orbit is frozen for all cells in  $B_n$  after at most  $t$  steps:

$$X_t = \{c \in Q^{\mathbb{G}} : \forall z \in B_n, \zeta_F(c, z) \leq t\}.$$

$(X_t)_t$  is monotone increasing and  $\bigcup_t X_t = Q^{\mathbb{G}}$  so the measure of these sets converges to 1.  $(F^{-t}([u]) \cap X_t)$  is also monotone increasing (if  $c \in F^{-t}([u]) \cap X_t$  then  $F^{t'}(c) \in [u]$  for any  $t' \geq t$ ), so the measure of these sets converges (from below) to some  $\alpha$ . Then, for any  $\epsilon > 0$ , there is some  $t_0 \in \mathbb{N}$  such that  $\mu(X_{t_0}) \geq 1 - \epsilon$  and  $\alpha - \epsilon \leq \mu(F^{-t_0}([u]) \cap X_{t_0}) \leq \alpha$ . For any  $t \geq t_0$ , we can write  $F^t(\mu)([u])$  as:

$$F^t(\mu)([u]) = \mu(F^{-t}([u]) \cap X_{t_0}) + \mu(F^{-t}([u]) \setminus X_{t_0}).$$

But it actually holds that  $F^{-t}([u]) \cap X_{t_0} = F^{-t_0}([u]) \cap X_{t_0}$ , so we have shown  $|F^t(\mu)([u]) - \alpha| \leq \epsilon$ , which concludes convergence of  $(\mathcal{M}(Q^{\mathbb{G}}), F)$  because  $\mu$  and  $u$  were chosen arbitrarily.  $\square$

A second observation on convergent CA is that they cannot be too chaotic. Precisely, a dynamical system  $(X, T)$  is said sensitive to initial conditions if there is  $\epsilon$  such that for all  $\delta$  and all  $x \in X$  there is some  $y \in X$  which is  $\delta$ -close to  $x$  but whose orbit will be  $\epsilon$ -far from that of  $x$  at some future step (see [31] for more details on this notion in the context of CA).

In the following lemma, we use the notation  $T_A(c)$  for some finite set  $A \subseteq \mathbb{G}$  to denote the trace of configuration  $c$  on  $A$ , i.e. the map  $t \mapsto F^t(c)|_A$ .

**Lemma 2.** *No convergent CA is sensitive to initial conditions.*

*Proof.* Suppose by contradiction that  $F$  on  $Q^{\mathbb{G}}$  is convergent and sensitive to initial conditions: there is  $N \in \mathbb{N}$  such that for any  $c \in Q^{\mathbb{G}}$  and any  $p \in \mathbb{N}$  there exists  $c' \in Q^{\mathbb{G}}$  such that  $c|_{B_p} = c'|_{B_p}$  and  $F^t(c)|_{B_N} \neq F^t(c')|_{B_N}$  for some  $t \in \mathbb{N}$ . Consider any configuration  $c_0 \in Q^{\mathbb{G}}$  and  $p_1 \geq N$ . By sensitivity there is  $c' \in Q^{\mathbb{G}}$  and  $t_1 \in \mathbb{N}$  such that either  $T_{B_N}(c_0)$  or  $T_{B_N}(c')$  is non-constant on time interval  $[0, t_1]$ . Denote by  $c_1$  the one among  $c_0$  and  $c'$  that corresponds to the non-constant trace. Let  $p_2 = p_1 + N + r(t_1 + 1)$  where  $r$  is the radius of  $F$ . Applying sensitivity again on  $c_1$  and  $p_2$  we know there exist  $c'$  and  $t_2$  such that:

- $c_1$  and  $c'$  are identical on  $B_{p_2}$ ;
- therefore, by choice of  $p_2$ ,  $T_{B_N}(c_1)$  and  $T_{B_N}(c')$  coincide on interval  $[0, t_1]$ ;
- $T_{B_N}(c_1)$  and  $T_{B_N}(c')$  differ at time  $t_2$ .

So one of  $c_1$  or  $c'$ , denoted  $c_2$ , is such that  $T_{B_N}(c_2)$  is not constant on interval  $[t_1, t_2]$ . Going on with the same reasoning we construct a converging sequence  $(c_n)_{n \in \mathbb{N}}$  of configurations such that  $T_{B_N}(c_n)$  is not constant on each interval  $[t_i, t_{i+1}]$  for  $0 \leq i < n$ . Taking  $c = \lim_n c_n$  we get a trace  $T_{B_N}(c)$  which is not eventually constant contradicting convergence.  $\square$

Our last observation on convergent CA is that they are always irreversible and even non-surjective (except the identity), at least  $\mathbb{G}$  is amenable. The proof below essentially relies on Poincaré recurrence theorem.

**Lemma 3.** *Let  $\mathbb{G}$  be an amenable group. If a convergent CA on  $Q^{\mathbb{G}}$  is surjective, then it is the identity map.*

*Proof.* Let  $F$  be surjective and convergent and let  $\mu$  denote the uniform product measure on configurations of  $F$ . By [4],  $\mu$  is preserved under  $F$ :  $\mu(X) = \mu(F^{-1}(X))$  for any Borel set  $X$  (see [8] for an overview of properties of surjective CA on amenable groups). If we suppose that  $F$  is not the identity map, then there is a word  $u \in Q^{B_n}$  such that for all  $x \in [u]$  it holds  $F(x)_0 \neq x_0$ . We claim that there is a configuration  $x$  such that  $F^t(x) \in [u]$  for infinitely many  $t$ . From this claim we deduce that  $F$  is not convergent because the orbit of  $x$  is not convergent, and the lemma follows. To prove the claim, let us denote  $R_t = \cup_{t' \geq t} F^{-t'}([u])$  and  $R = \cap_t R_t$ . By definition  $R$  is the set of configurations whose orbit visits  $[u]$  infinitely many times and we want to show that  $R \neq \emptyset$ . We actually show that  $\mu(R) > 0$ . Since  $R_{t+1} = F^{-1}(R_t)$  for all  $t \geq 0$  we have  $\mu(R_t) = \mu(R_0)$ . Moreover  $[u] \subseteq R_0$  so we deduce that  $\mu([u] \setminus R_t) \leq \mu(R_0 \setminus R_t) = 0$  since  $R_t \subseteq R_0$  and  $\mu(R_t) = \mu(R_0)$ . Finally, by Boole's inequality we have  $\mu([u] \setminus R) = 0$  so  $\mu(R) > 0$  since  $\mu([u]) > 0$ .  $\square$

## 4 Nilpotency and its Variants

This section explores examples where the asymptotic dynamics is reduced to a singleton. We will consider various properties using either the language of orbits of configurations or measures, or the language of traces.

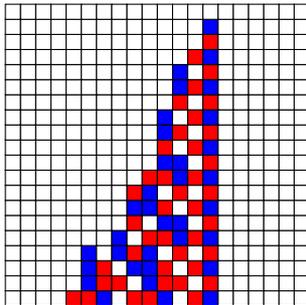
### 4.1 Extreme cases of convergence

Let us start by the strongest forms of nilpotency, which are strengthened forms of convergence.

**Definition 2.** *Let  $F$  be a CA on  $Q^{\mathbb{G}}$ .*

- *$F$  is said nilpotent if there is  $x \in Q^{\mathbb{G}}$  and  $t_0 \in \mathbb{N}$  such that for all  $y \in Q^{\mathbb{G}}$  it holds  $F^{t_0}(y) = x$ .*
- *$F$  is said asymptotically nilpotent if there is  $x \in Q^{\mathbb{G}}$  such that for all  $y \in Q^{\mathbb{G}}$  it holds  $F^t(y) \rightarrow_t x$ .*

**Example 2.** *A local evolution map which is a constant map yields a nilpotent CA, but let us give a non-trivial example. Consider the CA  $F : Q^{\mathbb{Z}} \rightarrow Q^{\mathbb{Z}}$  of radius 1 where  $Q = \{\square, \blacksquare, \blacksquare\}$  and defined by all the transitions appearing in the following space-time diagram (times goes from bottom to top) and such that any transition not appearing in it produces state  $\square$ :*



We claim that  $F^{19}(c)_0 = \square$  for any configuration  $c$  so that  $F$  is nilpotent (we verified this claim by a computer program). However, the diagram above clearly shows that  $F^{18}$  is not a constant map. We challenge the reader to find a nilpotent CA with same alphabet and radius but such that  $F^{19}$  is not a constant map.

The attracting configuration  $x$  in the definitions above must be uniform because it must attract in particular uniform configurations. Thus, these definitions can be expressed as properties of traces:

- $F$  is nilpotent if and only if there is  $t_0$  and  $q \in Q$  such that  $\mathcal{T}_x^F(t) = q$  for all  $t \geq t_0$  and all  $x \in Q^{\mathbb{G}}$ .
- $F$  is asymptotically nilpotent if and only if there is  $q \in Q$  such that all  $x \in Q^{\mathbb{G}}$  there is  $t_x \in \mathbb{N}$  such that  $\mathcal{T}_x^F(t) = q$  for all  $t \geq t_x$ .

The property of nilpotency is also well-known as a property of the *limit set*. The limit set of a CA  $F$  is the set  $\Omega_F = \bigcap_t F^t(Q^{\mathbb{G}})$  (see [13, 29]). Nilpotency is equivalent to  $\Omega_F$  being a singleton.

Asymptotic nilpotency can be reformulated through the action of the CA on measures.

**Lemma 4.** *A CA  $F$  on  $Q^{\mathbb{G}}$  is asymptotically nilpotent if and only there is  $q \in Q$  such that for any  $\mu \in \mathcal{M}(Q^{\mathbb{G}})$  it holds  $F^t(\mu) \rightarrow_t \delta_{\bar{q}}$ .*

*Proof.* The property on measures implies asymptotic nilpotency by the same argument as the first part of the proof of Lemma 1 with the additional fact that  $F^\omega(\delta_c) = \delta_{\bar{q}}$  which implies that  $F^\omega(c) = \bar{q}$ .

