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# STABILITY OF EQUIVARIANT LOGARITHMIC TANGENT SHEAVES ON TORIC VARIETIES OF PICARD RANK TWO

ACHIM NAPAME

ABSTRACT. For an equivariant log pair  $(X, D)$  where  $X$  is a normal toric variety and  $D$  a reduced Weil divisor, we study slope-stability of the logarithmic tangent sheaf  $\mathcal{T}_X(-\log D)$ . We give a complete description of divisors  $D$  and polarizations  $L$  such that  $\mathcal{T}_X(-\log D)$  is (semi)stable with respect to  $L$  when  $X$  has a Picard rank one or two.

## 1. INTRODUCTION

The notion of slope-stability for vector bundles was first introduced by Mumford [Mum63] in his attempt to construct the moduli spaces of vector bundles over a curve. This notion was generalized in higher dimension by Takemoto [Tak72]. The study of stability of coherent sheaves over polarized varieties is a difficult problem. This problem is simplified when additional structures are added on the sheaf.

A quasi-coherent sheaf  $\mathcal{E}$  on a toric variety  $X$  with torus  $T$  is said to be an *equivariant sheaf* if it admits a lift of the  $T$ -action on  $X$ , which is linear on the stalks of  $\mathcal{E}$ . Klyachko in [Kly90] gave a complete classification of equivariant vector bundles over toric varieties in terms of a family of filtrations of vector spaces. This classification was generalized in the case of quasi-coherent sheaves on toric varieties by Perling in [Per04].

By using the equivariant structure of the tangent bundle, Hering-Nill-Süss in [HNS22] and Dasgupta-Dey-Khan in [DDK20] studied slope-stability of the tangent bundle  $TX$  of a smooth projective toric variety  $X$  of Picard rank one or two. Inspired by *Itaka's philosophy*, in this paper, we extend the result of [HNS22] and [DDK20] to the case of log pairs  $(X, D)$ . More precisely, if  $X$  is a normal toric variety and  $D$  a reduced Weil divisor such that the logarithmic tangent sheaf  $\mathcal{T}_X(-\log D)$  is equivariant, we are interested by the set of polarizations  $L$  on  $X$  such that  $\mathcal{T}_X(-\log D)$  is (semi)stable with respect to  $L$ .

Assume that  $X$  is a Kähler manifold and  $D = \sum_i a_i D_i$  a simple normal crossing divisor with  $a_i \in [0; 1] \cap \mathbb{Q}$ . If  $K_X + D$  is ample, there exists a Kähler-Einstein metric  $\omega$  on the pair  $(X, D)$  by [BG14, Theorem C]; moreover, if  $D$  is reduced, then the logarithmic tangent sheaf  $\mathcal{T}_X(-\log D)$  is semi-stable with respect to  $K_X + D$  by [Gue16, Theorem A]. If the pair  $(X, D)$  is Fano, by [Ber16, Section 4.3], the pair  $(X, D)$  admits a Kähler-Einstein metric  $\omega$  if for all  $i$ ,  $0 < a_i < 1$ ; in this case,  $(X, D)$  is  $K$ -polystable. According to [Li20, Theorem 1.4], the orbifold tangent sheaf  $\mathcal{T}_X(-\log D)$  is slope semi-stable with respect to  $-(K_X + D)$ .

In this paper, we are interested in the case where  $a_i \in \{0, 1\}$ . According to [BB13, Section 1.2.1 and Section 3], when  $(X, D)$  is Fano, we don't have a natural notion of Kähler-Einstein metric on  $(X, D)$ . We will use another method to study the slope-stability of  $\mathcal{T}_X(-\log D)$ .

In Section 2, we recall some notions about toric varieties and equivariant sheaves. Let  $X$  be a normal toric variety over  $\mathbb{C}$  and  $T$  be its torus. We denote by  $\Sigma$  the fan of  $X$  and  $\Sigma(1)$  the set of rays of  $\Sigma$ . In Proposition 2.8, we show that : for a

reduced Weil divisor  $D$ , the logarithmic tangent sheaf  $\mathcal{T}_X(-\log D)$  is equivariant if and only if

$$D = \sum_{\rho \in \Delta} D_\rho$$

where  $\Delta$  is a subset of  $\Sigma(1)$  and  $D_\rho$  the Zariski closure of the orbit  $O(\rho)$  corresponding to the ray  $\rho$ . Using the family of multifiltrations defined in [Per04, Section 5], we give in Section 3.1 the family of multifiltrations corresponding to the equivariant sheaf  $\mathcal{T}_X(-\log D)$ . In Section 3.2, we give some conditions on  $\Sigma$  and  $\Delta$ , such that  $\mathcal{T}_X(-\log D)$  is decomposable. In Section 4, we recall some notions about slope-stability and we show that :

**Proposition 1.1.** *Let  $X$  be a toric variety without torus factor and  $p$  be the rank of its class group  $\text{Cl}(X)$ . If*

$$1 + p \leq \text{card}(\Delta) \leq \dim(X) + p - 1$$

*then  $\mathcal{T}_X(-\log D)$  is not semi-stable with respect to any polarizations.*

By Corollary 3.6, if  $\Delta = \emptyset$ , then  $\mathcal{T}_X(-\log D)$  is the tangent sheaf  $\mathcal{T}_X$  and when  $\Delta = \Sigma(1)$ ,  $\mathcal{T}_X(-\log D)$  is isomorphic to the trivial vector bundle of rank  $\dim X$ . If  $X$  has no torus factor, we have  $\text{card}(\Sigma(1)) = \dim(X) + p$  by [CLS11, Theorem 4.1.3]. According to Proposition 1.1, it is therefore sufficient to study the stability of  $\mathcal{T}_X(-\log D)$  when  $\text{card}(\Delta) \leq p$ . In this paper we will be interested in the case where  $p \in \{1, 2\}$ . As the case  $\text{card}(\Delta) = 0$  corresponds to the tangent sheaf, we will study the slope-stability of  $\mathcal{T}_X(-\log D)$  when  $1 \leq \text{card}(\Delta) \leq p$ .

In Section 4.4, we study slope-stability of  $\mathcal{T}_X(-\log D)$  when  $X$  is a toric orbifold of Picard rank one. If  $X = \mathbb{P}^n$  and  $\Delta \subset \Sigma(1)$ , we show that :

**Proposition 1.2.** *If  $\text{card}(\Delta) = 1$ , then  $\mathcal{T}_X(-\log D)$  is poly-stable but not stable with respect to  $\mathcal{O}_{\mathbb{P}^n}(1)$ .*

The Sections 5 and 6 form the core of this paper. In these sections, we study slope-stability of  $\mathcal{T}_X(-\log D)$  when  $X$  is a smooth toric variety of Picard rank 2. In the last part (Section 7), we apply different results of this paper to study the stability of  $\mathcal{T}_X(-\log D)$  with respect to  $-(K_X + D)$  when  $X$  is a toric surface and  $D$  a divisor such that  $-(K_X + D)$  is ample.

**1.1. Smooth toric varieties of Picard rank two.** Let  $X$  be a smooth toric variety of dimension  $n$  with fan  $\Sigma$  in  $\mathbb{R}^n$  such that  $\text{rk Pic}(X) = 2$ . By [CLS11, Theorem 7.3.7] due to Kleinschmidt (see [Kle88]), there exists  $r, s \in \mathbb{N}^*$  with  $r + s = n$  and  $a_1, \dots, a_r \in \mathbb{N}$  with  $a_1 \leq a_2 \leq \dots \leq a_r$  such that

$$(1) \quad X = \mathbb{P} \left( \mathcal{O}_{\mathbb{P}^s} \oplus \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^s}(a_i) \right).$$

We denote by  $\pi : X \rightarrow \mathbb{P}^s$  the projection to the base  $\mathbb{P}^s$ . By [CLS11, Section 7.3], the rays of  $\Sigma$  are given by the half-lines generated by  $w_0, w_1, \dots, w_s, v_0, v_1, \dots, v_r$  where  $(w_1, \dots, w_s)$  is the standard basis of  $\mathbb{Z}^s \times 0_{\mathbb{Z}^r}$ ,  $(v_1, \dots, v_r)$  the standard basis of  $0_{\mathbb{Z}^s} \times \mathbb{Z}^r$ ,

$$v_0 = -(v_1 + \dots + v_r) \quad \text{and} \quad w_0 = a_1 v_1 + \dots + a_r v_r - (w_1 + \dots + w_s).$$

We denote by  $D_{v_i}$  the divisor corresponding to the ray  $\text{Cone}(v_i)$  and  $D_{w_j}$  the divisor corresponding to the ray  $\text{Cone}(w_j)$ .

**Notation 1.3.** Let  $\text{Amp}(X) \subset N^1(X) \otimes_{\mathbb{Z}} \mathbb{R}$  be the ample cone of  $X$ . We define :

$$\text{Stab}(\mathcal{T}_X(-\log D)) = \left\{ L \in \text{Amp}(X) : \mathcal{T}_X(-\log(D)) \right. \\ \left. \text{is stable with respect to } L \right\}$$

and

$$\text{sStab}(\mathcal{T}_X(-\log D)) = \left\{ \begin{array}{l} L \in \text{Amp}(X) : \mathcal{T}_X(-\log(D)) \\ \text{is semi-stable with respect to } L \end{array} \right\}.$$

**1.2. Main results.** Let  $X$  be a smooth toric variety of Picard rank two and  $L = \pi^* \mathcal{O}_{\mathbb{P}^s}(\mu) \otimes \mathcal{O}_X(\lambda)$  be a polarization on  $X$ . We set  $\nu = \frac{\mu}{\lambda}$ . In Section 5, more precisely in Proposition 5.10, we give a criterion that allows to verify the stability of  $\mathcal{T}_X(-\log D)$ . For all divisors  $D = \sum_{\rho \in \Delta} D_\rho$  with  $1 \leq \text{card}(\Delta) \leq 2$ , we give in tables 1, 2, 3 and 4 the values of  $\nu$  for which  $\mathcal{T}_X(-\log D)$  is (semi)stable with respect to  $L$ . We give here some results.

If  $a_1 = \dots = a_r = 0$ , the variety  $X$  given in (1) is  $\mathbb{P}^s \times \mathbb{P}^r$ . Let

$$\mathcal{D}_1 = \{D_{v_i} : 0 \leq i \leq r\} \cup \{D_{w_j} : 0 \leq j \leq s\} \cup \{D_{v_i} + D_{w_j} : 0 \leq i \leq r, 0 \leq j \leq s\}$$

and

$$\mathcal{D}_2 = \{D_{v_i} + D_{v_j} : 0 \leq i < j \leq r\} \cup \{D_{w_i} + D_{w_j} : 0 \leq i < j \leq s\}.$$

We have :

**Theorem 1.4.** *If  $D \in \mathcal{D}_1 \cup \mathcal{D}_2$ , then  $\text{Stab}(\mathcal{T}_X(-\log D)) = \emptyset$ . But :*

- (1) *if  $D \in \mathcal{D}_2$ ,  $\text{sStab}(\mathcal{T}_X(-\log D)) = \emptyset$  ;*
- (2) *if  $D \in \mathcal{D}_1$ ,  $\mathcal{T}_X(-\log D)$  is poly-stable with respect to  $L$  if and only if  $L$  is a power of the polarization corresponding to  $-(K_X + D)$ .*

We now consider the general case, that is  $X$  given by (1) with  $a_r \geq 1$ .

**Theorem 1.5** (Stability of  $\mathcal{T}_X(-\log D_{v_r})$ ). *We have  $\text{Stab}(\mathcal{T}_X(-\log D_{v_r})) \neq \emptyset$  if and only if  $\text{sStab}(\mathcal{T}_X(-\log D_{v_r})) \neq \emptyset$  if and only if  $a_r = 1$  and  $a_{r-1} = 0$ . If  $a_r = 1$  and  $a_{r-1} = 0$ , then the logarithmic tangent sheaf  $\mathcal{T}_X(-\log D_{v_r})$  is stable (resp. semi-stable) with respect to  $\pi^* \mathcal{O}_{\mathbb{P}^s}(\mu) \otimes \mathcal{O}_X(\lambda)$  if and only if  $0 < \frac{\mu}{\lambda} < \nu_0$  (resp.  $0 < \frac{\mu}{\lambda} \leq \nu_0$ ) where  $\nu_0$  is the unique positive root of*

$$P_0(x) = \sum_{k=0}^{s-1} \binom{s+r-1}{k} x^k - s \binom{s+r-1}{s} x^s.$$

If  $a_r \geq 1$ , we see that this theorem is similar to [HNS22, Theorem 1.4]. If we fix  $r$  and  $s$ , there is only one smooth toric variety with Picard rank two such that  $\text{Stab}(\mathcal{T}_X(-\log D_{v_r})) \neq \emptyset$  or  $\text{Stab}(\mathcal{T}_X) \neq \emptyset$ ; this variety is given by (1) with  $a_1 = \dots = a_{r-1} = 0$  and  $a_r = 1$ . If  $a_r \geq 1$  and  $(a_1, \dots, a_r) \neq (0, \dots, 0, 1)$ , we have  $\text{sStab}(\mathcal{T}_X) = \emptyset$  by [HNS22, Theorem 1.4]. In the case of log pairs, with some conditions on the  $a_i$ , there exists another divisor  $D$  such that  $\text{Stab}(\mathcal{T}_X(-\log D)) \neq \emptyset$ .

**Theorem 1.6** (Stability of  $\mathcal{T}_X(-\log D_{v_0})$ ). *We assume that  $a_r \geq 1$ .*

- (1) *If  $r = 1$ , then  $\text{Stab}(\mathcal{T}_X(-\log D_{v_0})) \neq \emptyset$ .*
- (2) *If  $r \geq 2$ , then  $\text{Stab}(\mathcal{T}_X(-\log D_{v_0})) \neq \emptyset$  if and only if  $a_1 = \dots = a_r$  and  $a_r < \frac{s+1}{r-1}$ .*

If  $D \notin \{D_{v_0}, D_{v_r}\}$ , then  $\mathcal{T}_X(-\log D)$  is not stable with respect to any polarizations.

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## 2. TORIC VARIETIES AND EQUIVARIANT LOGARITHMIC TANGENT SHEAVES

**2.1. Toric varieties.** We give here some notions about toric varieties. We refer to [CLS11, Chapter 2, 3] and [CT22, Section 2.1].

**Definition 2.1** ([CLS11, Definition 3.1.1]). A *toric variety* is an irreducible variety  $X$  containing a torus  $T \simeq (\mathbb{C}^*)^n$  as a Zariski open subset such that the action of  $T$  on itself extends to an algebraic action of  $T$  on  $X$ .

Let  $N$  be a rank  $n$  lattice and  $M = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$  be its dual with pairing  $\langle \cdot, \cdot \rangle : M \times N \rightarrow \mathbb{Z}$ . The lattice  $N$  is the lattice of one-parameter subgroups of  $T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^*$ . For  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , we define  $N_{\mathbb{K}} = N \otimes_{\mathbb{Z}} \mathbb{K}$  and  $M_{\mathbb{K}} = M \otimes_{\mathbb{Z}} \mathbb{K}$ . For  $m \in M$ , we associate the character  $\chi^m : T_N \rightarrow \mathbb{C}^*$  and for  $u \in N$ , we associate the one-parameter subgroup  $\lambda^u : \mathbb{C}^* \rightarrow T_N$ .

**Definition 2.2** ([CLS11, Definition 3.1.2]). A *fan*  $\Sigma$  in  $N_{\mathbb{R}}$  is a finite collection of cones  $\sigma \subseteq N_{\mathbb{R}}$  such that :

- (1) Every  $\sigma \in \Sigma$  is a strongly convex rational polyhedral cone.
- (2) For all  $\sigma \in \Sigma$ , each face of  $\sigma$  is also in  $\Sigma$ .
- (3) For all  $\sigma_1, \sigma_2 \in \Sigma$ , the intersection  $\sigma_1 \cap \sigma_2$  is a face of each.

For a cone  $\sigma$ , we write  $\tau \preceq \sigma$  if  $\tau$  is a face of  $\sigma$ . If  $\Sigma$  is fan, the support of  $\Sigma$  is  $|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma$ . We say that  $\Sigma$  is *complete*, if  $|\Sigma| = N_{\mathbb{R}}$ . We denote by  $\Sigma(r)$  the set of  $r$ -dimensional cones of  $\Sigma$ . We call  $\Sigma(1)$  the set of rays of  $\Sigma$ .

Let  $X_{\Sigma}$  be a toric variety associated to a fan  $\Sigma$ . The variety  $X_{\Sigma}$  is obtained by gluing affine charts  $(U_{\sigma})_{\sigma \in \Sigma}$ , with  $U_{\sigma} = \text{Spec}(\mathbb{C}[S_{\sigma}])$  and  $\mathbb{C}[S_{\sigma}]$  is the semi-group algebra of

$$S_{\sigma} = \sigma^{\vee} \cap M = \{m \in M : \langle m, u \rangle \geq 0 \text{ for all } u \in \sigma\}.$$

By [CLS11, Theorem 3.1.5],  $X_{\Sigma}$  is a normal separated toric variety and its torus is  $T_N$ . From now on, a normal toric variety  $X$  will be defined by a fan  $\Sigma$ .

We denote by  $u_{\rho} \in N$  the minimal generator of  $\rho \in \Sigma(1)$ .

**Definition 2.3** ([CLS11, Proposition 3.3.9]). A normal toric variety  $X$  has a *torus factor* if and only if the  $u_{\rho}, \rho \in \Sigma(1)$  do not span  $N_{\mathbb{R}}$ .

A cone  $\sigma \in \Sigma$  gives the torus orbit  $O(\sigma) \cong \text{Hom}_{\mathbb{Z}}(\sigma^{\perp} \cap M, \mathbb{C}^*)$  where

$$\sigma^{\perp} \cap M = \{m \in M : \langle m, u \rangle = 0 \text{ for all } u \in \sigma\}.$$

We have :

**Theorem 2.4** (Orbit-Cone Correspondence, [CLS11, Theorem 3.2.6]).

- (1) *There is a bijective correspondence*

$$\begin{array}{ccc} \{\text{Cone } \sigma \text{ in } \Sigma\} & \longleftrightarrow & \{T\text{-orbits in } X\} \\ \sigma & \longleftrightarrow & O(\sigma) \end{array}$$

$$\text{and } \dim O(\sigma) = \dim N_{\mathbb{R}} - \dim \sigma.$$

- (2) *The affine open subset  $U_{\sigma}$  is the union of orbits  $U_{\sigma} = \bigcup_{\tau \preceq \sigma} O(\tau)$ .*

- (3)  *$\tau \preceq \sigma$  if and only if  $O(\sigma) \subset \overline{O(\tau)}$ , and  $\overline{O(\tau)} = \bigcup_{\tau \preceq \sigma} O(\sigma)$  where  $\overline{O(\tau)}$  denotes the closure in both the classical and Zariski topologies.*

For  $\rho \in \Sigma(1)$ , we define the  $T$ -invariant Weil divisor  $D_{\rho}$  as the closure in the Zariski topology of  $O(\rho)$ .

**Lemma 2.5** ([CLS11, Section 4.1]). *A Weil divisor  $D$  on  $X$  is invariant under the action of  $T$  if and only if*

$$D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho \quad \text{with } a_\rho \in \mathbb{Z} .$$

**Order Relation.** Let  $\Sigma$  be a fan in  $N_{\mathbb{R}}$  and  $\sigma \in \Sigma$ . We define an order relation  $\preceq_\sigma$  on  $M$  by setting  $m \preceq_\sigma m'$  if and only if  $m' - m \in S_\sigma$ . We write  $m \prec_\sigma m'$  if we have  $m \preceq_\sigma m'$  but not  $m' \preceq_\sigma m$ .

**Group action.** Let  $G$  be an algebraic group acting on the affine toric variety  $Y = \text{Spec}(R)$ . For any  $g \in G$ , let  $\phi_g : Y \rightarrow Y$  be the map defined by  $\phi_g(x) = g \cdot x$ . We define an action of  $G$  on  $R$  by setting : for  $g \in G$  and  $\varphi \in R$ ,

$$g \cdot \varphi = (\phi_{g^{-1}})^* \varphi$$

that is, for any  $y \in Y$ ,  $(g \cdot \varphi)(y) = \varphi(g^{-1} \cdot y)$ .

