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Barrier Function-Based Adaptive Higher Order Sliding Mode Controllers ⋆

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Abstract

In this paper, we present barrier function-based adaptive controllers for fast stabilization of a perturbed chain of integrators with bounded uncertainties. We refer to such controllers as adaptive higher order sliding mode controllers since they are designed for a perturbed chain of integrators of length r with bounded uncertainties such that the uncertainty bounds are unknown. We propose an adaptive control feedback law insuring the following: We provide a barrier function based adaptive controller ensuring the finite-time convergence of solutions to a desired vicinity of origin. Moreover, the homogeneity norm of solutions asymptotically tends to zero. The effectiveness of these controllers is illustrated through simulations.

Key words: Adaptive sliding modes ; Barrier function ; Higher order sliding modes.

1 Introduction

Sliding Mode Control is an efficient tool for matched uncertainties compensation. Higher Order Sliding Mode Controllers (HOSMCs) Levant (2001, 2003, 2005); Ding et al. (2016); Cruz-Zavala & Moreno (2017); Basin et al. (2016); Levant & Fridman (2010) allow to collapse in finite time the dynamics of the flat single input single output of order r. The homogeneity properties of discontinuous HOSMCs ensure the r -th order of asymptotic precision of the output with respect to the sampling step and parasitic dynamics Levant (2001, 2003, 2005); Ding et al. (2016); Cruz-Zavala & Moreno (2017); Levant & Fridman (2010).
To implement discontinuous HOSMCs, the knowledge of upper bounds for r -th derivatives is needed and usually such bounds are unknown. Moreover, the overestimation of HOSMCs leads to big chattering and energy losses Pèrez-Ventura & Fridman (2019). That is why the authors of the works devoted to HOSMCs adaptation Plestan et al. (2010); Bartolini et al. (2013); Moreno et al. (2016); Edwards & Shtessel (2016a,b); Oliveira et al. (2018); Hsu et al. (2018); Negrete-Chávez & Moreno (2016); Shtessel et al. (2012, 2017); Obeid et al. (2018); Hsu et al. (2019); Ferrara et al. (2019); Basin et al. (2016) should satisfy two contradictory requirements:

- ensure a finite-time exact convergence to the origin;
- avoid overestimation of the control gain.

Recently, adaptive sliding mode controllers have at-

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tracted the interest of many researchers dealing with adaptation of HOSMCs. The general goal of these techniques is to ensure a dynamical adaptation of the control gains in order to be as small as possible while still sufficient to counteract the disturbances and ensure a sliding mode. The adaptive sliding mode control approaches which exist in the literature can be broadly split into the following three classes:

a) approaches based on the usage of equivalent control value Edwards & Shtessel (2016a,b); Oliveira et al. (2018); Utkin & Poznyak (2013);
b) approaches based on monotonically increasing the gains (see, for example, Moreno et al. (2016); Negrete-Chávez & Moreno (2016));
c) approaches based on increasing and decreasing the gains Plestan et al. (2010); Bartolini et al. (2013); Shtessel et al. (2017); Plestan et al. (2013).

The approaches a) propose to use the filtered value of the equivalent control as an estimation of the disturbance. The latter consist in increasing the gain to enforce the sliding mode to be reached. Once the sliding mode is achieved, the high frequency control signal is low-pass filtered and used as an information about disturbance in controller gain. The sliding mode controller gain consists of the sum of filtered signal and some constant to compensate possible error between real disturbance and its value estimated by filter. However, the algorithm in Utkin & Poznyak (2013) requires the knowledge of the minimum and maximum allowed values of the adaptive gain, hence, it requires the information of the upper bound of disturbances derivatives. On the other hand, even the other algorithms Edwards & Shtessel (2016a,b); Oliveira et al. (2018) do not require theoretically the information of the disturbances derivatives, however, in practice, the usage of low-pass filter requires implicitly the information about this upper bound in order to adequately choose the filter time constant. With this information continuous HOSMCs Cruz-Zavala & Moreno (2017); Basin et al. (2016) can be applied successfully and no any adaptation will be needed. Strategies b) consist of increasing the gain until the sliding mode is reached, then the gain is fixed at this value, ensuring an ideal sliding mode for some interval. When the disturbance grows, the sliding mode can be lost, so the gain increases and reach it again. This second strategy has two main disadvantages: 1) the gain does not decrease, i.e., it will not follow disturbance when it is decreasing; 2) one cannot be sure that the sliding mode will never be lost because it is not ensured that the disturbance will not grow anymore.

To overcome the first of these disadvantages, approaches in c) have been developed. According to these approaches, the gain increases until the sliding mode is achieved and then decreases until the moment it is lost, i.e., the sliding mode is not reached any more. These approaches ensure the finite-time convergence of the sliding variable to some neighborhood of zero without big overestimation of the gain. The main drawback of these approaches is that the size of the above mentioned neighborhood and the time of convergence depend on the unknown upper bound of disturbance which are unknown a priori. Recently, novel approaches based on the usage of a monitoring function Hsu et al. (2018) have been proposed to adapt the control gains. However, this approach has been only applied for the first-order sliding mode controller.

To reach the finite-time convergence to a prescribed vicinity of the sliding set a barrier functions adaptive strategy is proposed in Obeid et al. (2018). Indeed, this strategy has been applied to adapt a first order sliding mode and super-twisting algorithm for a class of first order disturbed systems Obeid et al. (2018); Obeid et al. (2020).

In this paper, we present barrier function-based adaptive controllers ensuring the finite-time convergence of solutions to desired vicinity of origin for a perturbed chain of integrators of length r with bounded uncertainties whose bounds are unknown. By a minor extension of the definition (as explained in the next section), we refer to such controllers as adaptive HOSMCs. The proposed adaptive controller guarantees the asymptotic convergence to the origin i.e., it establishes real HOSM. Indeed, it ensures the convergence of the states and their maintenance in a decreasing domain which tends to zero. The main features of such adaptive controller can be summarized as follows:

- This controller can be extended to arbitrary order.
- The sliding variable and its $(r-1)$ first derivatives converge in finite time to a family of adaptive open sets $(D(t))_{t\geq 0}$ decreasing together with homogeneity norm of solutions to zero as t tends to infinity.
- Velocity of decreasing for the homogeneity norm could be chosen arbitrarily.
- We provide an explicit bound on the control which is best possible, i.e., (essentially) linear with respect to the uncertainty.

