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# Variational inequality solutions and finite stopping time for a class of thixotropic or shear-thinning flows

Laurent CHUPIN, Nicolae CÎNDEA and Geoffrey LACOUR <sup>\*1</sup>

<sup>1</sup>Université Clermont Auvergne, CNRS, LMBP, F-63000 Clermont-Ferrand, France

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## Abstract

The aim of this paper is to study the existence of variational inequality weak solutions and of a finite stopping time for a large class of generalized Newtonian fluids with thixotropic and shear-thinning flows. The existence of dissipative solutions for such flows is known since [1]. We submit here an alternative approach using variational inequality solutions as presented in [17] in the two-dimensional Bingham flow. In order to prove the existence of such solutions we regularize the non-linear term and then we apply a Galerkin method for finally passing to the limit with respect to both regularization and Galerkin discretization parameters. In a second time, we prove the existence of a finite stopping time for a class of fluids with threshold flows in dimension  $N \in \{2, 3\}$ . More exactly, we show that if the extra non-linear term is of the form  $F(s) = s^{-\alpha}$  with  $\alpha \in (0, 1]$  when  $s$  is large, then there exists a finite stopping time. This result extends the existing results concerning the existence of a finite stopping time for Bingham fluids in dimension two.

## 1 Introduction

The aim of this paper is to study the existence and some properties of solutions of the system

$$\left\{ \begin{array}{ll} \partial_t u + u \cdot \nabla u + \nabla p - \Delta u - \operatorname{div} (F(|D(u)|) D(u)) = f & \text{in } (0, T) \times \Omega \\ \operatorname{div}(u) = 0 & \text{in } (0, T) \times \Omega \\ u = 0 & \text{on } [0, T) \times \partial\Omega \\ u = u_0 & \text{on } \{0\} \times \Omega \end{array} \right. \quad (1.1)$$

in the form of nonlinear parabolic variational inequalities, where  $\Omega$  is an open bounded subset of  $\mathbb{R}^N$ , for  $N \in \{2, 3\}$  with a regular enough boundary  $\partial\Omega$ . Such nonlinear systems describe the flow of so-called generalized Newtonian fluids and give rise to several relevant models. Some possible choices of  $F$  giving different models are listed below:

- if  $F(t) = C$ , the system (1.1) is nothing else than the Navier-Stokes equations for a viscous incompressible fluid;
- if  $F(t) = \frac{1}{t}$ , we get the Bingham equations for a viscoplastic fluid;
- by choosing  $F(t) = (1 + t^2)^{-\frac{\alpha}{2}}$ , system (1.1) describes a Carreau flow. More generally, a such flow can be considered as a particular case as the one where the tensor  $M \mapsto F(|M^{\operatorname{sym}}|) M^{\operatorname{sym}} + M^{\operatorname{sym}}$ ,  $M \in \mathbb{R}^{3 \times 3}$ ,  $M^{\operatorname{sym}} = \frac{1}{2}(M + M^T)$ , has an  $(\alpha + 2, \mu)$ -structure (see definition below) for  $\mu > 0$ .

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\*Corresponding author: Geoffrey Lacour - [geoffrey.lacour@uca.fr](mailto:geoffrey.lacour@uca.fr)

Since the 1960s, the study of such systems has been the subject of numerous articles (see, for instance, [21] and references therein). On one hand, the existence of weak solutions of (1.1) is a difficult problem which has been studied in many particular cases. Roughly speaking, a fluid flow is said to be shear-thinning if its dynamic viscosity decreases in proportion to the increase in stresses applied to it. This behavior is to be distinguished from thixotropy: a fluid is said to be thixotropic when its viscosity decreases as a function of the increase in the velocity field. In the first case, the fluid behaves like a solid as soon as the stresses applied to it are null (or too low), while in the second case, we observe that the external forces applied to the fluid must be zero for a sufficiently long time for it to return to a state comparable to a solid. It is clear that the system (1.1) accounts for both behaviors depending on the choice of the function  $F$ . We mainly refer to [8] and [23], or more recently to [10] for the physical motivations of such models. When the shear tensor  $S$  has an  $(\alpha + 2, \mu)$ -structure, the existence of weak solutions has been established in special cases (see, for instance, [28], [17], or [32]).

Let us denote by  $\mathbb{R}^{3 \times 3}$  and respectively  $\mathbb{R}_{\text{sym}}^{3 \times 3}$  the space of three-dimensional square matrices and its subspace of symmetric matrices. We recall that a tensor  $S : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}_{\text{sym}}^{3 \times 3}$  is said to have an  $(\alpha + 2, \mu)$ -structure  $\alpha > -1$ ,  $\mu \geq 0$ , when it satisfies  $S(M) = S(M^{\text{sym}})$ ,  $S(0) = 0$ , and:

$$(P1) \quad S \in \mathcal{C}^0(\mathbb{R}^{3 \times 3}, \mathbb{R}_{\text{sym}}^{3 \times 3}) \cap \mathcal{C}^1(\mathbb{R}^{3 \times 3} \setminus \{0\}, \mathbb{R}_{\text{sym}}^{3 \times 3})$$

$$(P2) \quad \exists C_1 > 0, \forall (M, P) \in \mathbb{R}^{3 \times 3}, \quad \sum_{i,j,k,l=1}^N \partial_{k,l} S_{i,j}(M) P_{i,j} P_{k,l} \geq C_1 \Phi(|M^{\text{sym}}|) |P^{\text{sym}}|^2$$

$$(P3) \quad \exists C_2 > 0, \forall M \in \mathbb{R}^{3 \times 3}, |\partial_{k,l} S_{i,j}(M)| \leq C_2 \Phi(|M^{\text{sym}}|),$$

where  $\Phi$  is given by  $\Phi(t) = (\mu + t)^{\alpha-1}(\mu + (1 + \alpha)t)$ .

The existence of solutions when the shear tensor have such a structure can be proved under certain hypotheses for  $\alpha < \frac{4}{N+2}$  (see [18]). The study of the existence of solutions under other assumptions has been done in [21] in the case  $\alpha < \frac{2-N}{N+2}$ , in [27] for the case  $N = 3$  and  $\alpha \leq -\frac{1}{4}$ , and in [32] for  $\alpha < \frac{2}{N+2}$ , and in [6] for the three-dimensional periodic case in space with  $\alpha < \frac{3}{5}$ . One can avoid such hypotheses on  $\alpha$  by using dissipative solutions. The existence of dissipative solutions has been proved in [1] in the three-dimensional setting. Nevertheless, variational inequality solutions are particularly interesting in view of numerical simulations perspective, see for example [24, Chapter 4].

We recall that in the three-dimensional setting, it is known that we can prove the existence of a unique weak solution to the incompressible Navier-Stokes equations, as long as the norm of the initial velocity field is small enough and the force term is sufficiently regular. This theorem has been generalized for a large class of non-singular stress tensors in [3], as long as the initial velocity field  $u_0$  and the force term  $f$  are regular enough. The study of the regularity of solutions in a more general framework is a difficult problem, the case of the flow of an incompressible Newtonian fluid governed by the Navier-Stokes equations remaining open in the three dimensional case. However, the existence of regular solutions, sometimes giving rise to the uniqueness of solutions or even to the existence of strong solutions, has been established in the case of shear tensors having an  $(\alpha + 2, \mu)$ -structure in [15] and in [6] in a three-dimensional periodic in space case. In the steady case, some results have been obtained as in [20] (existence for  $\alpha < \frac{4}{N+2}$ ), in [33] (regularity), in [7] (regularity), or in [11] (existence and regularity). Note that a point that can be tricky in the case of a non-Newtonian fluid is to establish an energy equality, we quote as example the work of [5] on the subject.

Finally, a property of some non-Newtonian shear-thinning flows is the existence of a finite stopping time, that is, roughly speaking, a time from which the fluid is at the rest. This particularly surprising property has been established only in a very limited number of cases. Thus, it has been established in the case of a two-dimensional Bingham flow in [13], and in the case of some electrorheological fluids in [2]. In this last case, the ill-posedness of the problem studying the uniqueness of solutions backward in time is discussed.

In this paper, we firstly establish the existence of weak solutions by a parabolic variational inequality (see Theorem 3.1 and Definition 2.1) as used in [17] for shear tensors  $\tau$  of the form  $\tau(D(u)) = F(|D(u)|)D(u) + D(u)$ , by setting conditions directly on the viscosity coefficient  $F$ , restricting ourselves to the case of shear-thinning or thixotropic flows. More precisely, we will make the following assumptions:

- (C1)  $F : (0, +\infty) \rightarrow (0, +\infty)$ ;
- (C2)  $F \in W_{loc}^{1,\infty}((0, +\infty))$ ;
- (C3)  $t \mapsto tF(t)$  is non-decreasing on  $(0, +\infty)$ ;
- (C4) there exist  $\alpha \in [0, 1]$ ,  $t_0 > 0$  and  $K > 0$  such that for every  $t \geq t_0$ ,  $F(t) \leq Kt^{-\alpha}$ .

Some examples of functions verifying the above assumptions are given in Appendix A. We emphasize in particular that this takes into account many physical models, such as the Carreau, Bingham, Herschel-Bulkley, Cross, or power law flows.

**Remark 1.** *Assumption (C3) is equivalent to the fact that for all  $\varepsilon \geq 0$ , the function  $t \mapsto tF(\sqrt{\varepsilon + t^2})$  is non-decreasing. Indeed, we can write:*

$$\forall t \in (0, +\infty), tF(\sqrt{\varepsilon + t^2}) = \left( \frac{t}{\sqrt{\varepsilon + t^2}} \right) \sqrt{\varepsilon + t^2} F(\sqrt{\varepsilon + t^2}).$$

Hence,  $t \mapsto tF(\sqrt{\varepsilon + t^2})$  is the product of two non-negative and non-decreasing functions, so it is a non-decreasing function. The opposite implication being obvious by setting  $\varepsilon = 0$ .

Then, in Section 5 we establish the existence of a finite stopping time for solutions of (1.1) in the case of a function  $F$  verifying (C1)-(C4) and such that

$$F(t) \geq t^{-\alpha}. \tag{1.2}$$

This shows in particular the existence of a finite time from which the fluid is at rest, when the flow is comparable to that of a threshold fluid, thus for a shear-thinning flow.

We will note in a generic way the constants by the letter  $C$  throughout this article, and will omit their dependence on the parameters in the notations.

## 2 Weak characterization of solutions by a parabolic variational inequality

In this section we introduce a weak formulation of system (1.1) using a parabolic variational inequality (see Definition 2.1). Firstly, we point out that in the system (1.1), we do not consider any frictional force on  $\partial\Omega$ . Recalling that  $H_0^1(\Omega)$  is the closure of  $C_0^\infty(\Omega)$  into  $H^1(\Omega)$ , it is thus natural to assume that the initial velocity field  $u_0$  is of null trace on  $\partial\Omega$ , that is  $u_0$  belongs to  $H_{0,\sigma}^1(\Omega)$ , the space of functions  $v \in H_0^1(\Omega)$  such that  $\operatorname{div}(v) = 0$ , where  $H_0^1(\Omega)$  is endowed with the norm  $u \mapsto \|\nabla u\|_{L^2}$ . We denote  $H_\sigma^{-1}(\Omega)$  its dual and  $\langle \cdot, \cdot \rangle$  is the duality product between  $H_\sigma^{-1}(\Omega)$  and  $H_{0,\sigma}^1(\Omega)$ . Following the ideas employed for showing the existence of solution to Bingham equations in [17, 25], we define a functional  $j$  making appear the viscous non-linear term in (1.1) in its derivative.