**2.2. Equivariant sheaves.** We refer to [Per03, Section 2.2.2] for properties about equivariant coherent sheaf. Let  $T$  be the torus of  $X$ .

Let  $\sigma : T \times X \rightarrow X$  be an action of algebraic group  $T$  on  $X$ ,  $\mu : T \times T \rightarrow T$  the group multiplication,  $p_2 : T \times X \rightarrow X$  the projection onto the second factor and  $p_{23} : T \times T \times X \rightarrow T \times X$  the projection onto the second and the third factor.

We call a sheaf  $\mathcal{E}$  on  $X$  *equivariant* or  *$T$ -linearized* if there exists an isomorphism  $\Phi : \sigma^* \mathcal{E} \xrightarrow{\cong} p_2^* \mathcal{E}$  such that

$$(2) \quad (\mu \times \text{Id}_X)^* \Phi = p_{23}^* \Phi \circ (\text{Id}_T \times \sigma)^* \Phi .$$

For  $t \in T$ , let  $\alpha_t : X \rightarrow T \times X$  and  $\phi_t : X \rightarrow X$  be the morphisms given by

$$\alpha_t(x) = (t, x) \quad \text{and} \quad \phi_t(x) = \sigma(t, x) .$$

We get an isomorphism  $\Phi_t := \alpha_t^* \Phi : \phi_t^* \mathcal{E} \xrightarrow{\cong} \mathcal{E}$ . For any  $t, t' \in T$ , the cocycle condition (2) factors as follows :

$$(3) \quad \begin{array}{ccc} (\phi_{t' \cdot t})^* \mathcal{E} & \xrightarrow{\Phi_{t' \cdot t}} & \mathcal{E} \\ & \searrow \phi_t^* \Phi_t & \nearrow \Phi_t \\ & \phi_t^* \mathcal{E} & \end{array}$$

**2.3. Logarithmic tangent sheaves.** We recall here the definition of the logarithmic tangent sheaf of a log pair  $(X, D)$  where  $X$  is a normal projective variety of dimension  $n$  and  $D$  a reduced Weil divisor on  $X$ . We refer to [Gue16, Section 3.1].

**Definition 2.6.** We say that a pair  $(X, D)$  is log-smooth if  $X$  is smooth and  $D$  is reduced snc (simple normal crossing) divisor.

We denote by  $(X, D)_{\text{reg}}$  the snc locus of the pair  $(X, D)$ , that is, the locus of points  $x \in X$  where  $(X, D)$  is log-smooth in a neighborhood of  $x$ .

If  $(X, D)$  is log-smooth, we define the logarithmic tangent bundle  $T_X(-\log D)$  as the dual of the bundle of logarithmic differential form  $\Omega_X^1(\log D)$  where  $\Omega_X^1(\log D)$  is defined in [Iit76, §1]. By [Kaw78, Definition 4] and [Sai77, §1], we can see the space of sections of  $T_X(-\log D)$  as the set of vector fields on  $X$  which vanish along  $D$ . If  $D$  is locally given by  $(z_1 \cdots z_k = 0)$ , then  $T_X(-\log D)$  as a sheaf is the locally free  $\mathcal{O}_X$ -module generated by

$$z_1 \frac{\partial}{\partial z_1}, \dots, z_k \frac{\partial}{\partial z_k}, \frac{\partial}{\partial z_{k+1}}, \dots, \frac{\partial}{\partial z_n} .$$

**Definition 2.7** ([Gue16, Definition 3.4]). Let  $(X, D)$  be a log pair and  $X_0 = (X, D)_{\text{reg}}$ . The logarithmic tangent sheaf of  $(X, D)$ , denoted by  $\mathcal{T}_X(-\log D)$ , is defined as  $j_*T_{X_0}(-\log D|_{X_0})$  where  $j : X_0 \rightarrow X$  is the open immersion.

The sheaf  $\mathcal{T}_X(-\log D)$  (as well as its dual) is coherent ; by [Har80, Proposition 1.6], this sheaf is reflexive.

We now consider the case where  $X$  is a toric variety. We will give a condition on  $D$  such that  $\mathcal{T}_X(-\log D)$  is equivariant. Let  $\Sigma$  be the fan of  $X$  and  $X_0$  the toric variety corresponding to the fan  $\Sigma^1 = \Sigma(0) \cup \Sigma(1)$ . We denote by  $j : X_0 \rightarrow X$  the open immersion.

**Proposition 2.8.** *Let  $D$  be a reduced Weil divisor on  $X$ . The sheaf  $\Omega_X^1(\log D)$  is equivariant if and only if  $D$  is an invariant divisor under the torus action.*

*Proof.* We assume that  $D$  is an invariant divisor under the torus action. Let  $D_0$  be the restriction of  $D$  on  $X_0$ . For  $t \in T$ , let  $\phi_t : X \rightarrow X$  the map defined by  $\phi_t(x) = \sigma(t, x)$  and  $\Phi_t$  the map defined by  $\Phi_t = (d\phi_t)^{-1}$  where  $d\phi_t$  is the differential of  $\phi_t$ . If  $\mathcal{E} = T_{X_0}$ , we have an isomorphism  $\Phi_t : \phi_t^* \mathcal{E} \rightarrow \mathcal{E}$  and the diagram (3) is verified. Now if we replace  $\mathcal{E}$  by  $T_{X_0}(-\log D_0)$ , the diagram (3) remains true ; so  $T_{X_0}(-\log D_0)$  is equivariant. Hence  $\Omega_{X_0}^1(\log D_0)$  is equivariant. As

$$(4) \quad \Omega_X^1(\log D) \cong j_*\Omega_{X_0}^1(\log D_0) ,$$

we deduce that  $\Omega_X^1(\log D)$  is equivariant.

We now assume that  $\Omega_X^1(\log D)$  is equivariant. We write  $D = \sum_{j=1}^s D_j$ .

*First case.* We assume that  $X$  is smooth. By [EV92, Properties 2.3] we have an exact sequence

$$0 \longrightarrow \Omega_X^1 \longrightarrow \Omega_X^1(\log D) \longrightarrow \bigoplus_{j=1}^s \mathcal{O}_{D_j} \longrightarrow 0$$

where  $\mathcal{O}_{D_j}$  is viewed as a sheaf on  $X$  via extension by zero. The first part of the proof is to show that : for any  $t \in T$ ,  $t \cdot Z = Z$  where  $Z = X \setminus D$ .

Let  $x \in Z$  and assume that there is  $t \in T$  such that  $y = \sigma(t, x) \in D$ . We have two exact sequences

$$\begin{aligned} 0 \longrightarrow \Omega_{X,x}^1 \longrightarrow \Omega_X^1(\log D)_x \longrightarrow \bigoplus_{j=1}^s \mathcal{O}_{D_j,x} \longrightarrow 0 \\ 0 \longrightarrow \Omega_{X,y}^1 \longrightarrow \Omega_X^1(\log D)_y \longrightarrow \bigoplus_{j=1}^s \mathcal{O}_{D_j,y} \longrightarrow 0 \end{aligned}$$

As  $\Omega_X^1$  and  $\Omega_X^1(\log D)$  are equivariant, we have an isomorphism

$$\bigoplus_{j=1}^s \mathcal{O}_{D_j,x} \cong \bigoplus_{j=1}^s \mathcal{O}_{D_j,y} ;$$

this is absurd. Therefore, for any  $t \in T$ , we have  $t \cdot Z \subset Z$ , that is  $t \cdot Z = Z$ . As  $\Omega_X^1(\log D)$  is equivariant, by using the fact that  $D = X \setminus Z$ , for any  $t \in T$ , we have  $t \cdot D = D$ ; thus,  $D$  is a  $T$ -invariant divisor.

*Second case.* We assume that  $X$  is a normal variety. By (4), as  $\Omega_X^1(\log D)$  is equivariant, we also have the same property for  $\Omega_{X_0}^1(\log D_0)$ . By the first case,  $D_0$  is an invariant divisor under the action of  $T$  on  $X_0$ . As  $\text{codim}(X \setminus X_0) \geq 2$ , we deduce that  $D$  is the Zariski closure of  $D_0$  on  $X$ . Thus,  $D$  is an invariant divisor under the action of  $T$  on  $X$ .  $\square$

## 3. MULTIFILTRATIONS OF LOGARITHMIC TANGENT SHEAF

**3.1. Family of multifiltrations.** We give here the family of multifiltrations of equivariant logarithmic tangent sheaf on toric varieties. Let  $X$  be a toric variety of dimension  $n$  with fan  $\Sigma$ . We first give some results.

**Proposition 3.1** ([DDK20, Corollary 2.2.17]). *The family  $(\mathcal{T}, \{\mathcal{T}^\rho(i)\}_{\rho \in \Sigma(1), i \in \mathbb{Z}})$  of multifiltrations associated to the tangent sheaf  $\mathcal{T}_X$  are given by*

$$\mathcal{T}^\rho(i) = \begin{cases} 0 & \text{if } i \leq -2 \\ \text{Span}(u_\rho) & \text{if } i = -1 \\ N \otimes_{\mathbb{Z}} \mathbb{C} & \text{if } i \geq 0 \end{cases} .$$

*Remark 3.2.* If  $X$  is smooth, the tangent sheaf  $\mathcal{T}_X$  is the tangent bundle  $TX$ .

**Lemma 3.3.** *For every ray  $\rho \in \Sigma(1)$ , we have  $(U_\rho \cap D_\rho) \cap T = \emptyset$ .*

*Proof.* Let  $\rho \in \Sigma(1)$ . We assume that  $\rho = \text{Cone}(u_1)$  where  $u_1 \in N$ . Let  $(u_1, \dots, u_n)$  be a basis of  $N$  and  $(e_1, \dots, e_n)$  its dual basis. We set  $x_j = \chi^{e_j}$ , we have

$$U_\rho = \text{Spec}(\mathbb{C}[x_1, x_2^{\pm 1}, \dots, x_n^{\pm 1}]) \cong \mathbb{C} \times (\mathbb{C}^*)^{n-1}$$

and

$$T = \text{Spec}(\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]) \cong (\mathbb{C}^*)^n .$$

As on  $U_\rho$  the divisor  $D_\rho$  is defined by  $x_1 = 0$ , we deduce that  $(U_\rho \cap D_\rho) \cap T = \emptyset$ .  $\square$

**Lemma 3.4.** *Let  $\rho_1$  and  $\rho_2$  be two distinct rays of  $\Sigma$ . If  $i \neq j$ , then  $U_{\rho_i} \cap D_{\rho_j} = \emptyset$ .*

*Proof.* By the orbit-cone correspondence (Theorem 2.4), we have  $U_{\rho_i} = O(\rho_i) \cup T$  and  $O(\rho_i) \cap D_{\rho_j} = \emptyset$  if  $i \neq j$ . As

$$U_{\rho_i} \cap D_{\rho_j} = (O(\rho_i) \cap D_{\rho_j}) \cup (T \cap D_{\rho_j}) ,$$

by Lemma 3.3, we conclude that  $U_{\rho_i} \cap D_{\rho_j} = \emptyset$ .  $\square$

Let  $\Delta \subset \Sigma(1)$  and  $D$  be the  $T$ -invariant divisor of  $X$  defined by

$$D = \sum_{\rho \in \Delta} D_\rho .$$

For  $\rho \in \Sigma(1)$ , we set  $E^\rho = \Gamma(U_\rho, \mathcal{T}_X(-\log D))$ .

**Theorem 3.5.** *The family of multifiltrations  $(E, \{E^\rho(j)\}_{\rho \in \Sigma(1), j \in \mathbb{Z}})$  corresponding to the logarithmic tangent sheaf  $\mathcal{T}_X(-\log D)$  are given by*

$$E^\rho(j) = \begin{cases} 0 & \text{if } j \leq -1 \\ N_{\mathbb{C}} & \text{if } j \geq 0 \end{cases} \quad \text{if } \rho \in \Delta$$

and by

$$E^\rho(j) = \begin{cases} 0 & \text{if } j \leq -2 \\ \text{Span}(u_\rho) & \text{if } j = -1 \\ N_{\mathbb{C}} & \text{if } j \geq 0 \end{cases} \quad \text{if } \rho \notin \Delta .$$

*Proof.* According to Lemma 3.4, if  $\rho \in \Delta$ , we have  $U_\rho \cap D = U_\rho \cap D_\rho$  and for  $\rho \notin \Delta$ ,  $U_\rho \cap D = \emptyset$ . We can reduce the problem to the case where  $\Delta = \{\rho_1\}$  and  $D = D_{\rho_1}$ . For the rest of the proof, we assume that  $\Delta = \{\rho_1\}$ .

*First case :* We assume that  $\rho = \rho_1$ . Let  $(u_1, \dots, u_n)$  be a basis of  $N$  such that  $u_1 = u_\rho$ . We denote by  $(e_1, \dots, e_n)$  the dual basis of  $(u_1, \dots, u_n)$  and  $x_i = \chi^{e_i}$ . We have  $\mathbb{C}[S_\rho] = \mathbb{C}[x_1, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$  and  $U_\rho = \text{Spec}(\mathbb{C}[S_\rho])$ . As on  $U_\rho$  the divisor  $D$  is defined by the equation  $x_1 = 0$ , we have

$$E^\rho = \left( \mathbb{C}[S_\rho] \cdot x_1 \frac{\partial}{\partial x_1} \right) \oplus \left( \bigoplus_{i=2}^n \mathbb{C}[S_\rho] \cdot \frac{\partial}{\partial x_i} \right) .$$

We set

$$L_1^\rho = \bigoplus_{m \in S_\rho} \mathbb{C} \cdot \chi^{m+e_1} \frac{\partial}{\partial x_1} \quad \text{and for } i \in \{2, \dots, n\}, \quad L_i^\rho = \bigoplus_{m \in S_\rho} \mathbb{C} \cdot \chi^m \frac{\partial}{\partial x_i}.$$

We have  $E^\rho = \bigoplus_{i=1}^n L_i^\rho$ . For  $t \in T$  and  $m \in M$ , we have  $t \cdot \chi^m = \chi^{-m}(t)\chi^m$ , so  $t \cdot dx_i = \chi^{-e_i}(t) dx_i$ . Thus, we have  $t \cdot \frac{\partial}{\partial x_i} = \chi^{e_i}(t) \frac{\partial}{\partial x_i}$ . For  $i \in \{1, \dots, n\}$ , we write

$$L_i^\rho = \bigoplus_{m \in M} (L_i^\rho)_m \quad \text{where} \quad (L_i^\rho)_m = \{f \in L_i^\rho : t \cdot f = \chi^{-m}(t)f\}.$$

We have

$$(L_1^\rho)_m = \begin{cases} \mathbb{C} \cdot \chi^{m+e_1} \frac{\partial}{\partial x_1} & \text{if } 0 \preceq_\rho m \\ 0 & \text{otherwise} \end{cases}$$

and for  $i \in \{2, \dots, n\}$ ,

$$(L_i^\rho)_m = \begin{cases} \mathbb{C} \cdot \chi^{m+e_i} \frac{\partial}{\partial x_i} & \text{if } -e_i \preceq_\rho m \\ 0 & \text{otherwise} \end{cases}.$$

For  $i \in \{1, \dots, n\}$ , we set  $\mathbb{L}_i^\rho = \mathbb{C} \cdot \frac{\partial}{\partial x_i}$ . If  $i \in \{2, \dots, n\}$ , we have  $(L_i^\rho)_{-e_i} = \mathbb{L}_i^\rho$ .

For all  $m \in M$ :

- if  $i \geq 2$  and  $-e_i \preceq_\rho m$ , we identify  $(L_i^\rho)_m$  with  $\mathbb{L}_i^\rho$  through the multiplication by the character  $\chi^{-m-e_i}$ .
- If  $i = 1$  and  $0 \preceq_\rho m$ , we identify  $(L_1^\rho)_m$  with  $\mathbb{L}_1^\rho$  through the multiplication by  $\chi^{-m-e_1}$ .

For  $m \in M$ , we set  $j = \langle m, u_1 \rangle$ . The condition  $0 \preceq_\rho m$  is equivalent to  $j \geq 0$  and for  $i \in \{2, \dots, n\}$ ,  $-e_i \preceq_\rho m$  is equivalent to  $j \geq 0$ . As  $(L_i^\rho)_m$  is isomorphic to  $L_i^\rho(\langle m, u_\rho \rangle)$ , we deduce that : for all  $i \in \{1, \dots, n\}$ , the filtrations of  $L_i^\rho$  are given by

$$L_i^\rho(j) = \begin{cases} 0 & \text{if } j \leq -1 \\ \mathbb{L}_i^\rho & \text{if } j \geq 0 \end{cases}.$$

The torus  $T$  is a Lie group. The tangent space of  $T$  at the neutral element is isomorphic to  $N_{\mathbb{C}}$ . As the tangent space of  $T$  at the neutral element is generated by  $\left( \frac{\partial}{\partial x_i} \right)_{1 \leq i \leq n}$ , for all  $i \in \{1, \dots, n\}$ , we can identify  $\frac{\partial}{\partial x_i}$  with  $u_i$ . Thus

$$L_i^\rho(j) \cong \begin{cases} 0 & \text{if } j \leq -1 \\ \mathbb{C} u_i & \text{if } j \geq 0 \end{cases}.$$

As

$$E^\rho = \bigoplus_{m \in M} E^\rho(\langle m, u_1 \rangle) \otimes \chi^m \quad \text{where} \quad E^\rho(\langle m, u_1 \rangle) \cong \bigoplus_{i=1}^n L_i^\rho(\langle m, u_\rho \rangle),$$

$$\text{we get } E^\rho(j) \cong \begin{cases} 0 & \text{if } j \leq -1 \\ N_{\mathbb{C}} & \text{if } j \geq 0 \end{cases}.$$

*Second case :* We assume that  $\rho \in \Sigma(1) \setminus \Delta$ . We start as in the first case. We assume that  $\rho = \text{Cone}(u_1)$ . Let  $(u_1, \dots, u_n)$  be a basis of  $N$  and  $(e_1, \dots, e_n)$  its dual basis. As  $U_\rho \cap D = \emptyset$ , we have

$$E^\rho = \bigoplus_{i=1}^n \mathbb{C}[S_\rho] \cdot \frac{\partial}{\partial x_i} = \bigoplus_{i=1}^n \left( \bigoplus_{m \in S_\rho} \mathbb{C} \cdot \chi^m \frac{\partial}{\partial x_i} \right).$$

For all  $i \in \{1, \dots, n\}$ , we set  $L_i^\rho = \mathbb{C}[S_\rho] \cdot \frac{\partial}{\partial x_i}$ . We have

$$L_i^\rho = \bigoplus_{m \in M} (L_i^\rho)_m \quad \text{where} \quad (L_i^\rho)_m = \begin{cases} \mathbb{C} \cdot \chi^{m+e_i} \frac{\partial}{\partial x_i} & \text{if } -e_i \preceq_\rho m \\ 0 & \text{otherwise} \end{cases} .$$

For  $m \in M$ , we set  $j = \langle m, u_1 \rangle$ . The condition  $-e_i \preceq_\rho m$  is equivalent to  $j \geq -\langle e_i, u_1 \rangle$ . Thus, for all  $i \in \{2, \dots, n\}$ , the filtrations of  $L_i^\rho$  are given by

$$L_i^\rho(j) = \begin{cases} 0 & \text{if } j \leq -1 \\ \mathbb{L}_i^\rho & \text{if } j \geq 0 \end{cases}$$

and the filtrations of  $L_1^\rho$  are given by

$$L_1^\rho(j) = \begin{cases} 0 & \text{if } j \leq -2 \\ \mathbb{L}_i^\rho & \text{if } j \geq -1 \end{cases}$$

As in the first case, by identifying  $\mathbb{L}_i^\rho$  with  $\text{Span}(u_i)$ , we get

$$E^\rho(j) \cong \begin{cases} 0 & \text{if } j \leq -2 \\ \text{Span}(u_\rho) & \text{if } j = -1 \\ N \otimes_{\mathbb{Z}} \mathbb{C} & \text{if } j \geq 0 \end{cases} .$$

