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Jean-Philippe Anker, Effie Papageorgiou, Hong-Wei Zhang. Asymptotic behavior of solutions to the heat equation on noncompact symmetric spaces. *Journal of Functional Analysis*, 2023, 284 (6), pp.109828. 10.1016/j.jfa.2022.109828 . hal-03459571v2

**HAL Id: hal-03459571**

**<https://hal.science/hal-03459571v2>**

Submitted on 17 Jan 2023

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# ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO THE HEAT EQUATION ON NONCOMPACT SYMMETRIC SPACES

JEAN-PHILIPPE ANKER, EFFIE PAPAGEORGIOU AND HONG-WEI ZHANG

ABSTRACT. This paper is twofold. The first part aims to study the long-time asymptotic behavior of solutions to the heat equation on Riemannian symmetric spaces  $G/K$  of noncompact type and of general rank. We show that any solution to the heat equation with bi- $K$ -invariant  $L^1$  initial data behaves asymptotically as the mass times the fundamental solution, and provide a counterexample in the non bi- $K$ -invariant case. These answer problems recently raised by J.L. Vázquez. In the second part, we investigate the long-time asymptotic behavior of solutions to the heat equation associated with the so-called distinguished Laplacian on  $G/K$ . Interestingly, we observe in this case phenomena which are similar to the Euclidean setting, namely  $L^1$  asymptotic convergence with no bi- $K$ -invariance condition and strong  $L^\infty$  convergence.

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## 1. INTRODUCTION

The heat equation is one of the most fundamental partial differential equations in mathematics. After the celebrated work of *Joseph Fourier* in 1822, it has been extensively studied in various settings and is known to play a central role in several areas of mathematics (see for instance [Gri09]). The following classical long-time asymptotic convergence result corresponds to the *Central Limit Theorem* of probability in the PDE setting. We refer to the expository survey [Váz18] for more details on this property.

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2020 *Mathematics Subject Classification.* 22E30, 35B40, 35K05, 58J35.

*Key words and phrases.* Noncompact symmetric space, heat equation, asymptotic behavior, long-time convergence, distinguished Laplacian.

**Theorem 1.1.** *Consider the heat equation*

$$\begin{cases} \partial_t u(t, x) = \Delta_{\mathbb{R}^n} u(t, x), & t > 0, x \in \mathbb{R}^n \\ u(0, x) = u_0(x), \end{cases} \quad (1.1)$$

where the initial data  $u_0$  belongs to  $L^1(\mathbb{R}^n)$ . Denote by  $M = \int_{\mathbb{R}^n} dx u_0(x)$  the mass of  $u_0$  and by  $G_t(x) = (4\pi t)^{-n/2} e^{-|x|^2/4t}$  the heat kernel. Then the solution to (1.1) satisfies:

$$\|u(t, \cdot) - MG_t\|_{L^1(\mathbb{R}^n)} \longrightarrow 0 \quad (1.2)$$

and

$$t^{\frac{n}{2}} \|u(t, \cdot) - MG_t\|_{L^\infty(\mathbb{R}^n)} \longrightarrow 0 \quad (1.3)$$

as  $t \rightarrow \infty$ . The  $L^p$  ( $1 < p < \infty$ ) norm estimates follow by convexity.

To our knowledge, analogous properties on negatively curved manifolds were first investigated by Vázquez in his recent work [Váz19], which deals with real hyperbolic spaces and where he suggests possible generalizations, in particular to all Riemannian symmetric spaces of noncompact type (other hyperbolic spaces in rank one and noncompact symmetric spaces of higher rank). This is fully achieved in the present paper, where we also clarify some arguments in [Váz19] and consider a related Laplacian with Euclidean type properties. Let us elaborate. Given a symmetric space  $\mathbb{X} = G/K$  of non compact type, let us denote by  $\Delta$  the Laplace-Beltrami operator on  $\mathbb{X}$  and by  $h_t$  the associated heat kernel, i.e., the bi- $K$ -invariant convolution kernel of the semi-group  $e^{t\Delta}$ . Our first main result is about the long-time convergence of solutions to the heat equation:

$$\partial_t u(t, x) = \Delta_x u(t, x), \quad u(0, x) = u_0(x). \quad (1.4)$$

**Theorem 1.2.** *Let  $u_0 \in L^1(\mathbb{X})$  be a bi- $K$ -invariant initial data and  $M = \int_{\mathbb{X}} dx u_0(x)$  be its mass. Then the solution to the heat equation (1.4) satisfies*

$$\|u(t, \cdot) - Mh_t\|_{L^1(\mathbb{X})} \longrightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (1.5)$$

Moreover, this convergence fails in general without the bi- $K$ -invariance assumption.

**Remark 1.3.** *The convergence (1.5) generalizes the result previously obtained in [Váz19] on  $\mathbb{H}^n(\mathbb{R})$ . Vázquez also provided a counterexample in the non bi- $K$ -invariant case on  $\mathbb{H}^3(\mathbb{R})$ , where the heat kernel  $h_t$  has an explicit expression, and asked if it could be extended in other dimensions. We establish such a counterexample not only for  $\mathbb{H}^n(\mathbb{R})$ ,  $n \neq 3$ , but also for the symmetric spaces of general rank, see Sect. 3.4. Finally, our method sheds light to why this non-euclidean discrepancy occurs, see Remark 3.10.*

**Remark 1.4.** *If the initial data is in addition compactly supported, we obtain the better estimate*

$$\|u(t, \cdot) - Mh_t\|_{L^1(\mathbb{X})} \leq Ct^{-\frac{1}{2}+\varepsilon} \quad \forall t \geq 1,$$

where  $C > 0$  is a constant and  $\varepsilon$  is any small positive constant, see Sect. 3.1 and Sect. 3.2.

**Remark 1.5.** *We also provide the following sup norm (for which no bi- $K$ -invariance is needed) and  $L^p$  ( $1 < p < \infty$ ) norm estimates:*

$$\|u(t, \cdot) - Mh_t\|_{L^\infty(\mathbb{X})} = O(t^{-\frac{\nu}{2}} e^{-|\rho|^2 t}) \quad (1.6)$$

$$\|u(t, \cdot) - Mh_t\|_{L^p(\mathbb{X})} = o(t^{-\frac{\nu}{2p'}} e^{-\frac{|\rho|^2 t}{p'}}) \quad (1.7)$$

as  $t \rightarrow \infty$ . Here,  $p'$  denotes the dual exponent of  $p$ , defined by the formula  $\frac{1}{p} + \frac{1}{p'} = 1$ . As previously observed on  $\mathbb{H}^n(\mathbb{R})$ , the sup norm estimate in the present context is relatively weaker compared to (1.3) in the Euclidean setting, while the  $L^p$  norm estimate is similar. Here,  $\nu$  denotes the so-called dimension at infinity of  $\mathbb{X}$  and  $\rho$  is the half sum of positive roots with multiplicities, see Sect. 2.

Let  $S = N(\exp \mathfrak{a}) = (\exp \mathfrak{a})N$  be the solvable group occurring in the Iwasawa decomposition  $G = N(\exp \mathfrak{a})K$ . Then  $S$  is identifiable, as a manifold, with the symmetric space  $\mathbb{X} = G/K$ . Our second main contribution is to study the asymptotic convergence for solutions to the heat equation associated with the so-called distinguished Laplacian  $\tilde{\Delta}$  on  $S$ . In order to state the results, let us introduce some indispensable notation, which will be clarified in [Sect. 2](#) and [Sect. 4](#). Let  $\mathfrak{a}$  be the Cartan subspace of  $\mathbb{X}$ . Denote by  $\varphi_\lambda$ , where  $\lambda \in \mathfrak{a}$ , the spherical function occurring in the Harish-Chandra transform, by  $\tilde{\delta}$  the modular function on  $S$ , and by  $\tilde{h}_t = \tilde{\delta}^{1/2} e^{|\rho|^2 t} h_t$  the fundamental solution to the Cauchy problem

$$\partial_t \tilde{v}(t, g) = \tilde{\Delta}_g \tilde{v}(t, g), \quad \tilde{v}(0, g) = \tilde{v}_0(g). \quad (1.8)$$

Let  $\tilde{\varphi}_\lambda = \tilde{\delta}^{1/2} \varphi_\lambda$  be the modified spherical function and denote by  $\tilde{M} = \frac{\tilde{v}_0 * \tilde{\varphi}_\lambda}{\tilde{\varphi}_\lambda}$  the mass function on  $S$  which generalizes the mass in the Euclidean case (see [Remark 4.4](#)). Then, we show the following long-time asymptotic convergence results.

**Theorem 1.6.** *Let  $\tilde{v}_0$  be a continuous and compactly supported initial data on  $S$ . Then, the solution to the heat equation (1.8) satisfies*

$$\|\tilde{v}(t, \cdot) - \tilde{M} \tilde{h}_t\|_{L^1(S)} \longrightarrow 0 \quad (1.9)$$

and

$$t^{\frac{\ell + |\Sigma_r^+|}{2}} \|\tilde{v}(t, \cdot) - \tilde{M} \tilde{h}_t\|_{L^\infty(S)} \longrightarrow 0 \quad (1.10)$$

as  $t \rightarrow \infty$ . Here  $\ell$  denotes the rank of  $G/K$  and  $\Sigma_r^+$  the set of positive reduced roots. Analogous  $L^p$  ( $1 < p < \infty$ ) norm estimates follow by convexity.

**Remark 1.7.** *Let us comment about (1.9) and (1.10). Firstly, notice that the  $L^1$  convergence (1.9) holds with no bi- $K$ -invariance restriction, in contrast to [Theorem 1.2](#), and the sup norm estimate (1.10) is stronger than (1.6), as in the Euclidean setting. Secondly, the mass  $\tilde{M}$  is a bounded function and no more a constant in general. Thirdly, the power  $\ell + |\Sigma_r^+|$ , which occurs in the time factor, is always different from the dimension at infinity  $\nu = \ell + 2|\Sigma_r^+|$  and it is equal to the manifold dimension  $n = \ell + \sum_{\alpha \in \Sigma_+} m_\alpha$  if and only if the following equivalent conditions hold:*

- the root system  $\Sigma$  is reduced and all roots have multiplicity  $m_\alpha = 1$ .
- $G$  is a normal real form.

**Remark 1.8.** *The asymptotic convergences (1.9) and (1.10) hold for some larger classes of initial data, see [Sect. 4.4](#). These classes are not optimal and finding the right function space is an interesting question for further study.*

This paper is organized as follows. After the present introduction in [Sect. 1](#) and preliminaries in [Sect. 2](#), we deal with the long-time asymptotic behavior of solutions to the heat equation associated with the Laplace-Beltrami operator on symmetric spaces in [Sect. 3](#). We start with smooth bi- $K$ -invariant compactly supported initial data, and prove, on the one hand, the long-time convergence in  $L^1$  in the critical region where the heat kernel concentrates. On the other hand, we show that both the solution and the heat kernel vanish asymptotically outside that critical region. In the rest of this section, we discuss these problems for more general initial data in the  $L^p$  ( $p \geq 1$ ) setting, and provide a counterexample for  $L^1$  in the non bi- $K$ -invariant case. In [Sect. 4](#), we investigate the asymptotic behavior of solutions to the heat equation associated with the distinguished Laplacian. After specifying the critical region in this context, we study the long-time convergence in  $L^1$  and in  $L^\infty$  with compactly supported initial data. Questions associated with other initial data are discussed at the end of the paper.

Throughout this paper, the notation  $A \lesssim B$  between two positive expressions means that there is a constant  $C > 0$  such that  $A \leq CB$ . The notation  $A \asymp B$  means that  $A \lesssim B$  and  $B \lesssim A$ .

## 2. PRELIMINARIES

In this section, we first review spherical Fourier analysis on Riemannian symmetric spaces of noncompact type. The notation is standard and follows [Hel78; Hel00; GaVa88]. Next we recall the asymptotic concentration of the heat kernel. We refer to [AnJi99; AnOs03] for more details on the heat kernel analysis in this setting.

**2.1. Noncompact Riemannian symmetric spaces.** Let  $G$  be a semi-simple Lie group, connected, noncompact, with finite center, and  $K$  be a maximal compact subgroup of  $G$ . The homogeneous space  $\mathbb{X} = G/K$  is a Riemannian symmetric space of noncompact type. Let  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  be the Cartan decomposition of the Lie algebra of  $G$ . The Killing form of  $\mathfrak{g}$  induces a  $K$ -invariant inner product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{p}$ , hence a  $G$ -invariant Riemannian metric on  $G/K$ . We denote by  $d(\cdot, \cdot)$  the Riemannian distance on  $\mathbb{X}$ .

Fix a maximal abelian subspace  $\mathfrak{a}$  in  $\mathfrak{p}$ . The rank of  $\mathbb{X}$  is the dimension  $\ell$  of  $\mathfrak{a}$ . We identify  $\mathfrak{a}$  with its dual  $\mathfrak{a}^*$  by means of the inner product inherited from  $\mathfrak{p}$ . Let  $\Sigma \subset \mathfrak{a}$  be the root system of  $(\mathfrak{g}, \mathfrak{a})$  and denote by  $W$  the Weyl group associated with  $\Sigma$ . Once a positive Weyl chamber  $\mathfrak{a}^+ \subset \mathfrak{a}$  has been selected,  $\Sigma^+$  (resp.  $\Sigma_r^+$  or  $\Sigma_s^+$ ) denotes the corresponding set of positive roots (resp. positive reduced, i.e., indivisible roots or simple roots). Let  $n$  be the dimension and  $\nu$  be the pseudo-dimension (or dimension at infinity) of  $\mathbb{X}$ :

$$n = \ell + \sum_{\alpha \in \Sigma^+} m_\alpha \quad \text{and} \quad \nu = \ell + 2|\Sigma_r^+| \quad (2.1)$$

where  $m_\alpha$  denotes the dimension of the positive root subspace

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \langle \alpha, H \rangle X, \forall H \in \mathfrak{a}\}.$$

Let  $\mathfrak{n}$  be the nilpotent Lie subalgebra of  $\mathfrak{g}$  associated with  $\Sigma^+$  and let  $N = \exp \mathfrak{n}$  be the corresponding Lie subgroup of  $G$ . We have the decompositions

$$\begin{cases} G = N(\exp \mathfrak{a})K & \text{(Iwasawa),} \\ G = K(\exp \overline{\mathfrak{a}^+})K & \text{(Cartan).} \end{cases}$$

Denote by  $A(x) \in \mathfrak{a}$  and  $x^+ \in \overline{\mathfrak{a}^+}$  the middle components of  $x \in G$  in these two decompositions, and by  $|x| = |x^+|$  the distance to the origin. For all  $x, y \in G$ , we have

$$|A(xK)| \leq |x| \quad \text{and} \quad |x^+ - y^+| \leq d(xK, yK), \quad (2.2)$$

see for instance [AnJi99, Lemma 2.1.2]. In the Cartan decomposition, the Haar measure on  $G$  writes

$$\int_G dx f(x) = |K/\mathbb{M}| \int_K dk_1 \int_{\mathfrak{a}^+} dx^+ \delta(x^+) \int_K dk_2 f(k_1(\exp x^+)k_2),$$

with density

$$\delta(x^+) = \prod_{\alpha \in \Sigma^+} (\sinh \langle \alpha, x^+ \rangle)^{m_\alpha} \asymp \prod_{\alpha \in \Sigma^+} \left( \frac{\langle \alpha, x^+ \rangle}{1 + \langle \alpha, x^+ \rangle} \right)^{m_\alpha} e^{2\langle \rho, x^+ \rangle} \quad \forall x^+ \in \overline{\mathfrak{a}^+}. \quad (2.3)$$

Here  $K$  is equipped with its normalized Haar measure,  $\mathbb{M}$  denotes the centralizer of  $\exp \mathfrak{a}$  in  $K$  and the volume of  $K/\mathbb{M}$  can be computed explicitly, see [AnJi99, Eq (2.2.4)]. Recall that  $\rho \in \mathfrak{a}^+$  denotes the half sum of all positive roots  $\alpha \in \Sigma^+$  counted with their multiplicities  $m_\alpha$ :

$$\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha.$$

**2.2. Spherical Fourier analysis.** Let  $\mathcal{S}(K\backslash G/K)$  be the Schwartz space of bi- $K$ -invariant functions on  $G$ . The spherical Fourier transform (Harish-Chandra transform)  $\mathcal{H}$  is defined by

$$\mathcal{H}f(\lambda) = \int_G dx \varphi_{-\lambda}(x) f(x) \quad \forall \lambda \in \mathfrak{a}, \forall f \in \mathcal{S}(K\backslash G/K), \quad (2.4)$$

where  $\varphi_\lambda \in \mathcal{C}^\infty(K\backslash G/K)$  is the spherical function of index  $\lambda \in \mathfrak{a}$ . Denote by  $\mathcal{S}(\mathfrak{a})^W$  the subspace of  $W$ -invariant functions in the Schwartz space  $\mathcal{S}(\mathfrak{a})$ . Then  $\mathcal{H}$  is an isomorphism between  $\mathcal{S}(K\backslash G/K)$  and  $\mathcal{S}(\mathfrak{a})^W$ . The inverse spherical Fourier transform is given by

$$f(x) = C_0 \int_{\mathfrak{a}} d\lambda |\mathbf{c}(\lambda)|^{-2} \varphi_\lambda(x) \mathcal{H}f(\lambda) \quad \forall x \in G, \forall f \in \mathcal{S}(\mathfrak{a})^W, \quad (2.5)$$

where the constant  $C_0 = 2^{n-\ell}/(2\pi)^\ell |K/\mathbb{M}|||W|$  depends only on the geometry of  $\mathbb{X}$ , and  $|\mathbf{c}(\lambda)|^{-2}$  is the so-called Plancherel density. We next review some elementary facts about the Plancherel density and the elementary spherical functions.