We fix for the moment  $0 \leq \varepsilon \leq \delta$  and we define a function  $G_\varepsilon : (0, +\infty) \rightarrow (0, +\infty)$  and a functional  $j_\varepsilon : H_{0,\sigma}^1(\Omega) \rightarrow \mathbb{R}$  by

$$G_\varepsilon(t) = \int_0^t sF(\sqrt{\varepsilon + s^2}) ds \quad \text{for every } t \in (0, +\infty) \tag{2.1}$$

and

$$j_\varepsilon(v) = \int_\Omega G_\varepsilon(|D(v)|) dx, \quad (v \in H_{0,\sigma}^1(\Omega)), \tag{2.2}$$

respectively. We also denote  $j = j_0$  and  $G = G_0$ . It is easy to check that  $G_\varepsilon$  is a convex functional for  $\varepsilon$  small enough. Indeed,

$$G'_\varepsilon(t) = tF(\sqrt{\varepsilon + t^2}), \quad \text{for every } t \in (0, +\infty),$$

and applying the hypothesis (C3) the convexity of  $G$  follows immediately.

**Lemma 2.1.** *For every  $\varepsilon > 0$  the functional  $j_\varepsilon$  defined by (2.2) is convex and verifies*

$$\langle j'_\varepsilon(v), w \rangle_{-1,1} = \int_{\Omega} F\left(\sqrt{\varepsilon + |D(v)|^2}\right) (D(v) : D(w)) \, dx \quad (v, w \in H^1_{0,\sigma}(\Omega)). \quad (2.3)$$

*Proof.* The convexity of  $j_\varepsilon$  is immediately obtained from the hypothesis (C3) and (2.2). For every  $t \in \mathbb{R}$  we have

$$\begin{aligned} \frac{d}{dt} (G_\varepsilon(|D(v+tw)|)) &= G'_\varepsilon(|D(v+tw)|) \frac{d}{dt} (|D(v+tw)|) \\ &= F\left(\sqrt{\varepsilon + |D(v+tw)|^2}\right) |D(v+tw)| \left(\frac{D(v+tw) : D(w)}{|D(v+tw)|}\right) \\ &= F\left(\sqrt{\varepsilon + |D(v+tw)|^2}\right) D(v+tw) : D(w). \end{aligned}$$

Hence

$$\begin{aligned} \langle j'_\varepsilon(v+tw), w \rangle &= \frac{d}{dt} j_\varepsilon(v+tw) = \int_{\Omega} \frac{d}{dt} (G_\varepsilon(|D(v+tw)|)) \, dx \\ &= \int_{\Omega} F\left(\sqrt{\varepsilon + |D(v+tw)|^2}\right) D(v+tw) : D(w) \, dx. \end{aligned}$$

Letting  $t$  going to 0 we obtain (2.3).  $\square$

**Remark 2.** *We point out that  $j'$  is well defined. Firstly, by our assumptions (C2) and (C3), we can deduce that for all  $\beta \in (0, \frac{1}{2})$ , there exists  $\delta_0$  such that:*

$$F(t) \leq t^{-(1+\beta)} \quad \text{for every } t \in (0, \delta_0).$$

*Indeed, assume that this last inequality does not hold, then for every  $\delta_0 > 0$ , there exists  $t_0 \in (0, \delta_0)$  such that:*

$$F(t_0) > t_0^{-(1+\beta)}.$$

*We can consider without loss of generality that  $\delta_0 < \min\left(1, F(1)^{-\frac{1}{\beta}}\right)$ , which implies, using our assumption (C3):*

$$\delta_0^{-\beta} < t_0^{-\beta} < t_0 F(t_0) \leq F(1).$$

*This contradiction shows the result. We recall Korn's  $L^2$  equality for divergence free vector fields:*

$$\int_{\Omega} |D(\varphi)|^2 \, dx = \frac{1}{2} \|\varphi\|_{H^1_0}^2, \quad (\varphi \in H^1_{0,\sigma}(\Omega)).$$

*Using these last results and applying Cauchy Schwarz's and Hölder's inequalities, we get:*

$$\begin{aligned} |\langle j'(u), \varphi \rangle_{-1,1}| &= \left| \int_{\Omega} F(|D(u)|) D(u) : D(\varphi) \, dx \right| \\ &\leq \frac{1}{\sqrt{2}} \left( \int_{\Omega} F(|D(u)|)^2 |D(u)|^2 \, dx \right)^{\frac{1}{2}} \|\varphi\|_{H^1_0} \\ &= \frac{1}{\sqrt{2}} \left( \int_{\{|D(u)| \leq \delta_0\}} F(|D(u)|)^2 |D(u)|^2 \, dx + \int_{\{|D(u)| > \delta_0\}} F(|D(u)|)^2 |D(u)|^2 \, dx \right)^{\frac{1}{2}} \|\varphi\|_{H^1_0} \\ &\leq \frac{1}{\sqrt{2}} \left( \int_{\{|D(u)| \leq \delta_0\}} |D(u)|^{-2\beta} \, dx + \int_{\{|D(u)| > \delta_0\}} F(|D(u)|)^2 |D(u)|^2 \, dx \right)^{\frac{1}{2}} \|\varphi\|_{H^1_0} \\ &= \frac{1}{\sqrt{2}} \left( \frac{1}{1-2\beta} \int_{\{|D(u)| \leq \delta_0\}} \int_0^{|D(u)|} s^{1-2\beta} \, ds \, dx + \int_{\{|D(u)| > \delta_0\}} F(|D(u)|)^2 |D(u)|^2 \, dx \right)^{\frac{1}{2}} \|\varphi\|_{H^1_0}. \end{aligned}$$

And so  $j'$  is well-defined.

**Definition 2.1** (Weak solution of (1.1)). *We say that a function  $u \in L^2((0, T), H_{0,\sigma}^1(\Omega)) \cap \mathcal{C}_w((0, T), L_\sigma^2(\Omega))$  such that  $u' \in L^{\frac{4}{N}}((0, T), H_\sigma^{-1}(\Omega))$  is a weak solution of (1.1) if and only if  $u$  verifies  $u|_{t=0} = u_0 \in H_{0,\sigma}^1(\Omega)$ , and for all  $\varphi \in \mathcal{C}^\infty((0, T) \times \Omega)$ :*

$$\begin{aligned} & \int_0^T \langle u'(t), \varphi(t) \rangle dt + \frac{1}{2} \left( \|u_0\|_{L^2(\Omega)}^2 - \|u(T)\|_{L^2(\Omega)}^2 \right) + \int_0^T \int_\Omega D(u(t)) : D(\varphi(t) - u(t)) dx \\ & - \int_0^T \int_\Omega (u(t) \cdot \nabla u(t)) \cdot \varphi(t) dx dt + \int_0^T \int_\Omega G(|D(\varphi(t))|) - G(|D(u(t))|) dx dt \\ & \geq \int_0^T \langle f(t), \varphi(t) - u(t) \rangle dt. \end{aligned} \quad (2.4)$$

Let's quickly motivate this definition. First, we point out that since  $u$  belongs to  $\mathcal{C}_w((0, T), L_\sigma^2(\Omega))$ , Definition 2.1 makes sense. Then, if we consider that the Lebesgue measure of the set

$$\{(t, x) \in (0, T) \times \Omega \mid |D(u)(t, x)| \leq \delta\}$$

is equal to zero for a small  $\delta > 0$ , we have, from an argument similar to the one in Lemma 2.1 that:

$$\int_0^T \langle j'(u), \varphi \rangle dt = \int_0^T \int_\Omega F(|D(u)|) (D(u) : D(\varphi)) dx dt.$$

Now, if we replace  $\varphi$  by  $u + s\varphi$ , with  $s > 0$ , in the variational inequality (2.4), we obtain after dividing by  $s$ :

$$\begin{aligned} & \int_0^T \int_\Omega D(u) : D(\varphi) dx dt + \int_0^T \int_\Omega \frac{G(|D(u + s\varphi)|) - G(|D(u)|)}{s} dx dt \\ & \geq \int_0^T \int_\Omega \langle f - u', \varphi \rangle dt - \int_0^T \int_\Omega (u \cdot \nabla u) \cdot \varphi dx dt. \end{aligned}$$

Since  $j$  admits a Fréchet-derivative, it also admits a Gâteaux-derivative and both are the same. Hence, taking the limit as  $s \rightarrow 0$ :

$$\begin{aligned} & \int_0^T \int_\Omega D(u) : D(\varphi) dx dt + \int_0^T \int_\Omega F(|D(u)|) (D(u) : D(\varphi)) dx dt \\ & \geq \int_0^T \int_\Omega \langle f - u', \varphi \rangle dt - \int_0^T \int_\Omega (u \cdot \nabla u) \cdot \varphi dx dt. \end{aligned}$$

Repeating once again the previous reasoning but writing  $u - s\varphi$  instead of  $u + s\varphi$ , we get the following equality:

$$\begin{aligned} & \int_0^T \int_\Omega D(u) : D(\varphi) dx dt + \int_0^T \int_\Omega F(|D(u)|) (D(u) : D(\varphi)) dx dt \\ & = \int_0^T \int_\Omega \langle f - u', \varphi \rangle dt - \int_0^T \int_\Omega (u \cdot \nabla u) \cdot \varphi dx dt. \end{aligned}$$

Therefore, assuming that  $u$  is regular enough, we obtain

$$\begin{aligned} & -\frac{1}{2} \int_0^T \int_\Omega \Delta u \cdot \varphi dx dt - \int_0^T \int_\Omega \operatorname{div} (F(|D(u)|) D(u)) \varphi dx dt \\ & = \int_0^T \int_\Omega (f - u' - u \cdot \nabla u) \cdot \varphi dx dt. \end{aligned}$$

Furthermore De Rham's theorem for a domain with Lipschitz boundary states that there exists a pressure term  $p$  such that  $f = \nabla p$  into some well chosen Sobolev space (see [16, section 2] for details). Considering such a function and also the two previous observations, we can write:

$$\int_0^T \int_{\Omega} \left( u' + u \cdot \nabla u - \frac{1}{2} \Delta u + \nabla p - \operatorname{div} (F(|D(u)|) D(u)) - f \right) \varphi \, dx \, dt = 0, \quad (\varphi \in \mathcal{C}^\infty((0, T) \times \Omega)),$$

which is almost everywhere equivalent to the equation (1.1) up to the multiplicative dynamic viscosity constant  $\frac{1}{2}$ . We have omitted this constant in Definition 2.1 for convenience, and note that it is enough to add the constant 2 in front of the term  $\int_0^T \int_{\Omega} D(u) : D(u - \varphi) \, dx \, dt$  in order to find exactly (1.1).