The paper is organized as follows: problem formulation and adaptive controllers are presented in Section 2, simulation results which show the effectiveness of the proposed controllers are presented and discussed in Section 3. Some concluding remarks are given in Section 4.

2 Higher Order Sliding Mode Controllers

If r is a positive integer, the perturbed chain of integrators of length r corresponds to the (uncertain) control system given by

$$\dot{z}_i = z_{i+1},\ i = 1, ..., r-1, \quad \dot{z}_r = \varphi(t) + \gamma(t)u, \quad (1)$$

where $z = [z_1\ z_2\ ...z_r]^T \in \mathbb{R}^r$, $u \in \mathbb{R}$ and the functions φ and γ are any measurable functions defined almost everywhere (a.e. for short) on $\mathbb{R}_+$ and bounded by positive constants $\bar{\varphi}$, γ_m and γ_M, such that, for a.e. $t \geq 0$,

$$|\varphi(t)| \leq \bar{\varphi}, \quad 0 < \gamma_m \leq \gamma(t) \leq \gamma_M. \tag{2}$$

One can equivalently define a perturbed chain of integrators of length r as the differential inclusion $\dot{z}_r \in I_{\bar{\varphi}} + uI_\gamma$ where $I_{\bar{\varphi}} = [-\bar{\varphi}, \bar{\varphi}]$ and $I_\gamma = [\gamma_m, \gamma_M]$.

The usual objective regarding System (1) consists of stabilizing it with respect to the origin in finite time, i.e., determining feedback laws $u = U(z)$ so that the trajectories of the corresponding closed-loop system converge to the origin in finite time. Note that, in general, the controllers $U(\cdot)$ are discontinuous and then, solutions of (1) need to be understood here in Filippov's sense Filippov (1988), i.e., the right-hand vector set is enlarged at the discontinuity points of the differential inclusion to the convex hull of the set of velocity vectors obtained by approaching z from all directions in $\mathbb{R}^r$, while avoiding zero-measure sets. Several solutions for this problem exist Levant (2001, 2003, 2005); Ding et al. (2016); Cruz-Zavala & Moreno (2017). under the hypothesis that the bounds γ_m, γ_M and $\bar{\varphi}$ are known.

In case the bounds γ_m, γ_M and $\bar{\varphi}$ are unknown (one only assumes their existence) then it is obvious to see that finite time stabilization is not possible by a mere state feedback and therefore, one possible alternate objective consists in achieving asymptotic stabilization. This is the goal of this paper to establish such a result for System (1) and we provide next a precise definition of asymptotic stabilization.

Definition 1 *If r, p are positive integers and $f : \mathbb{R}^r \times \mathbb{R}^p \rightrightarrows \mathbb{R}^r$ is a continuous differential equation, we say that the system $\dot{z} = f(z, u)$ is asymptotic stabilizable if, there exists a continuous controller $u = U(\cdot, t)$ such that every trajectory of the closed-loop system $\dot{z} = f(z, U(\cdot, t))$ tends to the origin as t tends to infinity.*

The main result of that paper consists of designing controllers which asymptotically stabilize System (1) *independently* of the positive bounds $\bar{\varphi}$, γ_m and γ_M, i.e., the controllers $u = U(\cdot, t)$ which asymptotically stabilize System (1) does not depend on the bounds $\bar{\varphi}$, γ_m and γ_M.

We next recall the following definition needed in the sequel.

Definition 2 *(**Homogeneity.** cf. Levant (2011).) If r, m are positive integers, a function $f : \mathbb{R}^r \to \mathbb{R}^m$ (or a differential inclusion $F : \mathbb{R}^r \rightrightarrows \mathbb{R}^m$ respectively) is said to be homogeneous of degree $q \in \mathbb{R}$ with respect to the family of dilations $\delta_\varepsilon(z)$, $\varepsilon > 0$, defined by*

$$\delta_\varepsilon(z) = (z_1, \cdots, z_r) \mapsto (\varepsilon^{p_1} z_1, \cdots, \varepsilon^{p_r} z_r),$$

where $p_1, \cdots, p_r$ are positive real numbers (the weights), if for every positive ε and $z \in \mathbb{R}^r$, one has $f(\delta_\varepsilon(z)) = \varepsilon^q f(z)$ $\big(F(\delta_\varepsilon(z)) = \varepsilon^q \delta_\varepsilon(F(z))$ respectively$\big)$.

The following notations will be used throughout the paper. We define the function sgn as the multivalued function defined on $\mathbb{R}$ by $\operatorname{sgn}(x) = \frac{x}{|x|}$ for $x \neq 0$ and $\operatorname{sgn}(0) = [-1, 1]$. Similarly, for every $a \geq 0$ and $x \in \mathbb{R}$, we use $\lfloor x \rceil^a$ to denote $|x|^a \operatorname{sgn}(x)$. Note that $\lfloor \cdot \rceil^a$ is a continuous function for $a > 0$ and of class C^1 with derivative equal to $a\,|\cdot|^{a-1}$ for $a \geq 1$. Moreover, for every positive integer r, we use J_r to denote the r-th Jordan block, i.e., the $r \times r$ matrix whose (i, j)-coefficient is equal to 1 if $i = j - 1$ and zero otherwise. Finally, for $j = 1, \cdots, r$, we use e_j to denote the vector of $\mathbb{R}^r$ with zero coordinates except the j-th which is equal to one.

2.1 Adaptive Higher Order Sliding Mode Controller

We first define the system under study and provide parameters used later on.

Definition 3 *Let r be a positive integer. The r-th order chain of integrator $(CI)_r$ is the single-input control system given by*

$$(CI)_r \quad \dot{z} = J_r z + u e_r, \tag{3}$$

with $z = (z_1, \cdots, z_r)^T \in \mathbb{R}^r$ and $u \in \mathbb{R}$. For $\kappa < 0$ and $p > 0$ with $p + (r+1)\kappa \in [0, 1)$, set $p_i := p + (i-1)\kappa$, $1 \leq i \leq r+2$. For $\varepsilon > 0$, let $\delta_\varepsilon : \mathbb{R}^r \to \mathbb{R}^r$ be the family of dilations associated with $(p_1, \cdots, p_r)$.