**2.2.1. Plancherel density.** According to the Gindikin-Karpelevič formula for the Harish-Chandra  $\mathbf{c}$ -function, we can write the Plancherel density as

$$|\mathbf{c}(\lambda)|^{-2} = \prod_{\alpha \in \Sigma_r^+} \left| \mathbf{c}_\alpha \left( \frac{\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle} \right) \right|^{-2}$$

with

$$\mathbf{c}_\alpha(z) = \frac{\Gamma(\frac{\langle \alpha, \rho \rangle}{\langle \alpha, \alpha \rangle} + \frac{1}{2}m_\alpha)}{\Gamma(\frac{\langle \alpha, \rho \rangle}{\langle \alpha, \alpha \rangle})} \frac{\Gamma(\frac{1}{2}\frac{\langle \alpha, \rho \rangle}{\langle \alpha, \alpha \rangle} + \frac{1}{4}m_\alpha + \frac{1}{2}m_{2\alpha})}{\Gamma(\frac{1}{2}\frac{\langle \alpha, \rho \rangle}{\langle \alpha, \alpha \rangle} + \frac{1}{4}m_\alpha)} \frac{\Gamma(iz)}{\Gamma(iz + \frac{1}{2}m_\alpha)} \frac{\Gamma(\frac{i}{2}z + \frac{1}{4}m_\alpha)}{\Gamma(\frac{i}{2}z + \frac{1}{4}m_\alpha + \frac{1}{2}m_{2\alpha})}.$$

Notice that  $|\mathbf{c}_\alpha|^{-2}$  is a differential symbol on  $\mathbb{R}$ . But the Plancherel density  $|\mathbf{c}(\lambda)|^{-2}$ , as a product of one-dimensional symbols, is not a symbol on  $\mathfrak{a}$  in general. We shall use the expressions

$$|\mathbf{c}(\lambda)|^{-2} = \mathbf{c}(w.\lambda)^{-1} \mathbf{c}(-w.\lambda)^{-1} \quad \forall \lambda \in \mathfrak{a}, \forall w \in W$$

as well as

$$\mathbf{b}(\lambda) = \pi(i\lambda) \mathbf{c}(\lambda) = \prod_{\alpha \in \Sigma_r^+} \mathbf{b}_\alpha \left( \frac{\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle} \right)$$

where  $\pi(i\lambda) = \prod_{\alpha \in \Sigma_r^+} \langle \alpha, \lambda \rangle$  and

$$\mathbf{b}_\alpha(z) = |\alpha|^2 iz \mathbf{c}_\alpha(z) = |\alpha|^2 \frac{\Gamma(\frac{\langle \alpha, \rho \rangle}{\langle \alpha, \alpha \rangle} + \frac{1}{2}m_\alpha)}{\Gamma(\frac{\langle \alpha, \rho \rangle}{\langle \alpha, \alpha \rangle})} \frac{\Gamma(\frac{1}{2}\frac{\langle \alpha, \rho \rangle}{\langle \alpha, \alpha \rangle} + \frac{1}{4}m_\alpha + \frac{1}{2}m_{2\alpha})}{\Gamma(\frac{1}{2}\frac{\langle \alpha, \rho \rangle}{\langle \alpha, \alpha \rangle} + \frac{1}{4}m_\alpha)} \frac{\Gamma(iz+1)}{\Gamma(iz + \frac{1}{2}m_\alpha)} \frac{\Gamma(\frac{i}{2}z + \frac{1}{4}m_\alpha)}{\Gamma(\frac{i}{2}z + \frac{1}{4}m_\alpha + \frac{1}{2}m_{2\alpha})}.$$

For every root  $\alpha \in \Sigma_r^+$ , the function  $\mathbf{b}_\alpha(-iz)^{\pm 1}$  is holomorphic for  $\text{Im } z > -1/2$ , with

$$\frac{\Gamma(iz+1)}{\Gamma(iz + \frac{1}{2}m_\alpha)} \frac{\Gamma(\frac{i}{2}z + \frac{1}{4}m_\alpha)}{\Gamma(\frac{i}{2}z + \frac{1}{4}m_\alpha + \frac{1}{2}m_{2\alpha})} \sim 2^{\frac{m_{2\alpha}}{2}} z^{1 - \frac{m_\alpha}{2} - \frac{m_{2\alpha}}{2}} \quad \text{as } |z| \rightarrow \infty. \quad (2.6)$$

Hence  $\mathbf{b}(-\lambda)^{\pm 1}$  is a holomorphic function for  $\lambda \in \mathfrak{a} + i\overline{\mathfrak{a}^+}$ , which has the following behavior:

$$|\mathbf{b}(-\lambda)|^{\pm 1} \asymp \prod_{\alpha \in \Sigma_r^+} (1 + |\langle \alpha, \lambda \rangle|)^{\mp \frac{m_\alpha + m_{2\alpha}}{2} \pm 1} \quad (2.7)$$

and whose derivatives can be estimated by

$$p\left(\frac{\partial}{\partial \lambda}\right) \mathbf{b}(-\lambda)^{\pm 1} = O(|\mathbf{b}(-\lambda)|^{\pm 1}), \quad (2.8)$$

where  $p(\frac{\partial}{\partial \lambda})$  is any differential polynomial.

2.2.2. *Spherical functions.* For every  $\lambda \in \mathfrak{a}$ , the spherical function  $\varphi_\lambda$  is a smooth bi- $K$ -invariant eigenfunction of all  $G$ -invariant differential operators on  $\mathbb{X}$ , in particular of the Laplace-Beltrami operator:

$$-\Delta\varphi_\lambda(x) = (|\lambda|^2 + |\rho|^2)\varphi_\lambda(x).$$

It is symmetric in the sense that  $\varphi_\lambda(x^{-1}) = \varphi_{-\lambda}(x)$ , and is given by the integral representation

$$\varphi_\lambda(x) = \int_K dk e^{\langle i\lambda + \rho, A(kx) \rangle}. \quad (2.9)$$

We list below some of the known properties of spherical functions  $\varphi_\lambda$  and refer to [GaVa88, Chap.4] and [Hel00, Chap.IV] for more details. We say that a vector  $H \in \mathfrak{a}$  lies on a wall if there exists a root  $\alpha \in \Sigma$  such that  $\langle \alpha, H \rangle = 0$ . Otherwise, we say that  $H$  stays away from the walls. For all  $\lambda \in \mathfrak{a}$  *regular*, the following converging Harish-Chandra expansion is known to hold for all  $H \in \mathfrak{a}^+$

$$\varphi_\lambda(\exp H) = \sum_{w \in W} \mathbf{c}(w.\lambda) \Phi_{w.\lambda}(H), \quad (2.10)$$

where

$$\Phi_\lambda(H) = e^{\langle i\lambda - \rho, H \rangle} \sum_{q \in 2Q} \gamma_q(\lambda) e^{-\langle q, H \rangle}.$$

Here  $Q = \sum_{\alpha \in \Sigma_s^+} \mathbb{N}\alpha$  denotes the positive root lattice and the leading coefficient  $\gamma_0$  equals 1. For every  $q \geq 1$ ,  $\gamma_q(\lambda)$  are rational functions in  $\lambda \in \mathfrak{a}_{\mathbb{C}}$ , which have no poles for all  $\lambda \in \mathfrak{a} + i\overline{\mathfrak{a}^+}$  and satisfy there

$$|\gamma_q(\lambda)| \lesssim (1 + |q|)^{N_\gamma} \quad (2.11)$$

for some nonnegative constant  $N_\gamma$ . Their derivatives can be estimated as follows by using Cauchy's formula:

$$p\left(\frac{\partial}{\partial \lambda}\right)\gamma_q(\lambda) = \mathcal{O}(\gamma_q(\lambda)). \quad (2.12)$$

Moreover, all derivatives of  $\varphi_\lambda(\exp H)$  in  $H$  have the corresponding expansion

$$p\left(\frac{\partial}{\partial H}\right)\varphi_\lambda(\exp H) = e^{-\langle \rho, H \rangle} \sum_{q \in 2Q} e^{-\langle q, H \rangle} \sum_{w \in W} \mathbf{c}(w.\lambda) \gamma_q(w.\lambda) p(iw.\lambda - \rho - q) e^{i\langle w.\lambda, H \rangle}. \quad (2.13)$$

Recall that all the elementary spherical functions  $\varphi_\lambda$  with parameter  $\lambda \in \mathfrak{a}$  are controlled by the ground spherical function  $\varphi_0$ , which satisfies the global estimate

$$\varphi_0(\exp H) \asymp \left\{ \prod_{\alpha \in \Sigma_r^+} 1 + \langle \alpha, H \rangle \right\} e^{-\langle \rho, H \rangle} \quad \forall H \in \overline{\mathfrak{a}^+}. \quad (2.14)$$

2.3. **Heat kernel on symmetric spaces.** The heat kernel on  $\mathbb{X}$  is a positive bi- $K$ -invariant right convolution kernel, i.e.,  $h_t(xK, yK) = h_t(y^{-1}x) > 0$ , which is thus determined by its restriction to the positive Weyl chamber. According to the inversion formula of the spherical Fourier transform, the heat kernel is given by

$$h_t(x) = C_0 \int_{\mathfrak{a}} d\lambda |\mathbf{c}(\lambda)|^{-2} \varphi_\lambda(x) e^{-t(|\lambda|^2 + |\rho|^2)} \quad (2.15)$$

and satisfies the global estimate

$$h_t(\exp H) \asymp t^{-\frac{n}{2}} \left\{ \prod_{\alpha \in \Sigma_r^+} (1 + t + \langle \alpha, H \rangle)^{\frac{m_\alpha + m_{2\alpha}}{2} - 1} \right\} \varphi_0(\exp H) e^{-|\rho|^2 t - \frac{|H|^2}{4t}} \quad (2.16)$$

for all  $t > 0$  and  $H \in \overline{\mathfrak{a}^+}$ , see [AnJi99; AnOs03]. Recall that  $\int_{\mathbb{X}} dx h_t(x) = 1$ . Let  $r(t) = \frac{\sqrt{t}}{\varepsilon(t)}$ , where  $\varepsilon(t)$  is a positive function, such that  $\varepsilon(t) \searrow 0$  and  $\sqrt{t}\varepsilon(t) \rightarrow \infty$ , as  $t \rightarrow \infty$ . Let  $\gamma(t)$  be another positive function such that  $\sqrt{t}\gamma(t) \rightarrow \infty$  as  $t \rightarrow \infty$ . Denote by  $\Omega_{\text{annulus}} \in \mathfrak{a}$  the annulus  $2|\rho|t - r(t) \leq |H| \leq 2|\rho|t + r(t)$  and by  $\Omega_{\text{cone}} \in \mathfrak{a}$  the solid cone with angle  $\gamma(t)$

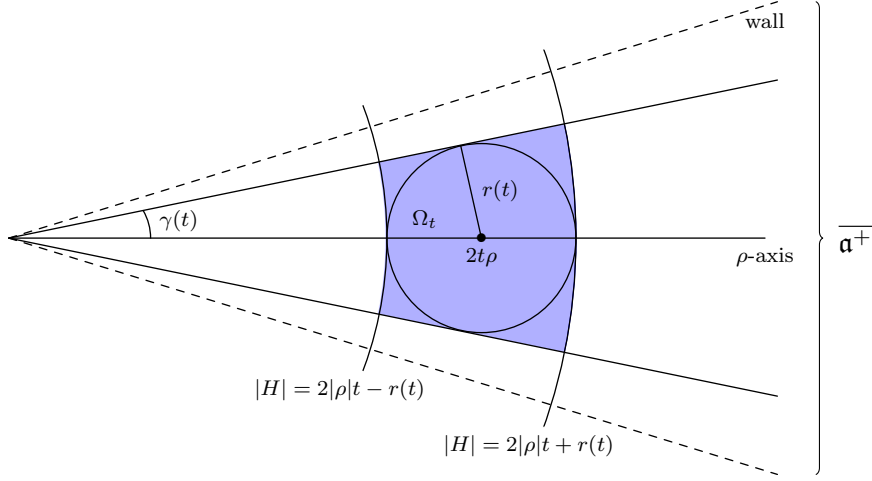


FIGURE 1. Critical region where the heat kernel concentrates.

around the  $\rho$ -axis. According to [AnSe92], the heat kernel  $h_t$  is asymptotically concentrated in  $K(\exp(\Omega_{\text{annulus}} \cap \Omega_{\text{cone}}))K$  (see the blue region in Fig. 1):

$$\int_{K(\exp(\Omega_{\text{annulus}} \cap \Omega_{\text{cone}}))K} dx h_t(x) \longrightarrow 1 \quad \text{as } t \rightarrow \infty.$$

The following lemma shows that the heat kernel  $h_t(x)$  concentrates indeed in  $K(\exp \Omega_t)K$ , where  $\Omega_t = B(2t\rho, r(t))$  denotes the ball in  $\mathfrak{a}$  with center  $2t\rho$  and radius  $r(t)$ .

**Lemma 2.1.** *The following estimate holds for any  $N \geq 0$ :*

$$\|h_t\|_{L^1(G \setminus K(\exp \Omega_t)K)} \lesssim \varepsilon(t)^N \quad \forall t > 1. \quad (2.17)$$

*Proof.* For simplicity, we denote by  $H = x^+$  the middle component of  $x \in G$  in the Cartan decomposition. Let  $\eta > 0$  be a constant such that  $\eta t > r(t)$ . The proof of (2.17) is based on the following large time heat kernel estimates, which follow from (2.16) for some  $L \geq 0$ :

$$h_t(\exp H) \lesssim \underbrace{e^{-|\rho|^2 t - \langle \rho, H \rangle - \frac{|H|^2}{4t}}}_{e^{-\frac{|H-2t\rho|^2}{4t} - 2\langle \rho, H \rangle}} \times \begin{cases} t^{-\frac{\ell}{2}} & \text{if } H \in B(2t\rho, \eta t), \\ t^L & \text{if } H \in B(0, 3|\rho|t) \setminus B(2t\rho, \eta t), \\ |H|^L & \text{if } H \geq 3|\rho|t. \end{cases} \quad (2.18)$$

Firstly, we have

$$\begin{aligned} \int_{K(\exp B(2t\rho, \eta t))K \setminus K(\exp \Omega_t)K} dx h_t(x) &\lesssim \int_{r(t) \leq |H-2t\rho| \leq \eta t} dH \delta(H) h_t(\exp H) \\ &\lesssim t^{-\frac{\ell}{2}} \int_{r(t) \leq |H-2t\rho| \leq \eta t} dH e^{-\frac{|H-2t\rho|^2}{4t}} \\ &\lesssim t^{-\frac{\ell}{2}} \int_{r(t)}^{\eta t} dR R^{\ell-1} e^{-\frac{R^2}{4t}} \\ &= \int_{\frac{r(t)}{2\sqrt{t}}}^{\frac{\eta\sqrt{t}}{2}} dR R^{\ell-1} e^{-R^2} \lesssim \left(\frac{\sqrt{t}}{r(t)}\right)^N \end{aligned}$$



for any  $N \geq 0$ . Similar computations imply secondly

$$\begin{aligned}
\int_{K(\exp B(0,3|\rho|t))K \setminus K(\exp B(2t\rho,\eta t))K} dx h_t(x) &\lesssim \int_{[\mathfrak{a}^+ \cap B(0,3|\rho|t)] \setminus B(2t\rho,\eta t)} dH \delta(H) h_t(\exp H) \\
&\lesssim t^L \int_{\eta t \leq |H-2\rho t| \leq 5|\rho|t} dH e^{-\frac{|H-2t\rho|^2}{4t}} \\
&\lesssim t^L \int_{\eta t}^{5|\rho|t} dR R^{\ell-1} e^{-\frac{R^2}{4t}} \\
&= t^{L+\frac{\ell}{2}} \int_{\frac{\eta\sqrt{t}}{2}}^{\frac{5|\rho|\sqrt{t}}{2}} dR R^{\ell-1} e^{-R^2} \lesssim t^{-N}
\end{aligned}$$

and thirdly

$$\begin{aligned}
\int_{G \setminus K(\exp B(0,3|\rho|t))K} dx h_t(x) &\lesssim \int_{\mathfrak{a}^+ \setminus B(0,3|\rho|t)} dH \delta(H) h_t(\exp H) \\
&\lesssim \int_{|H| \geq 3|\rho|t} dH |H|^L e^{-\frac{|H-2t\rho|^2}{4t}} \\
&\lesssim \int_{|H-2t\rho| \geq |\rho|t} dH |H-2t\rho|^L e^{-\frac{|H-2t\rho|^2}{4t}} \\
&\lesssim \int_{|\rho|t}^{+\infty} dR R^{L+\ell-1} e^{-\frac{R^2}{4t}} \lesssim t^{-N}
\end{aligned}$$

for any  $N \geq 0$ . The proposition follows by combining these three estimates.  $\square$

As the vectors in  $\Omega_t$  stay far away from walls, the large time behavior of the heat kernel can be described more accurately by the following asymptotics [AnJi99, Theorem 5.1.1]:

$$h_t(\exp H) \sim C_1 t^{-\frac{\nu}{2}} \mathbf{b}\left(-i\frac{H}{2t}\right)^{-1} \varphi_0(\exp H) e^{-|\rho|^2 t - \frac{|H|^2}{4t}} \quad (2.19)$$

as  $t \rightarrow \infty$  and  $\mu(H) = \min_{\alpha \in \Sigma^+} \langle \alpha, H \rangle \rightarrow \infty$ . Here  $C_1 = C_0 2^{-|\Sigma_r^+|} |W| \pi^{\frac{\ell}{2}} \boldsymbol{\pi}(\rho_0) \mathbf{b}(0)^{-1}$  is a positive constant, and  $\rho_0$  denotes the half sum of all positive reduced roots. Moreover, the ground spherical function satisfies

$$\varphi_0(\exp H) \sim C_2 \boldsymbol{\pi}(H) e^{-\langle \rho, H \rangle} \quad (2.20)$$

as  $\mu(H) \rightarrow \infty$ , where  $C_2 = \boldsymbol{\pi}(\rho_0)^{-1} \mathbf{b}(0)$ , see for instance [AnJi99, Proposition 2.2.12.(ii)].

