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► **To cite this version:**

Florian Bourgey, Emmanuel Gobet, Ying Jiao. Bridging socioeconomic pathways of CO₂ emission and credit risk. *Annals of Operations Research*, 2022, 10.1007/s10479-022-05135-y . hal-03458299v2

HAL Id: hal-03458299

<https://hal.science/hal-03458299v2>

Submitted on 14 Dec 2022

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Bridging socioeconomic pathways of CO₂ emission and credit risk

Florian Bourgey^{1,2}, Emmanuel Gobet¹ and Ying Jiao^{3*†}

¹CMAP, CNRS, Ecole Polytechnique, Institut Polytechnique de Paris, Palaiseau, 91128, France.

²Quantitative Research, Bloomberg L.P., London, EC4N 4TQ, United Kingdom.

³ISFA, Laboratoire SAF, Université Claude Bernard - Lyon 1, Lyon, 69007, France.

*Corresponding author(s). E-mail(s): ying.jiao@univ-lyon1.fr;

Contributing authors: bourgeyflorian@gmail.com;

emmanuel.gobet@polytechnique.edu;

†These authors contributed equally to this work.

Abstract

This paper investigates the impact of transition risk on a firm's low-carbon production. As the world is facing global climate change, the Intergovernmental Panel on Climate Change (IPCC) has set the idealized carbon-neutral scenario around 2050. In the meantime, many carbon reduction scenarios, known as *Shared Socioeconomic Pathways* (SSPs) have been proposed in the literature for different production sectors in a more comprehensive socio-economic context. We consider, on the one hand, a firm that aims to optimize its emission level under the double objectives of maximizing its production profit and respecting the emission mitigation scenarios. Solving the penalized optimization problem provides the optimal emission according to a given SSP benchmark. On the other hand, such transitions affect the firm's credit risk. We model the default time by using the structural default approach. We are particularly concerned with how, by following different SSPs scenarios, the adopted strategies may influence the firm's default probability.

Keywords: Climate risk, transition risk, credit risk, Shared Socioeconomic Pathways, carbon emission reduction, optimal production profit, structural credit model.

MSC Classification: 91B38 , 91G40

JEL Classification: Q54 , G32 , C61 , G38.

1 Introduction

The context of CO₂ emission scenario.

The continuous increase in anthropogenic greenhouse gas (GHG) emissions over the two last centuries is leading to climate change, characterized by an increase in the average surface temperature of the land and oceans. According to the Intergovernmental Panel on Climate Change [23], the global average temperature has already increased by 1°C since the pre-industrial era. Without proactive measures to limit these CO₂ and other GHG emissions, one can expect global warming of 3 or 4°C, and maybe more, by 2100. This current global warming is and will continue to profoundly disrupt environmental, geographical, and economic balances if no mitigation and adaptation measures are taken. The Paris Agreement at the twenty-first session of the Conference of the Parties (COP 21) is an important milestone in international climate policy as it establishes a global mitigation framework towards 2030 and sets the ground for global warming with stabilization at around 1.5°C only. This idealized scenario is based on carbon neutrality around 2050, with some variations depending on the countries and their Nationally Determined Contributions (NDC). There are many other scenarios, depending on the ecological transition trajectory that countries, economic actors, and populations will follow. In the most recent scientific literature, these scenarios are known as *Shared Socioeconomic Pathways* (SSPs), see [37] and references therein for an overview. In Figure 1, we plot the global CO₂ emissions of different sectors (energy, industrial, residential, and transportation) and scenarios for members of the Organization for Economic Co-operation and Development (OECD), [17, 37, 38]. The data is available on the SSP Public Database <https://tntcat.iiasa.ac.at/SspDb>. While the impacts of climate change are already perceptible, these scenarios are essential tools to help us to understand and anticipate mid-term and long-term consequences of near-term decisions. Our work aims to quantify how one firm's transition effort and strategy facing climate change will propagate to the firm's credit risk for a given scenario. Credit risk is the possibility of loss resulting from the default of a borrower (typically a company) to the lender (typically a bank). We refer to [28, 39] and references therein for possible modeling approaches to credit risk (without climate concerns).

Climate risks in finance.

Climate change generates new sources of risk (so-called climate risks), in particular, physical and transition risks as described by the solemn resounding speech

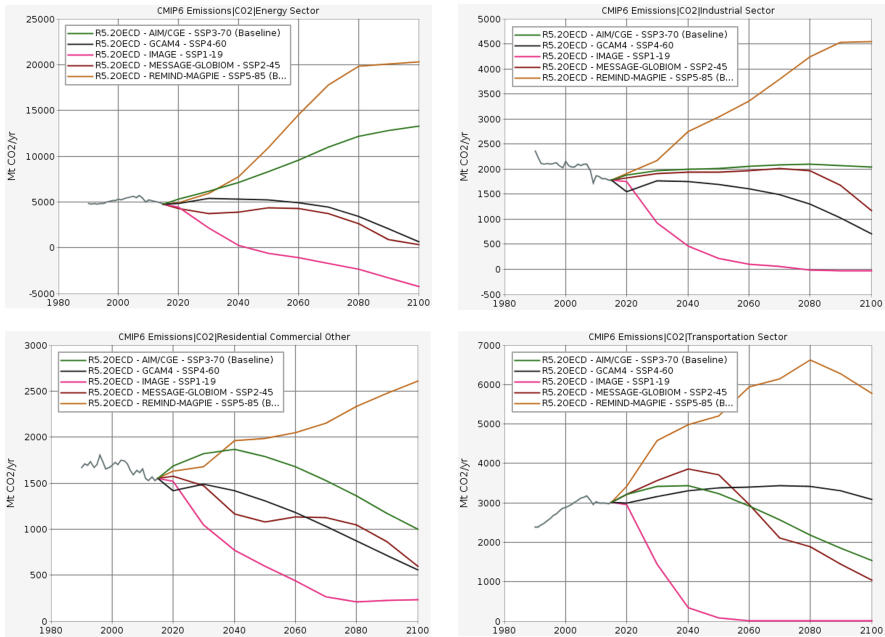


Fig. 1 Historical and scenario-based CO₂ emission, from 1980 to 2100, in Mt/yr in the OECD, according to the activity sectors: Energy (top left), Industry (top right), Residential Commercial (bottom left), and Transportation (bottom right). Source: Project CMIP 6, <https://tntcat.iiasa.ac.at/SspDb>.

by Mark Carney, the former Bank of England Governor [8]. Very quickly, regulators and financial institutions took up the issue and gathered under the aegis of the Network for the Greening of the Financial System (NGFS) [31]. The Basel Committee on Banking Supervision published on April 2021 a report [3] exploring how climate-related risk drivers, including physical risks and transition risks, can arise and affect both banks and the banking system via micro and macro-economic transmission channels. In this work, we mainly focus on the transition risks and explore how to link climate projection scenarios such as those described in the SSPs by Phase 6 of the Coupled Model Intercomparison Project (CMIP 6) into credit risk projections. More specifically, we provide a quantitative model where the inputs are some desired paths of CO₂ emission (e_t) _{$t \geq 0$} , the production characteristics of a company, its sensitivity to CO₂ emissions, its climate-free credit spread, and the outputs are the stochastic evolution of credit spread in an uncertain commercial demand. We consider a firm that aims to maximize its production profit and at the same time takes into consideration the CO₂ reduction plan described by SSPs. Over-emission compared to the target may induce a penalty. From the firm's perspective, the objective is to determine the optimal strategy for its effective emission by solving a penalized optimization problem. The credit quality of the firm can be impacted by such a carbon emission transition via its cash flow. In the classic structural credit approach of Merton [29] or Black–Cox [5], a default event

occurs when the firm's value is below its debt level. We describe the firm's value process as the discounted value of all its future cash flow according to the optimal emission and production profit and study the corresponding default probability.