In the spirit of Harmouche. et al. (2012), we put forwards geometric conditions on certain stabilizing feedbacks $u_0(\cdot)$ for $(CI)_r$ and corresponding Lyapunov functions V. These conditions will be instrumental for the latter developments.

Our construction of the feedback for practical stabilization relies on the following result.

Assumption 4 *Let r be a positive integer. There exists a feedback law $u_0 : \mathbb{R}^r \to \mathbb{R}$ homogeneous with respect to $(\delta_\varepsilon)_{\varepsilon>0}$ such that the closed-loop system $\dot{z} = J_r z + u_0(z) e_r$ is finite time globally asymptotically stable with respect to the origin and the following conditions hold true:*

(i) the function $z \mapsto J_r z + u_0(z) e_r$ is homogeneous of degree κ with respect to $(\delta_\varepsilon)_{\varepsilon>0}$ and there exists a continuous positive definite function $V(z) : \mathbb{R}^r \to \mathbb{R}_+$, C^1 except at the origin, homogeneous of positive degree with respect to $(\delta_\varepsilon)_{\varepsilon>0}$ such that there exists

$c > 0$ and $\alpha \in (0,1)$ for which the time derivative of $V(z)$ along non trivial trajectories of $\dot{z} = J_r z + u_0(z)e_r$ verifies

$$\dot{V}(z) \leq -cV^{\alpha}(z); \tag{4}$$

(ii) the function $z \mapsto u_0(z)\frac{\partial V(z)}{\partial z_r}$ is non positive over $\mathbb{R}^r$ and, for every non zero $z \in \mathbb{R}^r$ verifying $u_0(z) = 0$, one has $\frac{\partial V(z)}{\partial z_r} = 0$. As a consequence function $z \mapsto sgn(u_0(z))\frac{\partial V(z)}{\partial z_r}$ is well-defined over $\mathbb{R}^r \setminus \{0\}$ and non positive.

Remark 5 *Item (i) of the above proposition is classical, see for instance Hong (2002); Huang et al. (2005); Cruz-Zavala & Moreno (2014). Item (ii) considers a geometric condition on controllers verifying Item (i), which was introduced in Harmouche. et al. (2012) and used in Laghrouche et al. (2017). This geometric condition is indeed satisfied, for instance by Hong's controller, see Laghrouche et al. (2017) for other examples.*

Regarding our problem, we consider, for every $\varepsilon > 0$ the following controller:

$$u(z, z_0, t) = g(|u_0(z)|)u_0(z) + k\,\mathrm{sgn}\big(u_0(z)\big)\hat{\varphi}\big(t, V(z_0)\big), \tag{5}$$

where $z_0 = z(0)$, u_0 and $V(z)$ are provided by Assumption 4, $g : \mathbb{R}_+ \to \mathbb{R}_+^*$ is an arbitrary increasing C^1 function tending to infinity and the adaptive function $\hat{\varphi}$ is defined later. To proceed, let us first introduce, for every positive real ξ, the function F_ξ defined on $[0, \xi)$ by $F_\xi(x) = \frac{k\xi}{\xi - x}$ for $0 < x < \xi$ where k is a positive tuning parameter. Moreover, consider $\eta : \mathbb{R}_+ \to \mathbb{R}_+^*$, a non increasing C^1 function so that $\eta(0) = \varepsilon > 0$ and $-\dot{\eta} \leq M\eta$, where $\varepsilon > 0$ and $M \geq 0$ are chosen so that $M\varepsilon^{1-\alpha} < c$. Moreover, we define the following family of adaptive domains, for $t \geq 0$,

$$D(t) = \{z \in \mathbb{R}^r \;\mid\; V(z) < \eta(t)\}. \tag{6}$$

Assume first that $V(z_0) \leq \varepsilon/2$. Then $\hat{\varphi}\big(t, V(z_0)\big)$ is defined as $F_{\eta(t)}(V(z(t)))$ for any trajectory $z(\cdot)$ of System (1) closed-looped by the feedback control law (5) and starting at z_0, and so, as long as $z(\cdot)$ is defined. (Note that in this case there exists a non trivial interval of existence of such solutions since F_η is continuous.) In case $V(z_0) > \varepsilon/2$, then $\hat{\varphi}\big(t, V(z_0)\big) = t$ as long as any trajectory $z(\cdot)$ of System (1) closed-looped by the feedback control law (5) and starting at z_0 is defined and verifies $V(z(t)) > \eta(t)/2$. If there exists a first time $\bar{t}(V(z_0)) > 0$ such that $V(z(t)) = \eta(t)/2$, then $\hat{\varphi}\big(t, V(z_0)\big) = F_{\eta(t)}(V(z(t)))$ for $t \geq \bar{t}(V(z_0))$, and so, as long as $z(\cdot)$ is defined. Note that $\bar{t}(V(z_0))$ actually also depends on the trajectory $z(\cdot)$ since the latter may not be unique. Hence, as long as a trajectory $z(\cdot)$ of the closed-loop system and starting at z_0 is defined, the adaptive function $\hat{\varphi}\big(t, V(z_0)\big)$ is given by

$$\hat{\varphi}(t, x) = \begin{cases} t, & \text{if } 0 \leq t < \bar{t}(V(z_0)), \\ F_{\eta(t)}(x), & \text{if } t \geq \bar{t}(V(z_0)), \end{cases} \tag{7}$$

with the convention that $\bar{t}(V(z_0)) = 0$ if $V(z_0) \leq \varepsilon/2$ and $\bar{t}(V(z_0)) = \infty$ if $z(\cdot)$ is defined and verifies $V(z(t)) > \eta(t)$ for all non negative times t.
Here the positive function g and the positive constant k are gain parameter.

The following theorem provides the main result for the adaptive controller u.