Conversely, if we suppose that  $F$  is asymptotically nilpotent, we can use the same reasoning as in part two of the proof of Lemma 1. The additional fact in this case is that  $F^\omega(c) = \bar{q}$  for all  $c \in Q^{\mathbb{G}}$  so that  $F^{-t}([u]) \cap X_t = \emptyset$  for all pattern  $u$  except the uniform one:  $z \in B_n \mapsto q$ . We deduce that  $F^t(\mu)$  converges towards  $\delta_{\bar{q}}$ .  $\square$

It turns out that, under some conditions on  $\mathbb{G}$ , nilpotency and asymptotic nilpotency are equivalent. One might believe that this can be easily proved by a standard compactness argument, but it is not the case.

**Theorem 1** ([26, 41]). *If  $\mathbb{G} = \mathbb{Z}^d$  then nilpotency is equivalent to asymptotic nilpotency.*

*sketch.* Let  $q$  be the quiescent state involved in the asymptotic nilpotency property. First, we remark that for any ball  $B_n$  there is a uniform bound  $T_n$  such that in the orbit of any configuration  $c$  there is some time step  $t \leq T_n$  for which

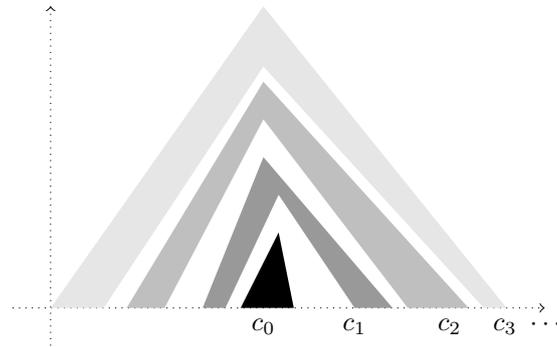


Figure 2: The infinite nested space-time diagram contradicting asymptotic nilpotency in Theorem 1. The white part represents state  $q$ , the grayed parts represent states other than  $q$ , and each nuance of gray correspond to a particular configuration  $c_i$ .

the cells inside  $B_n$  are all in state  $q$ :  $\forall z \in B_n, F^t(c)_z = q$ . Indeed, otherwise we would obtain by compactity a configuration whose orbit would not converge to  $\bar{q}$ . Call *finite* any configuration which is everywhere  $q$  except on a finite region, and call *mortal* any configuration  $c$  such that there is  $t$  with  $F^t(c) = \bar{q}$ .

Let us first establish the theorem for  $\mathbb{G} = \mathbb{Z}$  as in [26]. There are two key arguments:

- first, we show nilpotency on finite mortal configurations, *i.e.* there is a uniform  $T$  such that for all finite mortal  $c$  it holds  $F^T(c) = \bar{q}$ . This is proved using the remark above about times  $T_n$ : if there is no uniform time bound on the mortality of finite mortal configuration, then for arbitrary large  $B_n$  and arbitrary large  $t$  we can find a finite mortal configuration  $c$  such that  $c$  is uniformly  $q$  on  $B_n$  but  $F^t(c)_0 \neq q$ . From there, we can produce an infinite *nested space-time diagram* as in Figure 2 by taking a well-chosen sequence  $(c_i)$  of such configurations and considering their superposition  $c$ : it is sufficient that the non- $q$  portions of space-time diagrams are disjoint enough among the  $c_i$  (taking into account the radius of the CA). The orbit of this configuration  $c$  gives a contradiction with the convergence hypothesis.
- second, it can be shown that if the CA is not nilpotent, then we can extract finite mortal configurations that die arbitrarily late: indeed, by enclosing a finite configuration alive at time  $t$  with blocking words (whose existence is granted by Lemma 2, see [31]), we obtain a mortal configuration still alive at time  $t - b$  where  $b$  is a constant depending only on the choice of blocking words. This contradicts the previous item, so we deduce that the CA is nilpotent.

To extend the result to higher dimensions, it is enough to show that all finite configurations are mortal (this together with asymptotic nilpotency implies nilpotency, by the nested construction argument). To simplify we restrict to dimension 2 and consider any asymptotically nilpotent CA  $F$ . The idea is to reduce to the one-dimensional case by remarking that if a finite configuration  $c$  is periodized vertically to form a configuration  $c'$ , then  $c'$  is mortal because the

action of  $F$  on  $c'$  can be seen as the action of a one-dimensional asymptotically nilpotent CA on a one-dimensional configuration. The key step of the proof is then to show that no finite configuration can spread too much in the horizontal direction. This step is proved by contradiction using again a nested construction of an infinite space-time diagram, but this time a vertical periodization is applied at each step to guarantee mortality. Of course, this argument on the bounded horizontal spreading of finite configuration is also valid vertically, so we deduce that the orbit of any finite configuration remains inside a finite support forever. This is enough to conclude mortality since  $F$  is supposed asymptotically nilpotent.  $\square$

The above result has been greatly generalized by [45] in at least two ways: the class of dynamical systems considered (for instance, one can consider CA defined over a subshift rather than the whole space  $Q^{\mathbb{G}}$ ) and the class of groups  $\mathbb{G}$  considered. To obtain these results, a notion of tiered dynamical systems was developed which goes way beyond the scope of this tutorial. The authors also coined the term *nil-rigidity* and used it with various classes of dynamical systems. Let us stick to the simplest setting and say that a group  $\mathbb{G}$  is nilrigid if any asymptotically nilpotent CA on  $Q^{\mathbb{G}}$  is nilpotent. Despite the many results obtained about nilrigidity in [45] they left some interesting open questions (see section 11), some of which were already asked in [42].

**Question 1** ([45, 42]). *Is there any finitely generated group  $\mathbb{G}$  which is not nilrigid? In particular is the free group with 2 or more generators nilrigid?*

## 4.2 Variations with partial convergence and measures

Let us now consider variations of these strong notions of nilpotency. We can first consider a similar property of orbits but restricted to a subset of initial configurations.

A configuration  $x \in Q^{\mathbb{G}}$  is *totally periodic* if its orbit by translation  $\mathcal{O}_{\mathbb{G}}(x)$  is finite. When  $\mathbb{G} = \mathbb{Z}^d$  a totally periodic configuration is a configuration with  $d$  non-colinear periods.

**Definition 3.**  *$F$  is nilpotent on periodic configurations if there is a configuration  $c$  such that  $(F^t(x))_t$  converges to  $c$  for any totally periodic configuration  $x$ .*

Note again that the attracting configuration  $c$  must be uniform in this definition. An asymptotically nilpotent CA must be nilpotent on periodic configurations, but the converse is far from being true as we will see below.

As a second (seemingly unrelated) variation around nilpotency, we can change the point of view and switch from configurations to measures. A measure  $\mu \in \mathcal{M}(Q^{\mathbb{G}})$  is invariant for a CA on  $Q^{\mathbb{G}}$  if  $F(\mu) = \mu$ . The following definition is classical in dynamical system theory [50] and can be defined as a rigidity property of invariant measures.

**Definition 4.**  *$F$  is uniquely ergodic if it possesses a unique invariant measure.*

The invariant measures of  $F$  are exactly the limit points of Cesaro mean sequences  $(u_n)_{n \in \mathbb{N}}$  of the form ([50, Theorem 6.9])

$$u_n = \frac{1}{n} \sum_{t=0}^{n-1} F^t \mu$$

for  $\mu \in \mathcal{M}(Q^{\mathbb{G}})$ .

Orbit-wise unique ergodicity is equivalent to the convergence of all such Cesaro mean sequences to the same limit. Therefore we could also name this property 'Cesaro-nilpotency' to stress the dynamical side of it but it is best known as *unique ergodicity*. Note that by Lemma 4, any asymptotically nilpotent CA is also uniquely ergodic.

Nilpotency over periodic configuration and unique ergodicity are both implied by asymptotic nilpotency. The next lemma shows that the three properties actually form a hierarchy when  $\mathbb{G} = \mathbb{Z}^d$ .

**Lemma 5.** *A uniquely ergodic CA with  $\mathbb{G} = \mathbb{Z}^d$  is nilpotent on periodic configurations.*

*Proof.* Consider any uniquely ergodic CA  $F$  on  $Q^{\mathbb{G}}$ . Suppose first that there are two disjoint temporal cycles of totally periodic configurations, *i.e.* two sequences of totally periodic configurations  $(c_0, \dots, c_{k-1})$  and  $(d_0, \dots, d_{l-1})$  with  $F(c_i) = c_{i+1 \bmod k}$  and  $F(d_j) = d_{j+1 \bmod l}$  for  $0 \leq i < k$  and  $0 \leq j < l$ . Then we can define two measures  $\mu_c = \frac{1}{k} \sum_{i=0}^{k-1} \delta_{c_i}$  and  $\mu_d = \frac{1}{l} \sum_{j=0}^{l-1} \delta_{d_j}$  that are both invariant under  $F$  because  $(c_i)$  and  $(d_j)$  are cyclic orbits. Moreover since the two cycles are disjoint there must exist a cylinder  $[u]$  such that  $c_i \in [u]$  for some  $i$  but  $d_j \notin [u]$  whatever  $j$ . Thus  $\mu_c([u]) > 0$  and  $\mu_d([u]) = 0$  which proves that  $\mu_c$  and  $\mu_d$  are two distinct invariant measures: a contradiction.