Hence we get the theorem.  $\square$

The sheaf of regular section of the trivial vector bundle  $X \times \mathbb{C} \rightarrow X$  of rank 1 is  $\mathcal{O}_X$ . We denote by  $(F, \{F^\rho(j)\}_{\rho \in \Sigma(1), j \in \mathbb{Z}})$  the family of filtration of  $\mathcal{O}_X$ . For  $\rho \in \Sigma(1)$ , we set  $F^\rho = \mathcal{O}_X(U_\rho)$ . Let  $F_m^\rho = \{f \in F^\rho : t \cdot f = \chi^{-m}(t) f\}$ , as

$$F^\rho = \bigoplus_{m \in M} F_m^\rho = \mathbb{C}[S_\rho] = \bigoplus_{m \in S_\rho} \mathbb{C} \cdot \chi^m ,$$

we deduce that  $F_m^\rho = \mathbb{C} \cdot \chi^m$  if  $m \in S_\rho$  and  $F_m^\rho = 0$  if  $m \notin S_\rho$ . Hence,

$$F^\rho(j) = \begin{cases} 0 & \text{if } j \leq -1 \\ \mathbb{C} & \text{if } j \geq 0 \end{cases} .$$

**Corollary 3.6.** *Let  $\Delta \subset \Sigma(1)$  and  $D = \sum_{\rho \in \Delta} D_\rho$ .*

1. *If  $\Delta = \emptyset$ , then  $\mathcal{T}_X(-\log D)$  is the tangent sheaf  $\mathcal{T}_X$ .*
2. *If  $\Delta = \Sigma(1)$ , then  $\mathcal{T}_X(-\log D)$  is isomorphic to the trivial vector bundle of rank  $n$ .*

*Proof.* If  $\Delta = \emptyset$ , the family of multifiltrations of  $\mathcal{T}_X(-\log D)$  is identical to the family of multifiltrations given in Proposition 3.1. If  $\Delta = \Sigma(1)$ , for all  $\rho \in \Sigma(1)$ , we have

$$E^\rho(j) = \begin{cases} 0 & \text{if } j \leq -1 \\ N \otimes_{\mathbb{Z}} \mathbb{C} & \text{if } j \geq 0 \end{cases} .$$

Hence,  $\mathcal{T}_X(-\log D)$  is isomorphic to the trivial vector bundle of rank  $n$ .  $\square$

From now on, we will assume that  $\emptyset \subsetneq \Delta \subsetneq \Sigma(1)$  and  $D$  the  $T$ -invariant Weil divisor defined by

$$D = \sum_{\rho \in \Delta} D_\rho .$$

**Notation 3.7.** Let  $G$  be a sub-vector space of  $N_{\mathbb{C}}$ . We denote by  $\mathcal{E}_G$  the sub-sheaf of  $\mathcal{E} = \mathcal{T}_X(-\log D)$  defined by the family of multifiltrations  $(E_G, \{G^\rho(j)\}_{\rho \in \Sigma(1), j \in \mathbb{Z}})$  where  $E_G = G$  and  $G^\rho(j) = E^\rho(j) \cap G$ .

If  $\rho \in \Delta$  or  $u_\rho \notin G$ , then

$$G^\rho(j) = \begin{cases} 0 & \text{if } j \leq -1 \\ G & \text{if } j \geq 0 \end{cases} .$$

If  $\rho \notin \Delta$  and  $u_\rho \in G$ , then

$$G^\rho(j) = \begin{cases} 0 & \text{if } j \leq -2 \\ \text{Span}(u_\rho) & \text{if } j = -1 \\ G & \text{if } j \geq 0 \end{cases} .$$

**3.2. Decomposition of equivariant logarithmic tangent sheaf.** In this part, we give some conditions on  $\Sigma$  and  $\Delta$  which ensure that the logarithmic tangent sheaf is decomposable. We first recall the family of multifiltration of a direct sum of equivariant reflexive sheaves.

**Proposition 3.8** ([DDK20, Remark 2.2.15]). *Let  $\mathcal{F}$  and  $\mathcal{G}$  be two equivariant reflexive sheaves with  $(F, \{F^\rho(j)\}_{\rho \in \Sigma(1), j \in \mathbb{Z}})$  and  $(G, \{G^\rho(j)\}_{\rho \in \Sigma(1), j \in \mathbb{Z}})$  for family of filtrations. The family of multifiltrations of  $\mathcal{F} \oplus \mathcal{G}$  is given by*

$$(5) \quad (F \oplus G, \{(F \oplus G)^\rho(j)\}_{\rho \in \Sigma(1), j \in \mathbb{Z}}) \text{ where } (F \oplus G)^\rho(j) = F^\rho(j) \oplus G^\rho(j) .$$

We assume that  $X$  is toric variety without torus factor. We denote by  $p$  the rank of the class group  $\text{Cl}(X)$  of  $X$ . By [CLS11, Theorem 4.1.3], we have an exact sequence

$$(6) \quad 0 \longrightarrow M \longrightarrow \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z} \cdot D_\rho \longrightarrow \text{Cl}(X) \longrightarrow 0$$

and  $\text{card}(\Sigma(1)) = n + p$ .

**Theorem 3.9.** *We assume that  $\text{card}(\Delta) = p$ . We set  $\Sigma(1) \setminus \Delta = \{\rho_1, \dots, \rho_n\}$  where  $\rho_k = \text{Cone}(u_k)$  and  $u_k \in N$ . If  $N_{\mathbb{R}} = \text{Span}(u_1, \dots, u_n)$ , then  $\mathcal{E} = \mathcal{T}_X(-\log D)$  is decomposable and*

$$\mathcal{E} = \bigoplus_{k=1}^n \mathcal{E}_{F_k}$$

where  $\mathcal{E}_{F_k}$  is the sub-sheaf of  $\mathcal{E}$  corresponding to the vector space  $F_k = \text{Span}(u_k)$ .

*Proof.* For all  $k \in \{1, \dots, n\}$ , the family of multifiltration  $(F_k, \{F_k^\rho(j)\}_{\rho \in \Sigma(1), j \in \mathbb{Z}})$  of  $\mathcal{E}_{F_k}$  are given by

$$F_k^\rho(j) = \begin{cases} 0 & \text{if } j \leq -1 \\ F_k & \text{if } j \geq 0 \end{cases} \quad \text{if } \rho \neq \text{Cone}(u_k)$$

and

$$F_k^\rho(j) = \begin{cases} 0 & \text{if } j \leq -2 \\ \text{Span}(u_\rho) & \text{if } j = -1 \\ F_k & \text{if } j \geq 0 \end{cases} \quad \text{if } \rho = \text{Cone}(u_k) .$$

For all  $\rho \in \Sigma(1)$  and  $j \in \mathbb{Z}$ , we have

$$\bigoplus_{k=1}^n F_k^\rho(j) = \begin{cases} 0 & \text{if } j \leq -1 \\ N_{\mathbb{C}} & \text{if } j \geq 0 \end{cases} \quad \text{if } \rho \in \Delta$$

and

$$\bigoplus_{k=1}^n F_k^\rho(j) = \begin{cases} 0 & \text{if } j \leq -2 \\ \text{Span}(u_\rho) & \text{if } j = -1 \\ N_{\mathbb{C}} & \text{if } j \geq 0 \end{cases} \quad \text{if } \rho \notin \Delta .$$

Hence, by (5) and Theorem 3.5 we have  $\mathcal{E} = \bigoplus_{k=1}^n \mathcal{E}_{F_k}$ .  $\square$

**Proposition 3.10.** *We assume that  $\Delta$  satisfies  $1 + p \leq \text{card}(\Delta) \leq n + p - 1$ . The sheaf  $\mathcal{E} = \mathcal{T}_X(-\log D)$  is decomposable and  $\mathcal{E} = \mathcal{E}_G \oplus \mathcal{E}_F$  where  $G = \text{Span}(u_\rho : \rho \in \Sigma(1) \setminus \Delta)$  and  $F$  a sub-space of  $N_{\mathbb{C}}$  such that  $N_{\mathbb{C}} = G \oplus F$ .*

*Proof.* It suffices to work with the family of multifiltration as in the proof of Theorem 3.9.  $\square$

## 4. STABILITY OF EQUIVARIANT LOGARITHMIC SHEAF

**4.1. Some stability notions.** We recall now the notions of stability that we will consider in this paper. For this part, we refer to [Gue16, Section 3.1], [CT22, Section 4.1] and [Tak72].

**Definition 4.1.** Let  $\mathcal{E}$  be a torsion-free coherent sheaf on  $X$ . The *degree* of  $\mathcal{E}$  with respect to an ample class  $L \in \text{Amp}(X)$  is the real number obtained by intersection:

$$\deg_L(\mathcal{E}) = c_1(\mathcal{E}) \cdot L^{n-1}$$

and its *slope* with respect to  $L$  is given by

$$\mu_L(\mathcal{E}) = \frac{\deg_L(\mathcal{E})}{\text{rk}(\mathcal{E})}.$$

**Definition 4.2.** A torsion-free coherent sheaf  $\mathcal{E}$  is said to be  $\mu$ -*semi-stable* with respect to  $L \in \text{Amp}(X)$  if for any proper coherent sub-sheaf of lower rank  $\mathcal{F}$  of  $\mathcal{E}$  with  $0 < \text{rk } \mathcal{F} < \text{rk } \mathcal{E}$ , one has

$$\mu_L(\mathcal{F}) \leq \mu_L(\mathcal{E}).$$

When strict inequality always holds, we say that  $\mathcal{E}$  is  $\mu$ -*stable*. Finally,  $\mathcal{E}$  is said to be  $\mu$ -*polystable* if it is the direct sum of  $\mu$ -stable sub-sheaves of the same slope.

**Lemma 4.3** ([CT22, Lemma 4.3]). *Let  $\mathcal{E}$  be an equivariant reflexive sheaf with family of multifiltrations  $(E, \{E^\rho(j)\}_{\rho \in \Sigma(1), j \in \mathbb{Z}})$ . We have*

$$\mu_L(\mathcal{E}) = \frac{1}{\text{rk}(\mathcal{E})} \sum_{\rho \in \Sigma(1)} \sum_{i \in \mathbb{Z}} i e^\rho(i) \deg_L(D_\rho)$$

where  $e^\rho(i) = \dim E^\rho(i-1) - \dim E^\rho(i)$ .

Following [Koo11, Proposition 4.13] and [HNS22, Proposition 2.3], we have :

**Proposition 4.4** ([CT22, Proposition 4.2]). *Let  $\mathcal{E}$  be a  $T$ -equivariant reflexive sheaf on  $X$  with family of multifiltrations  $(E, \{E^\rho(j)\}_{\rho \in \Sigma(1), j \in \mathbb{Z}})$ . Then  $\mathcal{E}$  is  $\mu$ -semi-stable (resp.  $\mu$ -stable) with respect to  $L$  if and only if for all proper vector sub-spaces  $W \subset E$ ,  $\mu_L(\mathcal{E}_W) \leq \mu_L(\mathcal{E})$  (resp.  $\mu_L(\mathcal{E}_W) < \mu_L(\mathcal{E})$ ), where  $\mathcal{E}_W$  is defined in notation 3.7.*

**4.2. Polystability and semi-stability.** We give here a link between poly-stability and semi-stability.

**Proposition 4.5** ([Koo11, Claim 2 of Proposition 4.13]). *A reflexive  $\mu$ -polystable sheaf on  $X$  is a  $\mu$ -semi-stable sheaf on  $X$  isomorphic to a (finite, nontrivial) direct sum of reflexive  $\mu$ -stable sheaves. Let  $\mathcal{E}$  be a  $\mu$ -semi-stable reflexive sheaf on  $X$ . Then  $\mathcal{E}$  contains a unique maximal reflexive  $\mu$ -polystable sub-sheaf of the same slope as  $\mathcal{E}$ .*

**Corollary 4.6.** *Let  $\mathcal{E}$  be an equivariant reflexive sheaf such that*

$$\mathcal{E} = \bigoplus_{i=1}^r \mathcal{E}_i$$

with  $r \geq 2$ . Let  $L \in \text{Amp}(X)$  such that for all  $i \in \{1, \dots, r\}$ ,  $\mathcal{E}_i$  is stable with respect to  $L$ . Then,  $\mathcal{E}$  is  $\mu$ -polystable with respect to  $L$  if and only if  $\mathcal{E}$  is semi-stable with respect to  $L$ .

*Proof.* If  $\mathcal{E}$  is poly-stable with respect to  $L$ , then  $\mathcal{E}$  is semi-stable with respect to  $L$ . We now assume that  $\mathcal{E}$  is semi-stable with respect to  $L$ ; by using the identity

$$\mathrm{rk}(\mathcal{E}) \mu_L(\mathcal{E}) = \sum_{i=1}^r \mathrm{rk}(\mathcal{E}_i) \mu_L(\mathcal{E}_i) ,$$

we get  $\mu_L(\mathcal{E}_i) = \mu_L(\mathcal{E})$  for all  $i \in \{1, \dots, r\}$ . Hence,  $\mathcal{E}$  is poly-stable.  $\square$

**4.3. An instability condition for the logarithmic tangent sheaves.** Let  $\Delta \subset \Sigma(1)$  and

$$D = \sum_{\rho \in \Delta} D_\rho$$

a  $T$ -invariant Weil divisor on  $X$ . Let  $G$  be a sub-vector space of  $N_{\mathbb{C}}$  of dimension  $l$  with  $1 \leq l < n$ . Let  $(E, \{E^\rho(j)\}_{\rho \in \Sigma(1), j \in \mathbb{Z}})$  be the family of multifiltrations corresponding to  $\mathcal{E} = \mathcal{T}_X(-\log D)$  and  $(E_G, \{G^\rho(j)\}_{\rho \in \Sigma(1), j \in \mathbb{Z}})$  be the family of multifiltrations corresponding to  $\mathcal{E}_G$ . By Lemma 4.3, we have

$$(7) \quad \mu_L(\mathcal{E}) = \frac{1}{n} \sum_{\rho \notin \Delta} \deg_L(D_\rho)$$

and

$$(8) \quad \mu_L(\mathcal{E}_G) = \frac{1}{l} \sum_{\rho \notin \Delta \text{ and } u_\rho \in G} \deg_L(D_\rho) .$$

Therefore, we have

$$(9) \quad \mu_L(\mathcal{E}) - \mu_L(\mathcal{E}_G) = \left( \frac{1}{n} - \frac{1}{l} \right) \left( \sum_{\rho \notin \Delta, u_\rho \in G} \deg_L(D_\rho) \right) + \frac{1}{n} \left( \sum_{\rho \notin \Delta, u_\rho \notin G} \deg_L(D_\rho) \right) .$$

To study the stability of  $\mathcal{E}$  with respect to  $L \in \mathrm{Amp}(X)$ , it suffices to compare  $\mu_L(\mathcal{E})$  with  $\mu_L(\mathcal{E}_G)$  where  $G \subset \mathrm{Span}(u_\rho : \rho \notin \Delta)$  and  $1 \leq \dim G \leq n-1$ .

**Theorem 4.7.** *If  $1 \leq \mathrm{card}(\Sigma(1) \setminus \Delta) \leq n-1$ , then for any  $L \in \mathrm{Amp}(X)$ , the logarithmic tangent sheaf  $\mathcal{E} = \mathcal{T}_X(-\log D)$  is not semi-stable with respect to  $L$ .*

*Proof.* We assume that  $\Sigma(1) \setminus \Delta = \{\rho_1, \dots, \rho_k\}$  where  $1 \leq k \leq n-1$  and we denote by  $D_j$  the divisor corresponding to  $\rho_j = \mathrm{Cone}(u_j)$ . Let  $G = \mathrm{Span}(u_1, \dots, u_k)$  and  $l = \dim G$ . If  $L \in \mathrm{Amp}(X)$ , we have

$$\mu_L(\mathcal{E}) - \mu_L(\mathcal{E}_G) = \left( \frac{1}{n} - \frac{1}{l} \right) \sum_{j=1}^k \deg_L(D_j) < 0$$

because the numbers  $\deg_L(D_j)$  are positive and different to zero. Thus,  $\mathcal{E}$  is not semi-stable with respect to  $L$ .  $\square$

We assume that  $X$  has no torus factor. We have seen that  $\mathrm{card}(\Sigma(1)) = n + \mathrm{rk} \mathrm{Cl}(X)$ , so the Theorem 4.7 becomes :

**Corollary 4.8.** *We set  $p = \mathrm{rk} \mathrm{Cl}(X)$ . If  $1+p \leq \mathrm{card}(\Delta) \leq n+p-1$ , then for any  $L \in \mathrm{Amp}(X)$ , the logarithmic tangent sheaf  $\mathcal{T}_X(-\log D)$  is not semi-stable with respect to  $L$ .*

*Proof.* If  $1+p \leq \mathrm{card}(\Delta) \leq n+p-1$ , by using

$$\mathrm{card}(\Sigma(1)) = n+p = \mathrm{card}(\Delta) + \mathrm{card}(\Sigma(1) \setminus \Delta) ,$$

we get  $1 \leq \mathrm{card}(\Sigma(1) \setminus \Delta) \leq n-1$ ; we can conclude with Theorem 4.7.  $\square$

*Remark 4.9.* By Corollary 3.6, if  $\mathrm{card}(\Delta) = n+p$ ,  $\mathcal{T}_X(-\log D)$  is semi-stable with respect to any polarizations.

From now on, when we study the stability or semi-stability of  $\mathcal{T}_X(-\log D)$ , we will consider only the case where  $1 \leq \text{card}(\Delta) \leq p = \text{rk Cl}(X)$  and  $p \in \{1, 2\}$ .

**4.4. Stability on toric orbifolds of Picard rank one.** In this section, we will study the semi-stability of logarithmic tangent sheaf on toric orbifolds of Picard rank one.

Let  $q_0, q_1, \dots, q_n \in \mathbb{N}^*$  such that  $\gcd(q_0, \dots, q_n) = 1$ . We set

$$N = \mathbb{Z}^{n+1} / \mathbb{Z} \cdot (q_0, \dots, q_n) .$$

The dual lattice of  $N$  is  $M = \{(a_0, \dots, a_n) \in \mathbb{Z}^{n+1} : a_0 q_0 + \dots + a_n q_n = 0\}$ .

Let  $\{u_i : 0 \leq i \leq n\}$  be the images in  $N$  of the standard basis vectors in  $\mathbb{Z}^{n+1}$ , so the relation

$$q_0 u_0 + q_1 u_1 + \dots + q_n u_n = 0$$

holds in  $N$ . Let  $\Sigma$  be the fan in  $N_{\mathbb{R}}$  defined by

$$\Sigma = \{\text{Cone}(A) : A \subsetneq \{u_0, \dots, u_n\}\} .$$

We define  $X$  to be the toric variety of the fan  $\Sigma$ . As  $\Sigma$  is simplicial, by [CLS11, Proposition 4.2.7],  $\text{Pic}(X)$  has finite index in  $\text{Cl}(X)$ . By (6), we deduce that  $\text{rk Pic}(X) = 1$ . We denote by  $D_i$  the Weil divisor corresponding to the ray  $\text{Cone}(u_i)$ .

In the Corollary 4.8, we see that if  $2 \leq \text{card}(\Delta) \leq n$ , then  $\mathcal{T}_X(-\log D)$  is not semi-stable with respect to any  $L \in \text{Amp}(X)$ . We assume that  $\text{card}(\Delta) = 1$ . Let  $i \in \{0, \dots, n\}$  and  $A_i = \{0, \dots, n\} \setminus \{i\}$ . We set  $\mathcal{E} = \mathcal{T}_X(-\log D_i)$ .

