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# A conformal geometric point of view on the Caffarelli-Kohn-Nirenberg inequality

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## Abstract

We are interested in the Caffarelli-Kohn-Nirenberg inequality (CKN in short), introduced by these authors in 1984. We explain why the CKN inequality can be viewed as a Sobolev inequality on a weighted Riemannian manifold. More precisely, we prove that the CKN inequality can be interpreted in this way on three different and equivalent models, obtained as weighted versions of the standard Euclidean space, round sphere and hyperbolic space. This result can be viewed as an extension of conformal invariance to the weighted setting. Since the spherical CKN model we introduce has finite measure, the  $\Gamma$ -calculus introduced by Bakry and Émery provides an easy way to prove the Sobolev inequalities. This method allows us to recover the optimality of the region of parameters describing symmetry-breaking of minimizers of the CKN inequality, introduced by Felli and Schneider and proved by Dolbeault, Esteban and Loss in 2016. Finally, we develop the notion of  $n$ -conformal invariants, exhibiting a way to extend the notion of scalar curvature to weighted manifolds such as the CKN models.

## 1 Introduction and main results

### 1.1 The CKN Euclidean space

In their seminal paper [CKN84], Caffarelli, Kohn and Nirenberg found the optimal range of real parameters  $a, b, p$  for which the following inequality holds true:

$$\left( \int_{\mathbf{R}^d} \frac{|v|^p}{|x|^{bp}} dx \right)^{2/p} \leq C_{a,b} \int_{\mathbf{R}^d} \frac{|\nabla v|^2}{|x|^{2a}} dx, \quad v \in \mathcal{C}_c^\infty(\mathbf{R}^d \setminus \{0\}). \quad (1)$$

Here,  $|\cdot|$  is the Euclidean norm in  $\mathbf{R}^d$ ,  $d \in \mathbf{N}^*$  and  $C_{a,b}$  denotes the optimal constant, depending on  $a, b$  and  $d$  only. Note that the case  $a = b = 0$  (and  $p = 2d/(d-2)$ ) corresponds to Sobolev's inequality, while the case  $b = a + 1$  (and  $p = 2$ ) is Hardy's inequality, so that (1) is sometimes called the Hardy-Sobolev inequality. Note also that the inequality is achieved in the former case, while it is not in the latter.

Let us consider the measure

$$d\hat{\mu}(x) = |x|^{-bp} dx. \quad (2)$$

Then, the left-hand side of (1) is simply the  $L^p$ -norm of  $v$  with respect to the measure  $\hat{\mu}$  (squared). In addition, if we consider the metric<sup>1</sup> on the manifold  $M = \mathbf{R}^d \setminus \{0\}$  given by

$$\hat{\mathbf{g}}^{ij} = |x|^{bp-2a} \delta^{ij}, \quad (3)$$

then (1) takes the simpler form

$$\left( \int |v|^p d\hat{\mu} \right)^{2/p} \leq C_{a,b} \int |\nabla_{\hat{\mathbf{g}}} v|_{\hat{\mathbf{g}}}^2 d\hat{\mu}.$$

---

<sup>1</sup>If  $(M, g)$  is  $d$ -dimensional Riemannian manifold whose metric  $g$  is represented in a local system of coordinates at a point  $x \in M$  by the matrix  $G(x) = (g_{ij}(x))_{1 \leq i, j \leq d}$ , we use the letter  $\mathbf{g}$  to denote the bilinear form on the cotangent space of  $M$  represented by the inverse matrix  $G(x)^{-1} = (\mathbf{g}^{ij}(x))_{1 \leq i, j \leq d}$

By a standard scaling argument<sup>2</sup>, the following relation is necessary for the inequality to hold true:

$$p = \frac{2d}{d-2+2(b-a)} = \frac{d}{a_c - a + b}, \quad (4)$$

where  $a_c = \frac{d-2}{2}$ . Through the property of modified inversion symmetry (see Theorem 1.4(ii) in [CW01]), we may always assume that  $a < a_c = \frac{d-2}{2}$  since the case  $a > a_c$  is dual to it and the inequality fails to be true if  $a = a_c$  (see [CKN84]). For simplicity, we also focus on the case  $d \geq 3$  and refer to [DEL14] for the remaining cases  $d \in \{1, 2\}$ . Then, (1) holds true if and only if

$$a \leq b \leq a + 1$$

Accordingly, we define the set

$$\Theta = \{(a, b) \in \mathbf{R}^2, a \leq b \leq a + 1, a < a_c\} \quad (5)$$

where the CKN inequality (1) is valid (see Section A.3).

Observe that for  $(a, b) \in \Theta$ ,  $p \leq \frac{2d}{d-2}$  and so  $p$  can be rewritten as the critical Sobolev exponent associated to an *intrinsic* dimension  $n \in [d, +\infty]$  through the relations

$$p = \frac{2n}{n-2}, \quad n = \frac{d}{1+a-b}. \quad (6)$$

The fact that  $n$  is a meaningful number, entering in the classical Bakry-Emery curvature-dimension condition, will become transparent in a moment. To summarize, one can view inequality (1) exactly as Sobolev's inequality stated on the weighted Riemannian manifold<sup>3</sup> that we introduce now.

**Definition 1.1** (The Euclidean CKN space). The Euclidean CKN space is the triple  $(M, \hat{\mathbf{g}}, \hat{\mu})$ , where the manifold  $M = \mathbf{R}^d \setminus \{0\}$ , the metric<sup>4</sup>  $\hat{\mathbf{g}}^{ij} = |x|^{2(1-\alpha)}\delta^{ij}$  and  $\hat{\mu}$  is given by (2). The corresponding Riemannian volume is given by  $dV_{\hat{\mathbf{g}}} = |x|^{d(\alpha-1)}dx$ , the weight  $\hat{W}$ , verifying  $d\hat{\mu} = e^{-\hat{W}}dV_{\hat{\mathbf{g}}}$ , is given by  $\hat{W} = -\frac{\alpha(n-d)}{2} \ln|x|^2$  and the generator<sup>5</sup> is given by  $\hat{L} = \Delta_{\hat{\mathbf{g}}} - \nabla^{\hat{\mathbf{g}}}\hat{W} \cdot \hat{g} = |x|^{2(1-\alpha)}(\Delta - a\nabla \ln|x|^2 \cdot \nabla)$ .

For notational convenience, we introduced above the parameter<sup>6</sup>:

$$\alpha = 1 + a - \frac{pb}{2}, \quad (7)$$

where  $(a, b) \in \Theta$  (defined in (5)) and  $p$  is the critical exponent given by (4). In other words, returning to the parameters  $a, b, d$  (and  $a_c = (d-2)/2$ ),

$$\alpha = \frac{(a_c - a)(a + 1 - b)}{a_c - a + b}.$$

Note that for any  $(a, b) \in \Theta$ , we have  $\alpha \geq 0$ , see Section A.3 and Figure 1 for more information about parameters. Equivalently, and this is the notation adopted in this paper<sup>7</sup>, one can see the Euclidean CKN space as a Markov triple  $(M, \hat{\mu}, \hat{\Gamma})$ , where  $\hat{\mu}$  verifies (2) and the carré du champ operator is given by

$$\hat{\Gamma}(v) = |\nabla_{\hat{\mathbf{g}}}v|_{\hat{\mathbf{g}}}^2 = |x|^{bp-2a}|\nabla v|^2 = |x|^{2(1-\alpha)}|\nabla v|^2.$$

Inequality (1) now reads

$$\left( \int |v|^p d\hat{\mu} \right)^{2/p} \leq C_{a,b} \int \hat{\Gamma}(v) d\hat{\mu}.$$

<sup>2</sup>To see this, apply (1) to the function  $x \mapsto v(\lambda x)$  where  $\lambda > 0$  and let  $\lambda \rightarrow 0^+$  and  $\lambda \rightarrow +\infty$

<sup>3</sup>the words "smooth metric measure space" and "manifold with density" are also employed in the literature to designate the same object.

<sup>4</sup>The given expression of  $\hat{\mathbf{g}}$  is just a rewriting of (3)

<sup>5</sup>i.e. the operator such that  $-\int u \hat{L}v d\hat{\mu} = \int (\nabla_{\hat{\mathbf{g}}}u \cdot \hat{g} \nabla_{\hat{\mathbf{g}}}v) d\hat{\mu}$  for  $u, v \in C_c^\infty(\mathbf{R}^d \setminus \{0\})$

<sup>6</sup>The reader may check that  $\alpha$  turns out to be the same parameter as the one introduced in [DEL14] (for different reasons).

<sup>7</sup>See [BGL14] for an introduction to  $\Gamma$ -calculus.

## 1.2 Conformal invariance

As noticed earlier, when  $a = b = 0$ , we recover the standard Sobolev inequality on the standard Euclidean space. In that case, since the metric of the  $d$ -dimensional sphere  $\mathbf{S}^d$  and the metric of the  $d$ -dimensional hyperbolic space  $\mathbf{H}^d$  are both conformally equivalent to the Euclidean metric, Sobolev's inequality takes equivalent forms on these three model spaces. More precisely, the Euclidean Sobolev inequality applied to the function  $\varphi^{\frac{2-d}{2}}v$ , where  $\varphi(x) = \frac{1+|x|^2}{2}$  (respectively  $\varphi(x) = 1 - |x|^2$ ) and  $v \in C_c^\infty(\mathbf{R}^d)$  (resp.  $v \in C_c^\infty(B_1)$ ) yields

$$\left( \int |v|^p dV_{\mathfrak{g}} \right)^{2/p} \leq C \left[ \int S_{\mathfrak{g}} v^2 dV_{\mathfrak{g}} + \int |\nabla_{\mathfrak{g}} v|_{\mathfrak{g}}^2 dV_{\mathfrak{g}} \right], \quad (8)$$

where  $\mathfrak{g}$  is the round metric on the sphere  $\mathbf{S}^d$  expressed in stereographic coordinates (resp. the metric of the hyperbolic space in the Poincaré ball model),  $dV_{\mathfrak{g}}$  the associated Riemannian volume,  $S_{\mathfrak{g}} = \frac{d(d-2)}{4}$  (resp.  $S_{\mathfrak{g}} = -\frac{d(d-2)}{4}$ ) and  $C = \frac{4}{d(d-2)} |\mathbf{S}^d|^{-\frac{2}{d}}$  the best constant in the standard Euclidean Sobolev inequality. By analogy, we can extend the conformal invariance property to the setting of weighted manifolds as described next.

### The spherical CKN and the hyperbolic CKN spaces

Recall that the metric and reference measure of the Euclidean CKN space read

$$\hat{\mathfrak{g}}^{ij} = |x|^{2(1-\alpha)} \delta^{ij} \quad \text{and} \quad d\hat{\mu} = |x|^{-bp} dx.$$

Keeping in mind the expression of the standard stereographic projection and restricting to the case  $\alpha > 0$ , we define next the spherical and hyperbolic CKN spaces as follows.

**Definition 1.2** (The spherical and the hyperbolic CKN spaces).

- The spherical CKN space is the triple  $(M, \bar{\mathfrak{g}}, \bar{\mu})$ , where  $M = \mathbf{R}^d \setminus \{0\}$ ,

$$\bar{\mathfrak{g}}^{ij} = |x|^{2(1-\alpha)} \frac{(1 + |x|^{2\alpha})^2}{4} \delta^{ij} \quad \text{and} \quad d\bar{\mu} = 2^n \frac{|x|^{-bp}}{(1 + |x|^{2\alpha})^n} dx.$$

Associated objects are given by the following formulae:

- Riemannian volume:  $dV_{\bar{\mathfrak{g}}} = \frac{|x|^{d(\alpha-1)}}{(1+|x|^{2\alpha})^d} dx$
- weight:  $\bar{W} = (n-d) \log(1 + |x|^{2\alpha}) - \frac{\alpha(n-d)}{2} \log|x|^2$ ,
- Carré du champ operator:  $\bar{\Gamma}(v) = |\nabla_{\bar{\mathfrak{g}}} v|_{\bar{\mathfrak{g}}}^2 = |x|^{2(1-\alpha)} \frac{(1+|x|^{2\alpha})^2}{4} |\nabla v|^2$
- generator:  $\bar{L}(f) = |x|^{2(1-\alpha)} \frac{(1+|x|^{2\alpha})^2}{4} [\Delta f - a \nabla f \cdot \nabla \log|x|^2 - (n-2) \nabla f \cdot \nabla \log(1 + |x|^{2\alpha})]$ .

- The CKN hyperbolic space is the triple  $(\mathbf{B} \setminus \{0\}, \tilde{\mathfrak{g}}, \tilde{\mu})$ , where  $\mathbf{B}$  is the open unit ball in  $\mathbf{R}^d$ ,

$$\tilde{\mathfrak{g}}^{ij} = |x|^{2(1-\alpha)} \frac{(1 - |x|^{2\alpha})^2}{4} \delta^{ij} \quad \text{and} \quad d\tilde{\mu} = 2^n \frac{|x|^{-bp}}{(1 - |x|^{2\alpha})^n} dx.$$

Associated objects to this triple are given by the following formulae

- Riemannian volume:  $dV_{\tilde{\mathfrak{g}}} = \frac{|x|^{d(\alpha-1)}}{(1-|x|^{2\alpha})^d} dx$
- weight:  $\tilde{W} = (n-d) \log(1 - |x|^{2\alpha}) - \frac{\alpha(n-d)}{2} \log|x|^2$ ,
- Carré du champ operator:  $\tilde{\Gamma}(v) = |\nabla_{\tilde{\mathfrak{g}}} v|_{\tilde{\mathfrak{g}}}^2 = |x|^{2(1-\alpha)} \frac{(1-|x|^{2\alpha})^2}{4} |\nabla v|^2$
- generator:  $\tilde{L}(f) = |x|^{2(1-\alpha)} \frac{(1-|x|^{2\alpha})^2}{4} [\Delta f - a \nabla f \cdot \nabla \log|x|^2 - (n-2) \nabla f \cdot \nabla \log(1 - |x|^{2\alpha})]$ .