We have shown that  $F$  has the following synchronization property:

*there is a cycle of totally periodic configurations  $(c_0, \dots, c_{k-1})$  such that the orbit of any totally periodic configuration eventually enters this cycle  $(c_i)$ .*

We claim that  $k = 1$  which concludes the proof since it means that  $F$  is nilpotent on periodic configurations.

We are actually going to show the stronger claim that no CA  $F$  (not necessarily uniquely ergodic) with  $\mathbb{G} = \mathbb{Z}^d$  can have the synchronization property above with  $k \geq 2$ . The argument we give is due to G. Richard (see [17] for more background on the synchronization problem). First note that if this synchronization property holds for some CA  $F$  with  $\mathbb{G} = \mathbb{Z}^d$  and  $d > 1$  then it must also hold for some CA with  $\mathbb{G} = \mathbb{Z}$ . Indeed, considering  $F$  on configurations which are constant in  $d - 1$  directions and periodic in the last one, we actually define a one-dimensional CA which has the synchronization property as  $F$  does. We can thus suppose without loss of generality that  $d = 1$ . Clearly, each configuration  $c_i$  in the attracting cycle must be uniform because any uniform configuration has an orbit eventually entering this cycle. However, if the synchronization property holds with  $k \geq 2$  then  $F$  has no quiescent state. Now consider  $f : Q^{\mathbb{N}} \rightarrow Q$  the local map of  $F$  and let  $w \in Q^{\mathbb{N}}$  be a semi-infinite word verifying

$$w_{n+m} = f(w_n, \dots, w_{n+m-1})$$

for all  $n \in \mathbb{N}$ . There must exist  $n_1$  and  $n_2$  with  $n_1 + m < n_2$  such that

$$w_{[n_1, n_1+m-1]} = w_{[n_2, n_2+m-1]}.$$

Consider the word  $w = w_{[n_1, n_2-1]}$  of length at least  $m + 1$ . By construction and since  $F$  has no quiescent state  $w$  contains at least two distinct letters from  $Q$ . Moreover, if  $c$  is a periodic  $\mathbb{Z}$ -configuration of period  $w$  then  $F(c) = \sigma_p(c)$  for

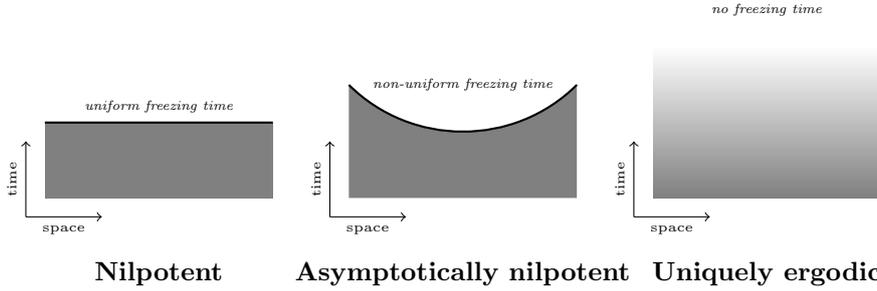


Figure 3: Properties of any space-time diagram of a nilpotent, asymptotically nilpotent and uniquely ergodic CA respectively. White color represents the quiescent state, gray color any state and various shades of gray corresponds to various density of the quiescent state (the lighter, the higher the density). Time goes from bottom to top.

some  $p$ . Therefore the orbit of  $c$  is a cycle of non-uniform configurations, which contradicts the synchronization property.  $\square$

Interestingly, we can also capture unique ergodicity as a property of traces of orbits on configurations: it is the property that there is some attracting state whose frequency goes to 1 when considering larger and larger prefixes of any trace. To be more precise, for any  $q \in Q$  and any word  $w \in Q^{\mathbb{N}}$  we denote by  $d_q(w)$  the (inferior) asymptotic density of occurrences of  $q$  in  $w$ :

$$d_q(w) = \liminf_n \frac{\#\{0 \leq i < n : w_i = q\}}{n}.$$

We then get this equivalent formulation of unique ergodicity [46, Proposition 3.2].

**Lemma 6.** *A CA  $F$  on  $Q^{\mathbb{G}}$  with  $\mathbb{G} = \mathbb{Z}^d$  is uniquely ergodic if and only if there is  $q \in Q$  such that for any  $x \in Q^{\mathbb{G}}$  it holds  $d_q(\mathcal{T}_x^F) = 1$ .*

*Proof.* Suppose first that  $F$  is uniquely ergodic, then by Lemma 5 it is nilpotent on periodic configurations and has a (unique) quiescent state  $q$  and its unique invariant measure is  $\delta_{\bar{q}}$ . Consider any  $x \in Q^{\mathbb{G}}$  and any  $\epsilon > 0$ . If there were infinitely many  $n \in \mathbb{N}$  such that  $\frac{\#\{0 \leq i < n : \mathcal{T}_x^F(i) \neq q\}}{n} > \epsilon$ , then we could extract a limit point  $\mu$  from the sequence  $(\frac{1}{n} \sum_{t=0}^{n-1} F^t(\delta_x))_n$  which would be an invariant measure [50, Theorem 6.9] such that  $\mu([q]) < 1 - \epsilon$  (recall that  $F^t(\delta_x) = \delta_{F^t(x)}$ ) hence different from  $\delta_{\bar{q}}$ : a contradiction. We deduce that  $d_q(\mathcal{T}_x^F) = 1$ .

Suppose now that  $d_q(\mathcal{T}_x^F) = 1$  for all  $x \in Q^{\mathbb{G}}$  for some  $q \in Q$ . Then  $q$  must be a quiescent state and thus the measure  $\delta_{\bar{q}}$  is invariant and also ergodic. If it were not the unique invariant measure then there would be another invariant ergodic measure  $\mu$  (see [50, Theorem 6.10]), which would necessarily verify  $\mu([q]) < 1$  in order to differ from  $\delta_{\bar{q}}$  (otherwise the Choquet's decomposition theorem on invariant measures would be contradicted). Then by the ergodic Theorem (see

[50, Lemma 6.13]), we would have some  $x \in Q^{\mathbb{G}}$  with

$$\frac{1}{n} \sum_{t=0}^{n-1} f(F^t(x)) \rightarrow_n \mu([q])$$

where  $f$  is the characteristic map of  $[q]$ . But then  $d_q(\mathcal{T}_x^F) < 1$  which contradicts the hypothesis. □

Uniquely ergodic CA form a strictly larger class than nilpotent CA. The example in the following result to show this separation is highly non-trivial and we will not present it here, but it is quite remarkable that such behaviors are possible with CA.

**Theorem 2** ([46]). *There exists a uniquely ergodic CA on  $\mathbb{Z}$  which is not nilpotent.*

**Remark 1.** *The CA from the above theorem cannot be convergent because then it would be asymptotically nilpotent by Lemma 6 and this would contradict Theorem 1. From Lemma 1 it is also non-convergent when starting from some measures. However, there is convergence in Cesaro mean starting from any  $\mu$  since otherwise, a Cesaro mean sequence would have several limit points and hence produce several invariant measure for the CA as already explained above.*

**Theorem 3.** *There exists a CA on  $\mathbb{Z}$  which is nilpotent over periodic configurations but not uniquely ergodic.*

*Sketch of proof.* It can be shown using the classical construction of [30] by taking a NE-deterministic aperiodic Wang tileset  $T$  and transforming it into a one-dimensional CA  $F$  with alphabet  $T \cup \{s\}$  where  $s$  is a spreading state.  $s$  being a spreading state,  $F$  admits  $\delta_{\bar{s}}$  as invariant measure. Because the tileset has no valid periodic tiling,  $F$  is nilpotent on periodic configurations. On the other hand, because the tileset admits a valid (aperiodic) tiling, it means that there is a configuration  $c$  whose orbit under  $F$  has no occurrence of  $s$ . Taking any limit point of the sequence of Cesaro means

$$\mu_n = \frac{1}{n} \sum_{t=0}^{n-1} F^t(\delta_c)$$

gives an invariant measure such that  $\mu([s]) = 0$  since  $\mu_n([s]) = 0$  for all  $n \in \mathbb{N}$ . We thus get a second invariant measure for  $F$ . □

So far we considered properties concerning all initial configurations, or a meager set of initial configuration. Natural variants of nilpotency have been considered in the literature to capture the behavior on 'most' or 'a large set' of initial configurations.

**Definition 5.** *Consider a CA  $F$  on  $Q^{\mathbb{G}}$  and  $\mu \in \mathcal{M}(Q^{\mathbb{G}})$ . Then  $F$  is:*

- *generically nilpotent if there is  $q \in Q$  and a comeager set  $B \subseteq Q^{\mathbb{G}}$  such that  $F^t(x) \rightarrow_t \bar{q}$  for all  $x \in B$ ;*

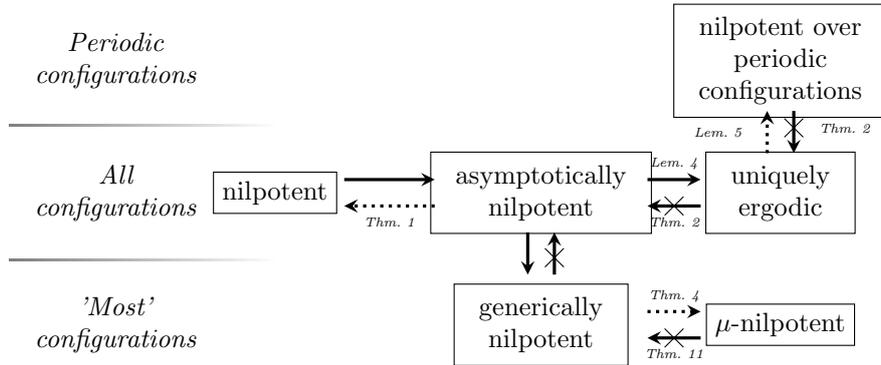


Figure 4: Nilpotency and its variants with (non-)implications between them as seen so far. Implications with dotted arrows refer to Theorems that require some hypothesis on either  $\mathbb{G}$  or  $\mu$ . Counter-examples (or non-implications) are represented with a cross on the arrow.