### 3. ASYMPTOTIC CONVERGENCE ASSOCIATED WITH THE LAPLACE-BELTRAMI OPERATOR

Consider the heat equation

$$\partial_t u(t, x) = \Delta_x u(t, x), \quad u(0, x) = u_0(x), \quad (3.1)$$

with  $u_0 \in L^1(\mathbb{X})$  and denote by  $M$  the mass of the initial data:

$$M = \int_G dx u_0(x).$$

In this section, we prove our first main result [Theorem 1.2](#). We start with smooth bi- $K$ -invariant compactly supported initial data and study the long-time asymptotic convergence in the critical region where the heat kernel concentrates. By completing the estimates outside the critical region and by using a standard density argument, we next establish (1.5) for all bi- $K$ -invariant  $L^1$  initial data. Finally, we provide a counterexample in the non bi- $K$ -invariant case.

**3.1. Heat asymptotics in the critical region for  $C_c^\infty(K \backslash G/K)$  initial data.** In order to treat heat asymptotics in the critical region, we pass to the frequency side. For an alternative proof of the following [Proposition 3.1](#), directly on the space side, see [Remark 3.10](#).

As both the heat kernel  $h_t$  and the initial data  $u_0$  are bi- $K$ -invariant, we have  $\mathcal{H}(u_0 * h_t) = (\mathcal{H}u_0)(\mathcal{H}h_t)$ , see for instance [[Ank91](#), p. 347]. Then, we can write the solution to [\(3.1\)](#) as

$$u(t, x) = (u_0 * h_t)(x) = C_0 \int_{\mathfrak{a}} d\lambda |\mathbf{c}(\lambda)|^{-2} e^{-t(|\lambda|^2 + |\rho|^2)} \varphi_\lambda(x) \mathcal{H}u_0(\lambda)$$

by using the inversion formula [\(2.5\)](#) of the spherical Fourier transform and [\(2.15\)](#). Notice that the mass  $M$  equals  $\mathcal{H}u_0(-i\rho)$  by the spherical Fourier transform and [\(2.9\)](#). Our aim in this subsection is to estimate the difference

$$u(t, x) - Mh_t(x) = C_0 e^{-|\rho|^2 t} \int_{\mathfrak{a}} d\lambda |\mathbf{c}(\lambda)|^{-2} e^{-t|\lambda|^2} \varphi_\lambda(x) \underbrace{\{\mathcal{H}u_0(\lambda) - \mathcal{H}u_0(-i\rho)\}}_{U(\lambda)} \quad (3.2)$$

in the critical region  $K(\exp \Omega_t)K$ . Recall that  $\Omega_t = B(2t\rho, r(t))$  where  $r(t)$  grows slightly faster than  $\sqrt{t}$  and slower than  $t$  as  $t \rightarrow \infty$ . To this end we resume the computations in [[AnJi99](#), Step 2, pp. 1054-1056].

**Proposition 3.1.** *Let  $u_0$  be a smooth bi- $K$ -invariant initial data with compact support. Then*

$$\|u(t, \cdot) - Mh_t\|_{L^1(K(\exp \Omega_t)K)} \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (3.3)$$

*Proof.* For simplicity, we denote by  $H = x^+$  the middle component of  $x \in G$  in the Cartan decomposition. By substituting [\(2.10\)](#) in [\(3.2\)](#) and by writing  $|\mathbf{c}(\lambda)|^{-2} = \mathbf{c}(\lambda)^{-1} \mathbf{c}(-\lambda)^{-1}$ , with  $\mathbf{c}(-\lambda)^{-1} = \boldsymbol{\pi}(-i\lambda) \mathbf{b}(-\lambda)^{-1}$ , we obtain first

$$u(t, \exp H) - Mh_t(\exp H) = C_0 |W| e^{-|\rho|^2 t - \langle \rho, H \rangle} \sum_{q \in Q} e^{-\langle q, H \rangle} E_q(t, H) \quad (3.4)$$

where

$$E_q(t, H) = \int_{\mathfrak{a}} d\lambda \boldsymbol{\pi}(-i\lambda) e^{-t|\lambda|^2} e^{i\langle \lambda, H \rangle} \mathbf{b}(-\lambda)^{-1} \gamma_q(\lambda) U(\lambda).$$

By using the formula

$$\boldsymbol{\pi}(-i\lambda) e^{-t|\lambda|^2} = \boldsymbol{\pi}\left(\frac{i}{2t} \frac{\partial}{\partial \lambda}\right) e^{-t|\lambda|^2}$$

and by integrating by parts, we obtain firstly

$$\begin{aligned} E_q(t, H) &= (2t)^{-|\Sigma_r^+|} \sum_{\Sigma_r^+ = \Sigma' \sqcup \Sigma''} \left\{ \prod_{\alpha' \in \Sigma'} \langle \alpha', H \rangle \right\} \\ &\quad \times \int_{\mathfrak{a}} d\lambda e^{-t|\lambda|^2} e^{i\langle \lambda, H \rangle} \left\{ \prod_{\alpha'' \in \Sigma''} (-i\partial_{\alpha''}) \right\} \left\{ \mathbf{b}(-\lambda)^{-1} \gamma_q(\lambda) U(\lambda) \right\}. \end{aligned}$$

After moving the contour of integration according to  $\lambda \mapsto \lambda + i\frac{H}{2t}$  and rescaling the integral according to  $\lambda \mapsto \frac{\lambda}{\sqrt{t}}$ , we obtain secondly

$$\begin{aligned} E_q(t, H) &= 2^{-|\Sigma_r^+|} t^{-\frac{\nu}{2}} e^{-\frac{|H|^2}{4t}} \sum_{\Sigma_r^+ = \Sigma' \sqcup \Sigma''} \left\{ \prod_{\alpha' \in \Sigma'} \langle \alpha', H \rangle \right\} \\ &\quad \times \int_{\mathfrak{a}} d\lambda e^{-|\lambda|^2} \left\{ \prod_{\alpha'' \in \Sigma''} (-i\sqrt{t}\partial_{\alpha''}) \right\} \left\{ \mathbf{b}\left(-\frac{\lambda}{\sqrt{t}} - i\frac{H}{2t}\right)^{-1} \gamma_q\left(\frac{\lambda}{\sqrt{t}} + i\frac{H}{2t}\right) U\left(\frac{\lambda}{\sqrt{t}} + i\frac{H}{2t}\right) \right\}. \end{aligned} \quad (3.5)$$

Now let us study [\(3.5\)](#) in different cases. We first assume that no derivative in [\(3.5\)](#) hits  $U$ . According to the Paley-Wiener theorem (see for instance [[Hel94](#), Chap.III, Theorem 5.1]), as

$u_0$  is a smooth bi- $K$ -invariant function with compact support on  $G$ ,  $\mathcal{H}u_0$  is a  $W$ -invariant holomorphic function on  $\mathfrak{a}_{\mathbb{C}}$  such that

$$\exists C' \geq 0, \forall j \in \mathbb{N}, \forall N \in \mathbb{N}, \exists C'' \geq 0, \forall \lambda \in \mathfrak{a}_{\mathbb{C}}, |\nabla_{\lambda}^j \mathcal{H}u_0(\lambda)| \leq C''(1 + |\lambda|)^{-N} e^{C'|\operatorname{Im} \lambda|}. \quad (3.6)$$

Hence

$$|U(\frac{\lambda}{\sqrt{t}} + i\frac{H}{2t})| \lesssim e^{\frac{C'}{2}\frac{|H|}{t}} \left(\frac{|\lambda|}{\sqrt{t}} + |\frac{H}{2t} - \rho|\right) \quad \forall \lambda, H \in \mathfrak{a}, \forall t > 0. \quad (3.7)$$

On the other hand, according to (2.7) and (2.8),

$$(\sqrt{t} \nabla_{\lambda})^j \mathbf{b}\left(-\frac{\lambda}{\sqrt{t}} - i\frac{H}{2t}\right)^{-1} = O\left(\left(1 + \frac{|\lambda|}{\sqrt{t}}\right)^m \prod_{\alpha \in \Sigma_r^+} \left(1 + \frac{\langle \alpha, H \rangle}{t}\right)^{\binom{m_{\alpha} + m_{2\alpha}}{2} - 1}\right) \quad (3.8)$$

where  $m = \sum_{\alpha \in \Sigma_r^+} \binom{m_{\alpha} + m_{2\alpha}}{2} - 1$ , and according to (2.11) and (2.12),

$$(\sqrt{t} \frac{\partial}{\partial \lambda})^j \gamma_q\left(\frac{\lambda}{\sqrt{t}} + i\frac{H}{2t}\right) = O((1 + |q|)^{N_{\gamma}}), \quad (3.9)$$

where  $N_{\gamma}$  is a positive constant. By using (3.7), (3.8) and (3.9), we estimate the terms in (3.5) where no derivative hits  $U$  by

$$\begin{aligned} t^{-\frac{\nu}{2}} e^{-\frac{|H|^2}{4t}} \overbrace{(1 + |H|)^{|\Sigma_r^+|}}^{\lesssim t^{|\Sigma_r^+|}} & \left\{ \prod_{\alpha \in \Sigma_r^+} \overbrace{\left(1 + \frac{\langle \alpha, H \rangle}{t}\right)^{\binom{m_{\alpha} + m_{2\alpha}}{2} - 1}}^{\lesssim 1} \right\} (1 + |q|)^{N_{\gamma}} \overbrace{e^{\frac{C'}{2}\frac{|H|}{t}}}^{\lesssim 1} \overbrace{\left(\frac{1}{\sqrt{t}} + \frac{r(t)}{t}\right)}^{\lesssim \frac{r(t)}{t}} \\ & \lesssim (1 + |q|)^{N_{\gamma}} t^{-\frac{\ell}{2} - 1} r(t) e^{-\frac{|H|^2}{4t}} \end{aligned} \quad (3.10)$$

when  $t$  is large and  $H \in \Omega_t$ . If some derivatives  $\partial_{\alpha'}$  in (3.5) hit  $U$ , then

$$\left| \prod_{\alpha' \in \Sigma'} \langle \alpha', H \rangle \right| \lesssim |H|^{|\Sigma'|} \lesssim t^{|\Sigma_r^+| - 1}$$

and, by using (3.6), (3.8) and (3.9), we obtain this time that (3.5) is bounded by

$$(1 + |q|)^{N_{\gamma}} t^{-\frac{\ell}{2} - 1} e^{-\frac{|H|^2}{4t}}. \quad (3.11)$$

By combining (3.4), (3.5), (3.10) and (3.11), we obtain

$$|u(t, \exp H) - Mh_t(\exp H)| \lesssim \underbrace{\left\{ \sum_{q \in 2Q} (1 + |q|)^{N_{\gamma}} e^{-\langle q, H \rangle} \right\}}_{\lesssim 1} t^{-\frac{\ell}{2} - 1} r(t) e^{-|\rho|^2 t - \langle \rho, H \rangle - \frac{|H|^2}{4t}}$$

when  $t$  is large and  $H \in \Omega_t$ . By integration, we conclude that

$$\begin{aligned} \int_{K(\exp \Omega_t)K} dx |u(t, x) - Mh_t(x)| &= C_0 \int_{\Omega_t} dH \delta(H) |u(t, \exp H) - Mh_t(\exp H)| \\ &\lesssim t^{-\frac{\ell}{2} - 1} r(t) \int_{\Omega_t} dH e^{-|\rho|^2 t + \langle \rho, H \rangle - \frac{|H|^2}{4t}} \\ &= \frac{r(t)}{t} t^{-\frac{\ell}{2}} \int_{B(2t\rho, r(t))} dH e^{-\frac{|H - 2t\rho|^2}{4t}} \\ &= \frac{r(t)}{t} \underbrace{\int_0^{\frac{r(t)}{\sqrt{t}}} dr r^{\ell-1} e^{-\frac{r^2}{4}}}_{< \infty} \lesssim \frac{r(t)}{t} \end{aligned}$$

tends to 0 at speed  $\frac{r(t)}{t} = \frac{1}{\sqrt{t}\varepsilon(t)}$ . □

**3.2. Estimates outside the critical region.** In this subsection, we show that the solution  $u(t, x)$  to the heat equation (3.1) vanishes asymptotically in  $L^1(G \setminus K(\exp \Omega_t)K)$  as  $t \rightarrow \infty$ .

**Proposition 3.2.** *The solution to the heat equation (3.1) satisfies*

$$\|u(t, \cdot)\|_{L^1(G \setminus K(\exp \Omega_t)K)} \lesssim \varepsilon(t)^N \quad (3.12)$$

for  $t > 0$  large enough and for any  $N \geq 0$ .

**Lemma 3.3.**  $d(K(\exp H_1)K, K(\exp H_2)K) = |H_1 - H_2|$  for all  $H_1, H_2 \in \overline{\mathfrak{a}^+}$ .

*Proof.* On the one hand,

$$d(K(\exp H_1)K, K(\exp H_2)K) = \inf_{k_1, k_2 \in K} d(k_1(\exp H_1)K, k_2(\exp H_2)K) \geq |H_1 - H_2|$$

according to (2.2). On the other hand,

$$\begin{aligned} d(K(\exp H_1)K, K(\exp H_2)K) &\leq d((\exp H_1)K, (\exp H_2)K) \\ &= d(\exp(H_1 - H_2)K, eK) = |H_1 - H_2| \end{aligned}$$

as  $(\exp \mathfrak{a})K$  is a flat subspace of  $G/K$ .  $\square$

*Proof of Proposition 3.2.* Since  $h_t$  and  $u_0$  are both bi- $K$ -invariant functions on  $G$ , we have

$$u(t, x) = (u_0 * h_t)(x) = (h_t * u_0)(x) = \int_G dy h_t(xy^{-1}) u_0(y).$$

Let  $\xi > 0$  be a constant such that the compact support of  $u_0$  belongs to  $K(\exp B(0, \xi))K$ . Then

$$\int_{G \setminus K(\exp \Omega_t)K} dx |u(t, x)| \lesssim \int_{K(\exp B(0, \xi))K} dy |u_0(y)| \int_{G \setminus K(\exp \Omega_t)K} dx h_t(xy^{-1}).$$

We notice from the Lemma 3.3 that

$$K(\exp \Omega_t)K = K(\exp B(2t\rho, r(t)))K = \{zK \in G/K \mid d(zK, K(\exp 2t\rho)K) < r(t)\}.$$

Hence

$$x \in G \setminus K(\exp \Omega_t)K \implies xy^{-1} \in G \setminus K(\exp B(2t\rho, r(t) - \xi))K.$$

Indeed, if  $xy^{-1} \in K(\exp B(2t\rho, r(t) - \xi))K$ , then

$$d(xK, K(\exp 2t\rho)K) \leq \underbrace{d(xK, xy^{-1}K)}_{d(yK, eK) < \xi} + \underbrace{d(xy^{-1}K, K(\exp 2t\rho)K)}_{< r(t) - \xi} < r(t),$$

which implies that  $x \in K(\exp \Omega_t)K$ . Therefore

$$\int_{G \setminus K(\exp \Omega_t)K} dx h_t(xy^{-1}) \leq \int_{G \setminus K(\exp B(2t\rho, r(t) - \xi))K} dz h_t(z) \quad (3.13)$$

for all  $y \in K(\exp B(0, \xi))K$ . As  $r(t) - \xi \geq \frac{r(t)}{2}$  for  $t$  large enough, we deduce from (2.17) that the right-hand side of (3.13) is also  $O(\varepsilon(t)^N)$ . In conclusion,

$$\int_{G \setminus K(\exp \Omega_t)K} dx |u(t, x)| \lesssim \varepsilon(t)^N \quad \forall N \geq 0.$$

$\square$

**Remark 3.4.** *The estimate (3.12) still holds without the bi- $K$ -invariance assumption. Indeed, by using the symmetries  $(K(\exp \Omega_t)K)^{-1} = K(\exp \Omega_t)K$  and  $h_t(y^{-1}x^{-1}) = h_t(xy)$ , we have*

$$\begin{aligned} \int_{G \setminus K(\exp \Omega_t)K} dx |u(t, x)| &= \int_{G \setminus K(\exp \Omega_t)K} dx |u(t, x^{-1})| \\ &\lesssim \int_{K(\exp B(0, \xi))K} dy |u_0(y)| \int_{G \setminus K(\exp \Omega_t)K} dx h_t(xy) \end{aligned}$$

and we can conclude similarly.

**3.3. Long-time convergence for general bi- $K$ -invariant data.** In the previous two subsections, we studied the long-time asymptotic behavior for the heat equation with smooth compactly supported bi- $K$ -invariant initial data. Using those estimates and a standard density argument, we first prove in this subsection [Theorem 1.2](#) for the whole class of  $L^1(\mathbb{X})$  functions that are bi- $K$ -invariant. Then we discuss the same problem in  $L^p(\mathbb{X})$  for  $p > 1$ .