*Remark 1.3.* • Note that in the case  $\alpha = 1$  (which is achieved in  $\Theta$  only when  $a = b = 0$ , see Lemma 1.9), the CKN sphere is the standard round sphere (punctured at both of its poles) viewed in the stereographic projection chart. Similarly, for  $\alpha = 1$ , the CKN hyperbolic space is the (punctured) hyperbolic space.

- Note that, letting  $\varphi(x) = \frac{1+|x|^{2\alpha}}{2}$ , we have  $\bar{\Gamma} = \varphi^2 \hat{\Gamma}$  and  $\bar{\mu} = \varphi^{-n} \hat{\mu}$ . We shall say that the CKN Euclidean and spherical spaces belong to the same  $n$ -conformal class ( $n$  not necessarily being equal to the topological dimension). Similarly, with  $\psi(x) = \frac{1-|x|^{2\alpha}}{2}$ , we have  $\tilde{\Gamma} = \psi^2 \hat{\Gamma}$  and  $\tilde{\mu} = \psi^{-n} \hat{\mu}$ , so that the hyperbolic CKN space also belongs to the same  $n$ -conformal class.
- When  $(a, b) \in \Theta$ ,  $\bar{\mu}$  has finite mass (see Remark A.4 in Section A.3). We prefer not to normalize the measure  $\bar{\mu}$ , a choice which makes the conformal invariance of Sobolev's inequality more transparent.

With these definitions at hand, we prove

**Theorem 1.4** (Conformal invariance of the three model spaces). *Let  $C > 0$  be an arbitrary constant. The three following Sobolev inequalities associated to each CKN model are equivalent:*

$$(i) \quad \forall v \in C_c^\infty(\mathbf{R}^d \setminus \{0\}), \quad \left( \int |v|^p d\hat{\mu} \right)^{2/p} \leq C \int \hat{\Gamma}(v) d\hat{\mu}, \quad (9)$$

$$(ii) \quad \forall v \in C_c^\infty(\mathbf{R}^d \setminus \{0\}), \quad \left( \int |v|^p d\bar{\mu} \right)^{2/p} \leq C \left( \int \bar{\Gamma}(v) d\bar{\mu} + \frac{n(n-2)}{4} \alpha^2 \int v^2 d\bar{\mu} \right), \quad (10)$$

$$(iii) \quad \forall v \in C_c^\infty(\mathbf{B} \setminus \{0\}), \quad \left( \int |v|^p d\tilde{\mu} \right)^{2/p} \leq C \left( \int \tilde{\Gamma}(v) d\tilde{\mu} - \frac{n(n-2)}{4} \alpha^2 \int v^2 d\tilde{\mu} \right). \quad (11)$$

*Remark 1.5.* • Inequality (9) is valid for some constant  $C = C_{a,b}$  if and only if  $(a, b) \in \Theta$  as proved in [CKN84]. Hence, so are (10) and (11).

- As we shall see, the value of the optimal constant  $C$  is known only in a restricted range of parameters, see Theorem 1.7 below.
- Since its set of test functions is smaller, inequality (11) need not be optimal even though (9) and (10) are. For example, in the absence of weights,  $C = \frac{4}{d(d-2)|\mathbf{S}^d|^{2/a}}$  is the optimal constant in (9) and (10) and extremals exist (and are classified), see Theorem 1.7 below. In contrast, inequality (11) holds with the same constant  $C = \frac{4}{d(d-2)|\mathbf{S}^d|^{2/a}}$  but when  $n = d = 3$  (and again  $\alpha = 1$ ), the constant  $-\frac{n(n-2)}{4}$  can be improved to  $-\frac{(n-1)^2}{4}$ , see [BFL08]. Using this fact and the proof of Theorem 1.4, it follows that the standard Sobolev inequality in  $\mathbf{R}^3$  improves to

$$\left( \int_{\mathbf{R}^3} |v|^6 dx \right)^{1/3} \leq \frac{1}{3} \left( \frac{2}{\pi} \right)^{4/3} \left( \int_{\mathbf{R}^3} |\nabla v|^2 dx - \int_{\mathbf{R}^3} \frac{v^2}{(1-|x|^2)^2} dx \right),$$

when restricted to functions  $v \in H_0^1(\mathbf{B})$ . When  $d \geq 4$  (and  $\alpha = 1$ ), inequality (11) is again optimal, but contrary to (9) and (10), the inequality is never attained<sup>8</sup>. We have not yet investigated the optimality of (11) in the general case.

### 1.3 Curvature-dimension conditions on the spherical CKN space

We just saw that the CKN inequality takes the forms (9), (10), (11) on the three CKN spaces. But what is the value of the best constant  $C$ ? To answer this question, let us first recall the following classical definition and result: a weighted manifold  $(M, \mathfrak{g}, \mu)$  is said to satisfy the  $CD(\rho, n)$  condition if for every  $f \in C^\infty(M)$ ,

$$\Gamma_2(f) \geq \rho \Gamma(f) + \frac{1}{n} (Lf)^2,$$

where  $(\rho, n) \in \mathbf{R} \times \bar{\mathbf{R}}$ ,  $d\mu = e^{-W} dV_g$  for some  $W \in C^\infty(M)$ ,  $\Gamma(f) = |\nabla_{\mathfrak{g}} f|_{\mathfrak{g}}^2$ ,  $Lf = \Delta_{\mathfrak{g}} f - \Gamma(W, f)$  and  $\Gamma_2(f) = \frac{1}{2} L(\Gamma(f)) - \Gamma(f, Lf)$ . The above curvature-dimension condition can in fact be stated in the somewhat more general framework of Markov triples (see [BGL14]) and the following theorem holds true:

<sup>8</sup>If it were, then (9) would also be attained by a compactly supported function.

**Theorem A** ([BL96, BGL14]). Assume that  $(E, \Gamma, \mu)$  is a full Markov triple satisfying the  $CD(\rho, n)$  condition with  $\rho > 0$  and  $n > 2$ . Let  $p = \frac{2n}{n-2}$  and normalize the measure  $\mu$  so that  $\mu(E) = 1$ . Then,

$$\left( \int |v|^p d\mu \right)^{2/p} \leq \frac{4}{n(n-2)} \frac{n-1}{\rho} \int \Gamma(v) d\mu + \int v^2 d\mu. \quad (12)$$

Hence, it suffices to determine whether the CKN-sphere is a full Markov triple satisfying the  $CD(\rho, n)$  condition in order to obtain an explicit value (which turns out to be optimal in our case) for the constant  $C$  in (10). This is what we do next.

**Proposition 1.6** (Curvature-dimension condition for the spherical CKN space).

Let  $(a, b) \in \Theta$  and

$$\rho = \alpha^2(n-1). \quad (13)$$

Then, the spherical CKN space satisfies the curvature-dimension condition  $CD(\rho, n)$  if and only if

$$\alpha^2 \leq \frac{d-2}{n-2} \quad (14)$$

Note that  $(\mathbf{R}^d \setminus \{0\}, \bar{\Gamma}, \bar{\mu})$  is a full Markov triple, see Section A.2, whence Sobolev's inequality (10) holds under the condition (14). However, a quick inspection of the proofs of Theorem A given in [DGZ20, BGL14] shows that Theorem A remains valid under the following *weaker* integrated form of the curvature-dimension condition:

$$\int \left( \Gamma_2(f) - \rho \Gamma(f) - \frac{1}{n} (Lf)^2 \right) f^{1-n} d\mu \geq 0, \quad (15)$$

for functions  $f \in C^\infty(\mathbf{R}^d \setminus \{0\})$  such that  $\inf f > 0$  and  $\sup f < +\infty$ . For the CKN sphere, this is of importance, as the following theorem demonstrates.

**Theorem 1.7** (Sobolev inequality for the spherical CKN space). Let  $(a, b) \in \Theta$ . Whenever

$$0 < \alpha \leq 1, \quad (16)$$

the following optimal Sobolev inequality holds

$$\left( \int v^p d\mu \right)^{2/p} \leq \frac{4}{n(n-2)\alpha^2} \int \bar{\Gamma}(v) d\mu + \int v^2 d\mu, \quad (17)$$

for any smooth function  $v$ , where  $\mu = \frac{1}{Z} \bar{\mu}$  and  $Z$  is a normalization constant<sup>9</sup> such that  $\mu$  is a probability measure. That is, inequality (10) is valid with optimal constant

$$C = \frac{4}{n(n-2)\alpha^2 Z^{\frac{2}{n}}}.$$

In addition,

- for  $\alpha \in (0, 1)$  equality holds in (17) if and only if

$$v(x) = (\lambda + \mu \tanh(\alpha s))^{-\frac{n-2}{2}}, \quad s = \ln |x|,$$

where  $\lambda, \mu$  are arbitrary constants such that  $\lambda > |\mu|$ .

- for  $\alpha = 1$ , equality holds in (17) if and only if

$$v = (\lambda + \mu \varphi_2)^{-\frac{d-2}{2}},$$

where  $\varphi_2$  is any eigenfunction of  $-\bar{L} = -\Delta_{\mathbf{S}^d}$  associated to the first nonzero eigenvalue  $\lambda_2 = d$  and  $\lambda, \mu$  are arbitrary constants such that  $\lambda > |\mu| \|\varphi_2\|_\infty$ .

---

<sup>9</sup> $Z = \bar{\mu}(\mathbf{R}^d \setminus \{0\}) = \frac{2}{\alpha} |\mathbf{S}^{d-1}| \int_0^{+\infty} (\cosh t)^{-n} dt$

*Remark 1.8.* • Note that  $\varphi_2(x) = \tanh(\alpha s)$  is a radial eigenfunction of  $\bar{L}$  associated to the eigenvalue  $\lambda_2 = \alpha^2 n$ . So, except for the round sphere (corresponding to the case  $\alpha = 1$ ), the extremals of Sobolev's inequality are obtained as a linear combination of radial extremals of Poincaré's inequality (19) (raised to the power  $-\frac{n-2}{2}$ ), provided this combination is bounded below by a positive constant.

- As we shall prove in Lemma 1.9, condition (16) is equivalent to  $\alpha^2 \leq \frac{d-1}{n-1}$ . In the limiting case  $\alpha^2 = \frac{d-1}{n-1}$ , extremals of Sobolev's inequality are radial, while extremals of Poincaré's inequality need not be, see Proposition 1.10 below.
- The extremals of Sobolev's inequality on the round sphere (i.e. when  $\alpha = 1$ ) were discovered by T. Aubin, see e.g. Theorem 5.1 in [Heb00]. They are more often written as constant multiples of

$$v = (\beta - \cos(r))^{-\frac{d-2}{2}},$$

where  $\beta > 1$  and  $r$  is the geodesic distance to an arbitrary point  $\omega_0 \in \mathbf{S}^d$ . Our notation puts forward the connection between the extremals of Sobolev's inequality and the extremals of Poincaré's inequality in Proposition 1.10: the former are obtained as a linear combination of the latter (raised to the power  $-\frac{d-2}{2}$ ), provided this combination is bounded below by a positive constant.

- Hardy's inequality (i.e. the case  $\alpha = 0$  in (9)) is optimal for the constant

$$C = \lim_{\alpha \rightarrow 0^+} \frac{4}{n(n-2)\alpha^2 Z_n^{\frac{2}{n}}} = \left( \frac{2}{d-2-2\alpha} \right)^2$$

but equality is never achieved.

- It follows from Theorem 1.4 and Theorem 1.7 that for  $\alpha \in (0, 1]$ , extremal functions for Sobolev's inequality on the Euclidean CKN space take the form

$$v(x) = \left( \frac{1 + |x|^{2\alpha}}{2} \right)^{-\frac{n-2}{2}},$$

up to normalization and dilation, providing thereby an alternative proof of the main result in [DEL14].

Condition (16) is strictly weaker than condition (14). It turns out to be equivalent to

$$\alpha^2 \leq \frac{d-1}{n-1}.$$

More precisely, consider the following Felli-Schneider region

$$\Theta_{FS} = \{(a, b) \in \Theta, b \geq b_{FS}(a) \text{ if } b \leq 0\}, \quad \text{where } b_{FS}(a) = \frac{d(a_c - a)}{2\sqrt{(a_c - a)^2 + d - 1}} - (a_c - a) \quad (18)$$

Let as well  $\Theta_{DGZ} \subset \Theta$  be the domain where Proposition 1.6 is valid:

$$\Theta_{DGZ} = \{(a, b) \in \Theta, \mathfrak{B}_{DGZ} \geq 0\}, \quad \text{where } \mathfrak{B}_{DGZ} = (d-2) - (n-2)\alpha^2.$$

Then, we prove that the region  $\Theta_{FS}$  corresponds exactly to the domain where Theorem 1.7 is valid:

**Lemma 1.9** (Comparison of the two regions).

$$\Theta_{FS} = \left\{ (a, b) \in \Theta, \mathfrak{B}_{DGZ} + \frac{n-d}{n-1} \geq 0 \right\} = \left\{ (a, b) \in \Theta, \alpha^2 \leq \frac{d-1}{n-1} \right\} = \{(a, b) \in \Theta, \alpha \in [0, 1]\}.$$

Hence,  $\Theta_{DGZ} \subsetneq \Theta_{FS}$ . Moreover, for any  $(a, b) \in \Theta$ ,  $\alpha = 1$  if and only if  $a = b = 0$ .

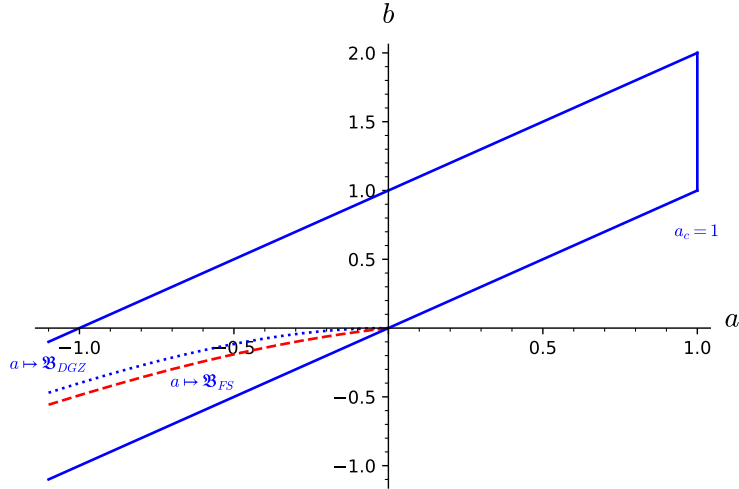


Figure 1:  $b_{FS}(a)$  (dashed style in red) and the curve  $\mathfrak{B}_{DGZ} = 0$  (dotted style in blue) with  $d = 4$ .