- $\mu$ -nilpotent if there is  $q \in Q$  such that  $F^t(\mu) \rightarrow_t \delta_{\bar{q}}$ .

Generic nilpotency and  $\mu$ -nilpotency both specify a behavior on 'most configurations', where 'most' is to be understood either topologically or according to a measure. It turns out that for well-behaved measures the topological version is stronger.

**Theorem 4** ([15]). *Let  $F$  be a generically nilpotent CA on  $Q^{\mathbb{G}}$  and  $\mu \in \mathcal{M}(Q^{\mathbb{G}})$  a full-support translation-ergodic measure. Then  $F$  is  $\mu$ -nilpotent.*

There are ways to characterize the previous variants of nilpotency by defining variants of limit sets, *i.e.* sets of (typical) asymptotic configurations [32, 15]. Each time, the nilpotency variant is equivalent to having such asymptotic sets being singletons (see [?, 6]).

The implication diagram of Figure 4 is not complete, in particular concerning the link between unique ergodicity and  $\mu$ -nilpotency.

**Question 2.** *What are the other implications or counter-examples in the diagram of Figure 4? In particular, for what measure  $\mu$  does unique ergodicity implies  $\mu$ -nilpotency? Said differently, for what measure  $\mu$  the convergence in Cesaro mean implied by unique ergodicity is actually a simple convergence?*

### 4.3 The undecidability garden

We conclude this section by undecidability results related to the notions presented above. Of course, an undecidability result is a negative result and this might appear repulsive to some. However, behind almost any undecidability result there is a positive one: a construction technique (a reduction) providing examples with rich and complex behaviors. Each of the result below rely on a non-trivial and nice construction idea that is of independent interest. A presentation of each technique would take too much place for this tutorial, but we invite the reader to dig into the corresponding references.

**Theorem 5** ([30, 1]). *The set of 1D nilpotent CA is  $\Sigma_1^0$ -complete.*

**Theorem 6** ([46]). *The set of 1D uniquely ergodic CA is  $\Pi_2^0$ -complete.*

**Theorem 7** ([?]). *The set of 1D generically nilpotent CA is  $\Sigma_2^0$ -complete.*

**Theorem 8** ([6]). *Let  $\mu$  denote the uniform Bernoulli measure. The set of 1D  $\mu$ -nilpotent CA is  $\Pi_3^0$ -complete.*

## 5 Proving Convergence

### 5.1 Bounded change CA

Proving convergence of a CA  $F$  means proving that there exists a time bound for each trace  $\mathcal{T}_c^F$  after which the trace is constant. This has been defined above as the freezing time  $\zeta_F(c, 0)$ . In general this time bound depends on the initial configuration (see Example 3), but, even if there is a global bound that doesn't depend on initial configuration like in nilpotent CA, it is generally uncomputable. Instead of bounding directly the time of the last change in traces, a way to tackle the problem is to bound the number of changes that can occur in a trace. The following definition is useful [49, 33].

**Definition 6.** *A CA  $F$  is  $k$ -change for some  $k \in \mathbb{N}$  if for any  $x \in Q^{\mathbb{Z}}$ , there are at most  $k$  state changes in  $\mathcal{T}_x^F$ , i.e.*

$$\#\{t \in \mathbb{N} : \mathcal{T}_x^F(t+1) \neq \mathcal{T}_x^F(t)\} \leq k.$$

*A CA is bounded-change if it is  $k$ -change for some  $k$ .*

A bounded-change CA is necessarily convergent: a trace which is not ultimately constant cannot exist since it would contain infinitely many changes. However the converse is false as shown by the following example (it is used extensively as a basic building block in [38]).

**Example 3.** *Let us consider the state set  $Q = \{\boxed{B}, \boxed{\rightarrow}, \boxed{\leftarrow}, \boxed{\leftarrow}, \boxed{\rightarrow}, e\}$  where  $\boxed{\leftarrow}$  and  $\boxed{\rightarrow}$  represent an active head moving left or right respectively,  $\boxed{\rightarrow}$  and  $\boxed{\leftarrow}$  are states indicating the position of the head (to the right and to the left respectively),  $\boxed{B}$  is a blank state and  $e$  is an error state. Let  $\Sigma$  be the set of configurations whose patterns of length two all appear in the space-time diagram of Figure 5, i.e. configurations made of zones with a unique active head (possibly none if the zone is infinite) in a blank background. Define the CA  $F : Q^{\mathbb{Z}} \rightarrow Q^{\mathbb{Z}}$  of radius 2, such that:*

- *$e$  is a spreading state and any cells turns into state  $e$  if there is some pattern not in  $L_2$  in its neighborhood,*
- *in the  $\Sigma$ -valid zones with a unique head, the only cells that change their states in one step are the one around the head, and they update in such a way that the head makes zigzag inside the zone and reduce the size of the zone by one unit at each bounce on a border. Precisely, all the transitions appear in the space-time diagram of Figure 5.*

*First, it is clear that  $F$  cannot be  $k$ -change whatever the value of  $k$ , because in a  $\Sigma$ -valid finite zone of size  $n$ , the orbit of the central cells contains at least  $n$  changes. On the other hand, if we consider any initial configuration  $c \in Q^{\mathbb{Z}}$  and any position  $z \in \mathbb{Z}$  there are only 3 possible cases:*

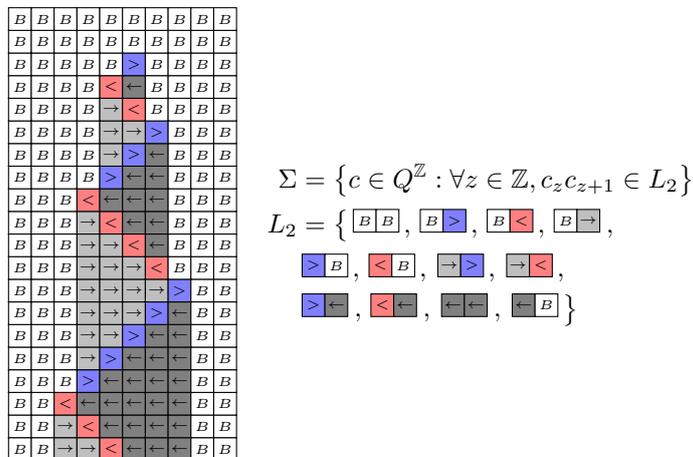


Figure 5: A convergent but not bounded-change CA (time from bottom to top in the space-time diagram on the left).

- either  $c \notin \Sigma$  and in this case the spreading state  $e$  will appear somewhere and reach position  $z$  at some step,
- or  $c \in \Sigma$  and  $z$  belongs to a finite valid zone, so after some number of zigzags of the head, the zone will shrink letting  $z$  outside in state  $\boxed{B}$ ,
- or  $c \in \Sigma$  and  $z$  belongs to an infinite valid zone, and after at most two passages of the head (at most one bounce on the boundary if any), the head will move towards infinity and cell  $z$  will remain unchanged forever.

In any case, we have that the trace at  $z$  starting from  $c$  is eventually constant. Hence  $F$  is convergent.

To use a metaphor inspired from physics, each cell of a bounded-change CA can be seen as a system that evolves according to information received from its neighbors, and for which each state change has a fixed energy cost. Then, in each orbit and at each cell, the instantaneous energy defined as the number of remaining potential state changes is non-increasing with time, and strictly decreasing at each state change. The number  $k$  in Definition 6 can be interpreted as the cell-wise capacity, *i.e.* the maximal energy an individual cell can hold initially.

As seen in previous section, a nilpotent CA  $F$  is such that  $F^t$  is a constant map for some  $t$ , so it is always bounded-change. Nilpotency for CA with a spreading state actually Turing reducible to the property of bounded-change as follows: consider  $F$  with a spreading state and add a  $\{0, 1\}$  component to states that constantly exchanges 0 and 1, except if there is a spreading state in the neighborhood in which case the additional component turns into 0. This new CA  $F'$  is bounded-change if and only if  $F$  is nilpotent: first, if  $F$  is nilpotent then  $F'$  also because the spreading state of  $F$  forces 0 everywhere on the second component; conversely, if  $F$  is not nilpotent then it must possess an orbit without any occurrence of the spreading state (by compactity), and then the

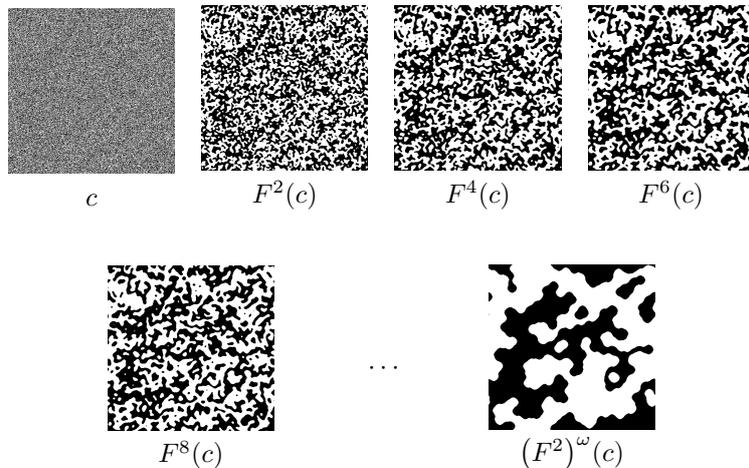


Figure 6: An orbit of  $F^2$  where  $F$  is some majority CA on  $\mathbb{Z}^2$  with symmetric neighborhood  $S$  with  $(0,0) \in S$  as in the conditions of Theorem 10, so that  $F^2$  is bounded-change and hence convergent.

corresponding orbit in  $F'$  is constantly changing the state on the second layer. By undecidability of nilpotency we deduce the following theorem.

