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# THE EXT-ALGEBRA OF THE BRAUER TREE ALGEBRA ASSOCIATED TO A LINE

OLIVIER DUDAS

Abstract. We compute the $\mathsf{Ext}$-algebra of the Brauer tree algebra associated to a line with no exceptional vertex.

## Introduction

This note provides a detailed computation of the $\mathsf{Ext}$-algebra for a very specific finite dimensional algebra, namely a Brauer tree algebra associated to a line, with no exceptional vertex. Such algebras appear for example as the principal $p$-block of the symmetric group $\mathfrak{S}_p$, and in a different context, as blocks of the Verlinde categories $\mathsf{Ver}_{p^2}$ studied by Benson–Etingof in [2] (our computation is actually motivated by [2, Conj. 1.3]).

Let us emphasise that $\mathsf{Ext}$-algebras for more general biserial algebras were explicitly computed by Green–Schroll–Snashall–Taillefer in [4], but under some assumption on the multiplicity of the vertices, assumption which is not satisfied for the simple example treated in this note. Other general results relying on Auslander–Reiten theory were obtained by Antipov–Generalov [1] and Brown [3]. However we did not manage to use their work to get an explicit description in our case. Nevertheless, the simple structure of the projective indecomposable modules for the line allows a straightforward approach using explicit projective resolutions of simple modules. The Poincaré series for the $\mathsf{Ext}$-algebra is given in Proposition 2.2theorem.2.2 and its structure as a path algebra with relations is given in Proposition 3.2theorem.3.2.

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## 1. Notation

Let $\mathbb{F}$ be a field, and $A$ be a self-injective finite dimensional $\mathbb{F}$-algebra. All $A$-modules will be assumed to be finitely generated. Given an $A$-module $M$, we denote by $\Omega(M)$ the kernel of a projective cover $P \twoheadrightarrow M$. Up to isomorphism it

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does not depend on the cover. We then define inductively $\Omega^n(M) = \Omega(\Omega^{n-1}(M))$ for $n \geq 2$.

To compute the extension groups between simple modules we will use the property that

$$\mathsf{Ext}^n_A(M, S) \simeq \mathsf{Hom}_A(\Omega^n(M), S)$$

for any simple $A$-module $S$ and any $n \geq 1$.

For computing the algebra structure on the various $\mathsf{Ext}$-groups it will be convenient to work in the homotopy category $\mathsf{Ho}(A)$ of the complexes of finitely generated $A$-modules. If $S$ (resp. $S'$) is a simple $A$-module, and $P_\bullet \to S$ (resp. $P'_\bullet \to S'$) is a projective resolution then

$$\mathsf{Ext}^n_A(S, S') \simeq \mathsf{Hom}_{\mathsf{Ho(A)}}(P_\bullet, P'_\bullet[n])$$

with the Yoneda product being given by the composition of maps in $\mathsf{Ho}(A)$.

Assume now that $A$ is an $\mathbb{F}$-algebra associated to the following Brauer tree with $N+1$ vertices:

Here, unlike in [4] we assume that there is no exceptional vertex. The edges are labelled by the simple $A$-modules $S_1, \ldots, S_N$. We will denote by $P_1, \ldots, P_N$ the corresponding indecomposable projective $A$-modules. The head and socle of $P_i$ are isomorphic to $S_i$ and $\mathsf{rad}(P_i)/S_i \simeq S_{i-1} \oplus S_{i+1}$ with the convention that $S_0 = S_{N+1} = 0$.

Given $1 \leq i \leq N-1$ we fix non-zero maps $f_i : P_i \longrightarrow P_{i+1}$ and $f_i^* : P_{i+1} \longrightarrow P_i$ such that $f_i^* \circ f_i + f_{i-1} \circ f_{i-1}^* = 0$ for all $2 \leq i \leq N-1$. This is possible since $f_i^* \circ f_i$ and $f_{i-1} \circ f_{i-1}^*$ are two non-zero elements of the Jacobson radical of $\mathrm{End}(P_i)$, which is isomorphic to $\mathbb{F}$. It follows that the algebra $A$ is Morita equivalent to the path algebra of the quiver

$$\begin{tikzcd} P_1 \arrow[r, bend left, "f_1"] & P_2 \arrow[l, bend left, "f_1^*"] \arrow[r, bend left, "f_2"] & P_3 \arrow[l, bend left, "f_2^*"] & \cdots & P_{N-2} \arrow[r, bend left, "f_{N-2}"] & P_{N-1} \arrow[l, bend left, "f_{N-2}^*"] \arrow[r, bend left, "f_{N-1}"] & P_N \arrow[l, bend left, "f_{N-1}^*"] \end{tikzcd}$$

subject to the relations $f_i^* \circ f_i + f_{i-1} \circ f_{i-1}^* = 0$ for all $2 \leq i \leq N-1$.