The two regions are represented in Figure 1 with  $d = 4$ . The condition  $\theta \in \Theta_{FS}$  is known in the literature as the Felli-Schneider condition. Felli and Schneider [FS03], building on the work of Catrina and Wang [CW01] initially proved that extremal functions for the optimal CKN inequality (1) cannot be radial whenever (16) fails. Conversely, in their work [DEL14], Dolbeault, Esteban and Loss computed the optimal constant in (1) and proved that extremals for the optimal CKN inequality (1) are radial and explicit whenever (16) holds. Combining Theorem 1.7 with Theorem 1.4 gives an immediate alternative proof of these latter facts. Our point of view may further clarify why the Felli-Schneider condition is optimal. Indeed, it is well-known that a tight Sobolev inequality implies a Poincaré inequality: precisely, applying (17) with  $v = 1 + \epsilon f$  and letting  $\epsilon \rightarrow 0$  leads to

$$\int f^2 d\mu - \left( \int f d\mu \right)^2 \leq \frac{1}{n\alpha^2} \int \bar{\Gamma}(f) d\mu. \quad (19)$$

We prove the following.

**Proposition 1.10** (Poincaré inequality for the spherical CKN space). *Let  $(a, b) \in \Theta$ . The Poincaré inequality (19) holds with optimal constant  $C = \frac{1}{n\alpha^2}$  if and only if the Felli-Schneider condition (16) holds. In addition,*

- if  $0 < \alpha < \frac{d-1}{n-1}$ , equality holds in (19) if and only if for some constants  $(\lambda, \mu) \in \mathbf{R}^2$ ,

$$f(x) = \lambda + \mu \tanh(\alpha s), \quad \text{where } s = \ln |x|.$$

- otherwise,  $\alpha = \frac{d-1}{n-1}$  and equality holds in (19) if and only if for some constants  $(\lambda, \mu, \nu) \in \mathbf{R}^3$  and some eigenfunction  $\varphi_2$  associated to the first nonzero eigenvalue  $\lambda_2 = d - 1$  of  $-\Delta_{\mathbf{S}^{d-1}}$ ,

$$f(x) = \lambda + \mu \tanh(\alpha s) + \nu \frac{\varphi_2(\omega)}{\cosh(\alpha s)}, \quad \text{where } s = \ln |x| \text{ and } \omega = \frac{x}{|x|}.$$

So, the Felli-Schneider condition cannot be improved in the statement of Theorem 1.7 and it is in fact equivalent to both Sobolev's and Poincaré's inequality with the given optimal constants on the spherical CKN space. In addition, Poincaré's inequality with constant  $C = \frac{n-1}{\rho n}$  is in fact equivalent to the following integrated  $CD(\rho, n)$  condition, where  $\rho, n > 0$ :

$$\int \left( \Gamma_2(f) - \rho \Gamma(f) - \frac{1}{n} (Lf)^2 \right) d\mu \geq 0,$$

see Proposition 4.8.3, Theorem 4.8.4 and their proofs in [BGL14]. Hence, the Felli-Schneider condition (16) can be interpreted as a curvature-dimension condition in integral form.



Let us also point out that if Sobolev's inequality (10) holds on a  $d$ -dimensional smooth manifold  $(M, \mathbf{g})$  *without weight* with optimal constant  $C = \frac{4}{d(d-2)|\mathbf{S}^{d-1}|^{2/d}}$ , then  $(M, \mathbf{g})$  must be isometric to the round sphere, as very recently demonstrated in [NV21]. Our analysis on the CKN spaces reveals that in the case of weighted manifolds, the optimal Sobolev constant can take a much different value. In particular, the standard assumption of Euclidean-volume growth on  $CD(0, n)$  spaces (see e.g. [NV21]) is not general enough to capture optimal constants in Sobolev-type inequalities on weighted manifolds. Determining all weighted manifolds which have the same optimal Sobolev constant  $\frac{4}{n(n-2)\alpha^2 Z^{2/n}}$  as the CKN sphere is an open problem that we plan to explore in future work.

## 1.4 The $n$ -conformal invariants

In this last introductory paragraph, we present expand on the conformal invariance of Sobolev's inequality in the setting of weighted manifolds and provide a deeper reason for why the three CKN model spaces satisfy equivalent conformal forms of the Sobolev inequality.

For the inequality (8) without weights, it turns out that  $S_{\mathbf{g}} = \frac{d-2}{4(d-1)}sc_{\mathbf{g}}$  is a constant multiple of the scalar curvature of  $\mathbf{g}$  (see Propositions 3.6.20, 3.6.21 and 6.2.2, as well as the second displayed formula on p. 63 in [Heb97] or [BGL14, Sec. 6.9.2] for proofs of this classical result). In other words, inequality (8) is valid on the whole conformal class of the round sphere, including the Euclidean (where  $S_{\mathbf{g}} = 0$ ) and hyperbolic (where  $S_{\mathbf{g}} = -\frac{d-2}{4(d-1)}sc_{\mathbf{g}}$ ) spaces.

For weighted manifolds, the notion of scalar curvature can be generalized as follows. As proposed in [BGL14, Sec. 6.9] (see also [CGY06] and [Cas12] for earlier perspectives<sup>10</sup>), given a  $d$ -dimensional ( $d \geq 2$ ) weighted Riemannian manifold  $(M, \mathbf{g}, \mu)$  with reference measure

$$d\mu = e^{-W} dV_{\mathbf{g}},$$

where  $W : M \rightarrow \mathbf{R}$  is a given weight and  $dV_{\mathbf{g}}$  the Riemannian volume, let

$$\Gamma(f) = |\nabla_{\mathbf{g}} f|_{\mathbf{g}}^2$$

denote the associated carré du champ operator, so that  $(M, \mu, \Gamma)$  is a Markov triple.

**Definition 1.11.** Take a real number  $n \in [d, +\infty]$ , which is not necessarily an integer. The  $n$ -conformal class of the triple  $(M, \mu, \Gamma)$  is the set of all Markov triples  $(M, c^{-n}\mu, c^2\Gamma)$ , where  $c : M \mapsto (0, \infty)$  is any smooth and positive function. An  $n$ -conformal invariant is a map  $S$  defined on the  $n$ -conformal class of  $(M, \mu, \Gamma)$  with values in the set of functions over  $M$ , such that for any positive smooth function  $c = e^{\tau}$ ,

$$S(c^{-n}\mu, c^2\Gamma) = c^2 \left[ S(\mu, \Gamma) + \frac{n-2}{2} \left( L\tau - \frac{n-2}{2} \Gamma(\tau) \right) \right], \quad (20)$$

where

$$L = \Delta_{\mathbf{g}} - \Gamma(W, \cdot).$$

It is important to notice that the operator  $L$  is uniquely determined by the carré du champ operator  $\Gamma$  and the measure  $\mu$  only. This is indeed the case since the operator  $\Delta_{\mathbf{g}}$  depends on the metric  $\mathbf{g}$  (which itself is uniquely determined by  $\Gamma$ ) and since  $W$  is related to the measure  $\mu$  and the metric  $\mathbf{g}$  through the Riemannian measure  $dV_{\mathbf{g}}$ . Also observe that setting  $u = c^{-\frac{n-2}{2}}$ ,  $s = S(\mu, \Gamma)$ ,  $\tilde{s} = S(c^{-n}\mu, c^2\Gamma)$ , then (20) can be reformulated as the following Yamabe-type equation:

$$-Lu + su = \tilde{s}u^{\frac{n+2}{n-2}} \quad \text{in } M.$$

Note that the case where  $n = d$ ,  $L = \Delta_{\mathbf{g}}$ ,  $s = \frac{d-2}{4(d-1)}sc_{\mathbf{g}}$  and  $\tilde{s}$  constant, is the standard Yamabe equation.

By a rather direct computation, see [BGL14, Prop. 6.9.2], whenever  $S = S(\mu, \Gamma)$  is an  $n$ -conformal invariant, the Sobolev inequality

$$\left( \int |v|^p d\mu \right)^{2/p} \leq C \left( \int \Gamma(v) d\mu + \int S v^2 d\mu \right), \quad (21)$$

<sup>10</sup>which correspond to the special case  $\gamma = -2$  in Proposition 1.12 below.

(with given constant  $C > 0$  and  $p = \frac{2n}{n-2}$ ) is invariant in the  $n$ -conformal class of the triple  $(M, \mu, \Gamma)$ . In other words, if the Sobolev inequality (21) holds for some constant  $C$ , then it also holds with the same constant  $C$  for all triples  $(M, c^{-n}\mu, c^2\Gamma)$  where  $c$  is any smooth and positive function.

Rephrasing what we said earlier, in the absence of weight,  $S_{\mathfrak{g}} = \frac{d-2}{4(d-1)}sc_{\mathfrak{g}}$  is an example of a  $d$ -conformal invariant (where in this case  $n = d$ ). The case of weighted Riemannian manifolds is a little bit more complicated and contains interesting examples. Let us recall Proposition 6.9.6 of [BGL14] (we will also provide a proof since the one in [BGL14] contains some mistakes).

**Proposition 1.12** ( $n$ -conformal invariant in a weighted manifold). *Let  $\gamma \in \mathbf{R}$  and  $n > d$ . Then,*

$$S_{\gamma}(\mu, \Gamma) = \theta_n(\gamma)[sc_{\mathfrak{g}} - \gamma\Delta_{\mathfrak{g}}W + \beta_n(\gamma)\Gamma(W)] \quad (22)$$

is an  $n$ -conformal invariant if

$$\beta_n(\gamma) = \frac{\gamma(n-2d+2) - 2(d-1)}{2(n-d)} \quad \text{and} \quad \theta_n(\gamma) = \frac{n-2}{4(d-1) - 2\gamma(n-d)}.$$

This being recalled, a natural question arises in the context of the Euclidean CKN space we introduced in Definition 1.1: does there exist a (unique) real number  $\gamma_0 \in \mathbf{R}$  such that this space satisfies

$$S_{\gamma_0}(\hat{\mu}, \hat{\Gamma}) = 0 \quad ?$$

This is indeed the case, as we are about to see. By Theorem 1.4, without any further computation, we deduce that for the same value of the parameter  $\gamma = \gamma_0$ ,  $S_{\gamma_0}(\bar{\mu}, \bar{\Gamma}) = \frac{n(n-2)}{4}\alpha^2 > 0$  is constant for the CKN sphere and  $S_{\gamma_0}(\tilde{\mu}, \tilde{\Gamma}) = -\frac{n(n-2)}{4}\alpha^2$  for the CKN hyperbolic space.

**Proposition 1.13.** *Let*

$$\mathfrak{B}_{DGZ} = (d-2) - (n-2)\alpha^2 \quad (23)$$

and

$$\gamma_0 = -\frac{2(d-1)}{\alpha^2(n-d)(n-2)}\mathfrak{B}_{DGZ}, \quad (24)$$

Then,  $S_{\gamma_0}(\hat{\mu}, \hat{\Gamma}) = 0$ ,  $S_{\gamma_0}(\bar{\mu}, \bar{\Gamma}) = \frac{n(n-2)}{4}\alpha^2$  and  $S_{\gamma_0}(\tilde{\mu}, \tilde{\Gamma}) = -\frac{n(n-2)}{4}\alpha^2$ .

As an immediate corollary of the CKN inequality (1) and the above lemma, we recover the validity of Sobolev's inequality on our three model spaces, stated in Theorem 1.4 above.

*Remark 1.14.* In a forthcoming report, we will further explain how a weighted version of Otto's calculus can be introduced in order to prove a wider class of optimal CKN inequalities, by working directly on the Euclidean CKN space, rather than the CKN sphere.

In Section 2 below, we prove the conformal invariance of Sobolev's inequality in the CKN spaces. Section 3 is dedicated to the characterization of the region of parameter  $\Theta_{DGZ}$  (resp.  $\Theta_{FS}$ ) for which the classical curvature-condition (resp. the integrated form (15)) holds, from which Sobolev's inequality follows. In Section 4, we prove all results pertaining to  $n$ -conformal invariance for general weighted manifolds. At last, an appendix contains lists of known formulas and constants, proofs of the numerology relating them as well as rigorous justification of the integrations by parts implicitly used in the proof of Sobolev's inequality.

## 2 Conformal invariance of Sobolev type inequalities for CKN models

### *Proof of Theorem 1.4*

◁ As in the unweighted case, the proof reduces to a simple change of unknown, once the proper notion of conformal invariance has been introduced. First we prove that the Sobolev inequality in the CKN Euclidean space is equivalent to the Sobolev inequality in the spherical CKN space.

