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PROBABILISTIC CONSTRUCTION OF SIMPLY-LACED TODA CONFORMAL FIELD THEORIES

BAPTISTE CERCLÉ, RÉMI RHODES, AND VINCENT VARGAS

Abstract. Following the 1984 seminal work of Belavin, Polyakov and Zamolodchikov on two-dimensional conformal field theories, Toda conformal field theories were introduced in the physics literature as a family of two-dimensional conformal field theories that enjoy, in addition to conformal symmetry, an extended level of symmetry usually referred to as *W-symmetry* or *higher-spin* symmetry. More precisely Toda conformal field theories provide a natural way to associate to a finite-dimensional simple and simply-laced Lie algebra a conformal field theory for which the algebra of symmetry *contains* the Virasoro algebra. In this document we use the path integral formulation of these models to provide a rigorous mathematical construction of simply-laced Toda conformal field theories based on probability theory. By doing so we recover expected properties of the theory such as the Weyl anomaly formula with respect to the change of background metric by a conformal factor and the existence of Seiberg bounds for the correlation functions.

1. Introduction

1.1. **Toda Conformal Field Theories in the physics literature.** In 1981, Polyakov presented in a pioneer work [32] a canonical way of defining the notion of random surface, usually called Liouville conformal field theory (LCFT hereafter), which is now considered to be an essential feature in the understanding of non-critical string theory and two-dimensional quantum gravity. A few years later Belavin, Polyakov and Zamolodchikov (BPZ) presented in their 1984 seminal work [2] a systematic procedure to solve models which like LCFT possess certain conformal symmetries, now referred to as *two-dimensional conformal field theories* (CFTs in the sequel). The main input of their method was to exploit the constraints imposed by conformal symmetry through the study of the algebra of its generators, the *Virasoro algebra*, which in turn completely determines (up to the so-called structure constants) the main quantities of interest, namely the correlation functions of certain special operators, thanks to a recursive procedure dubbed the *conformal bootstrap*.

A natural question which appeared shortly after these developments was: what happens when the algebra of symmetry *stricly contains* the Virasoro algebra? In other words, do the same techniques apply when Virasoro symmetry is extended to feature an additional level of symmetry? Certain extensions of the Virasoro algebra, called W-algebras, have been first studied by Zamolodchikov in his work [36] where was presented the notion of higher-spin symmetry, and following this work two-dimensional CFTs having this extended level of symmetry appeared in the physics literature in [11, 13]. In addition to being an object of interest in itself, the study of a CFT having what is called W-symmetry is crucial in the understanding of W-strings, W-gravity theories or certain statistical physics systems (the articles [12] and [23] provide explicit instances of such systems) and can be applied to

the understanding of some Wess-Zumino-Novikov-Witten models (such a link is explained in [14]). From the representation theory viewpoint the study of W-algebras has proved to be a seminal topic with numerous applications ranging from integrable hierarchies to the geometric Langlands program (see [1] and the references therein).

Toda Conformal Field Theories (TCFTs hereafter) may be thought of as realizations of these algebras of symmetry since they are assumed to provide highest-weight representations of W-algebras. In this context, the primary fields, defined as random fields on a Riemann surface Σ with special conformal covariance properties, are called Vertex Operators and their correlations are defined as an average of the product of the fields (taken at different points of the surface Σ) with respect to the law of a random map called the *Toda field*. This construction can be made somehow explicit thanks to the fundamental fact that TCFTs admit a path integral formulation based on the following Lagrangian

$$S_{T,\mathfrak{g}}(\varphi,g) := \frac{1}{4\pi}\int_\Sigma \Big(\langle \partial_g\varphi(x),\partial_g\varphi(x)\rangle_g + R_g\langle Q,\varphi(x)\rangle + 4\pi\sum_{i=1}^r \mu_i e^{\gamma\langle e_i,\varphi(x)\rangle}\Big)\,\mathrm{v}_g(dx), \tag{1.1}$$

where the Toda field φ is a map $\Sigma\to\mathfrak{h}$ and:

- g is a Riemannian metric on a Riemann surface Σ with associated scalar curvature R_g, gradient ∂_g and volume form v_g,
- $\mathfrak{h}$ is the Cartan subalgebra of some finite-dimensional simple and simply-laced complex Lie algebra $\mathfrak{g}$, equipped with its standard scalar product $\langle\cdot,\cdot\rangle$ and norm $|\cdot|$ (see Subsection 2.1.2),
- the e_i, $1\leqslant i\leqslant r$ are the simple roots of the Lie algebra $\mathfrak{g}$ relative to $\mathfrak{h}$,
- $\langle\cdot,\cdot\rangle_g$ is the scalar product associated to the tangent space of $\mathfrak{h}$-valued functions defined on Σ,
- the constants μ_i $(1\leqslant i\leqslant r)$ are positive and are dubbed the cosmological constants,
- $\gamma>0$ is the coupling constant,
- Q is the $\mathfrak{h}$-valued background charge.

In order to ensure conformal symmetry, the background charge is related to the coupling constant via the relation $Q := (\gamma+\frac{2}{\gamma})\rho$ where ρ is the Weyl vector associated to the Cartan subalgebra $\mathfrak{h}$. Let us emphasize that one recovers LCFT when $\mathfrak{g}$ is the Lie Algebra $\mathfrak{sl}_2$ (of 2×2 complex matrices with vanishing trace, in which case $r=1$) and that the convention for γ in this paper differs by a scaling factor of $\sqrt{2}$ from the standard convention for LCFT in the probabilistic literature. The next section 2 is devoted to providing more details on these notations. At this stage it is worth noticing that this action functional has a relevant geometrical meaning in the context of *W-geometry*, introduced by Gervais and Matsuo in [17]. Indeed, in the semi-classical limit $\gamma\to 0$ such that for all i, $\mu_i=\frac{\Lambda}{\gamma^2}$ with fixed $\Lambda>0$, the function which minimizes the Toda action (rescaled by γ) converges towards the solution $u:\Sigma\to\mathfrak{h}$ (provided that it exists and is unique) of the *Toda equation* :

$$2\Delta_g u = R_g\rho + 4\pi\Lambda\sum_{i=1}^r e_i e^{\langle e_i,u\rangle} \quad \text{on } \Sigma. \tag{1.2}$$

This means that, in the quantum theory, the Toda field (rescaled by γ) will have a tendency to remain close to the (classical) solution of this $\mathfrak{h}$-valued equation. It is explained in [17]

that when Σ is the two-dimensional sphere $\mathbb{S}^2$, $\mathfrak{sl}_n$ Toda equations (where one adds appropriate conical singularities) can in some sense be interpreted as compatibility equations for a meromorphic embedding of a two-sphere into a complex projective plane $\mathbb{C}\mathrm{P}^n$, and therefore establishes a correspondence between solutions of the Toda equation (1.2) and certain meromorphic embeddings from $\mathbb{C}\mathrm{P}^1$ to $\mathbb{C}\mathrm{P}^n$, a problem which somehow provides a generalization of the celebrated *uniformisation of Riemann surfaces*.

In the context of the quantum theory, the Toda field φ should be understood as a random map from a Riemannian surface (Σ, g) to $\mathfrak{h}$ whose law can be defined via the expression (here F is some real-valued functional on a space of maps $\Sigma \to \mathfrak{h}$):

$$\mathbb{E}^g_{\gamma,\boldsymbol{\mu}}[F(\varphi)] := \frac{\langle F\rangle_{T,g}}{\langle 1\rangle_{T,g}} \tag{1.3}$$

where the quantities that appear in this definition are defined by the (formal) path integral

$$\langle F\rangle_{T,g} := \int F(X) e^{-S_{T,\mathfrak{g}}(X,g)} DX \tag{1.4}$$

and DX refers to a "uniform measure" on the space of square integrable $\mathfrak{h}$-valued maps defined on Σ; the normalization factor $\langle 1\rangle_{T,g}$ corresponds to the partition function $\int e^{-S_{T,\mathfrak{g}}(X,g)} DX$ of TCFTs. In the sequel, we will focus on the case where Σ is the Riemann sphere $\mathbb{S}^2$ (though the present framework can be extended to other topologies). In this case, the partition function is not defined (*i.e.* infinite) and one can only define quantities with insertion of a certain number of Vertex Operators, whose parameters obey the *Seiberg bounds* [30] in order to ensure existence of the corresponding correlation function (see next subsection).

1.2. **A probabilistic construction.** Though an algebraic approach to CFTs was developped shortly after the BPZ paper (see the notion of Vertex Operator Algebra [3, 15]), a probabilistic approach to conformal invariance was only developed recently following the introduction by Schramm [34] of random curves, called Schramm-Loewner Evolutions (SLEs), which describe (conjecturally at least) the interfaces of critical models of statistical physics (such as percolation or the Ising model). More recently, there has been a huge effort in probability theory to make sense of LCFT within the realm of random conformal geometry and the scaling limit of random planar maps (see [28, 29, 20, 9, 6, 8]). Another approach, based on the path integral formulation of LCFT in the physics literature, was developed in [4, 5, 21, 19] to give a rigorous probabilistic construction of the correlation functions of LCFT. This construction initiated a program [25] to lay the mathematical foundations of the conformal bootstrap procedure envisioned in physics by BPZ, namely that one can express the LCFT correlation functions in terms of representation theoretical special functions. The building blocks are an explicit formula for the three point correlation functions (or equivalently the structure constants) and a recursive procedure for the higher correlations. The explicit formula for the three-point structure constants discovered in the physics literature, the celebrated DOZZ formula, was recovered probabilistically in [26] and a probabilistic justification of the conformal bootstrap formalism for the higher order correlations was provided recently in [18].

Building on [4], our goal here is to provide a probabilistic definition of TCFTs in the case where we consider the underlying Lie algebra to be a finite-dimensional simple and simply-laced complex Lie algebra, and when the manifold on which the theory is constructed is

the (Riemann) sphere. To do so we follow the ideas developed in the case of LCFT (which corresponds to the $\mathfrak{sl}_2$ case) in [4] and interpret the path integral formulation of the theory as a formal way of defining a measure on some functional space. More precisely we interpret the mapping defined via the path integral (1.4) as a measure $F \mapsto \langle F \rangle_{T,g}$ on the Sobolev space with negative index $\mathrm{H}^{-1}(\mathbb{S}^2 \to \mathfrak{h}, g)$ (which we define in (2.4)); in order to construct this measure we introduce a probabilistic framework which involves two objects: the Gaussian Free Field (GFF) and the exponential of the GFF called Gaussian Multiplicative Chaos (GMC).

The presence of the GFF is related to the presence of the square gradient term in the Toda field action and has proven to be particularly relevant in the context of constructive conformal field theory. But as opposed to LCFT where only one GFF is involved in the construction, in TCFTs we have to consider several GFFs that are coupled in a way that is prescribed by the underlying structure of the Lie algebra. For this coupling to be mathematically meaningful we need to consider simple Lie algebras that are simply-laced, that is for which the underlying Cartan matrix (which is used as a covariance matrix in our construction) is symmetric. Under this assumption, the fields of TCFTs are well-defined but non-regular since they exist only in the sense of Schwartz distributions; therefore the exponential terms that appear in (1.1) are not well-defined objects. However, GMC theory provides a way of making sense of these terms as (random) Radon measures.