*Proof of [Theorem 1.2](#).* Let  $\varepsilon > 0$  and  $U_0 \in C_c^\infty(K \backslash G / K)$  be such that  $\|u_0 - U_0\|_{L^1(\mathbb{X})} < \frac{\varepsilon}{3}$ . Denote by  $M_U = \int_{\mathbb{X}} dx U_0(x)$  the mass of  $U_0$ , then

$$|M - M_U| \leq \|u_0 - U_0\|_{L^1(\mathbb{X})} < \frac{\varepsilon}{3}.$$

Let  $U(t, x) = (U_0 * h_t)(x)$  be the solution to the heat equation with initial data  $U_0$ . We deduce from [\(2.17\)](#), [\(3.3\)](#) and [\(3.12\)](#) that, there exists  $T > 0$  such that

$$\begin{aligned} \|U(t, \cdot) - M_U h_t\|_{L^1(\mathbb{X})} &\leq \|U(t, \cdot) - M_U h_t\|_{L^1(K(\exp \Omega_t)K)} + \|U(t, \cdot)\|_{L^1(G \backslash K(\exp \Omega_t)K)} \\ &\quad + |M_U| \|h_t\|_{L^1(G \backslash K(\exp \Omega_t)K)} \\ &< \frac{\varepsilon}{3} \end{aligned}$$

for all  $t \geq T$ . In conclusion,

$$\begin{aligned} \|u(t, \cdot) - M h_t\|_{L^1(\mathbb{X})} &\leq \underbrace{\|u(t, \cdot) - U(t, \cdot)\|_{L^1(\mathbb{X})}}_{\leq \|u_0 - U_0\|_{L^1} \|h_t\|_{L^1}} + \underbrace{\|M_U h_t - M h_t\|_{L^1(\mathbb{X})}}_{\leq |M - M_U| \|h_t\|_{L^1}} \\ &\quad + \|U(t, \cdot) - M_U h_t\|_{L^1(\mathbb{X})} \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

for all  $\varepsilon > 0$  and  $t$  large enough.  $\square$

Let us turn to the long-time convergence in  $L^p(\mathbb{X})$  with  $p > 1$ . We first deal with the case  $p = \infty$  and reach the full range by convexity. The kernel estimate [\(2.16\)](#) gives us the sup norm estimate:

$$\|u(t, \cdot) - M h_t\|_{L^\infty(\mathbb{X})} \leq \|u_0\|_{L^1(\mathbb{X})} \|h_t\|_{L^\infty(\mathbb{X})} + |M| \|h_t\|_{L^\infty(\mathbb{X})} \lesssim t^{-\frac{\nu}{2}} e^{-|\rho|^2 t}$$

for  $t$  large and for all  $u_0 \in L^1(\mathbb{X})$ , see for instance [[AnJi99](#), Proposition 4.1.1]. In other words, we have

$$\|u(t, \cdot) - M h_t\|_{L^\infty(\mathbb{X})} = O(t^{-\frac{\nu}{2}} e^{-|\rho|^2 t}) \quad \text{as } t \rightarrow \infty. \quad (3.14)$$

Notice that such an estimate holds without the bi- $K$ -invariant assumption. By convexity, we obtain the following estimates in the  $L^p(\mathbb{X})$  setting.

**Corollary 3.5.** *Under the assumption of [Theorem 1.2](#), we have*

$$\|u(t, \cdot) - M h_t\|_{L^p(\mathbb{X})} = o\left(t^{-\frac{\nu}{2p}} e^{-\frac{|\rho|^2 t}{p}}\right) \quad \text{as } t \rightarrow \infty \quad (3.15)$$

for all  $1 < p < \infty$ .

**Remark 3.6.** *As previously observed by Vázquez on real hyperbolic spaces, the  $L^p$  norm estimate [\(3.15\)](#) is optimal, but the sup norm estimate [\(3.14\)](#) is weaker compared to the results in the Euclidean setting, where one has*

$$t^{\frac{n}{2}} \|u(t, \cdot) - M h_t\|_{L^\infty(\mathbb{R}^n)} \longrightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Such a convergence is no longer valid in our setting due to the spectral gap. A counterexample was provided on  $\mathbb{H}^3(\mathbb{R})$  in [Váz19, p.15]. Here, let us extend it to general noncompact symmetric spaces. Consider a delayed heat kernel  $h_{t+t'}$  for some  $t' > 0$  to be determined later. Then

$$t^{\frac{\nu}{2}} e^{|\rho|^2 t} \|h_{t+t'} - h_t\|_{L^\infty(\mathbb{X})} \geq t^{\frac{\nu}{2}} e^{|\rho|^2 t} (h_t(eK) - h_{t+t'}(eK))$$

since  $h_t(eK)$  is decreasing in  $t$ . According to (2.16), there exists a constant  $C \geq 1$  such that

$$\begin{aligned} t^{\frac{\nu}{2}} e^{|\rho|^2 t} (h_t(eK) - h_{t+t'}(eK)) &\geq t^{\frac{\nu}{2}} e^{|\rho|^2 t} \left\{ \frac{1}{C} t^{-\frac{\nu}{2}} e^{-|\rho|^2 t} - C(t+t')^{-\frac{\nu}{2}} e^{-|\rho|^2(t+t')} \right\} \\ &= C^{-1} - C \left( \frac{t}{t+t'} \right)^{\frac{\nu}{2}} e^{-|\rho|^2 t'} \geq \frac{1}{2C}, \end{aligned}$$

provided that  $t' > \frac{2 \ln C + \ln 2}{|\rho|^2}$ . Hence

$$t^{\frac{\nu}{2}} e^{|\rho|^2 t} \|h_{t+t'} - h_t\|_{L^\infty(\mathbb{X})} \not\rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

**3.4. Counterexample in the non bi- $K$ -invariant case.** Recall that the heat kernel concentrates in the region  $K(\exp \Omega_t)K$  where  $\Omega_t = B(2t\rho, r(t))$ . Here  $r(t)$  goes to infinity slightly faster than  $\sqrt{t}$  and slower than  $t$ , as  $t \rightarrow \infty$ . Fix  $y \in G \setminus K$  and consider the heat equation with the initial data  $u_0(x) = \delta_y(x)$ , where  $\delta_y$  denotes the Dirac measure at  $y$ . Since  $\delta_y$  is not bi- $K$ -invariant, it is sufficient to show that the property (1.5) breaks down with such initial data. We write

$$\begin{aligned} \|u(t, \cdot) - Mh_t\|_{L^1(\mathbb{X})} &= \|h_t(\cdot, yK) - h_t(\cdot, eK)\|_{L^1(\mathbb{X})} \\ &\geq \left\| h_t(\cdot, eK) \left( \frac{h_t(\cdot, yK)}{h_t(\cdot, eK)} - 1 \right) \right\|_{L^1(K(\exp \Omega_t)K)} \\ &= \left\{ \int_{\Omega_t} dg^+ \delta(g^+) h_t(\exp g^+) \right\} \left\{ \int_K dk |e^{\langle 2\rho, A(k^{-1}y) \rangle} - 1| + O\left(\frac{r(t)}{t}\right) \right\} \end{aligned} \quad (3.16)$$

according to the Cartan decomposition  $g = k(\exp g^+)k'$  and Proposition 3.7 which we will prove in a moment. Since the heat kernel  $h_t$  is bi- $K$ -invariant and concentrates in  $K(\exp \Omega_t)K$ , we have

$$\int_{\Omega_t} dg^+ \delta(g^+) h_t(\exp g^+) = \|h_t\|_{L^1(K(\exp \Omega_t)K)} \rightarrow 1 \quad \text{as } t \rightarrow \infty.$$

Hence, the right-hand side of (3.16) tends to

$$\int_K dk |e^{\langle 2\rho, A(k^{-1}y) \rangle} - 1|$$

as  $t \rightarrow \infty$ , which is not identically 0. Indeed, if  $y = k_1(\exp y^+)k_2$  in the Cartan decomposition, then the nonnegative continuous function  $k \mapsto |e^{\langle 2\rho, A(k^{-1}y) \rangle} - 1|$  does not vanish at  $k = k_1$ , where  $\langle \rho, A(k^{-1}y) \rangle = \langle \rho, y^+ \rangle > 0$ , since  $y \notin K$  implies  $y^+ \neq 0$ . In conclusion, Theorem 1.2 fails without the bi- $K$ -invariance assumption. It remains to prove Proposition 3.7.

**Proposition 3.7.** *Let  $y \in G \setminus K$ . Then, for every  $g$  in the critical region  $K(\exp \Omega_t)K$ , we have*

$$\frac{h_t(gK, yK)}{h_t(gK, eK)} = e^{\langle 2\rho, A(k^{-1}y) \rangle} + O\left(\frac{r(t)}{t}\right) \quad \text{as } t \rightarrow \infty,$$

where  $g = k(\exp g^+)k'$  in the Cartan decomposition.

For the proof of Proposition 3.7 we need the following lemma.

**Lemma 3.8.** *For all  $g$  in the critical region  $K(\exp \Omega_t)K$ , the following asymptotic behaviors hold as  $t \rightarrow \infty$ :*

$$(i) \quad \frac{|(y^{-1}g)^+|}{|g^+|} = 1 + O\left(\frac{r(t)}{t}\right).$$

(ii)  $\frac{g^+}{|g^+|}$  and  $\frac{(y^{-1}g)^+}{|(y^{-1}g)^+|}$  are both equal to  $\frac{\rho}{|\rho|} + O\left(\frac{r(t)}{t}\right)$ .

(iii) For every  $\alpha \in \Sigma^+$ ,  $\frac{\langle \alpha, (y^{-1}g)^+ \rangle}{\langle \alpha, g^+ \rangle} = 1 + O\left(\frac{r(t)}{t}\right)$ .

(iv)  $d(gK, eK) - d(gK, yK) = \langle \frac{\rho}{|\rho|}, A(k^{-1}y) \rangle + O\left(\frac{r(t)}{t}\right)$ .

*Proof of Lemma 3.8.* For all  $g \in K(\exp B(2t\rho, r(t)))K$  and for any fixed  $y \in G$ , we have

$$d(gK, eK) = 2|\rho|t + O(r(t)) \quad \text{and} \quad d(gK, yK) = 2|\rho|t + O(r(t))$$

as  $t \rightarrow \infty$ . We deduce first (i) and (ii) by using

$$\frac{|(y^{-1}g)^+|}{|g^+|} = \frac{d(gK, yK)}{d(gK, eK)} = 1 + O\left(\frac{r(t)}{t}\right), \quad \frac{g^+}{|g^+|} = \frac{2t\rho + O(r(t))}{2t|\rho| + O(r(t))} = \frac{\rho}{|\rho|} + O\left(\frac{r(t)}{t}\right),$$

and

$$|(y^{-1}g)^+ - g^+| = |(g^{-1}y)^+ - (g^{-1})^+| \leq d(g^{-1}yK, g^{-1}K) = |y| = O(1).$$

Here we have used (2.2) and the fact that

$$(x^{-1})^+ = -w_0 \cdot x^+ \quad \forall x \in G,$$

where  $w_0$  denotes the longest element in the Weyl group  $W$ , which exchanges the positive and the negative Weyl chambers. Let us next deduce (iii) from (i) and (ii). For every positive root  $\alpha$ ,

$$\begin{aligned} \frac{\langle \alpha, (y^{-1}g)^+ \rangle}{\langle \alpha, g^+ \rangle} &= \frac{\langle \alpha, \frac{(y^{-1}g)^+}{|(y^{-1}g)^+|} \rangle |(y^{-1}g)^+|}{\langle \alpha, \frac{g^+}{|g^+|} \rangle |g^+|} \\ &= \frac{\langle \alpha, \frac{\rho}{|\rho|} \rangle + O\left(\frac{r(t)}{t}\right)}{\langle \alpha, \frac{\rho}{|\rho|} \rangle + O\left(\frac{r(t)}{t}\right)} \left\{ 1 + O\left(\frac{r(t)}{t}\right) \right\} = 1 + O\left(\frac{r(t)}{t}\right). \end{aligned}$$

It remains us to prove (iv). Let  $g = k(\exp g^+)k'$  in the Cartan decomposition and consider the Iwasawa decomposition  $k^{-1}y = n(k^{-1}y)(\exp A(k^{-1}y))k''$  for some  $k'' \in K$ . Then

$$\begin{aligned} d(gK, yK) &= d(k(\exp g^+)K, kn(k^{-1}y)(\exp A(k^{-1}y))K) \\ &= d(\exp(-g^+)[n(k^{-1}y)]^{-1}(\exp g^+)K, \exp(A(k^{-1}y) - g^+)K). \end{aligned} \quad (3.17)$$

and we write

$$\begin{aligned} d(gK, eK) - d(gK, yK) &= \overbrace{d(gK, eK) - d(\exp(A(k^{-1}y) - g^+)K, eK)}^I \\ &\quad + \underbrace{d(\exp(A(k^{-1}y) - g^+)K, eK) - d(gK, yK)}_{II}. \end{aligned}$$

On the one hand, we have

$$\begin{aligned} I &= |g^+| - |A(k^{-1}y) - g^+| = \frac{2\langle g^+, A(k^{-1}y) \rangle - |A(k^{-1}y)|^2}{|g^+| + |A(k^{-1}y) - g^+|} \\ &= \left\langle \frac{g^+}{|g^+|}, A(k^{-1}y) \right\rangle + O\left(\frac{1}{|g^+|}\right) \\ &= \left\langle \frac{\rho}{|\rho|}, A(k^{-1}y) \right\rangle + O\left(\frac{r(t)}{t}\right) \end{aligned}$$

by using (ii) and the fact that  $\{A(k^{-1}y) \mid k \in K\}$  is a compact subset of  $\mathfrak{a}$ . On the other hand, we deduce from (3.17) that

$$|II| \leq d(\exp(-g^+)[n(k^{-1}y)]^{-1}(\exp g^+)K, eK)$$

where the right-hand side tends exponentially fast to 0, as  $\{n(k^{-1}y) \mid k \in K\}$  is a compact subset of  $N$  see for instance [Hel63, Lemma 3.1]. Indeed, the subgroup  $\exp \mathfrak{a}$  acts by conjugation on  $\mathfrak{n}$  and  $N$ . In particular  $\exp(-g^+)$  acts by contractions  $e^{-\langle \alpha, g^+ \rangle}$  on each subspace  $\mathfrak{g}_\alpha$ . Thus  $\text{Ad}(\exp(-g^+))(\log n)$  tends to 0 exponentially fast, for every  $\log n \in \mathfrak{n}$  and more generally uniformly on compact subsets of  $\mathfrak{n}$ . Via the exponential map,  $\exp(-g^+)n(\exp g^+)$  tends to  $e$  exponentially fast, for every  $n \in N$  and more generally uniformly on compact subsets of  $N$ . In conclusion, we obtain

$$d(gK, eK) - d(gK, yK) = \langle \frac{\rho}{|\rho|}, A(k^{-1}y) \rangle + O\left(\frac{r(t)}{t}\right). \quad (3.18)$$

□

**Remark 3.9.** Observe that (3.18) defines in the limit of a Busemann function. Let us elaborate. Any regular geodesic ray in  $G/K$  stemming from the origin is given by  $\gamma(r) = k(\exp rH_0)K$ , where  $k \in K$  and  $H_0$  is a unit vector in  $\mathfrak{a}^+$ . As in the proof of (iv) in Lemma 3.8, consider the Iwasawa decomposition  $k^{-1}y = n(k^{-1}y)(\exp A(k^{-1}y))k''$  for a given  $y \in G$ . We have, on the one hand  $d(\gamma(r), eK) = r$ . On the other hand,

$$d(\gamma(r), yK) = d(\exp(-rH_0)[n(k^{-1}y)]^{-1}(\exp rH_0)K, \exp(A(k^{-1}y) - rH_0)K).$$

Hence

$$|d(\gamma(r), yK) - d(\exp(A(k^{-1}y) - rH_0)K, eK)| \leq \underbrace{d(\exp(-rH_0)[n(k^{-1}y)]^{-1}(\exp rH_0)K, eK)}_{|A(k^{-1}y) - rH_0|} \underbrace{d(\exp(-rH_0)[n(k^{-1}y)]^{-1}(\exp rH_0)K, eK)}_{\rightarrow e}$$

as  $r \rightarrow \infty$ . Therefore,

$$\begin{aligned} d(\gamma(r), yK) - d(\gamma(r), eK) &= |A(k^{-1}y) - rH_0| - r + o(1) \\ &= \frac{|A(k^{-1}y)|^2 - 2r\langle H_0, A(k^{-1}y) \rangle}{2r + O(1)} + o(1) \\ &= -\langle H_0, A(k^{-1}y) \rangle + o(1) \end{aligned}$$

as  $r \rightarrow \infty$ . In conclusion, we have thus determined the Busemann function

$$B_\gamma(yK) = \lim_{r \rightarrow +\infty} \{d(\gamma(r), yK) - d(\gamma(r), eK)\} = -\langle H_0, A(k^{-1}y) \rangle.$$

Now, let us turn to the proof of Proposition 3.7.