The optimal production problem to maximize the expected profit of a company is well studied, see for example Guo and Pham [19]. We suppose that the firm's production depends on its energy consumption and in particular on the carbon emission level. We consider, in addition, the over-emission penalty under certain probabilistic risk measure constraints in the spirit of Föllmer and Leukert [14], where a loss function is concerned. The optimal emission strategy is obtained by adopting Pontryagin's maximum principle approach [35] which is a standard method in optimal control theory, especially for convex-type features. We then follow the classic structural models in credit risk to compute the climate-related default probability of the firm and show by numerical examples the impact of different relevant parameters and SSPs scenarios on the default probabilities and intensities. In this paper, we focus on the impact of transition risk and ignore at this stage the possible physical risk and related losses which may be caused by extreme weather. In other words, the joint influence of both transition and physical risks is not in the scope of this paper and will be explored in future works. Suggestions about how to combine both are given in section 3.

State of the art.

Quantitative modeling of the climate change impact on banking is in its infancy, see [7] for a review of the challenges. Several works have performed a thorough qualitative analysis, see for example [2], [30], [10]. We differ from them by providing a more quantitative description of the transmission mechanism from transition risk to credit risk. Very recently, NGFS has reported that transition scenarii should significantly impact the Gross Domestic Product [33], and presumably the credit risk [32] but without precise quantitative assessment for the latter: these references highlight the primary importance of the topics. In [16], the authors proposed an extended (top-down) approach to the usual credit models by incorporating new factors for physical and transition risks, with suggestions for estimating these extra factors' characteristics. On the other hand, bottom-up approaches attempt to start from financial statements and balance sheets (to be impacted by shocks or climate scenarios), to derive the firm's value, and then infer its default probability and credit risk. Very recently, a statistical approach has been conducted by the European Central Bank [1] on 4 million companies by calibrating a multivariate econometric model. Although intuitively meaningful, it leads to disappointing results on the default probability (the R squared is about 11.9%, see [1, Table 4, p.85]). This bottom-up approach has been developed by [4] in the first reference article on climate stress tests, see also [36] for the Dutch banking system. In [6], the authors study the credit risk sensitivity of nearly 800 companies to the price of carbon and look closely at the impact on the balance sheet. Such an

analysis implicitly assumes a lack of adaptation of the firm to carbon prices, see also the recent literature review summarized in [6, Figure 3]. The latter paper is presumably the closest to ours, but still, our contribution is different. Dealing also with a bottom-up approach, we model the firm's response (by optimizing its production accordingly) to a target CO₂ trajectory and an announced climate policy. The firm's response (see Proposition 2) may depend on its activity sector and parameters. As often pointed out in this field, to enable climate risk management, some actors need a simple but meaningful model (provided here) with a small number of features and parameters, as the data is often deficient or provided at an excessively low granularity.

Organization of the paper.

The models and the main results are presented in section 2. The production model and the optimization problem are introduced in section 2.1 while the optimal emission strategies are presented in section 2.2, first in the general case, and then in a special case where a closed-form formula can be obtained. Section 2.3 focuses on credit risk and default probability. Finally, numerical applications are given in section 3 illustrating the main results.

2 Model and results

2.1 Production and carbon emission constraint

Let us fix a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfying the usual assumptions. We consider a firm whose production $P = (P_t, t \geq 0)$ depends on its energy consumption and in particular on its effective CO₂-emission volume $\gamma = (\gamma_t, t \geq 0)$, and solves the following stochastic differential equation (SDE),

$$dP_t = P_t (\mu(t, P_t, \gamma_t) dt + \sigma dW_t), \quad P_0 > 0, \quad (1)$$

where γ_t is the instantaneous emission rate at time t , $W = (W_t, t \geq 0)$ is an \mathbb{F} -Brownian motion representing the demand and supply uncertainty for the production and σ is a positive constant volatility parameter. The emission rate is in general positive. The function $\mu : (t, x, y) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$ characterizes the production rate and satisfies the local Lipschitz condition on x , i.e., for all $t \geq 0$,

$$\forall y \in \mathbb{R}, \forall x, x' \in \mathbb{R}_+, \quad |\mu(t, x, y) - \mu(t, x', y)| \leq K|x - x'|, \quad (2)$$

for an independent positive constant K . We suppose in addition that for any $t \in \mathbb{R}_+$, $\mu(t, \cdot, \cdot)$ is of class C^1 . Empirical studies show that overproduction can lead to a decrease in production rate, for example, due to excess supply, whereas the effect of emissions on production growth is positive (see, e.g., [24]). We thus suppose that μ is decreasing with respect to the production P and increasing with respect to the emission γ .

In response to the greenhouse gas emission reduction target, the firm has an objective emission plan described by $e = (e_t, t \geq 0)$, which represents the emission benchmark suggested by an SSP projection or accredited to the firm by the European Commission. This quantity could be deterministic or stochastic, meaning that the allowances are fixed or recommended with a certain tolerance by regulation. The firm must monitor its emission evolution trajectory γ_t compared to the objective benchmark e_t at any time $t \geq 0$. In this paper, we consider the effective emission which is in general supposed to be positive. However, in other contexts such as carbon sequestration, or the emission allowance compensation within the European Emission Trading System (ETS), the SSPs may take negative values (see, e.g., Figure 1 Energy sector). In other words, the objective emission e_t may be negative, meaning that it has to be considered as net emission, once all processes related to compensation and carbon capture have been taken into account. We suppose, in addition, that e is bounded: this technical condition is anything but a restriction in practice. Exceeding the allocated benchmark can induce penalty losses to the firm as it may need to pay for the carbon tax or to purchase an extra quantity of emission allocation through ETS. In the meantime, if the effective emission is lower than the reference value, then the firm can obtain a certain form of award for its effort from regulators to stimulate additional future actions¹. These regulation rules suggest measuring the impact of effective emission transition trajectory on the firm's financial plan by using a loss function related to risk measures. In the literature, we often focus on the downside risk by using loss functions such as expected shortfall at a terminal date, see for example [11, 14]. In our case, we may also take into account the possibility of an upside award so that the loss function can take both positive and negative values, as in [15, section 4.9]. More precisely, concerning the regulation risk constraints, the evolving emission trajectory is tracked continuously and the penalty is described by a loss function $\ell : \mathbb{R} \rightarrow \mathbb{R}$ which is an increasing and convex function with standardization condition $\ell(0) = 0$. In addition, we assume the following technical assumption that the loss function ℓ is of quadratic growth at most, i.e., $\ell(x) = \mathcal{O}(|x|^2)$ as $|x| \rightarrow +\infty$.

The firm's goal is to maximize its production profit and, at the same time, manage its effective emission level by taking into account the advertised constraints. We let the instantaneous profit of the firm be described by a function $\pi : \mathbb{R}_+ \rightarrow \mathbb{R}$ on the production P . We suppose that it is increasing and concave, and belongs to C^1 , the class of all differentiable functions whose derivative is continuous, as in [19]. In addition, π satisfies the Inada conditions, i.e., $\lim_{x \rightarrow 0^+} \pi'(x) = +\infty$ and $\lim_{x \rightarrow +\infty} \pi'(x) = 0$, which is a standard assumption on production functions in economic growth theory, initially proposed by Inada [22].

We also consider the firm's production cost function $\mathcal{C} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ on the effective emission which represents the firm's energy-related costs such as supply and use, and the technical reform cost to reduce the emission during its

¹see https://ec.europa.eu/clima/policies/effort/regulation_en.

production process. Naturally, the cost function \mathcal{C} is increasing with respect to emission. In addition, we suppose that \mathcal{C} is convex, which is consistent with the principle that the marginal cost is higher and higher with the emissions because we use less and less efficient and more and more emitting energy sources. At this stage, we do not consider other types of costs such as the labor cost of the firm.

Given a benchmark emission projection, the firm chooses its optimal effective emission to maximize the expected profit by controlling the related production, cost, and emission constraints. The optimization problem is presented as below. We consider the profit maximization over all future time under the pathwise emission constraint and define the objective optimization function as

$$J(\gamma) := \mathbb{E} \left[\int_0^\infty e^{-rt} (\pi(P_t) - \mathcal{C}(\gamma_t) - \ell(\gamma_t - e_t)) dt \right] \quad (3)$$

where $r > 0$ is a constant² discount rate. We aim to solve

$$\hat{J} = \sup_{\gamma \in \mathcal{A}} J(\gamma) \quad (4)$$

where \mathcal{A} is the admissible strategy set for the positive progressively measurable processes γ such that for some $\eta \in (0, r)$,

$$\mathbb{E} \left[\int_0^\infty e^{-\eta t} \gamma_t^2 dt \right] < +\infty.$$

Note that a bounded emission process γ , as in Proposition 2, automatically fulfills the integrability condition above. We may also consider a fixed horizon time such as 2050, the year by which the COP wishes the world to be carbon-neutral. In this case, besides the path-wise constraint, an extra penalty function could be included for the total cumulative emission at the terminal time (see Remark 1 for details).