This interpretation allows to construct a regularized partition function by taking $F = 1$ in (1.4); however and similarly to the existence of Seiberg bounds in LCFT [30] this partition function will not converge in relation with an obstruction of geometrical nature: the Gauss-Bonnet theorem entails that classical Toda equations (*i.e* the equations of motion associated to this action) cannot admit solutions on the Riemann sphere. This difficulty can be overcome by looking at special functionals F that admit Vertex Operators as factors; by adding these extra terms to the measure the partition function becomes the correlation function of Vertex Operators and is predicted to exist as long as some conditions on these operators are satisfied. To define these Vertex Operators one relies on a regularization of the Toda field and introduces the regularized Vertex Operator $V_{\alpha,\varepsilon}(z)$, which is expressed for z on the Riemann sphere in terms of the Toda field φ and a *weight* $\alpha \in \mathfrak{h}$: up to constant terms, $V_{\alpha,\varepsilon}(z)$ is defined as $\varepsilon^{|\alpha|^2/2} e^{\langle \alpha, \varphi_\varepsilon \rangle}$ where φ_ε is the field φ smoothed up at scale ε (its definition will be made precise in Subsection 2.2.3). Our main result provides a necessary and sufficient condition that ensures the existence of the correlation functions (defined as limits of $\langle \prod_{k=1}^N V_{\alpha_k,\varepsilon}(z_k) \rangle_{T,g}$ when the cut-off ε is sent to 0). Moreover, the correlations are indeed conformally covariant as predicted by CFT:

Theorem 1.1. *Let $\mathfrak{g}$ be a finite-dimensional simple and simply-laced complex Lie algebra and assume that $\gamma \in (0, \sqrt{2})$. If g is any Riemannian metric in the conformal class of the standard round metric on the sphere $\hat{g}$, let $\langle \prod_{k=1}^N V_{\alpha_k,\varepsilon}(z_k) \rangle_{T,g}$ be the regularized correlation function of the $\mathfrak{g}$-Toda theory. Then:*

1. **(Seiberg bounds):** *The limit*

$$\langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle_{T,g} := \lim_{\varepsilon \to 0} \langle \prod_{k=1}^N V_{\alpha_k,\varepsilon}(z_k) \rangle_{T,g}$$

exists and is non trivial if and only if the two following conditions hold for all $i = 1, \dots, r$*:*

- $\langle \sum_{k=1}^{N} \alpha_k - 2Q, \omega_i \rangle > 0$ *where the* $(\omega_i)_{1 \leqslant i \leqslant r}$ *are the fundamental weights of* $\mathfrak{h}$*,*
- *for all* $1 \leqslant k \leqslant N$*,* $\langle \alpha_k, e_i \rangle < \langle Q, e_i \rangle = \gamma + \frac{2}{\gamma}$*.*

2. (Conformal covariance): *For any Möbius transform of the plane* ψ

$$\langle \prod_{k=1}^{N} V_{\alpha_k}(\psi(z_k)) \rangle_{T,g} = \prod_{k=1}^{N} |\psi'(z_k)|^{-2\Delta_{\alpha_k}} \langle \prod_{k=1}^{N} V_{\alpha_k}(z_k) \rangle_{T,g}.$$

where the conformal weights are given by $\Delta_{\alpha_j} := \langle \frac{\alpha_j}{2}, Q - \frac{\alpha_j}{2} \rangle$*.*

3. (Weyl anomaly): *For appropriate* φ *(more precisely* $\varphi \in \bar{C}^1(\mathbb{R}^2)$*: see notations in Section 2) then*

$$\langle \prod_{k=1}^{N} V_{\alpha_k}(z_k) \rangle_{T,e^{\varphi}\hat{g}} = e^{\frac{\mathbf{c}_T}{96\pi} S_L(\varphi,\hat{g})} \langle \prod_{k=1}^{N} V_{\alpha_k}(z_k) \rangle_{T,\hat{g}}$$

where S_L *is the Liouville functional (with vanishing cosmological constant)*

$$S_L(\varphi, \hat{g}) := \int_{\mathbb{S}^2} \left(|\partial_{\hat{g}} \varphi|^2_{\hat{g}} + 2R_{\hat{g}} \varphi \right) d\mathrm{v}_{\hat{g}},$$

and the central charge $\mathbf{c}_T$ *is given by* $\mathbf{c}_T = r + 6|Q|^2$*.*

The value of the central charge can be described explicitly in terms of the coupling constant γ and the underlying Lie algebra. Indeed, by the classification of finite-dimensional and simple complex Lie algebra we know that $\mathfrak{g}$, which is simply-laced, is necessarily isomorphic to $\mathfrak{sl}_n$ for $n \geqslant 2$, $\mathfrak{so}_{2n}$ for $n \geqslant 4$, E_6, E_7 or E_8 for which the central charge is explicit: see (2.18) or (3.6). Our main statement can be understood as a rigorous definition of the correlation functions of TCFTs, but also of the law of the Toda field φ (when we have fixed marked points $(\boldsymbol{z}, \boldsymbol{\alpha}) = (z_k, \alpha_k)_{1 \leqslant k \leqslant N}$ that satisfy the Seiberg bounds) by setting

$$\mathbb{E}^{(\boldsymbol{z},\boldsymbol{\alpha})}[F(\varphi)] := \frac{\langle F \prod_{i=1}^{N} V_{\alpha_i}(z_i) \rangle_{T,g}}{\langle \prod_{i=1}^{N} V_{\alpha_i}(z_i) \rangle_{T,g}} \tag{1.5}$$

where F is any bounded and continuous functional on $\mathrm{H}^{-1}(\mathbb{S}^2 \to \mathfrak{h}, g)$ and $\langle F \prod_{i=1}^{N} V_{\alpha_i}(z_i) \rangle_{T,g}$ is the limit of $\langle F \prod_{k=1}^{N} V_{\alpha_k,\varepsilon}(z_k) \rangle_{T,g}$ when the cut-off ε goes to 0 (this more general case can be handled similarly to the case $F = 1$).

Remark 1.2. *The above construction can be generalized when one considers as underlying Lie algebra any simply-laced,* **semisimple** *and complex Lie algebra. Indeed, from the classification of semisimple Lie algebras, such a Lie algebra can be written as a direct sum of* ***simple*** *Lie algebras* $\mathfrak{g} = \oplus_{k=1}^{p} \mathfrak{g}_k$*. Moreover a general property of TCFTs (which can be derived from the form of the Toda field action) is that for* A, B *two semisimple and simply-laced Lie algebras*

$$\langle \prod_{k=1}^{N} V_{(\alpha_k,\beta_k)}(z_k) \rangle^{A \oplus B}_{T,g} = \langle \prod_{k=1}^{N} V_{\alpha_k}(z_k) \rangle^{A}_{T,g} \langle \prod_{k=1}^{N} V_{\beta_k}(z_k) \rangle^{B}_{T,g}, \tag{1.6}$$

where with the notation $\langle\rangle^{\mathfrak{g}}_{T,g}$ we have stressed the dependence on the Lie Algebra $\mathfrak{g}$. This provides a way of constructing correlation functions for general finite-dimensional simply-laced and semisimple complex Lie algebras. The latter equation also implies that the central charges add up, in the sense that

$$\boldsymbol{c}_{T,A\oplus B} = \boldsymbol{c}_{T,A} + \boldsymbol{c}_{T,B} \tag{1.7}$$

where here again the notation $\boldsymbol{c}_{T,\mathfrak{g}}$ stresses the dependence on the Lie Algebra $\mathfrak{g}$.

2. Background and notations

2.1. Some reminders on conformal geometry and Lie algebras.

2.1.1. *Conformal geometry on the Riemann sphere.* The sphere $\mathbb{S}^2$ can be mapped by stereographic projection to the (compactified) plane (*i.e.* the Riemann sphere) which we view both as $\mathbb{R}^2\cup\{\infty\}$ and $\mathbb{C}\cup\{\infty\}$. We will work under this more convenient framework in the sequel.

Metrics on the Riemann sphere.

We will consider differentiable conformal metrics on the two-dimensional sphere $\mathbb{S}^2$; they can can be identified via stereographic projection with metrics on the plane of the form $g = e^{\varphi}\hat{g}$ with $\hat{g}$ is the standard round metric

$$\hat{g} \coloneqq \frac{4}{(1+|x|^2)^2}|dx|^2, \tag{2.1}$$

and $\varphi \in \bar{C}^1(\mathbb{R}^2)$ where, for $k \geqslant 0$, $\bar{C}^k(\mathbb{R}^2)$ stands for the space of functions $\varphi : \mathbb{R}^2 \to \mathbb{R}$ that are k-times differentiable with continuous derivatives as well as $x \mapsto \varphi(1/x)$ in a neighbourhood of $x = 0$. The reader may check that the metric $\hat{g}$ is the pushforward (via stereographic projection) of the standard metric on the Riemann sphere $\mathbb{S}^2$. We will thus work with such metrics g on the plane, for which we will denote by ∂_g the gradient, $\triangle_g$ the Laplace-Beltrami operator, $R_g = -\triangle_g \ln\sqrt{\det g}$ the Ricci scalar curvature and v_g the volume form. If $u, v \in \mathbb{R}^2$, we denote by $(u,v)_g$ the inner product with respect to the metric g ($|\cdot|_g$ stands for the associated norm). When no index is given, this means that the object has to be understood in terms of the usual Euclidean metric on the plane (*i.e.* ∂, $\triangle$, R, v and $(\cdot,\cdot)$). Since the stereographic projection is an isometry, we already know that the spherical metric $\hat{g}$ is such that $R_{\hat{g}} = 2$ (its Gaussian curvature is 1) with total mass $\mathrm{v}_{\hat{g}}(\mathbb{R}^2) = 4\pi$.

More generally, two metrics g and g' will be said to be conformally equivalent when

$$g = e^{\varphi} g'$$

for $\varphi \in \bar{C}^1(\mathbb{R}^2)$. It is readily seen that these conditions imply that $\int_{\mathbb{R}^2}\left(|\partial_{g'}\varphi|^2_{g'} + 2R_{g'}\varphi\right)\mathrm{d}\mathrm{v}_{g'} < \infty$ as soon as g' is in the conformal class of the spherical metric—that is when $g' = e^{\varphi}\hat{g}$ where φ is as above. Furthermore, for $\varphi \in \bar{C}^2(\mathbb{R}^2)$, the curvatures of two such metrics are related by the relation

$$R_g = e^{-\varphi}\big(R_{g'} - \Delta_{g'}\varphi\big). \tag{2.2}$$

In what follows and for given metrics g and $h \in \bar{C}^1(\mathbb{R}^2)$, we will denote by $m_g(h)$ the mean value of h in the metric g, that is the quantity

$$m_g(h) := \frac{1}{\mathrm{v}_g(\mathbb{R}^2)} \int_{\mathbb{R}^2} h(x)\, \mathrm{v}_g(dx) \tag{2.3}$$

and work in the Sobolev space $\mathrm{H}^1(\mathbb{R}^2, g)$, which is the closure of $C_c^\infty(\mathbb{R}^2)$ with respect to the Hilbert-norm

$$\int_{\mathbb{R}^2} h(x)^2\, \mathrm{v}_g(dx) + \int_{\mathbb{R}^2} |\partial_g h(x)|_g^2\, \mathrm{v}_g(dx). \tag{2.4}$$

The standard dual of $\mathrm{H}^1(\mathbb{R}^2, g)$ will be denoted $\mathrm{H}^{-1}(\mathbb{R}^2, g)$. It may be useful to note that the Dirichlet energy is a conformal invariant, that is to say is independent of the metric within a given conformal class:

$$\int_{\mathbb{R}^2} |\partial_{g'} h(x)|_{g'}^2\, \mathrm{v}_{g'}(dx) = \int_{\mathbb{R}^2} |\partial_g h(x)|_g^2\, \mathrm{v}_g(dx). \tag{2.5}$$