Finding a solution to the parabolic variational inequality thus amounts to giving meaning to the integral of the nonlinear viscosity coefficient term inherent in the problem, which can be a singular integral in the case of a threshold fluid flow, as a power law or a Bingham model.

### 3 Main results

We present in this section the main results of this article. The study of the existence of solutions by variational inequality has been developed following the classical Stampacchia's theorem and was further developed in [25]. Then, this method was successfully applied for some nonlinear parabolic problems, as the two dimensional Bingham equations in [17], or some power law systems in [26]. Following the same approach, we get the following existence theorem.

**Theorem 3.1.** *Assume that the function  $F$  satisfies the hypotheses (C1)-(C4) and that  $\Omega \subset \mathbb{R}^N$ ,  $N \in \{2, 3\}$ , is a bounded domain with a Lipschitz boundary,  $T > 0$  and consider an initial datum  $u_0 \in H_{0,\sigma}^1(\Omega)$  and a force term  $f \in L^2((0, T), H_{\sigma}^{-1}(\Omega))$ . Then, there exists a weak solution  $u$  of (1.1) having the following regularity*

$$u \in \mathcal{C}_w((0, T), L_{\sigma}^2(\Omega)) \cap L^2((0, T), H_{0,\sigma}^1(\Omega)) \quad \text{and} \quad u' \in L^{\frac{4}{N}}((0, T), H_{\sigma}^{-1}(\Omega)).$$

This theorem thus ensures the existence of suitable solutions in the two-dimensional and three-dimensional cases. It follows from classical arguments that the solutions are Hölder continuous in time, for a well-chosen Hölder coefficient. Moreover, we show in the appendix, in Corollary B.1 of Proposition B.1 that in some interesting cases as for threshold flows, that the solutions satisfy an energy equality.

Unlike the Navier-Stokes case, the nonlinear term in the Bingham equations allows us to obtain the rest of the fluid in finite time in the two-dimensional case. This has been demonstrated in [13], using the following approach: it is assumed that the force term will compensate the initial kinetic energy of the fluid, which amounts to establishing a relation between the norm  $\|u_0\|_{L^2}$  and an integral of  $\|f(t)\|_{L^2}$ . This argument is based on the use of the following Nirenberg-Strauss inequality:

$$\exists \gamma > 0, \forall u \in H_0^1(\Omega), \|u\|_{L^2} \leq \gamma \int_{\Omega} |D(u)| \, dx.$$

We note that such an inequality cannot be true in dimension greater than two, because it would contradict the optimality of Sobolev embeddings. We therefore propose to slightly adapt this approach to show the existence of a stopping finite time in both the two and the three-dimensional cases. Firstly, let us formalize the definition.

**Definition 3.1** (Finite stopping time). *Let  $u$  be a weak solution in the sense of Definition 2.1 of the system (1.1). We say that  $T_0 \in (0, T)$  is a finite stopping time for  $u$  if:*

$$\|u(T_0)\|_{L^2(\Omega)} = 0.$$

In order to prove the existence of a finite stopping time for the solution  $u$  provided by Theorem 3.1, we do not make any assumption on the initial velocity field, but we assume that after a certain time the fluid is no longer subjected to any external force. More exactly we make some more assumption on  $F$  as stated by the following theorem.

**Theorem 3.2** (Existence of a finite stopping time). *Assume that the hypotheses of Theorem 3.1 are verified, that  $T > 0$  is chosen large enough, and let  $\alpha \in (0, 1]$ . Moreover, we assume that there exists two positive constants  $\kappa$  and  $T_1 < T$  such that*

$$F(t) \geq \kappa t^{-\alpha} \text{ for every } t \in (0, +\infty) \quad \text{and} \quad f = 0 \quad \text{almost everywhere on } (T_1, T). \quad (3.1)$$

Then, there exists a finite stopping time  $T_0 \in (0, T)$  for  $u$  in the sense of Definition 3.1.

## 4 Proof of Theorem 3.1

In this section, we establish the proof of Theorem 3.1 in the bi-dimensional and three-dimensional settings. In order to prove this result, we begin by establishing an energy estimate for solutions obtained by the Galerkin method in order to obtain uniform bounds with respect to the parameters. We note here that we will have two parameters: a first parameter due to Galerkin's approximation, and a second one due to the regularization proper to the viscosity coefficient  $F$ .

### 4.1 Existence of a Galerkin weak solution

We apply here the usual Galerkin method using the Stokes operator in homogeneous Dirichlet setting, and we use its eigenfunctions  $(w_i)_{i \in \mathbb{N}}$  as an orthogonal basis of  $H_{0,\sigma}^1(\Omega)$  and orthonormal basis of  $L_\sigma^2(\Omega)$  (see [19] for details about this property, and [30, Section 2.3] for details concerning the Stokes operator).

For every positive integer  $m$ , we denote by  $P_m$  the projection of  $L_\sigma^2(\Omega)$  onto  $\text{Span}((w_i)_{1 \leq i \leq m})$ . We would like to formally define our Galerkin system as follows.

$$\begin{cases} \partial_t u_m + P_m(u_m \cdot \nabla u_m) + \nabla P_m(p) - \Delta u_m - P_m(\text{div}(F(|D(u_m)|)D(u_m))) = P_m f \\ \text{div}(u_m) = 0 & \text{on } (0, T) \times \Omega \\ u_m = 0 & \text{on } [0, T) \times \partial\Omega \\ u_m = P_m(u_0) & \text{on } \{0\} \times \Omega. \end{cases} \quad (4.1)$$

In order to avoid the issue posed by the nonlinear term in domains for which the fluid is not deformed we consider the following regularized Galerkin system:

$$\begin{cases} \partial_t u_{m,\varepsilon} + P_m(u_{m,\varepsilon} \cdot \nabla u_{m,\varepsilon}) + \nabla P_m(p) - \Delta u_{m,\varepsilon} - P_m\left(\text{div}\left(F\left(\sqrt{\varepsilon + |D(u_{m,\varepsilon})|^2}\right)D(u_{m,\varepsilon})\right)\right) = P_m f \\ \text{div}(u_{m,\varepsilon}) = 0 & \text{on } (0, T) \times \Omega \\ u_{m,\varepsilon} = 0 & \text{on } [0, T) \times \partial\Omega \\ u_{m,\varepsilon} = P_m(u_0) & \text{on } \{0\} \times \Omega, \end{cases} \quad (4.2)$$

with  $0 < \varepsilon < 1$ . Applying a Galerkin method, we can see that, writing  $u_{m,\varepsilon}(t) = \sum_{i=1}^m d_m^i(t)w_i$ , we obtain the ordinary differential system for all  $1 \leq i \leq m$ :

$$\begin{aligned} d_m^i{}'(t) &= \langle f, w_i \rangle - \int_\Omega \frac{1}{2} \|w_i\|_{H_0^1}^2 d_m^i(t) \, dx - \int_\Omega D(u_0) : D(w_i) \, dx \\ &\quad - \int_\Omega \frac{1}{2} \|w_i\|_{H_0^1}^2 F \left( \sqrt{\varepsilon + \sum_{j=1}^m \frac{1}{2} \|w_j\|_{H_0^1}^2 (d_m^j(t))^2 + 2(D(w_j) : D(u_0))d_m^j(t) + \frac{1}{2} \|u_0\|_{H_0^1}^2} \right) d_m^i(t) \, dx \\ &\quad - \int_\Omega F \left( \sqrt{\varepsilon + \sum_{j=1}^m \frac{1}{2} \|w_j\|_{H_0^1}^2 (d_m^j(t))^2 + 2(D(w_j) : D(u_0))d_m^j(t) + \frac{1}{2} \|u_0\|_{H_0^1}^2} \right) (D(u_0) : D(w_i)) \, dx \\ &\quad - \sum_{j=1}^m \int_\Omega w_j \cdot \nabla w_i d_m^i(t) d_m^j(t) \, dx, \end{aligned} \quad (4.3)$$



completed with initial condition  $d_m^i(0) = (u_0, w_i)_{H_0^1}$ . This system is described by a locally Lipschitz continuous function with respect to  $d_m$ . Indeed, applying the hypothesis (C2), the function  $\psi : \mathbb{R}^m \rightarrow \mathbb{R}$  defined by

$$\psi(x) = F \left( \sqrt{\varepsilon^2 + \sum_{j=1}^m \frac{1}{2} \|w_j\|_{H_0^1}^2 x_j^2 + 2(D(w_j) : D(u_0))x_j + \frac{1}{2} \|u_0\|_{H_0^1}^2} \right) \quad \forall x \in \mathbb{R}^m$$

is locally Lipschitz. The Picard-Lindelöf theorem shows the existence of a weak solution for system (4.2).

## 4.2 Energy estimate and consequences

We recall that the solution  $u_{m,\varepsilon}$  of (4.2) belongs to  $\text{Span}((w_i)_{1 \leq i \leq m})$ , for  $(w_i)_{i \in \mathbb{N}}$  the basis of  $H_{0,\sigma}^1(\Omega)$  which are the eigenfunctions of the Stokes operator in the homogeneous Dirichlet setting.

We consider the following notion of weak solution.

**Definition 4.1** (Weak solution of (4.2)). *We say that  $u_{m,\varepsilon} \in L^2((0, T), H_{0,\sigma}^1(\Omega))$ ,  $u'_{m,\varepsilon} \in L^2((0, T), H^{-1}(\Omega))$  is a weak solution of (4.2) if for every  $\varphi \in C^\infty((0, T) \times \Omega)$  and for a.e.  $t \in (0, T)$  it satisfies*

$$\langle u'_{m,\varepsilon}, \varphi \rangle + \int_{\Omega} D(u_{m,\varepsilon}) : D(\varphi) \, dx + \langle j'_\varepsilon(u_{m,\varepsilon}), \varphi \rangle - \int_{\Omega} (u_{m,\varepsilon} \cdot \nabla u_{m,\varepsilon}) \cdot \varphi \, dx = \langle f, \varphi \rangle. \quad (4.4)$$

We also say that (4.4) is the weak formulation in space of the weak solution of (4.2) when the time is fixed. We point out that this definition makes sense since we are studying smooth Galerkin weak solutions. Then, in order to obtain weak limits into the Galerkin formulation, we establish some estimates.

