STABILITY OF EQUIVARIANT LOGARITHMIC TANGENT SHEAVES ON TORIC VARIETIES OF PICARD RANK TWO

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Abstract. For an equivariant log pair (X, D) where X is a normal toric variety and D a reduced Weil divisor, we study slope-stability of the logarithmic tangent sheaf $\mathscr{T}_X(-\log D)$. We give a complete description of divisors D and polarizations L such that $\mathscr{T}_X(-\log D)$ is (semi)stable with respect to L when X has a Picard rank one or two.

1. Introduction

The notion of slope-stability for vector bundles was first introduced by Mumford [Mum62] in his attempt to construct the moduli spaces of vector bundles over a curve. This notion was generalized in higher dimension by Takemoto [Tak72]. The study of stability of coherent sheaves over polarized varieties is a difficult problem. This problem is simplified when additional structures are added on the sheaf.

A quasi-coherent sheaf $\mathscr{E}$ on a toric variety X with torus T is said to be an *equivariant sheaf* if it admits a lift of the T-action on X, which is linear on the stalks of $\mathscr{E}$. Klyachko in [Kly90] gave a complete classification of equivariant vector bundles over toric varieties in terms of a family of filtrations of vector spaces. This classification was generalized in the case of quasi-coherent sheaves on toric varieties by Perling in [Per04].

By using the equivariant structure of the tangent bundle, Hering-Nill-Süss in [HNS19] and Dasgupta-Dey-Khan in [DDK20] studied slope-stability of the tangent bundle TX of a smooth projective toric variety X of Picard rank one or two. Inspired by *Iitaka's philosophy*, in this paper, we extend the result of [HNS19] and [DDK20] to the case of log pairs (X, D). More precisely, if X is a normal toric variety and D a reduced Weil divisor such that the logarithmic tangent sheaf $\mathscr{T}_X(-\log D)$ is equivariant, we are interested by the set of polarizations L on X such that $\mathscr{T}_X(-\log D)$ is (semi)stable with respect to L.

Assume that X is a Kähler manifold and $D = \sum_i a_i D_i$ a simple normal crossing divisor with $a_i \in [0\,;1] \cap \mathbb{Q}$. If $K_X + D$ is ample, there exists a Kähler-Einstein metric ω on the pair (X, D) by [BG16, Theorem C]; moreover, if D is reduced, then the logarithmic tangent sheaf $\mathscr{T}_X(-\log D)$ is semi-stable with respect to $K_X + D$ by [Gue16, Theorem A]. If the pair (X, D) is Fano, by [Ber16, Section 4.3], the pair (X, D) admits a Kähler-Einstein metric ω if for all i, $0 < a_i < 1$. In this case, (X, D) is K-polysatble. According to [Li20, Theorem 1.4], the orbifold tangent sheaf $\mathscr{T}_X(-\log D)$ is slope semi-stable with respect to $-(K_X + D)$.

In this paper, we are interested in the case where $a_i \in \{0, 1\}$. According to [BB13, Section 1.2.1 and Section 3], when (X, D) is Fano, we don't have a natural notion of Kähler-Einstein metric on (X, D). We will use another method to study the slope-stability of $\mathscr{T}_X(-\log D)$.

In Section 2, we recall some notions about toric varieties and equivariant sheaves. Let X be a normal toric variety over $\mathbb{C}$ and T be its torus. We denote by Σ the fan of X and $\Sigma(1)$ the set of rays of Σ. In Proposition 2.10, we show that : for a

reduced Weil divisor D, the logarithmic tangent sheaf $\mathscr{T}_X(-\log D)$ is equivariant if and only if

$$D = \sum_{\rho\in\Delta} D_\rho$$

where Δ is a subset of $\Sigma(1)$ and D_ρ the Zariski closure of the orbit $O(\rho)$ corresponding to the ray ρ. Using the family of multifiltrations defined in [Per04, Section 5], we give in Section 3.1 the family of multifiltrations corresponding to the equivariant sheaf $\mathscr{T}_X(-\log D)$. In Section 3.2, we give some conditions on Σ and Δ, such that $\mathscr{T}_X(-\log D)$ is decomposable. In Section 4, we recall some notions about slope-stability and we show that :

Proposition 1.1. *Let X be a toric variety without torus factor and p be the rank of its class group $\mathrm{Cl}(X)$. If*

$$1 + p \leq \mathrm{card}(\Delta) \leq \dim(X) + p - 1$$

then $\mathscr{T}_X(-\log D)$ is not semi-stable with respect to any polarizations.

By Corollary 3.6, if $\Delta = \varnothing$, then $\mathscr{T}_X(-\log D)$ is the tangent sheaf $\mathscr{T}_X$ and when $\Delta = \Sigma(1)$, $\mathscr{T}_X(-\log D)$ is isomorphic to the trivial vector bundle of rank $\dim X$. If X has no torus factor, we have $\mathrm{card}(\Sigma(1)) = \dim(X) + p$ by [CLS11, Theorem 4.1.3]. According to Proposition 1.1, it is therefore sufficient to study the stability of $\mathscr{T}_X(-\log D)$ when $\mathrm{card}(\Delta) \leq p$. In this paper we will be interested in the case where $p \in \{1, 2\}$. As the case $\mathrm{card}(\Delta) = 0$ corresponds to the tangent sheaf, we will study the slope-stability of $\mathscr{T}_X(-\log D)$ when $1 \leq \mathrm{card}(\Delta) \leq p$.

In Section 4.4, we study slope-stability of $\mathscr{T}_X(-\log D)$ when X is a toric orbifold of Picard rank one. If $X = \mathbb{P}^n$ and $\Delta \subset \Sigma(1)$, we show that :

Proposition 1.2. *If $\mathrm{card}(\Delta) = 1$, then $\mathscr{T}_X(-\log D)$ is poly-stable but not stable with respect to any polarizations.*

In the last sections (Sections 5 and 6) we study slope-stability of $\mathscr{T}_X(-\log D)$ when X is a smooth toric variety of Picard rank 2. These parts form the core of this paper.

1.1. **Smooth toric varieties of Picard rank two.** Let X be a smooth toric variety of dimension n with fan Σ in $\mathbb{R}^n$ such that $\mathrm{rk}\,\mathrm{Pic}(X) = 2$. By [CLS11, Theorem 7.3.7] due to Kleinschmidt (see [Kle88]), there exists $r, s \in \mathbb{N}^*$ with $r + s = n$ and $a_1, \ldots, a_r \in \mathbb{N}$ with $a_1 \leq a_2 \leq \ldots \leq a_r$ such that

$$X = \mathbb{P}\left(\mathscr{O}_{\mathbb{P}^s} \oplus \bigoplus_{i=1}^{r} \mathscr{O}_{\mathbb{P}^s}(a_i)\right) . \tag{1}$$

We denote by $\pi : X \longrightarrow \mathbb{P}^s$ the projection to the base $\mathbb{P}^s$. By [CLS11, Section 7.3] and [DDK20, Section 4.2], the rays of Σ are given by the half-lines generated by $w_0, w_1, \ldots, w_s$, $v_0, v_1, \ldots, v_r$ where $(w_1, \ldots, w_s)$ is the standard basis of $\mathbb{Z}^s \times 0_{\mathbb{Z}^r}$, $(v_1, \ldots, v_r)$ the standard basis of $0_{\mathbb{Z}^s} \times \mathbb{Z}^r$,

$$v_0 = -(v_1 + \ldots + v_r) \quad \text{and} \quad w_0 = a_1 v_1 + \ldots + a_r v_r - (w_1 + \ldots + w_s) .$$

We denote by D_{v_i} the divisor corresponding to the ray $\mathrm{Cone}(v_i)$ and D_{w_j} the divisor corresponding to the ray $\mathrm{Cone}(w_j)$.

Notation 1.3. Let $\mathrm{Amp}(X) \subset N^1(X) \otimes_{\mathbb{Z}} \mathbb{R}$ be the *ample cone* of X. We define :

$$\mathrm{Stab}(\mathscr{T}_X(-\log D)) = \left\{ \begin{array}{c} L \in \mathrm{Amp}(X) : \mathscr{T}_X(-\log(D)) \\ \text{is stable with respect to } L \end{array} \right\}$$

and

$$\mathrm{sStab}(\mathscr{T}_X(-\log D)) = \left\{ \begin{array}{c} L \in \mathrm{Amp}(X) : \mathscr{T}_X(-\log(D)) \\ \text{is semi-stable with respect to } L \end{array} \right\} .$$

1.2. **Main results.** Let X be a smooth toric variety of Picard rank two and $L = \pi^*\mathscr{O}_{\mathbb{P}^s}(\mu) \otimes \mathscr{O}_X(\lambda)$ be a polarization on X. We set $\nu = \frac{\mu}{\lambda}$. In Section 5, more precisely in Proposition 5.10, we give a criterion that allows to verify the stability of $\mathscr{T}_X(-\log D)$. For all divisors $D = \sum_{\rho\in\Delta} D_\rho$ with $1 \leq \mathrm{card}(\Delta) \leq 2$, we give in tables 1, 2, 3 and 4 the values of ν for which $\mathscr{T}_X(-\log D)$ is (semi)stable with respect to L. We give here some results.

If $a_1 = \ldots = a_r = 0$, the variety X given in (1) is $\mathbb{P}^s \times \mathbb{P}^r$. Let

$$\mathscr{D}_1 = \{D_{v_i} : 0 \leq i \leq r\} \cup \{D_{w_j} : 0 \leq j \leq s\} \cup \{D_{v_i} + D_{w_j} : 0 \leq i \leq r,\ 0 \leq j \leq s\}$$

and

$$\mathscr{D}_2 = \{D_{v_i} + D_{v_j} : 0 \leq i < j \leq r\} \cup \{D_{w_i} + D_{w_j} : 0 \leq i < j \leq s\}\ .$$

We have :

Theorem 1.4. *If $D \in \mathscr{D}_1 \cup \mathscr{D}_2$, then* $\mathrm{Stab}(\,\mathscr{T}_X(-\log D)\,) = \varnothing$. *But :*

(1) *if $D \in \mathscr{D}_2$,* $\mathrm{sStab}(\mathscr{T}_X(-\log D)) = \varnothing$ *;*

(2) *if $D \in \mathscr{D}_1$, $\mathscr{T}_X(-\log D)$ is poly-stable with respect to L if and only if L is a power of the polarization corresponding to $-(K_X + D)$.*

We now consider the general case, that is X given by (1) with $a_r \geq 1$.

Theorem 1.5 (Stability of $\mathscr{T}_X(-\log D_{v_r})$)**.** *We have* $\mathrm{Stab}(\mathscr{T}_X(-\log D_{v_r})) \neq \varnothing$ *if and only if* $\mathrm{sStab}(\mathscr{T}_X(-\log D_{v_r})) \neq \varnothing$ *if and only if $a_r = 1$ and $a_{r-1} = 0$. If $a_r = 1$ and $a_{r-1} = 0$, then the logarithmic tangent sheaf $\mathscr{T}_X(-\log D_{v_r})$ is stable (resp. semi-stable) with respect to $\pi^*\mathscr{O}_{\mathbb{P}^s}(\mu) \otimes \mathscr{O}_X(\lambda)$ if and only if $0 < \frac{\mu}{\lambda} < \nu_0$ (resp. $0 < \frac{\mu}{\lambda} \leq \nu_0$) where ν_0 is the unique positive root of*

$$\mathsf{P}_0(x) = \sum_{k=0}^{s-1} \binom{s+r-1}{k} x^k - s\binom{s+r-1}{s} x^s\ .$$

If $a_r \geq 1$, we see that this theorem is similar to [HNS19, Theorem 1.4]. If we fix r and s, there is only one smooth toric variety with Picard rank two such that $\mathrm{Stab}(\mathscr{T}_X(-\log D_{v_r})) \neq \varnothing$ or $\mathrm{Stab}(\mathscr{T}_X) \neq \varnothing$; this variety is given by (1) with $a_1 = \ldots = a_{r-1} = 0$ and $a_r = 1$. If $a_r \geq 1$ and $(a_1, \ldots, a_r) \neq (0, \ldots, 0, 1)$, we have $\mathrm{sStab}(\mathscr{T}_X) = \varnothing$ by [HNS19, Theorem 1.4]. In the case of log pairs, with some conditions on the a_i, there exists another divisor D such that $\mathrm{Stab}(\mathscr{T}_X(-\log D)) \neq \varnothing$.

Theorem 1.6 (Stability of $\mathscr{T}_X(-\log D_{v_0})$)**.** *We assume that $a_r \geq 1$.*

(1) *If $r = 1$, then* $\mathrm{Stab}(\mathscr{T}_X(-\log D_{v_0})\,) \neq \varnothing$.

(2) *If $r \geq 2$, then* $\mathrm{Stab}(\mathscr{T}_X(-\log D_{v_0})\,) \neq \varnothing$ *if and only if $a_1 = \ldots = a_r$ and $a_r < \frac{s+1}{r-1}$.*

If $D \notin \{D_{v_0}, D_{v_r}\}$, then $\mathscr{T}_X(-\log D)$ is not stable with respect to any polarizations.

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2. Toric varieties and equivariant logarithmic tangent sheaves

2.1. **Toric varieties.** We give here some notions about toric varieties. We refer to [CLS11, Chapter 2, 3] and [CT20, Section 2.1].

Definition 2.1 ([CLS11, Definition 3.1.1])**.** A toric variety is an irreducible variety X containing a torus $T \simeq (\mathbb{C}^*)^n$ as a Zariski open subset such that the action of T on itself extends to an algebraic action of T on X.

Notation 2.2. Let T be a torus. We denote by M the lattice of characters of T and N the lattice of one-parameter subgroups of T; we have $T \cong N \otimes_{\mathbb{Z}} \mathbb{C}^*$ (it is a group isomorphism). We set $N_{\mathbb{K}} = N \otimes_{\mathbb{Z}} \mathbb{K}$ and $M_{\mathbb{K}} = M \otimes_{\mathbb{Z}} \mathbb{K}$ for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. For $m \in M$, we associate the character $\chi^m : T \longrightarrow \mathbb{C}^*$ and for $u \in N$, we associate the one-parameter subgroup $\lambda^u : \mathbb{C}^* \longrightarrow T$. There is a natural pairing $\langle \cdot, \cdot \rangle : M \times N \longrightarrow \mathbb{Z}$ between M and N.

Definition 2.3 ([CLS11, Definition 3.1.2])**.** A *fan* Σ in $N_{\mathbb{R}}$ is a finite collection of cones $\sigma \subseteq N_{\mathbb{R}}$ such that :

(1) Every $\sigma \in \Sigma$ is a strongly convex rational polyhedral cone.
(2) For all $\sigma \in \Sigma$, each face of σ is also in Σ.
(3) For all $\sigma_1, \sigma_2 \in \Sigma$, the intersection $\sigma_1 \cap \sigma_2$ is a face of each.

We will write $\tau \preceq \sigma$ if τ is a face of σ. If Σ is fan, the support of Σ is $|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma$. We say that Σ is *complete*, if $|\Sigma| = N_{\mathbb{R}}$. We denote by $\Sigma(r)$ the set of r-dimensional cones of Σ. We call $\Sigma(1)$ the set of rays of Σ. If $\rho \in \Sigma(1)$, we denote by $u_\rho \in N$ its minimal generator.

Let X_Σ be a toric variety associated to a fan Σ. The variety X_Σ is obtained by gluing affine charts $(U_\sigma)_{\sigma \in \Sigma}$, with $U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma])$ and $\mathbb{C}[S_\sigma]$ is the semi-group algebra of

$$S_\sigma = \sigma^\vee \cap M = \{m \in M \, : \, \langle m, u \rangle \geq 0 \text{ for all } u \in \sigma\} \, .$$

By [CLS11, Theorem 3.1.5], X_Σ is a normal separated toric variety.

Theorem 2.4 ([CLS11, Corollary 3.1.8])**.** *Let X be a normal separated toric variety with torus T. Then, there exists a fan Σ in $N_{\mathbb{R}}$ such that X is isomorphic to X_Σ.*

From now on, a normal toric variety X will be defined by a fan Σ.

Definition 2.5 ([CLS11, Proposition 3.3.9])**.** A normal toric variety X has a *torus factor* if and only if the u_ρ, $\rho \in \Sigma(1)$ do not span $N_{\mathbb{R}}$.

A cone $\sigma \in \Sigma$ gives the torus orbit $O(\sigma) \cong \mathrm{Hom}_{\mathbb{Z}}(\sigma^\perp \cap M, \mathbb{C}^*)$ where

$$\sigma^\perp \cap M = \{m \in M \, : \, \langle m, u \rangle = 0 \text{ for all } u \in \sigma\} \, .$$

We have :

Theorem 2.6 (Orbit-Cone Correspondence, [CLS11, Theorem 3.2.6])**.**

(1) *There is a bijective correspondence*

$$\begin{array}{ccc} \{\text{Cone } \sigma \text{ in } \Sigma\} & \longleftrightarrow & \{T - \text{orbits in } X\} \\ \sigma & \longleftrightarrow & O(\sigma) \end{array}$$

and $\dim O(\sigma) = \dim N_{\mathbb{R}} - \dim \sigma$.

(2) *The affine open subset U_σ is the union of orbits* $U_\sigma = \bigcup_{\tau \preceq \sigma} O(\tau)$.

(3) *$\tau \preceq \sigma$ if and only if $O(\sigma) \subset \overline{O(\tau)}$, and* $\overline{O(\tau)} = \bigcup_{\tau \preceq \sigma} O(\sigma)$ *where $\overline{O(\tau)}$ denotes the closure in both the classical and Zariski topologies.*

For $\rho \in \Sigma(1)$, we define the T-invariant Weil divisor D_ρ as the closure in the Zariski topology of $O(\rho)$.

Lemma 2.7 ([CLS11, Section 4.1]). *A Weil divisor D on X is invariant under the action of T if and only if*

$$D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho \quad \textit{with} \quad a_\rho \in \mathbb{Z} \ .$$

Order Relation. Let Σ be a fan in $N_\mathbb{R}$ and $\sigma \in \Sigma$. We define an order relation $\preceq_\sigma$ on M by setting $m \preceq_\sigma m'$ if and only if $m' - m \in S_\sigma$. We write $m \prec_\sigma m'$ if we have $m \preceq_\sigma m'$ but not $m' \preceq_\sigma m$.

Group action. Let G be an algebraic group acting on the affine toric variety $Y = \mathrm{Spec}(R)$. For any $g \in G$, let $\phi_g : Y \longrightarrow Y$ the map defined by $\phi_g(x) = g \cdot x$. We define an action of G on R by setting : for $g \in G$ and $\varphi \in R$,

$$g \cdot \varphi = \left(\phi_{g^{-1}}\right)^* \varphi$$

that is, for any $y \in Y$, $(g \cdot \varphi)(y) = \varphi(g^{-1} \cdot y)$.

2.2. **Equivariant sheaves.** We refer to [Per03, Section 2.2.2] for properties about equivariant coherent sheaf. Let T be the torus of X.