*Proof of Proposition 3.7.* Using (2.19) and (2.20), we write

$$\begin{aligned} \frac{h_t(gK, yK)}{h_t(gK, eK)} &= \frac{h_t(\exp(y^{-1}g)^+)}{h_t(\exp(g^+))} \\ &\sim \frac{\mathbf{b}\left(-i\frac{(y^{-1}g)^+}{2t}\right)^{-1}}{\mathbf{b}\left(-i\frac{g^+}{2t}\right)^{-1}} \frac{\pi((y^{-1}g)^+)}{\pi(g^+)} \frac{e^{-\langle \rho, (y^{-1}g)^+ \rangle}}{e^{-\langle \rho, g^+ \rangle}} \frac{e^{-\frac{|(y^{-1}g)^+|^2}{4t}}}{e^{-\frac{|g^+|^2}{4t}}} \end{aligned}$$

as  $t \rightarrow \infty$ . The asymptotic behaviors of the first two factors are based on Lemma 3.8.(iii). On the one hand,

$$\frac{\pi((y^{-1}g)^+)}{\pi(g^+)} = \prod_{\alpha \in \Sigma_r^+} \frac{\langle \alpha, (y^{-1}g)^+ \rangle}{\langle \alpha, g^+ \rangle} = 1 + O\left(\frac{r(t)}{t}\right). \quad (3.19)$$

On the other hand, by using (2.6), we have

$$\frac{\mathbf{b}\left(-i\frac{(y^{-1}g)^+}{2t}\right)^{-1}}{\mathbf{b}\left(-i\frac{g^+}{2t}\right)^{-1}} \sim \prod_{\alpha \in \Sigma_r^+} \left\{ \frac{\langle \alpha, g^+ \rangle}{\langle \alpha, (y^{-1}g)^+ \rangle} \right\}^{1 - \frac{m_\alpha}{2} - \frac{m_2 \alpha}{2}} = 1 + O\left(\frac{r(t)}{t}\right). \quad (3.20)$$



For the last two factors, we notice on the one hand that

$$\begin{aligned}
-\frac{|(y^{-1}g)^+|^2}{4t} + \frac{|g^+|^2}{4t} &= \frac{|g^+| + |(y^{-1}g)^+|}{4t} (|g^+| - |(y^{-1}g)^+|) \\
&= \underbrace{\frac{d(gK, eK) + d(gK, yK)}{4t}}_{=|\rho|+O(\frac{r(t)}{t})} \underbrace{\{d(gK, eK) - d(gK, yK)\}}_{=\langle \frac{\rho}{|\rho|}, A(k^{-1}y) \rangle + O(\frac{r(t)}{t})} \\
&= \langle \rho, A(k^{-1}y) \rangle + O\left(\frac{r(t)}{t}\right)
\end{aligned} \tag{3.21}$$

according to [Lemma 3.8](#). Therefore

$$e^{-\frac{|(y^{-1}g)^+|^2}{4t} + \frac{|g^+|^2}{4t}} = e^{\langle \rho, A(k^{-1}y) \rangle} + O\left(\frac{r(t)}{t}\right). \tag{3.22}$$

On the other hand, since  $(g^{-1})^+ = -w_0 \cdot g^+$  and  $\rho = -w_0 \cdot \rho$ , where  $w_0 \in W$  is the longest element in the Weyl group, we write

$$\begin{aligned}
-\langle \rho, (y^{-1}g)^+ \rangle + \langle \rho, g^+ \rangle &= -\langle \rho, (g^{-1}y)^+ \rangle + \langle \rho, (g^{-1})^+ \rangle \\
&= \underbrace{\frac{|(g^{-1})^+|^2 - |(g^{-1}y)^+|^2}{4t}}_{=\langle \rho, A(k^{-1}y) \rangle + O(\frac{r(t)}{t})} + \underbrace{\frac{|(g^{-1}y)^+ - 2t\rho|^2 - |(g^{-1})^+ - 2t\rho|^2}{4t}}_R.
\end{aligned}$$

where the first part of the right-hand side is estimated as [\(3.21\)](#), while the remainder

$$R = \{ |(g^{-1}y)^+ - 2t\rho| - |(g^{-1})^+ - 2t\rho| \} \frac{|(g^{-1}y)^+ - 2t\rho| + |(g^{-1})^+ - 2t\rho|}{4t}$$

is  $O(\frac{r(t)}{t})$ , according to (ii) in [Lemma 3.8](#). Hence

$$e^{-\langle \rho, (y^{-1}g)^+ \rangle + \langle \rho, g^+ \rangle} = e^{\langle \rho, A(k^{-1}y) \rangle} + O\left(\frac{r(t)}{t}\right). \tag{3.23}$$

In conclusion, we deduce from [\(3.20\)](#), [\(3.19\)](#), [\(3.22\)](#) and [\(3.23\)](#) that

$$\frac{h_t(gK, yK)}{h_t(gK, eK)} = e^{\langle 2\rho, A(k^{-1}y) \rangle} + O\left(\frac{r(t)}{t}\right)$$

with  $\frac{r(t)}{t} \rightarrow 0$  as  $t \rightarrow \infty$ . □

**Remark 3.10.** *Let us notice that [Proposition 3.7](#) also clarifies the role of bi- $K$ -invariance, and in addition gives an alternative proof for [Proposition 3.1](#). Let  $u_0 \in C_c(\mathbb{X})$  and recall that  $\mathbb{M}$  is the centralizer of  $\exp \mathfrak{a}$  in  $K$ . Recall that the Helgason-Fourier transform*

$$\mathcal{H}u_0(\lambda, k\mathbb{M}) = \int_G dg u_0(gK) e^{\langle -i\lambda + \rho, A(k^{-1}g) \rangle} \tag{3.24}$$

*boils down to the transform [\(2.4\)](#) when  $u_0$  is bi- $K$ -invariant. According to [Proposition 3.7](#), we know that, for all  $g$  in the critical region  $K(\exp \Omega_t)K$ ,*

$$\begin{aligned}
u(t, gK) - Mh_t(gK) &= \int_G dy u_0(y) (h_t(y^{-1}g) - h_t(gK)) \\
&= h_t(gK) \int_G dy u_0(y) \left( e^{\langle 2\rho, A(k^{-1}y) \rangle} - 1 + O\left(\frac{r(t)}{t}\right) \right) \\
&= h_t(gK) (\mathcal{H}u_0(i\rho, k\mathbb{M}) - \mathcal{H}u_0(-i\rho, k\mathbb{M}) + O\left(\frac{r(t)}{t}\right) \mathcal{H}u_0(-i\rho, k\mathbb{M})).
\end{aligned} \tag{3.25}$$

Notice that  $\mathcal{H}u_0(\pm i\rho, k\mathbb{M}) = \mathcal{H}u_0(\pm i\rho) = M$  when  $u_0$  is bi- $K$ -invariant. Then we deduce [Proposition 3.1](#) by integrating [\(3.25\)](#) over the critical region. On the other hand, we have

$$\begin{aligned} \int_{K(\exp \Omega_t)K} dg |u(t, gK) - Mh_t(gK)| &\longrightarrow \int_K dk \left| \int_G dy u_0(y) (e^{\langle 2\rho, A(k^{-1}y) \rangle} - 1) \right| \\ &= \int_K dk |\mathcal{H}u_0(i\rho, k\mathbb{M}) - \mathcal{H}u_0(-i\rho, k\mathbb{M})|, \end{aligned}$$

as  $t \rightarrow \infty$ . The last integral is not constantly zero when  $u_0$  is not bi- $K$ -invariant. For example, as we have seen in [\(3.16\)](#), if  $u_0$  is a Dirac measure supported on some point outside of  $K$ , the last integral does not vanish.

#### 4. ASYMPTOTIC CONVERGENCE ASSOCIATED WITH THE DISTINGUISHED LAPLACIAN

Let  $S = N(\exp \mathfrak{a}) = (\exp \mathfrak{a})N$  be the solvable group occurring in the Iwasawa decomposition  $G = N(\exp \mathfrak{a})K$ . Then  $S$  is identifiable, as a manifold, with the symmetric space  $\mathbb{X} = G/K$ . The distinguished Laplacian  $\tilde{\Delta}$  on  $S$  is given by the conjugation of the shifted Laplace-Beltrami operator  $\Delta + |\rho|^2$  on  $\mathbb{X}$ :

$$\tilde{\Delta} = \tilde{\delta}^{\frac{1}{2}} \circ (\Delta + |\rho|^2) \circ \tilde{\delta}^{-\frac{1}{2}} \quad (4.1)$$

where the modular function  $\tilde{\delta}$  of  $S$  is defined by

$$\tilde{\delta}(g) = \tilde{\delta}(n(\exp A)) = e^{-2\langle \rho, A \rangle} \quad \forall g \in S.$$

Here  $n = n(g)$  and  $A = A(g)$  denotes respectively the  $N$ -component and the  $\mathfrak{a}$ -component of  $g$  in the Iwasawa decomposition. The distinguished Laplacian  $\tilde{\Delta}$  is left- $S$ -invariant and self-adjoint with respect to the right-invariant Haar measure on  $S$ :

$$\int_S d_r g f(g) = \int_N dn \int_{\mathfrak{a}} dA f(n(\exp A)) = \int_{\mathfrak{a}} dA e^{2\langle \rho, A \rangle} \int_N dn f((\exp A)n).$$

Bear in mind the following different relations between the measures on  $S$  and the unimodular Haar measure on  $G$ :

$$\int_S d_r g f(g) = \int_G dg e^{2\langle \rho, A(g) \rangle} f(g) \quad \text{and} \quad \int_S d_\ell g f(g) = \int_G dg f(g). \quad (4.2)$$

This section aims to study the asymptotic behavior of solutions to the following Cauchy problem associated with the distinguished Laplacian:

$$\partial_t \tilde{v}(t, g) = \tilde{\Delta}_g \tilde{v}(t, g), \quad \tilde{v}(0, g) = \tilde{v}_0(g), \quad (4.3)$$

where the corresponding heat kernel is given by  $\tilde{h}_t = \tilde{\delta}^{\frac{1}{2}} e^{|\rho|^2 t} h_t$  in the sense that

$$(e^{t\tilde{\Delta}} f)(g) = (f * \tilde{h}_t)(g) = \int_S d_\ell y f(y) \tilde{h}_t(y^{-1}g) = \int_S d_r y f(gy^{-1}) \tilde{h}_t(y).$$

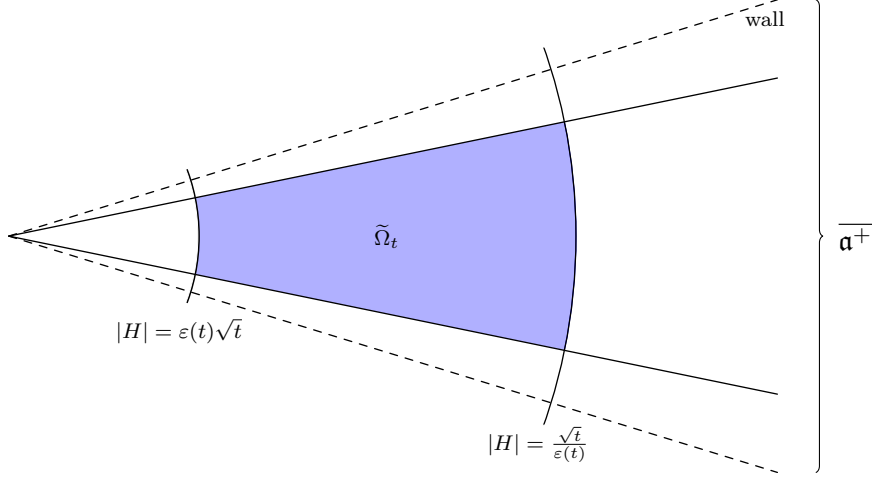
Here, we still denote by  $*$  the convolution product on  $S$  or on  $G$ . We refer to [\[Bou83; CGGM91\]](#) for more details about the distinguished Laplacian.

**Remark 4.1.** Notice that  $\tilde{h}_t(g) d_r g$  is a probability measure on  $S$ . Indeed, we know that the Abel transform of  $e^{|\rho|^2 t} h_t$  is the heat kernel on  $\mathfrak{a}$ , i.e.,

$$\mathcal{A}(e^{|\rho|^2 t} h_t)(A) = e^{-\langle \rho, A \rangle} \int_N dn e^{|\rho|^2 t} h_t(n(\exp A)) = (4\pi t)^{-\frac{\ell}{2}} e^{-\frac{|A|^2}{4t}},$$

hence

$$\int_N dn \tilde{h}_t(n(\exp A)) = (4\pi t)^{-\frac{\ell}{2}} e^{-\frac{|A|^2}{4t}}$$

FIGURE 2. Critical region  $\tilde{\Omega}_t$  in the positive Weyl chamber.

and

$$\int_S d_r g \tilde{h}_t(g) = \int_N dn \int_{\mathfrak{a}} dA \tilde{h}_t(n(\exp A)) = (4\pi t)^{-\frac{\ell}{2}} \int_{\mathfrak{a}} dA e^{-\frac{|A|^2}{4t}} = 1.$$

The first subsection is devoted to determine the critical region where the heat kernel  $\tilde{h}_t$  concentrates. In the next two subsections, we study respectively the  $L^1$  and the  $L^\infty$  asymptotic convergences of solutions to (4.3) with compactly supported initial data (no bi- $K$ -invariance required). We discuss the same questions for other initial data in the last subsection.

**4.1. Asymptotic concentration of the distinguished heat kernel.** Recall that the heat kernel  $h_t$  (associated with the heat equation (3.1)) concentrates in  $K(\exp \Omega_t)K$  where  $\Omega_t$  is a ball in the positive Weyl chamber. The following proposition shows that the heat kernel  $\tilde{h}_t$  (associated with the Cauchy problem (4.3)) concentrates in a different region. Recall that the function  $\mu : \bar{\mathfrak{a}}^+ \rightarrow \mathbb{R}^+$  is defined by  $\mu(H) = \min_{\alpha \in \Sigma^+} \langle \alpha, H \rangle$ .

**Proposition 4.2.** *Let  $t \mapsto \varepsilon(t)$  be a positive function such that  $\varepsilon(t) \searrow 0$  and  $\varepsilon(t)\sqrt{t} \rightarrow \infty$  as  $t \rightarrow \infty$ . Then the heat kernel associated with the distinguished Laplacian on  $S$  concentrates asymptotically in  $K(\exp \tilde{\Omega}_t)K$ , where*

$$\tilde{\Omega}_t = \{H \in \bar{\mathfrak{a}}^+ \mid \varepsilon(t)\sqrt{t} \leq |H| \leq \frac{\sqrt{t}}{\varepsilon(t)} \text{ and } \mu(H) \geq \varepsilon(t)\sqrt{t}\}.$$

In other words,

$$\lim_{t \rightarrow \infty} \int_{g \in S \text{ s.t. } g^+ \in \bar{\mathfrak{a}}^+ \setminus \tilde{\Omega}_t} d_r g \tilde{h}_t(g) = 0$$

where  $g^+$  denotes the middle component of  $g$  in the Cartan decomposition.

*Proof.* By using (4.2), let us write

$$I(t) = \int_{S \cap K(\exp \tilde{\Omega}_t)K} d_r g \tilde{h}_t(g) = e^{|\rho|^2 t} \int_{K(\exp \tilde{\Omega}_t)K} dg e^{\langle \rho, A(g) \rangle} h_t(g).$$

Since the heat kernel  $h_t$  is bi- $K$ -invariant on  $G$  and  $dk$  is the normalized Haar measure on the compact group  $K$ , we have

$$I(t) = e^{|\rho|^2 t} \int_{K(\exp \tilde{\Omega}_t)K} dg h_t(g) \int_K dk e^{\langle \rho, A(kg) \rangle} = e^{|\rho|^2 t} \int_{K(\exp \tilde{\Omega}_t)K} dg h_t(g) \varphi_0(g).$$

According to the Cartan decomposition, and to the estimates (2.3), (2.14) and (2.16), we obtain

$$\begin{aligned} I(t) &= e^{|\rho|^2 t} \int_{\tilde{\Omega}_t} dg^+ \delta(g^+) h_t(\exp g^+) \varphi_0(\exp g^+) \\ &\asymp t^{-\frac{n}{2}} \int_{\tilde{\Omega}_t} dg^+ e^{-\frac{|g^+|^2}{4t}} \omega_1(g^+) \omega_2(t, g^+) \end{aligned} \quad (4.4)$$

where

$$\omega_1(g^+) = \prod_{\alpha \in \Sigma^+} \left( \frac{\langle \alpha, g^+ \rangle}{1 + \langle \alpha, g^+ \rangle} \right)^{m_\alpha} \leq 1$$

and

$$\omega_2(t, g^+) = \prod_{\alpha \in \Sigma_r^+} (1 + \langle \alpha, g^+ \rangle)^2 (1 + t + \langle \alpha, g^+ \rangle)^{\frac{m_\alpha + m_{2\alpha}}{2} - 1}.$$

Next, let us study the right-hand side of (4.4) outside  $\tilde{\Omega}_t$ . On the one hand, let

$$I_0(t) = t^{-\frac{n}{2}} \int_{|g^+| < \varepsilon(t)\sqrt{t}} dg^+ e^{-\frac{|g^+|^2}{4t}} \omega_2(t, g^+).$$

By substituting  $g^+ = 2\sqrt{t}H$  and noticing that

$$\begin{aligned} \omega_2(t, 2\sqrt{t}H) &= (4t)^{\frac{n-\ell}{2}} \prod_{\alpha \in \Sigma_r^+} \left( \frac{1}{2\sqrt{t}} + \langle \alpha, H \rangle \right)^2 \left( \frac{1}{2t} + \frac{1}{2} + \frac{\langle \alpha, H \rangle}{\sqrt{t}} \right)^{\frac{m_\alpha + m_{2\alpha}}{2} - 1} \\ &\lesssim t^{\frac{n-\ell}{2}} \prod_{\alpha \in \Sigma_r^+} (1 + \langle \alpha, H \rangle)^{\frac{m_\alpha + m_{2\alpha} + 3}{2}} \end{aligned}$$

we get the upper bound of  $I_0(t)$  in large time:

$$I_0(t) \lesssim \int_{|H| < \frac{\varepsilon(t)}{2}} dH e^{-|H|^2} \prod_{\alpha \in \Sigma_r^+} (1 + \langle \alpha, H \rangle)^{\frac{m_\alpha + m_{2\alpha} + 3}{2}} \lesssim \varepsilon(t)^\ell. \quad (4.5)$$