**Proposition 4.1.** *Assume that  $u_{m,\varepsilon}$  is a weak solution of (4.2) in the sense of Definition 4.1. Then, there exists a positive constant  $C$  depending on  $\alpha, \Omega, N, T, \|u_0\|_{L^2(\Omega)}$  and  $\|f\|_{L^2((0,T),H^{-1}(\Omega))}$  such that the following estimates hold:*

1.  $\|u_{m,\varepsilon}\|_{L^\infty((0,T),L^2_\sigma)}^2 + \frac{1}{2} \|u_{m,\varepsilon}\|_{L^2((0,T),H_{0,\sigma}^1)}^2 \leq C \left( \|f\|_{L^2((0,T),H^{-1})}^2 + \|u_0\|_{L^2}^2 \right);$
2.  $\|j'_\varepsilon(u_{m,\varepsilon})\|_{L^{\frac{4}{N}}((0,T),H^{-1})} \leq C \left( 1 + \|f\|_{L^2((0,T),H^{-1})} + \|u_0\|_{L^2} \right)^{1-\alpha};$
3.  $\|u'_{m,\varepsilon}\|_{L^{\frac{4}{N}}((0,T),H^{-1})} \leq C \left( \|f\|_{L^2((0,T),H^{-1})}^2 + \|u_0\|_{L^2}^2 \right) + C \left( \|f\|_{L^2((0,T),H^{-1})}^2 + \|u_0\|_{L^2}^2 \right)^2 + C \left( 1 + \|f\|_{L^2((0,T),H^{-1})} + \|u_0\|_{L^2} \right)^{1-\alpha}.$

Before the proof of Proposition 4.1, we state some useful results. We start by recalling a well known Gagliardo-Nirenberg inequality (for the proof, see, for instance, [29] or [22]).

**Theorem 4.1** (Gagliardo-Nirenberg inequality on bounded Lipschitz domain). *Assume that  $\Omega$  is a bounded domain in  $\mathbb{R}^N$  with Lipschitz boundary. Moreover, assume that there exists a couple  $(q, r) \in [1, +\infty]$ ,  $\theta \in [0, 1]$  and  $(l, k) \in \mathbb{N}^2$  such that:*

$$\begin{cases} \frac{1}{p} = \frac{k}{N} + \left(\frac{1}{r} - \frac{l}{N}\right)\theta + \frac{1-\theta}{q} \\ \frac{k}{l} \leq \theta \leq 1. \end{cases}$$

*Then, there exists  $C := C(k, l, N, r, q, \theta, \Omega) > 0$  such that the following inequality holds:*

$$\|\nabla^k u\|_{L^p(\Omega)} \leq C \|u\|_{W^{l,r}(\Omega)}^\theta \|u\|_{L^q(\Omega)}^{1-\theta}.$$

The following result formalizes some other properties.

**Lemma 4.1.** *Let  $X$  be a Banach space, and  $\gamma \geq \frac{1}{2}$ . Then, the following inequality holds:*

$$\forall (u, v) \in X^2, \|u + v\|_X^\gamma \leq 2^{(\gamma - \frac{1}{2})} (\|u\|_X^\gamma + \|v\|_X^\gamma).$$

*Proof.* Using the convexity of  $t \mapsto t^{2\alpha}$  and triangle's inequality of the norm, we get:

$$\|u + v\|_X^{2\gamma} = 2^{2\gamma} \left\| \frac{u + v}{2} \right\|_X^{2\gamma} \leq 2^{2\gamma-1} \left( \|u\|_X^{2\gamma} + \|v\|_X^{2\gamma} \right).$$

Applying now the well-known inequality:  $\forall (a, b) \in [0, +\infty)^2$ ,  $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ , we get the result.  $\square$

*Proof of Proposition 4.1.*

1. Setting  $\varphi = u_{m,\varepsilon}$  in the weak formulation, we get:

$$\frac{1}{2} \frac{d}{dt} \|u_{m,\varepsilon}\|_{L^2}^2 + \int_{\Omega} |D(u_{m,\varepsilon})|^2 dx + \underbrace{\langle j'_\varepsilon(u_{m,\varepsilon}), u_{m,\varepsilon} \rangle}_{\geq 0} - \underbrace{\int_{\Omega} (u_{m,\varepsilon} \cdot \nabla u_{m,\varepsilon}) \cdot u_{m,\varepsilon} dx}_{=0} = \langle f, u_{m,\varepsilon} \rangle.$$

Using the well-known Korn's  $L^2$  equality for divergence free vectors fields, we get

$$\frac{d}{dt} \|u_{m,\varepsilon}(t)\|_{L^2}^2 + \|u_{m,\varepsilon}(t)\|_{H_0^1}^2 \leq 2 \langle f(t), u_{m,\varepsilon}(t) \rangle.$$

Moreover, we have:

$$2 \langle f(t), u_{m,\varepsilon}(t) \rangle \leq 2 \|f(t)\|_{H^{-1}}^2 + \frac{1}{2} \|u_{m,\varepsilon}(t)\|_{H_0^1}^2.$$

Then, using the above inequality and integrating on  $(0, t)$  we get

$$\|u_{m,\varepsilon}(t)\|_{L^2}^2 + \frac{1}{2} \int_0^t \|u_{m,\varepsilon}\|_{H_0^1}^2 dt \leq 2 \int_0^t \|f\|_{H^{-1}}^2 dt + \|u_0\|_{L^2}^2. \quad (4.5)$$

Indeed, we recall that  $(P_m(u_0), w_i)_{L^2} = (u_0, P_m w_i)_{L^2} = (u_0, w_i)_{L^2}$ , and the conclusion follows easily. From now on, we will omit to detail this last part which is usual.

2. We have, using Cauchy-Schwarz's inequality and Korn's equality in the divergence free  $L^2$  setting:

$$\begin{aligned} \langle j'_\varepsilon(u_{m,\varepsilon}), \varphi \rangle &= \int_{\Omega} F \left( \sqrt{\varepsilon + |D(u_{m,\varepsilon})|^2} \right) D(u_{m,\varepsilon}) : D(\varphi) dx \\ &\leq \frac{1}{\sqrt{2}} \left( \int_{\Omega} F \left( \sqrt{\varepsilon + |D(u_{m,\varepsilon})|^2} \right)^2 |D(u_{m,\varepsilon})|^2 dx \right)^{\frac{1}{2}} \|\varphi\|_{H_0^1}. \end{aligned} \quad (4.6)$$

From hypothesis (C4), setting  $A = \Omega \cap \{|D(u_{m,\varepsilon})| \leq t_0\}$  and  $B$  its complement in  $\Omega$ , we obtain

$$\begin{aligned} \int_{\Omega} F \left( \sqrt{\varepsilon + |D(u_{m,\varepsilon})|^2} \right)^2 |D(u_{m,\varepsilon})|^2 dx &= \int_A F \left( \sqrt{\varepsilon + |D(u_{m,\varepsilon})|^2} \right)^2 |D(u_{m,\varepsilon})|^2 dx \\ &\quad + \int_B F \left( \sqrt{\varepsilon + |D(u_{m,\varepsilon})|^2} \right)^2 |D(u_{m,\varepsilon})|^2 dx. \end{aligned}$$

Let's estimate these two integrals independently. By assumption (C3), we have that the application  $t \mapsto t^2 F \left( \sqrt{\varepsilon + t^2} \right)^2$  is non-decreasing, and we obtain directly:

$$\begin{aligned} \int_A F \left( \sqrt{\varepsilon + |D(u_{m,\varepsilon})|^2} \right)^2 |D(u_{m,\varepsilon})|^2 dx &\leq F \left( \sqrt{\varepsilon + t_0^2} \right)^2 t_0^2 |A| \\ &\leq F \left( \sqrt{\varepsilon + t_0^2} \right)^2 t_0^2 |\Omega| \\ &\leq F \left( \sqrt{1 + t_0^2} \right)^2 \sqrt{1 + t_0^2} |\Omega| \\ &\leq C. \end{aligned}$$

Then we have, using again (C4):

$$\begin{aligned}
\int_B F \left( \sqrt{\varepsilon + |D(u_{m,\varepsilon})|^2} \right)^2 |D(u_{m,\varepsilon})|^2 dx &\leq K \int_B \frac{|D(u_{m,\varepsilon})|^2}{(\varepsilon + |D(u_{m,\varepsilon})|^2)^\alpha} dx \\
&\leq K \int_B |D(u_{m,\varepsilon})|^{2-2\alpha} dx \\
&\leq K \int_B |\nabla u_{m,\varepsilon}|^{2-2\alpha} dx \\
&\leq C \|u_{m,\varepsilon}\|_{H_0^1}^{2-2\alpha},
\end{aligned}$$

where we used Jensen's inequality in the concave setting with  $t \mapsto t^{1-\alpha}$  in the last line. So, we obtain:

$$\left( \int_\Omega F \left( \sqrt{\varepsilon + |D(u_{m,\varepsilon})|^2} \right)^2 |D(u_{m,\varepsilon})|^2 dx \right)^{\frac{1}{2}} \leq \left( C + C \|u_{m,\varepsilon}\|_{H_0^1}^{2-2\alpha} \right)^{\frac{1}{2}}. \quad (4.7)$$

Thus, combining the inequality (4.6)-(4.7) and using Lemma 4.1 with  $\gamma = \frac{2}{N}$ , we get:

$$\|j'_\varepsilon(u_{m,\varepsilon})\|_{H^{-1}}^{\frac{4}{N}} \leq C + C \|u_{m,\varepsilon}\|_{H_0^1}^{\frac{4-4\alpha}{N}}.$$

Therefore, integrating in time over  $(0, T)$ :

$$\|j'_\varepsilon(u_{m,\varepsilon})\|_{L^{\frac{4}{N}}((0,T),H^{-1})}^{\frac{4}{N}} \leq C + C \|u_{m,\varepsilon}\|_{L^{\frac{4-4\alpha}{N}}((0,T),H_0^1)}^{\frac{4-4\alpha}{N}}.$$

Then, since  $0 < \frac{4-4\alpha}{N} \leq 2$ , we get, using the embedding  $L^2 \hookrightarrow L^{\frac{4-4\alpha}{N}}$  and Lemma 4.1 with  $X := H_0^1$ ,  $q = \frac{4-4\alpha}{N}$  and  $p = 2$  on  $\|u_{m,\varepsilon}\|_{L^{\frac{4-4\alpha}{N}}((0,T),H_0^1)}$ :

$$\|j'_\varepsilon(u_{m,\varepsilon})\|_{L^{\frac{4}{N}}((0,T),H^{-1})}^{\frac{4}{N}} \leq C + C \|u_{m,\varepsilon}\|_{L^2((0,T),H_0^1)}^{\frac{4-4\alpha}{N}}.$$

Using the first point of the proposition for  $t = T$ , and since  $\frac{4-4\alpha}{N} \geq 0$ , we get:

$$\|j'_\varepsilon(u_{m,\varepsilon})\|_{L^{\frac{4}{N}}((0,T),H^{-1})}^{\frac{4}{N}} \leq C + C (\|f\|_{L^2((0,T),H^{-1})} + \|u_0\|_{L^2})^{\frac{4-4\alpha}{N}}.$$

Then, using the exponent  $\frac{N}{4}$  on both sides and applying once again Lemma 4.1 with  $\gamma = \frac{N}{4}$  on the right-hand side in the inequality above leads us to:

$$\|j'_\varepsilon(u_{m,\varepsilon})\|_{L^{\frac{4}{N}}((0,T),H^{-1})} \leq C + C (\|f\|_{L^2((0,T),H^{-1})} + \|u_0\|_{L^2})^{1-\alpha}.$$

This is the wished result.