Theorem 6 *Let r be a positive integer and System (1) be the perturbed r-chain of integrators with unknown bounds γ_m, γ_M and $\bar{\varphi}$. Let $u_0, V : \mathbb{R}^r \to \mathbb{R}_+$ be the feedback law and the continuous positive definite function defined respectively in Assumption 4. For every $z_0 \in \mathbb{R}^r$, consider any trajectory $z(\cdot)$ of System (1) closed by the feedback control law (5) verifying $z(0) = z_0$. Then, $z(\cdot)$ is defined for all non negative times, there exists a first time $\bar{t}(V(z_0))$ for which $V(z(t)) \leq \eta(t)/2$ at $t = \bar{t}(V(z_0))$ and $z(t) \in D(t)$ for all $t > \bar{t}(V(z_0))$.*

Moreover, if $l > 0$ is the homogeneity degree of V, then there exist positive constants C_i, $1 \leq i \leq r$ (i.e., independent of $z(\cdot)$) such that

$$|z_i(t)| \leq C_i\, \eta^{p_i/l}(t), \quad 1 \leq i \leq r, \quad t > \bar{t}(V(z_0)). \tag{8}$$

PROOF. [**Proof of Theorem 6**] We refer to (S) as the closed-loop system defined by (1) and (5). The first issue we address is the existence of trajectories of (S) starting at any initial condition $z_0 \in \mathbb{R}^r$. Such an existence follows from the fact that the application $\mathbb{R}_+ \times \mathbb{R}^r \to \mathbb{R}$, $(t, z) \mapsto \hat{\varphi}(t, V(z))$ is continuous.

We next show that every trajectory of (S) is defined for all non negative times. For that purpose, consider a non trivial trajectory $z(\cdot)$ and let $I_{z(\cdot)}$ be its (non trivial) domain of definition. We obtain the following inequality for the time derivative of $V(z(\cdot))$ on $I_{z(\cdot)}$ by using Items (i) and (ii) of Assumption 4. For a.e. $t \in I_{z(\cdot)}$, one gets

$$\begin{aligned} \dot{V} &= \tfrac{\partial V}{\partial z_1} z_2 + \ldots + \tfrac{\partial V}{\partial z_r}\left(\gamma\left[g(|u_0|)u_0 + k\,\mathrm{sgn}(u_0)\hat{\varphi}\right] + \varphi\right), \\ &\leq -cV^{\alpha} - \left|\tfrac{\partial V}{\partial z_r}\right|\Big(\big(\gamma_m g(|u_0|) - 1\big)|u_0| + k\gamma_m\hat{\varphi} - \bar{\varphi}\Big), \\ &\leq -cV^{\alpha} - k\gamma_m\left|\tfrac{\partial V}{\partial z_r}\right|\left(\hat{\varphi} - \bar{\Phi}\right), \end{aligned} \tag{9}$$

with $\bar{\Phi} := \frac{1}{k\gamma_m}(\bar{\varphi} - h_m)$, where

$$h_m = \min\big(0, \min_{x\geq 0} x(\gamma_m g(x) - 1)\big).$$

We thus have the differential inequality a.e. for $t \in I_{z(\cdot)}$

$$\dot{V} \leq -cV^{\alpha} + C_1\bar{\varphi}V^{p_{r+2}}, \tag{10}$$

where C_1 is a positive constant independent of the trajectory $z(\cdot)$. Since $p_{r+2} \in [0, 1)$, it is therefore immediate to deduce that there is no blow-up in finite time and thus $I_{z(\cdot)} = \mathbb{R}_+$.
The next step consists in showing the existence of a finite time $\bar{t}(V(z_0))$ and, for that purpose, we can assume with no loss of generality that $V(z_0) > \varepsilon/2$. The corresponding trajectory is defined as long as $V(z(t)) > \eta(t)/2$, since in that case, the growth of the right-hand side of (1) is sublinear with respect to the state variable z. Arguing by contradiction, one gets that $V(z(t)) > \eta(t)/2$ for all non negative times, and hence $\dot{V} \leq -cV^{\alpha}$ for $t \geq \bar{\Phi}$. The latter inequality yields convergence to the origin in finite time, which is a contradiction. Then, the existence of a finite time $\bar{t}(V(z_0))$ is established.
One is left to prove that $z(t) \in D(t)$ for $t > \bar{t}(V(z_0))$. Since $\hat{\varphi} = F_\eta > 1$ on that time interval, one gets finite time convergence if $\bar{\Phi} \leq 1$, whatever the choice of η is. Assume that $\bar{\Phi} > 1$. For every $t > \bar{t}(V(z_0))$, let $V_*(t)$ be the unique solution in $(0, \eta(t)$ of the equation $F_{\eta(t)}(V_*(t)) = \bar{\Phi}$, i.e., $V_*(t) = (1 - 1/\bar{\Phi})\eta(t)$. Note that if $V(z(t)) \geq V_*(t)$, then $\dot{V} \leq -cV^{\alpha}$. We will actually prove that

$$V(z(t)) \leq \max(\tfrac{1}{2}, (1 - 1/\bar{\Phi}))\eta(t), \text{ for } t > \bar{t}(V(z_0)). \tag{11}$$

and, if $\bar{\Phi} \geq 2$, then there exists $t' \geq \bar{t}(V(z_0))$ such that $V(z(t)) \leq (1 - 1/\bar{\Phi})\eta(t)$ for $t > t'$. Assume first that $V(z(t)) \geq (1 - 1/\bar{\Phi})\eta(t)$ at $t = \bar{t}(V(z_0))$. Then, $\dot{V} \leq -cV^{\alpha}$ in a right neighborhood of $\bar{t}(V(z_0))$ and, by the previous argument by contradiction, there must exists a first time $t' \geq \bar{t}(V(z_0))$ so that $V(z(t)) = (1-1/\bar{\Phi})\eta(t)$ at $t = t'$. The time derivative of $t \mapsto V(z(t)) - (1-1/\bar{\Phi})\eta(t)$ at $t = t'$ is less than or equal to

$$-cV_*(t')^{\alpha} + MV_*(t') \leq -V_*(t')^{\alpha}(c - MV_*(t')^{1-\alpha})$$

which is negative since $V_*(t') < \eta(0) = \varepsilon$. Note also that the previous inequality holds true at every time s so that $V(z(s)) = (1 - 1/\bar{\Phi})\eta(s)$. Then, the zeros of $V - (1 - 1/\bar{\Phi})\eta$ on $[T_m, \infty)$ are isolated and $V - (1 - 1/\bar{\Phi})\eta > 0$ ($V - (1 - 1/\bar{\Phi})\eta < 0$ resp.) in a left (right resp.) neighborhood of s. Hence t' is the unique zero of $V - (1 - 1/\bar{\Phi})\eta$ on $[\bar{t}(V(z_0)), \infty)$ on $[\bar{\Phi})\eta(t), \infty)$ and the claim is proved.
Finally assume that $V(z(t)) < (1 - 1/\bar{\Phi})\eta(t)$ at $t = \bar{t}(V(z_0))$. By the previous computations, one gets that $V(z(t)) < (1 - 1/\bar{\Phi})\eta(t)$ for all $t \geq \bar{t}(V(z_0))$.