**Lemma 4.10.** *For all  $j \in A_i$ , the divisor  $q_i D_j$  is linearly equivalent to the divisor  $q_j D_i$ .*

*Proof.* We fix  $j \in A_i$ . Let  $m = (a_0, \dots, a_n) \in M$  defined by  $a_i = q_j$ ,  $a_j = -q_i$  and  $a_k = 0$  if  $k \in \{0, \dots, n\} \setminus \{i, j\}$ . As  $\text{div}(\chi^m) = q_j D_i - q_i D_j$ , we deduce that  $q_i D_j$  is linearly equivalent to  $q_j D_i$ .  $\square$

**Theorem 4.11.** *Let  $L \in \text{Amp}(X)$ . The sheaf  $\mathcal{E}$  is poly-stable with respect to  $L$  if and only if there is  $q \in \mathbb{N}^*$  such that for all  $j \in A_i$ ,  $q_j = q$ .*

*Proof.* The assumptions of Theorem 3.9 are verified. Hence,

$$\mathcal{E} = \bigoplus_{j \in A_i} \mathcal{E}_{F_j}$$

where  $F_j = \text{Span}(u_j)$ . By (8),  $\mu_L(\mathcal{E}_{F_j}) = \deg_L(D_j)$  for all  $j \in A_i$ . By Lemma 4.10, for all  $j \in A_i$ ,

$$\mu_L(\mathcal{E}_{F_j}) = \frac{q_j}{q_i} \deg_L(D_i) .$$

If  $\mathcal{E}$  is poly-stable with respect to  $L$ , there is  $r \in \mathbb{Q}$  such that for all  $j \in A_i$ ,  $q_j = r q_i$ . Hence, we have the existence of  $q \in \mathbb{N}^*$  such that for all  $j \in A_i$ ,  $q_j = q$ .

For the converse, if for all  $j \in A_i$ , we have  $q_j = q$ , then  $\mathcal{E}$  is poly-stable.  $\square$

According to Theorem 4.11 and Corollary 4.6, we have :

**Corollary 4.12.** *For all  $i \in \{0, \dots, n\}$ ,  $\text{sStab}(\mathcal{T}_X(-\log D_i)) \neq \emptyset$  if and only if there exists  $q \in \mathbb{N}^*$  such that for all  $j \in A_i$ ,  $q_j = q$ . Moreover, if for all  $j \in A_i$ , we have  $q_j = q$ , then*

$$\text{sStab}(\mathcal{T}_X(-\log D_i)) = \text{Amp}(X) .$$

**Corollary 4.13.** *For all  $i \in \{0, \dots, n\}$ , we have  $\text{Stab}(\mathcal{T}_X(-\log D_i)) = \emptyset$ .*

**Corollary 4.14.** *If  $q_0 = \dots = q_n = 1$ , that is  $X = \mathbb{P}^n$ , then for all  $i \in \{0, \dots, n\}$ , we have*

$$\emptyset = \text{Stab}(T_X(-\log D_i)) \subsetneq \text{sStab}(T_X(-\log D_i)) = \text{Amp}(X) .$$

## 5. DESCRIPTION OF SMOOTH PROJECTIVE TORIC VARIETIES OF PICARD RANK TWO

Let  $X$  be a smooth projective toric variety of dimension  $n$  such that  $\text{rk Pic}(X) = 2$ . There exists  $r, s \in \mathbb{N}^*$  with  $r + s = n$  and  $a_1, \dots, a_r \in \mathbb{N}$  with  $a_1 \leq a_2 \leq \dots \leq a_r$  such that

$$(10) \quad X = \mathbb{P} \left( \mathcal{O}_{\mathbb{P}^s} \oplus \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^s}(a_i) \right).$$

We keep the notation of Section 1.1. We denote by  $\pi : X \rightarrow \mathbb{P}^s$  the projection onto the basis  $\mathbb{P}^s$ . We have the following linear equivalence,

$$(11) \quad D_{v_i} \sim_{\text{lin}} D_{v_0} - a_i D_{w_0} \text{ for } i \in \{1, \dots, r\} \text{ and } D_{w_j} \sim_{\text{lin}} D_{w_0} \text{ for } j \in \{1, \dots, s\}.$$

By (11), we deduce that  $\text{Pic}(X)$  is generated by  $D_{v_0}$  and  $D_{w_0}$ .

**Proposition 5.1** ([DDK20, Proposition 4.2.1]). *Let  $D = \mu D_{w_0} + \lambda D_{v_0}$  be a  $T$ -invariant divisor of  $X$  with  $\mu, \lambda \in \mathbb{Z}$ . The divisor  $D$  is ample if and only if  $\mu > 0$  and  $\lambda > 0$ .*

*Remark 5.2.* Let  $\mu, \lambda \in \mathbb{N}^*$  and  $L = \mathcal{O}_X(\mu D_{w_0} + \lambda D_{v_0})$ . We have an isomorphism  $L \cong \pi^* \mathcal{O}_{\mathbb{P}^s}(\mu) \otimes \mathcal{O}_X(\lambda)$ .

By [CLS11, Theorem 8.1.2], the anti-canonical divisor of  $X$  is given by

$$(12) \quad -K_X = \sum_{i=0}^r D_{v_i} + \sum_{j=0}^s D_{w_j} \sim_{\text{lin}} (s+1 - a_1 - \dots - a_r) D_{w_0} + (r+1) D_{v_0}.$$

Hence,  $X$  is Fano if and only if  $a_1 + \dots + a_r \leq s$ .

Let  $D$  be a reduced  $T$ -invariant Weil divisor on  $X$ . To study the stability of  $\mathcal{E} = T_X(-\log D)$  with respect to  $L = \mathcal{O}_X(Z)$  where  $Z = \mu D_{w_0} + \lambda D_{v_0}$  ( $\mu, \lambda \in \mathbb{N}^*$ ), we will study the stability of  $\mathcal{E}$  with respect to the ample  $\mathbb{Q}$ -divisor  $Z' = \nu D_{w_0} + D_{v_0}$  where  $\nu = \frac{\mu}{\lambda}$ .

**5.1. Computation of the degree.** Let  $P \subset M_{\mathbb{R}}$  be a polytope such that the fan  $\Sigma_P$  of  $P$  is the fan  $\Sigma$  of  $X$ . For each  $\rho \in \Sigma(1)$  we denote by  $P^\rho$  the facet of  $P$  corresponding to the ray  $\rho$ .

We recall that a lattice  $M$  defines a measure  $\nu$  on  $M_{\mathbb{R}}$  as the pull-back of the Haar measure on  $M_{\mathbb{R}}/M$ . It is determined by the properties

- i.  $\nu$  is translation invariant,
- ii.  $\nu(M_{\mathbb{R}}/M) = 1$ .

For all  $\rho \in \Sigma(1)$ , we denote by  $\text{vol}(P^\rho)$  the volume of  $P^\rho$  with respect to the measure determined by the affine span of  $P^\rho \cap M$ .

**Proposition 5.3** ([Dan78, Section 11]). *Let  $(X, L)$  be a polarized toric variety corresponding to a lattice polytope  $P$ . For all ray  $\rho$ , we have  $\deg_L(D_\rho) = \text{vol}(P^\rho)$ .*

**5.2. Polytope corresponding to a  $\mathbb{Q}$ -ample divisor.** Here, we describe the polytope corresponding to the  $\mathbb{Q}$ -polarized variety  $(X, L)$  where

$$L = \pi^* \mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1).$$

We refer to [HNS22, Section 4].

Let  $\nu \in \mathbb{Q}^+$  such that  $\nu \neq 0$  and  $P_s = \text{Conv}(0, w_1, w_2, \dots, w_s)$ . Then,

$$P = \text{Conv}(\nu P_s \times \{0\} \cup (a_1 + \nu) P_s \times \{v_1\} \cup \dots \cup (a_r + \nu) P_s \times \{v_r\})$$

is the polytope corresponding to the  $\mathbb{Q}$ -polarized variety  $(X, L)$ . We denote by  $P^{v_i}$  (resp.  $P^{w_j}$ ) the facet of  $P$  corresponding to the ray  $\text{Cone}(v_i)$  (resp.  $\text{Cone}(w_j)$ ). The facet  $P^{v_i}$  is the convex hull of

$\nu P_s \times \{0\} \cup \dots \cup (a_{i-1} + \nu)P_s \times \{v_{i-1}\} \cup (a_{i+1} + \nu)P_s \times \{v_{i+1}\} \cup \dots \cup (a_r + \nu)P_s \times \{v_r\}$   
and  $P^{w_i}$  is isomorphic to

$$\nu P'_s \times \{0\} \cup (a_1 + \nu)P'_s \times \{v_1\} \cup \dots \cup (a_r + \nu)P'_s \times \{v_r\}$$

where  $P'_s = \text{Conv}(0, w_1, \dots, w_{s-1})$ .

**Notation 5.4.** For all  $i \in \{0, 1, \dots, r\}$ , we set  $V_i = \text{vol}(P^{v_i})$ . As for all  $j \in \{1, \dots, s\}$ ,  $\text{vol}(P^{w_j}) = \text{vol}(P^{w_0})$ , we set  $W = \text{vol}(P^{w_0})$ .

**Proposition 5.5** ([HNS22, Proposition 4.3]). *Let  $c_0, c_1, \dots, c_r \in \mathbb{N}$ ,  $\nu > 0$  and  $P_s = \text{Conv}(0, w_1, w_2, \dots, w_s)$ . The volume of the polytope*

$$P = \text{Conv}((c_0 + \nu)P_s \times \{0\} \cup (c_1 + \nu)P_s \times \{v_1\} \cup \dots \cup (c_r + \nu)P_s \times \{v_r\})$$

is given by

$$\sum_{k=0}^s \binom{s+r}{k} \left( \sum_{d_0+\dots+d_r=s-k} c_0^{d_0} \dots c_r^{d_r} \right) \nu^k.$$

By Proposition 5.5, we have

$$W = \sum_{k=0}^{s-1} \binom{s+r-1}{k} \left( \sum_{d_1+\dots+d_r=s-k-1} a_1^{d_1} \dots a_r^{d_r} \right) \nu^k$$

$$V_0 = \sum_{k=0}^s \binom{s+r-1}{k} \left( \sum_{d_1+\dots+d_r=s-k} a_1^{d_1} \dots a_r^{d_r} \right) \nu^k$$

and for  $i \in \{1, \dots, r\}$ ,

$$V_i = \sum_{k=0}^s \binom{s+r-1}{k} \left( \sum_{\substack{d_1+\dots+d_{i-1} \\ +d_{i+1}+\dots+d_r=s-k}} a_1^{d_1} \dots a_{i-1}^{d_{i-1}} a_{i+1}^{d_{i+1}} \dots a_r^{d_r} \right) \nu^k.$$

For  $\nu$  fixed, we give some relations between  $W$  and the numbers  $V_i$ .

If  $a_1 = \dots = a_r = 0$ , then

$$(13) \quad W = \binom{s+r-1}{s-1} \nu^{s-1} \quad \text{and} \quad V_i = \binom{s+r-1}{s} \nu^s.$$

We now assume that  $(a_1, \dots, a_r) \neq (0, \dots, 0)$  with  $a_1 \leq a_2 \leq \dots \leq a_r$ . Let  $z \in \{0, 1, \dots, r-1\}$  such that  $a_z = 0$  and  $a_{z+1} > 0$  (we set  $a_0 = 0$ ). For  $k \in \{0, \dots, s\}$  and  $i \in \{z+1, \dots, r\}$ , we set

$$W_k = \sum_{d_{z+1}+\dots+d_r=s-1-k} a_{z+1}^{d_{z+1}} \dots a_r^{d_r}$$

$$V_{0k} = \sum_{d_{z+1}+\dots+d_r=s-k} a_{z+1}^{d_{z+1}} \dots a_r^{d_r}$$

$$V_{ik} = \sum_{\substack{d_{z+1}+\dots+d_{i-1} \\ +d_{i+1}+\dots+d_r=s-k}} a_{z+1}^{d_{z+1}} \dots a_{i-1}^{d_{i-1}} a_{i+1}^{d_{i+1}} \dots a_r^{d_r}$$

where  $W_s = 0$ . If  $i \in \{1, \dots, z\}$ , we set  $V_{ik} = V_{0k}$ . We have  $W_{s-1} = 1$  and for  $i \in \{0, \dots, r\}$ ,  $V_{is} = 1$ .

*Remark 5.6.* If  $r = 1$ , we set  $V_{1s} = 1$  and for  $k \in \{0, \dots, s-1\}$ ,  $V_{1k} = 0$ .

**Lemma 5.7.** For  $k \in \{0, \dots, s-1\}$  and  $i \in \{1, \dots, r\}$ ,  $a_i W_k + V_{ik} = V_{0k}$ .

*Proof.* If  $i \in \{1, \dots, z\}$ , the equality is true because  $a_i = 0$ . We assume that  $i \in \{z+1, \dots, r\}$ , we have

$$\begin{aligned} V_{0k} &= \sum_{d_{z+1}+\dots+d_r=s-k} a_{z+1}^{d_{z+1}} \cdots a_r^{d_r} \\ &= \sum_{\substack{d_{z+1}+\dots+d_r=s-k \\ d_i=0}} a_{z+1}^{d_{z+1}} \cdots a_r^{d_r} + \sum_{\substack{d_{z+1}+\dots+d_r=s-k \\ d_i \geq 1}} a_{z+1}^{d_{z+1}} \cdots a_r^{d_r} \end{aligned}$$

The first term of the second line corresponds to the number  $V_{ik}$  and the second to  $a_i W_k$  (it suffices to replace  $d_i$  by  $d'_i + 1$ ). Hence,  $V_{0k} = V_{ik} + a_i W_k$ .  $\square$

**Corollary 5.8.** For all  $i \in \{1, \dots, r\}$ ,  $V_0 = a_i W + V_i$ .

**5.3. A necessary condition for stability.** In this part, we adapt some results of [HNS22, Section 4] for the study of the stability of  $T_X(-\log D)$ .

The following lemma will be useful in the proof of Proposition 5.10 which is the main result of this part. Let  $z \in \{0, \dots, r-1\}$  such that  $a_z = 0$  and  $a_{z+1} > 0$ . We have the following lemma.

**Lemma 5.9** ([HNS22, Lemma 4.2]).

Let  $I' \subset \{0, 1, \dots, r\}$  and  $G = \text{Span}(v_i : i \in I')$ . The vector  $a_1 v_1 + \dots + a_r v_r$  belongs to  $G$  if and only if

- i.  $\{z+1, \dots, r\} \subset I'$  or
- ii.  $\{0, \dots, z\} \subset I'$ ,  $\text{card}(\{z+1, \dots, r\} \setminus I') \geq 1$  and  $a_i = a_j$  for all  $i, j \in \{z+1, \dots, r\} \setminus I'$ .

Let  $\Delta \subset \Sigma(1)$  and  $D = \sum_{\rho \in \Delta} D_\rho$ .

Let  $I_\Sigma = \{\text{Cone}(v_0), \dots, \text{Cone}(v_r)\}$  and  $J_\Sigma = \{\text{Cone}(w_0), \dots, \text{Cone}(w_s)\}$ . We set

$$I = \{i \in \{0, 1, \dots, r\} : \text{Cone}(v_i) \in I_\Sigma \setminus (I_\Sigma \cap \Delta)\} \text{ and}$$

$$J = \{j \in \{0, 1, \dots, s\} : \text{Cone}(w_j) \in J_\Sigma \setminus (J_\Sigma \cap \Delta)\}.$$

Let  $L \in \text{Amp}(X)$ . To study the stability of  $\mathcal{E} = T_X(-\log D)$ , it suffices to compare  $\mu_L(\mathcal{E})$  and  $\mu_L(\mathcal{E}_G)$  where  $G = \text{Span}(v_i, w_j : i \in I', j \in J')$  with  $I' \subset I$ ,  $J' \subset J$  and  $1 \leq \dim G < (r+s)$ . By Proposition 5.3, and by using of (7), (8) we have

$$\mu_L(\mathcal{E}) = \frac{1}{r+s} \left( \sum_{i \in I} V_i + \text{card}(J) \cdot W \right)$$

and

$$\mu_L(\mathcal{E}_G) = \frac{1}{\dim G} \left( \sum_{i \in I'} V_i + \text{card}(J') \cdot W \right).$$

Here is a variant of [HNS22, Proposition 4.1] which will be useful for us in the study of stability of  $T_X(-\log D)$ .

**Proposition 5.10.** The logarithmic tangent bundle  $\mathcal{E} = T_X(-\log D)$  is stable (resp. semi-stable) with respect to  $L$  if and only if

$$\mu_L(\mathcal{E}) = \frac{1}{r+s} \left( \sum_{i \in I} V_i + \text{card}(J) \cdot W \right)$$

is greater than (resp. greater than or equal to) the maximum of

- (1)  $\mathbf{V}_{i_0}$  where  $i_0 = \min I$  if  $I \neq \emptyset$  ;
- (2)  $\frac{1}{r'} \sum_{i \in I} \mathbf{V}_i$  , if  $r' = \dim \text{Span}(v_i : i \in I) \neq 0$  ;
- (3)  $\frac{\text{card}(J) \cdot \mathbf{W}}{s'}$  , if  $s' = \dim \text{Span}(w_j : j \in J) \neq 0$  and  $s' < r + s$  ;
- (4)  $\frac{1}{s+k} \left( \sum_{i \in I'} \mathbf{V}_i + (s+1)\mathbf{W} \right)$  , if  $\text{card}(J') = s+1$  ,  $k = \text{card}(I') < r$  and  $\{z+1, \dots, r\} \subset I' \subset I$  ;
- (5)  $\frac{1}{s+k} \left( \sum_{i \in I'} \mathbf{V}_i + (s+1)\mathbf{W} \right)$  , if  $\text{card}(J') = s+1$  ,  $k = \text{card}(I') < r$  and  $I' \subset I$  such that the condition ii. of Lemma 5.9 is verified.

*Proof.* Let  $G = \text{Span}(v_i, w_j : i \in I', j \in J')$  where  $I' \subset I$  and  $J' \subset J$ . The expressions (1) of Proposition 5.10 corresponds to  $G = \text{Span}(v_{i_0})$ , (2) corresponds to  $G = \text{Span}(v_i : i \in I)$  and (3) corresponds to  $G = \text{Span}(w_j : j \in J)$ .

- If  $\text{card}(J') = 0$ , then for  $\emptyset \subsetneq I' \subseteq I$ , we have  $\dim G \leq r$  and

$$\mu_L(\mathcal{E}_G) = \frac{1}{\dim G} \sum_{i \in I'} \mathbf{V}_i ;$$

this number is less than or equal to the maximum of the numbers given in (1) and (2).

- If  $\text{card}(I') = 0$ , then for  $\emptyset \subsetneq J' \subseteq J$  such that  $\dim G < r + s$ , we have

$$\mu_L(\mathcal{E}_G) = \frac{\text{card}(J') \cdot \mathbf{W}}{\dim G} ;$$

this number is less than or equal to that given in (3).

- If  $\text{card}(I') = r + 1$ , then  $\dim G < r + s$  if and only if  $s' := \text{card}(J') < s$ . If  $1 \leq s' < s$ , then

$$\mu_L(\mathcal{E}_G) = \frac{1}{r+s'} \left( \sum_{i \in I'} \mathbf{V}_i + s'\mathbf{W} \right) \leq \max \left( \frac{1}{r} \sum_{i \in I'} \mathbf{V}_i, \mathbf{W} \right) ;$$

this number is less than or equal to the maximum of numbers given in (2) and (3).

- If  $1 \leq \text{card}(I') \leq r$  ,  $1 \leq \text{card}(J') \leq s$  and  $\dim G < r + s$ , then  $\mu_L(\mathcal{E}_G)$  is less than or equal to the maximum of numbers given in (1), (2) and (3).
- It remains to study the case where  $\text{card}(J') = s+1$  and  $1 \leq \text{card}(I') < r$  (because if  $\text{card}(I') \geq r$ , then  $\dim G = r + s$ ). We will treat it in two cases.

*First case :*  $a_r = 0$ . For all  $i \in \{1, \dots, r\}$ ,  $\mathbf{V}_i = \mathbf{V}_0$ . If  $r' = \text{card}(I')$  and  $1 \leq r' < r$ , then

$$\mu_L(\mathcal{E}_G) = \frac{1}{r'+s} \left( \sum_{i \in I'} \mathbf{V}_i + (s+1)\mathbf{W} \right) \leq \max \left( \mathbf{V}_0, \frac{(s+1)\mathbf{W}}{s} \right) .$$

*Second case :*  $a_r > 0$ . We set  $r' = \text{card}(I')$ . If  $I'$  satisfies the first (resp. second) condition of Lemma 5.9, we get (4) (resp. (5)).