3. From the weak formulation (4.4) we easily get

$$\langle u'_{m,\varepsilon}, \varphi \rangle = - \int_\Omega D(u_{m,\varepsilon}) : D(\varphi) dx - \langle j'_\varepsilon(u_{m,\varepsilon}), \varphi \rangle + \int_\Omega (u_{m,\varepsilon} \cdot \nabla u_{m,\varepsilon}) \cdot \varphi dx + \langle f, \varphi \rangle. \quad (4.8)$$

Let us point out that

$$\int_\Omega D(u_{m,\varepsilon}) : D(\varphi) dx = \frac{1}{2} \int_\Omega \nabla u_{m,\varepsilon} \cdot \nabla \varphi dx \leq \frac{1}{2} \|u_{m,\varepsilon}\|_{H_0^1} \|\varphi\|_{H_0^1}. \quad (4.9)$$

Also, setting  $p = 4$ ,  $k = 0$ ,  $l = 1$ ,  $r = q = 2$ , and  $s = 2$  into Theorem 4.1, we get the existence of a positive constant  $C$  which only depends on  $N$  and  $\Omega$  such that:

$$\|u\|_{L^4} \leq C \|\nabla u\|_{L^2}^{\frac{N}{4}} \|u\|_{L^2}^{\frac{4-N}{4}}$$

and, renoting  $C$ , we get:

$$\|u\|_{L^4}^2 \leq C \|\nabla u\|_{L^2}^{\frac{N}{2}} \|u\|_{L^2}^{\frac{4-N}{2}}. \quad (4.10)$$

Then, we have:

$$\begin{aligned} \left| \int_{\Omega} (u_{m,\varepsilon} \cdot \nabla u_{m,\varepsilon}) \cdot \varphi \, dx \right| &\leq \frac{1}{2} \int_{\Omega} |u_{m,\varepsilon}|^2 \cdot |\nabla \varphi| \, dx \\ &\leq \|u_{m,\varepsilon}\|_{L^4}^2 \|\nabla \varphi\|_{L^2} \\ &\leq C \|u_{m,\varepsilon}\|_{L^2}^{\frac{4-N}{2}} \|u_{m,\varepsilon}\|_{H_0^1}^{\frac{N}{2}} \|\varphi\|_{H_0^1}. \end{aligned} \quad (4.11)$$

So, putting (4.9)–(4.11) and the second estimate of the Proposition 4.1 in (4.8), we obtain

$$\begin{aligned} \langle u'_{m,\varepsilon}, \varphi \rangle &\leq \frac{1}{2} \|u_{m,\varepsilon}\|_{H_0^1} \|\varphi\|_{H_0^1} + \|j'_\varepsilon(u_{m,\varepsilon})\|_{H^{-1}} \|\varphi\|_{H_0^1} + C \|u_{m,\varepsilon}\|_{L^2}^{\frac{4-N}{2}} \|u_{m,\varepsilon}\|_{H_0^1}^{\frac{N}{2}} \|\varphi\|_{H_0^1} \\ &\quad + \|f\|_{H^{-1}} \|\varphi\|_{H_0^1}, \end{aligned}$$

and, therefore,

$$\|u'_{m,\varepsilon}(t)\|_{H^{-1}} \leq \frac{1}{2} \|u_{m,\varepsilon}\|_{H_0^1} + \|j'_\varepsilon(u_{m,\varepsilon})\|_{H^{-1}} + C \|u_{m,\varepsilon}\|_{L^2}^{\frac{4-N}{2}} \|u_{m,\varepsilon}\|_{H_0^1}^{\frac{N}{2}} + \|f\|_{H^{-1}}.$$

Now, using the following convexity inequality

$$\forall k \in \mathbb{N}, \forall (x_i)_{1 \leq i \leq k} \in (0, +\infty)^k, \exists C > 0, \left( \sum_{i=1}^k x_i \right)^{\frac{4}{N}} \leq C \sum_{i=1}^k x_i^{\frac{4}{N}}$$

we get:

$$\|u'_{m,\varepsilon}(t)\|_{H^{-1}}^{\frac{4}{N}} \leq C \left( \|u_{m,\varepsilon}\|_{H_0^1}^{\frac{4}{N}} + \|j'_\varepsilon(u_{m,\varepsilon})\|_{H^{-1}}^{\frac{4}{N}} + \|u_{m,\varepsilon}\|_{L^2}^{\frac{8-2N}{N}} \|u_{m,\varepsilon}\|_{H_0^1}^2 + \|f\|_{H^{-1}}^{\frac{4}{N}} \right).$$

Since  $N \in \{2, 3\}$ , we have  $\frac{4}{N} \leq 2$ . Hence, integrating in time over  $(0, T)$  and using the embedding  $L^2(\Omega) \hookrightarrow L^{\frac{4}{N}}(\Omega)$ :

$$\begin{aligned} \|u'_{m,\varepsilon}\|_{L^{\frac{4}{N}}((0,T),H^{-1})}^{\frac{4}{N}} &\leq C \left( \|u_{m,\varepsilon}\|_{L^2((0,T),H_0^1)}^{\frac{4}{N}} + \|j'_\varepsilon(u_{m,\varepsilon})\|_{L^{\frac{4}{N}}((0,T),H^{-1})}^{\frac{4}{N}} \right) \\ &\quad + C \|u_{m,\varepsilon}\|_{L^\infty((0,T),L^2)}^{\frac{8-2N}{N}} \|u_{m,\varepsilon}\|_{L^2((0,T),H_0^1)}^{\frac{4}{N}} + C \|f\|_{L^2((0,T),H^{-1})}^{\frac{4}{N}}. \end{aligned}$$

Using the previously given convexity inequality and the first and second points of the proposition we obtain the desired result.  $\square$

### 4.3 Weak convergence

We are now interested in the weak convergence of the estimates proven in Section 4.2. Here, we prove such convergences by passing to the limit with respect to the parameter  $\varepsilon$  in a first time, then by passing to the limit with respect to the Galerkin parameter  $m$ .

Before proving Theorem 3.1, we establish several useful lemmas.

**Lemma 4.2.** Consider that  $\varphi \in L^2((0, T), H_0^1(\Omega))$ , then there exists a constant  $C(\varepsilon, \varphi) > 0$  which goes to zero as  $\varepsilon$  does, such that the following inequality holds:

$$j_\varepsilon(\varphi) + C(\varepsilon, \varphi) \geq j(\varphi), \quad (4.12)$$

where  $j_\varepsilon$  and  $j$  are defined by (2.2).

*Proof.* Recalling that the assumption (C3) states that  $t \mapsto tF(t)$  is increasing, we get:

$$\begin{aligned} j(\varphi) &:= \int_{\Omega} \int_0^{|D(\varphi)|} sF(s) \, ds \, dx \\ &\leq \int_{\Omega} \int_0^{\sqrt{\varepsilon}} sF(s) \, ds \, dx + \int_{\Omega} \int_{\sqrt{\varepsilon}}^{\sqrt{\varepsilon} + |D(\varphi)|} sF(s) \, ds \, dx \\ &\leq \varepsilon \sqrt{\varepsilon} F(\varepsilon) |\Omega| + \int_{\Omega} \int_0^{\sqrt{2|D(\varphi)|\sqrt{\varepsilon} + |D(\varphi)|^2}} sF(\sqrt{\varepsilon + s^2}) \, ds \, dx \\ &\leq \underbrace{\varepsilon \sqrt{\varepsilon} F(\varepsilon) |\Omega| + \int_{\Omega} \int_{|D(\varphi)|}^{2^{\frac{1}{2}} \varepsilon^{\frac{1}{4}} |D(\varphi)|^{\frac{1}{2}} + |D(\varphi)|} sF(\sqrt{\varepsilon + s^2}) \, ds \, dx}_{:= C(\varepsilon, \varphi)} + j_\varepsilon(\varphi), \end{aligned}$$

which is the wished result.  $\square$

**Lemma 4.3.** Consider  $\Omega$  an open bounded subset of  $\mathbb{R}^N$  with Lipschitz boundary, and a sequence  $(w_n)_{n \in \mathbb{N}}$  such that  $w_n \xrightarrow{n \rightarrow +\infty} w$  in  $L^2((0, T), H_{0,\sigma}^1(\Omega))$ . Then, for almost all  $(t, x) \in (0, T) \times \Omega$ , the following inequality holds:

$$|D(w_n)(t, x)| \geq |D(w)(t, x)|.$$

*Proof.* Firstly, let us recall that since  $w_n \rightharpoonup w$  in  $L^2((0, T), H_0^1(\Omega))$  then, for all Lebesgue points  $t_0 \in (0, T)$  and  $x_0 \in \Omega$ , for all  $\delta > 0$  and  $R > 0$  small enough, we have  $w_n \rightharpoonup w$  in  $L^2((t_0 - \delta, t_0 + \delta), H^1(B(x_0, R)))$ . Indeed, we have for all test function  $\varphi$ :

$$\int_0^T \int_{\Omega} \nabla w_n \cdot \nabla \varphi \, dt \, dx \xrightarrow{n \rightarrow +\infty} \int_0^T \int_{\Omega} \nabla w \cdot \nabla \varphi \, dt \, dx.$$

Hence, we can take  $\varphi$ , which belongs to  $C_0^\infty((t_0 - \delta, t_0 + \delta) \times B(x_0, R))$  (up to arguing by density thereafter), satisfying:

$$\nabla \varphi = \begin{cases} \nabla \psi & \text{on } (t_0 - \delta, t_0 + \delta) \times B(x_0, R) \\ 0 & \text{on } (0, T) \times \Omega \setminus (t_0 - \delta, t_0 + \delta) \times B(x_0, R) \end{cases}$$

and so this leads to:

$$\int_{t_0 - \delta}^{t_0 + \delta} \int_{B(x_0, R)} \nabla w_n \cdot \nabla \psi \, dt \, dx \xrightarrow{n \rightarrow +\infty} \int_{t_0 - \delta}^{t_0 + \delta} \int_{B(x_0, R)} \nabla w \cdot \nabla \psi \, dt \, dx.$$

That is  $w_n \rightharpoonup w$  in  $L^2((t_0 - \delta, t_0 + \delta), H^1(B(x_0, R)))$ . Now, applying Korn's  $L^2$  equality and using that fact we get for all Lebesgue point  $t_0$  of  $(0, T)$  and  $x_0 \in \Omega$  that:

$$\int_{t_0 - \delta}^{t_0 + \delta} \int_{B(x_0, R)} |D(w_n)|^2 \, dx \, dt \geq \int_{t_0 - \delta}^{t_0 + \delta} \int_{B(x_0, R)} |D(w)|^2 \, dx \, dt.$$

Dividing each side by  $2\delta|B(x_0, R)|$ , we get:

$$\int_{t_0 - \delta}^{t_0 + \delta} \int_{B(x_0, R)} |D(w_n)|^2 \, dx \, dt \geq \int_{t_0 - \delta}^{t_0 + \delta} \int_{B(x_0, R)} |D(w)|^2 \, dx \, dt$$

then letting  $(\delta, R) \rightarrow (0, 0)$  leads to the result, applying Lebesgue's differentiation theorem.  $\square$

The following lemma gives the convergence of  $u_{m,\varepsilon}$  when  $\varepsilon$  goes to zero.