It remains to prove (8). It follows at once from the following set of equations: there exists constants C_i, $1 \leq i \leq r$, such that, for every $z = (z_1, \cdots, z_r) \in \mathbb{R}^r$, $|z_i| \leq C_i V(z)^{p_i/l}$. To see that, recall that V is homogeneous of degree l with respect to $(\delta_\varepsilon)_{\varepsilon>0}$ and assume that $z_i \neq 0$. Take ε so that $\varepsilon^{p_i}|z_i| = 1$, one has $V(z) = V(\cdot, \pm 1, \cdot)|z_i|^{l/p_i}$. Finally note that there exists $C_i > 0$ such that $V(\cdot, \pm 1, \cdot) \geq 1/C_i$, whatever $\cdot$ is. This concludes the proof of Theorem 6.

Remark 7 *At the light of the above argument, one can see that, if the bounds of the incertainties are known, then one can choose the gain parameter k in such a way that* $\bar{\Phi} \leq 1$ *and hence get finite time convergence to zero. In that way, our controller provides yet another finite time stabilizer of the perturbed integrator with known bounds on the perturbations.*

Remark 8 *From* (11) *and the choice* $\eta(t) = \varepsilon e^{-Mt}$, *one deduces that* $V(z(t)) < \varepsilon e^{-Mt}$ *for* $t \geq \bar{t}(V(z_0))$. *In particular, the trajectory converges to the origin with the exponential rate* $M > 0$ *which can be chosen at will. Moreover, it is easy to provide with the help of* (9) *upper bounds on* $\bar{t}(V(z_0))$ *in terms of* $\bar{\Phi}$ *and* $V(z_0)$.
To be complete, one should emphasize that the result in Theorem 6 does guarantee that a trajectory entering in the neighborhood $\{z,\ V(z) \leq \eta(t)/2\}$ *for the first time at* $t = \bar{t}(V(z_0))$ *will always remain in the larger neighborhood* $\{z,\ V(z) < \eta(t)\}$.

Remark 9 *One other way to diminish the delay time* $\bar{t}(V(z_0))$ *needed to enter into the neighborhoods* $\{z,\ V(z) \leq \eta(t)\}$ *consists in replacing the time t in the definition of* $\hat{\phi}$ *given in* (7) *by an increasing function* $l(t)$ *tending to infinity faster than a linear one. In that manner, inequalities such as* $t > \bar{\Phi}$ *is replaced by* $l(t) > \bar{\Phi}$. *The price to pay will be an larger upper bound for the gains.*

Remark 10 *Among different controllers that can fulfill Assumption 4, Hong's controller Hong (2002) can be used. This controller is defined as follows:*
Let $\kappa < 0$ *and* $l_1, \cdots, l_r$ *positive real numbers. For* $z = (z_1, \cdots, z_r)$, *we define* $u_0 = v_r$ *for* $i = 0, ..., r-1$:

$$v_0 = 0,\ v_{i+1} = -l_{i+1}\lfloor\lfloor z_{i+1}\rceil^{\beta_i} - \lfloor v_i\rceil^{\beta_i}\rceil^{(\alpha_{i+1}/\beta_i)}, \tag{12}$$

where $p_i = 1 + (i-1)\kappa,\ \beta_0 = p_2,\ (\beta_i + 1)p_{i+1} = \beta_0 + 1 > 0$ *and* $\alpha_i = \frac{p_{i+1}}{p_i}$.
Now, let $\psi_r(z_1, \cdots, z_r) = \lfloor z_r\rceil^{\beta_{r-1}} - \lfloor v_{r-1}\rceil^{\beta_{r-1}}$. *Then according to Harmouche et al. (2017), if* $\kappa = -1/r$, $u_0 = -l_r\, sgn\Big(\psi_r(z_1, \cdots, z_r)\Big)$ *is bounded and the adap-*

tive controller u can be written as

$$u(z,t) = g(|u_0(z)|)u_0(z) + k\, sgn\big(u_0(z)\big)\hat{\varphi}(t,V(z)\big) = -\Big(g(|u_0(z)|)l_r + k\hat{\varphi}(t,V(z))\Big)\, sgn\big(\psi_r(z_1,\cdots,z_r)\big) \tag{13}$$

Here the assumption that $\lim_{x\to+\infty} g(x) = +\infty$ can be removed. Indeed, in view of (9) and taking into account that $|u_0| = l_r$, one can remove this assumption and the result remains valid. Moreover, in the case when the bounds of the uncertainties are known, the functions g and $\hat{\varphi}$ can be chosen constants, which leads to homogeneous controller u.

2.2 Asymptotic bounds for the controller u

One deduces from Theorem 6 the following result which provides an asymptotic upper bound for the controller u that does not exhibit any overestimation.

Lemma 11 *The controller u defined in (5) verifies the following asymptotic upper bound, which is uniform with respect to trajectories of the closed-loop system (S):*

$$\limsup_{t\to\infty} |u| \le M(u_0) + k\max(1,\bar{\Phi}), \tag{14}$$

where $M(u_0) = g(l_r)l_r$ if u_0 is homogeneous of degree zero, where l_r is the supremum of $|u_0|$ over $\mathbb{R}^r$, and $M(u_0) = 0$ if u_0 is homogeneous of positive degree.

PROOF. It is enough to prove that $\hat{\varphi}(t,V(z(t))) \le \max(1,\bar{\Phi})$ for t large enough. Indeed, a careful examination of the argument of Theorem 6 shows that either $\bar{\Phi} \le 1$, in which case one has convergence to zero in finite time and $\hat{\varphi} = 1$ for t large enough, or $\bar{\Phi} > 1$, in which case (11) holds true and then $\hat{\varphi} \le \bar{\Phi}$ for t large enough.