If  $I'$  doesn't satisfy the conditions of Lemma 5.9, then  $\dim G = r' + (s+1)$ . Moreover, if  $r' + (s+1) < r + s$ , the number  $\mu_L(\mathcal{E}_G)$  is less than or equal to the maximum of the numbers given in (1) and (3).  $\square$

*Remark 5.11.* If  $a_1 = \dots = a_r = 0$ , to check the stability of  $\mathcal{E}$  with respect to  $L$ , it is enough to compare  $\mu_L(\mathcal{E})$  with the numbers given by the points 1, 2 and 3 of Proposition 5.10.

To check that  $\mathcal{E} = T_X(-\log D)$  is not semi-stable, we will need the lemma below. The purpose of the lemma is to avoid comparing  $\mu_L(\mathcal{E})$  with all the numbers given in the Proposition 5.10.

**Lemma 5.12.**

- (1) If  $a_r \geq 2$ , then  $sV_0 - (s+1)W \geq sV_r$ .
- (2) Let  $r \geq 2$  and  $i \in \{1, \dots, r-1\}$ . If  $a_r > a_i$ , then  $V_i - W \geq V_r$ .

*Proof.* If  $a_r \geq 2$ , then  $\left(s - \frac{s+1}{a_r}\right) = \frac{a_r s - (s+1)}{a_r} \geq \frac{2s - (s+1)}{a_r} \geq 0$  because  $s \geq 1$ . Hence,

$$\begin{aligned} sV_0 - (s+1)W &= sV_0 - \frac{s+1}{a_r}(V_0 - V_r) \quad \text{because } a_r W = V_0 - V_r \\ &= \left(s - \frac{s+1}{a_r}\right)V_0 + \frac{s+1}{a_r}V_r \\ &\geq \left(s - \frac{s+1}{a_r}\right)V_r + \frac{s+1}{a_r}V_r = sV_r. \end{aligned}$$

Since  $V_0 = a_i W + V_i = a_r W + V_r$ , we have  $V_i = (a_r - a_i)W + V_r$ . If  $a_r > a_i$ , then  $a_r - a_i \geq 1$ ; therefore  $V_i \geq W + V_r$ , i.e.  $V_i - W \geq V_r$ .  $\square$

**5.4. Stability of logarithmic tangent bundle on  $X = \mathbb{P}^r \times \mathbb{P}^s$ .** In this part, we assume that  $a_1 = a_2 = \dots = a_r = 0$ . We have  $X \cong \mathbb{P}^s \times \mathbb{P}^r$ . We denote by  $\pi_1 : X \rightarrow \mathbb{P}^s$  and  $\pi_2 : X \rightarrow \mathbb{P}^r$  the projection maps.

We first show that  $T_X(-\log D)$  is decomposable when  $\text{card}(\Delta) \in \{1, 2\}$ .

**Lemma 5.13.** For  $i \in \{0, \dots, r\}$ , the vector bundle  $\mathcal{E} = T_X(-\log D_{v_i})$  satisfies

$$\mathcal{E} = \left( \bigoplus_{k=0, k \neq i}^r \mathcal{E}_{F_k} \right) \oplus \mathcal{E}_G$$

where  $F_k = \text{Span}(v_k)$  and  $G = \text{Span}(w_0, \dots, w_s)$ .

**Lemma 5.14.** For  $j \in \{0, \dots, s\}$ , the vector bundle  $\mathcal{E} = T_X(-\log D_{w_j})$  satisfies

$$\mathcal{E} = \left( \bigoplus_{k=0, k \neq j}^s \mathcal{E}_{G_k} \right) \oplus \mathcal{E}_F$$

where  $G_k = \text{Span}(w_k)$  and  $F = \text{Span}(v_0, \dots, v_r)$ .

**Lemma 5.15.** Let  $i, i' \in \{0, \dots, r\}$  distinct and  $j, j' \in \{0, \dots, s\}$  distinct. We have

$$\begin{aligned} T_X(-\log(D_{v_i} + D_{v_{i'}})) &\cong T_{\mathbb{P}^s} \oplus T_{\mathbb{P}^r}(-\log(\pi_2(D_{v_i}) + \pi_2(D_{v_{i'}}))) \\ T_X(-\log(D_{w_j} + D_{w_{j'}})) &\cong T_{\mathbb{P}^s}(-\log(\pi_1(D_{w_j}) + \pi_1(D_{w_{j'}}))) \oplus T_{\mathbb{P}^r} \end{aligned}$$

For the proofs of these three lemmas it suffices to work with the family of multifiltrations as in the proof of Theorem 3.9. If  $i \in \{0, \dots, r\}$  and  $j \in \{0, \dots, s\}$ , by Theorem 3.9 the vector bundle  $T_X(-\log(D_{v_i} + D_{w_j}))$  is decomposable.

We now study (semi)stability. In this table, we give the values of  $\nu$  for which  $\mathcal{E} = T_X(-\log D)$  is (semi)stable with respect to  $\pi^* \mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$ .

TABLE 1. Stability of  $T_X(-\log D)$  when  $a_1 = \dots = a_r = 0$ 

Divisor $D$	$\text{Stab}(\mathcal{E})$	$\text{sStab}(\mathcal{E})$	References
$D_{v_i}, \quad 0 \leq i \leq r$	$\emptyset$	$\nu = \frac{s+1}{r}$	Theorem 5.16
$D_{w_j}, \quad 0 \leq j \leq s$	$\emptyset$	$\nu = \frac{s}{r+1}$	Theorem 5.16
$D_{v_j} + D_{w_j}$	$\emptyset$	$\nu = \frac{s}{r}$	Theorem 5.19
$D_{v_i} + D_{v_j}, \quad 0 \leq i < j \leq r$	$\emptyset$	$\emptyset$	Theorem 5.18
$D_{w_i} + D_{w_j}, \quad 0 \leq i < j \leq s$	$\emptyset$	$\emptyset$	Theorem 5.18

By (13), we have  $\mathbb{V} = \frac{r\nu}{s}\mathbb{W}$ .

**Theorem 5.16.** *Let  $i \in \{0, 1, \dots, r\}$  and  $j \in \{0, 1, \dots, s\}$ . For any  $L \in \text{Amp}(X)$ , the logarithmic tangent bundles  $T_X(-\log D_{v_i})$  and  $T_X(-\log D_{w_j})$  are not stable with respect to  $L$ . We have :*

- (1)  $T_X(-\log D_{v_i})$  is semi-stable with respect to  $\pi^*\mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$  if and only if  $\nu = \frac{s+1}{r}$ ;
- (2)  $T_X(-\log D_{w_j})$  is semi-stable with respect to  $\pi^*\mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$  if and only if  $\nu = \frac{s}{r+1}$ .

*Proof.* We start with  $\mathcal{E} = T_X(-\log D_{v_i})$ . We have :

$$\mu_L(\mathcal{E}) = \frac{1}{r+s}(r\mathbb{V} + (s+1)\mathbb{W}) = \frac{r^2\nu + s^2 + s}{s(r+s)}\mathbb{W} = \frac{r^2\nu + s^2 + s}{r\nu(r+s)}\mathbb{V}.$$

By Proposition 5.10, to have stability or semi-stability, it is enough to compare  $\mu_L(\mathcal{E})$  with  $\max\left(\frac{(s+1)\mathbb{W}}{s}, \mathbb{V}\right)$ .

If  $\mu_L(\mathcal{E}) \geq \mathbb{V}$ , then  $\frac{r^2\nu + s^2 + s}{r\nu(r+s)} \geq 1$ , i.e.  $(r^2\nu + s^2 + s) \geq (r^2\nu + r\nu s)$ ; hence,  $\nu \leq \frac{s+1}{r}$ . If  $\mu_L(\mathcal{E}) \geq \frac{(s+1)\mathbb{W}}{s}$ , then  $\frac{r^2\nu + s^2 + s}{s(r+s)} \geq \frac{s+1}{s}$ , i.e.  $\nu \geq \frac{s+1}{r}$ .

If  $\nu \neq \frac{s+1}{r}$ , the numbers  $\mu_L(\mathcal{E}) - \mathbb{V}$  and  $\mu_L(\mathcal{E}) - \frac{s+1}{s}\mathbb{W}$  are non-zero and have opposite signs. Therefore,  $T_X(-\log D_{v_i})$  is not semi-stable with respect to  $\pi^*\mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$ .

If  $\nu = \frac{s+1}{r}$ , then  $\mu_L(\mathcal{E}) = \mathbb{V} = \frac{s+1}{s}\mathbb{W}$ . Thus,  $T_X(-\log D_{v_i})$  is semi-stable but not stable with respect to  $\pi^*\mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$ .

If we regard the case where  $\mathcal{E} = T_X(-\log D_{w_j})$ , it is enough to compare  $\mu_L(\mathcal{E})$  with  $\max\left(\frac{(r+1)\mathbb{V}}{r}, \mathbb{W}\right)$ . A similar computation gives the result.  $\square$

**Corollary 5.17.** *With the decomposition given in Lemma 5.13 (resp. Lemma 5.14), the vector bundle  $T_X(-\log D_{v_i})$  (resp.  $T_X(-\log D_{w_i})$ ) is poly-stable with respect to  $\pi^*\mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$  if and only if  $\nu = \frac{s+1}{r}$  (resp.  $\nu = \frac{s}{r+1}$ ).*

**Theorem 5.18.** *Let  $i, i' \in \{0, \dots, r\}$  distinct and  $j, j' \in \{0, \dots, s\}$  distinct. For any  $L \in \text{Amp}(X)$ , the logarithmic tangent bundles  $T_X(-\log(D_{v_i} + D_{v_{i'}}))$  and  $T_X(-\log(D_{w_j} + D_{w_{j'}}))$  are not semi-stables with respect to  $L$ .*

*Proof.* Let  $\mathcal{E} = T_X(-\log(D_{v_i} + D_{v_{i'}}))$ . We have :

$$\mu_L(\mathcal{E}) = \frac{1}{r+s}((r-1)\mathbf{V} + (s+1)\mathbf{W}) = \frac{r(r-1)\nu + s^2 + s}{s(r+s)}\mathbf{W} = \frac{r(r-1)\nu + s^2 + s}{r\nu(r+s)}\mathbf{V}.$$

We assume that  $r = 1$ . We have  $\mu_L(\mathcal{E}) = \mathbf{W}$ . If we set  $F = \text{Span}(w_0, \dots, w_s)$ , we have  $\mu_L(\mathcal{E}_F) = \frac{(s+1)\mathbf{W}}{s}$ ; thus  $T_X(-\log(D_{v_i} + D_{v_{i'}}))$  is not semi-stable with respect to  $L$ .

We now assume that  $r \geq 2$ . To check the semi-stability, it is enough to compare  $\mu_L(\mathcal{E})$  with  $\max\left(\frac{(s+1)\mathbf{W}}{s}, \mathbf{V}\right)$ .

If  $\mu_L(\mathcal{E}) \geq \mathbf{V}$ , then  $\frac{r(r-1)\nu + s^2 + s}{r\nu(r+s)} \geq 1$ ; i.e  $\nu \leq \frac{s}{r}$ .

If  $\mu_L(\mathcal{E}) \geq \frac{(s+1)\mathbf{W}}{s}$ , then  $\frac{r(r-1)\nu + s^2 + s}{s(r+s)} \geq \frac{s+1}{s}$ ; i.e  $\nu \geq \frac{s+1}{r-1} > \frac{s}{r}$ .

As  $\nu$  cannot satisfy this two conditions, we deduce that  $T_X(-\log(D_{v_i} + D_{v_{i'}}))$  is not semi-stable with respect to  $L$ .  $\square$

**Theorem 5.19.** *Let  $i \in \{0, \dots, r\}$ ,  $j \in \{0, \dots, s\}$  and  $D = D_{v_i} + D_{w_j}$ . We have :*

- (1)  $T_X(-\log D)$  is not stable with respect to any  $L \in \text{Amp}(X)$ .
- (2)  $T_X(-\log D)$  is semi-stable with respect to  $\pi^*\mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$  if and only if  $\nu = \frac{s}{r}$ .

*Proof.* We have

$$\mu_L(\mathcal{E}) = \frac{1}{r+s}(r\mathbf{V} + s\mathbf{W}) = \frac{r^2\nu + s^2}{s(r+s)}\mathbf{W} = \frac{r^2\nu + s^2}{r\nu(r+s)}\mathbf{V}.$$

To check the semi-stability, it is enough to compare  $\mu_L(\mathcal{E})$  with  $\max(\mathbf{V}, \mathbf{W})$ .

If  $\mu_L(\mathcal{E}) \geq \mathbf{V}$ , then  $\frac{r^2\nu + s^2}{r\nu(r+s)} \geq 1$ ; i.e  $\nu \leq \frac{s}{r}$ .

If  $\mu_L(\mathcal{E}) \geq \mathbf{W}$ , then  $\frac{r^2\nu + s^2}{s(r+s)} \geq 1$ ; i.e  $\nu \geq \frac{s}{r}$ .

Hence,  $T_X(-\log D)$  is semi-stable with respect to  $\pi^*\mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$  if and only if  $\nu = \frac{s}{r}$ .  $\square$

**Corollary 5.20.** *With the decomposition given in theorem 3.9,  $T_X(-\log(D_{v_i} + D_{w_j}))$  is poly-stable with respect to  $\pi^*\mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$  if and only if  $\nu = \frac{s}{r}$ .*

According to (11) and (12), when  $a_1 = \dots = a_r = 0$ , we have :

$$D_{v_i} \sim_{\text{lin}} D_{v_0}, \quad D_{w_j} \sim_{\text{lin}} D_{w_0} \quad \text{and} \quad -K_X \sim_{\text{lin}} (s+1)D_{w_0} + (r+1)D_{v_0}.$$

By the above study when  $\text{sStab}(T_X(-\log D)) \neq \emptyset$ , we see that  $T_X(-\log D)$  is semi-stable with respect to  $L$  if and only if  $L \cong \mathcal{O}_X(-\alpha(K_X + D))$  with  $\alpha \in \mathbb{N}^*$ . This proves Theorem 1.4.

6. STABILITY OF  $T_X(-\log D)$  ON SMOOTH TORIC VARIETIES OF PICARD RANK TWO WHEN  $a_r \geq 1$ 

In this part, we will study the stability of  $T_X(-\log D)$  when  $X$  is given by (10) with  $a_r \geq 1$ . If  $D = \sum_{\rho \in \Delta} D_\rho$  with  $\Delta \subset \Sigma(1)$ , by Corollary 4.8, we will only study the case where  $\text{card } \Delta \in \{1, 2\}$ . The case  $\text{card } \Delta = 0$  was treated by HERING-NILL-SÜSS in [HNS22].

Let  $L = \pi^* \mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$  be an element of  $\text{Amp}(X) \subset N^1(X) \otimes_{\mathbb{Z}} \mathbb{R}$ . We recall that the numbers  $V_0, \dots, V_r$  defined on Section 5.2 are polynomials of  $\nu$  of degree  $s$  and  $W$  is polynomial of degree  $s - 1$ . If  $\mathcal{E} = T_X(-\log D)$ , the number  $\mu_L(\mathcal{E})$  is polynomial of degree at most  $s$ . Let  $P_0, P_1, P_2$  and  $Q$  be the polynomials of  $\nu$  defined by

$$P_0 = \mu_L(\mathcal{E}) - V_0, \quad P_1 = \mu_L(\mathcal{E}) - V_1, \quad P_2 = \mu_L(\mathcal{E}) - V_2 \quad \text{and} \quad Q = \mu_L(\mathcal{E}) - W.$$

We first recall the sign changes rule of Descartes. We refer to [Mig88, Chapter 5, Section 4.3].

**Theorem 6.1** (Descartes). *Let  $P = c_n X^n + c_{n-1} X^{n-1} + \dots + c_0$  be a polynomial with real coefficients where  $c_n c_0 \neq 0$ . Let  $p$  the number of sign changes in the sequence  $(c_0, \dots, c_n)$  of its coefficients and  $q$  the numbers of positive real roots, counted with their order of multiplicity. Then, there is  $m \in \mathbb{N}$  such that  $q = p - 2m$ .*

Under certain conditions on  $a_i, r$  and  $s$ , these polynomials ( $P_0, P_1, P_2$  and  $Q$ ) have respectively one or no positive root. If the positive root exists, we denote by

- $\nu_i$  the unique positive root of  $P_i$  where  $i \in \{0, 1, 2\}$
- $\nu_3$  the unique positive root of  $Q$ .

In these tables, we give the values of  $\nu$  for which  $\mathcal{E} = T_X(-\log D)$  is stable or semi-stable with respect to  $L = \pi^* \mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$ .

 TABLE 2. Stability of  $T_X(-\log D)$  when  $a_r \geq 1$ 

Divisor $D$	Condition on $r$ and $a_i$	Condition on $s$	$\text{Stab}(\mathcal{E})$	$\text{sStab}(\mathcal{E})$
$D_{w_j}$ $0 \leq j \leq s$ Theorem 6.2	$r \geq 1$ and $a_r \geq 1$	$s \geq 1$	$\emptyset$	$\emptyset$
$D_{v_i}$ $1 \leq i \leq r - 1$ Prop. 6.4	$r \geq 2$ and $a_r \geq 1$	$s \geq 1$	$\emptyset$	$\emptyset$
$D_{v_r}$ Theorem 6.6	$r \geq 1, a_r = 1$ and $a_{r-1} = 0$	$s \geq 1$	$0 < \nu < \nu_0$	$0 < \nu \leq \nu_0$
	$r \geq 1$ and ( $a_r \geq 2$ or $a_{r-1} \neq 0$ )	$s \geq 1$	$\emptyset$	$\emptyset$
$D_{v_0}$ Theorem 6.9 Lemma 6.8 Theorem 6.11	$r = 1$	$s \geq 1$	$0 < \nu < \nu_1$	$0 < \nu \leq \nu_1$
	$r \geq 2$ and $a_1 < a_r$	$s \geq 1$	$\emptyset$	$\emptyset$
	$r \geq 2$ and $a_1 = a_r = a$	$a \geq \frac{s+1}{r-1}$	$\emptyset$	$\emptyset$
		$\frac{s}{r} \leq a < \frac{s+1}{r-1}$	$0 < \nu < \nu_1$	$0 < \nu \leq \nu_1$
		$a r < s$	$\nu_3 < \nu < \nu_1$	$\nu_3 \leq \nu \leq \nu_1$

TABLE 3. Stability of  $\mathcal{F}_X(-\log D)$  when  $a_r \geq 1$ 

Divisor $D$	Condition on $r$ and $a_i$	Condition on $s$	$\text{Stab}(\mathcal{E})$	$\text{sStab}(\mathcal{E})$
$D_{w_i} + D_{w_j}$ $0 \leq i < j \leq s$ Theorem 6.2	$r \geq 1$ and $a_r \geq 1$	$s \geq 1$	$\emptyset$	$\emptyset$
$D_{v_i} + D_{v_j}$ $1 \leq i < j \leq r$ Corollary 6.5	$r \geq 2$ and $a_r \geq 1$	$s \geq 1$	$\emptyset$	$\emptyset$
$D_{v_i} + D_{w_j}$ , $j \geq 0$ and $1 \leq i \leq r-1$ Proposition 6.4	$r \geq 2$ and $a_r \geq 1$	$s \geq 1$	$\emptyset$	$\emptyset$
$D_{v_r} + D_{w_j}$ , $j \geq 0$ Corollary 6.7	$r \geq 1$ and $a_r \geq 1$	$s \geq 1$	$\emptyset$	$\emptyset$
$D_{v_0} + D_{w_j}$ , $0 \leq j \leq s$ Theorem 6.9 Lemma 6.8 Theorem 6.13	$r = 1$	$s \geq 1$	$\emptyset$	$\nu = \nu_3$
	$r \geq 2$ and $a_1 < a_r$	$s \geq 1$	$\emptyset$	$\emptyset$
	$r \geq 2$ and $a_1 = a_r = a$	$s \leq a(r-1)$	$\emptyset$	$\emptyset$
		$s > a(r-1)$	$\emptyset$	$\nu = \nu_3$
$D_{v_0} + D_{v_i}$ , $2 \leq i \leq r$ Lemma 6.8 Theorem 6.13	$r \geq 2$ and $a_1 < a_r$	$s \geq 1$	$\emptyset$	$\emptyset$
	$r \geq 2$ and $a_1 = a_r = a$	$s \leq a(r-1)$	$\emptyset$	$\emptyset$
		$s > a(r-1)$	$\emptyset$	$\nu = \nu_3$
		$s > a(r-1)$	$\emptyset$	$\nu = \nu_3$

TABLE 4. Stability of  $T_X(-\log D)$  when  $a_r \geq 1$ 

Divisor $D$	Condition on $r$ and $a_i$	Condition on $s$	$\text{Stab}(\mathcal{E})$	$\text{sStab}(\mathcal{E})$
$D_{v_0} + D_{v_1}$ Theorem 6.9 Prop. 6.14 Theorem 6.13 Prop. 6.17 Lemma 6.15 Prop. 6.19	$r = 1$	$s \geq 1$	$\emptyset$	$\nu > 0$
	$r \geq 2$ and $0 = a_1 < a_r$	$s \geq 1$	$\emptyset$	$\emptyset$
	$r \geq 2$ and $a_1 = a_r = a$	$s \leq a(r-1)$	$\emptyset$	$\emptyset$
		$s > a(r-1)$	$\emptyset$	$\nu = \nu_3$
	$r = 2$ and $0 < a_1 < a_2$	$s \leq \delta_2$	$\emptyset$	$\emptyset$
		$s > \delta_2$	$\emptyset$	$\nu = \nu_3$
	$r \geq 3$ and $a_2 < a_r$	$s \geq 1$	$\emptyset$	$\emptyset$
	$r \geq 3$ and $0 < a_1 < a_2 = \dots = a_r$	$s \leq \delta_r$	$\emptyset$	$\emptyset$
$s > \delta_r$		$\emptyset$	$\nu = \nu_3$	

6.1. **The case  $D = D_{w_j}$  and  $D = D_{w_j} + D_{w_i}$ .** Let  $i, j \in \{0, \dots, s\}$  distinct and  $\mathcal{E} = T_X(-\log D_{w_i})$ ,  $\mathcal{F} = T_X(-\log(D_{w_i} + D_{w_j}))$ .