2.2 Optimal emission strategy

In this section, we solve the optimization problems and characterize the optimal effective emission. We begin by using Pontryagin's maximum principle approach [35]. It provides a general set of necessary conditions for the optimal strategy based on perturbation arguments (first-order optimality condition). We then present an explicit model where the optimal emission strategy is obtained in closed form.

²This rate could be a deterministic function of time without significantly changing the following results.

2.2.1 Stochastic Pontryagin's maximum principle

We now deal with the optimization problem (4). From (1), the log-production $p := \log P$ solves

$$dp_t = \bar{\mu}(t, p_t, \gamma_t)dt + \sigma dW_t, \quad (5)$$

where the drift coefficient satisfies $\bar{\mu}(t, x, y) := \mu(t, e^x, y) - \frac{1}{2}\sigma^2$ for every $t \in \mathbb{R}_+$, $x \in \mathbb{R}$, $y \in \mathbb{R}_+$. We then define accordingly, for every $x \in \mathbb{R}$, the auxiliary cost function $\bar{\pi}(x) := \pi(e^x)$ so that

$$\pi(P_t) = \bar{\pi}(p_t). \quad (6)$$

We characterize the optimal effective emission and we are willingly a bit vague about the growth assumptions on the functions ℓ , \mathcal{C} , π , and μ (and their derivatives). These assumptions and some monotone conditions play an important role in justifying the next computations related to Pontryagin's maximum principle and ensuring the existence of the control and adjoint process Y in a suitable weighted L_2 space for the infinite horizon problem, see for instance the book of [34] for a broad account on the subject. The assumptions are similar to those of Proposition 1. The take-home message from the following derivation is rather to get the optimality condition equation characterizing the optimal emission in a fairly general situation. Checking these technical conditions would be done in a second step, according to the cost and dynamics coefficients.

Proposition 1. For any $\gamma \in \mathcal{A}$, let $Y(\gamma)$ denote the conditional expectation

$$Y_t(\gamma) = \mathbb{E} \left[\int_t^\infty e^{-ru + \int_t^u \partial_x \bar{\mu}(t, p_s, \gamma_s) ds} \bar{\pi}'(p_u) du \mid \mathcal{F}_t \right], \quad t \geq 0 \quad (7)$$

which is an \mathbb{F} -adapted càdlàg process and is supposed to satisfy $\mathbb{E} \left[\int_0^\infty Y_t(\gamma)^2 dt \right] < +\infty$. If $\hat{\gamma}$ is an optimal control, then

$$J(\hat{\gamma}) = \sup_{\gamma \in \mathcal{A}} J(\gamma),$$

so that it solves

$$\mathcal{C}'(\hat{\gamma}_t) + \ell'(\hat{\gamma}_t - e_t) = e^{rt} \partial_y \bar{\mu}(t, \hat{p}_t, \hat{\gamma}_t) Y_t(\hat{\gamma}), \quad dt \otimes d\mathbb{P} \text{ a.e.}, \quad (8)$$

where $d\hat{p}_t = \bar{\mu}(t, \hat{p}_t, \hat{\gamma}_t) dt + \sigma dW_t$.

Proof For two given controls $\gamma, \tilde{\gamma} \in \mathcal{A}$, define the directional derivatives for the log-production as

$$\dot{p}_t := \left. \partial_\varepsilon p_t^{\gamma + \varepsilon \tilde{\gamma}} \right|_{\varepsilon=0}.$$

From (5), it holds that

$$d\dot{p}_t = (\partial_x \bar{\mu}(t, p_t, \gamma_t) \dot{p}_t + \partial_y \bar{\mu}(t, p_t, \gamma_t) \tilde{\gamma}_t) dt, \quad \dot{p}_0 = 0. \quad (9)$$

Similarly, we define the objective value function

$$\dot{J}(\gamma, \tilde{\gamma}) := \partial_\varepsilon J(\gamma + \varepsilon \tilde{\gamma})|_{\varepsilon=0},$$

which from (3) equals

$$\dot{J}(\gamma, \tilde{\gamma}) = \mathbb{E} \left[\int_0^\infty e^{-rt} (\bar{\pi}'(p_t) \dot{p}_t - \mathcal{C}'(\gamma_t) \tilde{\gamma}_t - \ell'(\gamma_t - e_t) \tilde{\gamma}_t) dt \right]. \quad (10)$$

By the definition of $Y_t(\gamma)$, the process

$$M_t = Y_t(\gamma) + \int_0^t e^{-ru + \int_t^u \partial_x \bar{\mu}(t, p_s, \gamma_s) ds} \bar{\pi}'(p_u) du, \quad t \geq 0$$

is an \mathbb{F} -martingale and

$$\lim_{t \rightarrow +\infty} Y_t(\gamma) = 0.$$

Observing by (9) that $Y_0(\gamma) \dot{p}_0 = 0$ and that \dot{p} is a finite-variation process, we apply Ito's formula on $Y_t(\gamma) \dot{p}_t$,

$$\begin{aligned} 0 &= \mathbb{E} \left[\int_0^\infty (Y_t(\gamma) d\dot{p}_t + \dot{p}_t dY_t(\gamma)) \right] \\ &= \mathbb{E} \left[\int_0^\infty Y_t(\gamma) (\partial_x \bar{\mu}(t, p_t, \gamma_t) \dot{p}_t + \partial_y \bar{\mu}(t, p_t, \gamma_t) \tilde{\gamma}_t) dt \right. \\ &\quad \left. - \int_0^\infty \dot{p}_t (e^{-rt} \bar{\pi}'(p_t) + \partial_x \bar{\mu}(t, p_t, \gamma_t) Y_t(\gamma)) dt \right] \\ &= \mathbb{E} \left[\int_0^\infty Y_t(\gamma) \partial_y \bar{\mu}(t, p_t, \gamma_t) \tilde{\gamma}_t dt - \int_0^\infty e^{-rt} \bar{\pi}'(p_t) \dot{p}_t dt \right], \end{aligned}$$

using that $\mathbb{E} \left[\int_0^\infty \dot{p}_t dM_t \right] = 0$, which implies that

$$\mathbb{E} \left[\int_0^\infty e^{-rt} \bar{\pi}'(p_t) \dot{p}_t dt \right] = \mathbb{E} \left[\int_0^\infty Y_t(\gamma) \partial_y \bar{\mu}(t, p_t, \gamma_t) \tilde{\gamma}_t dt \right].$$

We can then rewrite (10) as

$$\dot{J}(\gamma, \tilde{\gamma}) = \mathbb{E} \left[\int_0^\infty (\partial_y \bar{\mu}(t, p_t, \gamma_t) Y_t(\gamma) - e^{-rt} \mathcal{C}'(\gamma_t) - e^{-rt} \ell'(\gamma_t - e_t)) \tilde{\gamma}_t dt \right],$$

which holds for any control process $\tilde{\gamma} \in \mathcal{A}$. Consequently, if $\hat{\gamma}$ is an optimal control, then necessarily $dt \otimes d\mathbb{P}$ a.e.

$$\partial_y \bar{\mu}(t, \hat{p}_t, \hat{\gamma}_t) Y_t(\hat{\gamma}) = e^{-rt} (\mathcal{C}'(\hat{\gamma}_t) + \ell'(\hat{\gamma}_t - e_t))$$

where $Y_t(\hat{\gamma})$ is written as

$$Y_t(\hat{\gamma}) = \mathbb{E} \left[\int_t^\infty e^{-ru + \int_t^u \partial_x \bar{\mu}(t, \hat{p}_s, \hat{\gamma}_s) ds} \bar{\pi}'(\hat{p}_u) du \mid \mathcal{F}_t \right].$$

In other words, $\hat{\gamma}$ is the solution to (8), which ends the proof. \square