Green kernels. Given a metric g on the Riemann sphere that is conformally equivalent to the spherical metric $\hat{g}$, we denote by G_g the Green function of the problem

$$\triangle_g u = -2\pi\,(f - m_g(f)) \quad \text{on } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} u(x)\, \mathrm{v}_g(dx) = 0$$

where f belongs to the space $L^2(\mathbb{R}^2, g)$ and u is in $\mathrm{H}^1(\mathbb{R}^2, g)$. This means that the solution u can be expressed as

$$u = \int_{\mathbb{R}^2} G_g(\cdot, x) f(x) \mathrm{v}_g(dx) =: G_g f \tag{2.6}$$

with $m_g(G_g(x, \cdot)) = 0$ for all $x \in \mathbb{R}^2$. The kernel G_g has an explicit expression given by (see [4, Equation (2.9)])

$$G_g(x, y) = \ln \frac{1}{|x - y|} - m_g\left(\ln \frac{1}{|x - \cdot|}\right) - m_g\left(\ln \frac{1}{|y - \cdot|}\right) + \theta_g \tag{2.7}$$

where

$$\theta_g := \frac{1}{\mathrm{v}_g(\mathbb{R}^2)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln \frac{1}{|x - y|} \mathrm{v}_g(dx) \mathrm{v}_g(dy).$$

For instance for the spherical metric this becomes

$$G_{\hat{g}}(x, y) = \ln \frac{1}{|x - y|} - \frac{1}{4}\left(\ln \hat{g}(x) + \ln \hat{g}(y)\right) + \ln 2 - \frac{1}{2}. \tag{2.8}$$

Another well-known property of these Green functions (see [4, Proposition 2.2] for instance) is that they are conformally covariant in the sense that:

Lemma 2.1 (Conformal covariance)**.** *Let ψ be a Möbius transform of the Riemann sphere and g be a Riemannian metric conformally equivalent to the spherical one. Then*

$$G_{g_\psi}(x, y) = G_g(\psi(x), \psi(y)) \tag{2.9}$$

where $g_\psi(z) = |\psi'(z)|^2\, g(\psi(z))$ is the pullback of the metric g by ψ.

Again let us register what happens for the spherical metric:

$$G_{\hat{g}}(\psi(x),\psi(y)) = G_{\hat{g}}(x,y) - \frac{1}{4}(\phi(x)+\phi(y)) \tag{2.10}$$

where ϕ is such that $e^{\phi} = \frac{\hat{g}_\psi}{\hat{g}}$.

2.1.2. *Lie algebras and the Toda field action.* .

Finite-dimensional simple Lie algebras. The special linear group $SL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}); \det(A) = 1\}$ can be endowed with the structure of a smooth manifold of (complex) dimension n^2-1. When equipped with the group operation of matrix multiplication, it becomes a Lie group. Its tangent space at identity $\mathfrak{sl}_n$ consists of all $n \times n$ complex matrices with trace 0 and becomes a Lie algebra, called the special linear Lie algebra, when equipped with the usual Lie bracket

$$[A,B] \coloneqq AB - BA \quad \text{for} \quad A,B \in \mathfrak{sl}_n. \tag{2.11}$$

The adjoint action ad is the map $\mathrm{ad}_A(B) \coloneqq [A,B]$ for $A,B \in \mathfrak{sl}_n$. The $\mathfrak{sl}_n$ Lie algebra is one of the simplest instances of *complex simple Lie algebras*. The interested reader may find more details on the notion of Lie algebra for instance in the textbook [22]; for the purpose of the present document we shed light on the fact that in the study of finite-dimensional complex simple Lie algebras there is a key subalgebra that naturally arises: the so-called *Cartan subalgebra* of the Lie algebra $\mathfrak{g}$, usually denoted $\mathfrak{h}$. In our context it can be defined as a maximal commutative subalgebra of $\mathfrak{g}$ for which the adjoint action ad_H is diagonalizable for each $H \in \mathfrak{h}$. Such Cartan algebras exist and are unique up to isomorphism: in the case of $\mathfrak{sl}_n$, it can be realized as the set of diagonal matrices whose diagonal entries sum up to 0, hence can be identified with $\mathfrak{h}_n \coloneqq \{x = (x_1,\dots,x_n) \in \mathbb{C}^n \mid \sum_{i=1}^n x_i = 0\}$.

When working with complex simple Lie algebras we are naturally led to introduce the *roots* of the Lie algebra $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{h}$; these are the linear functionals α acting on $\mathfrak{h}$ such that the set $\{x \in \mathfrak{g} | \quad \forall H \in \mathfrak{h}, \quad [H,x] = \alpha(H)x\}$ is not restricted to $\{0\}$. Among these roots it is possible to find a basis $(e_i)_{1 \leqslant i \leqslant n}$ of the space $\mathfrak{h}^*$ (of linear functionals on $\mathfrak{h}$) such that any root can be written as a linear combination of this basis using only integers with same sign: these are the so-called *simple roots of* $\mathfrak{g}$ *relative to* $\mathfrak{h}$. This Cartan subalgebra also comes equipped with a scalar product, the so-called *Killing form* B, inherited from the Lie algebra $\mathfrak{g}$ and which is defined by

$$B(x,y) \coloneqq \mathrm{tr}\left(\mathrm{ad}_x \circ \mathrm{ad}_y\right). \tag{2.12}$$

A convenient way to classify finite-dimensional and simple complex Lie algebras is to use their Cartan matrix. This $r \times r$ matrix (r is the rank of the simple Lie algebra, *i.e.* the dimension of $\mathfrak{h}$) is defined by

$$A_{i,j} \coloneqq 2\frac{B(e_i,e_j)}{B(e_i,e_i)}. \tag{2.13}$$

The entries of this matrix are integral, equal to 2 on the diagonal and non-positive elsewhere; the matrix is invertible. For instance the $\mathfrak{sl}_n$ Cartan matrix is tridiagonal with 2 on the diagonal and -1 on the entries (i,j) with $|i-j| = 1$. In the sequel and instead of working with the Killing form B we will use the scalar product on $\mathfrak{h}^*$ defined on its basis by $\langle e_i, e_j\rangle \coloneqq A_{i,j}$. These two scalar products only differ by a multiplicative constant, usually referred to as the

Dynkin index of the adjoint representation, and which is equal to $2g$, where g is the so-called *dual Coxeter number*.

It is very natural to introduce the basis of the *fundamental weights* $(\omega_i)_{1\leqslant i\leqslant r}$, that is the basis of $\mathfrak{h}^*$ dual to that of the simple roots:

$$\omega_i \coloneqq \sum_{l=1}^{r}(A^{-1})_{i,l}e_l. \tag{2.14}$$

They are defined so that (δ_{ij} is the Kronecker symbol)

$$\langle e_i^\vee,\omega_j\rangle=\delta_{ij},\quad \langle\omega_i,\omega_j\rangle=\sum_{l,l'=1}^{r}(A^{-1})_{i,l}A_{l,l'}(A^{-1})_{l',j}=(A^{-1})_{i,j} \tag{2.15}$$

where $e_i^\vee\coloneqq 2\frac{e_i}{\langle e_i,e_i\rangle}$ is the coroot associated to e_i. The *Weyl vector* which is defined by

$$\rho\coloneqq\sum_{i=1}^{r}\omega_i \tag{2.16}$$

naturally enjoys the property that $\langle\rho,e_i^\vee\rangle=1$ for all $1\leqslant i\leqslant r$. The squared norm of this vector can be expressed explicitly depending on the Lie algebra under consideration via the *Freudenthal-de Vries strange formula* for simple Lie algebras [16, Equation (47.11)][1]

$$|\rho|^2=\frac{g\dim\mathfrak{g}}{12}. \tag{2.17}$$

For finite-dimensional simple complex Lie algebras this quantity is given by

$$\frac{(n-1)n(n+1)}{12}\text{ for }\mathfrak{sl}_n,\quad \frac{(n-1)n(2n-1)}{6}\text{ for }\mathfrak{so}_{2n} \tag{2.18}$$

and 78, $\frac{399}{2}$, 620 for the exceptional Lie algebras E_6, E_7 and E_8. To finish this quick introductory part on simple Lie algebras let us mention the following duality relation between two vectors $\alpha,\beta\in\mathfrak{h}^*$

$$\langle\alpha,\beta\rangle=\sum_{i=1}^{r}\langle\alpha,\omega_i\rangle\langle\beta,e_i^\vee\rangle. \tag{2.19}$$

In the sequel we will work with simple Lie algebras whose Cartan matrices are symmetric (this is actually the definition of simply-laced simple Lie algebras); under this framework coroots and roots are equal, *i.e.* $e_i^\vee=e_i$. By the classification of finite-dimensional Lie algebras, the Lie algebra is then isomorphic to the special linear Lie algebra $\mathfrak{sl}_n$ (for $n\geqslant 2$), the even-dimensional special orthogonal Lie algebra $\mathfrak{so}_{2n}$ (for $n\geqslant 4$) or one of the exceptional Lie algebras E_6, E_7 and E_8.

[1]This equation differs from the one in [16] by a multiplicative factor $2g$. This is due to our normalization convention for the scalar product $\langle\cdot,\cdot\rangle$ on $\mathfrak{h}^*$.

Toda field action. Given φ and ϕ two differentiable maps from $\mathbb{R}^2$ to $\mathfrak{h}^*$[2], $\varphi = \sum_{i=1}^r \varphi_i \omega_i$ and $\phi = \sum_{i=1}^r \phi_i \omega_i$, let us set

$$\langle \partial_g \varphi, \partial_g \phi \rangle_g \coloneqq \sum_{i,j=1}^r \langle \omega_i, \omega_j \rangle (\partial_g \varphi_i, \partial_g \phi_j)_g. \tag{2.20}$$

Recall that we have defined the Toda field action $S_{T,\mathfrak{g}}$ in the metric g for the Lie algebra $\mathfrak{g}$ by the expression

$$S_{T,\mathfrak{g}}(\phi, g) \coloneqq \frac{1}{4\pi} \int_{\mathbb{R}^2} \Big(\langle \partial_g \phi(x), \partial_g \phi(x) \rangle_g + R_g \langle Q, \phi(x) \rangle + 4\pi \sum_{i=1}^r \mu_i e^{\gamma \langle e_i, \varphi(x) \rangle} \Big) \mathrm{v}_g(dx) \tag{2.21}$$

where

$$Q \coloneqq (\gamma + \frac{2}{\gamma})\rho \tag{2.22}$$

is the background charge, $\boldsymbol{\mu} \coloneqq (\mu_1 > 0, \cdots, \mu_r > 0)$ are the cosmological constants and $\gamma > 0$ is the coupling constant. In the sequel, we assume that γ satisfies the condition

$$\gamma \in (0, \sqrt{2}). \tag{2.23}$$

The condition on γ is the optimal condition ensuring that the probabilistic construction makes sense[3]: this will become clear later when connecting to GMC theory. From the definition of Q we already know that for all $1 \leqslant i \leqslant r$,

$$\langle Q, e_i \rangle = \gamma + \frac{2}{\gamma} \quad \text{and} \quad |Q|^2 \coloneqq \langle Q, Q \rangle = (\gamma + \frac{2}{\gamma})^2 |\rho|^2 .$$

The path integral we aim to construct corresponds to a measure on a suitable space of maps $\phi : \mathbb{R}^2 \to \mathfrak{h}^*$ formally corresponding to

$$e^{-S_{T,\mathfrak{g}}(\phi, g)} D\phi. \tag{2.24}$$

As we will explain below, this can be achieved thanks to the introduction of the GFF.