**Theorem 9.** *It is undecidable to determine whether a 1D CA is bounded-change or not.*

Despite the general undecidability, this approach turns out to be very fruitful to analyze some majority CA. A majority cellular automata consist in taking in each cell the state which has the majority of occurrences among the neighboring cells. The majority CA with neighborhood  $V \subseteq \mathbb{G}$  is defined on alphabet  $Q = \{0, 1\}$  by

$$F(x)_z = \begin{cases} 1 & \text{if } \sum_{i \in V} x_{z+i} > \#V/2, \text{ or } \sum_{i \in V} x_{z+i} = \#V/2 \text{ and } x_z = 1 \\ 0 & \text{otherwise.} \end{cases}$$

A subset  $S \subseteq \mathbb{Z}^d$  is symmetric if  $x \in S \implies -x \in S$ . The following result concerning majority CA is a specific case of a much more general result from [20] which deals with infinite graphs with some growth condition. A similar qualitative behavior for majority evolution rules was established before in the settings of automata networks (the lattice of cells is finite, the evolution rule is possibly non-uniform), see [23]. It is based on a decreasing energy function in both case.

**Theorem 10** ([20]). *Let  $F$  be any majority CA with  $\mathbb{G} = \mathbb{Z}^d$  and symmetric neighborhood  $S$  with  $0 \in S$ , then  $F^2$  is bounded-change.*

*Proof.* Denote by  $r$  the radius of the CA, i.e. the maximum  $\|u\|_\infty$  for  $u \in S$ . Let us fix any  $x \in Q^\mathbb{G}$ . The core of the argument is to consider the following “energy” function for any  $x \in Q^\mathbb{G}$  and any  $n, t \in \mathbb{N}$ :

$$E_n^t = \sum_{u \in B_n} \sum_{v \in B_n} a(u, v) |F^{t+1}(x)_u - F^t(x)_v|$$

where  $a(u, v) = 1$  if  $v \in u + S$ , and 0 else. Because  $S$  is symmetric  $a(u, v) = a(v, u)$  so we can write  $E_n^{t-1}$  as follows:

$$E_n^{t-1} = \sum_{u \in B_n} \sum_{v \in B_n} a(u, v) |F^{t-1}(x)_u - F^t(x)_v|.$$

For any  $t > 0$ , we will first give an upper bound on  $\Delta_n^t = E_n^t - E_n^{t-1}$  which intuitively means that the energy  $E_n^t$  essentially decreases with time, but only up to a ‘‘controlled perturbation’’ of the border of the ball  $B_n$ . Denoting

$$\Sigma_{n,t}^+(u) = \sum_{v \in B_n} a(u, v) |F^{t+1}(x)_u - F^t(x)_v|, \text{ and}$$

$$\Sigma_{n,t}^-(u) = \sum_{v \in B_n} a(u, v) |F^{t-1}(x)_u - F^t(x)_v|$$

we can rewrite  $\Delta_n^t$  as

$$\Delta_n^t = \sum_{0 \leq i \leq n-r} \sum_{u \in C_{i,t}} \Sigma_{n,t}^+ - \Sigma_{n,t}^- + \sum_{n-r < i \leq n} \sum_{u \in C_{i,t}} \Sigma_{n,t}^+ - \Sigma_{n,t}^- \quad (1)$$

where  $C_{i,t}$  denote the cells that have different states between step  $t+1$  and  $t-1$  (i.e.  $F^{t+1}(x)_u \neq F^{t-1}(x)_u$ ) and belong to the sphere of radius  $i$  (i.e.  $\|u\|_\infty = i$ ). This expression for  $\Delta_n^t$  holds because  $\Sigma_{n,t}^+(u) = \Sigma_{n,t}^-(u)$  when  $F^{t+1}(x)_u = F^{t-1}(x)_u$ .

We first claim that  $\Sigma_{n,t}^+(u) - \Sigma_{n,t}^-(u) \leq \frac{\#S-1}{2}$  for any  $u \in B_n$ : this holds because actually  $\Sigma_{n,t}^+(u) \leq \frac{\#S-1}{2}$  must hold to ensure that a majority of neighbors of  $u$  are in state  $F^{t+1}(x)_u$  at time  $t$ .

We now claim that  $\Sigma_{n,t}^+(u) - \Sigma_{n,t}^-(u) \leq -1$  for any  $u \in \bigcup_{0 \leq i \leq n-r} C_{i,t}$ . Indeed,

we can split the neighbors of any such  $u$  into two sets

$$S_u^+ = \{v \in u + S : F^t(x)_v = F^{t+1}(x)_u\}, \text{ and}$$

$$S_u^- = \{v \in u + S : F^t(x)_v = F^{t-1}(x)_u\},$$

and then  $S_u^+ = \Sigma_{n,t}^-(u)$  because  $u + S \subseteq B_n$ , and similarly  $S_u^- = \Sigma_{n,t}^+(u)$ . By the majority rule we must have  $\#S_u^+ > \#S_u^-$  (recall that  $S$  is of odd cardinality by hypothesis) which proves the claim.

From the two claims we can rewrite Equation (1) above as:

$$\Delta_n^t \leq -\# \bigcup_{0 \leq i \leq n-r} C_{i,t} + \#(B_n \setminus B_{n-r}) \frac{(\#S-1)}{2}. \quad (2)$$

We are now going to deduce a bound on the number of changes in  $\mathcal{T}_x^{F^2}$  from the above inequality in two steps: using a pondered sum spatially and then summing over time. Spatially, the trick is to use a telescoping sum to attenuate the perturbation of the second term in the upper bound of Equation (2) and put more weight on the central cell. Choose  $n = kr$ , let  $s = \frac{(\#S-1)}{2}$  and consider the sum

$$TE_n^t = E_n^t + \frac{1}{s} E_{n-r}^t + \left(1 + \frac{1}{s}\right) \frac{1}{s} E_{n-2r}^t + \left(1 + \frac{1}{s}\right)^2 \frac{1}{s} E_{n-3r}^t + \dots + \left(1 + \frac{1}{s}\right)^{k-1} \frac{1}{s} E_0^t.$$

By Equation (2), and by the identity

$$1 + \frac{1}{s} + \left(1 + \frac{1}{s}\right)\frac{1}{s} + \cdots + \left(1 + \frac{1}{s}\right)^{k-1}\frac{1}{s} = \left(1 + \frac{1}{s}\right)^k$$

we get

$$TE_n^t - TE_n^{t-1} \leq -\left(1 + \frac{1}{s}\right)^k C_{0,t} + s\#(B_n \setminus B_{n-r}).$$

By summing over time the successive differences we then obtain

$$TE_n^t - TE_n^0 \leq -\left(1 + \frac{1}{s}\right)^k \sum_{i < t} C_{0,i} + (t-1)s\#(B_n \setminus B_{n-r})$$

so

$$\sum_{0 \leq i < t} C_{0,i} \leq \frac{TE_n^0}{\left(1 + \frac{1}{s}\right)^k} + \frac{(t-1)s\#(B_n \setminus B_{n-r})}{\left(1 + \frac{1}{s}\right)^k}.$$

When  $n$  grows, the first term of this upper bound converges to a constant independent of  $t$  and of  $x$  (from the definition of  $TE_n^0$  because  $E_n^0$  is at most the square of the size of  $B_n$ ), and the second term vanishes. This concludes the proof because it shows that  $\sum_{i=0}^{\infty} C_{0,i}$  is bounded independently of  $x$ , and it counts the number of state changes in the trace  $\mathcal{T}_x^{F^2}$  and the trace  $\mathcal{T}_{F(x)}^{F^2}$ .  $\square$

Let us show two counter-examples to Theorem 10 when hypothesis on either  $\mathbb{G}$  or the symmetry of neighborhood are removed:

1. let  $\mathbb{G}$  be the free group with 2 generators,  $V$  the symmetric neighborhood made of all generators and their inverse and element  $0_{\mathbb{G}}$ , and denote by  $F$  the associated majority CA. Define the configuration  $c^b$  for any  $b : \mathbb{N} \rightarrow \{0, 1\}$  by

$$c_g^b = b(\|g\|_{\mathbb{G}}).$$

Because we are in the free group with 2 generators, if  $g \in B_n \setminus B_{n-1}$  then  $g + V$  contains 3 elements in  $B_{n+1} \setminus B_n$  and  $V$  is of size 5. Therefore we have  $F(c^b)_g = b(\|g\|_{\mathbb{G}} + 1)$ , and thus  $F(c^b) = c^{b'}$  with  $b'(n) = b(n+1)$  for all  $n \in \mathbb{N}$ . By choosing  $b$  such that  $b(i+1) \cdots b(i+2k) = 0^k 1^k$  for arbitrarily large  $k$  and for some  $i$ , we deduce that there is no  $p > 0$  such that  $F^p$  is convergent.

2. going back to  $\mathbb{G} = \mathbb{Z}^2$ , let us consider the non-symmetric neighborhood  $V = \{(0, 0), (0, 1), (1, 0)\}$  and let  $F$  be the associated majority CA. Define the configuration  $c^b$  for any  $b : \mathbb{N} \rightarrow \{0, 1\}$  by

$$c_{(x,y)}^b = b(x+y).$$

Again, for any  $(x, y) \in \mathbb{Z}^2$  there are two elements  $z$  in  $z + V$  such that  $c_{z'}^b = b(x+y+1)$ , so  $F(c^b)_{(x,y)} = b(x+y+1)$ . We can conclude as above that there is no  $p > 0$  such that  $F^p$  is convergent.