Let $\sigma : T \times X \longrightarrow X$ be an action of algebraic group T on X, $\mu : T \times T \longrightarrow T$ the group multiplication, $p_2 : T \times X \longrightarrow X$ the projection onto the second factor and $p_{23} : T \times T \times X \longrightarrow T \times X$ the projection onto the second and the third factor.

We call a sheaf $\mathscr{E}$ on X *equivariant* or *T-linearized* if there exists an isomorphism $\Phi : \sigma^*\mathscr{E} \xrightarrow{\cong} p_2^*\mathscr{E}$ such that

$$(\mu \times \mathrm{Id}_X)^*\Phi = p_{23}^*\Phi \circ (\mathrm{Id}_T \times \sigma)^*\Phi \ . \tag{2}$$

For $t \in T$, let $\alpha_t : X \longrightarrow T \times X$ and $\phi_t : X \longrightarrow X$ be the morphisms defined by

$$\alpha_t(x) = (t, x) \quad \text{and} \quad \phi_t(x) = \sigma(t, x) \ .$$

We get an isomorphism $\Phi_t := \alpha_t^*\Phi : \phi_t^*\mathscr{E} \xrightarrow{\cong} \mathscr{E}$. For any $t, t' \in T$, the cocyle condition (2) factors as follows :

(3)
$$\begin{array}{ccc} (\phi_{t'\cdot t})^*\mathscr{E} & \xrightarrow{\quad\Phi_{t'\cdot t}\quad} & \mathscr{E} \\ & {\scriptstyle \phi_t^*\Phi_{t'}} \searrow \quad \nearrow {\scriptstyle \Phi_t} & \\ & \phi_t^*\mathscr{E} & \end{array}$$

2.3. **Logarithmic tangent sheaves.** We recall here the definition of the logarithmic tangent sheaf of a log pair (X, D) where X is a normal projective variety and D a reduced Weil divisor on X. We refer to [Gue16, Section 3.1].

Definition 2.8. We say that a pair (X, D) is log-smooth if X is smooth and D is reduced snc (simple normal crossing) divisor.
We denote by $(X, D)_{\mathrm{reg}}$ the snc locus of the pair (X, D), that is, the locus of points $x \in X$ where (X, D) is log-smooth in a neighborhood of x.

If (X, D) is log-smooth, we define the logarithmic tangent bundle $T_X(-\log D)$ as the dual of the bundle of logarithmic differential form $\Omega^1_X(\log D)$ where $\Omega^1_X(\log D)$ is defined in [Ita76, §1]. By [Kaw78, Definition 4] and [Sai75, §1], we can see the space of sections of $T_X(-\log D)$ as the set of vector fields on X which vanish along D. If D is locally given by $(z_1 \cdots z_k = 0)$, then $T_X(-\log D)$ as a sheaf is the locally free $\mathscr{O}_X$-module generated by

$$z_1\frac{\partial}{\partial z_1}, \ldots, z_k\frac{\partial}{\partial z_k}, \frac{\partial}{\partial z_{k+1}}, \ldots, \frac{\partial}{\partial z_n} \ .$$

Definition 2.9 ([Gue16, Definition 3.4])**.** Let $(X,\, D)$ be a log pair and $X_0 = (X,\, D)_{\mathrm{reg}}$. The logarithmic tangent sheaf of $(X,\, D)$, denoted by $\mathscr{T}_X(-\log D)$, is defined as $j_*T_{X_0}(-\log D_{|X_0})$ where $j : X_0 \longrightarrow X$ is the open immersion.

The sheaf $\mathscr{T}_X(-\log D)$ (as well as its dual) is coherent ; by [Har80, Proposition 1.6], this sheaf is reflexive.

We now consider the case where X is a toric variety. We will give a condition on D such that $\mathscr{T}_X(-\log D)$ is equivariant. Let Σ be the fan of X and X_0 the toric variety corresponding to the fan $\Sigma^1 = \Sigma(0) \cup \Sigma(1)$. We denote by $j : X_0 \longrightarrow X$ the open immersion.

Proposition 2.10. *Let D be a reduced Weil divisor on X. The sheaf $\Omega^1_X(\log D)$ is equivariant if and only if D is an invariant divisor under the torus action.*

Proof. We assume that D is an invariant divisor under the torus action. Let D_0 be the restriction of D on X_0. For $t \in T$, let $\phi_t : X \longrightarrow X$ the map defined by $\phi_t(x) = \sigma(t,\, x)$ and Φ_t the map defined by $\Phi_t = (d\phi_t)^{-1}$ where $d\phi_t$ is the differential of ϕ_t . If $\mathscr{E} = TX_0$, we have an isomorphism $\Phi_t : \phi_t^*\mathscr{E} \longrightarrow \mathscr{E}$ and the diagram (3) is verified. Now if we replace $\mathscr{E}$ by $T_{X_0}(-\log D_0)$, the diagram (3) remains true ; so $T_{X_0}(-\log D_0)$ is equivariant. Hence $\Omega^1_{X_0}(\log D_0)$ is equivariant. Like

$$\Omega^1_X(\log D) \cong j_*\Omega^1_{X_0}(\log D_0)\ , \tag{4}$$

we deduce that $\Omega^1_X(\log D)$ is equivariant.

We now assume that $\Omega^1_X(\log D)$ is equivariant. We write $D = \sum_{j=1}^{s} D_j$.

First case. We assume that X is smooth. By [EV92, Section 2.3] we have an exact sequence

$$0 \longrightarrow \Omega^1_X \longrightarrow \Omega^1_X\,(\log D) \longrightarrow \bigoplus_{j=1}^{s} \mathscr{O}_{D_j} \longrightarrow 0$$

where $\mathscr{O}_{D_j}$ is viewing as a sheaf on X via extension by zero. The first part of the proof is to show that : for any $t \in T$, $t \cdot Z = Z$ where $Z = X \setminus D$.
Let $x \in Z$ and assume that there is $t \in T$ such that $y = \sigma(t,\, x) \in D$. We have two exact sequences

$$0 \longrightarrow \Omega^1_{X,\,x} \longrightarrow \Omega^1_X\,(\log D)_x \longrightarrow \bigoplus_{j=1}^{s} \mathscr{O}_{D_j,\,x} \longrightarrow 0$$

$$0 \longrightarrow \Omega^1_{X,\,y} \longrightarrow \Omega^1_X\,(\log D)_y \longrightarrow \bigoplus_{j=1}^{s} \mathscr{O}_{D_j,\,y} \longrightarrow 0$$

Like Ω^1_X and $\Omega^1_X(\log D)$ are equivariant, we have an isomorphism

$$\bigoplus_{j=1}^{s} \mathscr{O}_{D_j,\,x} \cong \bigoplus_{j=1}^{s} \mathscr{O}_{D_j,\,y}\ ;$$

this is absurd. Therefore, for any $t \in T$, we have $t \cdot Z \subset Z$, that is $t \cdot Z = Z$. As $\Omega^1_X(\log D)$ is equivariant, by using the fact that $D = X \setminus Z$, for any $t \in T$, we have $t \cdot D = D$; thus, D is a T-invariant divisor.

Second case. We assume that X is a normal variety. By (4), as $\Omega^1_X(\log D)$ is equivariant, we also have the same property for $\Omega^1_{X_0}(\log D_0)$. By the first case, D_0 is an invariant divisor under the action of T on X_0. Like $\mathrm{codim}(X \setminus X_0) \geq 2$, we deduce that D is the Zariski closure of D_0 on X. Thus, D is an invariant divisor under the action of T on X. □

3. Multifiltrations of logarithmic tangent sheaf

3.1. **Family of multifiltrations.** We give here the family of multifiltrations of equivariant logarithmic tangent sheaf on toric varieties. Let X be a toric variety of dimension n with fan Σ. We first give some results.

Proposition 3.1 ([DDK20, Corollary 2.2.17]). *The family* $\left(\mathscr{T}, \{\mathscr{T}^\rho(i)\}_{\rho\in\Sigma(1),\, i\in\mathbb{Z}}\right)$ *of multifiltrations associated to the tangent sheaf* $\mathscr{T}_X$ *are given by*

$$\mathscr{T}^\rho(i) = \begin{cases} 0 & \text{if } i \leq -2 \\ \mathrm{Span}(u_\rho) & \text{if } i = -1 \\ N \otimes_{\mathbb{Z}} \mathbb{C} & \text{if } i \geq 0 \end{cases} .$$

Remark 3.2. If X is smooth, the tangent sheaf $\mathscr{T}_X$ if the tangent bundle TX.

Lemma 3.3. *For every ray* $\rho \in \Sigma(1)$*, we have* $(U_\rho \cap D_\rho) \cap T = \varnothing$.

Proof. Let $\rho \in \Sigma(1)$. We assume that $\rho = \mathrm{Cone}(u_1)$ where $u_1 \in N$. Let $(u_1, \ldots, u_n)$ be a basis of N and $(e_1, \ldots, e_n)$ its dual basis. We set $x_j = \chi^{e_j}$, we have

$$U_\rho = \mathrm{Spec}(\mathbb{C}[x_1, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]) \cong \mathbb{C} \times (\mathbb{C}^*)^{n-1}$$

and

$$T = \mathrm{Spec}(\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]) \cong (\mathbb{C}^*)^n .$$

As on U_ρ the divisor D_ρ is defined by $x_1 = 0$, we deduce that $(U_\rho \cap D_\rho) \cap T = \varnothing$. □

Lemma 3.4. *Let* ρ_1 *and* ρ_2 *be two distinct rays of* Σ*. If* $i \neq j$*, then* $U_{\rho_i} \cap D_{\rho_j} = \varnothing$.

Proof. By the orbit-cone correspondence (Theorem 2.6), we have $U_{\rho_i} = O(\rho_i) \cup T$ and $O(\rho_i) \cap D_{\rho_j} = \varnothing$ if $i \neq j$. As

$$U_{\rho_i} \cap D_{\rho_j} = \left(O(\rho_i) \cap D_{\rho_j}\right) \cup \left(T \cap D_{\rho_j}\right) ,$$

by Lemma 3.3, we conclude that $U_{\rho_i} \cap D_{\rho_j} = \varnothing$. □

Let $\Delta \subset \Sigma(1)$ and D be the T-invariant divisor of X defined by

$$D = \sum_{\rho\in\Delta} D_\rho .$$

For $\rho \in \Sigma(1)$, we set $E^\rho = \Gamma(U_\rho, \mathscr{T}_X(-\log D))$.

Theorem 3.5. *The family of multifiltrations* $\left(E, \{E^\rho(j)\}_{\rho\in\Sigma(1),\, j\in\mathbb{Z}}\right)$ *corresponding to the logarithmic tangent sheaf* $\mathscr{T}_X(-\log D)$ *are given by*

$$E^\rho(j) = \begin{cases} 0 & \text{if } j \leq -1 \\ N_{\mathbb{C}} & \text{if } j \geq 0 \end{cases} \qquad \text{if } \rho \in \Delta$$

and by

$$E^\rho(j) = \begin{cases} 0 & \text{if } j \leq -2 \\ \mathrm{Span}(u_\rho) & \text{if } j = -1 \\ N_{\mathbb{C}} & \text{if } j \geq 0 \end{cases} \qquad \text{if } \rho \notin \Delta .$$

Proof. According to Lemma 3.4, if $\rho \in \Delta$, we have $U_\rho \cap D = D_\rho$ and for $\rho \notin \Delta$, $U_\rho \cap D = \varnothing$. We can reduce the problem to the case where $\Delta = \{\rho_1\}$ and $D = D_{\rho_1}$. For the rest of the proof, we assume that $\Delta = \{\rho_1\}$.

First case : We assume that $\rho = \rho_1$. Let $(u_1, \ldots, u_n)$ be a basis of N such that $u_1 = u_\rho$. We denote by $(e_1, \ldots, e_n)$ the dual basis of $(u_1, \ldots, u_n)$ and $x_i = \chi^{e_i}$. We have $\mathbb{C}[S_\rho] = \mathbb{C}[x_1, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$ and $U_\rho = \mathrm{Spec}(\mathbb{C}[S_\rho])$. Like on U_ρ the divisor D is defined by the equation $x_1 = 0$, we have

$$E^\rho = \left(\mathbb{C}[S_\rho] \cdot x_1 \frac{\partial}{\partial x_1}\right) \oplus \left(\bigoplus_{i=2}^{n} \mathbb{C}[S_\rho] \cdot \frac{\partial}{\partial x_i}\right)$$

that is

$$E^\rho = \bigoplus_{i=2}^{n} \left(\bigoplus_{m\in S_\rho} \mathbb{C}\cdot\chi^m \frac{\partial}{\partial x_i} \right) \oplus \left(\bigoplus_{m\in S_\rho} \mathbb{C}\cdot\chi^{m+e_1} \frac{\partial}{\partial x_1} \right) .$$

We set

$$L_1^\rho = \bigoplus_{m\in S_\rho} \mathbb{C}\cdot\chi^{m+e_1} \frac{\partial}{\partial x_1}$$

and for $i \in \{2, \dots, n\}$, we set

$$L_i^\rho = \bigoplus_{m\in S_\rho} \mathbb{C}\cdot\chi^{m} \frac{\partial}{\partial x_i} .$$

For $t \in T$ and $m \in M$, we have $t\cdot\chi^m = \chi^{-m}(t)\chi^m$, so $t\cdot dx_i = \chi^{-e_i}(t)\, dx_i$. Thus, we have $t\cdot\dfrac{\partial}{\partial x_i} = \chi^{e_i}(t)\dfrac{\partial}{\partial x_i}$. For $i \in \{1, \dots, n\}$, we write

$$L_i^\rho = \bigoplus_{m\in M} (L_i^\rho)_m \quad \text{where} \quad (L_i^\rho)_m = \{f \in L_i^\rho \,:\, t\cdot f = \chi^{-m}(t)\, f\} .$$

We have

$$(L_1^\rho)_m = \begin{cases} \mathbb{C}\cdot\chi^{m+e_1}\dfrac{\partial}{\partial x_1} & \text{if } 0 \preceq_\rho m \\ 0 & \text{otherwise} \end{cases}$$

and for $i \in \{2, \dots, n\}$, we have

$$(L_i^\rho)_m = \begin{cases} \mathbb{C}\cdot\chi^{m+e_i}\dfrac{\partial}{\partial x_i} & \text{if } -e_i \preceq_\rho m \\ 0 & \text{otherwise} \end{cases} .$$

For $i \in \{1, \dots, n\}$, we set $\mathbb{L}_i^\rho = \mathbb{C}\cdot\dfrac{\partial}{\partial x_i}$. If $i \in \{2, \dots, n\}$, we have $(L_i^\rho)_{-e_i} = \mathbb{L}_i^\rho$. For all $m \in M$:

- if $i \geq 2$ and $-e_i \preceq_\rho m$, we identify $(L_i^\rho)_m$ with $\mathbb{L}_i^\rho$ through the multiplication by the character χ^{-m-e_i}.
- If $i = 1$ and $0 \preceq_\rho m$, we identify $(L_i^\rho)_m$ with $\mathbb{L}_1^\rho$ through the multiplication by χ^{-m-e_1}.

For $m \in M$, we set $j = \langle m,\, u_1\rangle$.

- If $j \leq -1$, for all $i \in \{1, \dots, n\}$, we have $(L_i^\rho)_m = 0$.
- If $j \geq 0$, for all $i \in \{1, \dots, n\}$, $(L_i^\rho)_m \cong \mathbb{L}_i^\rho$.

The torus T is a Lie group. The tangent space of T at the neutral element is isomorphic to $N_\mathbb{C}$. Like the tangent space of T at the neutral element is generated by $\left(\dfrac{\partial}{\partial x_i}\right)_{1\leq i\leq n}$, for all $i \in \{1, \dots, n\}$, we can identify $\dfrac{\partial}{\partial x_i}$ with u_i. Like

$$E^\rho = \bigoplus_{m\in M} E^\rho(\langle m,\, u_1\rangle) \quad \text{where} \quad E^\rho(\langle m,\, u_1\rangle) = E^\rho_m = \bigoplus_{i=1}^{n} (L_i^\rho)_m ,$$

we get $E^\rho(j) \cong \begin{cases} 0 & \text{if } j \leq -1 \\ N_\mathbb{C} & \text{if } j \geq 0 \end{cases}$.

Second case : We assume that $\rho \in \Sigma(1) \setminus \Delta$. We start as in the first case. Like $U_\rho \cap D = \varnothing$, we have

$$E^\rho = \bigoplus_{i=1}^{n} \mathbb{C}[S_\rho]\cdot\frac{\partial}{\partial x_i} = \bigoplus_{i=1}^{n} \left(\bigoplus_{m\in S_\rho} \mathbb{C}\cdot\chi^m \frac{\partial}{\partial x_i} \right) .$$

For all $i \in \{1, \ldots, n\}$, we set

$$L_i^\rho = \bigoplus_{m \in S_\rho} \mathbb{C} \cdot \chi^m \frac{\partial}{\partial x_i} .$$

We have

$$L_i^\rho = \bigoplus_{m \in M} (L_i^\rho)_m \quad \text{where} \quad (L_i^\rho)_m = \begin{cases} \mathbb{C} \cdot \chi^{m+e_i} \dfrac{\partial}{\partial x_i} & \text{if } -e_j \preceq_\rho m \\ 0 & \text{oherwise} \end{cases} .$$

As in the first case, with a small modification, for all $i \in \{1, \ldots, n\}$, we identify $(L_i^\rho)_m$ with $\mathbb{L}_i^\rho$ if $m \in M$ satisfies $-e_i \preceq_\rho m$. By setting $j = \langle m, u_\rho \rangle$, we get

$$E^\rho(j) \cong \begin{cases} 0 & \text{if } j \leq -2 \\ \mathrm{Span}(u_\rho) & \text{if } j = -1 \\ N \otimes_{\mathbb{Z}} \mathbb{C} & \text{if } j \geq 0 \end{cases} .$$

Hence we get the theorem. □

The sheaf of regular section of the trivial vector bundle $X \times \mathbb{C} \longrightarrow X$ of rank 1 is $\mathscr{O}_X$. For $\rho \in \Sigma(1)$, we set $F^\rho = \mathscr{O}_X(U_\rho)$. Like $F^\rho = \mathbb{C}[S_\rho]$, we have

$$F^\rho = \bigoplus_{m \in S_\rho} \mathbb{C} \cdot \chi^m = \bigoplus_{m \in M} F_m^\rho \quad \text{where} \quad F_m^\rho = \{ f \in F^\rho \, : \, t \cdot f = \chi^{-m}(t)\, f \} .$$

Thus $F_m^\rho = 0$ if $m \notin S_\rho$ and $F_m^\rho = \mathbb{C} \cdot \chi^m$ if $m \in S_\rho$ (i.e $0 \preceq_\rho m$). For all $m \in S_\rho$, we identify F_m^ρ with $F_0^\rho = \mathbb{C}$. By setting $j = \langle m, u_\rho \rangle$, we get

$$F^\rho(j) \cong \begin{cases} 0 & \text{if } j \leq -1 \\ \mathbb{C} & \text{if } j \geq 0 \end{cases} .$$

Corollary 3.6. *Let $\Delta \subset \Sigma(1)$ and $D = \sum_{\rho \in \Delta} D_\rho$.*

1. *If $\Delta = \varnothing$, then $\mathscr{T}_X(-\log D)$ is the tangent sheaf $\mathscr{T}_X$.*
2. *If $\Delta = \Sigma(1)$, then $\mathscr{T}_X(-\log D)$ is isomorphic to the trivial vector bundle of rank n.*

Proof. If $\Delta = \varnothing$, the family of multifiltrations of $\mathscr{T}_X(-\log D)$ is identical to the family of multifiltrations given in Proposition 3.1. If $\Delta = \Sigma(1)$, for all $\rho \in \Sigma(1)$, we have

$$E^\rho(j) = \begin{cases} 0 & \text{if } j \leq -1 \\ N \otimes_{\mathbb{Z}} \mathbb{C} & \text{if } j \geq 0 \end{cases} .$$

Hence, $\mathscr{T}_X(-\log D)$ is isomorphic to the trivial vector bundle of rank n. □

From now on, we will assume that $\varnothing \subsetneq \Delta \subsetneq \Sigma(1)$ and D the T-invariant Weil divisor defined by

$$D = \sum_{\rho \in \Delta} D_\rho .$$

Notation 3.7. Let G be a sub-vector space of $N_{\mathbb{C}}$. We denote by $\mathscr{E}_G$ the sub-sheaf of $\mathscr{E} = \mathscr{T}_X(-\log D)$ defined by the family of multifiltrations $\big(E_G, \{G^\rho(j)\}_{\rho \in \Sigma(1),\, j \in \mathbb{Z}}\big)$ where $E_G = G$ and $G^\rho(j) = E^\rho(j) \cap G$.
If $\rho \in \Delta$ or $u_\rho \notin G$, then

$$G^\rho(j) = \begin{cases} 0 & \text{if } j \leq -1 \\ G & \text{if } j \geq 0 \end{cases} .$$

If $\rho \notin \Delta$ and $u_\rho \in G$, then

$$G^\rho(j) = \begin{cases} 0 & \text{if } j \leq -2 \\ \mathrm{Span}(u_\rho) & \text{if } j = -1 \\ G & \text{if } j \geq 0 \end{cases} .$$

3.2. **Decomposition of equivariant logarithmic tangent sheaf.** In this part, we give some conditions on Σ and Δ which ensure that the logarithmic tangent sheaf is decomposable. We first recall the family of multifiltration of a direct sum of equivariant reflexive sheaves.