2.2. **Probabilistic interpretation of the path integral.**

2.2.1. *Gaussian measure interpretation of the squared gradient.* The Toda field action can be decomposed as a sum of different terms. The first one is the quadratic term

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} \langle \partial_g \varphi(x), \partial_g \varphi(x) \rangle_g v_g(dx) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \langle \varphi(x), -\triangle_g \varphi(x) \rangle_g v_g(dx) =: \langle \varphi, -\triangle_g \varphi \rangle_g$$

which is reminiscent of a Gaussian measure. Indeed, the measure formally written as

$$e^{-\frac{1}{2} \langle \varphi, -\triangle_g \varphi \rangle_g} D\varphi, \tag{2.25}$$

[2]The scalar field φ being studied in TCFTs usually has values in $\mathfrak{h}$. To keep notations simple we adopt the convention that φ actually takes values in the space of roots $\mathfrak{h}^*$. This identification is possible thanks to the Riesz representation theorem.

[3]Recall that here our convention on γ is different from the standard convention for LCFT in the probabilistic literature by a factor of $\sqrt{2}$.

when restricted to the space

$$\Sigma := \{\varphi \in \mathrm{H}^{-1}(\mathbb{R}^2 \to \mathfrak{h}^*, g); \int_{\mathbb{R}^2} \varphi(x)\, \mathrm{v}_g(dx) = 0\}$$

where $\mathrm{H}^{-1}(\mathbb{R}^2 \to \mathfrak{h}^*, g)$ is the set of $\mathfrak{h}^*$-valued (generalized) functions with each component (with respect to a basis of $\mathfrak{h}^*$) in $\mathrm{H}^{-1}(\mathbb{R}^2, g)$, can be understood as the measure on a Gaussian space $(\langle \varphi, -\triangle_g f\rangle_g)_{f\in \mathrm{H}^1}$ with covariance kernel given by $\langle h, -\triangle_g f\rangle_g$. In other words we are looking for a Gaussian field enjoying the property that

$$\mathbb{E}\left[\langle \varphi, -\triangle_g f\rangle_g \langle \varphi, -\triangle_g h\rangle_g\right] = \langle h, -\triangle_g f\rangle_g$$

for $f, g \in \mathrm{H}^1(\mathbb{R}^2 \to \mathfrak{h}^*, g)$. When $r = 1$, this is achieved by introducing the GFF X_g with vanishing v_g-mean on the sphere, that is a centered Gaussian random distribution with covariance kernel given by the Green function G_g (see [7, 35] for more details on this object). An important feature of the GFF is that it is not defined pointwise but rather belongs to the distributional space $\mathrm{H}^{-1}(\mathbb{R}^2, g)$. For generic rank this is done by considering additional fields and setting

$$X^g := \sum_{i=1}^{r} X_i^g \omega_i, \tag{2.26}$$

where $X_1^g, \dots, X_r^g$ are r such GFFs with covariance structure given by

$$\mathbb{E}[X_i^g(x) X_j^g(y)] = A_{i,j} G_g(x, y). \tag{2.27}$$

Remark 2.2. *The reason why we must assume the underlying Lie algebra to be simply-laced is due to the fact that the Cartan matrix is symmetric only in that case, so that it is indeed possible to construct GFFs with covariance kernel given by* (2.27).

To summarize we may wish to interpret the formal Gaussian measure (2.25) restricted to Σ as

$$Z(g)^{-1} \int F(\varphi) e^{-\frac{1}{4\pi}\int_{\mathbb{R}^2} \langle \partial_g \varphi(x), \partial_g \varphi(x)\rangle_g \, \mathrm{v}_g(dx)} D\varphi = \mathbb{E}\Big[F(X^g)\Big]$$

for each continuous and bounded functional F on $\mathrm{H}^{-1}(\mathbb{R}^2 \to \mathfrak{h}^*, g)$, where $Z(g)$ stands for the total mass of the Gaussian integral (2.25)

$$\text{“}Z(g) := \det(\mathbf{\Sigma})^{-\frac{1}{2}}\text{”},$$

where $\mathbf{\Sigma}$ is the covariance matrix

$$\mathbf{\Sigma} := \frac{-\triangle_g}{\mathrm{Vol}_{\mathrm{g}}(\mathbb{R}^2)} A \tag{2.28}$$

with A the Cartan matrix of $\mathfrak{g}$ and $\det(\mathbf{\Sigma})$ is given by a regularized determinant. We know how this factor varies when we consider a conformal change of metric: it is proved in [31, Equation (1.13)] that

$$\log Z(e^{\varphi} g) = \log Z(g) + \frac{r}{96\pi} \int_{\mathbb{R}^2} \Big(|\mathrm{d}\varphi|_g^2 + 2R_g \varphi\Big)\, \mathrm{v}_g(dx). \tag{2.29}$$

As a consequence, up to a global factor, one has

$$Z(e^{\varphi}\hat{g}) = \det(A)^{-\frac{1}{2}} e^{\frac{r}{96\pi}\int_{\mathbb{R}^2}\left(|\mathrm{d}\varphi|_{\hat{g}}^2 + 2R_{\hat{g}}\varphi\right)\mathrm{v}_g(dx)} \tag{2.30}$$

within the conformal class of the spherical metric $\hat{g}$.

However in the above construction we do not take into account the fact that the GFF X_g has zero mean in the metric g; to overcome this issue we will introduce so-called *zero modes* in the interpretation of the squared gradient term as a Gaussian measure. To do so we introduce the Lebesgue measure $d\boldsymbol{c}$ on $\mathfrak{h}^*$ by setting for each positive measurable function $F : \mathfrak{h}^* \to \mathbb{R}$

$$\int_{\mathfrak{h}^*} F(\boldsymbol{c})\, d\boldsymbol{c} := \det(A)^{\frac{1}{2}} \int_{\mathbb{R}^r} F\Big(\sum_{i=1}^r c_i \omega_i\Big)\, dc_1 \dots dc_r, \tag{2.31}$$

where $dc_1, \dots, dc_r$ stands for the Lebesgue measure with respect to each variable c_i[4]. By doing so we are led to the following probabilistic interpretation of the formal (full) Gaussian measure (2.25)

$$\int F(\varphi) e^{-\frac{1}{2}\langle \varphi, -\triangle_g \varphi\rangle_g} D\varphi = Z(g) \int_{\mathfrak{h}^*} \mathbb{E}\Big[F\Big(X^g + \boldsymbol{c}\Big)\Big]\, d\boldsymbol{c} \tag{2.32}$$

for each continuous and bounded functional F on $\mathrm{H}^{-1}(\mathbb{R}^2 \to \mathfrak{h}^*, g)$ and g in the conformal class of the spherical metric.

2.2.2. *Gaussian Multiplicative Chaos interpretation of the exponential potential.* It remains to treat the other terms that appear in the Toda field action (1.1):

$$\frac{1}{4\pi} \int_{\mathbb{R}^2} \Big(R_g(x) \langle Q, \varphi(x)\rangle + 4\pi \sum_{i=1}^r \mu_i e^{\gamma \langle e_i, \varphi(x)\rangle} \Big)\, \mathrm{v}_g(dx).$$

The first term perfectly makes sense if we remember that the GFF has a meaning in the distributional sense. However the second term does not make sense because of the lack of regularity of the field. For it to be meaningful we need to make use of the notion of *Gaussian Multiplicative Chaos* (see [24, 33]) which relies on a proper renormalization of some regularization of the GFF.

Definition 2.3. *Let $\eta_\varepsilon := \frac{1}{\varepsilon^2}\eta(\frac{\cdot}{\varepsilon})$ be a smooth mollifier. We define the regularized field X_ε by considering the convolution approximation of X:*

$$X_\varepsilon := X * \eta_\varepsilon. \tag{2.33}$$

Usually one considers the regularization given by averaging the field on circles with radius ε. The reasoning is exactly the same.

Definition 2.4. *Assume that $\gamma < \sqrt{2}$. From (2.38) and basics of GMC theory [24, 33], we know that the following convergence holds in probability in the space of Radon measures (equipped with the weak topology):*

$$M^{\gamma e_i, g}(dx) := \lim_{\varepsilon \to 0} e^{\langle \gamma e_i, X^g_\varepsilon(x)\rangle - \frac{1}{2}\mathbb{E}\left[\langle \gamma e_i, X^g_\varepsilon(x)\rangle^2\right]} \mathrm{v}_g(dx). \tag{2.34}$$

The random measure $M^{\gamma e_i, g}(dx)$ is non trivial (i.e. different from 0) and is called the Gaussian Multiplicative Chaos (GMC) measure associated to the field $\langle \gamma e_i, X^g\rangle$.

[4]The prefactor $\det(A)^{\frac{1}{2}}$ in the equation comes from the fact that the basis $(\omega_i)_{1 \leqslant i \leqslant r}$ is not orthonormal.

More generally the GMC measure

$$M^{\alpha,g}(dx) := \lim_{\varepsilon\to 0} \varepsilon^{\frac{|\alpha|^2}{2}} e^{\langle \alpha, X^g_\varepsilon(x)\rangle} \mathrm{v}_g(dx)$$

exists and is non trivial if and only if $|\alpha|^2 := \langle \alpha, \alpha\rangle < 4$.

Remark 2.5. *The statement of* [4, Proposition (2.5)] *can be easily adapted to show that the regularized GFF thus defined has a variance which evolves as*

$$\mathbb{E}\left[X^{\hat g}_{i,\varepsilon}(x) X^{\hat g}_{j,\varepsilon}(x)\right] = A_{i,j}\left(-\ln\varepsilon - \frac{1}{2}\ln \hat g(x) + \theta_\eta + o(1)\right)$$

when ε goes to 0, and where we have set $\theta_\eta := \int_{\mathbb{C}}\int_{\mathbb{C}} \eta(x)\eta(y)\ln\frac{1}{|x-y|} v(dx)v(dy) + \ln 2 - \frac{1}{2}$ (recall that v denotes the standard Lebesgue measure). As a consequence the GMC measure defined above and the limiting measure defined by

$$\lim_{\varepsilon\to 0} \varepsilon^{\gamma^2} e^{\langle \gamma e_i, X^{\hat g}_\varepsilon(x) + \frac{Q}{2}\ln \hat g - \frac{\gamma e_i \theta_\eta}{2}\rangle} v(dx)$$

actually define the same random measure.

In the end we interpret the path integral of TCFTs in a probabilistic way by making the identification for $F \in \mathrm{H}^{-1}(\mathbb{R}^2 \to \mathfrak{h}^*, g)$:

$$\begin{aligned} &Z(g)^{-1}\int_\Sigma F(\varphi) e^{-S_{T,\mathfrak{g}}(\varphi,g)} D\varphi \\ &:= \lim_{\varepsilon\to 0}\int_{\mathfrak{h}^*} \mathbb{E}\Big[F\Big(X^g_\varepsilon + \frac{Q}{2}\ln g + \boldsymbol{c}\Big) e^{-\frac{1}{4\pi}\int_{\mathbb{R}^2} R_g(x)\langle Q, X^g_\varepsilon(x)+\boldsymbol{c}\rangle \mathrm{v}_g(dx) - \sum_{i=1}^r \mu_i e^{\gamma\langle e_i,\boldsymbol{c}\rangle} M^{\gamma e_i, g}_\varepsilon(\mathbb{C})}\Big] d\boldsymbol{c}, \end{aligned} \tag{2.35}$$

when the limit exists and where $g = e^\varphi \hat g$ is in the conformal class of the spherical metric.