## 2. Ext-groups

Given $1 \leq i \leq j \leq N$ with $i - j$ even, there is, up to isomorphism, a unique non-projective indecomposable module ${}^i\mathsf{X}^j$ such that

- $\mathsf{rad}({}^i\mathsf{X}^j) = S_{i+1} \oplus S_{i+3} \oplus \cdots \oplus S_{j-1}$
- $\mathsf{hd}({}^i\mathsf{X}^j) = S_i \oplus S_{i+2} \oplus \cdots \oplus S_j$.

In particular we have ${}^i\mathsf{X}^i = S_i$. The structure of ${}^i\mathsf{X}^j$ can be represented by the following diagram:

$$
{}^i\mathsf{X}^j = \begin{array}{ccccccccccc}
S_i & & S_{i+2} & & S_{i+4} & \cdots & S_{j-2} & & S_j \\
& \diagdown & & \diagdown & & \cdots & & \diagdown & \\
& & \diagup & & \diagup & & & & \\
& S_{i+1} & & S_{i+3} & & \cdots & & S_{j-1} &
\end{array}
$$

Similarly we denote by ${}_i\mathsf{X}_j$ the unique indecomposable module with the following structure:

$$
{}_i\mathsf{X}_j = \begin{array}{ccccccccccc}
& S_{i+1} & & S_{i+3} & & \cdots & & S_{j-1} & \\
& & \diagdown & & \diagdown & & \cdots & & \\
& \diagup & & \diagup & & & & \diagup & \\
S_i & & S_{i+2} & & S_{i+4} & \cdots & S_{j-2} & & S_j
\end{array}
$$

Note that ${}_i\mathsf{X}_i = S_i = {}^i\mathsf{X}^i$. Finally, in the case where $i-j$ is odd we define the modules ${}_i\mathsf{X}^j$ and ${}^i\mathsf{X}_j$ as the indecomposable modules with the following respective structure:

$$
{}_i\mathsf{X}^j = \begin{array}{ccccccccc}
& S_{i+1} & & S_{i+3} & & \cdots & & S_j \\
& & \diagdown & & \diagdown & & \cdots & & \\
& \diagup & & \diagup & & & & \diagup & \\
S_i & & S_{i+2} & & S_{i+4} & \cdots & S_{j-1} &
\end{array}
$$

$$
{}^i\mathsf{X}_j = \begin{array}{ccccccccc}
S_i & & S_{i+2} & & S_{i+4} & \cdots & S_{j-1} \\
& \diagdown & & \diagdown & & \cdots & & \diagdown \\
& & \diagup & & \diagup & & & \\
& S_{i+1} & & S_{i+3} & & \cdots & & S_j
\end{array}
$$

For convenience we will extend the notation ${}^i\mathsf{X}^j$, ${}_i\mathsf{X}_j$, ${}_i\mathsf{X}^j$ and ${}^i\mathsf{X}_j$ to any integers $i, j \in \mathbb{Z}$ (with the suitable parity condition on $i-j$) so that the following relations hold:

$$
{}^i\mathsf{X} = {}_{1-i}\mathsf{X}, \qquad {}^i\mathsf{X}^j = {}_j\mathsf{X}_i, \qquad {}^{i\pm 2N}\mathsf{X} = {}^i\mathsf{X}. \tag{1}
$$

Note that this also implies $\mathsf{X}^j = \mathsf{X}_{1-j}$, $\mathsf{X}^{j\pm 2N} = \mathsf{X}^j$ and ${}^i\mathsf{X}_j = {}_j\mathsf{X}^i$.

**Lemma 2.1.** *Let $i, j \in \mathbb{Z}$ with $i-j$ even. Then*

$$
\Omega({}^i\mathsf{X}^j) \simeq {}^{i-1}\mathsf{X}^{j+1}.
$$

*Proof.* Since ${}^i\mathsf{X}^j \simeq {}^{i\pm 2N}\mathsf{X}^{j\pm 2N}$ we can assume that both $i$ and $j$ are in $\{-N+1, \dots, N\}$. If $i \leq 0$ then $1-i \in \{1, \dots, N\}$, but $1-(i-1) = (1-i)+1$. Similarly if $j \leq 0$ then $1-j \in \{1, \dots, N\}$, but $1-(j+1) = (1-j)-1$. Therefore using the relations (1Ext-groupsequation.2.1) it is enough to prove that for $1 \leq k \leq l \leq N$ we have the following isomorphisms

$$
\Omega({}^k\mathsf{X}^l) \simeq {}^{k-1}\mathsf{X}^{l+1}, \quad \Omega({}_k\mathsf{X}^l) \simeq {}_{k+1}\mathsf{X}^{l+1}, \quad \Omega({}^k\mathsf{X}_l) \simeq {}^{k-1}\mathsf{X}_{l-1}, \quad \Omega({}_k\mathsf{X}_l) \simeq {}_{k+1}\mathsf{X}_{l-1}.
$$

We only consider the first one, the others are similar. If $1 \leq k \leq l \leq N$ a projective cover of ${}^k\mathsf{X}^l$ is given by $P_k \oplus P_{k+2} \oplus \cdots \oplus P_l \twoheadrightarrow {}^k\mathsf{X}^l$, whose kernel equals ${}^{k-1}\mathsf{X}^{l+1}$.