On the other hand, similar computations yield

$$\begin{aligned} I_\infty(t) &= t^{-\frac{n}{2}} \int_{|g^+| > \frac{\sqrt{t}}{\varepsilon(t)}} dg^+ e^{-\frac{|g^+|^2}{4t}} \omega_2(t, g^+) \\ &\lesssim \int_{|H| > \frac{1}{2\varepsilon(t)}} dH e^{-|H|^2} |H|^{\frac{n-\ell+3|\Sigma_r^+|}{2}} \lesssim \varepsilon(t)^N \end{aligned} \quad (4.6)$$

for any  $N \geq 0$ . In the case where  $\mu(g^+) = \min_{\alpha \in \Sigma^+} \langle \alpha, g^+ \rangle < \varepsilon(t)\sqrt{t}$ , i.e., when  $g^+$  is close to the walls, we have

$$I_\mu(t) = t^{-\frac{n}{2}} \int_{\mu(g^+) < \varepsilon(t)\sqrt{t}} dg^+ e^{-\frac{|g^+|^2}{4t}} \omega_2(t, g^+) = 2^\ell t^{-\frac{n-\ell}{2}} \int_{\mu(H) < \frac{\varepsilon(t)}{2}} dH e^{-|H|^2} \omega_2(t, 2\sqrt{t}H)$$

after substituting  $g^+ = 2\sqrt{t}H$ . Notice that, there exists at least one  $\alpha_0 \in \Sigma_r^+$  such that  $\langle \alpha_0, H \rangle < \frac{\varepsilon(t)}{2}$ . For such an  $\alpha_0$ , we estimate

$$\frac{1}{2\sqrt{t}} + \langle \alpha_0, H \rangle < \frac{1}{2\sqrt{t}} + \frac{\varepsilon(t)}{2} \lesssim \varepsilon(t) (1 + \langle \alpha_0, H \rangle).$$

For the other  $\alpha \in \Sigma_r^+ \setminus \{\alpha_0\}$ , we simply estimate

$$\frac{1}{2\sqrt{t}} + \langle \alpha, H \rangle \lesssim (1 + \langle \alpha, H \rangle)^2.$$

Altogether,

$$I_\mu(t) \lesssim \underbrace{\varepsilon(t)^2 \int_{\mu(H) < \frac{\varepsilon(t)}{2}} dH e^{-|H|^2} \prod_{\substack{\alpha \in \Sigma_r^+ \\ < \infty}} (1 + \langle \alpha, H \rangle)^{\frac{m_\alpha + m_{2\alpha} + 3}{2}}}_{< \infty}. \quad (4.7)$$

In conclusion, we deduce from (4.4), (4.5), (4.6) and (4.7), that

$$0 \leq \int_{g \in S \text{ s.t. } g^+ \in \overline{\mathfrak{a}^+} \setminus \tilde{\Omega}_t} d_r g \tilde{h}_t(g) \lesssim \varepsilon(t)$$

for  $t$  large enough. Since  $\varepsilon(t) \rightarrow 0$  as  $t \rightarrow \infty$ , we have proved that the heat kernel  $\tilde{h}_t$  associated with the distinguished Laplacian on  $S$  concentrates asymptotically in  $K(\exp \tilde{\Omega}_t)K$ .  $\square$

The asymptotic behaviors (2.19) and (2.20) of the heat kernel  $h_t(\exp H)$  and the ground spherical function  $\varphi_0(\exp H)$  hold when  $\mu(H) \rightarrow \infty$ , i.e., when  $H \in \overline{\mathfrak{a}^+}$  stays away from the walls. The following proposition sharpens these asymptotics when  $H$  lies in the present critical region  $\tilde{\Omega}_t$ .

**Proposition 4.3.** *Let  $C_1, C_2$  be the positive constants occurring in (2.19), (2.20), and let  $C_3 = C_1 \pi(\rho_0)^{-1}$ . Then the following asymptotics hold for  $H \in \tilde{\Omega}_t$  and  $t \rightarrow \infty$ :*

$$h_t(\exp H) = \left\{ C_3 + O\left(\frac{1}{\varepsilon(t)\sqrt{t}}\right) \right\} t^{-\frac{\nu}{2}} e^{-|\rho|^2 t} \pi(H) e^{-\langle \rho, H \rangle} e^{-\frac{|H|^2}{4t}} \quad (4.8)$$

and

$$\varphi_0(\exp H) = \left\{ C_2 + O\left(\frac{1}{\mu(H)}\right) \right\} \pi(H) e^{-\langle \rho, H \rangle}. \quad (4.9)$$

*Proof.* As  $H \in \tilde{\Omega}_t$  stays away from the walls in  $\overline{\mathfrak{a}^+}$ , we substitute the spherical function occurring in

$$e^{|\rho|^2 t} h_t(\exp H) = C_0 \int_{\mathfrak{a}} d\lambda |\mathbf{c}(\lambda)|^{-2} e^{-t|\lambda|^2} \varphi_\lambda(\exp H)$$

by its Harish-Chandra expansion (2.10), and we obtain firstly

$$e^{|\rho|^2 t} e^{\langle \rho, H \rangle} h_t(\exp H) = C_0 |W| \sum_{q \in 2Q} e^{-\langle q, H \rangle} \int_{\mathfrak{a}} d\lambda \mathbf{c}(-\lambda)^{-1} e^{-t|\lambda|^2} e^{i\langle \lambda, H \rangle} \gamma_q(\lambda).$$

By factorizing  $\mathbf{c}(-\lambda)^{-1} = \mathbf{b}(-\lambda)^{-1} \pi(-i\lambda)$  and by performing an integration by parts based on

$$\pi(-i\lambda) e^{-t|\lambda|^2} = \pi\left(\frac{i}{2t} \frac{\partial}{\partial \lambda}\right) e^{-t|\lambda|^2}$$

we obtain secondly

$$\begin{aligned} & (2t)^{|\Sigma_r^+|} e^{|\rho|^2 t} e^{\langle \rho, H \rangle} h_t(\exp H) \\ &= C_0 |W| \sum_{q \in 2Q} e^{-\langle q, H \rangle} \int_{\mathfrak{a}} d\lambda e^{-t|\lambda|^2} \pi\left(-i \frac{\partial}{\partial \lambda}\right) \{e^{i\langle \lambda, H \rangle} \mathbf{b}(-\lambda)^{-1} \gamma_q(\lambda)\} \\ &= C_0 |W| \sum_{q \in 2Q} e^{-\langle q, H \rangle} \sum_{\Sigma_r^+ = \Sigma_1 \sqcup \Sigma_2 \sqcup \Sigma_3} \int_{\mathfrak{a}} d\lambda e^{-t|\lambda|^2} \\ &\times \left\{ e^{i\langle \lambda, H \rangle} \prod_{\alpha_1 \in \Sigma_1} \langle \alpha_1, H \rangle \right\} \left\{ \prod_{\alpha_2 \in \Sigma_2} (-i \partial_{\alpha_2}) (\mathbf{b}(-\lambda)^{-1}) \right\} \left\{ \prod_{\alpha_3 \in \Sigma_3} (-i \partial_{\alpha_3}) (\gamma_q(\lambda)) \right\}. \end{aligned}$$

After the shift of contour  $\lambda \mapsto \lambda + i\frac{H}{2t}$  in  $\mathfrak{a}_{\mathbb{C}}$  and the rescaling  $\lambda \mapsto \frac{\lambda}{\sqrt{t}}$ , we obtain thirdly

$$\begin{aligned} & t^{\frac{\nu}{2}} e^{|\rho|^2 t} e^{\langle \rho, H \rangle} e^{\frac{|H|^2}{4t}} h_t(\exp H) \\ &= C_0 2^{-|\Sigma_r^+|} |W| \sum_{q \in 2Q} e^{-\langle q, H \rangle} \sum_{\Sigma_r^+ = \Sigma_1 \sqcup \Sigma_2 \sqcup \Sigma_3} \left\{ \prod_{\alpha_1 \in \Sigma_1} \langle \alpha_1, H \rangle \right\} \int_{\mathfrak{a}} d\lambda e^{-|\lambda|^2} \\ & \times \left\{ \prod_{\alpha_2 \in \Sigma_2} (-i\partial_{\alpha_2}) \mathbf{b}^{-1} \right\} \left(-\frac{\lambda}{\sqrt{t}} - i\frac{H}{2t}\right) \left\{ \prod_{\alpha_3 \in \Sigma_3} (-i\partial_{\alpha_3}) \gamma_q \right\} \left(-\frac{\lambda}{\sqrt{t}} - i\frac{H}{2t}\right). \end{aligned} \quad (4.10)$$

Recall that, for all  $H \in \tilde{\Omega}_t$ , we have  $\varepsilon(t)\sqrt{t} \leq |H| \leq \frac{\sqrt{t}}{\varepsilon(t)}$  and  $\mu(H) = \min_{\alpha \in \Sigma^+} \langle \alpha, H \rangle \geq \varepsilon(t)\sqrt{t}$ . Under these assumptions, we notice that the derivatives of  $\mathbf{b}^{-1}$  are bounded by

$$\begin{aligned} |\mathbf{b}(-\frac{\lambda}{\sqrt{t}} - i\frac{H}{2t})^{-1}| &\asymp \prod_{\alpha \in \Sigma_r^+} \left(1 + \frac{\langle \alpha, \lambda \rangle}{\sqrt{t}} + \frac{\langle \alpha, H \rangle}{t}\right)^{\frac{m_{\alpha} + m_{2\alpha}}{2} - 1} \\ &\lesssim \prod_{\alpha \in \Sigma_r^+} (1 + \langle \alpha, \lambda \rangle)^{\frac{m_{\alpha} + m_{2\alpha}}{2}} \lesssim (1 + |\lambda|)^{\frac{n-\ell}{2}}, \end{aligned} \quad (4.11)$$

according to (2.7) and (2.8), while the derivatives of  $\gamma_q$  are bounded by

$$|\gamma_q(-\frac{\lambda}{\sqrt{t}} - i\frac{H}{2t})| \lesssim (1 + |q|)^{N_{\gamma}}, \quad (4.12)$$

for some nonnegative constant  $N_{\gamma}$ , according to (2.11) and (2.12). Let us split up the the right-hand side of (4.10) as

$$I(t, H) + R(t, H)$$

where the leading term

$$I(t, H) = C_0 2^{-|\Sigma_r^+|} |W| \pi(H) \int_{\mathfrak{a}} d\lambda e^{-|\lambda|^2} \mathbf{b}(-\frac{\lambda}{\sqrt{t}} - i\frac{H}{2t})^{-1}$$

is the contribution of  $q = 0$  and  $\Sigma_1 = \Sigma_r^+$ , while  $R(t, H)$  denotes the remainder. On the one hand, noticing from (4.11) that

$$|\mathbf{b}(-\frac{\lambda}{\sqrt{t}} - i\frac{H}{2t})^{-1} - \mathbf{b}(0)^{-1}| \lesssim (1 + |\lambda|)^{\frac{n-\ell}{2}} \left(\frac{|\lambda|}{\sqrt{t}} + \frac{|H|}{2t}\right) \lesssim \frac{1}{\varepsilon(t)\sqrt{t}} (1 + |\lambda|)^{\frac{n-\ell}{2} + 1},$$

we obtain

$$I(t, H) = \left\{ C_3 + O\left(\frac{1}{\varepsilon(t)\sqrt{t}}\right) \right\} \pi(H) \quad (4.13)$$

where  $C_3 = C_0 2^{-|\Sigma_r^+|} |W| \pi^{\frac{\ell}{2}} \mathbf{b}(0)^{-1} = C_1 \pi(\rho_0)^{-1}$ . On the other hand, we deduce from (4.11) and (4.12) that

$$|R(t, H)| \lesssim \underbrace{\pi(H) \sum_{q \in 2Q, q \neq 0} e^{-\langle q, H \rangle} (1 + |q|)^{N_{\gamma}}}_{\lesssim e^{-\mu(H)}} \underbrace{\int_{\mathfrak{a}} d\lambda e^{-|\lambda|^2} (1 + |\lambda|)^{\frac{n-\ell}{2}}}_{< \infty}.$$

This proves the asymptotic behavior (4.8). Let us next turn to (4.9). Along the lines of [AnJi99], we multiply (2.10) by  $\pi(i\lambda)$  and obtain

$$\begin{aligned} e^{\langle \rho, H \rangle} \pi(i\lambda) \varphi_{\lambda}(\exp H) &= \sum_{w \in W} (\det w) \mathbf{b}(w.\lambda) \Phi_{w.\lambda}(H) \\ &= \sum_{q \in 2Q} e^{-\langle q, H \rangle} \sum_{w \in W} (\det w) \mathbf{b}(w.\lambda) \gamma_q(w.\lambda) e^{i\langle w.\lambda, H \rangle}. \end{aligned} \quad (4.14)$$

Here we used the factorization  $\mathbf{c}(w.\lambda) = \boldsymbol{\pi}(iw.\lambda)^{-1}\mathbf{b}(w.\lambda) = (\det w)\boldsymbol{\pi}(i\lambda)^{-1}\mathbf{b}(w.\lambda)$  for every  $w \in W$ . After applying  $\boldsymbol{\pi}(-i\frac{\partial}{\partial\lambda})|_{\lambda=0}$  to (4.14), the left-hand side becomes

$$|W| \boldsymbol{\pi}(\rho_0) e^{\langle \rho, H \rangle} \varphi_0(\exp H)$$

(see for instance the beginning of the proof of Proposition 2.2.12 in [AnJi99]). The leading term

$$I'(t, H) = |W| \mathbf{b}(0) \boldsymbol{\pi}(H)$$

in the right-hand side of (4.14) is obtained by taking  $q = 0$  and by applying  $\boldsymbol{\pi}(-i\frac{\partial}{\partial\lambda})|_{\lambda=0}$  to  $e^{i\langle w.\lambda, H \rangle}$ , while the remainder  $R'(t, H)$  is estimated by

$$|R'(t, H)| \lesssim \left\{ \frac{1}{\mu(H)} + e^{-\mu(H)} \right\} \boldsymbol{\pi}(H) \lesssim \frac{1}{\mu(H)} \boldsymbol{\pi}(H).$$

This proves the asymptotic behavior (4.9).  $\square$

**4.2. Heat asymptotics in  $L^1$  for compactly supported initial data.** In this subsection, we investigate the long-time asymptotic convergence in  $L^1(S)$  of solutions to the Cauchy problem (4.3), where the initial data  $\tilde{v}_0$  is assumed continuous and compactly supported in  $B(eK, \xi)$ . For every  $\lambda \in \mathfrak{a}$ , let  $\tilde{\varphi}_\lambda = \tilde{\delta}^{\frac{1}{2}} \varphi_\lambda$  be the modified spherical function. The mass function is defined by

$$\tilde{M}(g) = \frac{(\tilde{v}_0 * \tilde{\varphi}_0)(g)}{\tilde{\varphi}_0(g)} \quad \forall g \in S. \quad (4.15)$$

By using the fact that the modular function  $\tilde{\delta}$  is a character on  $S$ , we can also write the mass as

$$\tilde{M}(g) = \frac{1}{\tilde{\delta}(g)^{\frac{1}{2}} \varphi_0(g)} \int_S d_\ell y v_0(gK) \underbrace{\tilde{\delta}(y)^{\frac{1}{2}} \tilde{\delta}(y^{-1}g)^{\frac{1}{2}}}_{\tilde{\delta}(g)^{\frac{1}{2}}} \varphi_0(y^{-1}g) = \frac{(v_0 * \varphi_0)(g)}{\varphi_0(g)} \quad (4.16)$$

where  $v_0(gK) = \tilde{\delta}(g)^{-\frac{1}{2}} \tilde{v}_0(g)$  is a right  $K$ -invariant function on  $G$ , with compact support  $(\text{supp } \tilde{v}_0)K$ .

**Remark 4.4.** *The map*

$$\tilde{v}_0 \longmapsto \tilde{v}_0 * \tilde{\varphi}_0$$

can be interpreted as the spectral projection at the bottom 0 of the spectrum of  $\tilde{\Delta}$ . Thus  $\tilde{M}$  generalizes somehow the mass in the Euclidean case. Let us elaborate. We recall the Helgason-Fourier transform (3.24) and its inverse formula

$$f(gK) = |W|^{-1} \int_{\mathfrak{a}} \frac{d\lambda}{|\mathbf{c}(\lambda)|^2} \int_K dk \mathcal{H}f(\lambda, kM) e^{\langle -i\lambda - \rho, H(g^{-1}k) \rangle}. \quad (4.17)$$

By combining (3.24) with (4.17) and by using the formula

$$\varphi_\lambda(y^{-1}g) = \int_K dk e^{\langle i\lambda + \rho, A(kg) \rangle} e^{\langle -i\lambda + \rho, A(ky) \rangle} \quad (4.18)$$

(see for instance [Hel94, Chap.III, Theorem 1.1]), we obtain

$$v_0(gK) = |W|^{-1} \int_{\mathfrak{a}} d\lambda |\mathbf{c}(\lambda)|^2 (v_0 * \varphi_\lambda)(g)$$

with

$$\Delta(v_0 * \varphi_\lambda) = v_0 * (\Delta\varphi_\lambda) = -(|\rho|^2 + |\lambda|^2)(v_0 * \varphi_\lambda).$$

Hence

$$\tilde{v}_0(g) = |W|^{-1} \int_{\mathfrak{a}} d\lambda |\mathbf{c}(\lambda)|^2 (\tilde{v}_0 * \tilde{\varphi}_\lambda)(g)$$

with

$$\tilde{\Delta}(\tilde{v}_0 * \tilde{\varphi}_\lambda) = \tilde{v}_0 * (\tilde{\Delta}\tilde{\varphi}_\lambda) = -|\lambda|^2(\tilde{v}_0 * \tilde{\varphi}_\lambda).$$

**Remark 4.5.** If  $\tilde{v}_0 \in \mathcal{C}_c(S)$ , then the mass function  $\tilde{M}$  is bounded. This follows indeed from the local Harnack inequality:

$$\varphi_0(y^{-1}g) = \int_K dk e^{\langle \rho, A(kg) \rangle} e^{\langle \rho, A(ky) \rangle} \lesssim \int_K dk e^{\langle \rho, A(kg) \rangle} = \varphi_0(g) \quad (4.19)$$

which holds for every  $y \in \text{supp } v_0 = (\text{supp } \tilde{v}_0)K$  and for every  $g \in G$ . Here, we have used (4.18) and the fact that  $|A(ky)| \leq |y|$  is bounded. See also [GaVa88, Proposition 4.6.3].