3 Simulation Results

Consider the following third order system

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= z_3, \\ \dot{z}_3 &= \varphi + \gamma u, \end{aligned} \tag{15}$$

where γ and φ are *discontinuous* bounded uncertainties defined as

$$\begin{aligned} &\varphi = 5\,\mathrm{sgn}(\cos(t)) - 20\sin(2t), \quad |\varphi| \le 25, \\ &\gamma = 3 - 2\,\mathrm{sgn}(\sin(3t)), \quad 0 < 1 \le \gamma \le 5, \end{aligned} \tag{16}$$

and $z_0 = [4,4,-4]$.
In the following subsections, two cases will be considered: first the case when the bounds of the uncertainties are known, and second when they are unknown.
The first case is provided to show that the controller (13) can be considered as HOSMC due to the reason that it provides r-th order of asymptotic precision with respect to the sampling step. While in the second case, the effectiveness of the proposed approach to force the sliding variable and its $r-1$ first derivatives to the following family of adaptive domains

$$D(t) = \{z \in \mathbb{R}^r \;\mid\; V(z) < \eta(t)\}, \tag{17}$$

is studied using the controller (13).

3.1 Case when the bounds of uncertainties are known

In this subsection, the control parameters of u_0 in (13) are tuned to the following values

$$l_1 = 1,\ l_2 = 2,\ l_3 = 5,\ \kappa = -1/3, \tag{18}$$

the constant k and the function g, $\hat{\varphi}$ in (13) are selected as $k = \frac{\bar{\varphi}}{\gamma_m}$, $g = \frac{1}{\gamma_m}$, $\hat{\varphi} = 1$. Hence,

$$u = -\tfrac{l_3+\bar{\varphi}}{\gamma_m}\,\mathrm{sgn}\Big(\psi_3(z_1,z_2,z_3)\Big), \tag{19}$$

with $\psi_3(z_1,z_2,z_3)$ is given explicitly in appendix A.

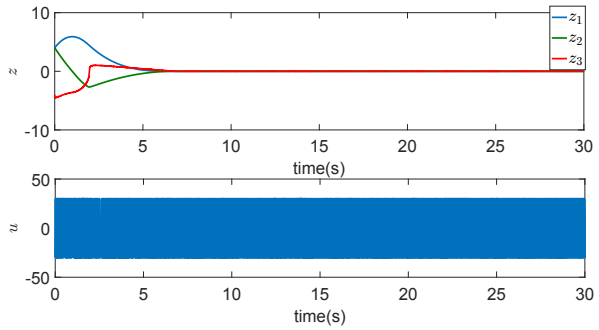

Fig. 1. System (15) in close loop with (19).

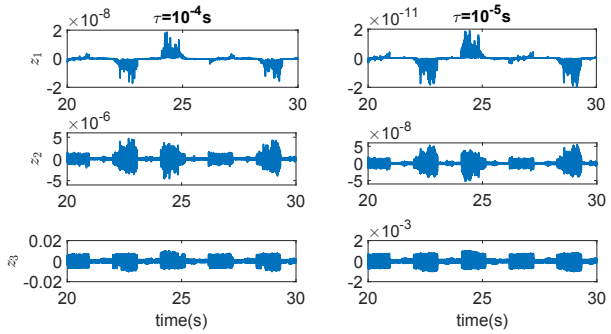

Fig. 2. Accuracy of System (15) in close loop with (19) for $\tau = 10^{-4}$ s and $\tau = 10^{-5}$ s.

Fig 1 shows the simulation results of system (15) with controller (19). We can see that the states z_1, z_2, z_3 converge in a finite time to the origin, and the control input u is discontinuous.

The system homogeneity degree is $\kappa = -1/3$ and the homogeneity weights of z_1, z_2, z_3 are 1, 2/3, 1/3 respectively. These weights can be multiplied by 3 in order to achieve homogeneity weights 3, 2, 1 with the closed loop system homogeneity degree $\kappa = -1$. This homogeneity is called 3-sliding homogeneity Levant (2005). According to these homogeneity weights, the controller (19) provides the following accuracy for the states w.r.t sampling step τ

$$|z_1| \leq \lambda_1 \tau^3, \quad |z_2| \leq \lambda_2 \tau^2, \quad |z_3| \leq \lambda_3 \tau, \tag{20}$$

where λ_i are constants. By simulations shown in Fig 2 with $\tau = 10^{-4}s$, constants λ_i are determined as $\lambda_1 = 20000$, $\lambda_2 = 500$, and $\lambda_3 = 200$. These constants have been confirmed by simulations with $\tau = 10^{-5}s$ also shown in Fig 2.

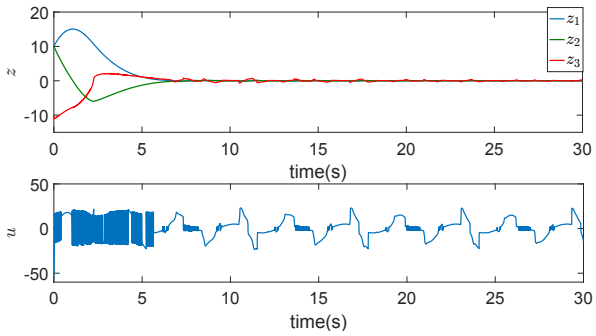

Fig. 3. System (15) in close loop with (21).