Note that this holds even when $k = 1$ since ${}^{0}\mathsf{X}^{l+1} = {}_{1}\mathsf{X}^{l+1}$ or when $l = N$ since ${}^{k-1}\mathsf{X}^{N+1} = {}^{k-1}\mathsf{X}^{-N+1} = {}^{k-1}\mathsf{X}_{N}$. $\square$

We deduce from Lemma 2.1theorem.2.1 that for any simple module $S_i$ and for all $k \geq 0$ we have

$$\Omega^k(S_i) = \Omega^k({}^{i}\mathsf{X}^{i}) \simeq {}^{i-k}\mathsf{X}^{i+k}$$

as $A$-modules. Consequently we have

$$\mathsf{Ext}^k_A(S_i, S_j) = \begin{cases} \mathbb{F} & \text{if } S_j \text{ appears in the head of } {}^{i-k}\mathsf{X}^{i+k}, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

From this description one can compute explicitly the Poincaré series of the $\mathsf{Ext}$-groups.

**Proposition 2.2.** *Given $1 \leq i, j \leq N$, the Poincaré series of $\mathsf{Ext}^\bullet_A(S_i, S_j)$ is given by*

$$\sum_{k\geq 0} \dim_{\mathbb{F}} \mathsf{Ext}^k_A(S_i, S_j)t^k = \frac{Q_{i,j}(t) + t^{2N-1}Q_{i,j}(t^{-1})}{1 - t^{2N}}$$

*where $Q_{i,j}(t) = t^{|j-i|} + t^{|j-i|+2} + \cdots + t^{N-1-|N+1-j-i|}$.*

*Proof.* First observe that

$$\Omega^N(S_i) = {}^{i-N}\mathsf{X}^{i+N} = {}_{1+N-i}\mathsf{X}_{1-N-i} = {}_{1+N-i}\mathsf{X}_{1+N-i} = S_{N+1-i}.$$

Then for all $k \geq 0$ we have $\mathsf{Ext}^k_A(S_i, S_j) = \mathsf{Ext}^k_A(S_{N+1-i}, S_{N+1-j})$. Moreover, $Q_{N+1-i,N+1-j} = Q_{i,j} = Q_{j,i}$ so that it is enough to prove the lemma under the assumption that $i \leq j$.

Now, assume that $i \leq j$ and let $k \in \{0, \ldots, N-1\}$. If $i + j \leq N + 1$, the simple module $S_j$ appears in the head of ${}^{i-k}\mathsf{X}^{i+k}$ if and only if $k = j-i, j-i+2, \ldots, j+i-2$. The limit cases are indeed ${}^{2i-j}\mathsf{X}^{j}$ for $k = j - i$ and ${}^{2-j}\mathsf{X}^{2i+j-2} = {}_{j-1}\mathsf{X}^{2i+j-2}$ for $k = j+i-2$. Note that if $j-i \leq k \leq i+j-2$ then $j \leq i+k$ and $j \leq 2N-i-k$ so that $S_j$ appears in the head of ${}^{i-k}\mathsf{X}^{i+k} = {}^{i-k}\mathsf{X}_{2N-i-k+1}$ whenever $k$ has the suitable parity. If $i+j > N+1$ one must ensure that $j \leq 2N-i-k$ and therefore $S_j$ appears in the head of ${}^{i-k}\mathsf{X}^{i+k}$ if and only if $k = j-i, j-i+2, \ldots, 2N-i-j$. Consequently, using the description of the $\mathsf{Ext}$-groups given in (2Ext-groupsequation.2.2) we have

$$\begin{aligned} \sum_{k=0}^{N-1} \dim_{\mathbb{F}} \mathsf{Ext}^k_A(S_i, S_j)t^k &= t^{j-i} + t^{j-i+2} + \cdots + t^{N-1-|N+1-j-i|} \\ &= t^{|j-i|} + t^{|j-i|+2} + \cdots + t^{N-1-|N+1-j-i|} \\ &= Q_{i,j}(t). \end{aligned} \tag{3}$$