**Remark 4.6.** If  $\tilde{v}_0 = \delta^{1/2}v_0$  with  $v_0$  bi- $K$ -invariant, we notice on the one hand that

$$\begin{aligned} \int_S d_r g \delta(g)^{\frac{1}{2}} v_0(g) &= \int_G dg v_0(g) e^{\langle \rho, A(g) \rangle} \\ &= \int_G dg v_0(g) \int_K dk e^{\langle \rho, A(kg) \rangle} = \int_G dg v_0(g) \varphi_0(g) \end{aligned} \quad (4.20)$$

and on the other hand that

$$\begin{aligned} \frac{(v_0 * \varphi_0)(g)}{\varphi_0(g)} &= \frac{1}{\varphi_0(g)} \int_G dy v_0(y) \varphi_0(y^{-1}g) \\ &= \frac{1}{\varphi_0(g)} \int_G dy v_0(y) \underbrace{\int_K dk \varphi_0(y^{-1}kg)}_{\varphi_0(g) \varphi_0(y)} = \int_G dy v_0(y) \varphi_0(y). \end{aligned} \quad (4.21)$$

Hence the mass function  $\tilde{M}$  is a constant if  $v_0$  is bi- $K$ -invariant and  $\tilde{v}_0 = \tilde{\delta}^{\frac{1}{2}}v_0$  belongs to  $L^1(S)$ :

$$\tilde{M} = \int_G dy v_0(y) \varphi_0(y) = \mathcal{H}v_0(0).$$

The following lemma plays a key role in the proof of [Theorem 1.6](#).

**Lemma 4.7.** For bounded  $y \in G$  and for all  $g$  in the critical region  $K(\exp \tilde{\Omega}_t)K$ , the following asymptotic behavior holds:

$$\frac{h_t(g^{-1}y)}{h_t(g^{-1})} - \frac{\varphi_0(g^{-1}y)}{\varphi_0(g^{-1})} = \mathcal{O}\left(\frac{1}{\varepsilon(t)\sqrt{t}}\right) \quad \text{as } t \rightarrow \infty.$$

*Proof.* Assume that  $|y| \leq \xi$  for some positive constant  $\xi$ . Recall that for every  $H \in \tilde{\Omega}_t$ , we have  $\varepsilon(t)\sqrt{t} \leq |H| \leq \frac{\sqrt{t}}{\varepsilon(t)}$  and  $\mu(H) \geq \varepsilon(t)\sqrt{t}$ , where  $\varepsilon(t) \rightarrow 0$  and  $\varepsilon(t)\sqrt{t} \rightarrow \infty$  as  $t \rightarrow \infty$ . Since we are interested in the asymptotic behavior when  $t$  goes to infinity, we can assume that  $t$  is large enough such that

$$\varepsilon(t) < \sqrt{2} \quad \text{and} \quad \varepsilon(t)\sqrt{t} > 2\xi(1 + \max_{\alpha \in \Sigma} |\alpha|).$$

Notice first that

$$g \in K(\exp \tilde{\Omega}_t)K \quad \iff \quad g^{-1} \in K(\exp \tilde{\Omega}_t)K$$

as

$$H \in \tilde{\Omega}_t \quad \iff \quad -w_0.H \in \tilde{\Omega}_t$$

where  $w_0$  denotes the longest element in  $W$ , which interchanges the positive and negative roots. Notice next that

$$|(g^{-1}y)^+ - (g^{-1})^+| \leq d(g^{-1}yK, g^{-1}K) = |y| < \xi$$



according to (2.2). Then we deduce the following estimates:

$$\begin{cases} |(g^{-1}y)^+| \leq |(g^{-1})^+| + \xi < \frac{\sqrt{t}}{\varepsilon(t)} + \frac{1}{2}\varepsilon(t)\sqrt{t} < \frac{2\sqrt{t}}{\varepsilon(t)}, \\ |(g^{-1}y)^+| \geq |(g^{-1})^+| - \xi > \varepsilon(t)\sqrt{t} - \frac{1}{2}\varepsilon(t)\sqrt{t} > \frac{\varepsilon(t)\sqrt{t}}{2}, \\ \langle \alpha, (g^{-1}y)^+ \rangle > \langle \alpha, (g^{-1})^+ \rangle - |\alpha|\xi > \varepsilon(t)\sqrt{t} - \frac{1}{2}\varepsilon(t)\sqrt{t} > \frac{\varepsilon(t)\sqrt{t}}{2} \quad \forall \alpha \in \Sigma^+. \end{cases}$$

In other words, we obtain

$$g^{-1}y \in K(\exp \tilde{\Omega}'_t)K \quad \forall g \in K(\exp \tilde{\Omega}_t)K, \quad \forall |y| < \xi,$$

where

$$\tilde{\Omega}'_t = \{H \in \bar{\mathfrak{a}}^+ \mid \varepsilon'(t)\sqrt{t} \leq |H| \leq \frac{\sqrt{t}}{\varepsilon'(t)} \text{ and } \mu(H) \geq \varepsilon'(t)\sqrt{t}\}.$$

and  $\varepsilon'(t) = \frac{1}{2}\varepsilon(t)$  is still a decreasing function satisfying  $\varepsilon'(t) \rightarrow 0$  and  $\varepsilon'(t)\sqrt{t} \rightarrow \infty$  as  $t \rightarrow \infty$ . Thus the asymptotics (4.8) and (4.9) yield

$$\frac{h_t(g^{-1}y)}{h_t(g^{-1})} = \underbrace{\frac{C_3 + O\left(\frac{1}{\varepsilon'(t)\sqrt{t}}\right)}{C_3 + O\left(\frac{1}{\varepsilon(t)\sqrt{t}}\right)}}_{1 + O\left(\frac{1}{\varepsilon(t)\sqrt{t}}\right)} \underbrace{\frac{\pi((g^{-1}y)^+)}{\pi((g^{-1})^+)}}_{R(g,y)} e^{-\langle \rho, (g^{-1}y)^+ \rangle + \langle \rho, (g^{-1})^+ \rangle} e^{-\frac{|(g^{-1}y)^+|^2}{4t} + \frac{|(g^{-1})^+|^2}{4t}}$$

and

$$\frac{\varphi_0(g^{-1}y)}{\varphi_0(g^{-1})} = \frac{C_2 + O\left(\frac{1}{\varepsilon'(t)\sqrt{t}}\right)}{C_2 + O\left(\frac{1}{\varepsilon(t)\sqrt{t}}\right)} R(g, y) = \left\{1 + O\left(\frac{1}{\varepsilon(t)\sqrt{t}}\right)\right\} R(g, y).$$

Hence

$$\frac{h_t(g^{-1}y)}{h_t(g^{-1})} - \frac{\varphi_0(g^{-1}y)}{\varphi_0(g^{-1})} = \left\{1 + O\left(\frac{1}{\varepsilon(t)\sqrt{t}}\right)\right\} \left\{e^{-\frac{|(g^{-1}y)^+|^2}{4t} + \frac{|(g^{-1})^+|^2}{4t}} - 1\right\} R(g, y)$$

On the one hand, since  $\frac{\pi((g^{-1}y)^+)}{\pi((g^{-1})^+)}$  and  $e^{-\langle \rho, (g^{-1}y)^+ \rangle + \langle \rho, (g^{-1})^+ \rangle}$  are uniformly bounded when  $g \in K(\exp \tilde{\Omega}_t)K$  and  $|y| < \xi$ , so is  $R(g, y)$ . On the other hand,

$$e^{-\frac{|(g^{-1}y)^+|^2}{4t} + \frac{|(g^{-1})^+|^2}{4t}} = e^{-\frac{|(g^{-1})^+| - |(g^{-1}y)^+|}{4\sqrt{t}} \frac{|(g^{-1})^+| + |(g^{-1}y)^+|}{\sqrt{t}}} = e^{O\left(\frac{1}{\varepsilon(t)\sqrt{t}}\right)} = 1 + O\left(\frac{1}{\varepsilon(t)\sqrt{t}}\right).$$

In conclusion,

$$\frac{h_t(g^{-1}y)}{h_t(g^{-1})} - \frac{\varphi_0(g^{-1}y)}{\varphi_0(g^{-1})} = O\left(\frac{1}{\varepsilon(t)\sqrt{t}}\right) \quad \forall g \in K(\exp \tilde{\Omega}_t)K, \quad \forall |y| < \xi.$$

□

Now, let us prove the first part of **Theorem 1.6**.

*Proof of (1.9) in Theorem 1.6.* By using

$$(\tilde{v}_0 * \tilde{\varphi}_\lambda)(g) = \int_S d_{ly} v_0(yK) \underbrace{\tilde{\delta}(y)^{\frac{1}{2}} \tilde{\delta}(y^{-1}g)^{\frac{1}{2}}}_{\tilde{\delta}(g)^{\frac{1}{2}}} \varphi_\lambda(y^{-1}g) = \tilde{\delta}(g)^{\frac{1}{2}} (v_0 * \varphi_\lambda)(gK),$$

let us write the solution  $\tilde{v}$  to (4.3) as

$$\tilde{v}(t, g) = (\tilde{v}_0 * \tilde{h}_t)(g) = e^{|\rho|^2 t} \tilde{\delta}(g)^{\frac{1}{2}} (v_0 * h_t)(g)$$

We aim to study the difference

$$\begin{aligned}
\tilde{v}(t, g) - \widetilde{M}(g)\tilde{h}_t(g) &= \tilde{h}_t(g) \frac{(v_0 * h_t)(g)}{h_t(g)} - \tilde{h}_t(g) \frac{(v_0 * \varphi_0)(g)}{\varphi_0(g)} \\
&= \tilde{h}_t(g) \int_G dy v_0(yK) \left\{ \frac{h_t(y^{-1}g)}{h_t(g)} - \frac{\varphi_0(y^{-1}g)}{\varphi_0(g)} \right\} \\
&= \tilde{h}_t(g) \int_G dy v_0(yK) \left\{ \frac{h_t(g^{-1}y)}{h_t(g^{-1})} - \frac{\varphi_0(g^{-1}y)}{\varphi_0(g^{-1})} \right\} \tag{4.22}
\end{aligned}$$

where the last expression is derived from the symmetries  $h_t(x^{-1}) = h_t(x)$  and  $\varphi_0(x^{-1}) = \varphi_0(x)$ . According to the previous lemma, we have

$$\frac{h_t(g^{-1}y)}{h_t(g^{-1})} - \frac{\varphi_0(g^{-1}y)}{\varphi_0(g^{-1})} = O\left(\frac{1}{\varepsilon(t)\sqrt{t}}\right) \quad \forall g \in K(\exp \tilde{\Omega}_t)K, \quad \forall y \in \text{supp } v_0,$$

and therefore the integral of  $\tilde{v}(t, \cdot) - \widetilde{M}\tilde{h}_t$  over the critical region

$$\int_{S \cap K(\exp \tilde{\Omega}_t)K} d_r g |\tilde{v}(t, g) - \widetilde{M}(g)\tilde{h}_t(g)| \lesssim \frac{1}{\varepsilon(t)\sqrt{t}} \underbrace{\int_S d_r g \tilde{h}_t(g)}_1 \underbrace{\int_G dy |v_0(yK)|}_{\text{const.}}$$

tends asymptotically to 0. It remains for us to check that the integral

$$\begin{aligned}
\int_{S \setminus K(\exp \tilde{\Omega}_t)K} d_r g |\tilde{v}(t, g) - \widetilde{M}(g)\tilde{h}_t(g)| &\leq \int_{S \setminus K(\exp \tilde{\Omega}_t)K} d_r g |\tilde{v}(t, g)| \\
&\quad + \int_{S \setminus K(\exp \tilde{\Omega}_t)K} d_r g |\widetilde{M}(g)\tilde{h}_t(g)|
\end{aligned}$$

tends also to 0. On the one hand, we know that  $\widetilde{M}$  is bounded and that the heat kernel  $\tilde{h}_t$  asymptotically concentrates in  $K(\exp \tilde{\Omega}_t)K$ , hence

$$\int_{S \setminus K(\exp \tilde{\Omega}_t)K} d_r g |\widetilde{M}(g)\tilde{h}_t(g)| \longrightarrow 0$$

as  $t \rightarrow \infty$ . On the other hand, notice that for all  $y \in \text{supp } v_0$  and for all  $g \in G$  such that  $g^+ \notin \tilde{\Omega}_t$ , we have

$$(y^{-1}g)^+ \notin \tilde{\Omega}_t'' = \{H \in \overline{\mathfrak{a}^+} \mid \varepsilon''(t)\sqrt{t} \leq |H| \leq \frac{\sqrt{t}}{\varepsilon''(t)} \text{ and } \mu(H) \geq \varepsilon''(t)\sqrt{t}\} \tag{4.23}$$

where  $\varepsilon''(t) = 2\varepsilon(t)$  (see the proof of the previous lemma). Hence

$$\begin{aligned}
\int_{S \setminus K(\exp \tilde{\Omega}_t)K} d_r g |\tilde{v}(t, g)| &\leq \int_G dy |v_0(yK)| \int_{G \setminus K(\exp \tilde{\Omega}_t)K} dg \delta(g)^{-\frac{1}{2}} e^{|\rho|^2 t} h_t(y^{-1}g) \\
&\lesssim \underbrace{\int_S d_r y |\tilde{v}_0(y)|}_{\|\tilde{v}_0\|_{L^1(S)}} \underbrace{\int_{S \setminus K(\exp \tilde{\Omega}_t'')K} d_r g \tilde{h}_t(g)}_{\rightarrow 0}.
\end{aligned}$$

This concludes the proof of the heat asymptotics in  $L^1$  for the distinguished Laplacian  $\tilde{\Delta}$  on  $S$  and for initial data  $\tilde{v}_0 \in \mathcal{C}_c(S)$ .  $\square$

**4.3. Heat asymptotics in  $L^\infty$  for compactly supported initial data.** Let us start with the following lemma, which allows us to compare the middle components occurring in the Iwasawa decomposition and in the Cartan decomposition.

**Lemma 4.8.** *For all  $g \in G$ , we have*

$$\langle \rho, A(g) \rangle \leq \langle \rho, g^+ \rangle \quad (4.24)$$

where  $A(g)$  denotes the  $\mathfrak{a}$ -component of  $g$  in the Iwasawa decomposition and  $g^+$  denotes its  $\overline{\mathfrak{a}^+}$ -component in the Cartan decomposition.

*Proof.* According to Kostant's convexity theorem [Hel00, Theorem IV.10.5], for every  $H \in \mathfrak{a}$ , the Iwasawa projection of  $K(\exp H)K$  is equal to the convex hull of  $W.H$ . Therefore  $A(K(\exp H)K) \subset \text{co}(W.H)$ , which implies that  $A(g) \in \text{co}(W.g^+)$  for all  $g \in G$ . The inequality (4.24) follows from the fact that  $\rho \in \mathfrak{a}^+$  while  $g^+ - \text{co}(W.g^+)$  is contained in the cone generated by the positive roots [Hel00, Lemma IV.8.3].  $\square$

In the following two propositions we collect some elementary properties of the distinguished heat kernel. The first one clarifies the lower and the upper bounds of  $\tilde{h}_t$ , while the second one describes its critical region for the  $L^\infty$  norm.

**Proposition 4.9.** *The heat kernel  $\tilde{h}_t$  associated with the distinguished Laplacian satisfies*

$$\|\tilde{h}_t\|_{L^\infty(S)} \asymp t^{-\frac{\ell+|\Sigma_r^+|}{2}} \quad (4.25)$$

for  $t$  large enough.