2.2.3. *Vertex Operators.* There is a class of functionals F which play a key role in the study of TCFTs; usually referred to as *Vertex Operators*, computing their correlation functions is often one of the main issue in the study of two-dimensional CFTs. As we will see below, they enjoy a certain conformal covariance identity which is the starting point in the understanding of the theory.

In TCFTs these Vertex Operators formally correspond to taking $F = e^{\langle \alpha, \varphi(z)\rangle}$ for $z \in \mathbb{R}^2$ and $\alpha \in \mathfrak{h}^*$ but since such F are not defined on $\mathrm{H}^{-1}(\mathbb{R}^2 \to \mathfrak{h}^*, g)$, one must use a regularization procedure in order to define their correlations. This motivates the following definition:

Definition 2.6. *For $z \in \mathbb{C}$ and $\alpha \in \mathfrak{h}^*$ the regularized Vertex Operator $V^g_{\alpha,\varepsilon}(z)$ is defined by*

$$V^g_{\alpha,\varepsilon}(z) := \varepsilon^{\frac{|\alpha|^2}{2}} e^{\langle \alpha, X^g_\varepsilon(z) + \frac{Q}{2}\ln g + \boldsymbol{c} - \frac{\alpha\theta_\eta}{2}\rangle} \tag{2.36}$$

where $X^g_\varepsilon(z)$ is the field regularized as above.

Similarly to the GMC measure, this regularized Vertex Operator has same limit when $\varepsilon \to 0$ as the Wick exponential

$$e^{\langle \alpha, X^{\hat g}_\varepsilon(x) + \boldsymbol{c}\rangle - \frac{1}{2}\mathbb{E}\left[\langle \alpha, X^{\hat g}_\varepsilon(x)\rangle^2\right]} \hat g(x)^{\langle \frac{\alpha}{2}, Q - \frac{\alpha}{2}\rangle}. \tag{2.37}$$

Indeed for $\alpha \in \mathfrak{h}^*$ the variable $\langle \alpha, X_\varepsilon^g \rangle$ is given by the following formula

$$\langle \alpha, X_\varepsilon^g \rangle = \sum_{i=1}^r \langle \alpha, \omega_i \rangle X_{i,\varepsilon}^g$$

and thus has the following covariance structure, for $\alpha, \beta \in \mathfrak{h}^*$

$$\mathbb{E}[\langle \alpha, X_\varepsilon^g(x) \rangle \langle \beta, X_\varepsilon^g(y) \rangle] = \langle \alpha, \beta \rangle G_{g,\varepsilon}(x,y) \tag{2.38}$$

where $G_{g,\varepsilon}$ is the covariance kernel of the field X_ε^g.

3. General definitions and existence theorems

Having introduced the probabilistic tools allowing to translate in mathematical terms the path integral formulation of TCFTs, we are now ready to investigate the existence of the partition function and of the correlation functions.

3.1. TCFT measure and correlation functions.

To start with, recall that we have given in (2.35) a meaning to the measure on the space $\mathrm{H}^{-1}(\mathbb{R}^2 \to \mathfrak{h}^*, g)$ formally defined by the expression (2.24) by setting for F a positive measurable function on $\mathrm{H}^{-1}(\mathbb{R}^2 \to \mathfrak{h}^*, g)$:

$$\langle F \rangle_{T,g} := Z(g) \lim_{\varepsilon \to 0} \int_{\mathfrak{h}^*} \mathbb{E}\Big[F\Big(X_\varepsilon^g + \frac{Q}{2} \ln g + \boldsymbol{c} \Big) e^{-\frac{1}{4\pi} \int_{\mathbb{R}^2} R_g(x) \langle Q, X_\varepsilon^g(x) + \boldsymbol{c} \rangle \mathrm{v}_g(dx) - \sum_{i=1}^r \mu_i e^{\gamma \langle e_i, \boldsymbol{c} \rangle} M_\varepsilon^{\gamma e_i, g}(\mathbb{C})} \Big] \, d\boldsymbol{c}.$$

The mapping $F \mapsto \langle F \rangle_{T,g}$ generates a measure on $\mathrm{H}^{-1}(\mathbb{R}^2 \to \mathfrak{h}^*, g)$. It is worth noticing that this measure has infinite mass because of the $\boldsymbol{c} \to -\infty$ behaviour of the integrand, but is non trivial as we will see below.

Indeed, there is a special family of functions F that deserve special attention: these are the Vertex Operators which allow to define the correlations function of TCFTs. Fix an integer $N \geqslant 1$ and N distinct points $z_1, \dots, z_N \in \mathbb{C}$ with respective associated weights $\alpha_1, \dots, \alpha_N \in \mathfrak{h}^*$. The regularized correlation function is defined by the expression :

$$\begin{aligned} &\langle V_{\alpha_1,\varepsilon}(z_1) \cdots V_{\alpha_N,\varepsilon}(z_N) \rangle_{T,g} := \\ &\quad Z(g) \int_{\mathfrak{h}^*} \mathbb{E}\left[\Big(\prod_{k=1}^N V_{\varepsilon,\alpha_k}^g(z_k) \Big) e^{-\frac{1}{4\pi} \int_{\mathbb{R}^2} R_g(x) \langle Q, X_\varepsilon^g(x) + \boldsymbol{c} \rangle \mathrm{v}_g(dx) - \sum_{i=1}^r \mu_i e^{\gamma \langle \boldsymbol{c}, e_i \rangle} M_\varepsilon^{\gamma e_i, g}(\mathbb{C})} \right] d\boldsymbol{c}. \end{aligned} \tag{3.1}$$

The correlation function is then set to be the limit when $\varepsilon \to 0$ of the above regularization:

$$\langle V_{\alpha_1}(z_1) \cdots V_{\alpha_N}(z_N) \rangle_{T,g} := \lim_{\varepsilon \to 0} \langle V_{\alpha_1,\varepsilon}(z_1) \cdots V_{\alpha_N,\varepsilon}(z_N) \rangle_{T,g}. \tag{3.2}$$

The next subsection is devoted to the study of the convergence of this correlation function.

3.2. Existence of the correlation function.

The form of the correlation function (3.1) is not really convenient when it comes to investigating its convergence as $\varepsilon \to 0$. To obtain a reformulation of the correlation functions, we introduce the random measures

$$Z_{(\boldsymbol{z},\boldsymbol{\alpha})}^{\gamma e_i}(dx) := e^{\gamma \sum_{j=1}^N \langle \alpha_j, e_i \rangle G_{\hat{g}}(x, z_j)} M^{\gamma e_i, \hat{g}}(dx) \tag{3.3}$$

and we define

$$\boldsymbol{s} := \frac{\sum_{j=1}^N \alpha_j - 2Q}{\gamma} \quad \text{as well as, for all } i, \quad s_i := \frac{\langle \sum_{j=1}^N \alpha_j - 2Q, \omega_i \rangle}{\gamma}. \tag{3.4}$$

The expression of the Vertex Operator as a Wick exponential (2.37) allows us to interpret the product in the regularized correlation function as a Girsanov transform (see Theorem 6.3), and has the effect of shifting the law of the GFF X^g. This reformulation is essential to prove the following result:

Theorem 3.1. *Existence and non triviality of the correlation function $\langle V_{\alpha_1}(z_1)\cdots V_{\alpha_N}(z_N)\rangle_{T,g}$ do not depend on the background metric g in the conformal class of the spherical metric. Furthermore:*

1. **(Seiberg bounds):** *The correlation function $\langle V_{\alpha_1}(z_1)\cdots V_{\alpha_N}(z_N)\rangle_{T,g}$ exists and is non trivial if and only if the two following conditions hold for all $i=1,\dots,r$:*
   - $s_i > 0$,
   - *for all* $1\leqslant k\leqslant N$, $\langle \alpha_k, e_i\rangle < \langle Q, e_i\rangle = \gamma+\frac{2}{\gamma}$.
2. **(Conformal covariance):** *Let ψ be a Möbius transform of the plane. Then*
$$\langle V_{\alpha_1}(\psi(z_1))\cdots V_{\alpha_N}(\psi(z_N))\rangle_{T,g} = \prod_{k=1}^N |\psi'(z_k)|^{-2\Delta_{\alpha_k}}\,\langle V_{\alpha_1}(z_1)\cdots V_{\alpha_N}(z_N)\rangle_{T,g}.$$
   where the conformal weights are given by $\Delta_{\alpha_j} := \langle \frac{\alpha_j}{2}, Q-\frac{\alpha_j}{2}\rangle$.
3. **(Weyl anomaly):** *If $\varphi\in \bar{C}^1(\mathbb{R}^2)$ then we have the following relation*
$$\langle V_{\alpha_1}(z_1)\cdots V_{\alpha_N}(z_N)\rangle_{T,e^{\varphi}\hat{g}} = e^{\frac{\mathbf{c}_T}{96\pi}S_L(\varphi,\hat{g})}\langle V_{\alpha_1}(z_1)\cdots V_{\alpha_N}(z_N)\rangle_{T,\hat{g}}$$
   where S_L is the Liouville functional
$$S_L(\varphi,\hat{g}) := \int_{\mathbb{R}^2}\left(|\partial_{\hat{g}}\varphi|^2_{\hat{g}} + 2R_{\hat{g}}\varphi\right)\mathrm{dv}_{\hat{g}},$$
   and the central charge $\mathbf{c}_T$ is given by $\mathbf{c}_T = r+6|Q|^2$.
4. **(GMC representation):** *In the particular case where $g=\hat{g}$ is the round metric, one gets the following explicit expression for the correlation function*

$$\begin{aligned}\langle V_{\alpha_1}(z_1)\cdots V_{\alpha_N}(z_N)\rangle_{T,\hat{g}} \\ = \left(\prod_{i=1}^r \frac{\Gamma(s_i)\mu_i^{-s_i}}{\gamma}\right)\prod_{k=1}^N \hat{g}(z_k)^{\Delta_{\alpha_k}} e^{\sum_{k<l}\langle\alpha_k,\alpha_l\rangle G_{\hat{g}}(z_k,z_l)}\mathbb{E}\left[\prod_{i=1}^r Z^{\gamma e_i}_{(\boldsymbol{z},\boldsymbol{\alpha})}(\mathbb{C})^{-s_i}\right].\end{aligned} \tag{3.5}$$

The items 2 and 3 above characterize the theory as a CFT with central charge $\mathbf{c}_T$, where $\boldsymbol{c}_T$ is given for $\mathfrak{g}$ simple by:

(3.6)

$\mathfrak{g}$	$\boldsymbol{c}_{T,\mathfrak{g}}$
$\mathfrak{sl}_n$	$n-1+\frac{(n-1)n(n+1)}{2}q^2$
$\mathfrak{so}_{2n}$	$n+(n-1)n(2n-1)q^2$
E_6	$6+468q^2$
E_7	$7+1197q^2$
E_8	$8+3720q^2$

where $q := \gamma+\frac{2}{\gamma}$.