## 5.2 Freezing CA

The argument of Theorem 10 to prove the bounded-change property is not immediate and the property is generally undecidable, even when  $\mathbb{G} = \mathbb{Z}$  (Theorem 9).

In some cases, however, the structure of the local rule of a CA directly implies the bounded-change property. The so called *freezing CA*, introduced in [21], are an example.

**Definition 7.** A CA  $F$  on  $Q^{\mathbb{G}}$  is freezing if there is some partial order  $(Q, <)$  on its state set such that for any  $c \in Q^{\mathbb{G}}$  and any  $z \in \mathbb{G}$  it holds:

$$F(c)_z < c_z.$$

Clearly, in a freezing CA the number of changes at any cell in any orbit is bound by the depth of the partial order, so all freezing CA are bounded-change. The freezing property can be tested efficiently by looking at the local transition table. Indeed, given any CA  $F$ , we can compute the canonical state change graph  $(Q, \rightarrow)$  defined by  $q_1 \rightarrow q_2$  if some transition changes state  $q_1$  into  $q_2$ . Then one can check that  $F$  is freezing if and only if this graph is acyclic.

Going back to our earlier metaphor, the partial order involved in the freezing property can be seen as a local non-increasing energy that serves as a certificate for the bounded change property. Of course, not all bounded-change CA are freezing: for instance  $F^2$ , where  $F$  is some CA from the hypothesis of Theorem 10, is a bounded-change CA but it cannot be freezing since  $F$  and hence  $F^2$  commute with the transformation that permutes states 0 and 1. Thus, if transition  $0 \rightarrow 1$  exists in the canonical state change graph of  $F^2$ , then transition  $1 \rightarrow 0$  also exists, which prevent any freezing order to satisfy Definition 7.

There are many example of freezing CA studied in the literature [25, 47, 18, 2, 24, 5].

A freezing CA  $F$  for some order  $<$  can also be *monotone*:

$$x < y \Rightarrow F(x) < F(y)$$

where  $<$  denotes the cellwise extension of  $<$  to configurations. Among these examples, the classical bootstrap percolation CA [11] has been the starting point of a rich branch of percolation theory, which can be view as the study of the qualitative behavior of monotone freezing CA initialized on random Bernoulli configurations [5, 28, 3]. This CA, denoted  $F_B$  is the sequel, is defined on  $\{0, 1\}^{\mathbb{Z}^2}$  as follows:

$$F_B(c)_z = \begin{cases} 0 & \text{if } c_z = 0 \text{ or } \#\{z' \in z + N : c_{z'} = 0\} \geq 2, \\ 1 & \text{else.} \end{cases}$$

where  $N = \{(1, 0), (0, 1), (-1, 0), (0, -1)\}$ .

$F_B$  is not nilpotent on periodic configurations (it admits both  $\bar{0}$  and  $\bar{1}$  as fixed points), and a classical results of [48] shows that it is  $\mu$ -nilpotent for any full-support Bernoulli measure<sup>1</sup>. Actually, much more is known on this CA started from random initial configurations [28].

<sup>1</sup>[48] don't use the notion of  $\mu$ -nilpotence but the property  $\mu(\{c : F^\omega(c) = \bar{0}\}) = 1$ . It can be shown that the two are equivalent for freezing CA in general [44, Lemma 3.6].

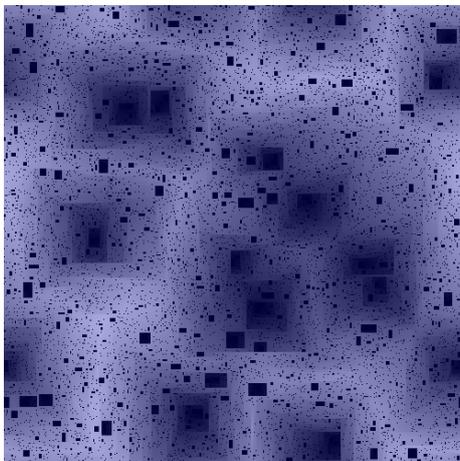


Figure 7: Representation of an orbit of  $F_B$  starting from a random initial configuration and converging to a uniform configuration: the darker the color, the earlier the freezing time.

**Theorem 11.**  $F_B$  is not generically nilpotent, however it is  $\mu$ -nilpotent for any Bernoulli measure  $\mu$  of full support.

*Proof.* The realm of attraction  $R_0$  of  $\bar{0}$  for  $F_B$  is a set of empty interior and hence cannot be comeagre. Indeed, for any finite subset  $X \subseteq \mathbb{Z}^2$ , a configuration  $c$  such that  $c_z = 1$  for any  $z \notin X$  is not attracted by  $\bar{0}$ , *i.e.*  $F^\omega(c) \neq \bar{0}$  (all cells outside a rectangle containing  $X$  are in state 1 and have at most one neighbor in state 0). This means that no cylinder set of domain  $X$  is included in  $R_0$ , whatever the finite set  $X$ , and hence no open set is included in  $R_0$ . Therefore  $F_B$  is not generically nilpotent (of course, the realm of attraction of  $\bar{1}$  is a singleton).

Let us now consider a Bernoulli measure  $\mu$  of full support, and denote by  $B$  the set of configurations converging to  $\bar{0}$ :

$$B = \{c : F_B^\omega(c) = \bar{0}\}.$$

As shown by [44, Lemma 3.6], it is enough to prove  $\mu(B) = 1$  to show that  $F_B$  is  $\mu$ -nilpotent.  $B$  is translation-invariant so by ergodicity of Bernoulli measures it must have measure 0 or 1. Let's call *rectangle area* any product of intervals  $A = I \times J \subseteq \mathbb{Z}^2$ . The boundary  $\mathcal{B}(A)$  of  $A$  is the set of  $z \notin A$  that have a neighbor in  $A$ . For a configuration  $c$  take a maximal rectangle area  $A$  containing  $(0, 0)$  and such that  $c_z = 0$  for any  $z \in A$ . If  $A \neq \mathbb{Z}^2$ , the rule  $F_B$  is such that there is a strictly larger rectangle area  $A'$  ( $A \subset A'$  and  $A \neq A'$ ) and a time step  $t$  with  $F_B^t(c)_z = 0$  for all  $z \in A'$  unless  $c$  is uniformly 1 on the boundary of  $A$ , *i.e.* unless  $c_z = 1$  for all  $z \in \mathcal{B}(A)$ . So the only way to stop the growth of a rectangle of state 0 is to have a rectangular boundary full of 1 around it.

Let  $X_n$  be the set configurations that have a rectangular boundary of perimeter  $n$  full of 1 around the origin (*i.e.* a boundary of a rectangular area containing  $(0, 0)$ ). It holds  $\mu(X_n) \leq p(n)2^{-n}$  where  $p(n)$  is a polynomial function (because there are only polynomially many such rectangular boundaries). Since the series

$\sum_n p(n)2^{-n}$  converges, there is some  $k$  such that

$$\mu\left(\bigcup_{k \leq n} X_n\right) < 1.$$

We deduce that the following set  $S$  has positive measure: configurations that have a rectangle area full of 0 of perimeter at least  $k$ , and containing the origin, and no rectangular boundary of 0 around it to prevent its growth. Any  $c \in S$  is such that  $F^\omega(c)$  contains an infinite rectangular area containing the origin and full of 0.

Finally, denote by  $A$  the set of configurations with infinitely many occurrences of 0 to the north, to the south, to the east and to the west on each row and each column:

$$A = \left\{ c : \forall n \in \mathbb{N}, \forall z \in \mathbb{Z}^2, \exists n_N > n, n_E > n, n_S > n, n_W > n : \right. \\ \left. c_{z+(0, n_N)} = c_{z+(n_E, 0)} = c_{z+(0, -n_S)} = c_{z+(-n_W, 0)} = 0 \right\}.$$

The set  $A$  is actually a countable intersection of sets of measure 1 of the form  $\{0, 1\}^{\mathbb{Z}^2} \setminus \bigcap_{z \in S} \{c : c_z = 1\}$  for  $S$  infinite. So  $\mu(A) = 1$ . Therefore  $\mu(B \cap A) > 0$ . To conclude the proof it is sufficient to note that  $S \cap A \subseteq B$ : indeed, for  $c \in S \cap A$ , take a maximal infinite rectangular area full of 0 and containing the origin in  $F^\omega(c)$ . If this area is not  $\mathbb{Z}^2$  then it must have some occurrence of 0 on its boundary because  $c \in A$ . This contradicts the fact that  $F^\omega(c)$  is a fixed-point. □

Although  $\mu$ -nilpotency is well understood for monotone freezing CA with 2 states by bootstrap percolation theory [3, 5], it remains hard for freezing CA in general.

**Theorem 12** ([44]). *The set of 2D freezing CA with 2 states which are  $\mu$ -nilpotent for all full-support Bernoulli measure  $\mu$  is recursively inseparable from the set of 2D freezing CA with 2 states which are  $\mu$ -nilpotent for no full-support Bernoulli measure  $\mu$ .*

A measure  $\mu$  is said limit-computable if there is a computable map  $\phi$  such that  $\lim_{n \rightarrow \infty} \phi(u, n) - \mu([u]) = 0$  for each cylinder  $[u]$ . One can check that if  $\mu' = \lim_t F^t(\mu)$  for some CA  $F$  and  $\mu$  a computable measure, then  $\mu'$  is limit computable. There is a remarkable converse to this observation.