Recall that for

$$\varphi(x) = \frac{1 + |x|^{2\alpha}}{2}, \quad x \in \mathbf{R}^d, \quad (25)$$

and that

$$\bar{\Gamma} = \varphi^2 \hat{\Gamma} \quad \text{and} \quad \bar{\mu} = \varphi^{-n} \hat{\mu}.$$

Apply (1) to the function  $f = \varphi^{\frac{2-n}{2}} g$ . On the one hand, we have

$$\int f^p d\hat{\mu} = \int f^{\frac{2n}{n-2}} d\hat{\mu} = \int g^p d\bar{\mu}.$$

On the other hand, letting  $V = \log \varphi$ ,

$$\begin{aligned} \hat{\Gamma}(f) &= \hat{\Gamma}(\varphi^{\frac{2-n}{2}} g) = \varphi^{2-n} \hat{\Gamma}(g) + 2\varphi^{\frac{2-n}{2}} g \hat{\Gamma}(\varphi^{\frac{2-n}{2}}, g) + \hat{\Gamma}(\varphi^{\frac{2-n}{2}}) g^2 \\ &= \varphi^{2-n} \left( \hat{\Gamma}(g) - \frac{n-2}{2} \hat{\Gamma}(g^2, V) + \frac{(n-2)^2}{4} \hat{\Gamma}(V) g^2 \right). \end{aligned}$$

An integration by parts with respect to  $\hat{\mu}$  yields

$$\int \hat{\Gamma}(g^2, V) \varphi^{2-n} d\hat{\mu} + \int \hat{\Gamma}(\varphi^{2-n}, V) g^2 d\hat{\mu} = - \int \hat{L} V g^2 \varphi^{2-n} d\hat{\mu},$$

so that we get

$$\begin{aligned} \int \hat{\Gamma}(f) d\hat{\mu} &= \int \bar{\Gamma}(g) d\bar{\mu} - \frac{n-2}{2} \int \hat{\Gamma}(g^2, V) \varphi^{2-n} d\hat{\mu} + \frac{(n-2)^2}{4} \int \hat{\Gamma}(V) g^2 \varphi^2 d\bar{\mu} \\ &= \int \bar{\Gamma}(g) d\bar{\mu} + \frac{n-2}{2} \int \left( \hat{L}(V) - \frac{n-2}{2} \hat{\Gamma}(V) \right) g^2 \varphi^2 d\bar{\mu}, \end{aligned}$$

The CKN inequality (1) becomes

$$\left( \int g^p d\bar{\mu} \right)^{2/p} \leq C \left( \int \bar{\Gamma}(g) d\bar{\mu} + \frac{(n-2)}{2} \int \left( \hat{L}(V) - \frac{n-2}{2} \hat{\Gamma}(V) \right) \varphi^2 g^2 d\bar{\mu} \right),$$

and it is enough to compute the quantity

$$\hat{L}(V) - \frac{n-2}{2} \hat{\Gamma}(V).$$

To that end, recall that  $\hat{L}$  is given by

$$\hat{L} = |x|^{2(1-\alpha)} [\Delta - a \nabla \log |x|^2 \cdot \nabla]. \quad (26)$$

Since  $V = \log \varphi$ , we have

$$\hat{L}(V) - \frac{n-2}{2} \hat{\Gamma}(V) = \frac{\hat{L}(\varphi)}{\varphi} - \frac{\hat{\Gamma}(\varphi)}{\varphi^2} - \frac{n-2}{2} \frac{\hat{\Gamma}(\varphi)}{\varphi^2} = \frac{\hat{L}(\varphi)}{\varphi} - \frac{n}{2} \frac{\hat{\Gamma}(\varphi)}{\varphi^2}.$$

Recalling the definition of  $\varphi$  given in (25), we find that

$$\frac{\hat{\Gamma}(\varphi)}{\varphi^2} = \frac{\alpha^2 |x|^{2\alpha}}{\varphi^2},$$

and

$$\frac{\hat{L}(\varphi)}{\varphi} = \frac{|x|^{2(1-\alpha)} [\Delta \varphi - a \nabla \log |x|^2 \cdot \nabla \varphi]}{\varphi} = \alpha \frac{d + 2(\alpha - 1) - 2a}{\varphi} = \frac{\alpha^2 n}{\varphi}.$$

So finally,

$$\hat{L}(V) - \frac{n-2}{2} \hat{\Gamma}(V) = \frac{\alpha^2 n}{2\varphi^2},$$

and the CKN inequality (1) takes the form (10), as announced.

Next, we prove that Sobolev's inequality in the CKN Euclidean space implies the Sobolev inequality on the CKN hyperbolic space. We mimic the previous proof. Define the function  $\psi$  on the punctured open unit ball  $\mathbf{B} \setminus \{0\} \subset \mathbf{R}^d$  by

$$\psi(x) = \frac{1 - |x|^{2\alpha}}{2}, \quad x \in \mathbf{B} \setminus \{0\}.$$

Then, on  $\mathbf{B} \setminus \{0\}$ ,

$$\tilde{\Gamma} = \psi^2 \hat{\Gamma} \quad \text{and} \quad \tilde{\mu} = \psi^{-n} \hat{\mu}.$$

Apply the CKN inequality (1) to the function  $f = \psi^{\frac{2-n}{2}} h$ , where  $h \in C_c^\infty(\mathbf{B} \setminus \{0\})$ . Again, we get

$$\int f^p d\hat{\mu} = \int f^{\frac{2n}{n-2}} d\hat{\mu} = \int h^p d\tilde{\mu}.$$

and (1) becomes, with  $U = \log \psi$ ,

$$\left( \int h^p d\tilde{\mu} \right)^{2/p} \leq C \left( \int \tilde{\Gamma}(h) d\tilde{\mu} + \frac{n-2}{2} \int \left( \hat{L}(U) - \frac{n-2}{2} \hat{\Gamma}(U) \right) \psi^2 h^2 d\tilde{\mu} \right).$$

We obtain

$$\hat{L}(U) - \frac{n-2}{2} \hat{\Gamma}(U) = -\frac{\alpha^2 n}{2\psi^2},$$

and so (11), as claimed.  $\triangleright$

### 3 Sobolev's inequality for the spherical CKN model

This section is devoted to the proof of the optimal Sobolev inequality for the spherical CKN space (Theorem 1.7) under the Felli-Schneider condition (16). It is convenient to introduce spherical coordinates  $\mathbf{R}^d \setminus \{0\} \ni x = r\theta$  with  $r > 0$  and  $\theta \in \mathbf{S}^{d-1}$ . The Sobolev inequality on the CKN sphere (17) then takes the form

$$\begin{aligned} \left( \int_{(0,\infty) \times \mathbf{S}^{d-1}} |v|^p \frac{r^{(d-1-pb)}}{(1+r^{2\alpha})^n} dr d\theta \right)^{2/p} &\leq C \int_{(0,\infty) \times \mathbf{S}^{d-1}} \left[ (\partial_r v)^2 + \frac{1}{r^2} |\nabla_\theta v|^2 \right] \frac{r^{(d-1-pb)}}{(1+r^{2\alpha})^{n-2}} dr d\theta \\ &\quad + 4Z^{-\frac{2}{n}} \int_{(0,\infty) \times \mathbf{S}^{d-1}} v^2 \frac{r^{(d-1-pb)}}{(1+r^{2\alpha})^n} dr d\theta, \end{aligned}$$

where  $|\nabla_\theta v|$  is the Riemannian length of the Riemannian gradient  $\nabla_\theta v$  on  $\mathbf{S}^{d-1}$  and  $d\theta$  is the associated Riemannian volume. Using the change of variable  $(0, \infty) \ni r = e^s$ , with  $s \in \mathbf{R}$ , the inequality becomes (with a different constant  $C$ ),

$$\left( \int_{\mathbf{R} \times \mathbf{S}^{d-1}} |v|^p \cosh(\alpha s)^{-n} ds d\theta \right)^{2/p} \leq C \int_{\mathbf{R} \times \mathbf{S}^{d-1}} [(\partial_s v)^2 + |\nabla_\theta v|^2] \cosh(\alpha s)^{2-n} ds d\theta + Z^{-\frac{2}{n}} \int_{\mathbf{R} \times \mathbf{S}^{d-1}} v^2 \cosh(\alpha s)^{-n} ds d\theta,$$

where we used the fact that  $d - n\alpha - pb = 0$ , see (53). In other words, in the cylindrical chart  $(s, \theta) \in \mathbf{R} \times \mathbf{S}^{d-1}$ , the spherical CKN space takes a new and nice form. Notice that the space is the same, it is only written in a new chart. More precisely, letting

$$\varphi(s) = \cosh(\alpha s), \quad s \in \mathbf{R}, \tag{27}$$

the metric becomes

$$\bar{\mathfrak{g}} = \varphi^2 \mathfrak{h} = e^{2\bar{\tau}} \mathfrak{h}, \tag{28}$$

where  $\bar{\tau} = \log \varphi$  and  $\mathfrak{h}$  is the standard product metric<sup>11</sup> on  $(0, \infty) \times \mathbf{S}^{d-1}$ , represented by the  $d$ -dimensional matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & G_\theta \end{pmatrix}$$

<sup>11</sup>on the cotangent space of  $(0, \infty) \times \mathbf{S}^{d-1}$ , where  $\mathbf{S}^{d-1}$  is viewed in a given chart

where  $G_\theta$  is the matrix of  $\mathfrak{g}_\theta$  in the chart  $(s, \theta)$ , and  $\mathfrak{g}_\theta$  is the round metric of  $\mathbf{S}^{d-1}$ . For convenience, in Lemma 3.2 and its proof, as well as the proof of Proposition 1.6, we will abuse the notations and identify the tensors with their coordinates in the chart  $(s, \theta)$ , since it will be the only chart used in all the calculations. The carré du champ operator takes the form

$$\bar{\Gamma}(f) = \varphi^2 [(\partial_s f)^2 + |\nabla_\theta f|^2] = \varphi^2 [(\partial_s f)^2 + \Gamma^\theta(f)],$$

where  $\Gamma^\theta(f) = |\nabla_\theta f|^2$  is the carré du champ operator associated to the the Laplace-Beltrami operator  $\Delta_\theta$  on  $\mathbf{S}^{d-1}$ . The Riemannian volume becomes  $dV_{\bar{\mathfrak{g}}} = \varphi^{-d} ds d\theta$  and the reference measure

$$d\bar{\mu} = \varphi^{-n} ds d\theta.$$

The corresponding weight  $\bar{W}$  is defined by  $d\bar{\mu} = e^{-\bar{W}} dV_{\bar{\mathfrak{g}}}$ , so that

$$\bar{W} = (n - d) \log \varphi$$

Finally, the associated generator takes the pleasant form

$$\bar{L}(f) = \varphi^2 \left[ \partial_{ss} f + (2 - n) \frac{\varphi'}{\varphi} \partial_s f + \Delta_\theta f \right].$$

Taking advantage of this chart, let us begin by proving that the spherical CKN space satisfies the  $CD(\rho, n)$  condition whenever condition (14) holds:

**Proof of Proposition 1.6**

◁ From [BGL14, Sec. C6], the generator  $\bar{L}$  satisfies a  $CD(\rho, n)$  condition (with  $n > d$ ) if and only if, as a symmetric tensor (with lower indices),

$$\text{Ric}(\bar{L}) - \rho \bar{g} \geq \frac{1}{n - d} \nabla^{\bar{\mathfrak{g}}} \bar{W} \otimes \nabla^{\bar{\mathfrak{g}}} \bar{W}.$$

Let us remark that, since  $\bar{\mathfrak{g}} = \varphi^2 \mathfrak{h}$ , the corresponding metric tensors (with lower indices) satisfy

$$\bar{g} = \frac{h}{\varphi^2}.$$

Compute first the r.h.s. of the above inequality. From the definition of  $\bar{W}$ , we have

$$\frac{\nabla^{\bar{\mathfrak{g}}} \bar{W} \otimes \nabla^{\bar{\mathfrak{g}}} \bar{W}}{n - d} = (n - d) \left( \frac{\varphi'}{\varphi} \right)^2 J.$$

where  $J$  is the  $d$ -dimensional matrix with all entries equal to zero but the first i.e.  $J_{ij} = \delta_{i1} \delta_{j1}$  or more visually, letting  $H$  (resp.  $G_\theta$ ) be the matrix representing the standard product metric  $h$  (resp.  $g_\theta$ ) in the coordinates  $(s, \theta)$  (resp.  $\theta$ ),

$$J = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = H - \begin{pmatrix} 0 & 0 \\ 0 & G_\theta \end{pmatrix}. \quad (29)$$

Now, applying formula (33) in Lemma 3.2 below, we get

$$\text{Ric}(\bar{L}) - \rho \bar{g} - \frac{1}{n - d} \nabla^{\bar{\mathfrak{g}}} \bar{W} \otimes \nabla^{\bar{\mathfrak{g}}} \bar{W} = -(n - d) \alpha^2 H + (d - 2)(1 - \alpha^2)(H - J) + (n - d) \alpha^2 J,$$

where, again, we conflate tensors and their matrices in the chart  $(s, \theta)$ . Hence, letting

$$\mathfrak{B}_{DGZ} = (d - 2) - \alpha^2(n - 2), \quad (30)$$

$$\text{Ric}(\bar{L}) - \rho \bar{g} - \frac{1}{n - d} \nabla^{\bar{\mathfrak{g}}} \bar{W} \otimes \nabla^{\bar{\mathfrak{g}}} \bar{W} = \mathfrak{B}_{DGZ} \begin{pmatrix} 0 & 0 \\ 0 & G_\theta \end{pmatrix}$$

and so,  $\bar{L}$  satisfies the curvature-dimension condition  $CD(\rho, n)$  if and only if  $\mathfrak{B}_{DGZ} \geq 0$ . ▷

*Remark 3.1.* Since the matrix  $H - J$  depends only on the variable  $\theta$ , when we restrict to functions depending on the variable  $s$  only, the corresponding model *always* satisfies the  $CD(\rho, n)$  condition, regardless of the sign of  $\mathfrak{B}_{DGZ}$ .

In the above proof, we made strong use of the following lemma.