Proof. To start with, and anticipating on the conformal anomaly formula, we can assume that we work with the spherical metric $\hat{g}$, which is such that, since $X^{\hat{g}}$ has zero mean value in the metric $\hat{g}$, $\frac{1}{4\pi}\int_{\mathbb{C}}\langle Q+X^{\hat{g}},\boldsymbol{c}\rangle R_{\hat{g}}(x)v_{\hat{g}}(dx)=2\langle Q,\boldsymbol{c}\rangle$. As a consequence the expression of the Vertex operators $V_{\alpha_k,\varepsilon}(z_k)$ as Wick exponentials (2.37) allow to interpret them as Girsanov weights that have the effect of shifitng the law of the GFF by a additive term $G_{\hat{g},\varepsilon}(\cdot,z_k)$ where $G_{\hat{g},\varepsilon}$ is a mollified version of $G_{\hat{g}}$ (see Theorem 6.3). This allows to rewrite the regularized correlation function as

$$\prod_{k=1}^{N}\hat{g}(z_k)^{\Delta_{\alpha_k}}e^{\sum_{k<l}\langle\alpha_k,\alpha_l\rangle G_{\hat{g},\varepsilon}(z_k,z_l)}\int_{\mathbb{R}^r}e^{\sum_{i=1}^r s_ic_i}\mathbb{E}\left[e^{-\sum_{i=1}^r\mu_ie^{\gamma\langle\boldsymbol{c},e_i\rangle}Z^{\gamma e_i}_{(\boldsymbol{z},\boldsymbol{\alpha}),\varepsilon}(\mathbb{C})}\right]dc_1...dc_r$$

where $Z^{\gamma e_i}_{(\boldsymbol{z},\boldsymbol{\alpha}),\varepsilon}(dx):=e^{\gamma\sum_{k=1}^N\langle\alpha_k,e_i\rangle G_{\hat{g},\varepsilon}(x,z_j)}M^{\gamma e_i,\hat{g}}(dx)$. Now, if one of the s_i is non-positive, then the whole integral can be lower-bounded by

$$\int_{\mathbb{R}^r}e^{\sum_{i=1}^r s_ic_i}e^{-\sum_{i=1}^r\mu_ie^{\gamma\langle\boldsymbol{c},e_i\rangle}M}dc_1...dc_r\mathbb{P}\left(\forall 1\leqslant i\leqslant r,Z^{\gamma e_i}_{(\boldsymbol{z},\boldsymbol{\alpha}),\varepsilon}(\mathbb{C})\leqslant M\right)=+\infty$$

where $M>0$ is taken so that $\mathbb{P}\left(\forall 1\leqslant i\leqslant r,Z^{\gamma e_i}_{(\boldsymbol{z},\boldsymbol{\alpha}),\varepsilon}(\mathbb{C})\leqslant M\right)>0$ (to see why this is possible note that $G_{\hat{g},\varepsilon}(x,\cdot)$ is bounded over $\mathbb{C}$ and apply Lemma 4.1 with all the α taken equal to zero). Therefore the ε-regularized partition function is infinite if one of the s_i is non-positive. Conversely if these s_i are all positive and using Lemma 4.1 we can make the change of variable $y_i=\mu_ie^{\gamma c_i}Z^{\gamma e_i}_{(\boldsymbol{z},\boldsymbol{\alpha}),\varepsilon}(\mathbb{C})$ in the integral so that we are left with

$$\left(\prod_{i=1}^r\frac{\Gamma(s_i)\mu_i^{-s_i}}{\gamma}\right)\prod_{k=1}^N\hat{g}(z_k)^{\Delta_{\alpha_k}}e^{\sum_{k<l}\langle\alpha_k,\alpha_l\rangle G_{\hat{g},\varepsilon}(z_k,z_l)}\mathbb{E}\left[\prod_{i=1}^r Z^{\gamma e_i}_{(\boldsymbol{z},\boldsymbol{\alpha}),\varepsilon}(\mathbb{C})^{-s_i}\right].$$

To conclude for the first and fourth item it remains to show that:

- If for all $1\leqslant i\leqslant r$, and $1\leqslant j\leqslant N$, $\langle\alpha_j,e_i\rangle<\langle Q,e_i\rangle$, then

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\prod_{i=1}^r Z^{\gamma e_i}_{(\boldsymbol{z},\boldsymbol{\alpha}),\varepsilon}(\mathbb{C})^{-s_i}\right]=\mathbb{E}\left[\prod_{i=1}^r Z^{\gamma e_i}_{(\boldsymbol{z},\boldsymbol{\alpha})}(\mathbb{C})^{-s_i}\right]>0.$$

- If for some $1\leqslant i\leqslant r$ and $1\leqslant k\leqslant N$, $\langle\alpha_k,e_i\rangle\geqslant\langle Q,e_i\rangle$, then

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\prod_{i=1}^r Z^{\gamma e_i}_{(\boldsymbol{z},\boldsymbol{\alpha}),\varepsilon}(\mathbb{C})^{-s_i}\right]=0.$$

Let us assume that for all $1\leqslant i\leqslant r$, and $1\leqslant k\leqslant N$, $\langle\alpha_k,e_i\rangle<\langle Q,e_i\rangle$. Then we know from Lemma 4.1 that the family of random variables $\left(\prod_{i=1}^r Z^{\gamma e_i}_{(\boldsymbol{z},\boldsymbol{\alpha}),\varepsilon}(\mathbb{C})^{-s_i}\right)_{\varepsilon\geqslant 0}$ have (uniformly

bounded in ε) positive moments of all orders. Thus we can write that

$$\mathbb{E}\left[\left|\prod_{i=1}^{r} Z^{\gamma e_i}_{(\boldsymbol{z},\boldsymbol{\alpha}),\varepsilon}(\mathbb{C})^{-s_i} - \prod_{i=1}^{r} Z^{\gamma e_i}_{(\boldsymbol{z},\boldsymbol{\alpha})}(\mathbb{C})^{-s_i}\right|\right] \leqslant \mathbb{E}\left[\left|Z^{\gamma e_1}_{(\boldsymbol{z},\boldsymbol{\alpha}),\varepsilon}(\mathbb{C})^{-s_1} - Z^{\gamma e_1}_{(\boldsymbol{z},\boldsymbol{\alpha})}(\mathbb{C})^{-s_1}\right| \prod_{i=2}^{r} Z^{\gamma e_i}_{(\boldsymbol{z},\boldsymbol{\alpha}),\varepsilon}(\mathbb{C})^{-s_i}\right]$$
$$+\mathbb{E}\left[\left|\prod_{i=2}^{r} Z^{\gamma e_i}_{(\boldsymbol{z},\boldsymbol{\alpha}),\varepsilon}(\mathbb{C})^{-s_i} - \prod_{i=2}^{r} Z^{\gamma e_i}_{(\boldsymbol{z},\boldsymbol{\alpha})}(\mathbb{C})^{-s_i}\right| Z^{\gamma e_1}_{(\boldsymbol{z},\boldsymbol{\alpha})}(\mathbb{C})^{-s_1}\right]$$
$$\leqslant \mathbb{E}\left[\left|Z^{\gamma e_1}_{(\boldsymbol{z},\boldsymbol{\alpha}),\varepsilon}(\mathbb{C})^{-s_1} - Z^{\gamma e_1}_{(\boldsymbol{z},\boldsymbol{\alpha})}(\mathbb{C})^{-s_1}\right|^p\right]^{\frac{1}{p}} \mathbb{E}\left[\prod_{i=2}^{r} Z^{\gamma e_i}_{(\boldsymbol{z},\boldsymbol{\alpha}),\varepsilon}(\mathbb{C})^{-qs_i}\right]^{\frac{1}{q}}$$
$$+\mathbb{E}\left[\left|\prod_{i=2}^{r} Z^{\gamma e_i}_{(\boldsymbol{z},\boldsymbol{\alpha}),\varepsilon}(\mathbb{C})^{-s_i} - \prod_{i=2}^{r} Z^{\gamma e_i}_{(\boldsymbol{z},\boldsymbol{\alpha})}(\mathbb{C})^{-s_i}\right|^p\right]^{\frac{1}{p}} \mathbb{E}\left[Z^{\gamma e_1}_{(\boldsymbol{z},\boldsymbol{\alpha})}(\mathbb{C})^{-qs_1}\right]^{\frac{1}{q}}$$

where we have used Hölder inequality with some $p = \frac{q}{q-1} > 1$. Therefore we can proceed by induction on r so that the only point to check is $\lim_{\varepsilon\to 0} \mathbb{E}\left[\left|Z^{\gamma e_1}_{(\boldsymbol{z},\boldsymbol{\alpha}),\varepsilon}(\mathbb{C})^{-s_1} - Z^{\gamma e_1}_{(\boldsymbol{z},\boldsymbol{\alpha})}(\mathbb{C})^{-s_1}\right|^p\right]^{\frac{1}{p}} = 0$. This fact has already been proved by the authors in [4, Lemma 3.3]. For the second bullet point, let us introduce the set $\mathcal{P} := \{i = 1, \dots, r \,|\exists 1 \leqslant k \leqslant N, \langle \alpha_i - Q, e_i\rangle \geqslant 0\}$ and assume that it is non-empty. Then we can write that, for positive $\frac{1}{p_i}$ and $\frac{1}{q}$ summing to one,

$$\mathbb{E}\left[\prod_{i=1}^{r} Z^{\gamma e_i}_{(\boldsymbol{z},\boldsymbol{\alpha}),\varepsilon}(\mathbb{C})^{-s_i}\right] \leqslant \prod_{i\in\mathcal{P}} \mathbb{E}\left[Z^{\gamma e_i}_{(\boldsymbol{z},\boldsymbol{\alpha}),\varepsilon}(\mathbb{C})^{-p_i s_i}\right]^{\frac{1}{p_i}} \mathbb{E}\left[\prod_{i\notin\mathcal{P}} Z^{\gamma e_i}_{(\boldsymbol{z},\boldsymbol{\alpha}),\varepsilon}(\mathbb{C})^{-qs_i}\right]^{\frac{1}{q}}.$$

Then we have already seen that the second expectation in the right-hand-side had a finite limit as $\varepsilon \to 0$ thanks to the results of Lemma 4.1. Conversely standard results of the GMC theory (see again [4, Lemma 3.3] for instance) imply that for any $i \in \mathcal{P}$, $\lim_{\varepsilon\to 0} \mathbb{E}\left[Z^{\gamma e_i}_{(\boldsymbol{z},\boldsymbol{\alpha}),\varepsilon}(\mathbb{C})^{-p_i s_i}\right]^{\frac{1}{p_i}} = 0$. This proves the first item in the statement of Theorem 3.1.