**Theorem 13** ([14, 36]). *Let  $\mathbb{G} = \mathbb{Z}^d$  for some  $d \geq 1$ . Let  $\mu$  be any translation-invariant and limit-computable measure on some alphabet. Then there exists a CA  $F$  (on a possibly different alphabet) such that  $\mu = \lim_t F^t(\mu_0)$  where  $\mu_0$  is the uniform Bernoulli measure on the alphabet of  $F$ .*

Interestingly, the construction in the above theorem relies on CA that are not convergent, although the particular orbit starting from  $\mu_0$  (and actually many other initial measures) is itself convergent. We have no idea about whether it is possible to realize the same limit measures with convergent CA.

**Question 3.** *What are the limit measures  $F^\omega(\mu_0)$  where  $F$  is convergent (or bounded change, or freezing) and  $\mu_0$  is the uniform Bernoulli measure (or some computable measure)?*

## 6 Computational Power of Convergent Cellular Automata

The convergence property dramatically restricts the possible dynamics of CA as we have seen so far. One can therefore legitimately ask whether the obvious fact that general CA are computationally universal still holds for convergent, bounded-change and freezing CA. This section tackles this question and outlines an answer in three steps: yes, even freezing CA in dimension 1 are computationally universal, but there is some loss of complexity, and for a convergent CA  $F$ , it is interesting to study the map  $c \mapsto F^\omega(c)$  from a computational point of view.

### 6.1 Embeddings of computationally universal systems

Let's start by the obvious observation that the space-time diagrams of any 1D CA can be grown by a 2D freezing CA.

**Example 4.** *Any 1D CA  $F$  with states  $Q$  and neighborhood  $V$  can be simulated by a 2D freezing CA  $F'$  with states  $Q \cup \{*\}$  as follow. Let  $V' = \{(v, -1) : v \in V\}$ . A cell in a state from  $Q$  never changes. A cell in state  $*$  looks at cells in its  $V'$  neighborhood: if they are all in a state from  $Q$  then it updates to the state given by applying  $F$  on them, otherwise it stays in  $*$ . Starting from a all- $*$  configuration except on one horizontal line where it is in a  $Q$ -configuration  $c_0$ , this 2D freezing CA will compute step by step the space-time diagram of  $F$  on configuration  $c_0$ .*

For 1D freezing CA, the embedding of computationally universal systems is much more constrained but still possible using Minsky machine [21, 9, 38].

**Definition 8.** *A  $k$ -counter Minsky machine is a 4-tuple  $M = (Q_M, q_0, h, \tau)$  where  $q_0, h \in Q_M$  are the initial and halting states and*

$$\tau : Q_M \times \{0, 1\}^k \rightarrow Q_M \times \{-1, 0, 1\}^k$$

*is its transition map, which verifies  $\tau(h, \cdot) = (h, (0, \dots, 0))$ . A configuration of  $M$  is an element of  $Q_M \times \mathbb{N}^k$ .  $M$  transforms any configuration  $c = (q, (\chi_i)_{1 \leq i \leq k})$  in one step into configuration*

$$M(c) = (q', (\max(0, \chi_i + \delta_i))_{1 \leq i \leq k})$$

*where  $(q', (\delta_i)_{1 \leq i \leq k}) = \tau(q, (\min(1, \chi_i))_{1 \leq i \leq k})$ .  $M$  halts on input  $(\chi_i)_{1 \leq i \leq k} \in \mathbb{N}^k$  if there is a time  $t$  such that  $M^t(q_0, (\chi_i)_{1 \leq i \leq k}) \in (h, \mathbb{N}^k)$ .*

**Example 5.** *Any  $k$ -counter Minsky machine  $M$  can be embedded into a 1D freezing CA  $F$  in the following sense. To simplify the exposition, let's take  $k = 1$ . A configuration of  $M$  at some step of the evolution is a pair  $(q, n)$  where  $q$  is a finite state and  $n \in \mathbb{N}$  the current value of the counter. This configuration will be encoded into the trace of some cell  $z \in \mathbb{Z}$  of  $F$ : the state  $q$  will appear at cell  $z$  at some time, and after the value  $n$  will be encoded by the time interval between specific state changes occurring at  $z$ . Successive configurations of an orbit of  $M$  are then encoded into successive cells of  $F$  through the temporal traces. In order to allow  $F$  to correctly produce an orbit where the trace at position  $z + 1$*

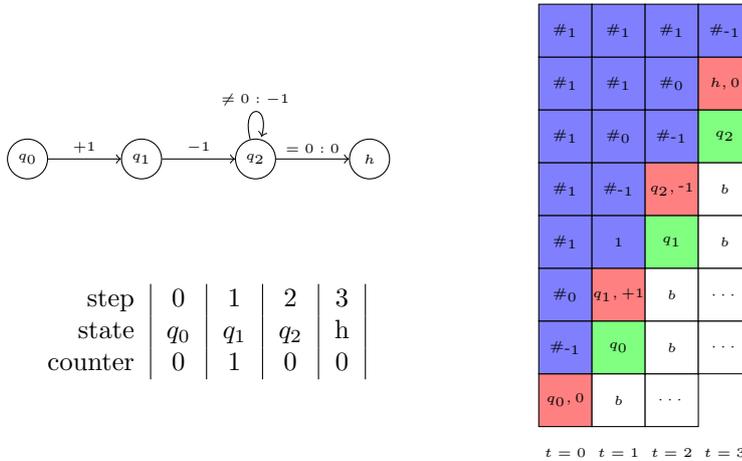


Figure 8: On the left, a Minsky machine with 1 counter and a few steps of the execution starting from counter value 0. On the right the corresponding space-time diagram of the freezing CA of Example 5 encoding it. Counter value is encoded as the blue part in each trace: value  $n$  is represented by state sequence  $1^n \#_{-1} \#_0 \#_1$ . Green cells in the space-time diagram indicate position where all the required information to compute a Minsky transition is available locally. Cells containing the result of Minsky transitions are represented in red.

encodes the configuration which is the image by  $M$  of the configuration encoded into the trace at position  $z$ , the key point is to shift temporally the time interval containing the encoded configuration from position to position  $z + 1$ . This technical condition allows to implement all the operations on a counter locally (zero test, increment and decrement). Concretely, if the configuration of  $M$  at time 0 is encoded into the time interval  $[0, \Delta_0]$  at cell 0, then the configuration at step  $t$  of machine  $M$  is encoded into the time interval  $[2t, 2t + \Delta_t]$  at cell  $t$ . See Figure 6.1 for an example with details.

From the embedding of Minsky machines as in the above example, one expects undecidability results to be transferred to 1D freezing CA. There are many ways to state such results, depending on the additional details implemented. Let us mention the following one which essentially relies on a Minsky machine embedding but requires a more technical construction than the above example. A freezing CA is always  $k$ -change for  $k$  no smaller than the cardinal of its alphabet, but determining the minimal  $k$  for which it is  $k$ -change is undecidable.

**Theorem 14** ([38]). *There exists a constant  $k$  such that the following problem is undecidable: given a freezing 1D CA  $F$ , decide whether  $F$  is  $k$ -change.*

## 6.2 Dimension 1: complexity gap between bounded change and convergent CA

There is a strong information flow bottleneck in bounded change CA that becomes critical in 1D. It is not the case for convergent CA. This can be formulated using communication complexity or classical computational complexity of the

prediction problem [38, 21] (see also [22] which generalizes the result to other  $\mathbb{G}$ ): the prediction problem has communication complexity at most  $\log(n)$  and is in class NL for 1D bounded change CA, while it can be  $\Omega(\sqrt{n})$  and P-complete for 1D convergent CA.

Instead of presenting the above results and their formal setting, we are going to illustrate this difference between bounded change and convergent CA through the problem of recognizing palindroms. Before stating the result, let's formalize the notion of language recognition by CA under time constraints. Say a language  $L \subseteq \{0, 1\}^*$  is recognized by a CA  $F$  on  $Q^{\mathbb{Z}}$  in time  $\tau : \mathbb{N} \rightarrow \mathbb{N}$  if  $\{0, 1, B, A, R\} \subseteq Q$  (where  $B$  is a blank state,  $A$  an accepting state and  $R$  a rejecting state) and for any  $n \in \mathbb{N}$  and any  $u \in \{0, 1\}^n$ , the orbit of configuration  $c^u$  defined by

$$c^u(z) = \begin{cases} u_z & \text{if } 0 \leq z < n, \\ B & \text{else,} \end{cases}$$

is such that:

- there is  $t \leq \tau(n)$  such that  $F^t(c^u)_0 \in \{A, R\}$ ,
- for the minimal such time  $t$  it holds

$$F^t(c^u)_0 = A \iff u \in L.$$

Finally, PAL is the language of palindromes on alphabet  $\{0, 1\}$ , *i.e.* words  $u$  such that  $u_i = u_{n-1-i}$  for  $0 \leq i < n = |u|$ .

**Theorem 15.** *Let  $\mathbb{G} = \mathbb{Z}$ . No bounded change CA can recognize PAL in exponential time, but there exists a convergent CA that recognizes PAL in quadratic time.*

*Proof.* Suppose by contradiction that a  $k$ -change CA  $F$  on  $Q^{\mathbb{Z}}$  and of radius  $r$  recognizes PAL in time  $\tau(n) = \alpha^n$ . For  $n \in \mathbb{N}$  and  $u \in \{0, 1\}^n$  denote by  $T_z^{u,n}$  the prefixes of traces of length  $\tau(n)$  and of width  $2r$  starting at position  $z$  in the orbit of  $c^u$ :

$$T_z^{u,n} : t \in \{0, \dots, \tau(n)\} \mapsto F^t(c^u)_z, \dots, F^t(c^u)_{z+2r}.$$

By the  $k$ -change property, there are at most  $(|Q| \log \tau(n))^k$  such prefixes in total including all choices of  $u$  and  $z$ . On the other hand, there are exponentially many words in PAL of each length, so for large enough even  $n$  there are  $u \neq v$  in PAL such that

$$T_{n/2}^{u,n} = T_{n/2}^{v,n} \tag{3}$$

Then, consider the word  $w = u_0 \cdots u_{n/2-1} v_{n/2} \cdots v_{n-1}$ . By the Equality (3) above, we have that  $\mathcal{T}_{c^u}^F(t) = \mathcal{T}_{c^w}^F(t)$  for all  $0 \leq t \leq \tau(n)$ . So in particular  $F$  must accept  $w$  in time at most  $\tau(n)$  because it does for  $u$ . However,  $w$  is not a palindrome, which yields the desired contradiction.