**Lemma 3.2** (Computation of  $Ric(\bar{L})$ ). *We have the following formulae*

$$Ric_{\bar{g}} = \frac{(d-1)\alpha^2}{\varphi^2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{\varphi^2} [(d-2)(1-\alpha^2)\varphi^2 + (d-1)\alpha^2] \begin{pmatrix} 0 & 0 \\ 0 & G_\theta \end{pmatrix}, \quad (31)$$

and

$$\nabla\nabla^{\bar{g}}\bar{W} = (n-d)\alpha^2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + (n-d)\alpha^2 \frac{1-\varphi^2}{\varphi^2} \begin{pmatrix} 0 & 0 \\ 0 & G_\theta \end{pmatrix}, \quad (32)$$

where  $G_\theta$  is the matrix of round metric on the sphere  $\mathbf{S}^{d-1}$ . With the constant  $\mathfrak{B}_{DGZ}$  given in (30), we obtain

$$Ric(\bar{L}) = Ric_{\bar{g}} + \nabla\nabla^{\bar{g}}\bar{W} = \frac{\alpha^2}{\varphi^2} [d-1 + \varphi^2(n-d)] \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{\varphi^2} [\alpha^2(n-1) + \varphi^2\mathfrak{B}_{DGZ}] \begin{pmatrix} 0 & 0 \\ 0 & G_\theta \end{pmatrix}. \quad (33)$$

**Proof**

$\triangleleft$  Let us start with  $Ric_{\bar{g}}$ , which is simply the Ricci tensor of the metric  $\bar{g}$ . Since  $\bar{g} = e^{2\bar{\tau}}\mathfrak{h}$  is conformal to  $\mathfrak{h}$ , we may apply (50) in the appendix to get (with lower indices)

$$Ric_{\bar{g}} = Ric_{\mathfrak{h}} + (\Delta_{\mathfrak{h}}\bar{\tau})\mathfrak{h} + (d-2)(\nabla\nabla^{\mathfrak{h}}\bar{\tau} + \nabla^{\mathfrak{h}}\bar{\tau} \odot_{\mathfrak{h}} \nabla^{\mathfrak{h}}\bar{\tau} - \nabla^{\mathfrak{h}}\bar{\tau} \cdot_{\mathfrak{h}} \nabla^{\mathfrak{h}}\bar{\tau} \mathfrak{h}). \quad (34)$$

Since  $Ric_{g_\theta} = (d-2)g_\theta$ , we have

$$Ric_{\mathfrak{h}} = (d-2)(H - J) = (d-2) \begin{pmatrix} 0 & 0 \\ 0 & G_\theta \end{pmatrix}.$$

Since  $\varphi$  depends only on the variable  $s$ , we have

$$\begin{aligned} \Delta_{\mathfrak{h}}(\bar{\tau}) &= \bar{\tau}'' = \frac{\varphi''}{\varphi} - \left(\frac{\varphi'}{\varphi}\right)^2 = \alpha^2 - \left(\frac{\varphi'}{\varphi}\right)^2, \\ \nabla^{\mathfrak{h}}\bar{\tau} \odot_{\mathfrak{h}} \nabla^{\mathfrak{h}}\bar{\tau} &= \left(\frac{\varphi'}{\varphi}\right)^2 J, \\ \nabla^{\mathfrak{h}}\bar{\tau} &= \frac{\varphi'}{\varphi} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \end{aligned}$$

and

$$(\nabla^{\mathfrak{h}}\bar{\tau} \cdot_{\mathfrak{h}} \nabla^{\mathfrak{h}}\bar{\tau}) \mathfrak{h} = \left(\frac{\varphi'}{\varphi}\right)^2 H.$$

Collecting the four terms and using (34), we get

$$Ric_g = H \left[ d-2 + \alpha^2 - (d-1) \left(\frac{\varphi'}{\varphi}\right)^2 \right] + J(d-2)(\alpha^2 - 1).$$

Since  $\varphi'^2 = \alpha^2(\varphi^2 - 1)$ , the equation can be written,

$$Ric_g = H \frac{(d-1)\alpha^2}{\varphi^2} + (H - J)(d-2)(1 - \alpha^2),$$

which is the desired result.

Let us now compute  $\nabla\nabla^{\bar{g}}\bar{W}$ , the Hessian with respect to the metric  $\bar{g}$ . We have (see (48)),

$$\nabla\nabla^{\bar{g}}\bar{W} = \nabla\nabla^{\mathfrak{h}}\bar{W} + 2\nabla\bar{W} \odot_{\mathfrak{h}} \nabla^{\mathfrak{h}}\bar{\tau} - (\nabla^{\mathfrak{h}}\bar{W} \cdot_{\mathfrak{h}} \nabla^{\mathfrak{h}}\bar{\tau}) \mathfrak{h}.$$

Since  $\bar{W}$  depends only on the variable  $s$ , we easily get that

$$\nabla \nabla^{\bar{g}} \bar{W} = J(n-d) \left[ \alpha^2 - \left( \frac{\varphi'}{\varphi} \right)^2 \right] + 2J(n-d) \left( \frac{\varphi'}{\varphi} \right)^2 - H(n-d) \left( \frac{\varphi'}{\varphi} \right)^2,$$

which is the expected result.  $\triangleright$

*Remark 3.3.* As an immediate consequence of Proposition 1.6, Lemma A.3 and Theorem A, we see that Sobolev's inequality (17) holds (and so Poincaré's inequality (19) too), as soon as (14) holds. Also note that  $f = \frac{\varphi'}{\varphi}$ , seen as a function of the first of the cylindrical coordinates  $(s, \theta)$ , solves

$$-\bar{L}f = n\alpha^2 f$$

and so equality in Poincaré's inequality (19) is achieved by  $f$ . In particular, the constant in Sobolev's inequality (17) is optimal.

In fact, one can do better and prove optimal inequalities in the optimal range of parameters given by the Felli-Schneider condition, as we describe next.

**Proof of Theorem 1.7**

$\triangleleft$  As explained in the introduction, Theorem A remains valid under the following *weaker* integrated form of the curvature-dimension condition (15):

$$\int \left( \bar{\Gamma}_2(f) - \rho \bar{\Gamma}(f) - \frac{1}{n} (\bar{L}f)^2 \right) f^{1-n} d\bar{\mu} \geq 0,$$

for any smooth function  $f$  which is bounded below and above by positive constants. From Lemma 3.6 below, we have

$$\int \left( \bar{\Gamma}_2(f) - \rho \bar{\Gamma}(f) - \frac{1}{n} (\bar{L}f)^2 \right) f^{1-n} d\bar{\mu} \geq \int \left( \|\nabla \nabla^\theta f\|^2 - \frac{1}{n-1} (\Delta_\theta f)^2 + \Gamma^\theta(f) \mathfrak{B}_{DGZ} \right) \varphi^4 f^{1-n} d\bar{\mu}.$$

Now,

$$\|\nabla \nabla_\theta f\|^2 - \frac{1}{n-1} (\Delta_\theta f)^2 = \frac{n-d}{n-1} \|\nabla \nabla_\theta f\|^2 + \frac{d-1}{n-1} \|\nabla \nabla_\theta f\|^2 - \frac{1}{n-1} (\Delta_\theta f)^2 \geq \frac{n-d}{n-1} \|\nabla \nabla_\theta f\|^2,$$

where we used the Cauchy-Schwarz inequality to infer that

$$\|\nabla \nabla_\theta f\|^2 \geq \frac{1}{d-2} (\Delta_\theta f)^2. \quad (35)$$

So,

$$\int \left( \bar{\Gamma}_2(f) - \rho \bar{\Gamma}(f) - \frac{1}{n} (\bar{L}f)^2 \right) f^{1-n} d\bar{\mu} \geq \int \left( \frac{n-d}{n-1} \|\nabla \nabla_\theta f\|^2 + \Gamma^\theta(f) \mathfrak{B}_{DGZ} \right) \varphi^4 f^{1-n} d\bar{\mu}.$$

Since, from (40) and (45),

$$\|\nabla \nabla_\theta f\|^2 = \Gamma_2^\theta(f) - \text{Ric}_{\mathfrak{S}^{d-1}}(\nabla_\theta f, \nabla_\theta f) = \Gamma_2^\theta(f) - (d-2)\Gamma^\theta(f),$$

the inequality becomes

$$\int \left( \bar{\Gamma}_2(f) - \rho \bar{\Gamma}(f) - \frac{1}{n} (\bar{L}f)^2 \right) f^{1-n} d\bar{\mu} \geq \frac{n-d}{n-1} \int \Gamma_2^\theta(f) \varphi^4 f^{1-n} d\bar{\mu} + \left[ \frac{n-d}{n-1} (d-2) + \mathfrak{B}_{DGZ} \right] \int \Gamma^\theta(f) \varphi^4 f^{1-n} d\bar{\mu}.$$

Using the estimate (44) in Lemma 3.8 below, we get

$$\int \left( \bar{\Gamma}_2(f) - \rho \bar{\Gamma}(f) - \frac{1}{n} (\bar{L}f)^2 \right) f^{1-n} d\bar{\mu} \geq \left[ \frac{n-d}{n-1} + \mathfrak{B}_{DGZ} \right] \int \Gamma^\theta(f) \varphi^4 f^{1-n} d\bar{\mu}. \quad (36)$$

That is,

$$\frac{n-d}{n-1} + \mathfrak{B}_{DGZ} \geq 0 \quad (37)$$

implies

$$\int \left( \bar{\Gamma}_2(f) - \rho \bar{\Gamma}(f) - \frac{1}{n} (\bar{L}f)^2 \right) f^{1-n} d\bar{\mu} \geq 0,$$

which is the needed inequality. It remains to study the case of equality. If  $v \in H^1(\bar{\mu})$  is an extremal function for (17), then by standard elliptic theory,  $v$  is smooth in  $\mathbf{R}^d \setminus \{0\}$ , bounded above and below by positive constants, and the function  $f = v^{-\frac{p-2}{2}}$  satisfies

$$\int \left( \bar{\Gamma}_2(f) - \rho \bar{\Gamma}(f) - \frac{1}{n} (\bar{L}f)^2 \right) f^{1-n} d\bar{\mu} = 0,$$

see the proofs of Theorem B in [DGZ20] and Theorem 6.8.3 in [BGL14] for these facts. In particular, if the parameters are such that inequality (37) is strict, it follows from (36) that  $f$  must be a function of  $s$  only. If  $\frac{n-d}{n-1} + \mathfrak{B}_{DGZ} = 0$ , then the estimate (42) in Lemma 3.8 provides the following improvement of (36):

$$\int \left( \bar{\Gamma}_2(f) - \rho \bar{\Gamma}(f) - \frac{1}{n} (\bar{L}f)^2 \right) f^{1-n} d\bar{\mu} \geq \left[ \frac{n-d}{n-1} + \mathfrak{B}_{DGZ} \right] \int \Gamma^\theta(f) \varphi^4 f^{1-n} d\bar{\mu} + \frac{n-d}{n-1} A \int \frac{\Gamma^\theta(f)^2}{f^2} \varphi^4 f^{1-n} d\bar{\mu}.$$

And so again,  $f$  is a function of  $s$  only, provided  $n > d$ . Using this information in (41), we deduce that if  $n > d$ ,

$$\partial_{ss}f + 2\frac{\varphi'}{\varphi}\partial_s f = 0,$$

while for  $n = d$  there must exist some function  $R : \mathbf{R} \rightarrow \mathbf{R}$  s.t.

$$\partial_{ss}f + 2\frac{\varphi'}{\varphi}\partial_s f - \frac{1}{d-1}\Delta_\theta f = 0 \quad \text{and} \quad \partial_s f + \frac{\varphi'}{\varphi}f = R(s) \quad (38)$$

In the former case, this means that  $f(s) = \lambda + \mu \tanh(\alpha s)$ , for some constants  $\lambda, \mu \in \mathbf{R}$  such that  $\lambda > |\mu|$ , since  $f$  is bounded below by a positive constant. In the latter case, the second equation in (38) implies that  $f$  can be written as  $f = \frac{f_1(\theta)}{\varphi(s)} + f_2(s)$ . Plugging this in the first equation implies that  $f_1 + \frac{\Delta_\theta f_1}{d-1}$  is constant i.e.  $f_1 = A_1 + B_1 \psi_2(\theta)$ , where  $A_1, B_1$  are constants and  $\psi_2$  is any eigenfunction of  $-\Delta_\theta$  associated to the eigenvalue  $d-1$ . This implies in turn that  $f_2$  takes the form  $f_2 = -\frac{A_1}{\varphi(s)} + A_3 + A_4 \tanh(s)$ . Summarizing, we have just proved that  $f = \lambda + \mu \varphi_2$  for some constants  $\lambda, \mu$  and some eigenfunction  $\varphi_2$  of  $-\Delta_{\mathbf{S}^d}$  associated to the eigenvalue  $d$  (and written in cylindrical coordinates). Again, we must have  $\lambda > |\mu| \|\varphi_2\|_\infty$  since  $f$  is bounded below by a positive constant.

Conversely, we need to check that  $f^{-\frac{n-2}{2}}$  where  $f(s) = \lambda + \mu \tanh(\alpha s)$  with  $\lambda > |\mu|$  if  $n > d$  (resp.  $f = \lambda + \mu \varphi_2$ ,  $\lambda > |\mu| \|\varphi_2\|_\infty$  if  $n = d$ ) is indeed an extremal function for Sobolev's inequality. Multiplying  $f$  by a constant if necessary, we may assume that  $\int f^{-n} d\mu = 1$ , where  $\mu$  is the normalized measure on the CKN sphere. By direct computation, recalling that  $\tanh(\alpha s)$  if  $n > d$  (resp.  $\varphi_2$  if  $n = d$ ) is an eigenfunction for the operator  $-\bar{L}$  associated to the eigenvalue  $n\alpha^2$ , we find that

$$f \bar{L}f - \frac{n}{2} \bar{\Gamma}(f) = \frac{n\alpha^2}{2} (1 - f^2)$$

This implies in turn that  $v = f^{-\frac{n-2}{2}}$  satisfies  $\int v^p d\mu = 1$  and solves

$$-\frac{4}{n(n-2)\alpha^2} \bar{L}v + v = v^{p-1}$$

Multiplying by  $v$  and integrating by parts, the result follows.  $\triangleright$

### **Proof of Proposition 1.10**

$\triangleleft$  As explained in the introduction, Poincaré's inequality (with constant  $C = \frac{n-1}{pn}$ ) follows from Sobolev's



inequality by linearization i.e. by applying (17) with  $v = 1 + \epsilon f$  and letting  $\epsilon \rightarrow 0$ . Also, Poincaré's inequality (with the same constant  $C$ ) is equivalent to the following integrated curvature-dimension condition

$$\int \left( \Gamma_2(f) - \rho \Gamma(f) - \frac{1}{n} (Lf)^2 \right) d\mu \geq 0.$$

Equality holds in Poincaré's inequality for some function  $f$  if and only if equality holds in the above inequality. So, extremals are characterized exactly as in the case of Sobolev's inequality except in the case  $\alpha^2 = \frac{d-1}{n-1}$ , in which we can no longer use (42) to deduce that  $f$  is radial. Still, we deduce from (41) that

$$\partial_{ss}f + 2\frac{\varphi'}{\varphi}\partial_s f - \frac{1}{n-1}\Delta_\theta f = 0 \quad \text{and} \quad \partial_s f + \frac{\varphi'}{\varphi}f = R(s)$$

The second equation in (38) implies that  $f$  can be written as  $f = \frac{f_1(\theta)}{\varphi(s)} + f_2(s)$ . Plugging this in the first equation implies that  $\alpha^2 f_1 + \frac{\Delta_\theta f_1}{n-1}$  is constant i.e.  $f_1 = A_1 + B_1 \psi_2(\theta)$ , where  $A_1, B_1$  are constants and  $\psi_2$  is any eigenfunction of  $-\Delta_\theta$  associated to the eigenvalue  $\alpha^2(n-1) = d-1$ . This implies in turn that  $f_2$  takes the form  $f_2 = -\frac{A_1}{\varphi(s)} + A_3 + A_4 \tanh(\alpha s)$ . Summarizing, we have just proved that extremals of Poincaré's inequality take the form  $f = \lambda + \mu \tanh(\alpha s) + \nu \frac{\varphi_2(\omega)}{\cosh(\alpha s)}$  for some constants  $\lambda, \mu, \nu$  and some eigenfunction  $\psi_2$  of  $-\Delta_{\mathbf{S}^{d-1}}$ , as desired.  $\triangleright$

*Remark 3.4.* Up to our knowledge, the CKN sphere is the first example where the optimal constants for both the Sobolev and the Poincaré inequalities are explicit functions of  $(\rho, n)$  yet the usual curvature-dimension condition doesn't hold, although the integral version (15) remains true. Beware though that the integral curvature-dimension needed for (and equivalent to) Poincaré's inequality, that is inequality (15) without the weight  $f^{1-n}$ , is in general much weaker, as evidenced by any space for which the Poincaré inequality holds but not the Sobolev inequality, such as, for instance, the Euclidean space equipped with the Gaussian measure.