If  $a_r \geq 1$ , the Lemma 5.14 is again true. The vector bundles  $\mathcal{E}$  and  $\mathcal{F}$  are decomposable and we have

$$\mathcal{E} = \left( \bigoplus_{k=0, k \neq i}^s \mathcal{E}_{G_k} \right) \oplus \mathcal{E}_F \quad \text{and} \quad \mathcal{F} = \left( \bigoplus_{k=0, k \neq i}^s \mathcal{F}_{G_k} \right) \oplus \mathcal{F}_F$$

where  $G_k = \text{Span}(w_k)$  and  $F = \text{Span}(v_0, \dots, v_r)$ .

**Theorem 6.2.** *Let  $(a_1, \dots, a_r) \neq (0, \dots, 0)$ . For any  $L \in \text{Amp}(X)$ , the vector bundles  $\mathcal{E}$  and  $\mathcal{F}$  are not semi-stable with respect to  $L$ .*

*Proof.* Let  $L = \pi^* \mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$ , we have

$$\mu_L(\mathcal{E}) = \frac{1}{r+s} (sW + (V_0 + \dots + V_r))$$

and

$$\mu_L(\mathcal{F}) = \frac{1}{r+s} ((s-1)W + (V_0 + \dots + V_r)) ;$$

thus,  $\mu_L(\mathcal{F}) < \mu_L(\mathcal{E})$ . The points 4 and 5 of Proposition 5.10 are not verifying. To check that  $\mathcal{E}$  (resp.  $\mathcal{F}$ ) is not semi-stable with respect to  $L$  it is enough to compare  $\mu_L(\mathcal{E})$  (resp.  $\mu_L(\mathcal{F})$ ) with

$$\max \left( V_0, \frac{1}{r} (V_0 + V_1 + \dots + V_r) \right) .$$

By Corollary 5.8, we have  $V_0 = a_r W + V_r$ . As  $a_r \geq 1$ , we have  $V_0 \geq W + V_r$ , i.e  $V_0 - W \geq V_r$ . Thus,

$$\begin{aligned} (r+s)(V_0 - \mu_L(\mathcal{E})) &= s(V_0 - W) + (rV_0 - (V_0 + \dots + V_{r-1})) - V_r \\ &\geq s(V_0 - W) - V_r \quad \text{because } V_i \leq V_0 \\ &\geq (s-1)V_r . \end{aligned}$$

If  $s \geq 2$ , we have  $V_0 - \mu_L(\mathcal{E}) > 0$  and  $V_0 - \mu_L(\mathcal{F}) > 0$ . Thus,  $\mathcal{E}$  and  $\mathcal{F}$  are not semi-stables with respect to  $L$ .

We now assume that  $s = 1$ . By using the formula defining  $V_i$  and  $W$  given in Section 5.2, we have

$$W = 1, \quad V_0 = (a_1 + \dots + a_r) + r\nu \quad \text{and} \quad V_i = V_0 - a_i \quad \text{for } i \in \{1, \dots, r\} .$$

We have

$$\mu_L(\mathcal{E}) = \frac{1}{r+1} (1 + (r+1)V_0 - (a_1 + \dots + a_r))$$

and

$$\frac{1}{r} (V_0 + \dots + V_r) = \frac{1}{r} ((r+1)V_0 - (a_1 + \dots + a_r)) .$$

Therefore,  $\frac{V_0 + \dots + V_r}{r} - \mu_L(\mathcal{E}) = \frac{(r+1)V_0 - (a_1 + \dots + a_r) - r}{r(r+1)}$ . We have

$$\begin{aligned} (r+1)V_0 - (a_1 + \dots + a_r) - r &= (r+1)(a_1 + \dots + a_r) + (r+1)r\nu \\ &\quad - (a_1 + \dots + a_r) - r \\ &= r(a_1 + \dots + a_r - 1) + (r+1)r\nu > 0 \end{aligned}$$

because  $(a_1 + \dots + a_r - 1) \geq 0$  and  $\nu > 0$ . Thus,  $\mathcal{E}$  and  $\mathcal{F}$  are not semi-stables with respect to  $L$ .  $\square$

**6.2. The case where  $\text{Cone}(v_0) \notin \Delta$ .** If  $i \in \{1, \dots, r\}$  and  $j \in \{0, \dots, s\}$ , by Theorem 3.9, the logarithmic tangent bundle  $T_X(-\log(D_{v_i} + D_{w_j}))$  is decomposable. Let  $r \geq 2$  and  $D \in \{D_{v_i} + D_{v_j} : 1 \leq i < j \leq r\}$ ; we set  $\mathcal{E} = T_X(-\log D)$ , we have the following lemma :

**Lemma 6.3.** *The logarithmic tangent bundle  $\mathcal{E}$  is decomposable. If  $a_i < a_j$ , then*

$$\mathcal{E} = \left( \bigoplus_{l=0}^s \mathcal{E}_{G_l} \right) \oplus \left( \bigoplus_{k=0, k \notin \{i, j\}}^r \mathcal{E}_{F_k} \right)$$

where  $G_l = \text{Span}(w_l)$  and  $F_k = \text{Span}(v_k)$ . If  $a_i = a_j$ , then

$$\mathcal{E} = \mathcal{E}_G \oplus \mathcal{E}_F$$

where  $G = \text{Span}(w_l, v_k : l \in \{0, \dots, s\}, k \in \{0, \dots, r\} \setminus \{i, j\})$  and  $F = \text{Span}(v_i)$  or  $F = \text{Span}(v_j)$ .

*Proof.* For the proof we use the families of multifiltrations. We use the Lemma 5.9 to explain the choice of this decomposition.  $\square$

If  $D \in \{D_{v_i} : 1 \leq i \leq r\}$ , we will not search to know if  $\mathcal{E} = T_X(-\log D)$  is decomposable because it depends on the numbers  $r, s$  and  $a_1, \dots, a_r$ . In particular, if we assume that  $r = 2, s = 1, a_1 = 0$  and  $a_2 = 1$ , then  $\mathcal{E} = T_X(-\log D_{v_1})$  is decomposable and  $\mathcal{E} = \mathcal{E}_F \oplus \mathcal{E}_G$  where  $F = \text{Span}(v_2, w_1)$  and  $G = \text{Span}(v_0)$ . But  $\mathcal{F} = T_X(-\log D_{v_2})$  is not decomposable.

We first study the stability of  $T_X(-\log D)$  when  $r \geq 2$  and

$$D \in \{D_{v_i} : 1 \leq i \leq r-1\} \cup \{D_{v_i} + D_{w_j} : 1 \leq i \leq r-1 \text{ and } 0 \leq j \leq s\} \\ \cup \{D_{v_i} + D_{v_j} : 1 \leq i < j \leq r\}.$$

**Proposition 6.4.** *Let  $r \geq 2, (a_1, \dots, a_r) \neq (0, \dots, 0), i \in \{1, \dots, r-1\}$  and  $j \in \{0, \dots, s\}$ . For any  $L \in \text{Amp}(X)$ , the logarithmic tangent bundles  $T_X(-\log D_{v_i})$  and  $T_X(-\log(D_{v_i} + D_{w_j}))$  are not semi-stables with respect to  $L$ .*

*Proof.* We set  $\mathcal{E} = T_X(-\log D_{v_i})$  and  $\mathcal{F} = T_X(-\log(D_{v_i} + D_{w_j}))$ . Let  $L \in \text{Amp}(X)$ , we have

$$\mu_L(\mathcal{E}) = \frac{1}{r+s} ((s+1)W + (V_0 + \dots + V_{i-1} + V_{i+1} + \dots + V_r))$$

and  $\mu_L(\mathcal{F}) < \mu_L(\mathcal{E})$ . We will show that  $V_0 > \mu_L(\mathcal{E})$ . We have

$$(r+s)(V_0 - \mu_L(\mathcal{E})) = (s+1)(V_0 - W) - V_r \\ + ((r-1)V_0 - (V_0 + \dots + V_{i-1} + V_{i+1} + \dots + V_{r-1})) \\ \geq (s+1)(V_0 - W) - V_r \\ \geq (s+1)V_r - V_r = sV_r$$

By Proposition 5.10,  $\mathcal{E}$  and  $\mathcal{F}$  are not semi-stables with respect to  $L$ .  $\square$

**Corollary 6.5.** *Let  $r \geq 2$  and  $(a_1, \dots, a_r) \neq (0, \dots, 0)$ . Let  $L \in \text{Amp}(X)$ ; for all  $i, j \in \{1, \dots, r\}$  distinct, the logarithmic tangent bundle  $T_X(-\log(D_{v_i} + D_{v_j}))$  is not semi-stable with respect to  $L$ .*

*Proof.* If we set  $\mathcal{G} = T_X(-\log(D_{v_i} + D_{v_j}))$ , by using the proof of Proposition 6.4, we have  $\mu_L(\mathcal{G}) < \mu_L(\mathcal{E}) < V_0$ . Thus,  $\mathcal{G}$  is not semi-stable with respect to  $L$ .  $\square$

We now study the stability of  $T_X(-\log D)$  when

$$D \in \{D_{v_r}\} \cup \{D_{v_r} + D_{w_j} : 0 \leq j \leq s\}.$$

**Theorem 6.6.** *Let  $r \geq 1$  and  $a_r \geq 1$ . We have  $\text{Stab}(T_X(-\log D_{v_r})) \neq \emptyset$  if and only if  $\text{sStab}(T_X(-\log D_{v_r})) \neq \emptyset$  if and only if  $a_r = 1$  and  $a_{r-1} = 0$ .*

*If  $a_r = 1$  and  $a_{r-1} = 0$ , then the logarithmic tangent bundle  $T_X(-\log D_{v_r})$  is stable (resp. semi-stable) with respect to  $\pi^* \mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$  if and only if  $0 < \nu < \nu_0$  (resp.  $0 < \nu \leq \nu_0$ ) where  $\nu_0$  is the unique positive root of*

$$P_0(x) = \sum_{k=0}^{s-1} \binom{s+r-1}{k} x^k - s \binom{s+r-1}{s} x^s.$$

*Proof.* Let  $\mathcal{E} = T_X(-\log D_{v_r})$  and  $L = \pi^* \mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$ . We have

$$(r+s)\mu_L(\mathcal{E}) = (s+1)W + V_0 + V_1 + \dots + V_{r-1}.$$

• If  $a_r \geq 2$ , by using the first point of Lemma 5.12 and the fact that  $V_i \leq V_0$ , we get :

$$(r+s)[V_0 - \mu_L(\mathcal{E})] = (sV_0 - (s+1)W) + rV_0 - (V_0 + \dots + V_{r-1}) \geq sV_r.$$

By Proposition 5.10,  $T_X(-\log D_{v_r})$  is not semi-stable with respect to  $L$ .

• We assume that  $r \geq 2$  and  $a_{r-1} = a_r = 1$ . As  $V_{r-1} = V_r$ , we have

$$\begin{aligned} (r+s)[V_0 - \mu_L(\mathcal{E})] &= (s+1)[V_0 - W] - V_{r-1} + [(r-1)V_0 - (V_0 + \dots + V_{r-2})] \\ &\geq (s+1)V_r - V_{r-1} \quad \text{because } V_0 - W \geq V_r \\ &\geq sV_r \end{aligned}$$

By Proposition 5.10,  $T_X(-\log D_{v_r})$  is not semi-stable with respect to  $L$ .

• Let  $r \geq 1$ . We now assume that  $a_{r-1} = 0$  and  $a_r = 1$ . By using the expressions of Section 5.2, we have  $V_0 = \dots = V_{r-1} = V$  where

$$V = \sum_{k=0}^s \binom{s+r-1}{k} \nu^k \quad \text{and} \quad W = \sum_{k=0}^{s-1} \binom{s+r-1}{k} \nu^k.$$

The points 4 and 5 of Proposition 5.10 are not verified in this case. To check the stability of  $\mathcal{E}$  it is enough to compare

$$\mu_L(\mathcal{E}) = \frac{1}{r+s}(rV + (s+1)W)$$

with  $\max(V, W)$ . We have

$$(r+s)[\mu_L(\mathcal{E}) - W] = rV - (r-1)W > 0 \quad \text{because } W < V$$

and

$$\begin{aligned} (r+s)[\mu_L(\mathcal{E}) - V] &= (s+1)W - sV \\ &= \sum_{k=0}^{s-1} \binom{s+r-1}{k} \nu^k - s \binom{s+r-1}{s} \nu^s = P_0(\nu). \end{aligned}$$

By the sign rule of Descartes (see Theorem 6.1), the polynomial  $P_0$  have a unique positive root  $\nu_0$ . If  $\nu > 0$ , then  $P_0(\nu) > 0$  (resp.  $P_0(\nu) \geq 0$ ) if and only if  $\nu < \nu_0$  (resp.  $\nu \leq \nu_0$ ). Thus,  $T_X(-\log D_{v_r})$  is stable (resp. semi-stable) with respect to  $\pi^* \mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$  if and only if  $0 < \nu < \nu_0$  (resp.  $0 < \nu \leq \nu_0$ ).  $\square$

**Corollary 6.7.** *We assume that  $r \geq 1$  and  $(a_1, \dots, a_r) \neq (0, \dots, 0)$ . Let  $j \in \{0, \dots, s\}$  and  $D = D_{v_r} + D_{w_j}$ . For any  $L \in \text{Amp}(X)$ , the logarithmic tangent bundle  $T_X(-\log D)$  is not semi-stable with respect to  $L$ .*

*Proof.* If  $\mathcal{E} = T_X(-\log D)$ , we have  $V_0 > \mu_L(\mathcal{E})$ . By Proposition 5.10,  $T_X(-\log D)$  is not semi-stable with respect to  $L$ .  $\square$

**6.3. The case where  $\text{Cone}(v_0) \in \Delta$ .** In this part we study the stability of the logarithmic tangent bundle  $T_X(-\log D)$  when  $r \geq 2$  and

$$D \in \{D_{v_0}\} \cup \{D_{v_0} + D_{w_j} : 0 \leq j \leq s\} \cup \{D_{v_0} + D_{v_i} : 2 \leq i \leq r\}.$$

The last case  $D = D_{v_0} + D_{v_1}$  will be study in the Section 6.4. We recall that  $a_0 = 0$ . By Lemma 6.3 and Theorem 3.9, for all

$$D \in \{D_{v_0} + D_{w_j} : 0 \leq j \leq s\} \cup \{D_{v_0} + D_{v_i} : 2 \leq i \leq r\}$$

$T_X(-\log D)$  is decomposable. If  $D = D_{v_0}$ ,  $r \geq 2$  and  $a_1 = 0$ , then  $\mathcal{E} = T_X(-\log D)$  is decomposable and  $\mathcal{E} = \mathcal{E}_G \oplus \mathcal{E}_F$  where  $G = \text{Span}(w_0, w_1, \dots, w_s, v_2, \dots, v_r)$

and  $F = \text{Span}(v_1)$ . By Lemma 5.9, if  $a_1 > 0$ , then  $T_X(-\log D_{v_0})$  cannot be written as a sum of two saturated equivariant sub-sheaf.

We now study the semi-stability of these bundles.

**Lemma 6.8.** *Let  $r \geq 2$ ,  $(a_1, \dots, a_r) \neq (0, \dots, 0)$  such that  $a_1 < a_r$ ,  $i \in \{2, \dots, r\}$  and  $j \in \{0, \dots, s\}$ . We set  $\mathcal{E} = T_X(-\log D_{v_0})$ ,  $\mathcal{F} = T_X(-\log(D_{v_0} + D_{v_i}))$  and  $\mathcal{G} = T_X(-\log(D_{v_0} + D_{w_j}))$ . For any  $L \in \text{Amp}(X)$ , the vector bundles  $\mathcal{E}$ ,  $\mathcal{F}$  and  $\mathcal{G}$  are not semi-stables with respect to  $L$ .*

*Proof.* We have  $\mu_L(\mathcal{E}) > \mu_L(\mathcal{F})$  and  $\mu_L(\mathcal{E}) > \mu_L(\mathcal{G})$ . We will show that  $V_1 > \mu_L(\mathcal{E})$ . By Lemma 5.12, we have  $V_1 - W \geq V_r$ . Therefore

$$\begin{aligned} (r+s)(V_1 - \mu_L(\mathcal{E})) &= (r+s)V_1 - (V_1 + \dots + V_r) - (s+1)W \\ &= (s+1)[V_1 - W] - V_r + [(r-1)V_1 - (V_1 + \dots + V_{r-1})] \\ &\geq (s+1)[V_1 - W] - V_r \\ &\geq sV_r \end{aligned}$$

By Proposition 5.10,  $\mathcal{E}$ ,  $\mathcal{F}$  and  $\mathcal{G}$  are not semi-stables with respect to  $L$ .  $\square$

Let  $a \in \mathbb{N}^*$ . We now study what happen in Lemma 6.8 when  $a_1 = \dots = a_r = a$ .

6.3.1. *We assume that  $r = 1$ .*

**Theorem 6.9.** *We assume that  $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^s} \oplus \mathcal{O}_{\mathbb{P}^s}(a))$ . Let  $P_1$  and  $Q$  the polynomials defined by*

$$P_1(x) = (s+1) \sum_{k=0}^{s-1} \binom{s}{k} a^{s-k-1} x^k - s x^s \quad \text{and} \quad Q(x) = x^s - \sum_{k=0}^{s-1} \binom{s}{k} a^{s-k-1} x^k .$$

*We have :*

- (1)  $T_X(-\log D_{v_0})$  is stable (resp. semi-stable) with respect to  $\pi^* \mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$  if and only if  $0 < \nu < \nu_1$  (resp.  $0 < \nu \leq \nu_1$ ) where  $\nu_1$  is the unique positive root of  $P_1$ .
- (2) If  $j \in \{0, \dots, s\}$ , then  $\text{Stab}(T_X(-\log(D_{v_0} + D_{w_j}))) = \emptyset$  ;  $T_X(-\log(D_{v_0} + D_{w_j}))$  is semi-stable with respect to  $\pi^* \mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$  if and only if  $\nu = \nu_3$  where  $\nu_3$  is the unique positive root of  $Q$ .
- (3)  $\emptyset = \text{Stab}(T_X(-\log(D_{v_0} + D_{v_1}))) \subsetneq \text{sStab}(T_X(-\log(D_{v_0} + D_{v_1}))) = \text{Amp}(X)$ .