Remark 1. We can consider an alternative problem with a finite horizon $T > 0$ and introduce the effective cumulative emission process up to time t as $\Gamma_t := \int_0^t \gamma_s ds$. It is then possible to incorporate a final regulation for the cumulative emission compared to the target $E_t = \int_0^t e_s ds$ at the horizon time. The objective function is then defined as

$$J_T(\gamma) := \mathbb{E} \left[\int_0^T e^{-rt} (\pi(P_t) - \mathcal{C}(\gamma_t) - \ell_1(\gamma_t - e_t)) dt - e^{-rT} \ell_2(\Gamma_T - E_T) \right], \quad (11)$$

where ℓ_1 and ℓ_2 are two loss functions and the optimization problem becomes

$$\widehat{J}_T = \sup_{\gamma \in \mathcal{A}_T} J_T(\gamma), \quad (12)$$

where \mathcal{A}_T is the admissible strategy set such that for any $T > 0$, $\mathbb{E}[\Gamma_T^2] < +\infty$, and that for any $x \geq 0$, $\int_0^T |\mu(t, x, \gamma_t)|^2 dt < +\infty$ almost surely. As Proposition 1, the solution to the finite time horizon can be characterized as below. The difference lies in the extra terminal constraint. More precisely, defining

$$Y_t^1(\gamma) = \mathbb{E} \left[\int_t^T e^{-ru + \int_t^u \partial_x \bar{\mu}(t, p_s, \gamma_s) ds} \bar{\pi}'(p_u) du \mid \mathcal{F}_t \right].$$

If $\widehat{\gamma}$ is an optimal solution to (12) then it must satisfy

$$\begin{aligned} e^{-rt} (\mathcal{C}'(\widehat{\gamma}_t) + \ell_1'(\widehat{\gamma}_t - e_t)) + \mathbb{E} [e^{-rT} \ell_2'(\widehat{\Gamma}_T - E_T) \mid \mathcal{F}_t] \\ = \partial_y \bar{\mu}(t, \widehat{p}_t, \widehat{\gamma}_t) Y_t^1(\widehat{\gamma}). \end{aligned} \quad (13)$$

2.2.2 Profit maximization in an explicit model

We now consider an explicit model with logarithmic profit function $\pi(x) = \log x$ and we let the cost and penalty be quadratic functions given respectively as

$$\mathcal{C}(x) = \frac{x^2}{2} \quad \text{and} \quad \ell(x) = \omega \frac{(x_+)^2}{2}. \quad (14)$$

The coefficient $\omega \geq 0$ characterizes the penalty force of the CO₂ emission constraint and the function x_+ denotes $\max(x, 0)$. We choose a quadratic penalty function to accentuate higher quantities of over-emission compared to the benchmark. The objective function (3) rewrites as

$$J(\gamma) = \mathbb{E} \left[\int_0^\infty e^{-rt} \left(\log P_t - \frac{\gamma_t^2}{2} - \omega \frac{(\gamma_t - e_t)_+^2}{2} \right) dt \right]. \quad (15)$$

We also assume that the drift function for the log-production (5) to have is affine, that is

$$\bar{\mu}(t, x, y) = a + bx + cy, \quad t \in \mathbb{R}_+, x \in \mathbb{R}, y \in \mathbb{R}_+. \quad (16)$$

The coefficient $a \geq 0$ corresponds to an average production level, $b \leq 0$ is a mean-reverting parameter with a negative sign meaning that over-production may deteriorate the production ability, and $c \geq 0$ represents the firm's production dependence on CO₂ emission.

Remark 2 (Comparison with the literature). As explained, the model above is inspired by intuitive modelling arguments. Some models in the literature also support this choice. For instance, in [9, section 1.3.2.], the authors, while

explaining the different mechanisms of carbon markets, propose a simple model similar to ours for the firm's profit. Using our notations, they assume that the firm's profit (see (25) for our expression) is

$$V(\gamma) = a + b\gamma - \gamma^2 - \theta\gamma.$$

The first linear term is linear in the emission γ as they also (see (16)) assume that a higher production generates more emission, and thus increases revenue. The third term corresponds to the cost of production, assumed also (see (14)) to be quadratic in production levels, and directly correlated with emissions. θ refers to the carbon price.

Proposition 2. Suppose $r > b$ and consider the optimization problem (15)-(16). The optimal emission strategy $\hat{\gamma}$ is given by

$$\hat{\gamma}_t = \min \left\{ \frac{c}{r-b}, \frac{1}{1+\omega} \left(\omega e_t + \frac{c}{r-b} \right) \right\}. \quad (17)$$

Proof From Proposition 1, we obtain that the conditional expectation (7) is a deterministic function and is given by $Y_t(\gamma) = e^{-rt}/(r-b)$ for any $\gamma \in \mathcal{A}$. Then, solving (8) leads to an explicit form for $\hat{\gamma}_t$, see (17).

We verify below that $\hat{\gamma}_t$ is indeed an optimal strategy for the optimization problem. Let $\gamma \in \mathcal{A}$ be any admissible strategy to the problem (15). Then

$$J(\hat{\gamma}) - J(\gamma) = \mathbb{E} \left[\int_0^\infty e^{-rt} \left((\hat{p}_t - p_t) - \frac{1}{2}(\hat{\gamma}_t^2 - \gamma_t^2) - \frac{\omega}{2}((\hat{\gamma}_t - e_t)_+^2 - (\gamma_t - e_t)_+^2) \right) dt \right] \quad (18)$$

where \hat{p}_t is given by

$$d\hat{p}_t = (a + b\hat{p}_t + c\hat{\gamma}_t)dt + \sigma dW_t.$$

By Ito's formula, we have

$$d(e^{-rt}(\hat{p}_t - p_t)) = e^{-rt} \left((b-r)(\hat{p}_t - p_t) + c(\hat{\gamma}_t - \gamma_t) \right) dt. \quad (19)$$

Since $\gamma \in \mathcal{A}$ and

$$\mathbb{E} \left[\left(\int_0^\infty e^{-\eta t} |\gamma_t| dt \right)^2 \right] \leq \left(\int_0^\infty (e^{-\eta t/2})^2 dt \right) \mathbb{E} \left[\int_0^\infty (e^{-\eta t/2} |\gamma_t|)^2 dt \right] < +\infty \quad (20)$$

for some $\eta \in (0, r)$, the integral $\int_0^\infty e^{-\eta t} |\gamma_t| dt$ is finite almost surely, the same holds obviously for $\hat{\gamma}$ since the benchmark e is bounded. By solving the linear equation (19), we get

$$\begin{aligned} e^{-rt}(\hat{p}_t - p_t) &= e^{-rt} \int_0^t e^{b(t-s)} c(\hat{\gamma}_s - \gamma_s) ds, \\ |e^{-rt}(\hat{p}_t - p_t)| &\leq ce^{-rt} \int_0^t |\hat{\gamma}_s - \gamma_s| ds \leq ce^{(\eta-r)t} \int_0^{+\infty} e^{-\eta s} |\hat{\gamma}_s - \gamma_s| ds \end{aligned}$$

using $b \leq 0$. Hence, since $r > \eta$,

$$\lim_{t \rightarrow +\infty} e^{-rt}(\hat{p}_t - p_t) = 0, \text{ a.s.}$$

which leads to

$$\int_0^\infty e^{-rt} \left((b-r)(\hat{p}_t - p_t) + c(\hat{\gamma}_t - \gamma_t) \right) dt = 0, \text{ a.s.} \quad (21)$$

owing to (19).