**Lemma 3.5** ( $\bar{\Gamma}_2$  in the cylindrical chart). *For any  $(a, b) \in \Theta$ , in the cylindrical coordinates, for any smooth function  $f$  on  $(0, \infty) \times \mathbf{S}^{d-1}$ ,*

$$\begin{aligned} \frac{\bar{\Gamma}_2(f)}{\varphi^4} &= (\partial_{ss}f)^2 + \|\nabla \nabla_\theta f\|^2 + 2\Gamma^\theta(\partial_s f) + 2\frac{\varphi'}{\varphi}\partial_{ss}f\partial_s f + 4\frac{\varphi'}{\varphi}\Gamma^\theta(\partial_s f, f) - 2\frac{\varphi'}{\varphi}\partial_s f\Delta_\theta f \\ &\quad (\partial_s f)^2 \left[ d \left( \frac{\varphi'}{\varphi} \right)^2 + \alpha^2 \left( \frac{d-1}{\varphi^2} + n - d \right) \right] + \Gamma^\theta(f) \left( 2 \left( \frac{\varphi'}{\varphi} \right)^2 + \alpha^2 \frac{n-1}{\varphi^2} + \mathfrak{B}_{DGZ} \right), \end{aligned} \quad (39)$$

where  $\|\nabla \nabla_\theta f\|^2$  is the Hilbert-Schmidt norm with respect to the variable  $\theta$ ,  $\Gamma^\theta(f) = |\nabla_\theta f|^2$  the carré du champ operator associated to  $\Delta_\theta$  and the function  $\varphi$  has been defined in (27).

**Proof**

$\triangleleft$  We can use the definition of the  $\Gamma_2$  operator to prove (39). But, since the Ricci curvature of  $\bar{L}$  has been computed in Lemma 3.2, we prefer to use the following Bochner-Lichnerowicz formula,

$$\bar{\Gamma}_2(f) = Ric(\bar{L})(\nabla f, \nabla f) + \|\nabla \nabla^{\bar{g}} f\|^2, \quad (40)$$

where  $\|\nabla \nabla^{\bar{g}} f\|^2$  is the Hilbert-Schmidt norm of the Hessian of  $f$  with respect to the metric  $\bar{g}$  (see for instance [BGL14, P. 71]). From Lemma 3.2, equation (33), we have first

$$\frac{Ric(\bar{L})(\nabla f, \nabla f)}{\varphi^4} = \frac{(\partial_s f)^2}{\varphi^2} \alpha^2 [d-1 + \varphi^2(n-d)] + \frac{\Gamma^\theta(f)}{\varphi^2} [\alpha^2(n-1) + \varphi^2 \mathfrak{B}_{DGZ}].$$

It remains to compute  $\|\nabla \nabla^{\bar{g}} f\|^2$ . From (28) we have  $\bar{g} = \varphi^2 \mathfrak{h} = e^{2\bar{\tau}} \mathfrak{h}$  and so we may apply formula (49) to get

$$\frac{\|\nabla \nabla^{\bar{g}} f\|^2}{\varphi^4} = \|\nabla \nabla^{\mathfrak{h}} f\|^2 + 2\Gamma^{\mathfrak{h}}(\bar{\tau}, \Gamma^{\mathfrak{h}}(f)) + 2\Gamma^{\mathfrak{h}}(f)\Gamma^{\mathfrak{h}}(\bar{\tau}) + (d-2)\Gamma^{\mathfrak{h}}(f, \bar{\tau})^2 - 2\Delta_{\mathfrak{h}} f \Gamma^{\mathfrak{h}}(f, \bar{\tau}).$$

Since  $\mathfrak{h}$  is the standard metric product and  $\tau$  depends only on the variable  $s$ , we have

$$\begin{aligned}\|\nabla\nabla^{\mathfrak{h}}f\|^2 &= (\partial_{ss}f)^2 + \|\nabla\nabla_{\theta}f\|^2 + 2\Gamma^{\theta}(\partial_s f), \\ \Gamma^{\mathfrak{h}}(\bar{\tau}, \Gamma^{\mathfrak{g}}(f)) &= 2\frac{\varphi'}{\varphi}\partial_{ss}f\partial_s f + 2\frac{\varphi'}{\varphi}\Gamma^{\theta}(\partial_s f, f),\end{aligned}$$

$$\Gamma^{\mathfrak{h}}(f) = (\partial_s f)^2 + \Gamma^{\theta}(f), \quad \Gamma^{\mathfrak{h}}(\bar{\tau}) = \left(\frac{\varphi'}{\varphi}\right)^2, \quad \Gamma^{\mathfrak{h}}(f, \bar{\tau}) = \frac{\varphi'}{\varphi}\partial_s f \quad \text{and} \quad \Delta_{\mathfrak{h}}f = \partial_{ss}f + \Delta_{\theta}f.$$

Collecting all the terms, we get

$$\begin{aligned}\frac{\|\nabla\nabla^{\bar{\mathfrak{g}}}f\|^2}{\varphi^4} &= (\partial_{ss}f)^2 + \|\nabla\nabla_{\theta}f\|^2 + 2\Gamma^{\theta}(\partial_s f) + 4\frac{\varphi'}{\varphi}\partial_{ss}f\partial_s f + 4\frac{\varphi'}{\varphi}\Gamma^{\theta}(\partial_s f, f) \\ &\quad + 2\left(\frac{\varphi'}{\varphi}\right)^2 [(\partial_s f)^2 + \Gamma^{\theta}(f)] + (d-2)\left(\frac{\varphi'}{\varphi}\right)^2 (\partial_s f)^2 - 2(\partial_{ss}f + \Delta_{\theta}f)\frac{\varphi'}{\varphi}\partial_s f,\end{aligned}$$

that is

$$\begin{aligned}\frac{\|\nabla\nabla^{\bar{\mathfrak{g}}}f\|^2}{\varphi^4} &= (\partial_{ss}f)^2 + \|\nabla\nabla_{\theta}f\|^2 + 2\Gamma^{\theta}(\partial_s f) + 2\frac{\varphi'}{\varphi}\partial_{ss}f\partial_s f + 4\frac{\varphi'}{\varphi}\Gamma^{\theta}(\partial_s f, f) \\ &\quad + 2\left(\frac{\varphi'}{\varphi}\right)^2 \Gamma^{\theta}(f) + d\left(\frac{\varphi'}{\varphi}\right)^2 (\partial_s f)^2 - 2\Delta_{\theta}f\frac{\varphi'}{\varphi}\partial_s f.\end{aligned}$$

Finally, by using (40), we get the expected formula (39).  $\triangleright$

**Lemma 3.6.** *In the cylindrical chart, for any smooth function  $f$  on  $(0, \infty) \times \mathbf{S}^{d-1}$ ,*

$$\begin{aligned}\frac{1}{\varphi^4}\left(\bar{\Gamma}_2(f) - \rho\bar{\Gamma}(f) - \frac{1}{n}(\bar{L}f)^2\right) &= \frac{n-1}{n}\left(\partial_{ss}f + 2\frac{\varphi'}{\varphi}\partial_s f - \frac{1}{n-1}\Delta_{\theta}f\right)^2 + \|\nabla\nabla^{\theta}f\|^2 - \frac{1}{n-1}(\Delta_{\theta}f)^2 \\ &\quad + 2\Gamma^{\theta}(\partial_s f + \frac{\varphi'}{\varphi}f) + \Gamma^{\theta}(f)\mathfrak{B}_{DGZ}.\end{aligned}\quad (41)$$

**Proof**

$\triangleleft$  In the cylindrical chart, the generator takes the following form, for a smooth function  $f$ :

$$\bar{L}(f) = \varphi^2\left[\partial_{ss}f + (2-n)\frac{\varphi'}{\varphi}\partial_s f + \Delta_{\theta}f\right],$$

and from Lemma 3.5 (formula (39)), we obtain

$$\begin{aligned}\frac{1}{\varphi^4}\left[\bar{\Gamma}_2(f) - \rho\bar{\Gamma}(f) - \frac{1}{n}(\bar{L}f)^2\right] &= \frac{n-1}{n}(\partial_{ss}f)^2 + \|\nabla\nabla_{\theta}f\|^2 - \frac{1}{n}(\Delta_{\theta}f)^2 + 4\frac{\varphi'}{\varphi}\frac{n-1}{n}\partial_{ss}f\partial_s f - \frac{4}{n}\frac{\varphi'}{\varphi}\partial_s f\Delta_{\theta}f \\ &\quad - \frac{2}{n}\partial_{ss}f\Delta_{\theta}f + 2\Gamma^{\theta}(\partial_s f) + 4\frac{\varphi'}{\varphi}\Gamma^{\theta}(\partial_s f, f) \\ &\quad + (\partial_s f)^2\left[\left(\frac{\varphi'}{\varphi}\right)^2\left(d - \frac{(n-2)^2}{n}\right) + (n-d)\alpha^2\frac{\varphi^2-1}{\varphi^2}\right] + \Gamma^{\theta}(f)\left(2\left(\frac{\varphi'}{\varphi}\right)^2 + \mathfrak{B}_{DGZ}\right).\end{aligned}$$

Since  $\varphi'^2 = \alpha^2(\varphi^2 - 1)$ , we get

$$\begin{aligned}\frac{1}{\varphi^4}\left[\bar{\Gamma}_2(f) - \rho\bar{\Gamma}(f) - \frac{1}{n}(\bar{L}f)^2\right] &= \frac{n-1}{n}(\partial_{ss}f)^2 + \|\nabla\nabla_{\theta}f\|^2 - \frac{1}{n}(\Delta_{\theta}f)^2 + 4\frac{\varphi'}{\varphi}\frac{n-1}{n}\partial_{ss}f\partial_s f - \frac{4}{n}\frac{\varphi'}{\varphi}\partial_s f\Delta_{\theta}f \\ &\quad + 4\frac{n-1}{n}\left(\frac{\varphi'}{\varphi}\right)^2 (\partial_s f)^2 - \frac{2}{n}\partial_{ss}f\Delta_{\theta}f + \Gamma^{\theta}(f)\left(2\left(\frac{\varphi'}{\varphi}\right)^2 + \mathfrak{B}_{DGZ}\right) + 2\Gamma^{\theta}(\partial_s f) + 4\frac{\varphi'}{\varphi}\Gamma^{\theta}(\partial_s f, f).\end{aligned}$$

Formula (41) follows then easily since  $\varphi$  depends only on the variable  $s$ .  $\triangleright$

*Remark 3.7.* By the Cauchy-Schwarz inequality,

$$\|\nabla\nabla_{\theta}f\|^2 \geq \frac{1}{d-1}(\Delta_{\theta}f)^2,$$

and so

$$\|\nabla\nabla_{\theta}f\|^2 - \frac{1}{n-1}(\Delta_{\theta}f)^2 \geq \frac{n-d}{(d-1)(n-1)}(\Delta_{\theta}f)^2 \geq 0$$

since  $n \geq d$ . We recover from (41) that, under the condition  $\mathfrak{B}_{DGS} \geq 0$ , the generator  $\bar{L}$  satisfies the  $CD(\rho, n)$  curvature-dimension condition.