Using the relation $\Omega^N(S_i) = S_{N+1-i}$ we obtain

$$\sum_{k=0}^{2N-1} \dim_{\mathbb{F}} \mathsf{Ext}^k_A(S_i, S_j)t^k = \sum_{k=0}^{N-1} \dim_{\mathbb{F}} \mathsf{Ext}^k_A(S_i, S_j)t^k + t^N \sum_{k=0}^{N-1} \dim_{\mathbb{F}} \mathsf{Ext}^k_A(S_{N+1-i}, S_j)t^k$$

which by (3Ext-groupsequation.2.3) equals $Q_{i,j}(t)+Q_{N+1-i,j}(t)$. Since $Q_{N+1-i,j}(t) = t^{N-1}Q_{i,j}(t^{-1})$ we finally get

$$\sum_{k=0}^{2N-1} \mathsf{dim}_{\mathbb{F}}\, \mathsf{Ext}_A^k(S_i,S_j)t^k = Q_{i,j}(t) + t^{N-1}Q_{i,j}(t^{-1})$$

and we conclude using the fact that $\mathsf{Ext}_A^{k+2N}(S_i,S_j) = \mathsf{Ext}_A^k(S_i,S_j)$. □

## 3. Algebra structure

We denote by $\mathsf{E}(A)$ the $\mathsf{Ext}$-algebra of $A$, that is the graded algebra

$$\mathsf{E}(A) := \bigoplus_{1\leq i,j\leq N} \mathsf{Ext}_A^\bullet(S_i,S_j)$$

endowed with the Yoneda product. We will give in Proposition 3.2theorem.3.2 a description of $\mathsf{E}(A)$ as the path algebra of a quiver with relations.

3.1. **Generation.** Let $1 \leq i,j \leq N$ and let $k \geq 1$. Assume that there is a non-zero map between $\Omega^k S_i$ and $S_j$, so that $S_j$ appears in the head of $\Omega^k S_i \simeq {}^{i-k}\mathsf{X}^{i+k}$. If $k \geq N$, any map between $\Omega^k S_i$ and $S_j$ factors through the (unique up to a scalar) isomorphism $\Omega^N S_{N+1-j} \xrightarrow{\sim} S_j$. If $0 < k < N$, one can use the relations (1Ext-groupsequation.2.1) to see that the module ${}^{i-k}\mathsf{X}^{i+k}$ is not simple. It follows from its structure that at least one of $S_{j-1}$ or $S_{j+1}$ appears in the socle. Consequently, any map between $\Omega^k S_i$ and $S_j$ will factor through a map $\Omega S_{j-1} \longrightarrow S_j$ (if $S_{j-1}$ appears in the socle of ${}^{i-k}\mathsf{X}^{i+k}$) or $\Omega S_{j+1} \longrightarrow S_j$ (if $S_{j+1}$ appears in the socle of ${}^{i-k}\mathsf{X}^{i+k}$). This shows that $\mathsf{E}(A)$ is generated in degree 1 and $N$ as a left module over itself, hence as an algebra.

3.2. **Minimal resolution.** Recall from §1Notationsection.1 that we have chosen non-zero maps $f_i : P_i \longrightarrow P_{i+1}$ and $f_i^* : P_{i+1} \longrightarrow P_i$ such that $f_i^* \circ f_i + f_{i-1} \circ f_{i-1}^* = 0$ for all $2 \leq i \leq N-1$. Given $1 \leq i \leq j \leq N$ with $j-i$ even we denote by ${}_iP_j$ the following projective $A$-module

$${}_iP_j := P_i \oplus P_{i+2} \oplus \cdots \oplus P_{j-2} \oplus P_j.$$

For $1 \leq i < j \leq N$ with $j-i$ even we let $d_{i,j} : {}_iP_j \longrightarrow {}_{i+1}P_{j-1}$ be the morphism of $A$-modules corresponding to the following matrix:

$$d_{i,j} = \begin{bmatrix} f_i & f_{i+1}^* & 0 & \cdots & \cdots & 0 \\ 0 & f_{i+2} & f_{i+3}^* & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & f_{j-2} & f_{j-1}^* \end{bmatrix}$$

The definition of ${}_iP_j$ extends to any integers $i,j \in \mathbb{Z}$ with the convention that

$${}_iP_j = {}_{j+1}P_{i-1}, \quad {}_iP_{-j} = {}_iP_j, \quad {}_iP_{j\pm 2N} = {}_iP_j. \tag{4}$$