Proposition 3.8 ([DDK20, Remark 2.2.15])**.** *Let $\mathscr{F}$ and $\mathscr{G}$ be two equivariant reflexive sheaves with $\big(F, \{F^\rho(j)\}_{\rho\in\Sigma(1),\, j\in\mathbb{Z}}\big)$ and $\big(G, \{G^\rho(j)\}_{\rho\in\Sigma(1),\, j\in\mathbb{Z}}\big)$ for family of filtrations. The family of multifiltrations of $\mathscr{F}\oplus\mathscr{G}$ is given by*

$$\big(F\oplus G, \{(F\oplus G)^\rho(j)\}_{\rho\in\Sigma(1),\, j\in\mathbb{Z}}\big) \text{ where } (F\oplus G)^\rho(j) = F^\rho(j)\oplus G^\rho(j)\ . \tag{5}$$

We assume that X is toric variety without torus factor. We denote by p the rank of the class group $\mathrm{Cl}(X)$ of X. By [CLS11, Theorem 4.1.3], we have an exact sequence

$$0 \longrightarrow M \longrightarrow \bigoplus_{\rho\in\Sigma(1)} \mathbb{Z}\cdot D_\rho \longrightarrow \mathrm{Cl}(X) \longrightarrow 0 \tag{6}$$

and $\mathrm{card}(\Sigma(1)) = n + p$.

Theorem 3.9. *We assume that* $\mathrm{card}(\Delta) = p$. *We set* $\Sigma(1)\backslash\Delta = \{\rho_1, \ldots, \rho_n\}$ *where* $\rho_k = \mathrm{Cone}(u_k)$ *and* $u_k \in N$. *If* $N_{\mathbb{R}} = \mathrm{Span}(u_1, \ldots, u_n)$, *then* $\mathscr{E} = \mathscr{T}_X(-\log D)$ *is decomposable and*

$$\mathscr{E} = \bigoplus_{k=1}^{n} \mathscr{E}_{F_k}$$

where $\mathscr{E}_{F_k}$ is the sub-sheaf of $\mathscr{E}$ corresponding to the vector space $F_k = \mathrm{Span}(u_k)$.

Proof. For all $k \in \{1, \ldots, n\}$, the family of multifiltration $\big(F_k, \{F_k^\rho(j)\}_{\rho\in\Sigma(1),\, j\in\mathbb{Z}}\big)$ of $\mathscr{E}_{F_k}$ are given by

$$F_k^\rho(j) = \begin{cases} 0 & \text{if } j \le -1 \\ F_k & \text{if } j \ge 0 \end{cases} \quad \text{if } \rho \ne \mathrm{Cone}(u_k)$$

and

$$F_k^\rho(j) = \begin{cases} 0 & \text{if } j \le -2 \\ \mathrm{Span}(u_\rho) & \text{if } j = -1 \\ F_k & \text{if } j \ge 0 \end{cases} \quad \text{if } \rho = \mathrm{Cone}(u_k)\ .$$

For all $\rho \in \Sigma(1)$ and $j \in \mathbb{Z}$, we have

$$\bigoplus_{k=1}^{n} F_k^\rho(j) = \begin{cases} 0 & \text{if } j \le -1 \\ N_{\mathbb{C}} & \text{if } j \ge 0 \end{cases} \qquad \text{if } \rho \in \Delta$$

and

$$\bigoplus_{k=1}^{n} F_k^\rho(j) = \begin{cases} 0 & \text{if } j \le -2 \\ \mathrm{Span}(u_\rho) & \text{if } j = -1 \\ N_{\mathbb{C}} & \text{if } j \ge 0 \end{cases} \qquad \text{if } \rho \notin \Delta\ .$$

Hence, by (5) and Theorem 3.5 we have $\mathscr{E} = \bigoplus_{k=1}^{n} \mathscr{E}_{F_k}$. □

Proposition 3.10. *We assume that Δ satisfies $1 + p \le \mathrm{card}(\Delta) \le n + p - 1$. The sheaf $\mathscr{E} = \mathscr{T}_X(-\log D)$ is decomposable and $\mathscr{E} = \mathscr{E}_G \oplus \mathscr{E}_F$ where $G = \mathrm{Span}(u_\rho : \rho \in \Sigma(1)\setminus\Delta)$ and F a sub-space of $N_{\mathbb{C}}$ such that $N_{\mathbb{C}} = G \oplus F$.*

Proof. It suffices to work with the family of multifiltration as in the proof of Theorem 3.9. □

4. Stability of equivariant logarithmic sheaf

4.1. **Some stability notions.** We recall now the notions of stability that we will consider in this paper. For this part, we refer to [Gue16, Section 3.1] and [CT20, Section 4.1].

Definition 4.1. Let $\mathscr{E}$ be a torsion-free coherent sheaf on X. The *degree* of $\mathscr{E}$ with respect to an ample class $L \in \mathrm{Amp}(X)$ is the real number obtained by intersection:

$$\deg_L(\mathscr{E}) = c_1(\mathscr{E}) \cdot L^{n-1}$$

and its *slope* with respect to L is given by

$$\mu_L(\mathscr{E}) = \frac{\deg_L(\mathscr{E})}{\mathrm{rk}(\mathscr{E})}.$$

Definition 4.2. A torsion-free coherent sheaf $\mathscr{E}$ is said to be *μ-semi-stable* with respect to $L \in \mathrm{Amp}(X)$ if for any proper coherent sub-sheaf of lower rank $\mathscr{F}$ of $\mathscr{E}$ with $0 < \mathrm{rk}\,\mathscr{F} < \mathrm{rk}\,\mathscr{E}$, one has

$$\mu_L(\mathscr{F}) \leq \mu_L(\mathscr{E}) \ .$$

When strict inequality always holds, we say that $\mathscr{E}$ is *μ-stable*. Finally, $\mathscr{E}$ is said to be *μ-polystable* if it is the direct sum of μ-stable sub-sheaves of the same slope.

Lemma 4.3 ([CT20, Lemma 4.3])**.** *Let $\mathscr{E}$ be an equivariant reflexive sheaf with family of multifiltrations $\big(E, \{E^\rho(j)\}_{\rho\in\Sigma(1),\, j\in\mathbb{Z}}\big)$. We have*

$$\mu_L(\mathscr{E}) = \frac{1}{\mathrm{rk}(\mathscr{E})} \sum_{\rho\in\Sigma(1)} \sum_{i\in\mathbb{Z}} i\, e^\rho(i)\, \deg_L(D_\rho)$$

where $e^\rho(i) = \dim E^\rho(i-1) - \dim E^\rho(i)$.

Following [Koo11, Proposition 4.13] and [HNS19, Proposition 2.3], we have :

Proposition 4.4 ([CT20, Proposition 4.2])**.** *Let $\mathscr{E}$ be a T-equivariant reflexive sheaf on X with family of multifiltrations $\big(E, \{E^\rho(j)\}_{\rho\in\Sigma(1),\, j\in\mathbb{Z}}\big)$. Then $\mathscr{E}$ is μ-semi-stable (resp. μ-stable) with respect to L if and only if for all proper vector sub-spaces $W \subset E$, $\mu_L(\mathscr{E}_W) \leq \mu_L(\mathscr{E})$ (resp. $\mu_L(\mathscr{E}_W) < \mu_L(\mathscr{E})$), where $\mathscr{E}_W$ is defined in notation 3.7.*

4.2. **Polystability and semi-stability.** We give here a link between poly-stability and semi-stability.

Proposition 4.5 ([Koo11, Claim 2 of Proposition 4.13])**.** *A reflexive μ-polystable sheaf on X is a μ-semi-stable sheaf on X isomorphic to a (finite, nontrivial) direct sum of reflexive μ-stable sheaves. Let $\mathscr{E}$ be a μ-semi-stable reflexive sheaf on X. Then $\mathscr{E}$ contains a unique maximal reflexive μ-polystable sub-sheaf of the same slope as $\mathscr{E}$.*

Corollary 4.6. *Let $\mathscr{E}$ be an equivariant reflexive sheaf such that*

$$\mathscr{E} = \bigoplus_{i=1}^{r} \mathscr{E}_i$$

with $r \geq 2$. Let $L \in \mathrm{Amp}(X)$ such that for all $i \in \{1, \ldots, r\}$, $\mathscr{E}_i$ is stable with respect to L. Then, $\mathscr{E}$ is poly-stable with respect to L if and only if $\mathscr{E}$ is semi-stable with respect to L.

Proof. If $\mathscr{E}$ is poly-stable with respect to L, then $\mathscr{E}$ is semi-stable with respect to L. We now assume that $\mathscr{E}$ is semi-stable with respect to L; by using the identity

$$\mathrm{rk}(\mathscr{E})\,\mu_L(\mathscr{E}) = \sum_{i=1}^{r} \mathrm{rk}(\mathscr{E}_i)\,\mu_L(\mathscr{E}_i)\ ,$$

we get $\mu_L(\mathscr{E}_i) = \mu_L(\mathscr{E})$ for all $i \in \{1, \ldots, r\}$. Hence, $\mathscr{E}$ is poly-stable. □

4.3. **An instability condition for the logarithmic tangent sheaves.** Let $\Delta \subset \Sigma(1)$ and

$$D = \sum_{\rho \in \Delta} D_\rho$$

a T-invariant Weil divisor on X. Let G be a sub-vector space of $N_{\mathbb{C}}$ of dimension l with $1 \leq l < n$. Let $\big(E, \{E^\rho(j)\}_{\rho\in\Sigma(1),\, j\in\mathbb{Z}}\big)$ be the family of multifiltrations corresponding to $\mathscr{E} = \mathscr{T}_X(-\log D)$ and $\big(E_G, \{G^\rho(j)\}_{\rho\in\Sigma(1),\, j\in\mathbb{Z}}\big)$ be the family of multifiltrations corresponding to $\mathscr{E}_G$. By Lemma 4.3, we have

$$\mu_L(\mathscr{E}) = \frac{1}{n} \sum_{\rho \notin \Delta} \deg_L(D_\rho) \tag{7}$$

and

$$\mu_L(\mathscr{E}_G) = \frac{1}{l} \sum_{\rho\notin\Delta \text{ and } u_\rho \in G} \deg_L(D_\rho)\ . \tag{8}$$

Therefore, we have

$$\mu_L(\mathscr{E}) - \mu_L(\mathscr{E}_G) = \left(\frac{1}{n} - \frac{1}{l}\right)\left(\sum_{\rho\notin\Delta,\, u_\rho\in G} \deg_L(D_\rho)\right) + \frac{1}{n}\left(\sum_{\rho\notin\Delta,\, u_\rho\notin G} \deg_L(D_\rho)\right). \tag{9}$$

To study the stability of $\mathscr{E}$ with respect to $L \in \mathrm{Amp}(X)$, it suffices to compare $\mu_L(\mathscr{E})$ with $\mu_L(\mathscr{E}_G)$ where $G \subset \mathrm{Span}(u_\rho : \rho \notin \Delta)$ and $1 \leq \dim G \leq n-1$.

Theorem 4.7. *If $1 \leq \mathrm{card}(\Sigma(1) \setminus \Delta) \leq n-1$, then for any $L \in \mathrm{Amp}(X)$, the logarithmic tangent sheaf $\mathscr{E} = \mathscr{T}_X(-\log D)$ is not semi-stable with respect to L.*

Proof. We assume that $\Sigma(1)\setminus\Delta = \{\rho_1, \ldots, \rho_k\}$ where $1 \leq k \leq n-1$ and we denote by D_j the divisor corresponding to $\rho_j = \mathrm{Cone}(u_j)$. Let $G = \mathrm{Span}(u_1, \ldots, u_k)$ and $l = \dim G$. If $L \in \mathrm{Amp}(X)$, we have

$$\mu_L(\mathscr{E}) - \mu_L(\mathscr{E}_G) = \left(\frac{1}{n} - \frac{1}{l}\right)\sum_{j=1}^{k} \deg_L(D_j) < 0$$

because the numbers $\deg_L(D_j)$ are positive and different to zero. Thus, $\mathscr{E}$ is not semi-stable with respect to L. □

We assume that X has no torus factor. We have seen that $\mathrm{card}(\Sigma(1)) = n + \mathrm{rk}\,\mathrm{Cl}(X)$, so the Theorem 4.7 becomes :

Corollary 4.8. *We set $p = \mathrm{rk}\,\mathrm{Cl}(X)$. If $1 + p \leq \mathrm{card}(\Delta) \leq n + p - 1$, then for any $L \in \mathrm{Amp}(X)$, the logarithmic tangent sheaf $\mathscr{T}_X(-\log D)$ is not semi-stable with respect to L.*

Proof. If $1 + p \leq \mathrm{card}(\Delta) \leq n + p - 1$, by using

$$\mathrm{card}(\Sigma(1)) = n + p = \mathrm{card}(\Delta) + \mathrm{card}(\Sigma(1) \setminus \Delta)\ ,$$

we get $1 \leq \mathrm{card}(\Sigma(1) \setminus \Delta) \leq n - 1$; we can conclude with Theorem 4.7. □

Remark 4.9. By Corollary 3.6, if $\mathrm{card}(\Delta) = n + p$, $\mathscr{T}_X(-\log D)$ is semi-stable.

From now on, when we study the stability or semi-stability of $\mathscr{T}_X(-\log D)$, we will consider only the case where $1 \leq \mathrm{card}(\Delta) \leq p = \mathrm{rk}\,\mathrm{Cl}(X)$ and $p \in \{1, 2\}$.

4.4. **Stability on toric orbifolds of Picard rank one.** In this section, we will study the semi-stability of logarithmic tangent sheaf on toric orbifolds of Picard rank one.
Let $q_0, q_1, \ldots, q_n \in \mathbb{N}^*$ such that $\gcd(q_0, \ldots, q_n) = 1$. We set

$$N = \mathbb{Z}^{n+1}/\mathbb{Z}(q_0, \ldots, q_n)\ .$$

The dual lattice of N is $M = \{(a_0, \ldots, a_n) \in \mathbb{Z}^{n+1} : a_0\, q_0 + \ldots + a_n\, q_n = 0\}$.
Let $\{u_i : 0 \leq i \leq n\}$ be the images in N of the standard basis vectors in $\mathbb{Z}^{n+1}$, so the relation

$$q_0\, u_0 + q_1\, u_1 + \ldots + q_n\, u_n = 0$$

holds in N. Let Σ be the fan in $N_{\mathbb{R}}$ defined by

$$\Sigma = \{\mathrm{Cone}(A) :\ A \subsetneq \{u_0, \ldots, u_n\}\}\ .$$

We define X to be the toric variety of the fan Σ. Like Σ is simplicial, by [CLS11, Proposition 4.2.7], $\mathrm{Pic}(X)$ has finite index in $\mathrm{Cl}(X)$. By (6), we deduce that $\mathrm{rk}\,\mathrm{Pic}(X) = 1$. We denote by D_i the Weil divisor corresponding to the ray $\mathrm{Cone}(u_i)$.

In the Corollary 4.8, we see that if $2 \leq \mathrm{card}(\Delta) \leq n$, then $\mathscr{T}_X(-\log D)$ is not semi-stable with respect to any $L \in \mathrm{Amp}(X)$. We assume that $\mathrm{card}(\Delta) = 1$. Let $i \in \{0, \ldots, n\}$ and $A_i = \{0, \ldots, n\} \setminus \{i\}$. We set $\mathscr{E} = \mathscr{T}_X(-\log D_i)$.