**Lemma 3.8.** *For any smooth function  $f$  bounded below and above by positive constants on  $(0, \infty) \times \mathbf{S}^{d-1}$ ,*

$$\int \Gamma_2^{\theta}(f)\varphi^4 f^{1-n}d\bar{\mu} \geq (d-1) \int \Gamma^{\theta}(f)\varphi^4 f^{1-n}d\bar{\mu} + A \int \frac{\Gamma^{\theta}(f)^2}{f^2}\varphi^4 f^{1-n}d\bar{\mu}, \quad (42)$$

where

$$A = \frac{n-1}{4(d+1)^2}(n(4d-5) + 3(4d+7)), \quad (43)$$

in particular,

$$\int \Gamma_2^{\theta}(f)\varphi^4 f^{1-n}d\bar{\mu} \geq (d-1) \int \Gamma^{\theta}(f)\varphi^4 f^{1-n}d\bar{\mu}. \quad (44)$$

**Proof**

◁ In the cylindrical chart, we have

$$\int \Gamma_2^{\theta}(f)f^{1-n}d\bar{\mu} = \int_0^{\infty} \varphi^{4-n} \int_{\mathbf{S}^{d-1}} \Gamma_2^{\theta}(f)f^{1-n}d\theta ds.$$

The operator  $\Delta_{\theta}$  is the Laplace-Beltrami operator on the  $(d-1)$ -dimensional sphere, therefore, it satisfies the  $CD(d-2, d-1)$  condition. Moreover,  $Ric_{\mathfrak{g}_{\theta}}$  (the Ricci tensor of  $\Delta_{\theta}$ ) satisfies

$$Ric_{\mathfrak{g}_{\theta}}(\nabla_{\theta}f, \nabla_{\theta}f) = (d-2)\Gamma^{\theta}(f). \quad (45)$$

In [GZ21, P. 767], it is proved that under the  $CD(K, m)$  condition, for a general operator  $L$  associated to the measure  $\mu$ , and the operators  $\Gamma$  and  $\Gamma_2$ , one has for any  $\chi \in \mathbf{R}$ ,

$$\int h^q \Gamma_2(h)d\mu \geq \frac{Km}{m-1} \int h^q \Gamma(h)d\mu + \int [Ah^{q-2}\Gamma(h)^2 + Bh^{q-1}\Gamma(h, \Gamma(h))]d\mu$$

where

$$\begin{cases} A = \frac{q(q-1)}{m-1} - \chi^2 - 2\chi \frac{q-1}{m-1}, \\ B = \frac{1}{m-1} \left( \frac{3q}{2} - \chi(m+2) \right). \end{cases}$$

Apply the previous inequality to our operator  $\Delta_{\theta}$  with parameters  $q = 1-n$ ,  $K = d-2$ ,  $m = d-1$  and  $\chi = \frac{3q}{2(m+2)}$  so that  $B = 0$ . We obtain

$$\begin{aligned} \int \Gamma_2^{\theta}(f)\varphi^4 f^{1-n}d\bar{\mu} &= \int_0^{\infty} \varphi^{4-n} \int_{\mathbf{S}^{d-1}} \Gamma_2^{\theta}(f)f^{1-n}d\theta ds \\ &\geq (d-1) \int_0^{\infty} \varphi^{4-n} \int_{\mathbf{S}^{d-1}} \Gamma^{\theta}(f)f^{1-n}d\theta ds + A \int_0^{\infty} \varphi^{4-n} \int_{\mathbf{S}^{d-1}} \frac{\Gamma^{\theta}(f)^2}{f^2} f^{1-n}d\theta ds \\ &\geq (d-1) \int \Gamma^{\theta}(f)\varphi^4 f^{1-n}d\bar{\mu} + A \int \frac{\Gamma^{\theta}(f)^2}{f^2}\varphi^4 f^{1-n}d\bar{\mu}. \end{aligned}$$

where  $A$  is given by (43) after a straightforward computation. In particular,  $A \geq 0$  and the estimation (44) follows. ▷

## 4 The $n$ -conformal invariant

### 4.1 The $n$ -conformal invariant on a weighted manifold

We begin this section by proving Proposition 1.12, which constructs a one-parameter family of  $n$ -conformal invariants on any given weighted manifold, thereby generalizing the notion of scalar curvature to this setting.

**Proof of Proposition 1.12**

◁ We want to check that  $S_\gamma(\mu, \Gamma)$  satisfies condition (20). Let  $c$  be a positive and smooth function on  $M$ ,  $\tau = \log c$  and  $\gamma \in \mathbf{R}$ . We are looking for the expression of the two numbers  $\theta_n(\gamma)$  and  $\beta_n(\gamma)$  in the definition of  $S_\gamma(\mu, \Gamma)$  which are such that

$$S_\gamma(c^{-n}\mu, c^2\Gamma) = c^2 \left[ S_\gamma(\mu, \Gamma) + \frac{n-2}{2} \left( L\tau - \frac{n-2}{2} \Gamma(\tau) \right) \right].$$

The measure  $\mu$  is transformed into  $\check{\mu} = c^{-n}\mu$ , and the carré du champ  $\Gamma$  into  $\check{\Gamma} = c^2\Gamma$ .

From (51),  $sc_{\mathfrak{g}}$  becomes

$$\check{sc}_{\mathfrak{g}} = c^2[sc_{\mathfrak{g}} + (d-1)(2\Delta_{\mathfrak{g}}\tau - (d-2)\Gamma(\tau))],$$

the weight  $W = -\log \frac{d\mu}{dV_{\mathfrak{g}}}$  becomes

$$\check{W} = -\log \frac{d\check{\mu}}{d\check{V}_{\mathfrak{g}}} = -\log \frac{c^{-n}d\mu}{c^{-d}dV_{\mathfrak{g}}} = -\log \left( c^{d-n} \frac{d\mu}{dV_{\mathfrak{g}}} \right) = W + (n-d)\tau,$$

and finally, from (47),  $\Delta_{\mathfrak{g}}$  becomes

$$\check{\Delta}_{\mathfrak{g}} = c^2[\Delta_{\mathfrak{g}} - (d-2)\Gamma(\tau, \cdot)].$$

So,

$$\begin{aligned} S_\gamma(c^{-n}\mu, c^2\Gamma) &= c^2\theta_n(\gamma) \left[ sc_{\mathfrak{g}} + (d-1)(2\Delta_{\mathfrak{g}}(\tau) - (d-2)\Gamma(\tau)) \right. \\ &\quad \left. - \gamma[\Delta_{\mathfrak{g}}(W + (n-d)\tau) - (d-2)\Gamma(\tau, W + (n-d)\tau)] + \beta_n(\gamma)\Gamma(W + (n-d)\tau) \right] \end{aligned}$$

that is

$$\begin{aligned} S_\gamma(c^{-n}\mu, c^2\Gamma) &= c^2\theta_n(\gamma) \left[ sc_{\mathfrak{g}} + [2(d-1) - \gamma(n-d)]\Delta_{\mathfrak{g}}(\tau) \right. \\ &\quad \left. + [\beta_n(\gamma)(n-d)^2 - (d-1)(d-2) + \gamma(d-2)(n-d)]\Gamma(\tau) \right. \\ &\quad \left. - \gamma\Delta_{\mathfrak{g}}(W) + [\gamma(d-2) + 2\beta_n(\gamma)(n-d)]\Gamma(\tau, W) + \beta_n(\gamma)\Gamma(W) \right]. \end{aligned}$$

It has to be equal to

$$\begin{aligned} c^2 \left[ S_\gamma(\mu, \Gamma) + \frac{n-2}{2} \left( \Delta_{\mathfrak{g}}(\tau) - \Gamma(W, \tau) - \frac{n-2}{2} \Gamma(\tau) \right) \right] &= \\ c^2 \left[ \theta_n(\gamma)[sc_{\mathfrak{g}} - \gamma\Delta_{\mathfrak{g}}(W) + \beta_n(\gamma)\Gamma(W)] + \frac{n-2}{2} \left( \Delta_{\mathfrak{g}}(\tau) - \Gamma(W, \tau) - \frac{n-2}{2} \Gamma(\tau) \right) \right], \end{aligned}$$

that is

$$\begin{cases} \theta_n(\gamma)[2(d-1) - \gamma(n-d)] = \frac{n-2}{2} \\ \theta_n(\gamma)[\beta_n(\gamma)(n-d)^2 - (d-1)(d-2) + \gamma(d-2)(n-d)] = -\frac{(n-2)^2}{4} \\ \theta_n(\gamma)[\gamma(d-2) + 2\beta_n(\gamma)(n-d)] = -\frac{n-2}{2} \end{cases} \quad (46)$$

which implies that

$$\begin{cases} \theta_n(\gamma) = \frac{n-2}{4(d-1) - 2\gamma(n-d)} \\ \beta_n(\gamma) = \frac{\gamma(n-2d+2) - 2(d-1)}{2(n-d)}. \end{cases}$$

Let us notice that the second equation in (46) is automatically valid for this choice of parameters  $\theta_n(\gamma)$  and  $\beta_n(\gamma)$  and so we are done.  $\triangleright$

*Remark 4.1.* As explained in the introduction, when  $W = 0$  the  $d$ -conformal invariant is, up to a multiplicative constant, the scalar curvature. In a weighted Riemannian manifold, the  $n$ -conformal invariant is given by (22) and is a way to extend the definition of the scalar curvature in the weighted case.

## 4.2 The $n$ -conformal invariant for the CKN spaces

In this section, we would like to prove that the three CKN spaces enjoy, for some  $\gamma \in \mathbf{R}$ , a *constant*  $n$ -conformal invariant. By construction, the three CKN models (Euclidean, spherical and hyperbolic) belong to the same  $n$ -conformal class. So, in virtue of Theorem 1.4 and Proposition 1.12, it suffices to prove that there exists a unique  $\gamma \in \mathbf{R}$  such that  $S_\gamma = 0$  for the Euclidean CKN space in order to prove Proposition 1.13.

### **Proof of Proposition 1.13**

$\triangleleft$  Let  $\gamma \in \mathbf{R}$ . Then,

$$S_\gamma(\hat{\mu}, \hat{\Gamma}) = \theta_n(\gamma)(sc_{\hat{g}} - \gamma\Delta_{\hat{g}}\widehat{W} + \beta_n(\gamma)\hat{\Gamma}(\widehat{W})).$$

So, we need to find  $\gamma$  such that  $sc_{\hat{g}} - \gamma\Delta_{\hat{g}}\widehat{W} + \beta_n(\gamma)\hat{\Gamma}(\widehat{W}) = 0$ .

- Computation of the scalar curvature. From the identity (51),

$$sc_{\hat{g}} = |x|^{2(1-\alpha)}(0 + (d-1)(2\Delta\hat{\tau} - (d-2)|\nabla\hat{\tau}|^2),$$

hence,

$$sc_{\hat{g}} = |x|^{-2\alpha}(d-1)(d-2)(1-\alpha^2).$$

- Computation of  $\Delta_{\hat{g}}\widehat{W}$ . First, from the identity (47),

$$\Delta_{\hat{g}}\widehat{W} = |x|^{2(1-\alpha)\kappa}(\Delta\widehat{W} - (d-2)\nabla\hat{\tau} \cdot \nabla\widehat{W}),$$

so

$$\Delta_{\hat{g}}\widehat{W} = |x|^{-2\alpha}(d-2)\alpha^2(n-d).$$

- Computation of  $\hat{\Gamma}(\widehat{W})$ . We have

$$\hat{\Gamma}(\widehat{W}) = |x|^{-2\alpha}\alpha^h a^2(n-d)^2.$$

So, in the end,

$$S_\gamma(\mu, \Gamma) = \theta_n(\gamma)|x|^{-2\alpha}((d-1)(d-2)(1-\alpha^2) + \gamma\alpha^2(n-d)(d-2) + \beta_n(\gamma)\alpha^2(n-d)^2),$$

and we need to find  $\gamma \in \mathbf{R}$  such that

$$(d-1)(d-2)(1-\alpha^2) + \gamma\alpha^2(n-d)(d-2) + \beta_n(\gamma)\alpha^2(n-d)^2 = 0.$$

Since

$$\beta_n(\gamma) = \frac{\gamma(n-2d+2) - 2(d-1)}{2(n-d)}$$

we have

$$\gamma = 2 \frac{(d-1)(d-2 + \alpha^2(2-n))}{\alpha^2(n-d)(2-n)}.$$

or by using the constant  $\mathfrak{B}_{DGZ}$ ,

$$\gamma = \frac{2(d-1)}{\alpha^2(n-d)(2-n)} \mathfrak{B}_{DGZ}.$$

$\triangleright$

*Remark 4.2.* It is interesting to notice that the  $n$ -conformal invariant for the CKN spaces does not depend on the sign of  $\mathfrak{B}_{DGZ}$  or the Felli-Schneider region.

# A Appendix

## A.1 Some Riemannian formulas

We recall here some general formulas on conformal transformations of a  $d$ -dimensional Riemannian manifold  $(M, \mathbf{g})$ . All formulas can be found for example in [BGL14, Sec. 6.9]. We transform the metric (with upper indices)  $\mathbf{g}^{ij}$  into the conformal metric  $\mathbf{h}^{ij} = c^2 \mathbf{g}^{ij}$ , where  $c$  is any positive and smooth function. We let  $\tau = \log c$ . Then,

- The carré du champ operator is given by

$$\Gamma^{\mathbf{h}} = c^2 \Gamma^{\mathbf{g}}.$$

- The Laplace-Beltrami operator is given by

$$\Delta_{\mathbf{h}} = c^2 (\Delta_{\mathbf{g}} - (d-2) \Gamma^{\mathbf{g}}(\tau, \cdot)). \quad (47)$$

- For any smooth function  $\psi$ , the Hessian of  $\psi$  with respect to the metric  $\mathbf{h}$ , denoted  $\nabla \nabla^{\mathbf{h}} \psi$  is given by

$$\nabla \nabla^{\mathbf{h}} \psi = \nabla \nabla^{\mathbf{g}} \psi + 2 \nabla^{\mathbf{g}} \psi \odot_{\mathbf{g}} \nabla^{\mathbf{g}} \tau - \Gamma^{\mathbf{g}}(\psi, \tau) \mathbf{g}, \quad (48)$$

Here and below,  $\nabla \nabla^{\mathbf{g}} \psi$  is the Hessian of  $\psi$  with respect to  $\mathbf{g}$  and  $\nabla^{\mathbf{g}} \psi \odot_{\mathbf{g}} \nabla^{\mathbf{g}} \tau$  is the symmetric tensor product, that is for any functions  $f, g$ ,

$$(\nabla^{\mathbf{g}} \psi \odot_{\mathbf{g}} \nabla^{\mathbf{g}} \tau)(\nabla^{\mathbf{g}} f, \nabla^{\mathbf{g}} g) = \frac{1}{2} [\Gamma^{\mathbf{g}}(f, \psi) \Gamma^{\mathbf{g}}(g, \tau) + \Gamma^{\mathbf{g}}(f, \tau) \Gamma^{\mathbf{g}}(g, \psi)].$$