So (18) can be rewritten as

$$\begin{aligned} & J(\hat{\gamma}) - J(\gamma) \\ &= \mathbb{E} \left[\int_0^\infty e^{-rt} \left(\frac{c}{r-b} (\hat{\gamma}_t - \gamma_t) - \frac{1}{2} (\hat{\gamma}_t^2 - \gamma_t^2) - \frac{\omega}{2} ((\hat{\gamma}_t - e_t)_+^2 - (\gamma_t - e_t)_+^2) \right) dt \right] \\ &= \mathbb{E} \left[\int_0^\infty e^{-rt} \left(f(\hat{\gamma}_t, e_t) - f(\gamma_t, e_t) \right) dt \right] \end{aligned}$$

where

$$f(x, e) := \frac{c}{r-b} x - \frac{x^2}{2} - \frac{\omega}{2} (x - e)_+^2. \quad (22)$$

We note that for a given e , the maximal value of $x \mapsto f(x, e)$ is attained at

$$x = \min \left\{ \frac{c}{r-b}, \frac{1}{1+\omega} \left(\omega e + \frac{c}{r-b} \right) \right\},$$

which implies that $J(\hat{\gamma}) - J(\gamma) \geq 0$ for any $\gamma \in \mathcal{A}$ and concludes the proof. \square

Remark 3 (On the estimation of the model's parameters). Assume that we are given an n -dimensional vector of the firm's log-production $(p_{t_i})_{i=1, \dots, n}$ along with its carbon emission $(\gamma_{t_i})_{i=1, \dots, n}$. The typical time-step $t_{i+1} - t_i$ would be one year. Such data can be retrieved through standard financial data providers: this is easy for companies with a typical flagship product (for example, a car manufacturer), this is more complicated for companies with numerous sales products. Then, estimating a, b, c, σ in (16) boils down to estimate the parameters of a linear model, using for instance log-likelihood type methods such as the Shoji–Ozaki method, see [20, section 2.3.]; see also [25] for an overview of statistical methods for stochastic processes. Observe that getting a, b, c is equivalent to a regression problem of the response $(p_{t_{i+1}} - p_{t_i})$ using the co-variables p_{t_i}, γ_{t_i} .

In Proposition 2, we note that the constant value

$$\bar{\gamma} := \frac{c}{r-b} \quad (23)$$

appearing in (17), corresponds to a critical level for the firm's optimal emission strategy. It represents its desired emission level without any carbon penalty, i.e., when $\omega = 0$.

- If the benchmark e_t is above $\bar{\gamma}$, then the optimal strategy is to remain at the constant level $\bar{\gamma}$. No efforts are required for the company. In particular, when $c = 0$, the firm's production does not depend on emission at all, and the optimal emission is expected to be zero.
- On the contrary, when e_t is below $\bar{\gamma}$, the regulation requires a stricter emission reduction plan. The optimal strategy is then an affine function of the benchmark.

The penalty weight ω also plays an essential role. A larger value of ω implies that the firm will adopt a stronger mitigation strategy. In particular, when ω tends to infinity, the optimal emission converges to the benchmark.

Remark 4. The positive function ℓ defined in (14) focuses on the downside penalty corresponding to the out-performance part of the effective emission compared to the SSP benchmark. It is possible to describe the compensation or award when the firm's emission is below the benchmark. In such a case, one may consider instead a more general function ℓ that can take both positive and negative values such as

$$\ell(x) = \omega_1 \frac{(x_+)^2}{2} - \omega_2 \frac{(x_-)^2}{2}$$

where ω_1 and ω_2 are two positive real numbers and $x_- = \max(-x, 0)$. We note that when $\omega_2 < 1$, $x \mapsto \ell(x) + \frac{1}{2}x^2$ is a convex function as assumed in section 2.1. Similar to the proof of Proposition 2, we replace the function (22) by

$$f(x, e) := \frac{c}{r-b}x - \frac{x^2}{2} - \frac{\omega_1}{2}(x-e)_+^2 + \frac{\omega_2}{2}(x-e)_-^2.$$

Then, when $0 \leq \omega_2 < 1$, the explicit optimal effective emission reads as

$$\hat{\gamma}_t = \begin{cases} \frac{1}{1+\omega_1}(\omega_1 e_t + \bar{\gamma}), & \text{if } e_t \leq \bar{\gamma}, \\ \frac{1}{1-\omega_2}(\bar{\gamma} - \omega_2 e_t), & \text{if } e_t > \bar{\gamma}. \end{cases} \quad (24)$$

In particular, when e_t is above $\bar{\gamma}$, the optimal emission $\hat{\gamma}$ is mitigated with respect to $\bar{\gamma}$ (instead of remaining constant $\bar{\gamma}$ as in Proposition 2) when the reward force parameter ω_2 becomes strictly positive, i.e., $\omega_2 \in (0, 1)$, since

$$\frac{1}{1-\omega_2}(\bar{\gamma} - \omega_2 e_t) < \bar{\gamma} < e_t.$$

The comparison with Proposition 2 shows that adding the reward mechanism when the firm achieves better mitigation results (besides the penalty upon over-emission) can urge firms to attain carbon neutrality.

2.3 Credit risk under emission transition

In this section, we study the firm's credit risk induced by the transition towards the low-carbon emission and production pattern. We use the effective production obtained in the previous optimization problem to deduce the firm's value process and then compute the default probability in a structural modeling approach.

2.3.1 Firm's value process

In the classical structural credit models such as Merton [29] and Black-Cox [5] models, a default event is triggered when the firm's value no longer covers the debt reimbursement. In the following, we describe the value process of the firm by using the discounted cash flow approach which traces back to Keynesian economics. More precisely, let the firm's value $V^\gamma = (V_t^\gamma, t \geq 0)$ be defined as the conditional discounted value of all future cash flows including the global production income deduced by the cost and penalty depending on the effective emission γ , that is

$$V_t^\gamma = \mathbb{E} \left[\int_t^\infty e^{-r(u-t)} (\pi(P_u) - \mathcal{C}(\gamma_u) - \ell(\gamma_u - e_u)) du \mid \mathcal{F}_t \right]. \quad (25)$$

At the initial time $t = 0$, we have $V_0^\gamma = J(\gamma)$. The firm would produce according to the optimal production quantity associated with the emission strategy $\hat{\gamma}$ obtained from the maximization procedure in Propositions 1 or 2.

For any $\nu \in \mathcal{A}$ and any $t \geq 0$, the firm's dynamic optimal value (viewed at time t) is given by

$$\begin{aligned} \hat{V}_t(\gamma) &= \operatorname{ess\,sup}_{\gamma \in \mathcal{A}(t, \nu)} V_t^\gamma \\ &= \operatorname{ess\,sup}_{\gamma \in \mathcal{A}(t, \nu)} \mathbb{E} \left[\int_t^\infty e^{-r(u-t)} (\bar{\pi}(p_u) - \mathcal{C}(\gamma_u) - \ell(\gamma_u - e_u)) du \mid \mathcal{F}_t \right] \end{aligned} \quad (26)$$

where the set $\mathcal{A}(t, \nu) := \{\gamma \in \mathcal{A} \text{ such that } \gamma_{\cdot \wedge t} = \nu_{\cdot \wedge t}\}$ represents all controls that coincide with ν up to time t . By the dynamic programming principle exposed in El Karoui and Quenez [13], we have that for any $\gamma \in \mathcal{A}$, the process $\left(\hat{V}_t(\gamma) + \int_0^t e^{-r(u-t)} (\bar{\pi}(p_u) - \mathcal{C}(\gamma_u) - \ell(\gamma_u - e_u)) du, t \geq 0 \right)$ is a supermartingale. In particular, for an optimal $\hat{\gamma}$, $\left(\hat{V}_t(\hat{\gamma}) + \int_0^t e^{-r(u-t)} (\bar{\pi}(\hat{p}_u) - \mathcal{C}(\hat{\gamma}_u) - \ell(\hat{\gamma}_u - e_u)) du, t \geq 0 \right)$ is a martingale. The value process associated with $\hat{\gamma}$ by Proposition 1 or 2 corresponds to the firm's optimal value.