Lemma 4.10. *For all $j \in A_i$, the divisor $q_i D_j$ is linearly equivalent to the divisor $q_j D_i$.*

Proof. We fix $j \in A_i$. Let $m = (a_0, \ldots, a_n) \in M$ defined by $a_i = q_j$, $a_j = -q_i$ and $a_k = 0$ if $k \in \{0, \ldots, n\} \setminus \{i, j\}$. Like $\mathrm{div}(\chi^m) = q_j\, D_i - q_i\, D_j$, we deduce that $q_i\, D_j$ is linearly equivalent to $q_j\, D_i$. □

Theorem 4.11. *Let $L \in \mathrm{Amp}(X)$. The sheaf $\mathscr{E}$ is poly-stable with respect to L if and only if there is $q \in \mathbb{N}^*$ such that for all $j \in A_i$, $q_j = q$.*

Proof. The assumptions of Theorem 3.9 are verified. Hence,

$$\mathscr{E} = \bigoplus_{j \in A_i} \mathscr{E}_{F_j}$$

where $F_j = \mathrm{Span}(u_j)$. By (8), $\mu_L(\mathscr{E}_{F_j}) = \deg_L(D_j)$ for all $j \in A_i$. By Lemma 4.10, for all $j \in A_i$,

$$\mu_L(\mathscr{E}_{F_j}) = \frac{q_j}{q_i} \deg_L(D_i)\ .$$

If $\mathscr{E}$ is poly-stable with respect to L, there is $r \in \mathbb{Q}$ such that for all $j \in A_i$, $q_j = r\, q_i$. Hence, we have the existence of $q \in \mathbb{N}^*$ such that for all $j \in A_i$, $q_j = q$.
For the converse, if for all $j \in A_i$, we have $q_j = q$, then $\mathscr{E}$ is poly-stable. □

According to Theorem 4.11 and Corollary 4.6, we have :

Corollary 4.12. *For all $i \in \{0, \ldots, n\}$, $\mathrm{sStab}(\mathscr{T}_X(-\log D_i)) \neq \varnothing$ if and only if there exists $q \in \mathbb{N}^*$ such that for all $j \in A_i$, $q_j = q$. Moreover, if for all $j \in A_i$, we have $q_j = q$, then*

$$\mathrm{sStab}(\mathscr{T}_X(-\log D_i)) = \mathrm{Amp}(X)\ .$$

Corollary 4.13. *For all $i \in \{0, \ldots, n\}$, we have $\mathrm{Stab}(\mathscr{T}_X(-\log D_i)) = \varnothing$.*

Corollary 4.14. *If $q_0 = \ldots = q_n = 1$, that is $X = \mathbb{P}^n$, then for all $i \in \{0, \ldots, n\}$, we have*

$$\varnothing = \mathrm{Stab}(T_X(-\log D_i)) \subsetneq \mathrm{sStab}(T_X(-\log D_i)) = \mathrm{Amp}(X)\ .$$

5. Description of smooth projective toric varieties of Picard rank two

Let X be a smooth projective toric variety of dimension n such that $\operatorname{rk}\operatorname{Pic}(X) = 2$. There exists $r, s \in \mathbb{N}^*$ with $r + s = n$ and $a_1, \ldots, a_r \in \mathbb{N}$ with $a_1 \leq a_2 \leq \ldots \leq a_r$ such that

$$X = \mathbb{P}\left(\mathscr{O}_{\mathbb{P}^s} \oplus \bigoplus_{i=1}^{r} \mathscr{O}_{\mathbb{P}^s}(a_i)\right) . \tag{10}$$

We keep the notation of Section 1.1. We denote by $\pi : X \longrightarrow \mathbb{P}^s$ the projection onto the basis $\mathbb{P}^s$. We have the following linear equivalence,

$$D_{v_i} \sim_{\text{lin}} D_{v_0} - a_i D_{w_0} \text{ for } i \in \{1, \ldots, r\} \text{ and } D_{w_j} \sim_{\text{lin}} D_{w_0} \text{ for } j \in \{1, \ldots, s\} . \tag{11}$$

By (11), we deduce that $\operatorname{Pic}(X)$ is generated by D_{v_0} and D_{w_0}.

Proposition 5.1 ([DDK20, Proposition 4.2.1])**.** *Let $D = \mu D_{w_0} + \lambda D_{v_0}$ be a T-invariant divisor of X with $\mu, \lambda \in \mathbb{Z}$. The divisor D is ample if and only if $\mu > 0$ and $\lambda > 0$.*

Remark 5.2. Let $\mu, \lambda \in \mathbb{N}^*$ and $L = \mathscr{O}_X(\mu D_{w_0} + \lambda D_{v_0})$. We have an isomorphism $L \cong \pi^*\mathscr{O}_{\mathbb{P}^s}(\mu) \otimes \mathscr{O}_X(\lambda)$.

By [CLS11, Theorem 8.1.2], the anti-canonical divisor of X is given by

$$-K_X = \sum_{i=0}^{r} D_{v_i} + \sum_{j=0}^{s} D_{w_j} \sim_{\text{lin}} (s + 1 - a_1 - \ldots - a_r)D_{w_0} + (r + 1)D_{v_0} . \tag{12}$$

Hence, X is Fano if and only if $a_1 + \ldots + a_r \leq s$.

Let D be a reduced T-invariant Weil divisor on X. To study the stability of $\mathscr{E} = T_X(-\log D)$ with respect to $L = \mathscr{O}_X(Z)$ where $Z = \mu D_{w_0} + \lambda D_{v_0}$ ($\mu, \lambda \in \mathbb{N}^*$), we will study the stability of $\mathscr{E}$ with respect to the ample $\mathbb{Q}$-divisor $Z' = \nu D_{w_0} + D_{v_0}$ where $\nu = \dfrac{\mu}{\lambda}$.

5.1. Computation of the degree.

Let $P \subset M_{\mathbb{R}}$ be a polytope such that the fan Σ_P of P is the fan Σ of X. For each $\rho \in \Sigma(1)$ we denote by P^ρ the facet of P corresponding to a ray ρ.
We recall that a lattice M defines a measure ν on $M_{\mathbb{R}}$ as the pull-back of Haar measure on $M_{\mathbb{R}}/M$. It is determined by the properties

i. ν is translation invariant,
ii. $\nu(M_{\mathbb{R}}/M) = 1$.

For all $\rho \in \Sigma(1)$, we denote by $\operatorname{vol}(P^\rho)$ the volume of P^ρ with respect to the measure determined by the lattice $\operatorname{Span}(P^\rho) \cap M$.

Proposition 5.3 ([Dan78, Section 11])**.** *Let (X, L) be a polarized toric variety corresponding to a lattice polytope P. For all ray ρ, we have $\deg_L(D_\rho) = \operatorname{vol}(P^\rho)$.*

5.2. Polytope corresponding to a $\mathbb{Q}$-ample divisor.

Here, we describe the polytope corresponding to the $\mathbb{Q}$-polarized variety (X, L) where

$$L = \pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \times \mathscr{O}_X(1) .$$

We refer to [HNS19, Section 4].
Let $\nu \in \mathbb{Q}^+$ such that $\nu \neq 0$ and $P_s = \operatorname{Conv}(0, w_1, w_2, \ldots, w_s)$. Then,

$$P = \operatorname{Conv}\left(\nu P_s \times \{0\} \cup (a_1 + \nu)P_s \times \{v_1\} \cup \ldots \cup (a_r + \nu)P_s \times \{v_r\}\right)$$

is the polytope corresponding to the $\mathbb{Q}$-polarized variety $(X,\, L)$. We denote by P^{v_i} (resp. P^{w_j}) the facet of P corresponding to the ray $\mathrm{Cone}(v_i)$ (resp. $\mathrm{Cone}(w_j)$). The facet P^{v_i} is the convex hull of

$$\nu P_s\times\{0\}\cup\ldots\cup(a_{i-1}+\nu)P_s\times\{v_{i-1}\}\cup(a_{i+1}+\nu)P_s\times\{v_{i+1}\}\cup\ldots\cup(a_r+\nu)P_s\times\{v_r\}$$

and P^{w_i} is isomorphic to

$$\nu P'_s\times\{0\}\cup(a_1+\nu)P'_s\times\{v_1\}\cup\ldots\cup(a_r+\nu)P'_s\times\{v_r\}$$

where $P'_s=\mathrm{Conv}(0,\, w_1,\ldots,\, w_{s-1})$.

Notation 5.4. For all $i\in\{0,\,1,\ldots,\,r\}$, we set $\mathsf{V}_i=\mathrm{vol}(P^{v_i})$. Like for all $j\in\{1,\ldots,\,s\}$, $\mathrm{vol}(P^{w_j})=\mathrm{vol}(P^{w_0})$, we set $\mathsf{W}=\mathrm{vol}(P^{w_0})$.

Proposition 5.5 ([HNS19, Proposition 4.3])**.** *Let $c_0,\, c_1,\ldots,\, c_r\in\mathbb{N}$, $\nu>0$ and $P_s=\mathrm{Conv}(0,\, w_1,\, w_2,\ldots,\, w_s)$. The volume of the polytope*

$$P=\mathrm{Conv}\left((c_0+\nu)P_s\times\{0\}\cup(c_1+\nu)P_s\times\{v_1\}\cup\ldots\cup(c_r+\nu)P_s\times\{v_r\}\right)$$

is given by

$$\sum_{k=0}^{s}\binom{s+r}{k}\left(\sum_{d_0+\ldots+d_r=s-k}c_0^{d_0}\cdots c_r^{d_r}\right)\nu^k\ .$$

By Proposition 5.5, we have

$$\mathsf{W}=\sum_{k=0}^{s-1}\binom{s+r-1}{k}\left(\sum_{d_1+\ldots+d_r=s-k-1}a_1^{d_1}\cdots a_r^{d_r}\right)\nu^k$$

$$\mathsf{V}_0=\sum_{k=0}^{s}\binom{s+r-1}{k}\left(\sum_{d_1+\ldots+d_r=s-k}a_1^{d_1}\cdots a_r^{d_r}\right)\nu^k$$

and for $i\in\{1,\ldots,\,r\}$,

$$\mathsf{V}_i=\sum_{k=0}^{s}\binom{s+r-1}{k}\left(\sum_{\substack{d_1+\ldots+d_{i-1}\\+d_{i+1}+\ldots+d_r=s-k}}a_1^{d_1}\cdots a_{i-1}^{d_{i-1}}a_{i+1}^{d_{i+1}}\cdots a_r^{d_r}\right)\nu^k\ .$$

For ν fixed, we give some relations between W and the numbers V_i.
If $a_1=\ldots=a_r=0$, then

$$\mathsf{W}=\binom{s+r-1}{s-1}\nu^{s-1}\quad\text{and}\quad\mathsf{V}_i=\binom{s+r-1}{s}\nu^s\ . \tag{13}$$

We now assume that $(a_1,\ldots,\,a_r)\neq(0,\ldots,\,0)$ with $a_1\leq a_2\leq\ldots\leq a_r$. Let $z\in\{0,\,1,\ldots,\,r-1\}$ such that $a_z=0$ and $a_{z+1}>0$ (we set $a_0=0$). For $k\in\{0,\ldots,\,s\}$ and $i\in\{z+1,\ldots,\,r\}$, we set

$$\mathsf{W}_k=\sum_{d_{z+1}+\ldots+d_r=s-1-k}a_{z+1}^{d_{z+1}}\cdots a_r^{d_r}$$

$$\mathsf{V}_{0k}=\sum_{d_{z+1}+\ldots+d_r=s-k}a_{z+1}^{d_{z+1}}\cdots a_r^{d_r}$$

$$\mathsf{V}_{ik}=\sum_{\substack{d_{z+1}+\ldots+d_{i-1}\\+d_{i+1}+\ldots+d_r=s-k}}a_{z+1}^{d_{z+1}}\cdots a_{i-1}^{d_{i-1}}a_{i+1}^{d_{i+1}}\cdots a_r^{d_r}$$

where $\mathsf{W}_s=0$. If $i\in\{1,\ldots,\,z\}$, we set $\mathsf{V}_{ik}=\mathsf{V}_{0k}$. We have $\mathsf{W}_{s-1}=1$ and for $i\in\{0,\ldots,\,r\}$, $\mathsf{V}_{is}=1$.

Remark 5.6. If $r=1$, we set $\mathsf{V}_{1s}=1$ and for $k\in\{0,\ldots,\,s-1\}$, $\mathsf{V}_{1k}=0$.

Lemma 5.7. *For $k \in \{0, \dots, s-1\}$ and $i \in \{1, \dots, r\}$, $a_i \mathsf{W}_k + \mathsf{V}_{ik} = \mathsf{V}_{0k}$.*

Proof. If $i \in \{1, \dots z\}$, the equality is true because $a_i = 0$. We assume that $i \in \{z+1, \dots, r\}$, we have

$$\begin{aligned}\mathsf{V}_{0k} &= \sum_{d_{z+1}+\dots+d_r=s-k} a_{z+1}^{d_{z+1}} \cdots a_r^{d_r} \\ &= \sum_{\substack{d_{z+1}+\dots+d_r=s-k \\ d_i=0}} a_{z+1}^{d_{z+1}} \cdots a_r^{d_r} + \sum_{\substack{d_{z+1}+\dots+d_r=s-k \\ d_i\geq 1}} a_{z+1}^{d_{z+1}} \cdots a_r^{d_r}\end{aligned}$$

The first term of the second line corresponds to the number V_{ik} and the second to $a_i \mathsf{W}_k$ (it suffices to replace d_i by $d_i' + 1$). Hence, $\mathsf{V}_{0k} = \mathsf{V}_{ik} + a_i \mathsf{W}_k$. □

Corollary 5.8. *For all $i \in \{1, \dots, r\}$, $\mathsf{V}_0 = a_i \mathsf{W} + \mathsf{V}_i$.*

5.3. **A necessary condition for stability.** In this part, we adapt some results of [HNS19, Section 4] for the study of the stability of $T_X(-\log D)$.

The following lemma will be useful in the proof of Proposition 5.10 which is the main result of this part. Let $z \in \{0, \dots, r-1\}$ such that $a_z = 0$ and $a_{z+1} > 0$. We have the following lemma.

Lemma 5.9 ([HNS19, Lemma 4.2])**.**
Let $I' \subset \{0, 1 \dots, r\}$ and $G = \mathrm{Span}(v_i : i \in I')$. The vector $a_1 v_1 + \dots + a_r v_r$ belongs to G if and only if

i. *$\{z+1, \dots, r\} \subset I'$ or*
ii. *$\{0, \dots, z\} \subset I'$, $\mathrm{card}(\{z+1, \dots, r\} \setminus I') \geq 1$ and $a_i = a_j$ for all $i, j \in \{z+1, \dots, r\} \setminus I'$.*

Let $\Delta \subset \Sigma(1)$ and $D = \sum_{\rho \in \Delta} D_\rho$.
Let $I_\Sigma = \{\mathrm{Cone}(v_0), \dots, \mathrm{Cone}(v_r)\}$ and $J_\Sigma = \{\mathrm{Cone}(w_0), \dots, \mathrm{Cone}(w_s)\}$. We set

$$I = \{i \in \{0, 1, \dots, r\} : \mathrm{Cone}(v_i) \in I_\Sigma \setminus (I_\Sigma \cap \Delta)\} \text{ and}$$

$$J = \{j \in \{0, 1, \dots, s\} : \mathrm{Cone}(w_j) \in J_\Sigma \setminus (J_\Sigma \cap \Delta)\}.$$

Let $L \in \mathrm{Amp}(X)$. To study the stability of $\mathscr{E} = T_X(-\log D)$, it suffices to compare $\mu_L(\mathscr{E})$ and $\mu_L(\mathscr{E}_G)$ where $G = \mathrm{Span}(v_i, w_j : i \in I', j \in J')$ with $I' \subset I$, $J' \subset J$ and $1 \leq \dim G < (r+s)$. By Proposition 5.3, and by using of (7), (8) we have

$$\mu_L(\mathscr{E}) = \frac{1}{r+s}\left(\sum_{i \in I} \mathsf{V}_i + \mathrm{card}(J) \cdot \mathsf{W}\right)$$

and

$$\mu_L(\mathscr{E}_G) = \frac{1}{\dim G}\left(\sum_{i \in I'} \mathsf{V}_i + \mathrm{card}(J') \cdot \mathsf{W}\right).$$

Here is a variant of [HNS19, Proposition 4.1] which will be useful for us in the study of stability of $T_X(-\log D)$.

Proposition 5.10. *The logarithmic tangent bundle $\mathscr{E} = T_X(-\log D)$ is stable (resp. semi-stable) with respect to L if and only if*

$$\mu_L(\mathscr{E}) = \frac{1}{r+s}\left(\sum_{i \in I} \mathsf{V}_i + \mathrm{card}(J) \cdot \mathsf{W}\right)$$

is greater than (resp. greater than or equal to) the maximum of

(1) V_{i_0} *where* $i_0 = \min I$ *if* $I \neq \varnothing$;

(2) $\dfrac{1}{r'}\displaystyle\sum_{i\in I}\mathsf{V}_i$, *if* $r' = \dim \mathrm{Span}(v_i : i \in I) \neq 0$;

(3) $\dfrac{\mathrm{card}(J)\cdot \mathsf{W}}{s'}$, *if* $s' = \dim \mathrm{Span}(w_j : j \in J) \neq 0$ *and* $s' < r + s$;

(4) $\dfrac{1}{s+k}\left(\displaystyle\sum_{i\in I'}\mathsf{V}_i + (s+1)\mathsf{W}\right)$, *if* $\mathrm{card}(J') = s+1$, $k = \mathrm{card}(I') < r$ *and* $\{z+1,\dots,r\} \subset I' \subset I$;

(5) $\dfrac{1}{s+k}\left(\displaystyle\sum_{i\in I'}\mathsf{V}_i + (s+1)\mathsf{W}\right)$, *if* $\mathrm{card}(J') = s+1$, $k = \mathrm{card}(I') < r$ *and* $I' \subset I$ *such that the condition ii. of Lemma 5.9 is verified.*

Proof. Let $G = \mathrm{Span}(v_i, w_j : i \in I', j \in J')$ where $I' \subset I$ and $J' \subset J$. The expressions (1) of Proposition 5.10 corresponds to $G = \mathrm{Span}(v_{i_0})$, (2) corresponds to $G = \mathrm{Span}(v_i : i \in I)$ and (3) corresponds to $G = \mathrm{Span}(w_j : j \in J)$.

• If $\mathrm{card}(J') = 0$, then for $\varnothing \subsetneq I' \subseteq I$, we have $\dim G \leq r$ and

$$\mu_L(\mathscr{E}_G) = \frac{1}{\dim G}\sum_{i\in I'}\mathsf{V}_i \ ;$$

this number is less than or equal to the maximum of the numbers given in (1) and (2).

• If $\mathrm{card}(I') = 0$, then for $\varnothing \subsetneq J' \subseteq J$ such that $\dim G < r+s$, we have

$$\mu_L(\mathscr{E}_G) = \frac{\mathrm{card}(J')\cdot W}{\dim G} \ ;$$

this number is less than or equal to that given in (3).

• If $\mathrm{card}(I') = r+1$, then $\dim G < r+s$ if and only if $s' := \mathrm{card}(J') < s$. If $1 \leq s' < s$, then

$$\mu_L(\mathscr{E}_G) = \frac{1}{r+s'}\left(\sum_{i\in I'}\mathsf{V}_i + s'\mathsf{W}\right) \leq \max\left(\frac{1}{r}\sum_{i\in I'}\mathsf{V}_i\,,\,\mathsf{W}\right) \ ;$$

this number is less than or equal to the maximum of numbers given in (2) and (3).

• If $1 \leq \mathrm{card}(I') \leq r$, $1 \leq \mathrm{card}(J') \leq s$ and $\dim G < r+s$, then $\mu_L(\mathscr{E}_G)$ is less than or equal to the maximum of numbers given in (1), (2) and (3).