*Proof.* By the sign rule of Descartes (Theorem 6.1),  $P_1$  and  $Q$  have respectively one positive root. Let  $L = \pi^* \mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$ , by using the expressions of Section 5.2, we have :

$$V_1 = \nu^s \quad \text{and} \quad W = \sum_{k=0}^{s-1} \binom{s}{k} a^{s-k-1} \nu^k .$$

Let  $\mathcal{E} = T_X(-\log D_{v_0})$ . By Proposition 5.10, to check the stability of  $\mathcal{E}$ , it is enough to compare

$$\mu_L(\mathcal{E}) = \frac{V_1 + (1+s)W}{1+s}$$

with  $\max(V_1, W)$ . We don't use the points 4 and 5 because the hypothesis are not verifying. We have

$$\mu_L(\mathcal{E}) > W \quad \text{and} \quad (1+s)(\mu_L(\mathcal{E}) - V_1) = P_1(\nu) .$$

Thus,  $\mathcal{E}$  is stable (resp. semi-stable) with respect to  $\pi^* \mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$  if and only if  $0 < \nu < \nu_1$  (resp.  $0 < \nu \leq \nu_1$ ).

Let  $\mathcal{F} = T_X(-\log(D_{v_0} + D_{w_j}))$ . By Proposition 5.10, we compare

$$\mu_L(\mathcal{F}) = \frac{V_1 + sW}{1 + s}$$

with  $\max(V_1, W)$ . We have

$$(1 + s)(\mu_L(\mathcal{F}) - W) = Q(\nu) \quad \text{and} \quad (1 + s)(\mu_L(\mathcal{F}) - V_1) = -sQ(\nu).$$

Thus, for any  $\nu > 0$ ,  $\mathcal{F}$  is not stable with respect to  $L$ . The vector bundle  $\mathcal{F}$  is semi-stable with respect to  $L$  if and only if  $\nu = \nu_3$ .

Let  $\mathcal{G} = T_X(-\log(D_{v_0} + D_{v_1}))$ . We have  $\mu_L(\mathcal{G}) = W$ . By Proposition 5.10,  $\mathcal{G}$  is semi-stable but not stable with respect to  $L$ .  $\square$

6.3.2. *We assume that  $r \geq 2$ . We assume that  $a_1 = \dots = a_r = a$  with  $a \in \mathbb{N}^*$ .*

**Lemma 6.10.** *We have*

$$\text{card}\{(\alpha_1, \dots, \alpha_p) \in \mathbb{N}^p : \alpha_1 + \dots + \alpha_p = m\} = \binom{m + p - 1}{m}.$$

By Lemma 6.10, for all  $k \in \{0, \dots, s - 1\}$ , we have

$$W_k = \sum_{d_1 + \dots + d_r = s - k - 1} a_1^{d_1} \dots a_r^{d_r} = \binom{s - k + r - 2}{s - k - 1} a^{s - k - 1}$$

and

$$V_{1k} = \sum_{d_2 + \dots + d_r = s - k} a_2^{d_2} \dots a_r^{d_r} = \binom{s - k + r - 2}{s - k} a^{s - k}.$$

We recall that  $V_{1s} = 1$ . By using the equality  $\binom{n}{p - 1} = \frac{p}{n - p + 1} \binom{n}{p}$ , for all  $k \in \{0, \dots, s - 1\}$ ,

$$W_k = \binom{s - k + r - 2}{s - k - 1} a^{s - k - 1} = \frac{s - k}{r - 1} \binom{s - k + r - 2}{s - k} a^{s - k - 1} = \frac{s - k}{a(r - 1)} V_{1k}.$$

Let  $P_1$  and  $Q$  the polynomials defined by :

$$P_1(x) = \sum_{k=0}^{s-1} \left[ \left( -s + \frac{(s-k)(s+1)}{a(r-1)} \right) \binom{s+r-1}{k} V_{1k} \right] x^k - s \binom{s+r-1}{s} x^s$$

$$Q(x) = \sum_{k=0}^{s-1} \left[ \left( r - \frac{s-k}{a} \right) \binom{s+r-1}{k} V_{1k} \right] x^k + r \binom{s+r-1}{s} x^s$$

**Theorem 6.11.** *Let  $r \geq 2$  and  $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^s} \oplus \mathcal{O}_{\mathbb{P}^s}(a_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^s}(a_r))$  with  $a_1 = \dots = a_r = a$  where  $a \in \mathbb{N}^*$ . We set  $\mathcal{E} = T_X(-\log D_{v_0})$ .*

- (1) *If  $a < \frac{s}{r}$ , then  $\mathcal{E}$  is stable (resp. semi-stable) with respect to  $\pi^* \mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$  if and only if  $\nu_3 < \nu < \nu_1$  (resp.  $\nu_3 \leq \nu \leq \nu_1$ ) where  $\nu_1$  and  $\nu_3$  are respectively the positive roots of  $P_1$  and  $Q$ .*
- (2) *If  $\frac{s}{r} \leq a < \frac{s+1}{r-1}$ , then  $\mathcal{E}$  is stable (resp. semi-stable) with respect to  $\pi^* \mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$  if and only if  $0 < \nu < \nu_1$  (resp.  $0 < \nu \leq \nu_1$ ) where  $\nu_1$  is the unique positive root of  $P_1$ .*
- (3) *If  $a \geq \frac{s+1}{r-1}$ , then for any  $L \in \text{Amp}(X)$ ,  $\mathcal{E}$  is not semi-stable with respect to  $L$ .*

Before giving the proof of this theorem, we will explain the condition which ensure the existence of positive roots on  $P_1$  and  $Q$ . We will use the Descartes sign rule.

We write  $P_1$  as  $P_1(x) = \sum_{k=0}^s \alpha_k x^k$ . We have  $\alpha_s < 0$ . If  $-s + \frac{(s-k)(s+1)}{a(r-1)} > 0$ ,

then  $s-k > \frac{as(r-1)}{s+1}$ , i.e  $k < \left(1 - \frac{a(r-1)}{s+1}\right)s$ .

If  $\frac{a(r-1)}{s+1} \geq 1$ , then for any  $x \geq 0$ ,  $P_1(x) < 0$ .

If  $a < \frac{s+1}{r-1}$ , then for all  $k < \left(1 - \frac{a(r-1)}{s+1}\right)s$ , we have  $\alpha_k > 0$  and for all  $k \geq \left(1 - \frac{a(r-1)}{s+1}\right)s$ ,  $\alpha_k \leq 0$ . Hence  $P_1$  has only one positive root  $\nu_1$ .

We write  $Q(x) = \sum_{k=0}^s \beta_k x^k$ . We have  $\beta_s > 0$ . If  $r - \frac{s-k}{a} < 0$ , then  $ra - (s-k) < 0$ ,

i.e  $k < s - ra$ . If  $a \geq \frac{s}{r}$ , then for any  $x \geq 0$ ,  $Q(x) > 0$ .

If  $a < \frac{s}{r}$ , then for all  $k < s - ra$ , we have  $\beta_k < 0$  and for all  $k \geq s - ra$ ,  $\beta_k \geq 0$ . Thus,  $Q$  has only one positive root  $\nu_3$ .

**Lemma 6.12.** *If  $a < \frac{s}{r}$ , then  $\nu_3 < \nu_1$ .*

*Proof.* If  $a < \frac{s}{r}$ , then  $a < \frac{s+1}{r-1}$ . By the above analysis,  $\nu_1$  and  $\nu_3$  exist. For  $x \geq 0$ , we have  $P_1(x) > 0$  if and only if  $0 \leq x < \nu_1$ . As

$$\begin{aligned} \frac{P_1(x)}{-s} - \frac{Q(x)}{r} &= \sum_{k=0}^{s-1} \left[ \left(1 - \frac{(s-k)(s+1)}{as(r-1)} - 1 + \frac{s-k}{ra}\right) \binom{s+r-1}{k} V_{1k} \right] x^k \\ &= \frac{-(r+s)}{asr(r-1)} \sum_{k=0}^{s-1} (s-k) \binom{s+r-1}{k} V_{1k} x^k = P(x) \end{aligned}$$

we have  $\frac{P_1(\nu_3)}{-s} - \frac{Q(\nu_3)}{r} = P(\nu_3) < 0$ . Hence,  $P_1(\nu_3) > 0$ , i.e  $\nu_3 < \nu_1$ .  $\square$

Now, we can prove Theorem 6.11.

**Proof of Theorem 6.11.** As  $a_1 = \dots = a_r$ , we have  $V_1 = \dots = V_r$ . Therefore

$$\mu_L(\mathcal{E}) = \frac{rV_1 + (s+1)W}{r+s}.$$

In this case the point 4 and 5 are not verified. To check the stability of  $\mathcal{E} = T_X(-\log D_{v_0})$ , it is enough to compare  $\mu_L(\mathcal{E})$  with  $\max(V_1, W)$ . We have

$$\begin{aligned} (r+s)(\mu_L(\mathcal{E}) - V_1) &= -sV_1 + (s+1)W \\ &= -s \sum_{k=0}^s \binom{s+r-1}{k} V_{1k} \nu^k + (s+1) \sum_{k=0}^{s-1} \binom{s+r-1}{k} W_k \nu^k \\ &= P_1(\nu) \end{aligned}$$

and

$$\begin{aligned}
(r+s)(\mu_L(\mathcal{E}) - W) &= rV_1 - (r-1)W \\
&= r \sum_{k=0}^s \binom{s+r-1}{k} V_{1k} \nu^k - (r-1) \sum_{k=0}^{s-1} \binom{s+r-1}{k} W_k \nu^k \\
&= Q(\nu)
\end{aligned}$$

We have :

- i. If  $a \geq \frac{s+1}{r-1}$ , then for any  $\nu > 0$ , we have  $P_1(\nu) < 0$ .
- ii. If  $a < \frac{s+1}{r-1}$ , then  $P_1(\nu) > 0$  (resp.  $P_1(\nu) \geq 0$ ) if and only if  $0 < \nu < \nu_1$  (resp.  $0 < \nu \leq \nu_1$ ).
- iii. If  $a \geq \frac{s}{r}$ , then for any  $\nu > 0$ , we have  $Q(\nu) > 0$ .
- iv. If  $a < \frac{s}{r}$ , then  $Q(\nu) > 0$  (resp.  $Q(\nu) \geq 0$ ) if and only if  $\nu > \nu_3$  (resp.  $\nu \geq \nu_3$ ).

The point *i.* shows the third point of the theorem. By using the points *ii.*, *iv.* and the Lemma 6.12, we get the first point of theorem. The points *ii.* and *iii.* give the second point of the theorem.  $\square$

Let  $i \in \{1, \dots, r\}$  and  $j \in \{0, \dots, s\}$ . We now study the stability of  $\mathcal{F}_j = T_X(-\log(D_{v_0} + D_{w_j}))$  and  $\mathcal{G}_i = T_X(-\log(D_{v_0} + D_{v_i}))$ . Let  $Q$  the polynomial defined by

$$Q(x) = \sum_{k=0}^{s-1} \left[ \left( 1 - \frac{s-k}{a(r-1)} \right) \binom{s+r-1}{k} V_{1k} \right] x^k + \binom{s+r-1}{s} x^s.$$

**Theorem 6.13.** *We assume that  $r \geq 2$  and  $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^s} \oplus \mathcal{O}_{\mathbb{P}^s}(a_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^s}(a_r))$  with  $a_1 = \dots = a_r = a$  where  $a \in \mathbb{N}^*$ . Let  $i \in \{1, \dots, r\}$  and  $j \in \{0, \dots, s\}$ .*

- (1) *For any  $L \in \text{Amp}(X)$ ,  $\mathcal{F}_j$  and  $\mathcal{G}_i$  are not stables with respect to  $L$ .*
- (2) *If  $a \geq \frac{s}{r-1}$ , then for any  $L \in \text{Amp}(X)$ ,  $\mathcal{F}_j$  and  $\mathcal{G}_i$  are not semi-stables with respect to  $L$ .*
- (3) *If  $a < \frac{s}{r-1}$ , then  $\mathcal{F}_j$  and  $\mathcal{G}_i$  are semi-stables with respect to  $\pi^* \mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$  if and only if  $\nu = \nu_3$  where  $\nu_3$  is the unique root of  $Q$ .*

*Proof.* We first study the polynomial  $Q$ . We write  $Q(x) = \sum_{k=0}^s \alpha_k x^k$ . We have

$$\alpha_s > 0. \text{ If } 1 - \frac{s-k}{a(r-1)} < 0, \text{ then } a(r-1) - (s-k) < 0, \text{ i.e. } k < s - a(r-1).$$

If  $a \geq \frac{s}{r-1}$ , then for any  $x \geq 0$ ,  $Q(x) > 0$ .

If  $a < \frac{s}{r-1}$ , then for all  $k < s - a(r-1)$ , we have  $\alpha_k < 0$  and for  $k \geq s - a(r-1)$ ,  $\alpha_k \geq 0$ . Thus,  $Q$  has a unique positive root  $\nu_3$ .

As  $a_1 = \dots = a_r$ , we have  $V_1 = \dots = V_r$ . Thus,

$$\mu_L(\mathcal{F}_j) = \frac{rV_1 + sW}{r+s} \quad \text{and} \quad \mu_L(\mathcal{G}_i) = \frac{(r-1)V_1 + (s+1)W}{r+s}.$$

By using Proposition 5.10, to check the stability of  $\mathcal{F}_j$  (resp.  $\mathcal{G}_i$ ), it is enough to compare  $\mu_L(\mathcal{F}_j)$  (resp.  $\mu_L(\mathcal{G}_i)$ ) with  $\max(V_1, W)$ . We have

$$\begin{cases} (r+s)(\mu_L(\mathcal{F}_j) - V_1) = s(W - V_1) = -sQ(\nu) \\ (r+s)(\mu_L(\mathcal{F}_j) - W) = r(V_1 - W) = rQ(\nu) \end{cases}$$

and

$$\begin{cases} (r+s)(\mu_L(\mathcal{G}_i) - V_1) = (s+1)(W - V_1) = -(s+1)Q(\nu) \\ (r+s)(\mu_L(\mathcal{G}_i) - W) = (r-1)(V_1 - W) = (r-1)Q(\nu) \end{cases}$$

If  $a \geq \frac{s}{r-1}$ , then for any  $\nu > 0$ ,  $Q(\nu) > 0$ ; thus,  $\mu_L(\mathcal{F}_j) < V_1$  and  $\mu_L(\mathcal{G}_i) < V_1$ . Hence, for any  $\nu > 0$ ,  $\mathcal{F}_j$  and  $\mathcal{G}_i$  are not semi-stables with respect to  $L$ .

We now assume that  $a < \frac{s}{r-1}$ . Let  $\nu_3$  the positive root of  $Q$ .

If  $\nu \neq \nu_3$ , by the above equalities,  $\mathcal{F}_j$  and  $\mathcal{G}_i$  are not semi-stables with respect to  $L$ . If  $\nu = \nu_3$ , the logarithmic tangent bundles  $\mathcal{F}_j$  and  $\mathcal{G}_i$  are semi-stables but not stable with respect to  $\pi^* \mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$ .  $\square$

**6.4. The case  $D = D_{v_0} + D_{v_1}$ .** In this part, we assume that  $D = D_{v_0} + D_{v_1}$  and  $\mathcal{E} = T_X(-\log D)$ . In the beginning of Section 6.3, we see that  $\mathcal{E}$  is decomposable. In this part, we will study the stability of  $\mathcal{E}$  when  $r \geq 2$  and  $a_1 < a_r$ . The stability of  $\mathcal{E}$  when  $r = 1$  was treated in Theorem 6.9. When  $r \geq 2$ , in Theorem 6.13, we study the stability of  $\mathcal{E}$  when  $a_1 = \dots = a_r$  with  $a_1 \in \mathbb{N}^*$ .

**Proposition 6.14.** *If  $a_1 = 0$ , then for any  $L \in \text{Amp}(X)$ ,  $\mathcal{E}$  is not semi-stable with respect to  $L$ .*

*Proof.* We have  $\mu_L(\mathcal{E}) = \frac{1}{r+s}((s+1)W + V_2 + \dots + V_r)$ . As  $\text{card}\{2, \dots, r\} = r-1$ , by using the point 4 of Proposition 5.10 with  $I' = \{2, \dots, r\}$ , we have

$$\frac{1}{r+s-1} \left( \sum_{i \in I'} V_i + (s+1)W \right) = \frac{1}{r+s-1} (V_2 + \dots + V_r + (s+1)W) .$$

Thus,  $T_X(-\log(D_{v_0} + D_{v_1}))$  is not semi-stable with respect to  $L$ .  $\square$

We also have :

**Lemma 6.15.** *Let  $r \geq 3$ ,  $(a_1, \dots, a_r) \neq (0, \dots, 0)$  such that  $a_2 < a_r$ . For any  $L \in \text{Amp}(X)$ , the vector bundle  $\mathcal{E}$  is not semi-stable with respect to  $L$ .*

*Proof.* We have  $\mu_L(\mathcal{E}) = \frac{1}{r+s}(V_2 + \dots + V_r + (s+1)W)$ . We will show that  $V_2 > \mu_L(\mathcal{E})$ . By Lemma 5.12, we have  $V_2 - W \geq V_r$ , thus

$$\begin{aligned} (r+s)(V_2 - \mu_L(\mathcal{E})) &= (r+s)V_2 - (V_2 + \dots + V_r) - (s+1)W \\ &= (s+1)[V_2 - W] + [(r-1)V_2 - (V_2 + \dots + V_r)] \\ &\geq (s+1)[V_2 - W] \\ &\geq (s+1)V_r \end{aligned}$$

Hence, by Proposition 5.10,  $\mathcal{E}$  is not semi-stable with respect to  $L$ .  $\square$

We now assume that  $0 < a_1 < a_2 = \dots = a_r$ . For the stability of  $\mathcal{E}$ , we see that: the point 4 and 5 of Proposition 5.10 are not verified here. To check the stability of  $\mathcal{E}$ , it is enough to compare  $\mu_L(\mathcal{E})$  with  $\max(V_2, W)$ . We have

$$\mu_L(\mathcal{E}) = \frac{(r-1)V_2 + (s+1)W}{r+s}$$

and

$$(14) \quad \begin{cases} (r+s)[\mu_L(\mathcal{E}) - V_2] = (s+1)(W - V_2) \\ (r+s)[\mu_L(\mathcal{E}) - W] = -(r-1)(W - V_2) \end{cases}$$

We have :

**Proposition 6.16.** *Let  $r \geq 2$ . If  $0 < a_1 < a_2 = \dots = a_r$ , then  $\text{Stab}(\mathcal{E}) = \emptyset$ .*

The vector bundle  $\mathcal{E} = T_X(-\log(D_{v_0} + D_{v_1}))$  is semi-stable with respect to  $L = \pi^* \mathcal{O}_{\mathbb{P}^s}(\nu_3) \otimes \mathcal{O}_X(1)$  if and only if  $\nu_3$  is a positive root of the polynomial  $Q(\nu) = W - V_2$  ( $W$  and  $V_2$  depend on  $\nu$ ).