Note that these relations imply ${}_{1-i}P_j = {}_{1+i}P_j$ and ${}_{i\pm 2N}P_j = {}_iP_j$. Furthermore, the definition of $d_{i,j}$ extends naturally to any pair $i, j$ if we set in addition

$$d_{i,i} = (-1)^i f_i^* \circ f_i = (-1)^{i-1} f_{i-1} \circ f_{i-1}^*,$$

a map from ${}_iP_i = P_i$ to ${}_{i+1}P_{i-1} = P_i$. With this notation one checks that for all $k > 0$ the image of the map $d_{i-k,i+k} : {}_{i-k}P_{i+k} \longrightarrow {}_{i-k+1}P_{i+k-1}$ is isomorphic to ${}^{i-k}\mathsf{X}^{i+k} \simeq \Omega^k(S_i)$ and its kernel to ${}^{i-k-1}\mathsf{X}^{i+k+1} \simeq \Omega^{k+1}(S_i)$ so that the bounded above complex

$$R_i := \cdots \xrightarrow{d_{i-k-1,i+k+1}} {}_{i-k}P_{i+k} \xrightarrow{d_{i-k,i+k}} \cdots \xrightarrow{d_{i-2,i+2}} {}_{i-1}P_{i+1} \xrightarrow{d_{i-1,i+1}} P_i \longrightarrow 0$$

forms a minimal projective resolution of $S_i$.

3.3. **Generators and relations.** We have seen in Section 3.1Generationsubsection.3.1 that the $\mathsf{Ext}$-algebra is generated in degrees 1 and $N$. Here we will construct explicit generators using the minimal resolutions defined above.

We start by defining a map $z_i \in \mathsf{Hom}_{\mathsf{Ho}(A)}(R_i, R_{i+1}[1])$ for any $1 \leq i \leq N-1$. Let $k$ be a positive integer. If $k \notin N\mathbb{Z}$, the projective modules ${}_{i-k}P_{i+k}$ and ${}_{i+1-(k-1)}P_{i+1+(k-1)} = {}_{i-k+2}P_{i+k}$ have at least one indecomposable summand in common and we can consider the map $Z_{i,k} : {}_{i-k}P_{i+k} \longrightarrow {}_{i-k+2}P_{i+k}$ given by the identity map on the common factors, followed by multiplication by $(-1)^k$. If $k \in N + 2N\mathbb{Z}$ then from the relations (4Minimal resolutionequation.3.4) we have

$${}_{i-k}P_{i+k} = {}_{i-N}P_{i+N} = {}_{i+N+1}P_{i-N-1} = {}_{-i-N+1}P_{-i+N+1} = P_{N+1-i}$$

and

$${}_{i-k+2}P_{i+k} = {}_{i-N+2}P_{i+N} = {}_{N-i}P_{-N-i} = P_{N-i}.$$

In that case we set $Z_{i,k} := (-1)^i f_{N-i}^*$. If $k \in 2N\mathbb{Z}$ then ${}_{i-k}P_{i+k} = P_i$, ${}_{i-k+2}P_{i+k} = {}_{i+2}P_i = P_{i+1}$ and we set $Z_{i,k} := (-1)^i f_i$. If $k \geq 0$ we set $Z_{i,k} := 0$. Then the family of morphisms of $A$-modules $Z_i := (Z_{i,k})_{k\in\mathbb{Z}}$ defines a morphism of complexes of $A$-modules from $R_i$ to $R_{i+1}[1]$ and we denote by $z_i$ its image in $\mathsf{Ho}(A)$. Note that $z_i$ is non-zero; indeed, the composition of $Z_i$ with the the natural map $R_{i+1}[1] \longrightarrow S_{i+1}[1]$ is already not null-homotopic since $\mathsf{Ext}^1_A(S_i, S_{i+1}) \neq 0$.

Similarly we define a map $Z_i^* : R_{i+1} \longrightarrow R_i[1]$ by exchanging the role of $f$ and $f^*$. More precisely we consider in that case $Z^*_{i,-N} := (-1)^i f_{N-i}$ and $Z^*_{i,-2N} := (-1)^i f_i^*$. We denote by $z_i^*$ the image of $Z_i^*$ in $\mathsf{Ho}(A)$.

Assume now that $1 \leq i \leq N$. The modules ${}_{i-k}P_{i+k}$ and ${}_{(N+1-i)-(k-N)}P_{(N+1-i)+(k-N)}$ are equal, which means that starting from the degree $-N$, the terms of the complexes $R_i$ and $R_{N+1-i}[N]$ coincide. In addition, the differentials only differ by $(-1)^N$. We denote by $Y_i : R_i \longrightarrow R_{N+1-i}[N]$ the natural projection between $R_i$ and its obvious truncation at degrees $\leq -N$, followed by the multiplication by $(-1)^{Nk}$ in each degree $k$. We will write $y_i$ for its image in $\mathsf{Ho}(A)$. Again, $y_i$ is non-zero since $\mathsf{Ext}^N_A(S_i, S_{N+1-i}) \neq 0$.