*Proof.* Using the global estimates (2.14) and (2.16), we have

$$\tilde{h}_t(g) \asymp t^{-\frac{\ell+|\Sigma_r^+|}{2}} e^{-\langle \rho, A(g) \rangle} e^{-\langle \rho, g^+ \rangle} e^{-\frac{|g^+|^2}{4t}} \left\{ \prod_{\alpha \in \Sigma_r^+} \frac{1+\langle \alpha, g^+ \rangle}{\sqrt{t}} \left(1 + \frac{\langle \alpha, g^+ \rangle}{t}\right)^{\frac{m_\alpha+m_{2\alpha}}{2}-1} \right\} \quad (4.26)$$

We obtain first the lower bound in (4.25) by evaluating the right hand side of (4.26) at  $g_0 = \exp(-\sqrt{t}\rho)$  and by observing that

$$A(g_0) = -\sqrt{t}\rho \quad \text{and} \quad g_0^+ = \sqrt{t}\rho.$$

For the upper bound, notice that

$$e^{-\langle \rho, A(g) \rangle} e^{-\langle \rho, g^+ \rangle} \leq 1 \quad (4.27)$$

according to (4.24), and that

$$\frac{1+\langle \alpha, g^+ \rangle}{\sqrt{t}} \left(1 + \frac{\langle \alpha, g^+ \rangle}{t}\right)^{\frac{m_\alpha+m_{2\alpha}}{2}-1} \leq \left(1 + \frac{\langle \alpha, g^+ \rangle}{\sqrt{t}}\right) \left(1 + \frac{\langle \alpha, g^+ \rangle}{t}\right)^{\frac{m_\alpha+m_{2\alpha}}{2}} \leq \left(1 + \frac{\langle \alpha, g^+ \rangle}{\sqrt{t}}\right)^{\frac{m_\alpha+m_{2\alpha}}{2}+1}$$

for  $t$  large enough. We deduce from (4.26) that

$$\tilde{h}_t(g) \lesssim t^{-\frac{\ell+|\Sigma_r^+|}{2}} e^{-\frac{1}{4}\frac{|g^+|^2}{\sqrt{t}}} \underbrace{\prod_{\alpha \in \Sigma_r^+} \left(1 + \frac{\langle \alpha, g^+ \rangle}{\sqrt{t}}\right)^{\frac{m_\alpha+m_{2\alpha}}{2}+1}}_{=O(1)} \lesssim t^{-\frac{\ell+|\Sigma_r^+|}{2}}. \quad (4.28)$$

$\square$

**Proposition 4.10.** *The heat kernel  $\tilde{h}_t$  concentrates asymptotically in the same critical region for the  $L^\infty$  norm as for the  $L^1$  norm. In other words,*

$$t^{\frac{\ell+|\Sigma_r^+|}{2}} \|\tilde{h}_t\|_{L^\infty(S \setminus K(\exp \tilde{\Omega}_t)K)} \longrightarrow 0 \quad \text{as } t \rightarrow \infty.$$

*Proof.* Let us study the sup norm of  $\tilde{h}_t$  outside the critical region. Recall that

$$\tilde{\Omega}_t = \left\{ H \in \overline{\mathfrak{a}^+} \mid \varepsilon(t)\sqrt{t} \leq |H| \leq \frac{\sqrt{t}}{\varepsilon(t)} \text{ and } \mu(H) \geq \varepsilon(t)\sqrt{t} \right\}$$

where  $\mu(H) = \min_{\alpha \in \Sigma^+} \langle \alpha, H \rangle$ , and  $\varepsilon(t) \rightarrow 0$  satisfies  $\frac{\sqrt{t}}{\varepsilon(t)} \rightarrow \infty$  and  $\varepsilon(t)\sqrt{t} \rightarrow \infty$  as  $t \rightarrow \infty$ .

We deduce from (4.26) and (4.27) that

$$t^{\frac{\ell+|\Sigma_r^+|}{2}} \tilde{h}_t(g) \lesssim e^{-\frac{|g^+|^2}{4t}} \prod_{\alpha \in \Sigma_r^+} \frac{1+\langle \alpha, g^+ \rangle}{\sqrt{t}} \left(1 + \frac{\langle \alpha, g^+ \rangle}{t}\right)^{\frac{m_\alpha+m_{2\alpha}}{2}-1} \quad (4.29)$$

*Case 1:* Assume that  $|g^+| < \varepsilon(t)\sqrt{t}$ . Then we deduce easily from (4.29) that

$$t^{\frac{\ell+|\Sigma_r^+|}{2}} \tilde{h}_t(g) \lesssim \varepsilon(t)^{|\Sigma_r^+|}$$

tends to 0.

*Case 2:* Assume that  $|g^+| > \frac{\sqrt{t}}{\varepsilon(t)}$ . The estimate (4.28) implies that, for any  $N > 0$ ,

$$t^{\frac{\ell+|\Sigma_r^+|}{2}} \tilde{h}_t(g) \lesssim e^{-\frac{1}{4}\left|\frac{g^+}{\sqrt{t}}\right|^2} \prod_{\alpha \in \Sigma_r^+} (1 + \langle \alpha, \frac{g^+}{\sqrt{t}} \rangle)^{\frac{m_\alpha + m_{2\alpha}}{2} + 1} \lesssim \varepsilon(t)^N,$$

which tends to 0.

*Case 3:* Assume that  $\mu(g^+) < \varepsilon(t)\sqrt{t}$ . By using the estimate

$$\frac{1 + \langle \alpha, g^+ \rangle}{\sqrt{t}} \lesssim \varepsilon(t) \lesssim \varepsilon(t) \left( \frac{1 + \langle \alpha, g^+ \rangle}{\sqrt{t}} \right)$$

in (4.29) for the roots  $\alpha \in \Sigma_r^+$  satisfying  $\langle \alpha, g^+ \rangle \leq \varepsilon(t)\sqrt{t}$ , we obtain

$$t^{\frac{\ell+|\Sigma_r^+|}{2}} \tilde{h}_t(g) \lesssim \varepsilon(t) \underbrace{e^{-\frac{1}{4}\left|\frac{g^+}{\sqrt{t}}\right|^2} \prod_{\alpha \in \Sigma_r^+} (1 + \langle \alpha, \frac{g^+}{\sqrt{t}} \rangle)^{\frac{m_\alpha + m_{2\alpha}}{2} + 1}}_{=O(1)},$$

which tends to 0.

In conclusion, we obtain

$$\tilde{h}_t(g) = o(t^{-\frac{\ell+|\Sigma_r^+|}{2}}) \quad \text{as } t \rightarrow \infty$$

outside the critical region  $S \cap K(\exp \tilde{\Omega}_t)K$ . In other words, the heat kernel  $\tilde{h}_t(g)$  concentrates asymptotically in the same critical region for the  $L^\infty$  norm as for the  $L^1$  norm.  $\square$

Finally, let us prove the remaining part of [Theorem 1.6](#).

*Proof of (1.10) in Theorem 1.6.* Let us resume the proof of [Theorem 1.6](#). In the critical region  $S \cap K(\exp \tilde{\Omega}_t)K$ , we have

$$|\tilde{v}(t, g) - \tilde{M}(g)\tilde{h}_t(g)| \leq \tilde{h}_t(g) \int_{|y| < \xi} dg |v_0(yK)| \left| \frac{h_t(g^{-1}y)}{h_t(g^{-1})} - \frac{\varphi_0(g^{-1}y)}{\varphi_0(g^{-1})} \right|$$

with

$$\left| \frac{h_t(g^{-1}y)}{h_t(g^{-1})} - \frac{\varphi_0(g^{-1}y)}{\varphi_0(g^{-1})} \right| \lesssim \frac{1}{\varepsilon(t)\sqrt{t}}$$

according to (4.22) and to [Lemma 4.7](#). Then we deduce from (4.25) that

$$t^{\frac{\ell+|\Sigma_r^+|}{2}} |\tilde{v}(t, g) - \tilde{M}(g)\tilde{h}_t(g)| \lesssim \frac{1}{\varepsilon(t)\sqrt{t}} \quad \forall g \in S \cap K(\exp \tilde{\Omega}_t)K$$

where the right-hand side tends to 0 as  $t \rightarrow \infty$ . Outside the critical region, we estimate separately  $\tilde{v}(t, g)$  and  $\tilde{M}(g)\tilde{h}_t(g)$ . On the one hand, we know that  $\tilde{M}(g)$  is a bounded function and that  $\tilde{h}_t(g) = o(t^{-(\ell+|\Sigma_r^+|)/2})$ . Then  $t^{(\ell+|\Sigma_r^+|)/2}\tilde{M}(g)\tilde{h}_t(g)$  tends to 0 as  $t \rightarrow \infty$ . On the other hand, since  $g \notin K(\exp \tilde{\Omega}_t)K$  and  $|y| < \xi$  imply that  $g^{-1}y \notin K(\exp \tilde{\Omega}'_t)K$  (see (4.23)), we obtain

$$|\tilde{v}(t, g)| \lesssim \int_G dy |\tilde{v}_0(yK)| |\tilde{h}_t(g^{-1}y)|$$

which is  $o(t^{-(\ell+|\Sigma_r^+|)/2})$  outside the critical region. In conclusion,

$$t^{\frac{\ell+|\Sigma_r^+|}{2}} \|\tilde{v}(t, \cdot) - \tilde{M}\tilde{h}_t\|_{L^\infty(S)} \rightarrow 0$$

as  $t \rightarrow \infty$ .  $\square$

By convexity we obtain easily the corresponding result for the  $L^p$  norm.

**Corollary 4.11.** *The solution  $\tilde{v}$  to the Cauchy problem (4.3) with initial data  $\tilde{v}_0 \in \mathcal{C}_c(S)$  satisfies*

$$t^{\frac{\ell+|\Sigma_r^+|}{2p'}} \|\tilde{v}(t, \cdot) - \widetilde{M} \tilde{h}_t\|_{L^p(S)} \longrightarrow 0 \quad \text{as } t \rightarrow \infty, \quad (4.30)$$

for all  $1 < p < \infty$ .

**4.4. Heat asymptotics for other initial data.** We have obtained above the long-time asymptotic convergence in  $L^p$  ( $1 \leq p \leq \infty$ ) for the distinguished heat equation with compactly supported initial data. It is natural and interesting to ask whether the expected convergence still holds when the initial data lie in larger functional spaces? The following corollaries give positive examples, but the optimal answer remains open.

**Corollary 4.12.** *The asymptotic convergences (1.9) and (1.10), hence (4.30), still hold with initial data  $\tilde{v}_0 = \tilde{\delta}^{\frac{1}{2}} v_0 \in L^1(S)$  when  $v_0$  is bi- $K$ -invariant.*

*Proof.* Notice from (4.20) and (4.21) that the mass function  $\widetilde{M} = (\mathcal{H}v_0)(0)$  is a constant under the present assumption. Let us start with the  $L^1$  convergence. Given  $\varepsilon > 0$ , let  $\tilde{V}_0 = \tilde{\delta}^{\frac{1}{2}} V_0 \in \mathcal{C}_c(S)$ , with  $V_0$  bi- $K$ -invariant, be such that  $\|\tilde{v}_0 - \tilde{V}_0\|_{L^1(S)} < \frac{\varepsilon}{3}$ . Then the corresponding mass function  $(\mathcal{H}V_0)(0)$  is also a constant. We observe firstly that the solution to the Cauchy problem

$$\partial_t \tilde{V}(t, g) = \tilde{\Delta}_g \tilde{V}(t, g), \quad \tilde{V}(0, g) = \tilde{V}_0(g)$$

satisfies

$$\|\tilde{v}(t, \cdot) - \tilde{V}(t, \cdot)\|_{L^1(S)} = \|\tilde{v}_0 * \tilde{h}_t - \tilde{V}_0 * \tilde{h}_t\|_{L^1(S)} \leq \|\tilde{v}_0 - \tilde{V}_0\|_{L^1(S)} \|\tilde{h}_t\|_{L^1(S)} < \frac{\varepsilon}{3}, \quad (4.31)$$

since  $\|\tilde{h}_t\|_{L^1(S)} = 1$ , and secondly that there exists  $T > 0$  such that for all  $t \geq T$ ,

$$\|\tilde{V}(t, \cdot) - (\mathcal{H}V_0)(0) \tilde{h}_t\|_{L^1(S)} < \frac{\varepsilon}{3} \quad (4.32)$$

according to Theorem 1.6. Under the bi- $K$ -invariance assumption, we deduce from (4.20) and (4.21) that

$$(\mathcal{H}V_0)(0) - (\mathcal{H}v_0)(0) = \int_G dg (V_0(g) - v_0(g)) \varphi_0(g) = \int_S d_r g \tilde{\delta}(g)^{\frac{1}{2}} (V_0(g) - v_0(g)).$$

Hence, we have thirdly

$$\|((\mathcal{H}V_0)(0) - (\mathcal{H}v_0)(0)) \tilde{h}_t\|_{L^1(S)} \leq \|\tilde{v}_0 - \tilde{V}_0\|_{L^1(S)} \|\tilde{h}_t\|_{L^1(S)} < \frac{\varepsilon}{3}. \quad (4.33)$$

In conclusion, by putting (4.31), (4.32) and (4.33) altogether, we obtain

$$\|\tilde{v}(t, \cdot) - \widetilde{M} \tilde{h}_t\|_{L^1(S)} < \varepsilon$$

for all  $\varepsilon > 0$ . Let us turn to the  $L^\infty$  convergence. According to (4.25) and to Theorem 1.6, we have this time

$$t^{\frac{\ell+|\Sigma_r^+|}{2}} \|\tilde{v}(t, \cdot) - \tilde{V}(t, \cdot)\|_{L^\infty(S)} \leq \|\tilde{v}_0 - \tilde{V}_0\|_{L^1(S)} t^{\frac{\ell+|\Sigma_r^+|}{2}} \|\tilde{h}_t\|_{L^\infty(S)} < \frac{\varepsilon}{3},$$

$$t^{\frac{\ell+|\Sigma_r^+|}{2}} \|\tilde{V}(t, \cdot) - (\mathcal{H}V_0)(0) \tilde{h}_t\|_{L^\infty(S)} < \frac{\varepsilon}{3},$$

and

$$t^{\frac{\ell+|\Sigma_r^+|}{2}} \|((\mathcal{H}V_0)(0) - (\mathcal{H}v_0)(0)) \tilde{h}_t\|_{L^\infty(S)} \leq \|\tilde{v}_0 - \tilde{V}_0\|_{L^1(S)} t^{\frac{\ell+|\Sigma_r^+|}{2}} \|\tilde{h}_t\|_{L^\infty(S)} < \frac{\varepsilon}{3}.$$

Altogether,

$$t^{\frac{\ell+|\Sigma_r^+|}{2}} \|\tilde{v}(t, \cdot) - \widetilde{M} \tilde{h}_t\|_{L^\infty(S)} \longrightarrow 0$$

as  $t \rightarrow \infty$ . The  $L^p$  convergence follows from convexity.  $\square$

**Corollary 4.13.** *The asymptotic convergences (1.9) and (1.10), hence (4.30), still hold with no bi- $K$ -invariance condition but under the assumption*

$$\int_G dg |v_0(gK)| e^{\langle \rho, g^+ \rangle} < \infty. \quad (4.34)$$

*Proof.* Notice first that

$$\int_S d_r g |v_0(g)| |\tilde{\delta}(g)|^{\frac{1}{2}} = \int_G dg |v_0(gK)| e^{\langle \rho, A(g) \rangle} \leq \int_G dg |v_0(gK)| e^{\langle \rho, g^+ \rangle}$$

according to (4.24). Hence, the assumption (4.34) is indeed stronger than  $\tilde{v}_0 \in L^1(S)$ . Under this assumption, the mass function is bounded:

$$\begin{aligned} |\tilde{M}(g)| &\leq \frac{1}{\varphi_0(y)} \int_G dg |v_0(yK)| \varphi_0(y^{-1}g) \\ &= \frac{1}{\varphi_0(g)} \int_G dy |v_0(yK)| \int_K dk e^{\langle \rho, A(kg) \rangle} e^{\langle \rho, A(ky) \rangle} \\ &\leq \underbrace{\frac{1}{\varphi_0(g)} \int_K dk e^{\langle \rho, A(kg) \rangle}}_{=1} \underbrace{\int_G dy |v_0(yK)| e^{\langle \rho, y^+ \rangle}}_{=C} = C. \end{aligned}$$

Here, we used the inequality (4.24) and the fact that  $(ky)^+ = y^+$  for all  $k \in K$ . For proving the  $L^1$  and the  $L^\infty$  convergences, we argue again by density. Since  $v_0(gK) e^{\langle \rho, g^+ \rangle}$  belongs to  $L^1(\mathbb{X})$ , there exists a function  $V_0(gK) e^{\langle \rho, g^+ \rangle}$  in  $\mathcal{C}_c(\mathbb{X})$  such that

$$\int_G dg |v_0(gK) - V_0(gK)| e^{\langle \rho, g^+ \rangle} < \frac{\varepsilon}{3}$$

for every  $\varepsilon > 0$ . This implies that the function  $\tilde{V}_0 = \tilde{\delta}^{\frac{1}{2}} V_0$  approximates the initial data  $\tilde{v}_0 = \tilde{\delta}^{\frac{1}{2}} v_0$  in  $L^1(S)$ . Indeed,

$$\begin{aligned} \int_S d_r g |\tilde{v}_0(g) - \tilde{V}_0(g)| &= \int_G dg |\tilde{v}_0(gK) - \tilde{V}_0(gK)| e^{\langle \rho, A(g) \rangle} \\ &\leq \int_G dg |v_0(gK) - V_0(gK)| e^{\langle \rho, g^+ \rangle} < \frac{\varepsilon}{3}. \end{aligned}$$

Let  $\tilde{V} = \tilde{V}_0 * \tilde{h}_t$  be the corresponding solution to the distinguished heat equation, and denote by  $\tilde{M}_V(g) = \frac{(V_0 * \varphi_0)(gK)}{\varphi_0(g)}$  the corresponding mass of  $V_0$ . On the one hand, there exists  $T > 0$  such that for all  $t > T$ , we have

$$\|\tilde{V}(t, \cdot) - \tilde{M}_V \tilde{h}_t\|_{L^1(S)} < \frac{\varepsilon}{3}$$

since  $\tilde{V}_0 \in \mathcal{C}_c(S)$ . On the other hand, for every  $g \in G$ , we have

$$\begin{aligned} |\tilde{M}(g) - \tilde{M}_V(g)| &\leq \frac{1}{\varphi_0(y)} \int_G dg |v_0(yK) - V_0(gK)| \varphi_0(y^{-1}g) \\ &\leq \int_G dy |v_0(yK) - V_0(gK)| e^{\langle \rho, y^+ \rangle} < \frac{\varepsilon}{3}. \end{aligned}$$

We conclude by resuming the proof of the previous corollary.  $\square$

**Acknowledgements.** The authors are grateful to the referees for checking this manuscript carefully and making helpful suggestions of improvement. The second author is supported by the Hellenic Foundation for Research and Innovation, Project HFRI-FM17-1733. She acknowledges the SSHN fellowship by the French Institute of Greece and the French Embassy in Greece, which allowed a visit to Institut Denis Poisson, where this work was initiated. Finally, she is

thankful to the Institut for the warm hospitality, as well to M. Kolountzakis for its help during this stay. The last author acknowledges financial support from the Methusalem Programme *Analysis and Partial Differential Equations (Grant number 01M01021)* during his postdoc stay at Ghent University.

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