We now turn to the proofs of the second and third items. Let us start with the third item. By making a standard change of variable in the zero mode $\boldsymbol{c}$, and using the fact that $X_g - m_{\hat{g}}(X_g)$ and $X_{\hat{g}}$ have same law, we can assume that X is a GFF with vanishing mean with respect to the round metric. Since $g = e^{\varphi}\hat{g}$, we have that

$$\langle V_{\alpha_1,\varepsilon}(z_1)\cdots V_{\alpha_N,\varepsilon}(z_N)\rangle_g = Z(g)\prod_{k=1}^{N} e^{\frac{\langle \alpha_k, Q\rangle}{2}\varphi(x_k)} \times \lim_{\varepsilon\to 0}$$
$$\int_{\mathfrak{h}^*} e^{\gamma\langle \boldsymbol{s},\boldsymbol{c}\rangle}\mathbb{E}\left[\prod_{j=1}^{k} \widetilde{V}^{\hat{g}}_{\varepsilon,\alpha_j}(z_j) e^{-\frac{1}{4\pi}\int_{\mathbb{C}} R_g\langle Q,X\rangle(x)v_g(dx) - \sum_{i=1}^{r}\mu_i e^{\gamma c_i}\int_{\mathbb{C}} e^{\gamma^2\frac{\varphi(x)}{2}} V^{\hat{g}}_{\varepsilon,e_i}(x)v_g(dx)}\right] d\boldsymbol{c}$$

where this time regularization is done with respect to the round metric, and $\widetilde{V}$ is the vertex operator without constant mode. Recall that for all i, $\langle Q, e_i\rangle = \gamma + \frac{2}{\gamma}$; therefore:

$$\begin{aligned}&\mathbb{E}\left[\left(\int_{\mathbb{C}} R_g(y)\langle Q, X\rangle(y)v_g(dy)\right)\langle e_i, X\rangle(x)\right]\\&= \langle Q, e_i\rangle \int_{\mathbb{C}} R_g(y)G_{\hat{g}}(x,y)v_g(dy) = (\gamma + \frac{2}{\gamma})2\pi\left(\varphi(x) - m_{\hat{g}}(\varphi)\right)\end{aligned}$$

where we have used that $R_g(y)v_g(dy) = (-\Delta_{\hat{g}}\varphi(y) + 2)v_{\hat{g}}(dy)$ (here in the weak sense since $\varphi \in \bar{C}^1(\mathbb{R}^2)$) and the definition of the Green function $G_{\hat{g}}$. More generally when $\alpha \in \mathfrak{h}^*$:

$$\mathbb{E}\left[\left(\frac{1}{4\pi}\int_{\mathbb{C}} R_g(y)\langle Q, X\rangle(y)v_g(dy)\right)\langle \alpha, X\rangle(x)\right] = \langle Q, \frac{\alpha}{2}\rangle\left(\varphi(x) - m_{\hat{g}}(\varphi)\right).$$

Next we want to consider the exponential of this term $\int_{\mathbb{C}} R_g(y)\langle Q, X\rangle(y)v_g(dy)$ as a Girsanov transform. Its variance is given by

$$\begin{aligned}&\mathbb{E}\left[\left(\frac{1}{4\pi}\int_{\mathbb{C}} R_g(x)\langle Q, X\rangle(x)v_g(dx)\right)^2\right]\\&= \frac{1}{16\pi^2}\int_{\mathbb{C}\times\mathbb{C}} R_g(x)R_g(y)\mathbb{E}[\langle Q, X\rangle(x)\langle Q, X\rangle(y)]v_g(dx)v_g(dy)\\&= \frac{1}{16\pi^2}|Q|^2\int_{\mathbb{C}\times\mathbb{C}} R_g(x)R_g(y)G_{\hat{g}}(x,y)v_g(dx)v_g(dy)\\&= \frac{1}{16\pi^2}|Q|^2\int_{\mathbb{C}} R_g(x)\left(\int_{\mathbb{C}} R_g(y)G_{\hat{g}}(x,y)v_g(dy)\right)v_g(dx)\\&= \frac{1}{8\pi}|Q|^2\int_{\mathbb{C}} R_g(x)\left(\varphi(x) - m_{\hat{g}}(\varphi)\right)v_g(dx)\\&= \frac{1}{8\pi}|Q|^2\int_{\mathbb{C}} \left(-\Delta_{\hat{g}}\varphi(x) + 2\right)\left(\varphi(x) - m_{\hat{g}}(\varphi)\right)v_{\hat{g}}(dx)\\&= \frac{1}{8\pi}|Q|^2\int_{\mathbb{C}} |\partial_{\hat{g}}\varphi|^2 v_{\hat{g}}(dx)\end{aligned}$$

and as seen above it has the effect of shifting the law of X by an additional factor $\frac{Q}{2}\left(\varphi(x) - m_{\hat{g}}(\varphi)\right)$.

Therefore $\langle V_{\alpha_1}(z_1)\cdots V_{\alpha_N}(z_N)\rangle_g = Z(g)e^{\frac{|Q|^2}{16\pi}\int_{\mathbb{C}}|\partial_{\hat{g}}\varphi|^2 v_{\hat{g}}}\times$

$$\lim_{\varepsilon\to 0}\int_{\mathfrak{h}^*} e^{\gamma\langle \boldsymbol{s},\boldsymbol{c}\rangle}\mathbb{E}\left[\prod_{k=1}^N \widetilde{V}_{\varepsilon,\alpha_k}(z_k)e^{-\sum_{i=1}^r \mu_i e^{\gamma c_i}\int_{\mathbb{C}} e^{\varphi(x)(\gamma^2\frac{|e_i|^2}{4} - \frac{\gamma\langle Q, e_i\rangle}{2})}\widetilde{V}_{\varepsilon,e_i}(x)v_g(dx)}\right]d\boldsymbol{c}.$$

Since for all i we have $\frac{|\gamma e_i|^2}{4} - \frac{\gamma\langle Q, e_i\rangle}{2} = -1$, by the change of variable $c_i \to c_i - \langle Q, \frac{e_i}{2}\rangle m_{\hat{g}}(\varphi)$ we get that

$$\langle V_{\alpha_1}(z_1)\cdots V_{\alpha_N}(z_N)\rangle_g = Z(g)e^{\frac{|Q|^2}{16\pi}\int_{\mathbb{C}}|\partial_{\hat{g}}\varphi|^2 v_{\hat{g}} + \frac{|Q|^2}{4\pi}\int_{\mathbb{C}}\varphi v_{\hat{g}}}\times\lim_{\varepsilon\to 0}\int_{\mathfrak{h}^*}\mathbb{E}\left[\prod_{k=1}^N V^{\hat{g}}_{\varepsilon,\alpha_k}(z_k)e^{-\sum_{i=1}^r \mu_i\int_{\mathbb{C}} V^{\hat{g}}_{\varepsilon,e_i}(x)v_{\hat{g}}(dx)}\right]d\boldsymbol{c},$$

whence the result, by using the expression (2.30) for the regularized determinant $Z(g)$.

For the second item, we see that according to our proof of the first item, $\langle F(\varphi_{\hat{g}})\prod_{k=1}^N V_{\alpha_k}(z_k)\rangle$, where F is continuous and bounded on $\mathrm{H}^{-1}(\mathbb{S}^2\to\mathfrak{h}^*,\hat{g})$, is actually given by

$$\prod_{k=1}^N \hat{g}(z_k)^{\Delta_{\alpha_k}} e^{\sum_{k<l}\langle\alpha_k,\alpha_l\rangle G_{\hat{g}}(z_k,z_l)}\int_{\mathfrak{h}^*} e^{\langle \boldsymbol{s},\boldsymbol{c}\rangle}\mathbb{E}\left[F\left(X_{\hat{g}}+\frac{Q}{2}\ln\hat{g}+\boldsymbol{c}+\sum_{k=1}^N\alpha_k G_{\hat{g}}(\cdot,z_k)\right)e^{-\sum_{i=1}^r\mu_i e^{\gamma\langle\boldsymbol{c},e_i\rangle}Z^{\gamma e_i}_{(\boldsymbol{z},\boldsymbol{\alpha})}(\mathbb{C})}\right]d\boldsymbol{c}.$$

Now since ψ is a conformal map we know that the Riemannian metric $\psi^*\hat{g}$ (that we have denoted $\hat{g}_\psi$ before) is in the conformal class of $\hat{g}$; as a consequence the free fields $X_{\hat{g}_\psi}-m_{\hat{g}}(X_{\hat{g}_\psi})$ and $X_{\hat{g}}$ have same law. Moreover from Lemma 2.1 we know that $X_{\hat{g}_\psi}$ has same law as $X_{\hat{g}}\circ\psi$. As a consequence and using (2.10) we see that the laws of

$$X_{\hat{g}}+\sum_{k=1}^N\alpha_k G_{\hat{g}}(\cdot,z_k)\quad\text{and}\quad\left(X_{\hat{g}}+\sum_{k=1}^N\alpha_k G_{\hat{g}}(\cdot,z_k)\right)\circ\psi+\frac{1}{4}\sum_{k=1}^N\alpha_k(\phi+\phi(z_k))-m_{\hat{g}}(X_{\hat{g}}\circ\psi)$$

are actually the same. Therefore shifting the zero mode $\boldsymbol{c}$ by the additional constant $m_{\hat{g}}(X_{\psi^*\hat{g}})-\frac{1}{4}\sum_{k=1}^N\alpha_k\phi(z_k)-\frac{1}{4}\boldsymbol{s}m_{\hat{g}}(\phi)$ and collecting up terms using (2.10) yields:

$$\prod_{k=1}^N \hat{g}_\psi(z_k)^{\Delta_{\alpha_k}} e^{\sum_{k<l}\langle\alpha_k,\alpha_l\rangle G_{\hat{g}}(\psi(z_k),\psi(z_l))}$$
$$\int_{\mathfrak{h}^*} e^{\langle \boldsymbol{s},\boldsymbol{c}\rangle}\mathbb{E}\left[e^{\langle\boldsymbol{s},m_{\hat{g}}(X_{\psi^*\hat{g}})\rangle-\frac{m_{\hat{g}}(\phi)\langle\boldsymbol{s},\boldsymbol{s}\rangle}{4}}F\left(\boldsymbol{\Phi}_{\hat{g}}\circ\psi+Q\ln|\psi'|+\frac{1}{4}\boldsymbol{s}(\phi-m_{\hat{g}}(\phi))\right)e^{-\sum_{i=1}^r\mu_i e^{\gamma\langle\boldsymbol{c},e_i\rangle}Z^{\gamma e_i}_{(\boldsymbol{z},\boldsymbol{\alpha})}(\mathbb{C})}\right]d\boldsymbol{c},$$

where $\boldsymbol{\Phi}_{\hat{g}}$ is a shorthand for $X_{\hat{g}}+\frac{Q}{2}\ln\hat{g}+\boldsymbol{c}+\sum_{k=1}^N\alpha_k G_{\hat{g}}(\cdot,z_k)$. To conclude the proof it remains to apply the same reasoning as in the proof of the third item to see that the exponential term $e^{\langle\boldsymbol{s},m_{\hat{g}}(X_{\psi^*\hat{g}})\rangle-\frac{m_{\hat{g}}(\phi)\langle\boldsymbol{s},\boldsymbol{s}\rangle}{4}}$ will shift the law of $\boldsymbol{\Phi}_{\hat{g}}\circ\psi$ by $-\frac{1}{4}\boldsymbol{s}(\phi-m_{\hat{g}}(\phi))$. □

In the proof of the conformal covariance property we have shown a slightly more general result. Indeed we have proved that under the Seiberg bounds and for any Möbius transform of the plane, the following was true for any continuous and bounded map F on $\mathrm{H}^{-1}(\mathbb{S}^2\to\mathfrak{h}^*,\hat{g})$:

$$\langle F(\varphi\circ\psi+Q\ln|\psi'|)\prod_{k=1}^N V_{\alpha_k}(\psi(z_k))\rangle_g=\prod_{k=1}^N|\psi'(z_k)|^{-2\Delta_{\alpha_k}}\langle F(\varphi)\prod_{k=1}^N V_{\alpha_k}(z_k)\rangle_g. \tag{3.7}$$

This statement is usually referred to as the conformal covariance of the Toda field.