**Lemma 4.4.** *With the hypotheses of Proposition 4.1 there exists  $v_m \in L^2((0, T), H_{0,\sigma}^1(\Omega)) \cap L^\infty((0, T), L_\sigma^2(\Omega))$  with  $v'_m \in L^{\frac{4}{N}}((0, T), H_\sigma^{-1}(\Omega))$  such that, up to subsequences:*

1.  $u'_{m,\varepsilon} \rightharpoonup v'_m$  in  $L^{\frac{4}{N}}((0, T), H_\sigma^{-1}(\Omega))$ ;
2.  $u_{m,\varepsilon} \rightharpoonup v_m$  in  $L^2((0, T), H_{0,\sigma}^1(\Omega))$ ;
3.  $u_{m,\varepsilon} \rightarrow v_m$  in  $L^2((0, T), L_\sigma^2(\Omega))$ ;
4.  $u_{m,\varepsilon} \xrightarrow{*} v_m$  in  $L^\infty((0, T), L_\sigma^2(\Omega))$ .

Moreover,  $v_m$  satisfies, for all  $\psi \in C^\infty((0, T) \times \Omega)$ :

$$\begin{aligned} \int_0^T \langle v'_m, v_m - \psi \rangle dt + \int_0^T \int_\Omega D(v_m) : D(v_m - \psi) dx dt + \int_0^T j(v_m) - j(\psi) dt \\ - \int_0^T \int_\Omega (v_m \cdot \nabla v_m) \cdot \psi dx dt \leq \int_0^T \langle f, v_m - \psi \rangle dt. \end{aligned} \quad (4.13)$$

*Proof.* The first and second points follow from the reflexivity of  $L^{\frac{4}{N}}((0, T), H_\sigma^{-1}(\Omega))$  and  $L^2((0, T), H_{0,\sigma}^1(\Omega))$  respectively, the third one from Aubin-Lions' Lemma, and the last one by Banach-Alaoglu-Bourbaki's theorem.

Then, since  $u_{m,\varepsilon}$  is a weak solution of (4.2), it satisfies (4.4). Testing against  $\varphi = u_{m,\varepsilon} - \psi$  in (4.4) for a test function  $\psi$ , we have:

$$\begin{aligned} \langle u'_{m,\varepsilon}, u_{m,\varepsilon} - \psi \rangle + \int_\Omega D(u_{m,\varepsilon}) : D(u_{m,\varepsilon} - \psi) dx + \langle j'_\varepsilon(u_{m,\varepsilon}), u_{m,\varepsilon} - \psi \rangle \\ - \int_\Omega (u_{m,\varepsilon} \cdot \nabla u_{m,\varepsilon}) \cdot \psi dx = \langle f, u_{m,\varepsilon} - \psi \rangle. \end{aligned} \quad (4.14)$$

Applying Lemma 2.1 leads to the well-known convexity inequality:

$$j_\varepsilon(u_{m,\varepsilon}) - j_\varepsilon(\psi) \leq \langle j'_\varepsilon(u_{m,\varepsilon}), u_{m,\varepsilon} - \psi \rangle. \quad (4.15)$$

Using now Lemma 4.2 for  $u_{m,\varepsilon}$  in (4.15), we get:

$$j(u_{m,\varepsilon}) - C(\varepsilon, u_{m,\varepsilon}) - j_\varepsilon(\psi) \leq \langle j'_\varepsilon(u_{m,\varepsilon}), u_{m,\varepsilon} - \psi \rangle$$

and then, by (C3) and Lemma 4.3 applied to  $u_{m,\varepsilon}$  for the convergence toward  $v_m$ , we get:

$$j(v_m) - C(\varepsilon, u_{m,\varepsilon}) - j_\varepsilon(\psi) \leq \langle j'_\varepsilon(u_{m,\varepsilon}), u_{m,\varepsilon} - \psi \rangle.$$

Then, we can write (see [19] part 5.9. for details):

$$\forall \varphi \in H_{0,\sigma}^1(\Omega), \int_\Omega u_{m,\varepsilon}(T) \varphi dx = \langle u_{m,\varepsilon}(T), \varphi \rangle = \int_0^T \langle u'_{m,\varepsilon}(t), \varphi \rangle dt + \langle u_0, \varphi \rangle. \quad (4.16)$$

Now, we also have, using Proposition 4.1:

$$\begin{aligned} \int_0^T \langle u'_{m,\varepsilon}(t), \varphi \rangle dt + \langle u_0, \varphi \rangle \leq \|u'_{m,\varepsilon}\|_{L^{\frac{4}{N}}((0,T), H^{-1})} \left( \int_0^T \|\varphi\|_{H_0^1}^{\frac{4}{4-N}} dt \right)^{\frac{4-N}{4}} + C \|u_0\|_{L^2} \|\varphi\|_{H_0^1} \\ \leq C \left( T^{\frac{4-N}{4}} + \|u_0\|_{L^2} \right) \|\varphi\|_{H_0^1}. \end{aligned}$$

In the above inequality we considered  $\varphi$  as a function in  $L^\infty((0, T), H_0^1(\Omega))$ , so it belongs to  $L^{\frac{4}{4-N}}((0, T), H_0^1(\Omega))$  and its left-hand side defines a linear form over  $L^{\frac{4}{N}}((0, T), H^{-1}(\Omega))$ .

Also, the weak convergence leads to:

$$\int_0^T \langle u'_{m,\varepsilon}(t), \varphi \rangle dt \xrightarrow{\varepsilon \rightarrow 0} \int_0^T \langle v'_m(t), \varphi \rangle dt. \quad (4.17)$$

Finally, (4.16) and (4.17) imply, up to apply a dominated convergence theorem, to:

$$u_{m,\varepsilon}(T) \xrightarrow{\varepsilon \rightarrow 0} v_m(T) \quad \text{in } L^2(\Omega). \quad (4.18)$$

Then, (4.18) implies:

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{2} (\|u_{m,\varepsilon}(T)\|_{L^2}^2 - \|P_m(u_0)\|_{L^2}^2) \geq \frac{1}{2} (\|v_m(T)\|_{L^2}^2 - \|P_m(u_0)\|_{L^2}^2) \quad (4.19)$$

Also, from usual estimates (see [30, Chapter 4]), since  $u_{m,\varepsilon} \xrightarrow{\varepsilon} v_m$  in  $L^2((0, T), H_{0,\sigma}^1(\Omega))$ , we have:

$$\int_0^T \int_{\Omega} |D(u_{m,\varepsilon})|^2 dx dt \xrightarrow{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega} |D(v_m)|^2 dx dt \quad (4.20)$$

and

$$\int_0^T \int_{\Omega} (u_{m,\varepsilon} \cdot \nabla u_{m,\varepsilon}) \cdot \psi dx dt \xrightarrow{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega} (v_m \cdot \nabla v_m) \cdot \psi dx dt. \quad (4.21)$$

Integrating in time (4.14), and passing to the limit over  $\varepsilon$ , combining with (4.20), (4.21), and (4.19) leads to (4.13).  $\square$

Arguing in the same way, we obtain the following result.

**Lemma 4.5.** *Under the assumptions of Proposition 4.1, there exists  $u \in L^2((0, T), H_{0,\sigma}^1(\Omega)) \cap L^\infty((0, T), L_\sigma^2(\Omega))$  with  $u' \in L^{\frac{4}{N}}((0, T), H_\sigma^{-1}(\Omega))$  such that the function  $v_m$  given by Lemma 4.4 verifies.*

1.  $v'_m \rightharpoonup u'$  in  $L^{\frac{4}{N}}((0, T), H_\sigma^{-1}(\Omega))$ ;
2.  $v_m \rightarrow u$  in  $L^2((0, T), L_\sigma^2(\Omega))$ ;
3.  $v_m \rightharpoonup u$  in  $L^2((0, T), H_{0,\sigma}^1(\Omega))$ ;
4.  $v_m \overset{*}{\rightharpoonup} u$  in  $L^\infty((0, T), L_\sigma^2(\Omega))$ .

Moreover, we point out that  $u \in C_w((0, T), L_\sigma^2(\Omega))$  from the above estimates (see [9, Proposition V.1.7. p.363] for details).

*Proof of Theorem 3.1.* We point out that the coefficients of  $v_m$  given by Lemma 4.4 satisfy an ODE as (4.3) with  $\varepsilon = 0$ , then  $v_m$  is still smooth in space and time. Moreover, we can take up again the method previously used, that is we can write :

$$\forall \varphi \in H_{0,\sigma}^1(\Omega), \int_{\Omega} v_m(T) \varphi dx = \langle v_m(T), \varphi \rangle = \int_0^T \langle v'_m(t), \varphi \rangle dt + \langle P_m(u_0), \varphi \rangle. \quad (4.22)$$

Using Proposition 4.1 then leads to:

$$\begin{aligned} \int_0^T \langle v'_m(t), \varphi \rangle dt + \langle u_0, \varphi \rangle &\leq \|v'_m\|_{L^{\frac{4}{N}}((0,T), H^{-1})} \left( \int_0^T \|\varphi\|_{H_0^1}^{\frac{4}{4-N}} dt \right)^{\frac{4-N}{4}} + C \|u_0\|_{L^2} \|\varphi\|_{H_0^1} \\ &\leq C \left( T^{\frac{4-N}{4}} + \|u_0\|_{L^2} \right) \|\varphi\|_{H_0^1}. \end{aligned} \quad (4.23)$$

Then, the weak convergence leads to:

$$\int_0^T \langle v'_m(t), \varphi \rangle dt \xrightarrow{\varepsilon \rightarrow 0} \int_0^T \langle v'_m(t), \varphi \rangle dt. \quad (4.24)$$

Finally, (4.22) and (4.24) imply:

$$v_m(T) \xrightarrow{\varepsilon \rightarrow 0} u(T) \quad \text{in } L^2(\Omega). \quad (4.25)$$

Then, (4.18) implies:

$$\begin{aligned} \liminf_{m \rightarrow +\infty} \frac{1}{2} (\|v_m(T)\|_{L^2}^2 - \|P_m(u_0)\|_{L^2}^2) &\geq \frac{1}{2} (\|u(T)\|_{L^2}^2 - \|u_0\|_{L^2}^2) \\ \Leftrightarrow \liminf_{m \rightarrow +\infty} \int_0^T \langle v'_m, v_m \rangle dt &\geq \int_0^T \langle u', u \rangle dt. \end{aligned} \quad (4.26)$$