3.2 Case when the bounds of uncertainties are unknown

In this case, the control parameters of u_0 are the same as in subsection 3.1, while the constant k and the functions g and $\hat{\varphi}$ in (13) are selected as $k = 1$, $g = 1$ and $\hat{\varphi}$ follows (7) where $\eta(t) = e^{-0.2t}$. Hence, u can be written as

$$u = -\Big(l_3 + \hat{\varphi}(t, V(z))\Big)\, \mathrm{sgn}\Big(\psi_3(z_1, z_2, z_3)\Big), \tag{21}$$

with the Lyapunov Function candidate $V(z)$ is given by (see appendix A)

$$V(z) = \frac{3}{5}|z_1|^{\frac{5}{3}} + \frac{2}{5}|z_2|^{\frac{5}{2}} + \frac{3}{5}l_1^{\frac{5}{2}}|z_1|^{\frac{5}{3}} + l_1^{\frac{3}{2}} z_2 \lfloor z_1 \rceil + \frac{1}{5}|z_3|^5 + \frac{4}{5}l_2^5 \Big\lfloor \lfloor z_2 \rceil^{\frac{3}{2}} + l_1^{\frac{3}{2}} \lfloor z_1 \rceil \Big\rceil^{\frac{5}{3}} + z_3 l_2^4 \Big\lfloor \lfloor z_2 \rceil^{\frac{3}{2}} + l_1^{\frac{3}{2}} \lfloor z_1 \rceil \Big\rceil^{\frac{4}{3}}.$$

Figs 3-4-5 illustrate the simulation results of system (15) with the adaptive controller (21). It can be noticed in

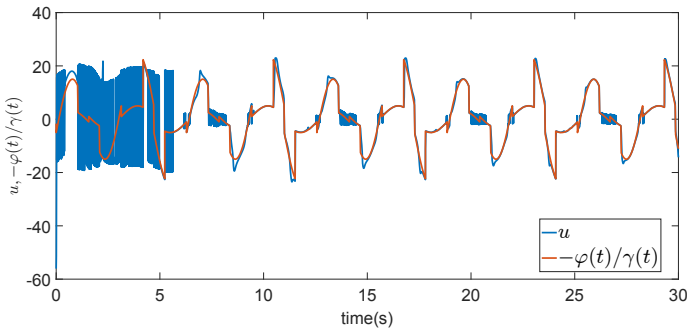

Fig. 4. Evolution of the control signal and the uncertainties for System (15) with (21)

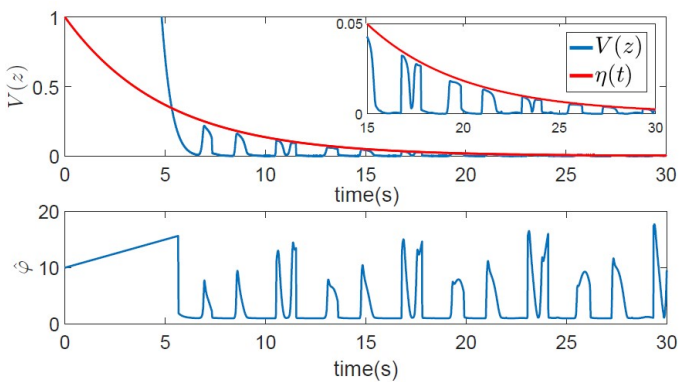

Fig. 5. Evolution of the Lyapunov function $V(z)$ and the adaptive function $\hat{\varphi}(t, V(z))$ for System (15) with (21).

Fig 3 that the proposed controller provides the practical stabilization for system (15), and the control input is not continuous in general. Moreover, it is confirmed in Fig 4 that the control signal is not overestimated. Indeed, after the time instant $\bar{t}(V(z_0))$ the control signal closely follows the uncertainties. On the other hand, the fulfillment of the control objective can be shown in Fig 5, where the sliding variable and its $r-1$ first derivatives enter in a finite time in the family of adaptive domains $D(t)$; i.e. $V(z) < \eta(t)$ and cannot exceed it. It can be seen also that the adaptive function $\hat{\varphi}(t, V(z))$ increases linearly until the time instant $\bar{t}(V(z_0))$ and then starts to follow the function $F_{\eta(t)}(x)$.

In this section, it is supposed that the sliding variable and its $r-1$ derivatives are measurable. However, it should be mentioned that in the case where the sliding variable is only the measurable state, the proposed controller can be combined with the differentiator proposed in Oliveira et al. (2017) yielding output-feedback control.

4 Conclusions

This paper has proposed a new Lyapunov-based adaptive scheme for higher-order sliding mode controller ap-

plied for a class of perturbed chain of integrators with unknown bounded uncertainties. The proposed adaptive controller guarantees (a) the finite time convergence to a family of adaptive open sets $(D(t))_{t\geq 0}$ decreasing to the origin as t tends to infinity; (b) once a trajectory enters some $D(t_*)$ at time t_*, it remains trapped in the $D(t)$'s, i.e., $z(t) \in D(t)$ for $t \geq t_*$. Another advantage of this adaptive controller consists in the fact that the homogeneity norm of solutions asymptotically tends to zero.

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A u_0 and the Lyapunov function V design Hong (2002)

Firstly, the controller u_0 is determined explicitly.
Let the system homogeneity degree $\kappa = -\frac{1}{3}$. Then the homogeneity weights for the states z_1, z_2, z_3 are 1, $\frac{2}{3}$, $\frac{1}{3}$ respectively and the constants β_i are $\beta_0 = \frac{2}{3}$, $\beta_1 = \frac{3}{2}$, $\beta_2 = 4$. According to Hong (2002), $u_0 = v_3$ with

$$\begin{cases} v_0 = 0, \\ v_{i+1} = -l_{i+1}\Big\lfloor \lfloor z_{i+1} \rceil^{\beta_i} - \lfloor v_i \rceil^{\beta_i} \Big\rceil^{\frac{p_{(i+1)}+\kappa}{p_{(i)}} \times \frac{1}{\beta_i}}, \ i = 0, 1, 2. \end{cases} \tag{A.1}$$

From (A.1)

$$v_1 = -l_1 \lfloor z_1 \rceil^{\frac{p_2}{p_1}} = -l_1 \lfloor z_1 \rceil^{\frac{2}{3}}. \tag{A.2}$$

Then

$$v_2 = -l_2\Big\lfloor \lfloor z_2 \rceil^{\beta_1} - \lfloor v_1 \rceil^{\beta_1} \Big\rceil^{\frac{p_2+\kappa}{p_2} \times \frac{1}{\beta_1}} \tag{A.3}$$

$$= -l_2\Big\lfloor \lfloor z_2 \rceil^{\frac{3}{2}} + l_1^{\frac{3}{2}} \lfloor z_1 \rceil \Big\rceil^{\frac{1}{3}}. \tag{A.4}$$