• It remains to study the case where $\mathrm{card}(J') = s+1$ and $1 \leq \mathrm{card}(I') < r$ (because if $\mathrm{card}(I') \geq r$, then $\dim G = r+s$). We will treat it in two cases.

First case : $a_r = 0$. For all $i \in \{1,\dots,r\}$, $\mathsf{V}_i = \mathsf{V}_0$. If $r' = \mathrm{card}(I')$ and $1 \leq r' < r$, then

$$\mu_L(\mathscr{E}_G) = \frac{1}{r'+s}\left(\sum_{i\in I'}\mathsf{V}_i + (s+1)\mathsf{W}\right) \leq \max\left(\mathsf{V}_0\,,\,\frac{(s+1)\mathsf{W}}{s}\right) \ .$$

Second case : $a_r > 0$. We set $r' = \mathrm{card}(I')$. If I' satisfies the first (resp. second) condition of Lemma 5.9, we get (4) (resp. (5)).
If I' doesn't satisfy the conditions of Lemma 5.9, then $\dim G = r' + (s+1)$. Moreover, if $r' + (s+1) < r+s$, the number $\mu_L(\mathscr{E}_G)$ is less than or equal to the maximum of the numbers given in (1) and (3). □

Remark 5.11. If $a_1 = \ldots = a_r = 0$, to check the stability of $\mathscr{E}$ with respect to L, it is enough to compare $\mu_L(\mathscr{E})$ with the numbers given by the points 1, 2 and 3 of Proposition 5.10.

To check that $\mathscr{E} = T_X(-\log D)$ is not semi-stable, we will need the lemma below. The purpose of the lemma is to avoid comparing $\mu_L(\mathscr{E})$ with all the numbers given in the Proposition 5.10.

Lemma 5.12.

(1) *If* $a_r \geq 2$, *then* $s\,\mathsf{V}_0 - (s+1)\mathsf{W} \geq s\,\mathsf{V}_r$.
(2) *Let* $r \geq 2$ *and* $i \in \{1, \ldots, r-1\}$. *If* $a_r > a_i$, *then* $\mathsf{V}_i - \mathsf{W} \geq \mathsf{V}_r$.

Proof. If $a_r \geq 2$, then $\left(s - \dfrac{s+1}{a_r}\right) = \dfrac{a_r\, s - (s+1)}{a_r} \geq \dfrac{2s - (s+1)}{a_r} \geq 0$ because $s \geq 1$. Hence,

$$\begin{aligned} s\,\mathsf{V}_0 - (s+1)\mathsf{W} &= s\,\mathsf{V}_0 - \frac{s+1}{a_r}(\mathsf{V}_0 - \mathsf{V}_r) \quad \text{because } a_r\,\mathsf{W} = \mathsf{V}_0 - \mathsf{V}_r \\ &= \left(s - \frac{s+1}{a_r}\right)\mathsf{V}_0 + \frac{s+1}{a_r}\mathsf{V}_r \\ &\geq \left(s - \frac{s+1}{a_r}\right)\mathsf{V}_r + \frac{s+1}{a_r}\mathsf{V}_r = s\,\mathsf{V}_r\ . \end{aligned}$$

Since $\mathsf{V}_0 = a_i\,\mathsf{W} + \mathsf{V}_i = a_r\,\mathsf{W} + \mathsf{V}_r$, we have $\mathsf{V}_i = (a_r - a_i)\mathsf{W} + \mathsf{V}_r$. If $a_r > a_i$, then $a_r - a_i \geq 1$; therefore $\mathsf{V}_i \geq \mathsf{W} + \mathsf{V}_r$, i.e $\mathsf{V}_i - \mathsf{W} \geq \mathsf{V}_r$. □

5.4. **Stability of logarithmic tangent bundle on $X = \mathbb{P}^r \times \mathbb{P}^s$.** In this part, we assume that $a_1 = a_2 = \ldots = a_r = 0$. We have $X \cong \mathbb{P}^s \times \mathbb{P}^r$. We denote by $\pi_1 : X \longrightarrow \mathbb{P}^s$ and $\pi_2 : X \longrightarrow \mathbb{P}^r$ the projection maps.

We first show that $T_X(-\log D)$ is decomposable when $\mathrm{card}(\Delta) \in \{1, 2\}$.

Lemma 5.13. *For* $i \in \{0, \ldots, r\}$, *the vector bundle* $\mathscr{E} = T_X(-\log D_{v_i})$ *satisfies*

$$\mathscr{E} = \left(\bigoplus_{k=0,\, k\neq i}^{r} \mathscr{E}_{F_k}\right) \oplus \mathscr{E}_G$$

where $F_k = \mathrm{Span}(v_k)$ *and* $G = \mathrm{Span}(w_0, \ldots, w_s)$.

Lemma 5.14. *For* $j \in \{0, \ldots, s\}$, *the vector bundle* $\mathscr{E} = T_X(-\log D_{w_j})$ *satisfies*

$$\mathscr{E} = \left(\bigoplus_{k=0,\, k\neq j}^{s} \mathscr{E}_{G_k}\right) \oplus \mathscr{E}_F$$

where $G_k = \mathrm{Span}(w_k)$ *and* $F = \mathrm{Span}(v_0, \ldots, v_r)$.

Lemma 5.15. *Let* $i, i' \in \{0, \ldots, r\}$ *distinct and* $j, j' \in \{0, \ldots, s\}$ *distinct. We have*

$$\begin{aligned} T_X(-\log(D_{v_i} + D_{v_{i'}})) &\cong T_{\mathbb{P}^s} \oplus T_{\mathbb{P}^r}(-\log(\pi_2(D_{v_i}) + \pi_2(D_{v_{i'}}))) \\ T_X(-\log(D_{w_j} + D_{w_{j'}})) &\cong T_{\mathbb{P}^s}(-\log(\pi_1(D_{w_j}) + \pi_1(D_{w_{j'}}))) \oplus T_{\mathbb{P}^r} \end{aligned}$$

For the proofs of these three lemmas it suffices to work with the family of multifiltrations as in the proof of Theorem 3.9. If $i \in \{0, \ldots, r\}$ and $j \in \{0, \ldots, s\}$, by Theorem 3.9 the vector bundle $T_X(-\log(D_{v_i} + D_{w_j}))$ is decomposable.

We now study (semi)stability. In this table, we give the values of ν for which $\mathscr{E} = T_X(-\log D)$ is (semi)stable with respect to $\pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$.

TABLE 1. Stability of $T_X(-\log D)$ when $a_1 = \ldots = a_r = 0$

Divisor D	Stab($\mathscr{E}$)	sStab($\mathscr{E}$)	References
$D_{v_i}\ ,\quad 0 \le i \le r$	$\varnothing$	$\nu = \dfrac{s+1}{r}$	Theorem 5.16
$D_{w_j}\ ,\quad 0 \le j \le s$	$\varnothing$	$\nu = \dfrac{r+1}{s}$	Theorem 5.16
$D_{v_j} + D_{w_j}$	$\varnothing$	$\nu = \dfrac{s}{r}$	Theorem 5.19
$D_{v_i} + D_{v_j}\ ,\quad 0 \le i < j \le r$	$\varnothing$	$\varnothing$	Theorem 5.18
$D_{w_i} + D_{w_j}\ ,\quad 0 \le i < j \le s$	$\varnothing$	$\varnothing$	Theorem 5.18

By (13), we have $\mathsf{V} = \dfrac{r\,\nu}{s}\mathsf{W}$.

Theorem 5.16. *Let $i \in \{0, 1, \ldots, r\}$ and $j \in \{0, 1, \ldots, s\}$. For any $L \in \mathrm{Amp}(X)$, the logarithmic tangent bundles $T_X(-\log D_{v_i})$ and $T_X(-\log D_{w_j})$ are not stable with respect to L. We have :*

(1) *$T_X(-\log D_{v_i})$ is semi-stable with respect to $\pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$ if and only if $\nu = \dfrac{s+1}{r}$;*
(2) *$T_X(-\log D_{w_j})$ is semi-stable with respect to $\pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$ if and only if $\nu = \dfrac{r+1}{s}$.*

Proof. We start with $\mathscr{E} = T_X(-\log D_{v_i})$. We have :

$$\mu_L(\mathscr{E}) = \frac{1}{r+s}(r\,\mathsf{V} + (s+1)\mathsf{W}) = \frac{r^2\nu + s^2 + s}{s(r+s)}\,\mathsf{W} = \frac{r^2\nu + s^2 + s}{r\nu(r+s)}\,\mathsf{V}\ .$$

By Proposition 5.10, to have stability or semi-stability, it is enough to compare $\mu_L(\mathscr{E})$ with $\max\left(\dfrac{(s+1)\mathsf{W}}{s}, \mathsf{V}\right)$.

If $\mu_L(\mathscr{E}) \ge \mathsf{V}$, then $\dfrac{r^2\nu + s^2 + s}{r\nu(r+s)} \ge 1$, i.e $(r^2\nu + s^2 + s) \ge (r^2\nu + rs\nu)$; hence, $\nu \le \dfrac{s+1}{r}$. If $\mu_L(\mathscr{E}) \ge \dfrac{(s+1)\mathsf{W}}{s}$, then $\dfrac{r^2\nu + s^2 + s}{s(r+s)} \ge \dfrac{s+1}{s}$, i.e $\nu \ge \dfrac{s+1}{r}$.

If $\nu \ne \dfrac{s+1}{r}$, the numbers $\mu_L(\mathscr{E}) - \mathsf{V}$ and $\mu_L(\mathscr{E}) - \dfrac{s+1}{s}\mathsf{W}$ are non-zero and have opposite signs. Therefore, $T_X(-\log D_{v_i})$ is not semi-stable with respect to $\pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$.

If $\nu = \dfrac{s+1}{r}$, then $\mu_L(\mathscr{E}) = \mathsf{V} = \dfrac{s+1}{s}\mathsf{W}$. Thus, $T_X(-\log D_{v_i})$ is semi-stable but not stable with respect to $\pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$.

If we regard the case where $\mathscr{E} = T_X(-\log D_{w_j})$, it is enough to compare $\mu_L(\mathscr{E})$ with $\max\left(\dfrac{(r+1)\mathsf{V}}{r}, \mathsf{W}\right)$. We get the result by exchanging the roles of r and s and also the roles of V and W in the computation above. □

Corollary 5.17. *With the decomposition given in Lemma 5.13 (resp. Lemma 5.14), the vector bundle $T_X(-\log D_{v_i})$ (resp. $T_X(-\log D_{w_i})$) is poly-stable with respect to $\pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$ if and only if $\nu = \dfrac{s+1}{r}$ (resp. $\nu = \dfrac{r+1}{s}$).*

Theorem 5.18. *Let $i, i' \in \{0, \dots, r\}$ distinct and $j, j' \in \{0, \dots, s\}$ distinct. For any $L \in \mathrm{Amp}(X)$, the logarithmic tangent bundles $T_X(-\log(D_{v_i} + D_{v_{i'}}))$ and $T_X(-\log(D_{w_j} + D_{w_{j'}}))$ are not semi-stable with respect to L.*

Proof. Let $\mathscr{E} = T_X(-\log(D_{v_i} + D_{v_{i'}})$. We have :

$$\mu_L(\mathscr{E}) = \frac{1}{r+s}((r-1)\mathsf{V}+(s+1)\mathsf{W}) = \frac{r(r-1)\nu + s^2 + s}{s(r+s)}\,\mathsf{W} = \frac{r(r-1)\nu + s^2 + s}{r\nu(r+s)}\,\mathsf{V}\,.$$

We assume that $r = 1$. We have $\mu_L(\mathscr{E}) = \mathsf{W}$. If we set $F = \mathrm{Span}(w_0, \dots, w_s)$, we have $\mu_L(\mathscr{E}_F) = \dfrac{(s+1)\mathsf{W}}{s}$; thus $T_X(-\log(D_{v_i} + D_{v_{i'}}))$ is not semi-stable with respect to L.

We now assume that $r \geq 2$. To check the stability or semi-stability, it is enough to compare $\mu_L(\mathscr{E})$ with $\max\left(\dfrac{(s+1)\mathsf{W}}{s}, \mathsf{V}\right)$.

If $\mu_L(\mathscr{E}) \geq \mathsf{V}$, then $\dfrac{r(r-1)\nu + s^2 + s}{r\nu(r+s)} \geq 1$; i.e $\nu \leq \dfrac{s}{r}$.

If $\mu_L(\mathscr{E}) \geq \dfrac{(s+1)\mathsf{W}}{s}$, then $\dfrac{r(r-1)\nu + s^2 + s}{s(r+s)} \geq \dfrac{s+1}{s}$; i.e $\nu \geq \dfrac{s+1}{r-1} > \dfrac{s}{r}$.

Like ν cannot satisfy this two conditions, we deduce that $T_X(-\log(D_{v_i} + D_{v_{i'}}))$ is not semi-stable with respect to L. □

Theorem 5.19. *Let $i \in \{0, \dots, r\}$, $j \in \{0, \dots, s\}$ and $D = D_{v_i} + D_{w_j}$. We have :*

1. *$T_X(-\log D)$ is not stable with respect to any $L \in \mathrm{Amp}(X)$.*
2. *$T_X(-\log D)$ is semi-stable with respect to $\pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$ if and only if $\nu = \dfrac{s}{r}$.*

Proof. We have

$$\mu_L(\mathscr{E}) = \frac{1}{r+s}(r\mathsf{V} + s\mathsf{W}) = \frac{r^2\nu + s^2}{s(r+s)}\,\mathsf{W} = \frac{r^2\nu + s^2}{r\nu(r+s)}\,\mathsf{V}\,.$$

To check stability or semi-stability, it is enough to compare $\mu_L(\mathscr{E})$ with $\max(\mathsf{V}, \mathsf{W})$.

If $\mu_L(E) \geq \mathsf{V}$, then $\dfrac{r^2\nu + s^2}{r\nu(r+s)} \geq 1$; i.e $\nu \leq \dfrac{s}{r}$.

If $\mu_L(E) \geq \mathsf{W}$, then $\dfrac{r^2\nu + s^2}{s(r+s)} \geq 1$; i.e $\nu \geq \dfrac{s}{r}$.

Hence, $T_X(-\log D)$ is semi-stable with respect to $\pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$ if and only if $\nu = \dfrac{s}{r}$. □

Corollary 5.20. *With the decomposition given in theorem 3.9, $T_X(-\log(D_{v_i} + D_{w_j}))$ is poly-stable with respect to $\pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$ if and only if $\nu = \dfrac{s}{r}$.*

According to (11) and (12), when $a_1 = \ldots = a_r = 0$, we have :

$$D_{v_i} \sim_{\mathrm{lin}} D_{v_0}\ ,\ D_{w_j} \sim_{\mathrm{lin}} D_{w_0} \quad \text{and} \quad -K_X \sim_{\mathrm{lin}} (s+1)D_{w_0} + (r+1)D_{v_0}\ .$$

By the above study when $\mathrm{sStab}(T_X(-\log D)) \neq \varnothing$, we see that $T_X(-\log D)$ is semi-stable with respect to L if and only if $L \cong \mathscr{O}_X(-\alpha\,(K_X + D))$ with $\alpha \in \mathbb{N}^*$.

6. Stability of $T_X(-\log D)$ on smooth toric varieties of Picard rank two when $a_r \geq 1$

In this part, we will study the stability of $T_X(-\log D)$ when X is given by (10) with $a_r \geq 1$. If $D = \sum_{\rho\in\Delta} D_\rho$ with $\Delta \subset \Sigma(1)$, by Corollary 4.8, we will only study the case where $\operatorname{card}\Delta \in \{1, 2\}$. The case $\operatorname{card}\Delta = 0$ was treated by Hering-Nill-Süss in [HNS19].

Let $L = \pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$ be an element of $\operatorname{Amp}(X) \subset N^1(X) \otimes_{\mathbb{Z}} \mathbb{R}$. We recall that the numbers $\mathsf{V}_0, \ldots, \mathsf{V}_r$ defined on Section 5.2 are polynomials of ν of degree s and W is polynomial of degree $s-1$. If $\mathscr{E} = T_X(-\log D)$, the number $\mu_L(\mathscr{E})$ is polynomial of degree at most s. Let P_0, P_1, P_2 and Q be the polynomials of ν defined by

$$\mathsf{P}_0 = \mu_L(\mathscr{E}) - \mathsf{V}_0\,, \quad \mathsf{P}_1 = \mu_L(\mathscr{E}) - \mathsf{V}_1\,, \quad \mathsf{P}_2 = \mu_L(\mathscr{E}) - \mathsf{V}_2 \quad \text{and} \quad \mathsf{Q} = \mu_L(\mathscr{E}) - \mathsf{W}\,.$$

We first recall the sign changes rule of Descartes. We refer to [Mig89, Chapter 5, Section 4.3].

Theorem 6.1 (Descartes)**.** *Let $\mathsf{P} = c_n X^n + c_{n-1}X^{n-1} + \ldots + c_0$ be a polynomial with real coefficients where $c_n\, c_0 \neq 0$. Let p the number of sign changes in the sequence $(c_0, \ldots, c_n)$ of its coefficients and q the numbers of positive real roots, counted with their order of multiplicity. Then, there is $m \in \mathbb{N}$ such that $q = p - 2m$.*

Under certain conditions on a_i, r and s, these polynomials (P_0, $\mathsf{P}_1, \mathsf{P}_2$ and Q) have respectively one or no positive root. If the positive root exists, we denote by

- ν_i the unique positive root of P_i where $i \in \{0, 1, 2\}$
- ν_3 the unique positive root of Q.

In these tables, we give the values of ν for which $\mathscr{E} = T_X(-\log D)$ is stable or semi-stable with respect to $L = \pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$.