**Lemma 3.1.** *The following relations hold in* $\mathrm{End}^\bullet_{\mathsf{Ho}(A)}(\bigoplus R_i)$*:*

(a) $z_1^*[1] \circ z_1 = 0$, $z_{N-1}[1] \circ z_{N-1}^* = 0$*;*
(b) $z_i[1] \circ z_i^* = z_{i+1}^*[1] \circ z_{i+1}$ *for all* $i = 1, \ldots, N-2$*;*

(c) $y_{i+1}[1] \circ z_i = z^*_{N-i}[N] \circ y_i$ *for all* $i = 1, \dots, N-1$*;*
(d) $y_i[1] \circ z^*_i = z_{N-i}[N] \circ y_{i+1}$ *for all* $i = 1, \dots, N-1$.

*Proof.* If $N = 1$ there are no relation to check. Note that in that case the algebra $A$ is isomorphic to $\mathbb{F}[t]/(t^2)$. It is a Koszul algebra whose dual is isomorphic to $\mathbb{F}[t]$. Therefore we assume $N \geq 2$. The relations in (a) follow from the fact that $\mathsf{Ext}^2_A(S_1, S_1) = \mathsf{Ext}^2_A(S_N, S_N) = 0$, which is for example a consequence of Proposition 2.2theorem.2.2.

To show (c), we observe that the morphism of complexes $Z_i : R_i \longrightarrow R_{i+1}[1]$ defined above coincide with $Z^*_{N-i}[N] : R_{N+1-i}[N] \longrightarrow R_{N-i}[N+1]$ in degrees less than $-N$. Since $Y_i$ and $Y_{i+1}$ are just obvious truncations with suitable signs we actually have $Y_{i+1}[1] \circ Z_i = Z^*_{N-i}[N] \circ Y_i$. The relation (d) is obtained by a similar argument.

We now consider (b). The morphism of complexes $Z_i[1] \circ Z^*_i$ and $Z^*_{i+1}[1] \circ Z_{i+1}$ coincide at every degree $k$ except when $k$ is congruent to 0 or $-1$ modulo $N$. Let us first look in details at the degrees $-N$ and $-N-1$. The map $Z_i[1] \circ Z^*_i$ is as follows:

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & P_{N-1-i} \oplus P_{N+1-i} \arrow[r, "\begin{bmatrix} f_{N-1-i} & f^*_{N-i} \end{bmatrix}"] \arrow[d, "\begin{bmatrix} 0 & (-1)^{N+1} \end{bmatrix}"] & P_{N-i} \arrow[r, "(-1)^{N-i} f^*_{N-i} \circ f_{N-i}"] \arrow[d, "(-1)^i f_{N-i}"] & P_{N-i} \arrow[d, "\begin{bmatrix} (-1)^{N-1} \\ 0 \end{bmatrix}"] \\
P_{N-i} \oplus P_{N+2-i} \arrow[r, "-\begin{bmatrix} f_{N-i} & f^*_{N+1-i} \end{bmatrix}"] \arrow[d, "\begin{bmatrix} (-1)^{N+1} & 0 \end{bmatrix}"] & P_{N+1-i} \arrow[r, "(-1)^{N-i+1} f_{N-i} \circ f^*_{N-i}"] \arrow[d, "(-1)^i f^*_{N-i}"] & P_{N+1-i} \arrow[r, "-\begin{bmatrix} f^*_{N-i} \\ f_{N+1-i} \end{bmatrix}"] \arrow[d, "\begin{bmatrix} 0 \\ (-1)^{N-1} \end{bmatrix}"] & P_{N-i} \oplus P_{N+2-i} \arrow[d] \\
P_{N-i} \arrow[r, "(-1)^{N-i} f^*_{N-i} \circ f_{N-i}"] & P_{N-i} \arrow[r, "\begin{bmatrix} f^*_{N-1-i} \\ f_{N-i} \end{bmatrix}"] & P_{N-1-i} \oplus P_{N+1-i} \arrow[r] & \cdots
\end{tikzcd}$$

whereas the map $Z_{i+1}^*[1] \circ Z_{i+1}$ corresponds to the following composition:

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & P_{N-1-i} \oplus P_{N+1-i} \arrow[r, "\begin{bmatrix} f_{N-1-i} & f_{N-i}^* \end{bmatrix}"] \arrow[d, "\begin{bmatrix} (-1)^{N+1} & 0 \end{bmatrix}"] & P_{N-i} \arrow[r, "(-1)^{N-i} f_{N-i}^* \circ f_{N-i}"] \arrow[d, "(-1)^{i+1} f_{N-1-i}^*"] & P_{N-i} \arrow[d, "\begin{bmatrix} 0 \\ (-1)^{N-1} \end{bmatrix}"] \\
P_{N-2-i} \oplus P_{N-i} \arrow[r, "-\begin{bmatrix} f_{N-2-i} & f_{N-1-i}^* \end{bmatrix}"] \arrow[d, "\begin{bmatrix} 0 & (-1)^{N+1} \end{bmatrix}"] & P_{N-1-i} \arrow[r, "(-1)^{N-i} f_{N-1-i}^* \circ f_{N-1-i}"] \arrow[d, "(-1)^{i+1} f_{N-1-i}"] & P_{N-1-i} \arrow[r, "-\begin{bmatrix} f_{N-2-i}^* \\ f_{N-1-i} \end{bmatrix}"] \arrow[d, "\begin{bmatrix} (-1)^{N-1} \\ 0 \end{bmatrix}"] & P_{N-2-i} \oplus P_{N-i} \arrow[d] \\
P_{N-i} \arrow[r, "(-1)^{N-i} f_{N-i}^* \circ f_{N-i}"] & P_{N-i} \arrow[r, "\begin{bmatrix} f_{N-1-i}^* \\ f_{N-i} \end{bmatrix}"] & P_{N-1-i} \oplus P_{N+1-i} \arrow[r] & \cdots
\end{tikzcd}$$

We deduce that at the degrees $-N$ and $-N-1$ the map $Z_i[1] \circ Z_i^* - Z_{i+1}^*[1] \circ Z_{i+1}$ is given by

$$\begin{tikzcd}
P_{N-1-i} \oplus P_{N+1-i} \arrow[r, "\begin{bmatrix} f_{N-1-i} & f_{N-i}^* \end{bmatrix}"] \arrow[d, "(-1)^{N+1-i} \begin{bmatrix} f_{N-1-i} & f_{N-i}^* \end{bmatrix}"'] & P_{N-i} \arrow[d, "(-1)^{N+1-i} \begin{bmatrix} f_{N-1-i}^* \\ f_{N-i} \end{bmatrix}"] \\
P_{N-i} \arrow[r, "\begin{bmatrix} f_{N-1-i}^* \\ f_{N-i} \end{bmatrix}"] & P_{N-1-i} \oplus P_{N+1-i}
\end{tikzcd}$$

A similar picture holds at the degrees $-2N$ and $-2N-1$:

$$\begin{tikzcd}
P_i \oplus P_{i+2} \arrow[r, "\begin{bmatrix} f_i & f_{i+1}^* \end{bmatrix}"] \arrow[d, "(-1)^{i+1} \begin{bmatrix} f_i & f_{i+1}^* \end{bmatrix}"'] & P_{i+1} \arrow[d, "(-1)^{i+1} \begin{bmatrix} f_i^* \\ f_{i+1} \end{bmatrix}"] \\
P_{i+1} \arrow[r, "\begin{bmatrix} f_i^* \\ f_{i+1} \end{bmatrix}"] & P_i \oplus P_{i+2}
\end{tikzcd}$$

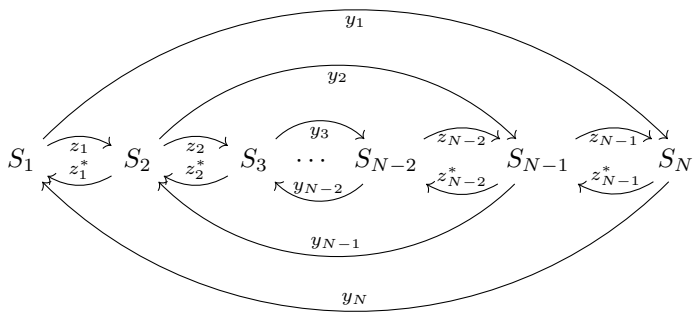
Using the map $s : R_{i+1} \to R_{i+1}[1]$ defined by

$$s_k := \begin{cases} (-1)^{N+1-i} \mathrm{Id}_{P_{N-i}} & \text{if } -k \in N + 2N\mathbb{N}, \\ (-1)^{i+1} \mathrm{Id}_{P_{i+1}} & \text{if } -k \in 2N + 2N\mathbb{N}, \\ 0 & \text{otherwise,} \end{cases}$$

we see that $Z_i[1] \circ Z_i^* - Z_{i+1}^*[1] \circ Z_{i+1}$ is null-homotopic, which proves that $z_i[1] \circ z_i^* - z_{i+1}^*[1] \circ z_{i+1}$ is zero in $\mathsf{Hom}_{\mathsf{Ho}(A)}(R_{i+1}, R_{i+1}[2])$. □

The next proposition shows that the relations given in Lemma 3.1theorem.3.1 are actually enough to describe the $\mathsf{Ext}$-algebra. We use here the concatenation of paths as opposed to the composition of arrows, which explains the discrepancy in the relations.