Proposition 3. Let $\hat{\gamma}$ be an optimal emission strategy solving (3). Then, for any $t \geq 0$,

$$\mathbb{E} \left[\hat{V}_t(\hat{\gamma}) \right] = \mathbb{E} \left[V_t^{\hat{\gamma}} \right]. \quad (27)$$

Proof By definition of the essential supremum in (26), we have $\mathbb{E}[\hat{V}_t(\hat{\gamma})] \geq \mathbb{E}[V_t^{\hat{\gamma}}]$ for any $t \geq 0$. To prove the converse inequality, fix $t \geq 0$ and observe that for any

$\gamma \in \mathcal{A}(t, \hat{\gamma})$,

$$\begin{aligned} & \mathbb{E} \left[V_t^\gamma + \int_0^t e^{-r(u-t)} (\bar{\pi}(p_u) - \mathcal{C}(\gamma_u) - \ell(\gamma_u - e_u)) du \right] \\ &= \mathbb{E} \left[\int_0^\infty e^{-r(u-t)} (\bar{\pi}(p_u) - \mathcal{C}(\gamma_u) - \ell(\gamma_u - e_u)) du \right] \\ &\leq \mathbb{E} \left[\int_0^\infty e^{-r(u-t)} (\bar{\pi}(\hat{p}_u) - \mathcal{C}(\hat{\gamma}_u) - \ell(\hat{\gamma}_u - e_u)) du \right] \end{aligned}$$

where the last inequality stems from the optimality of $\hat{\gamma}$ in the optimization problem (3). By the definition of $\mathcal{A}(t, \hat{\gamma})$, it holds $\hat{\gamma}_s = \gamma_s$ for any $s \leq t$, we then obtain from the above inequality

$$\mathbb{E}[V_t^\gamma] \leq \mathbb{E} \left[\int_t^\infty e^{-r(u-t)} (\bar{\pi}(\hat{p}_u) - \mathcal{C}(\hat{\gamma}_u) - \ell(\hat{\gamma}_u - e_u)) du \right] = \mathbb{E}[V_t^{\hat{\gamma}}], \quad (28)$$

since

$$\begin{aligned} & \mathbb{E} \left[\int_0^t e^{-r(u-t)} (\bar{\pi}(p_u) - \mathcal{C}(\gamma_u) - \ell(\gamma_u - e_u)) du \right] \\ &= \mathbb{E} \left[\int_0^t e^{-r(u-t)} (\bar{\pi}(\hat{p}_u) - \mathcal{C}(\hat{\gamma}_u) - \ell(\hat{\gamma}_u - e_u)) du \right]. \end{aligned}$$

Note by (26) that $\mathbb{E}[\hat{V}_t(\hat{\gamma})] = \mathbb{E}[\text{ess sup}_{\gamma \in \mathcal{A}(t, \hat{\gamma})} V_t^\gamma]$. By using a measurable selection argument (see, e.g., [40]), for any $\epsilon > 0$, there exists $\gamma_\epsilon \in \mathcal{A}(t, \hat{\gamma})$ such that

$$\text{ess sup}_{\gamma \in \mathcal{A}(t, \hat{\gamma})} V_t^\gamma \leq V_t^{\gamma_\epsilon} + \epsilon.$$

We then have from (28) that

$$\mathbb{E} \left[\text{ess sup}_{\gamma \in \mathcal{A}(t, \hat{\gamma})} V_t^\gamma \right] \leq \mathbb{E} [V_t^{\gamma_\epsilon}] + \epsilon \leq \mathbb{E}[V_t^{\hat{\gamma}}] + \epsilon.$$

From the arbitrariness of $\epsilon > 0$, we obtain the following inequality

$$\mathbb{E} [\hat{V}_t(\hat{\gamma})] = \mathbb{E} \left[\text{ess sup}_{\gamma \in \mathcal{A}(t, \hat{\gamma})} V_t^\gamma \right] \leq \mathbb{E}[V_t^{\hat{\gamma}}],$$

and hence the required equality (27). \square

2.3.2 Structural credit model and default probability

In a structural credit model, the firm is considered to default when its value process gets below a default threshold. We let the default barrier be described by a deterministic function $L(t)$ which depends on time and represents the minimal level of the firm's liability payment such as the debt reimbursement together with labor and other functioning costs (as the operational and capital expenditure) at time t . When the firm's value is higher than the threshold, it is in a financially sustainable situation and can function normally. On the contrary, when the firm encounters fiscal difficulty, a default event may be triggered. We are interested in the probability of default (PD in short) at a certain date t , defined as

$$\text{PD}_t = \mathbb{P} \left(V_t^{\hat{\gamma}} \leq L(t) \right). \quad (29)$$

This corresponds to the classic Merton model where the default-or-not state of the firm is determined by the instantaneous value of the firm's value and the default threshold. From a regulation viewpoint, financial and insurance institutions are required by the Solvency II Directive to evaluate their solvency risk and associated default probability at a certain given time. From a computational viewpoint, the default probability depends on the distribution of the firm's value process $V^{\hat{\gamma}}$ at time t . In the next section, we give quasi-explicit formulas for PD_t in the model of Proposition 2. It is expressed as a function of the SSP describing the target emission scenario $(e_t)_{t \geq 0}$.

In other cases such as the Black–Cox model, the default event depends on the trajectory of the value process. At a given time t , the firm defaults if its value has crossed the threshold during the period $[0, t]$, that is,

$$\text{PD}_t = \mathbb{P} \left(\exists s \in [0, t] \text{ such that } V_s^{\hat{\gamma}} \leq L(s) \right). \quad (30)$$

We are then concerned with the random default time or first-hitting time τ with respect to the filtration \mathbb{F} given as

$$\tau := \inf \{ t \geq 0, V_t^{\hat{\gamma}} \leq L(t) \}, \quad (31)$$

with the convention $\inf \emptyset = \infty$. In this case, the first-passage time to the curved boundary is a deterministic function of time. Such problems are studied in theoretical and applied probability theory using different approaches (see for example the book of [26]). Since explicit expressions for the probability density of the first-passage time can be obtained only in very specific cases, a computational approach is often required either to approximate the density function, e.g., [12] or to simulate directly the first-passage time, e.g., [18, 21].

2.3.3 Default probability in the explicit model

We now present the emission-related default probability in the explicit model as in Proposition 2. We use here a slightly different definition for the value process associated with the optimal emission $V^{\hat{\gamma}}$. Instead of choosing a logarithmic function, the firm's profit is defined as the average price $N > 0$ multiplied by the total production so that

$$\hat{V}_t = \mathbb{E} \left[\int_t^\infty e^{-r(u-t)} \left(N \hat{P}_u - \mathcal{C}(\hat{\gamma}_u) - \ell(\hat{\gamma}_u - e_u) \right) du \mid \mathcal{F}_t \right]. \quad (32)$$

Suppose that $\hat{V}_0 > L(0)$ and recall the expression for the probability of default (29). We aim to compute the firm's value (32) by specifying a benchmark e_t . Note that if the benchmark is given as a deterministic function, such as in an SSP scenario, then from (17), the optimal emission $\hat{\gamma}$ is also deterministic. We

then have

$$\widehat{V}_t = N \int_t^\infty e^{-r(u-t)} \mathbb{E}[\widehat{P}_u | \mathcal{F}_t] du - \int_t^\infty e^{-r(u-t)} (\mathcal{C}(\widehat{\gamma}_u) + \ell(\widehat{\gamma}_u - e_u)) du. \quad (33)$$

From (5) and (16), the optimal log-production rewrites as

$$\widehat{p}_u = e^{b(u-t)} \widehat{p}_t + \frac{a}{b} (e^{b(u-t)} - 1) + c \int_t^u e^{b(u-v)} \widehat{\gamma}_v dv + \sigma \int_t^u e^{b(u-v)} dW_v.$$

Hence for every $u \geq t \geq 0$, conditionally on \mathcal{F}_t , the optimal log-production \widehat{p}_u is Gaussian with mean

$$e^{b(u-t)} \widehat{p}_t + \frac{a}{b} (e^{b(u-t)} - 1) + c \int_t^u e^{b(u-v)} \widehat{\gamma}_v dv =: e^{b(u-t)} \widehat{p}_t + m_{u,t},$$

and variance

$$\sigma_{u,t}^2 := \frac{\sigma^2}{2b} (e^{2b(u-t)} - 1).$$

Therefore, concerning the optimal production \widehat{P} , we have

$$\mathbb{E}[\widehat{P}_u | \mathcal{F}_t] = \exp \left(e^{b(u-t)} \widehat{p}_t + m_{u,t} + \frac{\sigma_{u,t}^2}{2} \right).$$

From (33), the firm's value with an optimal emission is thus given by

$$\begin{aligned} \widehat{V}_t &= N \int_t^\infty e^{-r(u-t)} \exp \left(e^{b(u-t)} \widehat{p}_t + m_{u,t} + \frac{\sigma_{u,t}^2}{2} \right) du \\ &\quad - \int_t^\infty \frac{e^{-r(u-t)}}{2} (\widehat{\gamma}_u^2 + \omega ((\widehat{\gamma}_u - e_u)_+)^2) du \\ &=: h(t, \widehat{p}_t). \end{aligned} \quad (34)$$

We can then compute the emission-related default probabilities with the explicit form of the firm's value \widehat{V} according to the Merton model. Here, the default probability rewrites as

$$\begin{aligned} \text{PD}_t &= \mathbb{P}(\widehat{V}_t \leq L(t)) = \mathbb{P}(\widehat{p}_t \leq (h(t, \cdot))^{-1}(L(t))) \\ &= \Phi \left(\frac{(h(t, \cdot))^{-1}(L(t)) - e^{bt} p_0 - m_{t,0}}{\sigma_{t,0}} \right), \end{aligned} \quad (35)$$

where Φ is the cumulative distribution function of a standard normal random variable.