Table 2. Stability of $T_X(-\log D)$ when $a_r \geq 1$

Divisor D	Condition on r and a_i	Condition on s	$\operatorname{Stab}(\mathscr{E})$	$\operatorname{sStab}(\mathscr{E})$
D_{w_j} $0 \leq j \leq s$ Theorem 6.2	$r \geq 1$ and $a_r \geq 1$	$s \geq 1$	$\varnothing$	$\varnothing$
D_{v_i} $1 \leq i \leq r-1$ Prop. 6.4	$r \geq 2$ and $a_r \geq 1$	$s \geq 1$	$\varnothing$	$\varnothing$
D_{v_r} Theorem 6.6	$r \geq 1$, $a_r = 1$ and $a_{r-1} = 0$	$s \geq 1$	$0 < \nu < \nu_0$	$0 < \nu \leq \nu_0$
	$r \geq 1$ and ($a_r \geq 2$ or $a_{r-1} \neq 0$)	$s \geq 1$	$\varnothing$	$\varnothing$
D_{v_0} Theorem 6.9 Lemma 6.8 Theorem 6.11	$r = 1$	$s \geq 1$	$0 < \nu < \nu_1$	$0 < \nu \leq \nu_1$
	$r \geq 2$ and $a_1 < a_r$	$s \geq 1$	$\varnothing$	$\varnothing$
	$r \geq 2$ and $a_1 = a_r = a$	$a \geq \dfrac{s+1}{r-1}$	$\varnothing$	$\varnothing$
		$\dfrac{s}{r} \leq a < \dfrac{s+1}{r-1}$	$0 < \nu < \nu_1$	$0 < \nu \leq \nu_1$
		$a\,r < s$	$\nu_3 < \nu < \nu_1$	$\nu_3 \leq \nu \leq \nu_1$

TABLE 3. Stability of $\mathscr{T}_X(-\log D)$ when $a_r \geq 1$

Divisor D	Condition on r and a_i	Condition on s	Stab($\mathscr{E}$)	sStab($\mathscr{E}$)
$D_{w_i} + D_{w_j}$ $0 \leq i < j \leq s$ Theorem 6.2	$r \geq 1$ and $a_r \geq 1$	$s \geq 1$	$\varnothing$	$\varnothing$
$D_{v_i} + D_{v_j}$ $1 \leq i < j \leq r$ Corollary 6.5	$r \geq 2$ and $a_r \geq 1$	$s \geq 1$	$\varnothing$	$\varnothing$
$D_{v_i} + D_{w_j}$, $j \geq 0$ and $1 \leq i \leq r-1$ Proposition 6.4	$r \geq 2$ and $a_r \geq 1$	$s \geq 1$	$\varnothing$	$\varnothing$
$D_{v_r} + D_{w_j}$, $j \geq 0$ Corollary 6.7	$r \geq 1$ and $a_r \geq 1$	$s \geq 1$	$\varnothing$	$\varnothing$
$D_{v_0}+D_{w_j}$, $0 \leq j \leq s$ Theorem 6.9	$r = 1$	$s \geq 1$	$\varnothing$	$\nu = \nu_3$
	$r \geq 2$ and $a_1 < a_r$	$s \geq 1$	$\varnothing$	$\varnothing$
Lemma 6.8	$r \geq 2$ and	$s \leq a(r-1)$	$\varnothing$	$\varnothing$
Theorem 6.13	$a_1 = a_r = a$	$s > a(r-1)$	$\varnothing$	$\nu = \nu_3$
$D_{v_0}+D_{v_i}$, $2 \leq i \leq r$ Lemma 6.8	$r \geq 2$ and $a_1 < a_r$	$s \geq 1$	$\varnothing$	$\varnothing$
	$r \geq 2$ and	$s \leq a(r-1)$	$\varnothing$	$\varnothing$
Theorem 6.13	$a_1 = a_r = a$	$s > a(r-1)$	$\varnothing$	$\nu = \nu_3$

TABLE 4. Stability of $T_X(-\log D)$ when $a_r \geq 1$

Divisor D	Condition on r and a_i	Condition on s	Stab($\mathscr{E}$)	sStab($\mathscr{E}$)
$D_{v_0} + D_{v_1}$	$r = 1$	$s \geq 1$	$\varnothing$	$\nu > 0$
	$r \geq 2$ and $0 = a_1 < a_r$	$s \geq 1$	$\varnothing$	$\varnothing$
	$r \geq 2$ and	$s \leq a(r-1)$	$\varnothing$	$\varnothing$
Theorem 6.9	$a_1 = a_r = a$	$s > a(r-1)$	$\varnothing$	$\nu = \nu_3$
Prop. 6.14	$r = 2$ and	$s \leq \delta_2$	$\varnothing$	$\varnothing$
Theorem 6.13	$0 < a_1 < a_2$	$s > \delta_2$	$\varnothing$	$\nu = \nu_3$
Prop. 6.17	$r \geq 3$ and $a_2 < a_r$	$s \geq 1$	$\varnothing$	$\varnothing$
Lemma 6.15	$r \geq 3$ and	$s \leq \delta_r$	$\varnothing$	$\varnothing$
Prop. 6.19	$0 < a_1 < a_2 = \ldots = a_r$	$s > \delta_r$	$\varnothing$	$\nu = \nu_3$

6.1. **The case $D = D_{w_j}$ and $D = D_{w_j} + D_{w_i}$.** Let $i, j \in \{0, \ldots, s\}$ distinct and $\mathscr{E} = T_X(-\log D_{w_i})$, $\mathscr{F} = T_X(-\log(D_{w_i} + D_{w_j}))$.
If $a_r \geq 1$, the Lemma 5.14 is again true. The vector bundles $\mathscr{E}$ and $\mathscr{F}$ are decomposable and we have

$$\mathscr{E} = \left(\bigoplus_{k=0,\, k\neq i}^{s} \mathscr{E}_{G_k} \right) \oplus \mathscr{E}_F \quad \text{and} \quad \mathscr{F} = \left(\bigoplus_{k=0,\, k\neq i}^{s} \mathscr{F}_{G_k} \right) \oplus \mathscr{F}_F$$

where $G_k = \mathrm{Span}(w_k)$ and $F = \mathrm{Span}(v_0, \ldots, v_r)$.

Theorem 6.2. *Let* $(a_1, \ldots, a_r) \neq (0, \ldots, 0)$. *For any* $L \in \mathrm{Amp}(X)$, *the vector bundles* $\mathscr{E}$ *and* $\mathscr{F}$ *are not semi-stable with respect to* L.

Proof. Let $L = \pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$, we have

$$\mu_L(\mathscr{E}) = \frac{1}{r+s}\,(\,s\mathsf{W} + (\mathsf{V}_0 + \ldots + \mathsf{V}_r)\,)$$

and

$$\mu_L(\mathscr{F}) = \frac{1}{r+s}\,(\,(s-1)\mathsf{W} + (\mathsf{V}_0 + \ldots + \mathsf{V}_r)\,)\ ;$$

thus, $\mu_L(\mathscr{F}) < \mu_L(\mathscr{E})$. The points 4 and 5 of Proposition 5.10 are not verifying. To check that $\mathscr{E}$ (resp. $\mathscr{F}$) is not semi-stable with respect to L it is enough to compare $\mu_L(\mathscr{E})$ (resp. $\mu_L(\mathscr{F})$) with

$$\max\left(\mathsf{V}_0,\ \frac{1}{r}(\mathsf{V}_0 + \mathsf{V}_1 + \ldots + \mathsf{V}_r)\right)\ .$$

By Corollary 5.8, we have $\mathsf{V}_0 = a_r\,\mathsf{W} + \mathsf{V}_r$. Like $a_r \geq 1$, we have $\mathsf{V}_0 \geq \mathsf{W} + \mathsf{V}_r$, i.e $\mathsf{V}_0 - \mathsf{W} \geq \mathsf{V}_r$. Thus,

$$\begin{aligned}(r+s)\,(\mathsf{V}_0 - \mu_L(\mathscr{E})) &= s(\mathsf{V}_0 - \mathsf{W}) + (r\,\mathsf{V}_0 - (\mathsf{V}_0 + \ldots + \mathsf{V}_{r-1})) - \mathsf{V}_r \\ &\geq s(\mathsf{V}_0 - \mathsf{W}) - \mathsf{V}_r \quad \text{because } \mathsf{V}_i \leq \mathsf{V}_0 \\ &\geq (s-1)\mathsf{V}_r\ .\end{aligned}$$

If $s \geq 2$, we have $\mathsf{V}_0 - \mu_L(\mathscr{E}) > 0$ and $\mathsf{V}_0 - \mu_L(\mathscr{F}) > 0$. Thus, $\mathscr{E}$ and $\mathscr{F}$ are not semi-stables with respect to L.
We now assume that $s = 1$. By using the formula defining V_i and W given in Section 5.2, we have

$$\mathsf{W} = 1\ ,\ \mathsf{V}_0 = (a_1 + \ldots + a_r) + r\nu \quad \text{and} \quad \mathsf{V}_i = \mathsf{V}_0 - a_i \quad \text{for } i \in \{1, \ldots, r\}\ .$$

We have

$$\mu_L(\mathscr{E}) = \frac{1}{r+1}\,(1 + (r+1)\mathsf{V}_0 - (a_1 + \ldots + a_r))$$

and

$$\frac{1}{r}(\mathsf{V}_0 + \ldots + \mathsf{V}_r) = \frac{1}{r}\,((r+1)\mathsf{V}_0 - (a_1 + \ldots + a_r))\ .$$

Therefore, $\dfrac{\mathsf{V}_0 + \ldots + \mathsf{V}_r}{r} - \mu_L(\mathscr{E}) = \dfrac{(r+1)\mathsf{V}_0 - (a_1 + \ldots + a_r) - r}{r(r+1)}$. We have

$$\begin{aligned}(r+1)\mathsf{V}_0 - (a_1 + \ldots + a_r) - r =& (r+1)(a_1 + \ldots + a_r) + (r+1)r\nu \\ &- (a_1 + \ldots + a_r) - r \\ =& r(a_1 + \ldots + a_r - 1) + (r+1)r\nu > 0\end{aligned}$$

because $(a_1 + \ldots + a_r - 1) \geq 0$ and $\nu > 0$. Thus, $\mathscr{E}$ and $\mathscr{F}$ are not semi-stables with respect to L. □

6.2. **The case where** $\mathrm{Cone}(v_0) \notin \Delta$**.** If $i \in \{1, \ldots, r\}$ and $j \in \{0, \ldots, s\}$, by Theorem 3.9, the logarithmic tangent bundle $T_X(-\log(D_{v_i} + D_{w_j}))$ is decomposable. Let $r \geq 2$ and $D \in \{D_{v_i} + D_{v_j} : 1 \leq i < j \leq r\}$; we set $\mathscr{E} = T_X(-\log D)$, we have the following proposition :

Lemma 6.3. *The logarithmic tangent bundle* $\mathscr{E}$ *is decomposable. If* $a_i < a_j$, *then*

$$\mathscr{E} = \left(\bigoplus_{l=0}^{s} \mathscr{E}_{G_l}\right) \oplus \left(\bigoplus_{k=0,\, k\notin\{i,\,j\}}^{r} \mathscr{E}_{F_k}\right)$$

where $G_l = \mathrm{Span}(w_l)$ *and* $F_k = \mathrm{Span}(v_k)$. *If* $a_i = a_j$, *then*

$$\mathscr{E} = \mathscr{E}_G \oplus \mathscr{E}_F$$

where $G = \mathrm{Span}(w_l,\, v_k \,:\, l \in \{0,\dots,s\},\, k \in \{0,\dots,r\}\setminus\{i,j\})$ *and* $F = \mathrm{Span}(v_i)$ *or* $F = \mathrm{Span}(v_j)$.

Proof. For the proof we use the families of multifiltrations. We use the Lemma 5.9 to explain the choice of this decomposition. □

If $D \in \{D_{v_i} \,:\, 1 \le i \le r\}$, we will not search to know if $\mathscr{E} = T_X(-\log D)$ is decomposable because it depends on the numbers r, s and $a_1,\dots,a_r$. In particular, if we assume that $r = 2$, $s = 1$, $a_1 = 0$ and $a_2 = 1$, then $\mathscr{E} = T_X(-\log D_{v_1})$ is decomposable and $\mathscr{E} = \mathscr{E}_F \oplus \mathscr{E}_G$ where $F = \mathrm{Span}(v_2,\, w_1)$ and $G = \mathrm{Span}(v_0)$. But $\mathscr{F} = T_X(-\log D_{v_2})$ is not decomposable.

We first study the stability of $T_X(-\log D)$ when $r \ge 2$ and

$$\begin{aligned} D \in &\{D_{v_i} \,:\, 1 \le i \le r-1\} \cup \{D_{v_i} + D_{w_j} \,:\, 1 \le i \le r-1 \text{ and } 0 \le j \le s\} \\ &\cup \{D_{v_i} + D_{v_j} \,:\, 1 \le i < j \le r\}\ . \end{aligned}$$

Proposition 6.4. *Let* $r \ge 2$, $(a_1,\dots,a_r) \ne (0,\dots,0)$, $i \in \{1,\dots,r-1\}$ *and* $j \in \{0,\dots,s\}$. *For any* $L \in \mathrm{Amp}(X)$, *the logarithmic tangent bundles* $T_X(-\log D_{v_i})$ *and* $T_X(-\log(D_{v_i} + D_{w_j}))$ *are not semi-stables with respect to* L.

Proof. We set $\mathscr{E} = T_X(-\log D_{v_i})$ and $\mathscr{F} = T_X(-\log(D_{v_i} + D_{w_j}))$.
Let $L \in \mathrm{Amp}(X)$, we have

$$\mu_L(\mathscr{E}) = \frac{1}{r+s}\left((s+1)\mathsf{W} + (\mathsf{V}_0 + \ldots + \mathsf{V}_{i-1} + \mathsf{V}_{i+1} + \ldots + \mathsf{V}_r)\right)$$

and $\mu_L(\mathscr{F}) < \mu_L(\mathscr{E})$. We will show that $\mathsf{V}_0 > \mu_L(\mathscr{E})$. We have

$$\begin{aligned} (r+s)(\mathsf{V}_0 - \mu_L(\mathscr{E})) =& (s+1)(\mathsf{V}_0 - \mathsf{W}) - \mathsf{V}_r \\ &+ \left((r-1)\mathsf{V}_0 - (\mathsf{V}_0 + \ldots + \mathsf{V}_{i-1} + \mathsf{V}_{i+1} + \ldots + \mathsf{V}_{r-1})\right) \\ \ge& (s+1)(\mathsf{V}_0 - \mathsf{W}) - \mathsf{V}_r \\ \ge& (s+1)\mathsf{V}_r - \mathsf{V}_r = s\,\mathsf{V}_r \end{aligned}$$

By Proposition 5.10, $\mathscr{E}$ and $\mathscr{F}$ are not semi-stables with respect to L. □

Corollary 6.5. *Let* $r \ge 2$ *and* $(a_1,\dots,a_r) \ne (0,\dots,0)$. *Let* $L \in \mathrm{Amp}(X)$; *for all* $i,\, j \in \{1,\dots,r\}$ *distinct, the logarithmic tangent bundle* $T_X(-\log(D_{v_i} + D_{v_j}))$ *is not semi-stable with respect to* L.

Proof. If we set $\mathscr{G} = T_X(-\log(D_{v_i} + D_{v_j}))$, by using the proof of Proposition 6.4, we have $\mu_L(\mathscr{G}) < \mu_L(\mathscr{E}) < \mathsf{V}_0$. Thus, $\mathscr{G}$ is not semi-stable with respect to L. □

We now study the stability of $T_X(-\log D)$ when

$$D \in \{D_{v_r}\} \cup \{D_{v_r} + D_{w_j} \,:\, 0 \le j \le s\}\ .$$

Theorem 6.6. *Let* $r \ge 1$ *and* $a_r \ge 1$. *We have* $\mathrm{Stab}(T_X(-\log D_{v_r})) \ne \varnothing$ *if and only if* $\mathrm{sStab}(T_X(-\log D_{v_r})) \ne \varnothing$ *if and only if* $a_r = 1$ *and* $a_{r-1} = 0$.
If $a_r = 1$ *and* $a_{r-1} = 0$, *then the logarithmic tangent bundle* $T_X(-\log D_{v_r})$ *is stable (resp. semi-stable) with respect to* $\pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$ *if and only if* $0 < \nu < \nu_0$ *(resp.* $0 < \nu \le \nu_0$*) where* ν_0 *is the unique positive root of*

$$\mathsf{P}_0(x) = \sum_{k=0}^{s-1}\binom{s+r-1}{k}x^k - s\binom{s+r-1}{s}x^s\ .$$

Proof. Let $\mathscr{E} = T_X(-\log D_{v_r})$ and $L = \pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$. We have

$$(r+s)\mu_L(\mathscr{E}) = (s+1)\mathsf{W} + \mathsf{V}_0 + \mathsf{V}_1 + \ldots + \mathsf{V}_{r-1}\ .$$

• If $a_r \geq 2$, by using the first point of Lemma 5.12 and the fact that $\mathsf{V}_i \leq \mathsf{V}_0$, we get :

$$(r+s)[\mathsf{V}_0 - \mu_L(\mathscr{E})] = (s\mathsf{V}_0 - (s+1)\mathsf{W}) + r\mathsf{V}_0 - (\mathsf{V}_0 + \ldots + \mathsf{V}_{r-1}) \geq s\mathsf{V}_r \ .$$

By Proposition 5.10, $T_X(-\log D_{v_r})$ is not semi-stable with respect to L.

• We assume that $r \geq 2$ and $a_{r-1} = a_r = 1$. Like $\mathsf{V}_{r-1} = \mathsf{V}_r$, we have

$$\begin{aligned}(r+s)[\mathsf{V}_0 - \mu_L(\mathscr{E})] &= (s+1)[\mathsf{V}_0 - \mathsf{W}] - \mathsf{V}_{r-1} + [(r-1)\mathsf{V}_0 - (\mathsf{V}_0 + \ldots + \mathsf{V}_{r-2})] \\ &\geq (s+1)\mathsf{V}_r - \mathsf{V}_{r-1} \quad \text{because } \mathsf{V}_0 - \mathsf{W} \geq \mathsf{V}_r \\ &\geq s\mathsf{V}_r\end{aligned}$$

By Proposition 5.10, $T_X(-\log D_{v_r})$ is not semi-stable with respect to L.