**Proposition 3.2.** *The* $\mathsf{Ext}$*-algebra of* $A$ *is isomorphic to the path algebra associated with the following quiver*




*with* $z_i$*'s of degree* 1 *and* $y_i$*'s of degree* $N$*, subject to the relations*

(a) $z_1 z_1^* = z_{N-1}^* z_{N-1} = 0$*;*
(b) $z_i^* z_i = z_{i+1} z_{i+1}^*$ *for all* $i = 1, \dots, N-2$*;*
(c) $z_i y_{i+1} = y_i z_{N-i}^*$ *for all* $i = 1, \dots, N-1$*;*
(d) $z_i^* y_i = y_{i+1} z_{N-i}$ *for all* $i = 1, \dots, N-1$.

*Proof.* Let $Q$ (resp. $I$) be the quiver (resp. the ideal generated by the set of relations) given in the proposition. Let $\Gamma = \mathbb{F}Q/I$ be the corresponding path algebra. By Section 3.1Generationsubsection.3.1 and Lemma 3.1theorem.3.1, the $\mathsf{Ext}$-algebra $\mathsf{E}(A)$ of $A$ is a quotient of $\Gamma$. To show that $\mathsf{E}(A) \simeq \Gamma$ it is enough to show that the graded dimension of $\Gamma$ is smaller than that of $\mathsf{E}(A)$.

Let $1 \leq i, j \leq N$ and $\gamma$ be a path between $S_i$ and $S_j$ in $Q$ containing only $z_l$'s and $z_l^*$'s. Let $k$ be the length of $\gamma$. We have $k \geq |i-j|$, which is the length of the minimal path from $S_i$ to $S_j$. Using the relations, there exist cycles $\gamma_1$ and $\gamma_2$ around $S_i$ and $S_j$ respectively such that

$$\gamma = \begin{cases} \gamma_1 z_i z_{i+1} \cdots z_{j-1} = z_i z_{i+1} \cdots z_{j-1} \gamma_2 & \text{if } i \leq j; \\ \gamma_1 z_{i-1}^* z_{i-2}^* \cdots z_j^* = z_{i-1}^* z_{i-2}^* \cdots z_j^* \gamma_2 & \text{otherwise.} \end{cases}$$

Maximal non-zero cycles starting and ending at $S_i$ are either $z_{i-1}^* z_{i-2}^* \cdots z_1^* z_1 z_2 \cdots z_{i-1}$ or $z_i z_{i+1} \cdots z_{N-1} z_{N-1}^* \cdots z_{i+1}^* \cdots z_i^*$ depending on whether $S_i$ is closer to $S_1$ or $S_N$. Indeed, any longer cycle will involve $z_1 z_1^*$ or $z_{N-1}^* z_{N-1}$, which are zero by (a). Therefore if $\mathsf{deg}(\gamma_1) > 2(i-1)$ or $\mathsf{deg}(\gamma_1) > 2(N-i)$ then $\gamma_1 = 0$. Using a similar

argument for cycles around $S_j$ we deduce that $\gamma$ is zero whenever

$$k = \mathsf{deg}(\gamma) > |i-j| + 2\,\mathsf{min}(i-1, j-1, N-i, N-j)$$

which is equivalent to $k = \mathsf{deg}(\gamma) > N-1-|N+1-j-i|$. This proves that $\gamma$ is zero unless $|i-j| \leq k \leq N-1-|N+1-j-i|$ in which case it equals

$$\gamma = z_i z_{i+1} \cdots z_{r-1} z_{r-1}^* z_{r-2}^* \cdots z_j^*$$

where $k = 2r-i-j$. In particular, $k-|i-j|$ must be even. Consequently, the subspace of $\Gamma$ spanned by such paths has graded dimension at most equal to $t^{|i-j|} + t^{|i-j|+2} + \cdots + t^{N-1-|N+1-j-i|} = Q_{i,j}(t)$.

Assume now that $\gamma$ is any path of length $k$ between $S_i$ and $S_j$ in $Q$. Using the relations one can write $\gamma$ as $\gamma = y_i^a \gamma_1 \gamma_2$ where $\gamma_2$ is a cycle around $S_j$ containing only $y_l$'s (therefore a power of $y_j y_{N-j}$), $\gamma_1$ is a product of $z_l$'s and $a \in \{0,1\}$. Note that $\mathsf{deg}(\gamma_2)$ is a multiple of $2N$ and $\gamma_1$ is either a path from $S_i$ to $S_j$ if $a=0$ or a path from $S_{N+1-i}$ to $S_j$ if $a=1$. From the previous discussion and Proposition 2.2theorem.2.2 we conclude that $\gamma$ is zero if $\mathsf{dim}_{\mathbb{F}}\, \mathsf{Ext}_A^k(S_i, S_j) = 0$ or unique modulo $I$ otherwise. By (2Ext-groupsequation.2.2) and §3.1Generationsubsection.3.1 this shows that the projection of $\Gamma$ to the $\mathsf{Ext}$-algebra of $A$ must be an isomorphism. □

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