3 Numerical illustrations

3.1 SSPs scenarios and optimal emission

We consider the following CO₂ emission scenarios which correspond to different socioeconomic reference pathways provided by CMIP6: SSP1-2.6, SSP2-4.5, SSP3-LowNTCF, SSP4-6.0, and SSP5-3.4-OS. These scenarios are illustrative pathways adopted by the IPCC in the sixth Assessment Report indicating the CO₂ concentrations in the atmosphere from lowest (SSP1) to highest (SSP5). In other words, SSP1 and SSP5 describe respectively economic growth patterns via sustainable and fossil-fuel pathways. We choose two sectors: Transportation (Figure 2) and Industrial (Figure 3) sectors for which the year 2015 is our starting point. For each sector, we consider the above five SSPs including two baseline scenarios (Tier 1, c.f. [27]): SSP1-2.6 which is the most mitigated scenario corresponding approximately to the previous scenario generation Representative Concentration Pathway (RCP) 2.6, and SSP2-4.5 with is a moderate scenario similar to RCP-4.5. We also consider three (Tier 2) supplementary scenarios: SSP3-LowNTCF (Near-Term Climate Forcing) which provides a comparison scenario with high NTCF emissions (notably SO_x and methane), SSP4-6.0 focusing on a socio-economic context of inequality, and SSP5-34-OS (Overshoot) which allows for a large overshoot by mid-century followed by substantive policy tools in the latter half of the century.

The numerical computations are based on the explicit model of Proposition 2. We start with the transportation sector. The different emission benchmarks (left) $t \mapsto e_t$ are normalized so that $e_0 = \hat{\gamma}_0 = \frac{c}{r-b} (= \bar{\gamma})$. The associated optimal emission (right) is provided by (17) which is equal to $\bar{\gamma}$ if the benchmark is above and is given as an affine function of e_t on the contrary case. As one can observe, for both sectors, the scenario SSP1-2.6 is the hardest benchmark which imposes immediate reduction from the start date of 2015. For the transportation sector, the scenario SSP4-6.0 imposes no emission constraints and the optimal emission remains constant at level $\bar{\gamma}$. The situation is similar to the scenario SSP3-LowNTCF in the industrial sector. Other scenarios correspond to relatively soft emission constraints with overshoot where the reduction may begin from a later date. Figure 4 shows the impact of the parameter ω . More precisely, the optimal emission is decreasing with respect to the penalty force. In other words, a stronger penalty policy will induce larger emission reductions from the firm.

3.2 Default probability and intensity

We now plot for different sectors the default probability PD_t given by (35). For our experiments, the default boundary $L(t)$ is determined as follows. We consider the firm's default probability without climate concern (that is, $\omega = 0$) as a baseline where the default intensity λ_{ref} is supposed to be a fixed reference

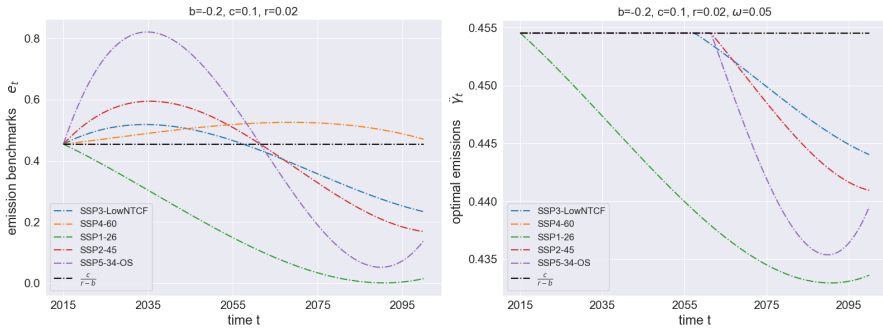


Fig. 2 Transportation sector: SSPs emission scenarios v.s. associated optimal emission.



Fig. 3 Industrial sector: SSPs emission scenarios v.s. associated optimal emission .

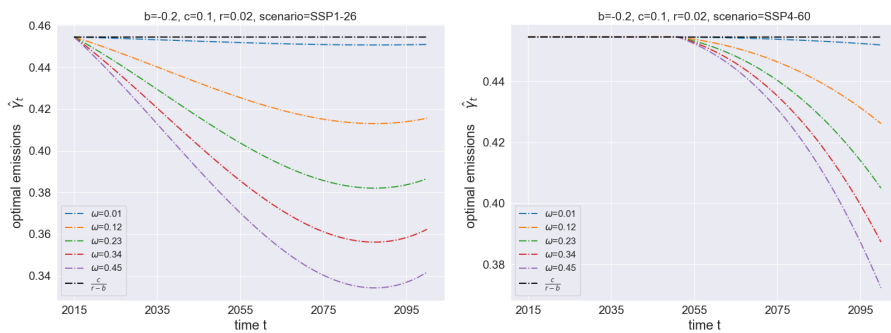


Fig. 4 Industrial sector: different optimal emissions w.r.t. ω for scenarios SSP1-26 and SSP4-60.

value in our numerical tests and is set at 3%. More precisely, we let

$$\mathbb{P}(\widehat{V}_t^{\text{ref}} \leq L(t)) = 1 - e^{-\lambda_{\text{ref}} t},$$

where $\widehat{V}_t^{\text{ref}}$ corresponds to the optimal value without any emission constraint, i.e., $\omega = 0$, so that the value of $L(t)$ is obtained. We have by (34) that

$$L(t) = h \left(t, \Phi^{-1}(1 - e^{-\lambda_{\text{ref}} t}) \sigma_{t,0} + \frac{e^{bt} - 1}{b} \left(a + \frac{c^2}{r - b} \right) + e^{bt} p_0 \right),$$

using that $m_{t,0} = \frac{e^{bt} - 1}{b} \left(a + \frac{c^2}{r - b} \right)$ when $\omega = 0$ and $\sigma_{t,0}$ is independent of e_t .

In Figures 5 and 6, we plot for the two sectors Transportation and Industrial the default probability PD_t given by (35) and the respective intensity $t \mapsto \lambda_t$ for the different scenarios. The intensity λ_t is computed as

$$\mathbb{P}(\widehat{V}_t \leq L(t)) = 1 - \exp \left(- \int_0^t \lambda_s ds \right).$$

The integral $\int_0^t \lambda_s ds$ is approximated with a left-point rectangle scheme, i.e., between two times $0 \leq s_{i-1} < s_i \leq t$, we have

$$\lambda_{s_{i-1}} = \frac{1}{s_i - s_{i-1}} \log \left(\frac{\mathbb{P}(\widehat{V}_{s_{i-1}} > L(s_{i-1}))}{\mathbb{P}(\widehat{V}_{s_i} > L(s_i))} \right).$$

As expected the initial value of the intensity coincides with the prefixed value $\lambda_0 = 3\%$. When time evolves, the more constrained scenarios are associated with larger default probabilities and higher intensities. The SSP1-2.6 scenario is the most impacted one, which is quite natural given its immediate and hard reduction strategy. The scenario which follows is SSP5-3.4-OS: although this benchmark allows for a large overshoot up to 2060, the relatively strict mitigation during the latter period makes the default probability and intensity increase significantly. Observe that SSP4-6.0 corresponds to a fixed intensity of $\lambda_0 = 3\%$ in the Transportation sector and the same phenomenon appears for the SSP3-LowNTCF scenario in the Industrial sector, as the optimal emission is unconstrained in these two cases. We note that in this study we only investigate the transition risk related to the firm's mitigation strategy and ignore the possible physical risks under each scenario for example the more frequent damage and natural catastrophes under scenarios with higher temperature increases such as an SSP5 scenario (see for example [1] for more discussions). Although physical climate risk is not the main modeling concern of this work, it could easily be incorporated consistently with our approach, by adding a suitable physical climate damage contribution in the firm's value (32)-(34), to broadly account for climate risks.