In particular, one can deduce the Hilbert-Schmidt norm of  $W = \nabla \nabla^{\mathbf{h}} \psi$  with respect to the new metric  $\mathbf{h}$ :

$$\|\nabla \nabla^{\mathbf{h}} \psi\|^2 = c^4 \left[ \|\nabla \nabla^{\mathbf{g}} \psi\|^2 + 2 \Gamma^{\mathbf{h}}(\tau, \Gamma^{\mathbf{g}}(\psi)) + 2 \Gamma^{\mathbf{g}}(\psi) \Gamma^{\mathbf{g}}(\tau) + (d-2) \Gamma^{\mathbf{g}}(\psi, \tau)^2 - 2 (\Delta_{\mathbf{g}} \psi) \Gamma^{\mathbf{g}}(\psi, \tau) \right]. \quad (49)$$

- The Ricci curvature reads

$$Ric_{\mathbf{h}} = Ric_{\mathbf{g}} + (\Delta_{\mathbf{g}} \tau) \mathbf{g} + (d-2) (\nabla \nabla^{\mathbf{g}} \tau + \nabla \tau \odot_{\mathbf{g}} \nabla \tau - \Gamma^{\mathbf{g}}(\tau) \mathbf{g}) \quad (50)$$

- At last, the scalar curvature is given by

$$sc_{\mathbf{h}} = c^2 [sc_{\mathbf{g}} + (d-1) (2 \Delta_{\mathbf{g}} \tau - (d-2) \Gamma^{\mathbf{g}}(\tau))]. \quad (51)$$

## A.2 The triple $(\mathbf{R}^d \setminus \{0\}, \bar{\Gamma}, \bar{\mu})$ is a full Markov triple

Using the notions in Section 3.4.3 of [BGL14], we define the algebra  $\mathcal{A}_0 = C_c^\infty(\mathbf{R}^d \setminus \{0\})$  and extended algebra  $\mathcal{A} = C^\infty(\mathbf{R}^d \setminus \{0\})$ . The main points necessary to prove that  $(\mathbf{R}^d \setminus \{0\}, \bar{\Gamma}, \bar{\mu})$  is a full Markov triple are the following

1. The integration by parts formula

$$\int -\bar{L} u v d\bar{\mu} = \int \bar{\Gamma}(u, v) d\bar{\mu} = \int -\bar{L} v u d\bar{\mu},$$

has to hold for any  $u \in \mathcal{A}$ ,  $v \in \mathcal{A}_0$ . That point is straightforward.

2. We need to check that the associated semi-group  $P_t$  sends  $\mathcal{A}_0$  into  $\mathcal{A}$  for  $t \in \mathbf{R}_+$ , which follows from standard parabolic regularity theory.
3. Lastly, we need to prove essential self-adjointness of the operator  $\bar{L}$ , which we detail in the next series of lemmas.

### A.2.1 $\bar{L}$ is essentially self-adjoint

Let  $H$  denote the closure of  $C_c^\infty(\mathbf{R}^d \setminus \{0\})$  with respect to the norm

$$\|u\|_H^2 = \int (\bar{\Gamma}(u) + u^2) d\bar{\mu}$$

Let  $u \in L^2(\bar{\mu})$ . Then,  $u \in \mathcal{D}'(\mathbf{R}^d \setminus \{0\})$ . And so  $|x|^{1-\alpha} \frac{1+|x|^{2\alpha}}{2} \nabla u$  and  $\bar{L}u$  are well-defined distributions on  $\mathbf{R}^d \setminus \{0\}$  and we may ask whether they are actual functions in  $L^2(\bar{\mu})$ , that is, we may consider  $\bar{L}$  as an unbounded operator in  $L^2(\bar{\mu})$  with domain

$$D(\bar{L}) = \{u \in H : \bar{L}u \in L^2(\bar{\mu})\}.$$

**Lemma A.1.** *Let  $u, v \in D(\bar{L})$ . Then,*

$$\int -\bar{L}u v d\bar{\mu} = \int \bar{\Gamma}(u, v) d\bar{\mu} \tag{52}$$

**Proof**

◁ Assume first that  $u, v \in C_c^\infty(\mathbf{R}^d \setminus \{0\})$ . Then, (52) follows by standard integration by parts. Next, if  $u \in D(\bar{L})$  and  $v \in C_c^\infty(\mathbf{R}^d \setminus \{0\})$ , take  $u_n \in C_c^\infty(\mathbf{R}^d \setminus \{0\})$  s.t.  $u_n \rightarrow u$  in  $H$ . Using successively the definition of distributional derivatives, the convergence  $u_n \rightarrow u$  in  $L^2(\bar{\mu})$ , standard integration by parts and the convergence  $u_n \rightarrow u$  in  $H$ , we find

$$\int -\bar{L}u v d\bar{\mu} = \int u(-\bar{L}v) d\bar{\mu} = \lim_{n \rightarrow +\infty} \int u_n(-\bar{L}v) d\bar{\mu} = \lim_{n \rightarrow +\infty} \int \bar{\Gamma}(u_n, v) d\bar{\mu} = \int \bar{\Gamma}(u, v) d\bar{\mu}$$

Finally, if  $u \in D(\bar{L})$  and if  $v \in D(\bar{L})$ , take  $v_n \in C_c^\infty(\mathbf{R}^d \setminus \{0\})$  s.t.  $v_n \rightarrow v$  in  $H$ . Then, according to what we just proved,

$$\int -\bar{L}u v d\bar{\mu} = \lim_{n \rightarrow +\infty} \int -\bar{L}u v_n d\bar{\mu} = \lim_{n \rightarrow +\infty} \int \bar{\Gamma}(u, v_n) d\bar{\mu} = \int \bar{\Gamma}(u, v) d\bar{\mu}$$

▷

**Lemma A.2.** *Let  $f \in L^2(\bar{\mu})$ . There exists a unique function  $u \in D(\bar{L})$  s.t. for every  $v \in D(\bar{L})$ ,*

$$\int u(-\bar{L}v + v) d\bar{\mu} = \int f v d\bar{\mu}$$

**Proof**

◁ By the Riesz representation theorem, there exists a unique function  $u \in H$  s.t. for every  $v \in H$

$$\int (\bar{\Gamma}(u, v) + uv) d\bar{\mu} = \int f v d\bar{\mu}$$

Taking  $v \in C_c^\infty(\mathbf{R}^d \setminus \{0\})$ , we deduce that  $-\bar{L}u = f - u \in L^2(\bar{\mu})$  and so  $u \in D(\bar{L})$ . We may then apply the integration by parts formula (52). ▷

We can now prove the following lemma.

**Lemma A.3.**  *$\bar{L}$  is a self-adjoint operator in  $L^2(\bar{\mu})$ .*

**Proof**

◁ Take first  $u \in D(\bar{L})$ . By (52), we have for any  $v \in D(\bar{L})$ ,

$$\left| \int u(-\bar{L}v) d\bar{\mu} \right| = \left| \int (-\bar{L}u)v d\bar{\mu} \right| \leq \|\bar{L}u\|_{L^2(\bar{\mu})} \|v\|_{L^2(\bar{\mu})}$$

and so  $u \in D(\bar{L}^*)$  and  $\bar{L}^*u = \bar{L}u$ . Now, take  $u \in D(\bar{L}^*)$  i.e. there exists  $C > 0$  s.t. for any  $v \in D(\bar{L})$ ,

$$\left| \int u(-\bar{L}v) d\bar{\mu} \right| \leq C \|v\|_{L^2(\bar{\mu})}$$

Hence, there exists a unique function  $f \in L^2(\bar{\mu})$  s.t. for any  $v \in D(\bar{L})$ ,

$$\int u(-\bar{L}v) d\bar{\mu} = \int f v d\bar{\mu}$$

Letting  $g = f + u \in L^2(\bar{\mu})$ , this can be rewritten

$$\int u(-\bar{L}v + v) d\bar{\mu} = \int g v d\bar{\mu}$$

By Lemma A.2, there exist a unique function  $\tilde{u} \in D(\bar{L})$  s.t. for all  $v \in D(\bar{L})$ ,

$$\int \tilde{u}(-\bar{L}v + v) d\bar{\mu} = \int g v d\bar{\mu}$$

and so

$$\int (\tilde{u} - u)(-\bar{L}v + v) d\bar{\mu} = 0$$

Take at last  $\psi \in L^2(\bar{\mu})$ . Applying Lemma A.2 again, we find  $v \in D(\bar{L})$  s.t.  $(-\bar{L}v + v) = \psi$ . Choosing  $\psi = u - \tilde{u}$ , it follows that  $\tilde{u} = u$  and so  $u \in D(\bar{L})$ .

▷

### A.3 List of constants and regions of parameters

We recall in this section the definition of the parameters and also some useful properties. Recall that  $d \in \mathbf{N}$  is the topological dimension of the considered spaces, and that we assume that  $d \geq 3$ . Recall from the introduction the definition of the parameter range

$$\Theta = \{(a, b) \in \mathbf{R}^2, a \leq b \leq a + 1, a < a_c\},$$

where  $a_c = (d - 2)/2$ . This is the set of parameters  $(a, b)$  where the CKN inequality (1) holds for all test functions  $v \in C_c^\infty(\mathbf{R}^d)$  which need not vanish near the origin. We also defined the number  $\alpha = 1 + a - \frac{pb}{2}$ , that is

$$\alpha = \frac{(a_c - a)(a + 1 - b)}{a_c - a + b}.$$

Clearly  $\alpha \geq 0$ , for any  $(a, b) \in \Theta$ , including the limiting case  $a = b = 0$  for which  $\alpha = 1$ . For any  $(a, b) \in \Theta$ , the exponent  $p$  is given by

$$p = \frac{d}{a_c - a + b} < 2^* = \frac{2d}{d - 2}$$

and

$$p = \frac{2n}{n - 2},$$

that is

$$n = d + \frac{d(b - a)}{1 + a - b} = \frac{d}{1 + a - b}.$$

We always have  $n \geq d$ , and we call  $n$  the intrinsic dimension of the considered model spaces. From a straightforward computation, we have

$$d - n\alpha - pb = 0 \tag{53}$$

The constant  $\mathfrak{B}_{DGZ} = \alpha^2(2 - n) + d - 2$  which appears throughout the paper takes the following form with respect to  $a$  and  $b$ :

$$\mathfrak{B}_{DGZ} = -2 \frac{(a_c - a)^2(1 + a - b)}{a_c + b - a} + 2a_c$$



Let us also recall the definition of the Felli-Schneider region: for  $a \leq 0$ ,

$$b_{FS}(a) = \frac{d(a_c - a)}{2\sqrt{(a_c - a)^2 + d - 1}} + a - a_c, \quad \text{and } \Theta_{FS} = \{(a, b) \in \Theta, b \geq b_{FS}(a) \text{ if } a \leq 0\}$$

Let us prove Lemma 1.9, which simplifies the expression of the Felli-Schneider region and shows its relation to our region  $\Theta_{DGZ} = \{(a, b) \in \Theta, \mathfrak{B}_{DGZ} \geq 0\}$ .

**Proof of Lemma 1.9**

◁ Since

$$\mathfrak{B}_{DGZ} + \frac{n-d}{n-1} = (n-2) \left( -\alpha^2 + \frac{d-1}{n-1} \right),$$

we have

$$\left\{ (a, b) \in \Theta, \mathfrak{B}_{DGZ} + \frac{n-d}{n-1} \geq 0 \right\} = \left\{ (a, b) \in \Theta, \alpha^2 \leq \frac{d-1}{n-1} \right\}.$$

The fact that

$$\Theta_{FS} = \left\{ (a, b) \in \Theta, \alpha^2 \leq \frac{d-1}{n-1} \right\}$$

is more delicate and is proved in [DEL16, Sec. 3].

Finally, the identity  $\Theta_{FS} = \{(a, b) \in \Theta, \alpha \in [0, 1]\}$  is a little trickier. Since  $(a, b) \in \Theta$ , then

$$0 \leq \alpha = \frac{(a_c - a)(a + 1 - b)}{a_c - a + b} \leq 1 - \frac{a}{a_c}. \quad (54)$$

In the case where  $a \geq 0$  it follows that  $0 \leq \alpha \leq 1$ . Assume now that  $a < 0$ . By definition of  $\alpha$ ,

$$b = \frac{(a_c - a)(a + 1 - \alpha)}{\alpha + a_c - a}.$$

Then the inequality  $b \geq \mathfrak{B}_{FS}$  is equivalent to

$$\frac{1 + a_c}{\alpha + a_c - a} \geq \frac{d}{2\sqrt{(a_c - a)^2 + d - 1}}.$$

Since  $1 + a_c = d/2$ , the previous condition becomes

$$a \geq \frac{\alpha^2 + \alpha(d-2) - d + 1}{2\alpha} = \frac{(\alpha - 1 + d)(\alpha - 1)}{2\alpha}.$$

If  $\alpha > 1$ , this inequality implies  $a > 0$ , contradicting our assumption. We just proved that  $\Theta_{FS} \subset \{(a, b) \in \Theta, \alpha \in [0, 1]\}$ . Since  $\alpha = 0$  for  $(a, b) = (0, 1)$ ,  $\alpha = 1$  for  $(a, b) = (0, 0)$ , since  $\Theta$  is connected and since  $\alpha$  depends continuously on  $(a, b)$ , we deduce that  $\Theta_{FS} = \{(a, b) \in \Theta, \alpha \in [0, 1]\}$ . At last, if  $\alpha = 1$ , then, from the previous displayed inequality,  $a \geq 0$  and from (54),  $a \leq 0$ . That is,  $a = b = 0$ . ▷

*Remark A.4.* The normalizing constant  $Z$  defined in Theorem 1.7 is finite.

Indeed, using (53) and the change of variable  $|x| = e^t$ , we find

$$Z = \int d\bar{\mu} = \int_{\mathbf{R}^d \setminus \{0\}} \left( \frac{2}{1 + |x|^{2\alpha}} \right)^n |x|^{-bp} dx = \int_{\mathbf{R}^d \setminus \{0\}} \left( \frac{2}{|x|^{-\alpha} + |x|^\alpha} \right)^n |x|^{-d} dx = |\mathbf{S}^{d-1}| \int_{\mathbf{R}} \cosh(\alpha t)^{-n} dt < \infty$$

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