Using once again usual estimates for Navier-Stokes equation, since  $v_m \xrightarrow{m \rightarrow +\infty} u$  in  $L^2((0, T), H_{0,\sigma}^1(\Omega))$ , we have:

$$\int_0^T \int_{\Omega} |D(v_m)|^2 dx dt \xrightarrow{m \rightarrow +\infty} \int_0^T \int_{\Omega} |D(u)|^2 dx dt \quad (4.27)$$

and

$$\int_0^T \int_{\Omega} (v_m \cdot \nabla v_m) \cdot \psi dx dt \xrightarrow{m \rightarrow +\infty} \int_0^T \int_{\Omega} (u \cdot \nabla u) \cdot \psi dx dt. \quad (4.28)$$

Applying lemma 4.3 with our assumption (C3) and passing to the limit over  $m$ , we get:

$$\liminf_{m \rightarrow +\infty} \int_0^T j(v_m) dt \geq j(u). \quad (4.29)$$

Passing to the limit over  $m$  in (4.23), combining with (4.27), (4.28), (4.29) and (4.26) leads to:

$$\begin{aligned} \frac{1}{2} (\|u(T)\|_{L^2(\Omega)}^2 - \|u_0\|_{L^2(\Omega)}^2) - \int_0^T \langle u', \psi \rangle dt + \int_0^T \int_{\Omega} D(u) : D(u - \psi) dx dt + \int_0^T j(u) - j(\psi) dt \\ - \int_0^T \int_{\Omega} (u \cdot \nabla u) \cdot \psi dx dt \leq \int_0^T \langle f, u - \psi \rangle dt \end{aligned} \quad (4.30)$$

which is the desired result, that is  $u$  is a weak solution of (1.1).  $\square$

## 5 Existence of a finite stopping time for shear-thinning flows

In this part, we assume that hypotheses of the Theorem 3.2 are fulfilled. We are interested to show the existence of a finite stopping time of weak solutions of (1.1) for a viscosity coefficient  $F$  which behaves at least as a power-law model, following classical methods for proving such an extinction profile. In fact, one can observe that the nonlinearity proper to threshold flows in some special cases implies the existence of such a finite stopping time, as it has already been proved for the two-dimensional Bingham equation under some assumptions in [13]. Moreover, the study of such a profile has been proved in the case of the parabolic  $p$ -Laplacian, see [14, section VII.2] for a bounded initial datum or [4, Theorem 4.6] for the case  $p = 1$  and with initial datum belonging to  $L^2(\Omega)$ . In this section, we will moreover assume for convenience that the force term belongs to  $L^2((0, T), L^2(\Omega))$  or, if necessary, we will identify the duality bracket  $\langle \cdot, \cdot \rangle$  with the  $L^2$  inner product. Note that this assumption is not necessary, the results remain valid for  $f \in L^2((0, T), H_{\sigma}^{-1}(\Omega))$ .

Before proving the Theorem 3.2, we need to prove the following useful lemmas.

**Lemma 5.1.** *The solution  $v_m$  provided by Lemma 4.4 verifies, for every function  $\varphi \in C_{\sigma}^{\infty}((0, T) \times \Omega)$  and for almost all  $t \in (0, T)$ :*

$$\begin{aligned} \langle v'_m(t), \varphi(t) - v_m(t) \rangle + \int_{\Omega} D(v_m(t)) : D(\varphi(t) - v_m(t)) dx - \int_{\Omega} (v_m(t) \cdot \nabla v_m(t)) \cdot \varphi(t) dx \\ + \int_{\Omega} G(|D(\varphi(t))|) - G(|D(v_m(t))|) dx \geq \langle f(t), \varphi(t) - v_m(t) \rangle. \end{aligned} \quad (5.1)$$



*Proof.* Assume that  $t_0$  is a Lebesgue point of  $(0, T)$ . Let  $0 < \delta < t_0$  and consider a test function  $\psi$  in (4.13) having the following form:

$$\psi(t) := \begin{cases} \varphi(t) & \text{in } (t_0 - \delta, t_0 + \delta) \\ v_m(t) & \text{in } (0, T) \setminus (t_0 - \delta, t_0 + \delta), \end{cases}$$

$\varphi$  being such that  $\psi \in \mathcal{C}_\sigma^\infty((0, T) \times \Omega)$ . Multiplying the inequality obtained in this way by  $\frac{1}{2\delta}$  and applying Lebesgue's differentiation theorem, leads to:

$$\begin{aligned} & \langle v'_m(t_0), \varphi(t_0) - v_m(t_0) \rangle + \int_{\Omega} D(v_m(t_0)) : D(\varphi(t_0) - v_m(t_0)) \, dx - \int_{\Omega} (v_m(t_0) \cdot \nabla v_m(t_0)) \cdot \varphi(t_0) \, dx \\ & + \int_{\Omega} G(|D(\varphi(t_0))|) - G(|D(v_m(t_0))|) \, dx \geq \langle f(t_0), \varphi(t_0) - v_m(t_0) \rangle. \end{aligned}$$

Since almost every point in  $(0, T)$  is a Lebesgue point, the inequality (5.1) follows directly from the above estimate.  $\square$

**Lemma 5.2.** *Assume that  $u \in L^6(\Omega)$ . Then, for all  $r \in (0, 3)$ , the following inequality holds:*

$$\|u\|_{L^2(\Omega)}^r \leq \left(\frac{3-r}{3}\right) \|u\|_{L^{\frac{3}{2}}(\Omega)}^{\frac{4r}{3-r}} + \left(\frac{r}{3}\right) \|u\|_{L^6(\Omega)}^2. \quad (5.2)$$

*Proof of Lemma 5.2.* First, we write for  $s \in (0, 2)$ :

$$\|u\|_{L^2(\Omega)}^2 := \int_{\Omega} |u|^s |u|^{2-s} \, dx. \quad (5.3)$$

Now, passing to the power  $\frac{r}{2}$  and using Hölder's inequality in (5.3) leads to:

$$\|u\|_{L^2(\Omega)}^r \leq \left(\int_{\Omega} |u|^{sp} \, dx\right)^{\frac{r}{2p}} \left(\int_{\Omega} |u|^{(2-s)q} \, dx\right)^{\frac{r}{2q}}. \quad (5.4)$$

Finally, we apply Young's inequality into (5.4) to obtain:

$$\|u\|_{L^2(\Omega)}^r \leq \frac{1}{a} \left(\int_{\Omega} |u|^{sp} \, dx\right)^{\frac{ar}{p}} + \frac{1}{b} \left(\int_{\Omega} |u|^{(2-s)q} \, dx\right)^{\frac{br}{q}}. \quad (5.5)$$

Fixing successively  $p = \frac{3}{2s}$ ,  $s = \frac{4}{3}$ , and  $b = \frac{3}{r}$  leads to (5.2), that is the result is proved. We point out that it is necessary to have  $r \in (0, 3)$  in order to satisfy the necessary conditions in the inequalities used above:

$$\begin{cases} s \in (0, \frac{3}{2}), \\ r > 0, \\ \frac{1}{p} + \frac{1}{q} = 1, \\ \frac{1}{a} + \frac{1}{b} = 1. \end{cases}$$

$\square$

Moreover, we recall the Nirenberg-Strauss inequality:

**Lemma 5.3** ([31, Theorem 1]). *Let  $\Omega$  be an open bounded subset of  $\mathbb{R}^N$  with Lipschitz boundary, then there exists a constant  $C > 0$  which depends of  $N$  and  $\Omega$  such that for all  $u \in W_0^{1, \frac{N}{N-1}}(\Omega)$  the following inequality holds:*

$$\|u\|_{L^{\frac{N}{N-1}}(\Omega)} \leq C \|D(u)\|_{L^1(\Omega)}. \quad (5.6)$$

We are now able to prove Theorem 3.2. We point out that the proof being well-known in the two-dimensional case (see [13]) and can be in that last case a direct application of the Korn's inequality and Sobolev's embedding theorem. For this reason, we only give a proof in the three-dimensional setting.

*Proof of Theorem 3.2.* Let  $v_m$  be the function provided by Lemma 4.4. From Lemma 5.1 this function verifies estimate (5.1). Choosing  $\varphi = 0$  in this estimate we get:

$$\langle v'_m, v_m \rangle + \int_{\Omega} |D(v_m)|^2 dx + j(v_m) \leq \langle f, v_m \rangle. \quad (5.7)$$

Combining (2.2) and (3.1), we easily obtain

$$j(v_m) \geq C \|D(v_m)\|_{L^{2-\alpha}}^{2-\alpha}. \quad (5.8)$$

From (5.7) and (5.8), we get:

$$\frac{1}{2} \frac{d}{dt} \left( \|v_m(t)\|_{L^2(\Omega)}^2 \right) + \int_{\Omega} |D(v_m)|^2 dx + C \|D(v_m)\|_{L^{2-\alpha}(\Omega)}^{2-\alpha} \leq \langle f, v_m \rangle. \quad (5.9)$$

Then, using successively the embedding  $L^{2-\alpha}(\Omega) \hookrightarrow L^1(\Omega)$ , assumption (3.1) and the Theorem 5.3, we get from (5.9), for  $t \in (T_1, t)$ :

$$\frac{1}{2} \frac{d}{dt} \left( \|v_m(t)\|_{L^2(\Omega)}^2 \right) + \int_{\Omega} |D(v_m)|^2 dx + C \|v_m\|_{L^{\frac{3}{2}}(\Omega)}^{2-\alpha} \leq 0. \quad (5.10)$$

Now, from the embedding  $\{u \in H_0^1(\Omega) / \|D(u)\|_{L^2(\Omega)} < +\infty\} \hookrightarrow L^6(\Omega)$  which can be obtained using Korn's  $L^2$  equality and Sobolev embedding  $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$  we get from (5.10):

$$\frac{1}{2} \frac{d}{dt} \left( \|v_m(t)\|_{L^2(\Omega)}^2 \right) + C \left( \|v_m\|_{L^6(\Omega)}^2 + \|v_m\|_{L^{\frac{3}{2}}(\Omega)}^{2-\alpha} \right) \leq 0. \quad (5.11)$$

Now, we can apply Lemma 5.2 with  $r = \frac{6-3\alpha}{6-\alpha}$  to get:

$$\|v_m\|_{L^2(\Omega)}^{1-2\left(\frac{\alpha}{6-\alpha}\right)} \leq \left( \frac{4}{6-\alpha} \right) \|v_m\|_{L^{\frac{3}{2}}(\Omega)}^{2-\alpha} + \left( \frac{2-\alpha}{6-\alpha} \right) \|v_m\|_{L^6(\Omega)}^2. \quad (5.12)$$

Then, (5.12) combined with (5.11) leads to, for  $t \in (T_1, T)$ :

$$\frac{1}{2} \frac{d}{dt} \left( \|v_m(t)\|_{L^2(\Omega)}^2 \right) + C \|v_m\|_{L^2(\Omega)}^{1-2\left(\frac{\alpha}{6-\alpha}\right)} \leq 0. \quad (5.13)$$