• Let $r \geq 1$. We now assume that $a_{r-1} = 0$ and $a_r = 1$. By using the expressions of Section 5.2, we have $\mathsf{V}_0 = \ldots = \mathsf{V}_{r-1} = \mathsf{V}$ where

$$\mathsf{V} = \sum_{k=0}^{s} \binom{s+r-1}{k} \nu^k \quad \text{and} \quad \mathsf{W} = \sum_{k=0}^{s-1} \binom{s+r-1}{k} \nu^k \ .$$

The points 4 and 5 of Proposition 5.10 are not verified in this case. To check the stability of $\mathscr{E}$ it is enough to compare

$$\mu_L(\mathscr{E}) = \frac{1}{r+s}(r\mathsf{V} + (s+1)\mathsf{W})$$

with $\max(V, W)$. We have

$$(r+s)[\mu_L(\mathscr{E}) - \mathsf{W}] = r\mathsf{V} - (r-1)\mathsf{W} > 0 \quad \text{because } \mathsf{W} < \mathsf{V}$$

and

$$\begin{aligned}(r+s)[\mu_L(\mathscr{E}) - \mathsf{V}] =& (s+1)\mathsf{W} - s\mathsf{V} \\ =& \sum_{k=0}^{s-1} \binom{s+r-1}{k} \nu^k - s\binom{s+r-1}{s} \nu^s = \mathsf{P}_0(\nu) \ .\end{aligned}$$

By the sign rule of Descartes (see Theorem 6.1), the polynomial P_0 have a unique positive root ν_0. If $\nu > 0$, then $\mathsf{P}_0(\nu) > 0$ (resp. $\mathsf{P}_0(\nu) \geq 0$) if and only if $\nu < \nu_0$ (resp. $\nu \leq \nu_0$). Thus, $T_X(-\log D_{v_r})$ is stable (resp. semi-stable) with respect to $\pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$ if and only if $0 < \nu < \nu_0$ (resp. $0 < \nu \leq \nu_0$). □

Corollary 6.7. *We assume that $r \geq 1$ and $(a_1, \ldots, a_r) \neq (0, \ldots, 0)$. Let $j \in \{0, \ldots, s\}$ and $D = D_{v_r} + D_{w_j}$. For any $L \in \mathrm{Amp}(X)$, the logarithmic tangent bundle $T_X(-\log D)$ is not semi-stable with respect to L.*

Proof. If $\mathscr{E} = T_X(-\log D)$, we have $\mathsf{V}_0 > \mu_L(\mathscr{E})$. By Proposition 5.10, $T_X(-\log D)$ is not semi-stable with respect to L. □

6.3. **The case where** $\mathrm{Cone}(v_0) \in \Delta$**.** In this part we study the stability of the logarithmic tangent bundle $T_X(-\log D)$ when $r \geq 2$ and

$$D \in \{D_{v_0}\} \cup \{D_{v_0} + D_{w_j} \,:\, 0 \leq j \leq s\} \cup \{D_{v_0} + D_{v_i} \,:\, 2 \leq i \leq r\} \ .$$

The last case $D = D_{v_0} + D_{v_1}$ will be study in the Section 6.4. We recall that $a_0 = 0$. By Lemma 6.3 and Theorem 3.9, for all

$$D \in \{D_{v_0} + D_{w_j} \,:\, 0 \leq j \leq s\} \cup \{D_{v_0} + D_{v_i} \,:\, 2 \leq i \leq r\}$$

$T_X(-\log D)$ is decomposable. If $D = D_{v_0}$, $r \geq 2$ and $a_1 = 0$, then $\mathscr{E} = T_X(-\log D)$ is decomposable and $\mathscr{E} = \mathscr{E}_G \oplus \mathscr{E}_F$ where $G = \mathrm{Span}(w_0, w_1, \ldots, w_s, v_2, \ldots, v_r)$ and $F = \mathrm{Span}(v_1)$. By Lemma 5.9, if $a_1 > 0$, then $T_X(-\log D_{v_0})$ cannot be written as a sum of two equivariant vector bundles.

We now study the semi-stability of these bundles.

Lemma 6.8. *Let $r \geq 2$, $(a_1, \ldots, a_r) \neq (0, \ldots, 0)$ such that $a_1 < a_r$, $i \in \{2, \ldots, r\}$ and $j \in \{0, \ldots, s\}$. We set $\mathscr{E} = T_X(-\log D_{v_0})$, $\mathscr{F} = T_X(-\log(D_{v_0} + D_{v_i}))$ and $\mathscr{G} = T_X(-\log(D_{v_0} + D_{w_j}))$. For any $L \in \mathrm{Amp}(X)$, the vector bundles $\mathscr{E}$, $\mathscr{F}$ and $\mathscr{G}$ are not semi-stables with respect to L.*

Proof. We have $\mu_L(\mathscr{E}) > \mu_L(\mathscr{F})$ and $\mu_L(\mathscr{E}) > \mu_L(\mathscr{G})$. We will show that $V_1 > \mu_L(\mathscr{E})$. By Lemma 5.12, we have $V_1 - W \geq V_r$. Therefore

$$\begin{aligned}(r+s)(V_1 - \mu_L(\mathscr{E})) &= (r+s)V_1 - (V_1 + \ldots + V_r) - (s+1)W \\ &= (s+1)[V_1 - W] - V_r + [(r-1)V_1 - (V_1 + \ldots + V_{r-1})] \\ &\geq (s+1)[V_1 - W] - V_r \\ &\geq sV_r\end{aligned}$$

By Proposition 5.10, $\mathscr{E}$, $\mathscr{F}$ and $\mathscr{G}$ are not semi-stables with respect to L. □

Let $a \in \mathbb{N}^*$. We now study what happen in Lemma 6.8 when $a_1 = \ldots = a_r = a$.

6.3.1. *We assume that $r = 1$.*

Theorem 6.9. *We assume that $X = \mathbb{P}(\mathscr{O}_{\mathbb{P}^s} \oplus \mathscr{O}_{\mathbb{P}^s}(a))$. Let P_1 and Q the polynomials defined by*

$$\mathsf{P}_1(x) = (s+1)\sum_{k=0}^{s-1}\binom{s}{k}a^{s-k-1}x^k - s\,x^s \quad \text{and} \quad \mathsf{Q}(x) = x^s - \sum_{k=0}^{s-1}\binom{s}{k}a^{s-k-1}x^k\ .$$

We have :

1. *$T_X(-\log D_{v_0})$ is stable (resp. semi-stable) with respect to $\pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$ if and only if $0 < \nu < \nu_1$ (resp. $0 < \nu \leq \nu_1$) where ν_1 is the unique positive root of P_1.*
2. *If $j \in \{0, \ldots, s\}$, then $\mathrm{Stab}(T_X(-\log(D_{v_0} + D_{w_j}))) = \varnothing$; $T_X(-\log(D_{v_0} + D_{w_j}))$ is semi-stable with respect to $\pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$ if and only if $\nu = \nu_3$ where ν_3 is the unique positive root of Q.*
3. *$\varnothing = \mathrm{Stab}(T_X(-\log(D_{v_0} + D_{v_1}))) \subsetneq \mathrm{sStab}(T_X(-\log(D_{v_0} + D_{v_1}))) = \mathrm{Amp}(X)$.*

Proof. By the sign rule of Descartes (cf. Theorem 6.1), P_1 and Q have respectively one positive root. Let $L = \pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$, by using the expressions of Section 5.2, we have :

$$\mathsf{V}_1 = \nu^s \qquad \text{and} \qquad \mathsf{W} = \sum_{k=0}^{s-1}\binom{s}{k}a^{s-k-1}\nu^k\ .$$

Let $\mathscr{E} = T_X(-\log D_{v_0})$. By Proposition 5.10, to check the stability of $\mathscr{E}$, it is enough to compare

$$\mu_L(\mathscr{E}) = \frac{\mathsf{V}_1 + (1+s)\mathsf{W}}{1+s}$$

with $\max(\mathsf{V}_1, \mathsf{W})$. We don't use the points 4 and 5 because the hypothesis are not verifying. We have

$$\mu_L(\mathscr{E}) > \mathsf{W} \qquad \text{and} \qquad (1+s)(\mu_L(\mathscr{E}) - \mathsf{V}_1) = \mathsf{P}_1(\nu)\ .$$

Thus, $\mathscr{E}$ is stable (resp. semi-stable) with respect to $\pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$ if and only if $0 < \nu < \nu_1$ (resp. $0 < \nu \leq \nu_1$).

Let $\mathscr{F} = T_X(-\log(D_{v_0} + D_{w_j}))$. By Proposition 5.10, we compare

$$\mu_L(\mathscr{F}) = \frac{\mathsf{V}_1 + s\mathsf{W}}{1+s}$$

with $\max(\mathsf{V}_1, \mathsf{W})$. We have

$$(1+s)(\mu_L(\mathscr{F}) - \mathsf{W}) = \mathsf{Q}(\nu) \quad \text{and} \quad (1+s)(\mu_L(\mathscr{F}) - \mathsf{V}_1) = -s\,\mathsf{Q}(\nu)\ .$$

Thus, for any $\nu > 0$, $\mathscr{F}$ is not stable with respect to L. The vector bundle $\mathscr{F}$ is semi-stable with respect to L if and only if $\nu = \nu_3$.

Let $\mathscr{G} = T_X(-\log(D_{v_0} + D_{v_1}))$. We have $\mu_L(\mathscr{G}) = \mathsf{W}$. By Proposition 5.10, $\mathscr{G}$ is semi-stable not stable with respect to L. □

6.3.2. *We assume that* $r \geq 2$. We assume that $a_1 = \ldots = a_r = a$ with $a \in \mathbb{N}^*$.

Lemma 6.10. *We have*

$$\mathrm{card}\{(\alpha_1, \ldots, \alpha_p) \in \mathbb{N}^p : \alpha_1 + \ldots + \alpha_p = m\} = \binom{m+p-1}{m}\ .$$

By Lemma 6.10, for all $k \in \{0, \ldots, s-1\}$, we have

$$\mathsf{W}_k = \sum_{d_1+\ldots+d_r = s-k-1} a_1^{d_1} \cdots a_r^{d_r} = \binom{s-k+r-2}{s-k-1} a^{s-k-1}$$

and

$$\mathsf{V}_{1k} = \sum_{d_2+\ldots+d_r = s-k} a_2^{d_2} \cdots a_r^{d_r} = \binom{s-k+r-2}{s-k} a^{s-k}\ .$$

We recall that $\mathsf{V}_{1s} = 1$. By using the equality $\binom{n}{p-1} = \dfrac{p}{n-p+1}\binom{n}{p}$, for all $k \in \{0, \ldots, s-1\}$,

$$\mathsf{W}_k = \binom{s-k+r-2}{s-k-1} a^{s-k-1} = \frac{s-k}{r-1}\binom{s-k+r-2}{s-k} a^{s-k-1} = \frac{s-k}{a(r-1)}\mathsf{V}_{1k}\ .$$

Let P_1 and Q the polynomials defined by :

$$\mathsf{P}_1(x) = \sum_{k=0}^{s-1} \left[\left(-s + \frac{(s-k)(s+1)}{a(r-1)}\right)\binom{s+r-1}{k}\mathsf{V}_{1k}\right] x^k - s\binom{s+r-1}{s} x^s$$

$$\mathsf{Q}(x) = \sum_{k=0}^{s-1} \left[\left(r - \frac{s-k}{a}\right)\binom{s+r-1}{k}\mathsf{V}_{1k}\right] x^k + r\binom{s+r-1}{s} x^s$$

Theorem 6.11. *Let* $r \geq 2$ *and* $X = \mathbb{P}(\mathscr{O}_{\mathbb{P}^s} \oplus \mathscr{O}_{\mathbb{P}^s}(a_1) \oplus \ldots \oplus \mathscr{O}_{\mathbb{P}^s}(a_r))$ *with* $a_1 = \ldots = a_r = a$ *where* $a \in \mathbb{N}^*$. *We set* $\mathscr{E} = T_X(-\log D_{v_0})$.

1. *If* $a < \dfrac{s}{r}$, *then* $\mathscr{E}$ *is stable (resp. semi-stable) with respect to* $\pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$ *if and only if* $\nu_3 < \nu < \nu_1$ *(resp.* $\nu_3 \leq \nu \leq \nu_1$*) where* ν_1 *and* ν_3 *are respectively the positive roots of* P_1 *and* Q.
2. *If* $\dfrac{s}{r} \leq a < \dfrac{s+1}{r-1}$, *then* $\mathscr{E}$ *is stable (resp. semi-stable) with respect to* $\pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$ *if and only if* $0 < \nu < \nu_1$ *(resp.* $0 < \nu \leq \nu_1$*) where* ν_1 *is the unique positive root of* P_1.
3. *If* $a \geq \dfrac{s+1}{r-1}$, *then for any* $L \in \mathrm{Amp}(X)$, $\mathscr{E}$ *is not semi-stable with respect to* L.

Before giving the proof of this theorem, we will explain the condition which ensure the existence of positive roots on P_1 and Q. We will use the Descartes sign rule.

We write P_1 as $\mathsf{P}_1(x) = \sum_{k=0}^{s} \alpha_k\, x^k$. We have $\alpha_s < 0$. If $-s + \dfrac{(s-k)(s+1)}{a(r-1)} > 0$,

then $s-k > \dfrac{a\,s(r-1)}{s+1}$, i.e $k < \left(1-\dfrac{a(r-1)}{s+1}\right)s$.
If $\dfrac{a(r-1)}{s+1} \geq 1$, then for any $x \geq 0$, $\mathsf{P}_1(x) < 0$.
If $a < \dfrac{s+1}{r-1}$, then for all $k < \left(1-\dfrac{a(r-1)}{s+1}\right)s$, we have $\alpha_k > 0$ and for all $k \geq \left(1-\dfrac{a(r-1)}{s+1}\right)s$, $\alpha_k \leq 0$. Hence P_1 has only one positive root ν_1.

We write $\mathsf{Q}(x) = \sum_{k=0}^{s} \beta_k\, x^k$. We have $\beta_s > 0$. If $r - \dfrac{s-k}{a} < 0$, then $r\,a-(s-k) < 0$, i.e $k < s - r\,a$. If $a \geq \dfrac{s}{r}$, then for any $x \geq 0$, $\mathsf{Q}(x) > 0$.
If $a < \dfrac{s}{r}$, then for all $k < s - r\,a$, we have $\beta_k < 0$ and for all $k \geq s - r\,a$, $\beta_k \geq 0$. Thus, Q has only one positive root ν_3 .

Lemma 6.12. *If* $a < \dfrac{s}{r}$, *then* $\nu_3 < \nu_1$.

Proof. If $a < \dfrac{s}{r}$, then $a < \dfrac{s+1}{r-1}$. By the above analysis, ν_1 and ν_3 exist. For $x \geq 0$, we have $\mathsf{P}_1(x) > 0$ if and only if $0 \leq x < \nu_1$. Like

$$\begin{aligned}\frac{\mathsf{P}_1(x)}{-s} - \frac{\mathsf{Q}(x)}{r} &= \sum_{k=0}^{s-1}\left[\left(1 - \frac{(s-k)(s+1)}{a\,s(r-1)} - 1 + \frac{s-k}{r\,a}\right)\binom{s+r-1}{k}\mathsf{V}_{1k}\right]x^k \\ &= \frac{-(r+s)}{a\,s\,r(r-1)}\sum_{k=0}^{s-1}(s-k)\binom{s+r-1}{k}\mathsf{V}_{1k}\,x^k = \mathsf{P}(x)\end{aligned}$$

we have $\dfrac{\mathsf{P}_1(\nu_3)}{-s} - \dfrac{\mathsf{Q}(\nu_3)}{r} = \mathsf{P}(\nu_3) < 0$. Hence, $\mathsf{P}_1(\nu_3) > 0$, i.e $\nu_3 < \nu_1$. □

Now, we can prove Theorem 6.11.

Proof of Theorem 6.11. Like $a_1 = \ldots = a_r$, we have $\mathsf{V}_1 = \ldots = \mathsf{V}_r$. Therefore

$$\mu_L(\mathscr{E}) = \frac{r\,\mathsf{V}_1 + (s+1)\mathsf{W}}{r+s}\ .$$

In this case the point 4 and 5 are not verified. To check the stability of $\mathscr{E} = T_X(-\log D_{v_0})$, it is enough to compare $\mu_L(\mathscr{E})$ with $\max(\mathsf{V}_1, \mathsf{W})$. We have

$$\begin{aligned}(r+s)(\mu_L(\mathscr{E}) - \mathsf{V}_1) &= -s\,\mathsf{V}_1 + (s+1)\mathsf{W} \\ &= -s\sum_{k=0}^{s}\binom{s+r-1}{k}\mathsf{V}_{1k}\,\nu^k + (s+1)\sum_{k=0}^{s-1}\binom{s+r-1}{k}\mathsf{W}_k\,\nu^k \\ &= \mathsf{P}_1(\nu)\end{aligned}$$

and

$$\begin{aligned}(r+s)(\mu_L(\mathscr{E}) - \mathsf{W}) &= r\,\mathsf{V}_1 - (r-1)\mathsf{W} \\ &= r\sum_{k=0}^{s}\binom{s+r-1}{k}\mathsf{V}_{1k}\,\nu^k - (r-1)\sum_{k=0}^{s-1}\binom{s+r-1}{k}\mathsf{W}_k\,\nu^k \\ &= \mathsf{Q}(\nu)\end{aligned}$$

We have :

i. If $a \geq \dfrac{s+1}{r-1}$, then for any $\nu > 0$, we have $\mathsf{P}_1(\nu) < 0$.

ii. If $a < \dfrac{s+1}{r-1}$, then $\mathsf{P}_1(\nu) > 0$ (resp. $\mathsf{P}_1(\nu) \geq 0$) if and only if $0 < \nu < \nu_1$ (resp. $0 < \nu \leq \nu_1$).

iii. If $a \geq \dfrac{s}{r}$, then for any $\nu > 0$, we have $\mathsf{Q}(\nu) > 0$.

iv. If $a < \dfrac{s}{r}$, then $\mathsf{Q}(\nu) > 0$ (resp. $\mathsf{Q}(\nu) \geq 0$) if and only if $\nu > \nu_3$ (resp. $\nu \geq \nu_3$).

The point *i.* shows the third point of the theorem. By using the points *ii.* , *iv.* and the Lemma 6.12, we get the first point of theorem. The points *ii.* and *iii.* give the second point of the theorem. □

Let $i \in \{1, \ldots, r\}$ and $j \in \{0, \ldots, s\}$. We now study the stability of $\mathscr{F}_j = T_X(-\log(D_{v_0} + D_{w_j}))$ and $\mathscr{G}_i = T_X(-\log(D_{v_0} + D_{v_i}))$. Let Q the polynomial defined by

$$\mathsf{Q}(x) = \sum_{k=0}^{s-1} \left[\left(1 - \frac{s-k}{a(r-1)}\right) \binom{s+r-1}{k} \mathsf{V}_{1k} \right] x^k + \binom{s+r-1}{s} x^s \ .$$

Theorem 6.13. *We assume that $r \geq 2$ and $X = \mathbb{P}\left(\mathscr{O}_{\mathbb{P}^s} \oplus \mathscr{O}_{\mathbb{P}^s}(a_1) \oplus \ldots \oplus \mathscr{O}_{\mathbb{P}^s}(a_r)\right)$ with $a_1 = \ldots = a_r = a$ where $a \in \mathbb{N}^*$. Let $i \in \{1, \ldots, r\}$ and $j \in \{0, \ldots, s\}$.*

(1) *For any $L \in \mathrm{Amp}(X)$, $\mathscr{F}_j$ and $\mathscr{G}_i$ are not stables with respect to L.*

(2) *If $a \geq \dfrac{s}{r-1}$, then for any $L \in \mathrm{Amp}(X)$, $\mathscr{F}_j$ and $\mathscr{G}_i$ are not semi-stables with respect to L.*

(3) *If $a < \dfrac{s}{r-1}$, then $\mathscr{F}_j$ and $\mathscr{G}_i$ are semi-stables with respect to $\pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$ if and only if $\nu = \nu_3$ where ν_3 is the unique root of Q.*

Proof. We first study the polynomial Q. We write $\mathsf{Q}(x) = \sum_{k=0}^{s} \alpha_k \, x^k$. We have $\alpha_s > 0$. If $1 - \dfrac{s-k}{a(r-1)} < 0$, then $a(r-1) - (s-k) < 0$, i.e $k < s - a(r-1)$.
If $a \geq \dfrac{s}{r-1}$, then for any $x \geq 0$, $\mathsf{Q}(x) > 0$.
If $a < \dfrac{s}{r-1}$, then for all $k < s - a(r-1)$, we have $\alpha_k < 0$ and for $k \geq s - a(r-1)$, $\alpha_k \geq 0$. Thus, Q has a unique positive root ν_3 .
Like $a_1 = \ldots = a_r$, we have $\mathsf{V}_1 = \ldots = \mathsf{V}_r$. Thus,

$$\mu_L(\mathscr{F}_j) = \frac{r\,\mathsf{V}_1 + s\mathsf{W}}{r+s} \qquad \text{and} \qquad \mu_L(\mathscr{G}_i) = \frac{(r-1)\mathsf{V}_1 + (s+1)\mathsf{W}}{r+s} \ .$$

By using Proposition 5.10, to check the stability of $\mathscr{F}_j$ (resp. $\mathscr{G}_i$), it is enough to compare $\mu_L(\mathscr{F}_j)$ (resp. $\mu_L(\mathscr{G}_i)$) with $\max(V_1, W)$. We have

$$\begin{cases} (r+s)(\mu_L(\mathscr{F}_j) - \mathsf{V}_1) = s(\mathsf{W} - \mathsf{V}_1) = -s\,\mathsf{Q}(\nu) \\ (r+s)(\mu_L(\mathscr{F}_j) - \mathsf{W}) = r(\mathsf{V}_1 - \mathsf{W}) = r\,\mathsf{Q}(\nu) \end{cases}$$

and

$$\begin{cases} (r+s)(\mu_L(\mathscr{G}_i) - \mathsf{V}_1) = (s+1)(\mathsf{W} - \mathsf{V}_1) = -(s+1)\,\mathsf{Q}(\nu) \\ (r+s)(\mu_L(\mathscr{G}_i) - \mathsf{W}) = (r-1)(\mathsf{V}_1 - \mathsf{W}) = (r-1)\,\mathsf{Q}(\nu) \end{cases}$$

If $a \geq \dfrac{s}{r-1}$, then for any $\nu > 0$, $Q(\nu) > 0$; thus, $\mu_L(\mathscr{F}_j) < \mathsf{V}_1$ and $\mu_L(\mathscr{G}_i) < \mathsf{V}_1$.
Hence, for any $\nu > 0$, $\mathscr{F}_j$ and $\mathscr{G}_i$ are not semi-stables with respect to L.