It is not difficult to adapt Example 3 to make a convergent CA that recognizes PAL in quadratic time. Roughly, using extra states, any input word  $BuB$  is turned in one step into a segment of state with two components: the binary component that keeps the information of  $u$  and the zigzag component of the form  $\boxed{B} \boxed{\triangleright} \boxed{\leftarrow} \cdots \boxed{\leftarrow} \boxed{B}$ , where the head state  $\boxed{\triangleright}$  holds an extra bit, initialized

to  $u_0$ . Then, with the zigzag movement of the head, this bit held inside the right-moving head  $\blacktriangleright$  is compared to the bit in the binary component when the head bounces on the right border (*i.e.*  $u_0$  is compared to  $u_{n-1}$  at the end of the first zig). If the comparison fails, an error state  $e$  appears which is interpreted as a rejecting state. If not, the head turns into  $\blacktriangleleft$  (which does not hold any additional information) and goes back to the left boundary as in the rule of Example 3. Once the left boundary is reached again, the head becomes  $\blacktriangleright$  and copy the bit from the binary component at this position (*i.e.*  $u_1$  at the start of the second zig). When the zone is shrunk down to a single position, an accepting state is generated that spreads to the left to reach the position which initially was holding value  $u_0$  (the accepting state spreads over any other state except  $e$ , which is not present in the orbit of  $c^u$  if  $u$  is in PAL). The additional mechanism on top of the rule of Example 3 does not compromise the convergent property: the analysis is the same as in Example 3 with the presence of two spreading states instead of one (rejecting state  $e$  spreading over the accepting state which spreads over any other state).  $\square$

### 6.3 Complexity of limit configurations

Any convergent CA  $F$  on  $Q^{\mathbb{G}}$  induces a map  $F^\omega : Q^{\mathbb{G}} \rightarrow Q^{\mathbb{G}}$ . We say that a configuration  $c \in Q^{\mathbb{G}}$  is computable if it is computable as a map. For any  $t$ , the map  $F^t$  transforms computable configurations into computable configurations, however there is no reason to expect that the same holds for  $F^\omega$ .

Let us first make the following observation: the computability of the limit configuration is linked to the computability of freezing times.

**Lemma 7.** *Let  $F$  be a convergent CA over  $Q^{\mathbb{G}}$  and  $c \in Q^{\mathbb{G}}$  be any computable configuration. Then the map  $z \mapsto F^\omega(c)_z$  is Turing reducible to the map  $z \mapsto \zeta_F(c, z)$ . Moreover, if  $F$  is freezing, both maps are Turing-equivalent.*

*Proof.* To compute  $F^\omega(c)_z$  from  $\zeta_F(c, z)$ , it is sufficient to compute  $F^{\zeta_F(c, z)}(c)_z$ . Reciprocally, if  $F$  is supposed freezing,  $\zeta_F(c, z)$  can be computed from  $F^\omega(c)_z$  as it is the first time step  $t$  such that  $F^t(c)_z = F^\omega(c)_z$ .  $\square$

For bounded change 1D CA, it is impossible to produce uncomputable limits from computable initial configurations. The argument of the following proof is due to G. Richard.

**Theorem 16** ([38]). *For any 1D bounded-change  $F$  and any computable configuration  $c$ ,  $F^\omega(c)$  is computable.*

*sketch.* We restrict without loss of generality to  $F$  of radius 1 and fix a computable configuration  $c$ . If we know the number of state changes in the the orbit of  $c$  at cells  $z_1$  and  $z_2$ , with  $z_1 < z_2$ , then we can compute the value  $F^\omega(c)_z$  for all  $z \in [z_1, z_2]$ . Indeed, we can compute  $F^t(c)$  on cells  $[z_1, z_2]$  for increasing values of  $t$  until the correct number of changes is observed at  $z_1$  and  $z_2$ . From that time on, the evolution of cells in the segment  $[z_1, z_2]$  is independent of the context since cells  $z_1$  and  $z_2$  no longer change and  $F$  has radius 1. So we can compute the evolution until this set of cells reaches a fixed point, which is then the value they have in  $F^\omega(c)$ .

The number of changes at any cell is bounded, and there is a maximal value  $L$  for which there are infinitely many positions  $z < 0$  with exactly  $L$  changes.

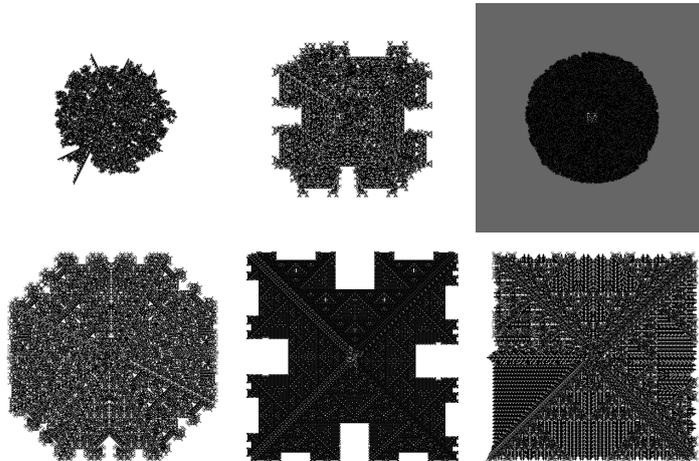


Figure 9: Examples of configurations obtained after some time from a finite seed by randomly chosen 2D freezing CA. All CA have the same neighborhood and the same number of state, with the same freezing order. All finite seed have the same size (the non-uniform portion of the configuration is a small centered square), and all example configurations were obtained after the same number of steps. The difference in the uniform background comes from the fact that not all examples have the same quiescent states.

Moreover, there is a limit position  $z_L$  to the left of which no cell has more than  $L$  changes. The same is true for positions  $z > 0$  giving the corresponding constants  $R$  and  $z_R$ .

Then the algorithm to compute  $F^\omega(c)_z$  given  $z$  is the following: compute larger and larger portions of the space-time diagram around position  $z$  until finding  $z_1 \leq z \leq z_2$ :

1.  $z_1 \leq z_L$  and  $z_R \leq z_2$ ,
2. the state of  $z_1$  has changed  $L$  times and the state of  $z_2$  has changed  $R$  times.

Then it is sufficient to apply the algorithm of the above to compute  $F^\omega(c)_{[z_1, z_2]}$  and therefore obtain  $F^\omega(c)_z$ .  $\square$

Interestingly, the above limitation for bounded change CA disappears when considering convergent CA.

**Theorem 17** ([38]). *There exists a 1D convergent CA  $F$  and a computable configuration  $c$  such that  $F^\omega(c)$  is uncomputable.*

In 2D, it is easy to find a freezing CA  $F$  and a computable configuration  $c$  such that  $F^\omega(c)$  is uncomputable: let  $c(i, j)$  be 1 if  $i, j \in \mathbb{N}$  and Turing machine  $i$  halts in at most  $j$  steps on the empty tape, and 0 else. Then, if  $F$  is the freezing CA that spreads state 1 vertically, we get that  $F^\omega(c)_{(i, 0)} = 1$  for  $i \in \mathbb{N}$  if and only if machine  $i$  halts.

A more interesting question is to ask whether  $F^\omega(c)$  can be uncomputable for some finite or eventually periodic configuration  $c$  (*i.e.* a configuration which

is totally periodic up to a finite region of  $\mathbb{Z}^2$ ). The behavior of freezing CA from finite configuration is quite rich (see Figure 9) and it is actually possible to realize uncomputable limits with a freezing 1-change CA and slightly more with a 2-change CA.

**Theorem 18.** [34, 38] For a configuration  $c \in Q^{\mathbb{Z}^2}$  and  $q \in Q$ , define the set of cells of  $c$  in state  $q$ :  $\chi_q(c) = \{z \in \mathbb{Z}^2 : c(z) = q\}$ . The following holds:

- There exists a 1-change freezing CA  $F_1$ , a state  $q$  of  $F_1$ , and a finite configuration  $c^1$  such that  $\chi_q(F_1^\omega(c^1))$  is not computable.
- For any 1-change freezing CA  $F$ , any state  $q$  and any finite configuration  $c$ ,  $\chi_q(F^\omega(c))$  is either recursively enumerable or co-recursively enumerable.
- There exists a 2-change freezing CA  $F_2$ , a state  $q$  of  $F_2$ , and a finite configuration  $c^2$  such that  $\chi_q(F_2^\omega(c^2))$  is neither recursively enumerable, nor co-recursively enumerable.

*Proof.* The first item is the main result of [34]. The second item is [38, Proposition 9]. The third item is [38, Corollary 1].  $\square$

Recall from Lemma 7 that in order to produce an uncomputable limit configuration from a computable one by a freezing CA, the freezing times of cells must be uncomputable. We stress that for this reason it is not sufficient for a 2D freezing CA to be 'computationally universal' in order to produce uncomputable limit configurations from eventually periodic initial ones. For instance, in the orbits described in the simple embedding of Example 4, the freezing times are computable by construction since cells are frozen progressively line by line.

**Question 4.** Let  $F$  be the 'life without death' CA [25]. Is there a finite or eventually periodic configuration  $c$  such that  $F^\omega(c)$  is uncomputable?

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