Finally

$$v_3 = -l_3\Big\lfloor \lfloor z_3 \rceil^{\beta_2} - \lfloor v_2 \rceil^{\beta_2} \Big\rceil^{\frac{p_3+\kappa}{p_3} \times \frac{1}{\beta_2}} \tag{A.5}$$

$$= -l_3\Big\lfloor \lfloor z_3 \rceil^4 + {l_2}^4 \Big\lfloor \lfloor z_2 \rceil^{\frac{3}{2}} + l_1^{\frac{3}{2}} \lfloor z_1 \rceil \Big\rceil^{\frac{4}{3}} \Big\rceil^0 \tag{A.6}$$

$$= -l_3 sign\Big(\psi_3(z_1, z_2, z_3)\Big). \tag{A.7}$$

where $\psi_3(z_1, z_2, z_3) = \lfloor z_3 \rceil^4 + {l_2}^4 \Big\lfloor \lfloor z_2 \rceil^{\frac{3}{2}} + l_1^{\frac{3}{2}} \lfloor z_1 \rceil \Big\rceil^{\frac{4}{3}}$. Therefore, the controller u_0 can be expressed as

$$\Rightarrow u_0 = -l_3 sign\Big(\psi_3(z_1, z_2, z_3)\Big)$$

Now, the Lyapunov function V is given. From Hong (2002), the Lyapunov function is defined in the following form

$$\begin{cases} V_i = w_i + V_{i-1}, \quad i = 1, 2, 3 \\ w_i = \int\limits_{v_{i-1}}^{z_i} \lfloor s \rceil^{\beta_{i-1}} - \lfloor v_{i-1} \rceil^{\beta_{i-1}} ds. \end{cases} \tag{A.8}$$

In view of (A.8)

$$w_1 = \frac{1}{1+p_2}|z_1|^{1+r_2} = \frac{1}{1+\frac{2}{3}}|z_1|^{1+\frac{2}{3}},$$

which leads

$$V_1 = w_1 = \frac{3}{5}|z_1|^{\frac{5}{3}}. \tag{A.9}$$

Then

$$\begin{aligned} w_2 &= \int\limits_{v_1}^{z_2} \lfloor s \rceil^{\beta_1} - \lfloor v_1 \rceil^{\beta_1} ds \\ &= \frac{1}{\beta_1+1}(|z_2|^{\beta_1+1} + \beta_1|v_1|^{\beta_1+1}) - z_2 \lfloor v_1 \rceil^{\beta_1} \\ &= \frac{1}{\frac{3}{2}+1}(|z_2|^{\frac{3}{2}+1} + \frac{3}{2}\lfloor v_1 \rceil^{\frac{3}{2}+1}) - z_2 \lfloor v_1 \rceil^{\frac{3}{2}}, \end{aligned} \tag{A.10}$$

with $v_1 = -l_1 \lfloor z_1 \rceil^{\frac{2}{3}}$, it implies that

$$w_2 = \frac{2}{5}|z_2|^{\frac{5}{2}} + \frac{3}{5}l_1^{\frac{5}{2}}|z_1|^{\frac{5}{3}} + l_1^{\frac{3}{2}} z_2 \lfloor z_1 \rceil. \tag{A.11}$$

Therefore,

$$\begin{aligned} V_2 &= V_1 + w_2 \\ &= \frac{3}{5}|z_1|^{\frac{5}{3}} + \frac{2}{5}|z_2|^{\frac{5}{2}} + \frac{3}{5}l_1^{\frac{5}{2}}|z_1|^{\frac{5}{3}} + l_1^{\frac{3}{2}} z_2 \lfloor z_1 \rceil. \end{aligned} \tag{A.12}$$

Finally,

$$\begin{aligned} w_3 &= \int_{v_2}^{z_3} \lfloor s \rceil^{\beta_2} - \lfloor v_2 \rceil^{\beta_2} ds \\ &= \frac{1}{\beta_2 + 1}(|z_3|^{\beta_2+1} + \beta_2 |v_2|^{\beta_2+1}) - z_3 \lfloor v_2 \rceil^{\beta_2}, \end{aligned} \tag{A.13}$$

with $\beta_2 = 4$; $v_2 = -l_2 \left\lfloor \lfloor z_2 \rceil^{\frac{3}{2}} + l_1^{\frac{3}{2}} \lfloor z_1 \rceil \right\rceil^{\frac{1}{3}}$, it leads

$$\begin{aligned} w_3 &= \frac{1}{5}\left(|z_3|^5 + 4l_2^5 \left\lfloor \lfloor z_2 \rceil^{\frac{3}{2}} + l_1^{\frac{3}{2}} \lfloor z_1 \rceil \right\rceil^{\frac{5}{3}}\right) \\ &\quad + z_3 l_2^4 \left\lfloor \lfloor z_2 \rceil^{\frac{3}{2}} + l_1^{\frac{3}{2}} \lfloor z_1 \rceil \right\rceil^{\frac{4}{3}} \\ &= \frac{1}{5}|z_3|^5 + \frac{4}{5}l_2^5 \left\lfloor \lfloor z_2 \rceil^{\frac{3}{2}} + l_1^{\frac{3}{2}} \lfloor z_1 \rceil \right\rceil^{\frac{5}{3}} \\ &\quad + z_3 l_2^4 \left\lfloor \lfloor z_2 \rceil^{\frac{3}{2}} + l_1^{\frac{3}{2}} \lfloor z_1 \rceil \right\rceil^{\frac{4}{3}}. \end{aligned} \tag{A.14}$$

Thus, the Lyapunov function V is given by

$$\begin{aligned} V = V_3 = \frac{3}{5}|z_1|^{\frac{5}{3}} + \frac{2}{5}|z_2|^{\frac{5}{2}} + \frac{3}{5}l_1^{\frac{5}{2}}|z_1|^{\frac{5}{3}} + l_1^{\frac{3}{2}} z_2 \lfloor z_1 \rceil + \frac{1}{5}|z_3|^5 + \\ \frac{4}{5}l_2^5 \left\lfloor \lfloor z_2 \rceil^{\frac{3}{2}} + l_1^{\frac{3}{2}} \lfloor z_1 \rceil \right\rceil^{\frac{5}{3}} + z_3 l_2^4 \left\lfloor \lfloor z_2 \rceil^{\frac{3}{2}} + l_1^{\frac{3}{2}} \lfloor z_1 \rceil \right\rceil^{\frac{4}{3}}. \end{aligned} \tag{A.15}$$