Assume that there does not exist any finite stopping time, that is for almost all  $t \in (0, T)$ , we have:

$$\|v_m(t)\|_{L^2(\Omega)} > 0.$$

Then dividing by  $\|v_m(t)\|_{L^2(\Omega)}$  the both sides of (5.13), we obtain for almost all  $t \in (T_1, T)$ :

$$\frac{d}{dt} \left( \|v_m(t)\|_{L^2(\Omega)} \right) + C \|v_m(t)\|_{L^2(\Omega)}^{-2\left(\frac{\alpha}{6-\alpha}\right)} \leq 0,$$

which is equivalent to:

$$\frac{d}{dt} \left( \|v_m(t)\|_{L^2(\Omega)}^{\frac{6+\alpha}{6-\alpha}} \right) \leq -C.$$

Integrating over  $(T_1, t)$ , we have for almost all  $t \in (T_1, T)$ :

$$\|v_m(t)\|_{L^2(\Omega)}^{\frac{6+\alpha}{6-\alpha}} \leq \|v_m(T_1)\|_{L^2(\Omega)}^{\frac{6+\alpha}{6-\alpha}} + C(T_1 - t),$$

which leads to  $\|v_m(t)\|_{L^2(\Omega)}^{\frac{6+\alpha}{6-\alpha}} < 0$  for  $t$  large enough, up to take  $T$  large enough. This is a contradiction, so, there exists a time  $T_0 \in (0, T)$  such that  $\|v_m(t)\|_{L^2(\Omega)} = 0$  for all  $t \in [T_0, T)$ . Indeed, assume that  $t_s$  is the time for which we have  $\|v_m(t_s)\|_{L^2(\Omega)} = 0$ . Then one can easily verify that  $t \mapsto \|v_m(t)\|_{L^2(\Omega)}^2$  is non-increasing on  $[\max(t_s, T_1), T)$ , and we get the result setting  $T_0 = \max(t_s, T_1)$ .

So we get that there exists  $\tilde{t} \in [T_0, T)$  such that  $\|u(\tilde{t})\|_{L^2(\Omega)} = 0$ . If it was not the case, we would have:

$$\forall t \in [T_0, T), \|v_m(t)\|_{L^2(\Omega)} < \|u(t)\|_{L^2(\Omega)},$$

which implies:

$$\int_{T_0}^T \|v_m(t)\|_{L^2(\Omega)}^2 dt < \int_{T_0}^T \|u(t)\|_{L^2(\Omega)}^2 dt.$$

This would be a contradiction with the uniform bound of  $(v_m)_{m \in \mathbb{N}}$  in  $L^2((T_0, T), L^2_\sigma(\Omega))$  and the convergence  $v_m \rightharpoonup u$  in  $L^2((T_0, T), L^2_\sigma(\Omega))$ . Also, there exists  $\tilde{t}$  in  $[T_0, T)$  such that  $\|u(\tilde{t})\|_{L^2(\Omega)} = 0$ , and, arguing as we have already done for  $(v_m)_{m \in \mathbb{N}}$ , we have that  $t \mapsto \|u(t)\|_{L^2(\Omega)}^2$  is non-increasing on  $[\tilde{t}, T)$ . Finally, since  $u \in \mathcal{C}_w((0, T), L^2_\sigma(\Omega))$ , we get:

$$\forall t \in [\tilde{t}, T), \|u(t)\|_{L^2(\Omega)} = 0,$$

which is the desired result and concludes the proof. □

## 6 Conclusions and problems remaining open

In this paper, we have been able to establish the existence of variational inequality solutions for a large class of generalized Newtonian flows in both two- and three-dimensional settings, including threshold fluid flows. This result completes that of [17] and should be related to that obtained in [1] in the context of dissipative solutions. The issue of global regularity in the three-dimensional setting remains open, the question being still widely open for the Leray-Hopf (and Leray) solutions of the incompressible Navier-Stokes equations. Moreover, we note that the addition of a nonlinear term complicates the study of regularity in the framework of generalized Newtonian flows, particularly in the case of threshold flows.

An important question would be to know if it is possible to control this extinction profile, the question of the control of solutions of quasilinear parabolic equations of  $p$ -Laplacian type remaining currently open.

## A Some examples of functions $F$

In this section, we give some examples of functions  $F$  satisfying the conditions (C1)-(C4), most of which correspond to models of non-Newtonian coherent flows in the physical sense. This is the case for quasi-Newtonian fluids such as blood, threshold fluids such as mayonnaise, or more generally in the case of polymeric liquids.

1. Firstly, in order to describe power-law fluids, we can consider functions  $(F_\alpha)_{0 \leq \alpha \leq 1}$  given by:

$$F_\alpha(t) : \begin{cases} (0, +\infty) \rightarrow (0, +\infty) \\ t \mapsto t^{-\alpha}. \end{cases}$$

2. Considering functions  $(F_{\mu, \alpha})_{\mu > 0, \alpha \in (0, 1]}$  of the form

$$F_{\mu, \alpha}(t) : \begin{cases} (0, +\infty) \rightarrow (0, +\infty) \\ t \mapsto (\mu + t^2)^{-\frac{\alpha}{2}} \end{cases}$$

leads to Carreau flows.

3. Cross fluids are obtained by choosing function  $(F_{\gamma, \alpha})_{\gamma > 0, \alpha \in (0, 1]}$  given by:

$$F_{\gamma, \alpha}(t) : \begin{cases} (0, +\infty) \rightarrow (0, +\infty) \\ t \mapsto \gamma + t^{-\alpha}. \end{cases}$$

4. Another possible choice is to take functions  $(F_{\alpha, \beta, \gamma})$  given

$$F_{\alpha,\beta,\gamma}(t) : \begin{cases} (0, +\infty) \rightarrow (0, +\infty) \\ t \mapsto \begin{cases} t^{-\alpha} \log(1+t)^{-\beta} & \text{If } t \in (0, \gamma] \\ \log(1+\gamma)^{-\beta} t^{-\alpha} & \text{If } t \in (\gamma, +\infty) \end{cases} \end{cases}$$

for  $0 < \alpha < 1$  and some  $\beta, \gamma > 0$  with  $\gamma$  small enough.

## B An energy equality

This appendix is aimed to prove a useful estimate, interesting by itself, stating that the weak solutions of (1.1) are continuous in time. More exactly, we have the following results.

**Proposition B.1.** *Assume that  $u$  is a weak solution to (1.1) satisfying assumptions of Theorem 3.1. Moreover, assume that there exists  $\theta > 0$  such that for all  $t \in (0, T)$ , we have:*

$$G(t) = \theta t^2 F(\theta t). \quad (\text{B.1})$$

Then we have for almost all  $t \in (0, T)$  that there exists  $\eta(t) \in (0, 1]$  such that:

$$\|u(t)\|_{L^2}^2 + \frac{1}{2} \|u\|_{L^2((0,T), H_0^1)}^2 + \int_0^t \frac{1}{\eta(s)} \langle j'(\eta(s)u), u \rangle ds = \|u_0\|_{L^2}^2 + \int_0^t \langle f, u \rangle ds.$$

*Proof.* As previously mentioned, we can apply an analogue of the Lemma 5.1 to  $u$ , that is we have for almost all time  $t \in (0, T)$ :

$$\langle u', \varphi - u \rangle + \int_{\Omega} D(u) : D(\varphi - u) dx - \int_{\Omega} (u \cdot \nabla u) \cdot \varphi dx + j(\varphi) - j(u) \geq \langle f, \varphi - u \rangle. \quad (\text{B.2})$$

Testing against  $\varphi = 0$  in (B.2), we get:

$$\langle u', u \rangle + \int_{\Omega} |D(u)|^2 dx + \int_{\Omega} G(|D(u)|) dx \leq \langle f, u \rangle. \quad (\text{B.3})$$

Now, we write:

$$\forall t \in (0, +\infty), G(t) = \int_0^t sF(s) ds = t^2 \int_0^1 yF(ty) dy.$$

We point out that the result still holds for  $t = 0$ . Combining the above equality and (B.1), we deduce that:

$$\int_{\Omega} G(|D(u)|) dx = \int_{\Omega} \theta |D(u)|^2 F(\theta |D(u)|) dx = \frac{1}{\theta} \langle j'(\theta u), \theta u \rangle.$$

So, (B.3) leads to:

$$\langle u', u \rangle + \int_{\Omega} |D(u)|^2 dx + \frac{1}{\theta} \langle j'(\theta u), \theta u \rangle \leq \langle f, u \rangle. \quad (\text{B.4})$$

Using now  $\varphi = (1 + \delta)u$  (for every  $\delta > 0$ ) as a test function in B.2 leads to:

$$\delta \langle u', u \rangle + \delta \int_{\Omega} |D(u)|^2 dx + \int_{\Omega} G((1 + \delta)|D(u)|) - G(|D(u)|) dx \geq \delta \langle f, u \rangle. \quad (\text{B.5})$$

Dividing each side of (B.5) by  $\delta > 0$  and passing to the limit with respect to the parameter  $\delta \rightarrow 0$  we obtain, recalling that  $G$  admits a Gâteaux-derivative, that:

$$\langle u', u \rangle + \int_{\Omega} |D(u)|^2 dx + \langle j'(u), u \rangle \geq \langle f, u \rangle. \quad (\text{B.6})$$

Finally, using once again the assumption (C2), we obtain from the inequalities (B.4) and (B.6), that there exists  $\eta \in [\theta, 1]$  such that:

$$\langle u', u \rangle + \int_{\Omega} |D(u)|^2 dx + \frac{1}{\eta} \langle j'(\eta u), \eta u \rangle - \langle f, u \rangle = 0.$$

The result of the proposition follows integrating over  $(0, t)$ . □

**Remark 3.** Clearly, the case of threshold flows is contained into the previous proposition, since we can write, for  $t > 0$ :

$$\int_0^t s^{1-\alpha} ds = \frac{1}{2-\alpha} t^{2-\alpha} = \theta t^2 (\theta t)^{-\alpha}$$

with  $\theta = (2-\alpha)^{-\frac{1}{1-\alpha}}$  if  $0 < \alpha < 1$ ,  $\theta = 1$  if  $\alpha = 1$ . Also, Proposition B.1 directly leads to the next result, which is important by itself.

**Corollary B.1.** Assume that  $u$  is a weak solution of (1.1) satisfying assumptions of Proposition B.1, then  $u$  belongs to  $\mathcal{C}((0, T), L^2_{\sigma}(\Omega))$ . Moreover, the function  $t \mapsto \|u(t)\|_{L^2}$  is differentiable.

*Proof.* Applying Proposition B.1 for  $s$  and  $t$  such that  $s < t$ , up to redefine  $u$  on a set of null measure, we can get:

$$\|u(t)\|_{L^2}^2 - \|u(s)\|_{L^2}^2 + \int_s^t \frac{1}{\eta(r)} \langle j'(\eta(r)u), u \rangle dr = \int_s^t \langle f, u \rangle dr.$$

Making  $s \rightarrow t$  leads to the result. Up to fix an  $s \in [0, t)$ , it directly follows from the previous equality that  $t \mapsto \|u(t)\|_{L^2}^2$  admits a derivative, using the fundamental theorem of calculus. □

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