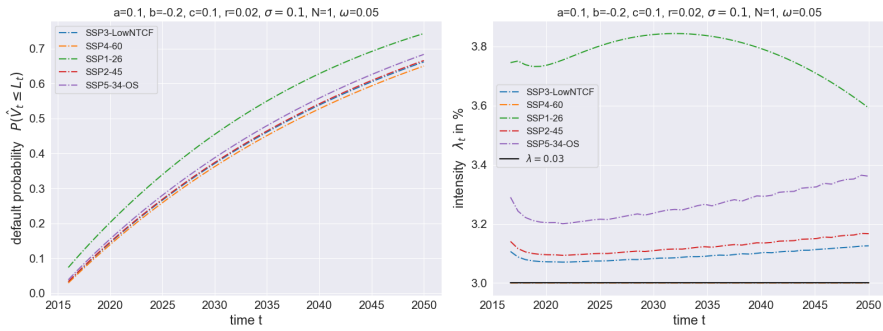


Fig. 5 Transportation sector: default probability and associated intensity.

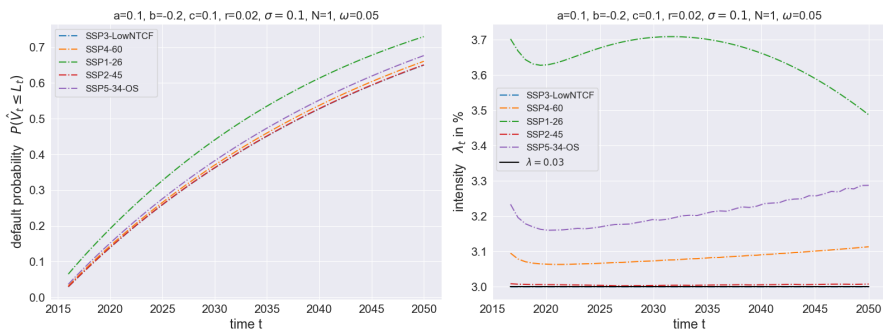


Fig. 6 Industrial sector: default probability and associated intensity.

Figure 7 illustrates the default intensity for the industrial sector under the impact of parameter c representing the firm's dependence on emission and ω representing the penalty force. We consider the hardest scenario SSP1-2.6 and a moderate one SSP4-6.0. For both scenarios, the increase of one of the two parameters implies a higher default intensity and this phenomenon is particularly accentuated when the firm's production is highly dependent on the CO₂ emission (when c is large). For such a firm, a strong penalty policy together with a hard mitigation scenario such as SSP1-2.6 (left) could have a significant impact on the firm's default probability.

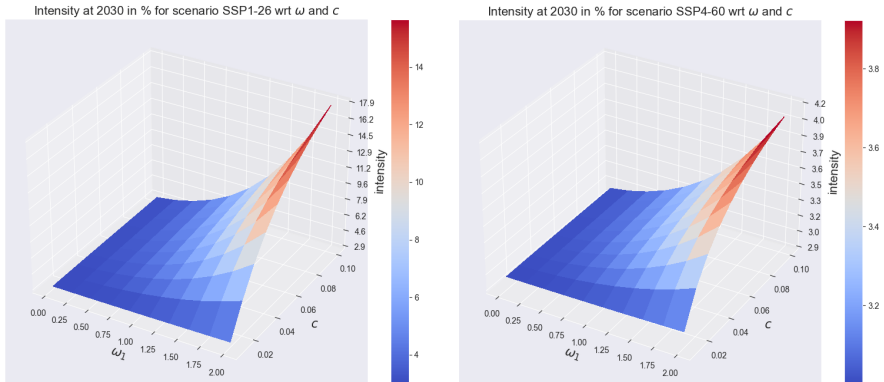


Fig. 7 Industrial sector: intensity surface at 2030 for the scenarios SSP1-26 and SSP4-60

We finally present in Figure 8 the default intensity obtained in the path-dependent Black-Cox model (30) in comparison with the Merton model (29): since in the Black-Cox approach the default event is monitored throughout a given period and not only at the end, this naturally gives rise to a more important measure of credit risk. For our experiments, the default event is considered at annual intermediate dates, and the default probability is then computed as a discrete approximation of (30) given by the following definition of default probability:

$$t \mapsto \mathbb{P} \left(\exists s = t_0, t_1, \dots, t_n \in [0, t] : \widehat{V}_s \leq L(s) \right) = \text{PD}_t.$$

The Black-Cox default intensity is then obtained as

$$1 - \text{PD}_{t_{k+1}} = (1 - \text{PD}_{t_k}) \exp(-\lambda_{t_k}(t_{k+1} - t_k)).$$

The two scenarios in Figure 8 are SSP1-2.6 and SSP5-3.4-OS (industrial sector) which are the most impacted scenarios by the mitigation strategy (as shown in Figure 6). As expected, we observe that the possibility to include intermediate default will induce a gap – for both scenarios – between the default probabilities and intensities obtained in the two credit models. Anyway, choosing a Merton or Black-Cox approach is up to the end-user of credit risk; in our “SSP emission-to-firm production” modeling both are possible and allow in all cases to bridge a projected transition scenario and the credit risk of a company in a given sector.

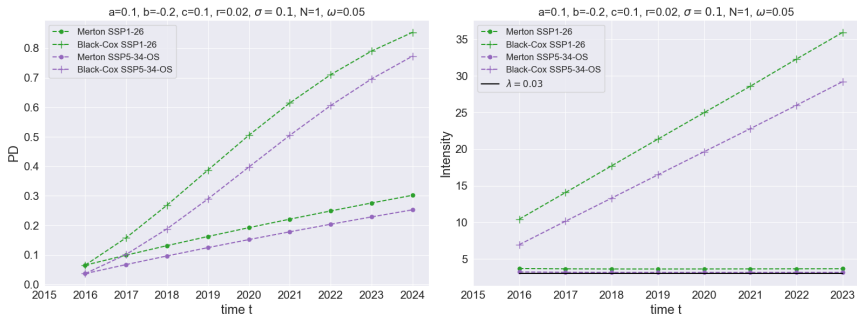


Fig. 8 Industrial sector: Black-Cox default probability and associated intensity for the industrial sector.

4 Conclusion

This paper focuses on a firm's transition risk towards low-carbon production and shows how to quantify the credit risk embedded in the CO₂ emission mitigation strategy. We consider a profit maximization problem under some loss constraints (related to CO₂ emission) and then use the obtained optimal emission to construct the firm's value process and investigate default probabilities and intensities under different SSPs scenarios. An explicit model of production is proposed to derive closed-form computation for related quantities: this is a valuable building block in the climate risk-credit risk nexus.

We conclude that the loss penalty imposed on over-emission compared to the benchmark may urge the firm to decrease its effective CO₂ emission. The adopted strategy depends on the relevant SSP scenario and the firm's production characteristics. A suitable choice for the loss function, such as incorporating some kind of reward when under-emitting, appears to stimulate more efficiently the firm to achieve the carbon-neutral objective. Concerning credit risk, the more constrained mitigation scenarios are associated with higher default probability and intensity, especially for firms whose production is closely related to emission and when the penalty force is strong. A higher value of penalty force enforces the firm to perform a larger emission reduction but increases its default risk.

In this work, we do not investigate the physical risk related to different SSPs scenarios and their impact on the credit quality and default probability: however, the model we have proposed could be adapted to analyze the presence of joint impact by both physical and transition risks, see remarks in section 3. Moreover, in the numerical examples, we have presented, we consider the reference intensity to be constant for illustration, it could straightforwardly be extended to account for the credit intensity curve. A more comprehensive default model including stochastic intensity could also be more realistic. These points are left to future works.

Statements and Declarations

This action benefited from the support of the Chair Stress Test, Risk Management and Financial Steering, led by the French École Polytechnique and its foundation and sponsored by BNP Paribas. The views expressed in this document are the author's own and do not necessarily reflect those endorsed by BNP Paribas. Ying Jiao also thanks Institut Europlace de Finance for partial support.

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