6.4.1. **We assume that  $r = 2$  and  $0 < a_1 < a_2$ .**

**Proposition 6.17.** *Let  $r = 2$  and  $0 < a_1 < a_2$ . We define  $\delta = \frac{\ln(1 + a_2 - a_1)}{\ln(a_2) - \ln(a_1)}$  and  $Q$  the polynomial*

$$Q(x) = \sum_{k=0}^{s-1} \left[ \frac{a_1^{s-k}}{a_2 - a_1} \left( \left( \frac{a_2}{a_1} \right)^{s-k} - 1 - a_2 + a_1 \right) \binom{s+1}{k} \right] x^k - (s+1)x^s.$$

We have :

- (1) If  $s \leq \delta$ , then  $\text{sStab}(T_X(-\log(D_{v_0} + D_{v_1}))) = \emptyset$ ;
- (2) If  $s > \delta$ , then  $T_X(-\log(D_{v_0} + D_{v_1}))$  is semi-stable with respect to  $\pi^* \mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$  if and only if  $\nu = \nu_3$  where  $\nu_3$  is the positive root of  $Q$ .

*Proof.* We have

$$V_2 = \sum_{k=0}^s \binom{s+1}{k} a_1^{s-k} \nu^k$$

$$W = \sum_{k=0}^{s-1} \binom{s+1}{k} \left( \sum_{d_1+d_2=s-k-1} a_1^{d_1} a_2^{d_2} \right) \nu^k = \sum_{k=0}^{s-1} \frac{a_2^{s-k} - a_1^{s-k}}{a_2 - a_1} \binom{s+1}{k} \nu^k$$

hence,

$$W - V_2 = \sum_{k=0}^{s-1} \left[ \frac{a_1^{s-k}}{a_2 - a_1} \left( \left( \frac{a_2}{a_1} \right)^{s-k} - 1 - a_2 + a_1 \right) \binom{s+1}{k} \right] \nu^k - (s+1)\nu^s = Q(\nu).$$

We write  $Q(x) = \sum_{k=0}^s \alpha_k x^k$ . We have  $\alpha_s < 0$ . The inequality

$$\left( \frac{a_2}{a_1} \right)^{s-k} - 1 - a_2 + a_1 > 0$$

gives  $k < s - \frac{\ln(1 + a_2 - a_1)}{\ln(a_2) - \ln(a_1)} = s - \delta$ . If  $s \leq \delta$ , then for any  $x \geq 0$ ,  $Q(x) < 0$ .

If  $s > \delta$ , then by the Descartes rule,  $Q$  has a unique positive root  $\nu_3$ .

Hence, if  $s > \delta$ , then  $\mathcal{E}$  is semi-stable with respect to  $\pi^* \mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$  if and only if  $\nu = \nu_3$ .  $\square$

6.4.2. **We assume that  $r \geq 3$  and  $a_1 < a_r$ .** Let  $a, b \in \mathbb{N}^*$  such that  $a < b$ . We assume that  $a_1 = a$  and  $a_2 = \dots = a_r = b$ . By Lemma 6.10, for all  $k \in \{0, \dots, s-1\}$ , we have :

$$W_k = \sum_{\substack{d_1+\dots+d_r \\ =s-k-1}} a_1^{d_1} \dots a_r^{d_r} = \sum_{j=0}^{s-k-1} a^{s-k-1-j} \left( \sum_{d_2+\dots+d_r=j} b^j \right)$$

$$= \sum_{j=0}^{s-k-1} \binom{j+r-2}{j} b^j a^{s-k-1-j}$$

and

$$\begin{aligned} V_{2k} = V_{rk} &= \sum_{\substack{d_1 + \dots + d_{r-1} \\ = s-k}} a_1^{d_1} \dots a_{r-1}^{d_{r-1}} = \sum_{j=0}^{s-k} a^{s-k-j} \left( \sum_{d_2 + \dots + d_{r-1} = j} b^j \right) \\ &= \sum_{j=0}^{s-k} \binom{j+r-3}{j} b^j a^{s-k-j}. \end{aligned}$$

For  $k \in \{0, \dots, s-1\}$ , we have  $(s-k) \in \{1, \dots, s\}$ . For  $p \in \{1, \dots, s\}$ , we set  $\alpha_p = W_{s-p} - V_{2, s-p}$ ; we have

$$\alpha_p = \sum_{j=0}^{p-1} \binom{j+r-2}{j} b^j a^{p-1-j} - \sum_{j=0}^p \binom{j+r-3}{j} b^j a^{p-j}.$$

Let  $Q_s$  be the polynomial defined by

$$Q_s(x) = \sum_{k=0}^{s-1} \binom{s+r-1}{k} \alpha_{s-k} x^k - \binom{s+r-1}{s} x^s.$$

We have  $W - V_2 = Q_s(\nu)$ . We now search a condition on  $s$  which ensure the existence of positive root on  $Q$ . By using the identity  $\binom{n}{p-1} = \frac{p}{n-p+1} \binom{n}{p}$ , we have

$$\begin{aligned} \alpha_p &= \sum_{j=0}^{p-1} \frac{j+1}{r-2} \binom{j+r-2}{j+1} b^j a^{p-1-j} - \sum_{j=1}^p \binom{j+r-3}{j} b^j a^{p-j} - a^p \\ &= \sum_{j=0}^{p-1} \left[ \left( \frac{j+1}{r-2} - b \right) \binom{j+r-2}{j+1} b^j a^{p-1-j} \right] - a^p. \end{aligned}$$

If  $1 \leq p \leq b(r-2)$ , then for all  $j \in \{0, \dots, p-1\}$ , we have

$$\frac{j+1}{r-2} - b \leq \frac{p}{r-2} - b = \frac{p-b(r-2)}{r-2} \leq 0;$$

thus  $\alpha_p < 0$ . Hence, if  $\alpha_p > 0$ , then we must have  $p > b(r-2)$ .

If there is  $p > b(r-2)$  such that  $\alpha_p > 0$ , then for any  $q \geq p$ , we have  $\alpha_q > 0$ ; this follows from these equalities.

$$\begin{aligned} \alpha_{p+1} &= \sum_{j=0}^{p-1} \left[ \left( \frac{j+1}{r-2} - b \right) \binom{j+r-2}{j+1} b^j a^{p-j} \right] - a^{p+1} + \left( \frac{p+1}{r-2} - b \right) \binom{p+r-2}{p+1} b^p \\ &= a \alpha_p + \left( \frac{p+1}{r-2} - b \right) \binom{p+r-2}{p+1} b^p \end{aligned}$$

We denote by  $\lfloor x \rfloor$  the floor of  $x \in \mathbb{R}$ .

**Lemma 6.18.** *Let  $m = b(r-2)$ . There is a unique integer  $\delta_r \in [m+1; \lfloor 3.2m \rfloor + 1]$  such that : if  $p \leq \delta_r$ , then  $\alpha_p \leq 0$  and if  $p > \delta_r$ , then  $\alpha_p > 0$ .*

Let  $\delta_r$  be the integer given in Lemma 6.18. If  $s \leq \delta_r$ , then all coefficients of  $Q_s$  are negative; thus, for any  $x > 0$ ,  $Q_s(x) < 0$ . If  $s > \delta_r$ , then by the Descartes rule (see Theorem 6.1),  $Q_s$  has a only one positive root  $\nu_3$ . We deduce :

**Proposition 6.19.** *Let  $r \geq 3$  and  $a, b \in \mathbb{N}^*$  such that  $a < b$  and  $a_1 = a$ ,  $a_2 = \dots = a_r = b$ .*

- (1) *If  $s \leq \delta_r$ , then  $\text{sStab}(T_X(-\log(D_{v_0} + D_{v_1}))) = \emptyset$ ;*
- (2) *If  $s > \delta_r$ , then  $T_X(-\log(D_{v_0} + D_{v_1}))$  is semi-stable with respect to  $\pi^* \mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$  if and only if  $\nu = \nu_3$ .*

We now give the proof of Lemma 6.18.

*Proof.* We have  $\alpha_p = \sum_{j=0}^{m-1} \left[ \binom{j+1}{r-2} b \binom{j+r-2}{j+1} b^j a^{p-1-j} \right] - a^p + \beta_p$  where

$$\begin{aligned} \beta_p &= \sum_{l=m}^{p-1} \left[ \binom{l+1-(r-2)b}{r-2} \binom{l+r-2}{l+1} b^l a^{p-1-l} \right] \\ &= \sum_{l=0}^{p-1-m} \left[ \frac{l+1}{r-2} \binom{l+m+r-2}{l+m+1} b^{l+m} a^{p-1-(l+m)} \right] \end{aligned}$$

The goal of this proof is to find an integer  $p$  such that  $\alpha_p > 0$ . We will search an integer  $p$  such that

$$(15) \quad \beta_p \geq a^p + \sum_{j=0}^{m-1} b \binom{j+r-2}{j+1} b^j a^{p-1-j} .$$

We have  $\binom{l+m+r-2}{l+m+1} = \binom{l+m+r-2}{r-3} = \frac{r-2}{l+m+1} \binom{l+m+r-2}{r-2}$ . From the equality

$$\sum_{j=n}^r \binom{j}{n} = \binom{r+1}{n+1} \quad \text{we have} \quad \binom{r+1}{n+1} = 1 + \sum_{j=0}^{r-n-1} \binom{j+n+1}{n} ;$$

Hence,

$$\begin{aligned} \binom{l+m+r-2}{r-2} &= 1 + \sum_{j=0}^{l+m-1} \binom{j+r-2}{r-3} = 1 + \sum_{j=0}^{l+m-1} \binom{j+r-2}{j+1} \\ &\geq 1 + \sum_{j=0}^{m-1} \binom{j+r-2}{j+1} . \end{aligned}$$

Thus,

$$\begin{aligned} \beta_p &\geq \left( 1 + \sum_{j=0}^{m-1} \binom{j+r-2}{j+1} \right)^{p-1-m} \sum_{l=0}^{p-1-m} \frac{l+1}{l+1+m} b^{l+m} a^{p-1-(l+m)} \\ &\geq b^m a^{p-1-m} \left( 1 + \sum_{j=0}^{m-1} \binom{j+r-2}{j+1} \right)^{p-1-m} \sum_{l=0}^{p-1-m} \frac{l+1}{l+1+m} \left( \frac{b}{a} \right)^l . \end{aligned}$$

We have

$$\begin{aligned} \sum_{l=0}^{p-1-m} \frac{l+1}{l+1+m} &\geq \sum_{l=0}^{p-1-m} \int_l^{l+1} \frac{x dx}{x+m} = \int_0^{p-m} \frac{x}{x+m} dx \\ &= \left[ \frac{x}{1} - m \ln(x+m) \right]_0^{p-m} \\ &= p - m - m \ln \left( \frac{p}{m} \right) \end{aligned}$$

We set  $k = \frac{p}{m}$ . If  $k \geq \frac{17}{5} = 3.2$ , then  $(k-1-\ln(k)) > 1$ . If we set  $p = [3.2m] + 1$ , we get

$$\sum_{l=0}^{p-1-m} \frac{l+1}{l+1+m} \left( \frac{b}{a} \right)^l \geq \sum_{l=0}^{p-1-m} \frac{l+1}{l+1+m} \geq m \geq b .$$

For  $j \in \{0, \dots, m-1\}$ , we have  $b^m a^{p-1-m} \geq b^j a^{p-1-j}$ . If  $p = \lfloor 3.2m \rfloor + 1$ , we get

$$\begin{aligned} \beta_p &\geq b b^m a^{p-1-m} \left( 1 + \sum_{j=0}^{m-1} \binom{j+r-2}{j+1} \right) \\ &\geq b^{m+1} a^{p-1-m} + \sum_{j=0}^{m-1} b \binom{j+r-2}{j+1} b^j a^{p-1-j} ; \end{aligned}$$

this proves the inequality (15). Thus,  $\alpha_p > 0$  for  $p = \lfloor 3.2m \rfloor + 1$ . Hence, we deduce the existence of the integer  $\delta_r$  in the interval  $[m+1; \lfloor 3.2m \rfloor + 1]$ .  $\square$

## 7. STABILITY OF THE LOGARITHMIC TANGENT SHEAF OF LOG SMOOTH DEL PEZZO PAIRS

The goal of this part is to study the stability of the logarithmic tangent bundle  $T_X(-\log D)$  with respect to  $-(K_X + D)$  when the pair  $(X, D)$  is Fano and equivariant.

**7.1. Log smooth toric del Pezzo pairs.** For this part, we refer to [Nap22]. We assume that  $N = \mathbb{Z}^2$ ,  $M = \mathbb{Z}^2$  and the pairing  $\langle \cdot, \cdot \rangle : M \times N \rightarrow \mathbb{Z}$  is the dot product. Let  $\Sigma$  be a smooth complete fan in  $\mathbb{R}^2$  and  $X$  the toric surface associated to  $\Sigma$ . We denote by  $T$  the torus of  $X$ . There is a family of primitive vectors  $\{u_i \in N : 0 \leq i \leq n-1\}$  with  $n \geq 3$  such that

$$\Sigma = \{0\} \cup \{\text{Cone}(u_i) : 0 \leq i \leq n-1\} \cup \{\text{Cone}(u_i, u_{i+1}) : 0 \leq i \leq n-1\}$$

where  $u_n = u_0$ . For all  $i \in \{0, \dots, n-1\}$ , we denote by  $D_i$  the divisor corresponding to the ray  $\text{Cone}(u_i)$ .

**Theorem 7.1.** *Let  $X$  be a complete smooth toric surface such that  $\text{rk Pic}(X) \geq 3$ . For any reduced invariant divisor  $D$  of  $X$ , the pair  $(X, D)$  is not toric log del Pezzo.*

Let  $(e_1, e_2)$  be the standard basis of  $\mathbb{Z}^2$ . The rays of the fan of  $\mathbb{P}^2$  are given by the half-line generated by  $u_1 = e_1$ ,  $u_2 = e_2$  and  $u_0 = -(e_1 + e_2)$ .

**Theorem 7.2.** *If  $X = \mathbb{P}^2$ , then the pair  $(X, D)$  is toric log del Pezzo if and only if  $D \in \{D_0, D_1, D_2\} \cup \{D_0 + D_1, D_0 + D_2, D_1 + D_2\}$ .*

By (1), a toric surface  $X$  of Picard rank two is given by  $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(r))$  with  $r \in \mathbb{N}$ . The rays of the fan of  $X$  are given by the half-line generated by  $u_1 = e_1$ ,  $u_2 = e_2$ ,  $u_3 = -e_1 + r e_2$  and  $u_0 = -e_2$ . We have :

**Theorem 7.3.** *Let  $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(r))$  with  $r \in \mathbb{N}$ . Then :*

- (1)  $-K_X$  or  $-(K_X + D_0)$  are ample if and only if  $r \in \{0, 1\}$ .
- (2) If  $D \in \{D_1, D_3, D_0 + D_1, D_0 + D_3\}$ ,  $-(K_X + D)$  is ample if and only if  $r = 0$ .
- (3) If  $D \in \{D_2, D_2 + D_1, D_2 + D_3\}$ ,  $-(K_X + D)$  is ample if and only if  $r \in \mathbb{N}$ .
- (4) if  $D \in \{D_0 + D_2, D_1 + D_3\}$ ,  $-(K_X + D)$  is not ample for any  $r \in \mathbb{N}$ .

**7.2. Stability with respect to  $-(K_X + D)$ .** According to Section 7.1, we study in this part the stability of the logarithmic tangent bundle  $T_X(-\log D)$  with respect to  $-(K_X + D)$  when  $-(K_X + D)$  is ample.

**Proposition 7.4.** *Let  $X = \mathbb{P}^2$  and  $D_0, D_1, D_2$  the irreducible invariant divisors of  $\mathbb{P}^2$  as in Theorem 7.2.*

- (1) If  $D \in \{D_0, D_1, D_3\}$ , then  $T_X(-\log D)$  is poly-stable but not stable with respect to  $\mathcal{O}_{\mathbb{P}^2}(1)$ .
- (2) If  $D \in \{D_0 + D_1, D_0 + D_2, D_1 + D_2\}$ , then  $T_X(-\log D)$  is not semi-stable with respect to  $\mathcal{O}_{\mathbb{P}^2}(1)$ .

*Proof.* The second point follows from Corollary 4.8 and the first point follows from Theorem 4.11.  $\square$

We now consider the case where  $X$  is a toric surface of Picard rank two. Let  $D_0, D_1, D_2, D_3$  be the irreducible invariant divisors of  $X$  as in Theorem 7.3. We denote by  $\pi : X \rightarrow \mathbb{P}^1$  the projection on the base  $\mathbb{P}^1$ . The Picard group of  $X$  is generated by  $D_0$  and  $D_3$ . A Cartier divisor  $\mu D_3 + \lambda D_0$  is ample if and only if  $\lambda, \mu \in \mathbb{N}^*$ . Let  $L = \mathcal{O}_X(\mu D_3 + \lambda D_0)$  with  $\lambda, \mu \in \mathbb{N}^*$ , we have an isomorphism  $L \cong \pi^* \mathcal{O}_{\mathbb{P}^1}(\mu) \otimes \mathcal{O}_X(\lambda)$ . We set  $\nu = \frac{\mu}{\lambda}$ . The divisors  $D_0, D_1, D_2, D_3$  of  $X$  defined here are given in Section 1.1 by

$$D_0 = D_{v_0} \quad D_1 = D_{w_1} \quad D_2 = D_{v_1} \quad D_3 = D_{w_0}$$

where  $v_1 = e_2$  and  $w_1 = e_1$ . By (12), we have  $-K_X = D_0 + D_1 + D_2 + D_3$ .

**Proposition 7.5.** *Let  $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(0)) \cong \mathbb{P}^1 \times \mathbb{P}^1$ . If*

$$D \in \{D_i : 0 \leq i \leq 3\} \cup \{D_0 + D_1, D_0 + D_3\} \cup \{D_2 + D_1, D_2 + D_3\},$$

*then  $T_X(-\log D)$  is poly-stable with respect to  $-(K_X + D)$ .*

*Proof.* By (11), we have  $D_1 \sim_{\text{lin}} D_3$  and  $D_2 \sim_{\text{lin}} D_0$ . As  $-(K_X + D_1) \sim_{\text{lin}} D_3 + 2D_0$  and  $-(K_X + D_2) \sim_{\text{lin}} 2D_3 + D_0$ , by Theorem 5.16, we deduce that  $T_X(-\log D)$  is poly-stable with respect to  $-(K_X + D)$  if  $D \in \{D_i : 0 \leq i \leq 3\}$ .

If  $D = D_i + D_{i+1}$  for  $i \in \{0, \dots, 3\}$ , then  $-(K_X + D) \sim_{\text{lin}} D_3 + D_0$ . By Theorem 5.19, we deduce that  $T_X(-\log D)$  is poly-stable with respect to  $-(K_X + D)$   $\square$

We now consider the case where  $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(r))$  with  $r \geq 1$ . By (11), we have  $D_1 \sim_{\text{lin}} D_3$  and  $D_2 \sim_{\text{lin}} D_0 - rD_3$ .

**Proposition 7.6.** *If  $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ , then  $T_X(-\log D_0)$  is stable with respect to  $-(K_X + D_0)$ .*

*Proof.* We have  $-(K_X + D_0) \sim_{\text{lin}} D_0 + D_3$  and  $\nu = 1$ . The polynomial  $P_1$  defined in Theorem 6.9 is  $P_1 = 2 - x$ . As  $0 < \nu < 2$ , we deduce that  $T_X(-\log D_0)$  is stable with respect to  $-(K_X + D_0)$ .  $\square$

By Corollary 6.7, we have :

**Proposition 7.7.** *Let  $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(r))$ . For any  $r \in \mathbb{N}^*$ , if  $D \in \{D_2 + D_1, D_2 + D_3\}$ , then  $T_X(-\log D)$  is not semi-stable with respect to any polarizations.*

**Proposition 7.8** (Stability of  $T_X(-\log D_2)$ ). *Let  $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(r))$ . For any  $r \in \mathbb{N}^*$ ,  $T_X(-\log D_2)$  is not semi-stable with respect to  $-(K_X + D_2)$ .*

*Proof.* We have  $-(K_X + D_2) \sim_{\text{lin}} 2D_3 + D_0$  and  $\nu = 2$ . By Theorem 6.6, if  $r \geq 2$ , then  $T_X(-\log D_2)$  is not semi-stable with respect to any polarizations. If  $r = 1$ , the polynomial  $P_0$  is  $P_0 = 1 - x$ . As  $\nu > 1$ , we deduce that  $T_X(-\log D_2)$  is not semi-stable with respect to  $-(K_X + D_2)$ .  $\square$

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