We now assume that $a < \dfrac{s}{r-1}$. Let ν_3 the positive root of Q.
If $\nu \neq \nu_3$, by the above equalities, $\mathscr{F}_j$ and $\mathscr{G}_i$ are not semi-stables with respect to

L. If $\nu = \nu_3$, the logarithmic tangent bundles $\mathscr{F}_j$ and $\mathscr{G}_i$ are semi-stables but not stable with respect to $\pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$. □

6.4. **The case $D = D_{v_0} + D_{v_1}$.** In this part, we assume that $D = D_{v_0} + D_{v_1}$ and $\mathscr{E} = T_X(-\log D)$. In the beginning of Section 6.3, we see that $\mathscr{E}$ is decomposable. In this part, we will study the stability of $\mathscr{E}$ when $r \geq 2$ and $a_1 < a_r$. The stability of $\mathscr{E}$ when $r = 1$ was treated in Theorem 6.9. When $r \geq 2$, in Theorem 6.13, we study the stability of $\mathscr{E}$ when $a_1 = \ldots = a_r$ with $a_1 \in \mathbb{N}^*$.

Proposition 6.14. *If $a_1 = 0$, then for any $L \in \mathrm{Amp}(X)$, $\mathscr{E}$ is not semi-stable with respect to L.*

Proof. We have $\mu_L(\mathscr{E}) = \dfrac{1}{r+s}\left((s+1)\mathsf{W} + \mathsf{V}_2 + \ldots + \mathsf{V}_r\right)$. Like $\mathrm{card}\{2, \ldots, r\} = r-1$, by using the point 4 of Proposition 5.10 with $I' = \{2, \ldots, r\}$, we have

$$\frac{1}{r+s-1}\left(\sum_{i\in I'} \mathsf{V}_i + (s+1)\mathsf{W}\right) = \frac{1}{r+s-1}\left(\mathsf{V}_2 + \ldots + \mathsf{V}_r + (s+1)\mathsf{W}\right) .$$

Thus, $T_X(-\log(D_{v_0} + D_{v_1}))$ is not semi-stable with respect to L. □

We also have :

Lemma 6.15. *Let $r \geq 3$, $(a_1, \ldots, a_r) \neq (0, \ldots, 0)$ such that $a_2 < a_r$. For any $L \in \mathrm{Amp}(X)$, the vector bundle $\mathscr{E}$ is not semi-stable with respect to L.*

Proof. We have $\mu_L(\mathscr{E}) = \dfrac{1}{r+s}(\mathsf{V}_2 + \ldots + \mathsf{V}_r + (s+1)\mathsf{W})$. We will show that $\mathsf{V}_2 > \mu_L(\mathscr{E})$. By Lemma 5.12, we have $\mathsf{V}_2 - \mathsf{W} \geq \mathsf{V}_r$, thus

$$\begin{aligned}(r+s)(\mathsf{V}_2 - \mu_L(\mathscr{E}) &= (r+s)\mathsf{V}_2 - (\mathsf{V}_2 + \ldots + \mathsf{V}_r) - (s+1)\mathsf{W} \\ &= (s+1)[\mathsf{V}_2 - \mathsf{W}] + [(r-1)\mathsf{V}_2 - (\mathsf{V}_2 + \ldots + \mathsf{V}_r)] \\ &\geq (s+1)[\mathsf{V}_2 - \mathsf{W}] \\ &\geq (s+1)\mathsf{V}_r\end{aligned}$$

Hence, by Proposition 5.10, $\mathscr{E}$ is not semi-stable with respect to L. □

We now assume that $0 < a_1 < a_2 = \ldots = a_r$. For the stability of $\mathscr{E}$, we see that: the point 4 and 5 of Proposition 5.10 are not verified here. To check the stability of $\mathscr{E}$, it is enough to compare $\mu_L(\mathscr{E})$ with $\max(\mathsf{V}_2, \mathsf{W})$. We have

$$\mu_L(\mathscr{E}) = \frac{(r-1)\mathsf{V}_2 + (s+1)\mathsf{W}}{r+s}$$

and

$$\text{(14)} \qquad \begin{cases} (r+s)[\mu_L(\mathscr{E}) - \mathsf{V}_2] = (s+1)(\mathsf{W} - \mathsf{V}_2) \\ (r+s)[\mu_L(\mathscr{E}) - \mathsf{W}] = -(r-1)(\mathsf{W} - \mathsf{V}_2) \end{cases}$$

We have :

Proposition 6.16. *Let $r \geq 2$. If $0 < a_1 < a_2 = \ldots = a_r$, then $\mathrm{Stab}(\mathscr{E}) = \varnothing$.*

The vector bundle $\mathscr{E} = T_X(-\log(D_{v_0} + D_{v_1}))$ is semi-stable with respect to $L = \pi^*\mathscr{O}_{\mathbb{P}^s}(\nu_3) \otimes \mathscr{O}_X(1)$ if and only if ν_3 is a positive root of the polynomial $\mathsf{Q}(\nu) = \mathsf{W} - \mathsf{V}_2$ (W and V_2 depend on ν).

6.4.1. **We assume that $r = 2$ and $0 < a_1 < a_2$.**

Proposition 6.17. *Let $r = 2$ and $0 < a_1 < a_2$. We define $\delta = \dfrac{\ln(1 + a_2 - a_1)}{\ln(a_2) - \ln(a_1)}$ and Q the polynomial*

$$\mathsf{Q}(x) = \sum_{k=0}^{s-1} \left[\frac{a_1^{s-k}}{a_2 - a_1} \left(\left(\frac{a_2}{a_1} \right)^{s-k} - 1 - a_2 + a_1 \right) \binom{s+1}{k} \right] x^k - (s+1)x^s \ .$$

We have :

(1) *If $s \leq \delta$, then $\mathrm{sStab}(\, T_X(-\log(D_{v_0} + D_{v_1}))\,) = \varnothing$;*
(2) *If $s > \delta$, then $T_X(-\log(D_{v_0} + D_{v_1}))$ is semi-stable with respect to $\pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$ if and only if $\nu = \nu_3$ where ν_3 is the positive root of Q.*

Proof. We have

$$\mathsf{V}_2 = \sum_{k=0}^{s} \binom{s+1}{k} a_1^{s-k}\, \nu^k$$

$$\mathsf{W} = \sum_{k=0}^{s-1} \binom{s+1}{k} \left(\sum_{d_1 + d_2 = s-k-1} a_1^{d_1} a_2^{d_2} \right) \nu^k = \sum_{k=0}^{s-1} \frac{a_2^{s-k} - a_1^{s-k}}{a_2 - a_1} \binom{s+1}{k} \nu^k$$

hence,

$$\mathsf{W} - \mathsf{V}_2 = \sum_{k=0}^{s-1} \left[\frac{a_1^{s-k}}{a_2 - a_1} \left(\left(\frac{a_2}{a_1} \right)^{s-k} - 1 - a_2 + a_1 \right) \binom{s+1}{k} \right] \nu^k - (s+1)\nu^s = \mathsf{Q}(\nu) \ .$$

We write $\mathsf{Q}(x) = \sum_{k=0}^{s} \alpha_k\, x^k$. We have $\alpha_s < 0$. The inequality

$$\left(\frac{a_2}{a_1} \right)^{s-k} - 1 - a_2 + a_1 > 0$$

gives $k < s - \dfrac{\ln(1 + a_2 - a_1)}{\ln(a_2) - \ln(a_1)} = s - \delta$. If $s \leq \delta$, then for any $x \geq 0$, $\mathsf{Q}(x) < 0$.
If $s > \delta$, then by the Descartes rule, Q has a unique positive root ν_3.
Hence, if $s > \delta$, then $\mathscr{E}$ is semi-stable with respect to $\pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$ if and only if $\nu = \nu_3$. □

6.4.2. **We assume that $r \geq 3$ and $a_1 < a_r$.** Let $a, b \in \mathbb{N}^*$ such that $a < b$. We assume that $a_1 = a$ and $a_2 = \ldots = a_r = b$. By Lemma 6.10, for all $k \in \{0, \ldots, s-1\}$, we have :

$$\mathsf{W}_k = \sum_{\substack{d_1 + \ldots + d_r \\ = s-k-1}} a_1^{d_1} \cdots a_r^{d_r} = \sum_{j=0}^{s-k-1} a^{s-k-1-j} \left(\sum_{d_2 + \ldots + d_r = j} b^j \right)$$

$$= \sum_{j=0}^{s-k-1} \binom{j+r-2}{j} b^j\, a^{s-k-1-j}$$

and

$$\mathsf{V}_{2k} = \mathsf{V}_{rk} = \sum_{\substack{d_1 + \ldots + d_{r-1} \\ = s-k}} a_1^{d_1} \cdots a_{r-1}^{d_{r-1}} = \sum_{j=0}^{s-k} a^{s-k-j} \left(\sum_{d_2 + \ldots + d_{r-1} = j} b^j \right)$$

$$= \sum_{j=0}^{s-k} \binom{j+r-3}{j} b^j\, a^{s-k-j} \ .$$

For $k \in \{0, \dots, s-1\}$, we have $(s-k) \in \{1, \dots, s\}$. For $p \in \{1, \dots, s\}$, we set $\alpha_p = \mathsf{W}_{s-p} - \mathsf{V}_{2,\, s-p}$; we have

$$\alpha_p = \sum_{j=0}^{p-1} \binom{j+r-2}{j} b^j\, a^{p-1-j} - \sum_{j=0}^{p} \binom{j+r-3}{j} b^j\, a^{p-j}\ .$$

Let Q_s be the polynomial defined by

$$\mathsf{Q}_s(x) = \sum_{k=0}^{s-1} \binom{s+r-1}{k} \alpha_{s-k}\, x^k - \binom{s+r-1}{s} x^s\ .$$

We have $\mathsf{W} - \mathsf{V}_2 = \mathsf{Q}_s(\nu)$. We now search a condition on s which ensure the existence of positive root on Q. By using the identity $\binom{n}{p-1} = \dfrac{p}{n-p+1}\binom{n}{p}$, we have

$$\begin{aligned}\alpha_p &= \sum_{j=0}^{p-1} \frac{j+1}{r-2} \binom{j+r-2}{j+1} b^j\, a^{p-1-j} - \sum_{j=1}^{p} \binom{j+r-3}{j} b^j\, a^{p-j} - a^p \\ &= \sum_{j=0}^{p-1} \left[\left(\frac{j+1}{r-2} - b \right) \binom{j+r-2}{j+1} b^j\, a^{p-1-j} \right] - a^p\ .\end{aligned}$$

If $1 \le p \le b(r-2)$, then for all $j \in \{0, \dots, p-1\}$, we have

$$\frac{j+1}{r-2} - b \le \frac{p}{r-2} - b = \frac{p - b(r-2)}{r-2} \le 0\ ;$$

thus $\alpha_p < 0$. Hence, if $\alpha_p > 0$, then we must have $p > b(r-2)$.
If there is $p > b(r-2)$ such that $\alpha_p > 0$, then for any $q \ge p$, we have $\alpha_q > 0$; this follows from these equalities.

$$\begin{aligned}\alpha_{p+1} &= \sum_{j=0}^{p-1} \left[\left(\frac{j+1}{r-2} - b \right) \binom{j+r-2}{j+1} b^j\, a^{p-j} \right] - a^{p+1} + \left(\frac{p+1}{r-2} - b \right) \binom{p+r-2}{p+1} b^p \\ &= a\, \alpha_p + \left(\frac{p+1}{r-2} - b \right) \binom{p+r-2}{p+1} b^p\end{aligned}$$

We denote by $\lfloor x \rfloor$ the floor of $x \in \mathbb{R}$.

Lemma 6.18. *Let $m = b(r-2)$. There is a unique integer $\delta_r \in [m+1\ ;\ \lfloor 3.2\, m \rfloor + 1]$ such that : if $p \le \delta_r$, then $\alpha_p \le 0$ and if $p > \delta_r$, then $\alpha_p > 0$.*

Let δ_r be the integer given in Lemma 6.18. If $s \le \delta_r$, then all coefficients of Q_s are negative; thus, for any $x > 0$, $\mathsf{Q}_s(x) < 0$. If $s > \delta_r$, then by the Descartes rule (see Theorem 6.1), Q_s has a only one positive root ν_3. We deduce :

Proposition 6.19. *Let $r \ge 3$ and $a, b \in \mathbb{N}^*$ such that $a < b$ and $a_1 = a$, $a_2 = \ldots = a_r = b$.*

1. *If $s \le \delta_r$, then $\mathrm{sStab}(T_X(-\log(D_{v_0} + D_{v_1}))) = \varnothing$;*
2. *If $s > \delta_r$, then $T_X(-\log(D_{v_0} + D_{v_1}))$ is semi-stable with respect to $\pi^* \mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$ if and only if $\nu = \nu_3$.*

We now give the proof of Lemma 6.18.

Proof. We have $\alpha_p = \sum_{j=0}^{m-1}\left[\left(\frac{j+1}{r-2}-b\right)\binom{j+r-2}{j+1}b^j\,a^{p-1-j}\right] - a^p + \beta_p$ where

$$\begin{aligned}\beta_p &= \sum_{l=m}^{p-1}\left[\left(\frac{l+1-(r-2)b}{r-2}\right)\binom{l+r-2}{l+1}b^l\,a^{p-1-l}\right]\\ &= \sum_{l=0}^{p-1-m}\left[\frac{l+1}{r-2}\binom{l+m+r-2}{l+m+1}b^{l+m}\,a^{p-1-(l+m)}\right]\end{aligned}$$

The goal of this proof is to find an integer p such that $\alpha_p > 0$. We will search an integer p such that

$$\beta_p \geq a^p + \sum_{j=0}^{m-1} b\binom{j+r-2}{j+1}b^j\,a^{p-1-j}\ . \tag{15}$$

We have $\binom{l+m+r-2}{l+m+1} = \binom{l+m+r-2}{r-3} = \frac{r-2}{l+m+1}\binom{l+m+r-2}{r-2}$. From the equality

$$\sum_{j=n}^{r}\binom{j}{n} = \binom{r+1}{n+1} \quad \text{we have} \quad \binom{r+1}{n+1} = 1 + \sum_{j=0}^{r-n-1}\binom{j+n+1}{n}\ ;$$

Hence,

$$\begin{aligned}\binom{l+m+r-2}{r-2} = 1 + \sum_{j=0}^{l+m-1}\binom{j+r-2}{r-3} &= 1 + \sum_{j=0}^{l+m-1}\binom{j+r-2}{j+1}\\ &\geq 1 + \sum_{j=0}^{m-1}\binom{j+r-2}{j+1}\ .\end{aligned}$$

Thus,

$$\begin{aligned}\beta_p &\geq \left(1 + \sum_{j=0}^{m-1}\binom{j+r-2}{j+1}\right)\sum_{l=0}^{p-1-m}\frac{l+1}{l+1+m}\,b^{l+m}\,a^{p-1-(l+m)}\\ &\geq b^m\,a^{p-1-m}\left(1 + \sum_{j=0}^{m-1}\binom{j+r-2}{j+1}\right)\sum_{l=0}^{p-1-m}\frac{l+1}{l+1+m}\left(\frac{b}{a}\right)^l\ .\end{aligned}$$

We have

$$\begin{aligned}\sum_{l=0}^{p-1-m}\frac{l+1}{l+1+m} \geq \sum_{l=0}^{p-1-m}\int_l^{l+1}\frac{x\,dx}{x+m} &= \int_0^{p-m}\frac{x}{x+m}dx\\ &= \left[\frac{x}{1} - m\ln(x+m)\right]_0^{p-m}\\ &= p - m - m\ln\left(\frac{p}{m}\right)\end{aligned}$$

We set $k = \frac{p}{m}$. If $k \geq \frac{17}{5} = 3.2$, then $(k-1-\ln(k)) > 1$. If we set $p = \lfloor 3.2\,m\rfloor + 1$, we get

$$\sum_{l=0}^{p-1-m}\frac{l+1}{l+1+m}\left(\frac{b}{a}\right)^l \geq \sum_{l=0}^{p-1-m}\frac{l+1}{l+1+m} \geq m \geq b\ .$$

For $j \in \{0, \ldots, m-1\}$, we have $b^m\, a^{p-1-m} \geq b^j\, a^{p-1-j}$. If $p = \lfloor 3.2\, m \rfloor + 1$, we get

$$\begin{aligned}\beta_p &\geq b\, b^m\, a^{p-1-m} \left(1 + \sum_{j=0}^{m-1} \binom{j+r-2}{j+1}\right) \\ &\geq b^{m+1}\, a^{p-1-m} + \sum_{j=0}^{m-1} b \binom{j+r-2}{j+1} b^j\, a^{p-1-j}\ ;\end{aligned}$$

this proves the inequality (15). Thus, $\alpha_p > 0$ for $p = \lfloor 3.2\, m \rfloor + 1$. Hence, we deduce the existence of the integer δ_r in the interval $[m+1\ ;\ \lfloor 3.2\, m \rfloor + 1]$. □

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