4. The conditional volume measures

We wish to extend here the validity of the probabilistic representation (3.5). The point is that the explicit expression (3.5) allows us to isolate the constraints $s_i>0$ in the product of Γ functions. This term can obviously be analytically removed. The question is then to determine whether the expectation in (3.5) makes sense beyond the range of parameters permitted by the Seiberg bounds. So we claim

Lemma 4.1 (Extended Seiberg bounds)**.** *The bound*

$$\mathbb{E}\left[\prod_{i=1}^r(Z^{\gamma e_i}_{(\boldsymbol{z},\boldsymbol{\alpha})}(\mathbb{R}^2))^{-s_i}\right]<\infty \tag{4.1}$$

holds if and only if for all $i = 1, \cdots, r$ one has

$$-s_i < \frac{2}{\gamma^2} \wedge \min_{k=1,\dots,N} \frac{1}{\gamma}\langle Q - \alpha_k, e_i\rangle \tag{4.2}$$

Proof. We suppose that condition (4.2) holds. Let us consider the families of indices

$$\mathcal{P} := \{i = 1, \dots, r \,|\, s_i \geqslant 0\} \quad \text{and} \quad \mathcal{N} := \{i = 1, \dots, r \,|\, s_i < 0\}.$$

Choose $p > 1$ such that for all $i \in \mathcal{N}$, $-ps_i < \frac{2}{\gamma^2} \wedge \min_{k=1,\dots,N} \frac{2}{\gamma}\langle Q - \alpha_k, e_i\rangle$ and fix the conjugate exponent $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. By Hölder inequality we can write that

$$\mathbb{E}\left[\prod_{i=1}^{r} (Z^{\gamma e_i}_{(\boldsymbol{z},\boldsymbol{\alpha})}(\mathbb{R}^2))^{-s_i}\right] \leqslant \mathbb{E}\left[\prod_{i\in\mathcal{P}} (Z^{\gamma e_i}_{(\boldsymbol{z},\boldsymbol{\alpha})}(\mathbb{R}^2))^{-ps_i}\right]^{1/p} \mathbb{E}\left[\prod_{i\in\mathcal{N}} (Z^{\gamma e_i}_{(\boldsymbol{z},\boldsymbol{\alpha})}(\mathbb{R}^2))^{-qs_i}\right]^{1/q}.$$

The product running over $i \in \mathcal{P}$ is finite because GMC admits negative moments of all order (see [33, Theorem 2.12]). For the product running over $i \in \mathcal{N}$, we use Corollary 6.2 in appendix as well as the relation (2.38), which shows that the GFFs $\langle \gamma e_i, X_g\rangle$ and $\langle \gamma e_j, X_g\rangle$, for $i \neq j$, are negatively correlated since $\langle \gamma e_i, \gamma e_j\rangle = \gamma^2 A_{i,j}$ (recall that A is the Cartan matrix) and all off-diagonal elements of A are nonpositive. Hence

$$\mathbb{E}\left[\prod_{i\in\mathcal{N}} (Z^{\gamma e_i}_{(\boldsymbol{z},\boldsymbol{\alpha})}(\mathbb{R}^2))^{-qs_i}\right] \leqslant \prod_{i\in\mathcal{N}} \mathbb{E}\left[(Z^{\gamma e_i}_{(\boldsymbol{z},\boldsymbol{\alpha})}(\mathbb{R}^2))^{-qs_i}\right]. \tag{4.3}$$

We conclude that each expectation in the product above is finite because [4, Lemma A.1] ensures that it is indeed the case provided that $-ps_i < \frac{2}{\gamma^2} \wedge \min_{k=1,\dots,N} \frac{1}{\gamma}\langle Q - \alpha_k, e_i\rangle$ for all i, which we assumed to hold.

Conversely, assume that the expectation (4.1) is finite. By Corollary 6.2 in the appendix applied to the function

$$G(x_1, \dots, x_d) = \prod_{i\in\mathcal{N}} x_i^{-s_i} \prod_{i\in\mathcal{P}} x_i^{-s_i},$$

to $r = |\mathcal{N}|$, and to the GFFs $(\langle \gamma e_i, X_g\rangle)_{i=1,\dots,r}$ we deduce

$$\mathbb{E}\left[\prod_{i=1}^{r} (Z^{\gamma e_i}_{(\boldsymbol{z},\boldsymbol{\alpha})}(\mathbb{R}^2))^{-s_i}\right] \geqslant \mathbb{E}\left[\prod_{i\in\mathcal{P}} (Z^{\gamma e_i}_{(\boldsymbol{z},\boldsymbol{\alpha})}(\mathbb{R}^2))^{-s_i}\right] \mathbb{E}\left[\prod_{i\in\mathcal{N}} (Z^{\gamma e_i}_{(\boldsymbol{z},\boldsymbol{\alpha})}(\mathbb{R}^2))^{-s_i}\right].$$

Since GMC admits negative moments of all order [33, Theorem 2.12], the first expectation in the right-hand side above is a finite constant $C > 0$. This implies that the second expectation is finite too. From now on, we fix $i_0 \in \mathcal{N}$ and $j \in \{1, \dots, N\}$. Without loss of generality and for the sake of simplicity, we may assume that $z_j = 0$. Then we can choose $\delta > 0$ such that $\min_{j'\neq j} |z'_j| > 10 \times \delta$ and we can choose non empty balls $(B_i)_{i\neq i_0, i\in\mathcal{N}}$ all of them at distance at least $10 \times \delta > 0$ from each other and all of them at distance at least $10 \times \delta$ from all the z_j's. Set $B_{i_0} := B(0, \delta)$. Obviously we have

$$\mathbb{E}\left[\prod_{i\in\mathcal{N}} (Z^{\gamma e_i}_{(\boldsymbol{z},\boldsymbol{\alpha})}(\mathbb{R}^2))^{-s_i}\right] \geqslant \mathbb{E}\left[\prod_{i\in\mathcal{N}} (Z^{\gamma e_i}_{(\boldsymbol{z},\boldsymbol{\alpha})}(B_i))^{-s_i}\right].$$

Consider the mean value of the field $Y := \frac{1}{2\pi i}\oint_{|x|=\delta} X^g(x)\frac{dx}{x}$. A simple check of covariances shows that the law of the field $X^g - Y$ is the independent sum of the field X_g^h—which coincides with $X_g - Y$ outside of $B(0,2\delta)$ and corresponds inside $B(0,2\delta)$ to the harmonic extension (component by component) of the field $X_g - Y$ restricted to the boundary $\partial B(0,2\delta)$—plus the Dirichlet field X_d defined by

$$X_d = \sum_{i=1}^{r} \omega_i X_{d,i}, \tag{4.4}$$

where $(X_{d,1}^g, \dots, X_{d,r}^g)$ is a family of centered correlated Dirichlet GFFs inside $B(0,2\delta)$ with covariance structure given by

$$\mathbb{E}[X_{d,i}^g(z)X_{d,j}^g(z')] = A_{i,j}G_d(z,z')$$

and $G_d(z,z')$ stands for the Dirichlet Green function inside $B(0,2\delta)$. From now on we will write $Z_{(\boldsymbol{z},\boldsymbol{\alpha})}^{\langle\gamma e_i, X^g\rangle}(d^2x)$ instead of $Z_{(\boldsymbol{z},\boldsymbol{\alpha})}^{\gamma e_i}(d^2x)$ to indicate in the notations the dependence on the underlying Gaussian field. This means that, generally speaking, we will write $Z_{(\boldsymbol{z},\boldsymbol{\alpha})}^{\langle\gamma e_i, X\rangle}$ for

$$Z_{(\boldsymbol{z},\boldsymbol{\alpha})}^{\langle\gamma e_i, X\rangle}(d^2x) := \lim_{\varepsilon\to 0} e^{\sum_{j=1}^N \langle\alpha_j,\gamma e_i\rangle G_{\hat{g}}(x,z_j)}\varepsilon^{\frac{|\gamma e_i|^2}{2}} e^{\langle\gamma e_i, X_\varepsilon(x)\rangle}\mathrm{v}_g(dx)$$

where X_ε stands for the ε-regularization of the field X in the metric g. So we can write

$$\mathbb{E}\left[\prod_{i\in\mathcal{N}}(Z_{(\boldsymbol{z},\boldsymbol{\alpha})}^{\langle\gamma e_i, X^g\rangle}(B_i))^{-s_i}\right] = \mathbb{E}\left[e^{-\sum_{i\in\mathcal{N}} s_i\langle\gamma e_i, Y\rangle}\prod_{i\in\mathcal{N}}(Z_{(\boldsymbol{z},\boldsymbol{\alpha})}^{\langle\gamma e_i, X^g - Y\rangle}(B_i))^{-s_i}\right].$$

Now we use the Girsanov transform to remover the factor $e^{-\sum_{i\in\mathcal{N}} s_i\langle\gamma e_i, Y\rangle}$, which we renormalize by its variance. We do not need to compute explicitly the variance, nor the covariance of Y with $X_g - Y$. Indeed, the variance is bounded as well as the covariance of Y with $X_g - Y$. This entails the existence of some constant $C > 0$ such that

$$\mathbb{E}\left[e^{-\sum_{i\in\mathcal{N}} s_i\langle\gamma e_i, Y\rangle}\prod_{i\in\mathcal{N}}(Z_{(\boldsymbol{z},\boldsymbol{\alpha})}^{\langle\gamma e_i, X^g - Y\rangle}(B_i))^{-s_i}\right] \geqslant C\mathbb{E}\left[\prod_{i\in\mathcal{N}}(Z_{(\boldsymbol{z},\boldsymbol{\alpha})}^{\langle\gamma e_i, X^g - Y\rangle}(B_i))^{-s_i}\right].$$

Using the decomposition of the law of $X^g - Y = X_d + X_h^g$ and independence of X_d and X_h^g, we get

$$\mathbb{E}\left[\prod_{i\in\mathcal{N}}(Z_{(\boldsymbol{z},\boldsymbol{\alpha})}^{\langle\gamma e_i, X^g - Y\rangle}(B_i))^{-s_i}\right] \geqslant \mathbb{E}\left[(Z_{(\boldsymbol{z},\boldsymbol{\alpha})}^{\langle\gamma e_{i_0}, X_d\rangle}(B_{i_0}))^{-s_{i_0}}\right]\mathbb{E}\left[e^{\min_{x\in B_{i_0}} X_h^g(x)}\prod_{i\neq i_0, i\in\mathcal{N}}(Z_{(\boldsymbol{z},\boldsymbol{\alpha})}^{\langle\gamma e_i, X^g - Y\rangle}(B_i))^{-s_i}\right].$$

This implies that both expectations in the right-hand side are finite (they are obviously nonzero). Finiteness of the first expectation above entails that $-s_{i_0} < \frac{2}{\gamma^2} \wedge \frac{1}{\gamma}\langle Q - \alpha_j, e_i\rangle$ (see [4, Lemma A.1]). Since the argument is valid for all $i_0 \in \mathcal{N}$ and all j, this yields the result. □

5. Perspectives

TCFTs provide natural extensions of LCFT with a higher level of symmetry in addition to the Weyl anomaly which encodes the local conformal structure. In this document we have constructed TCFTs but we have not really shed light on where this W-symmetry does appear in the model and how useful it can be. We review below some interesting questions related to this observation.

5.1. W-symmetry and local conformal structure of Toda theories.

5.1.1. *Characterizing the $\mathfrak{sl}_3$ Lie algbera using W-symmetry.* Let us consider the first non-trivial extension of the Liouville theory, *i.e.* when one works with two negatively correlated GFFs—like in the construction of the $\mathfrak{sl}_3$ Toda theory. Then we can create a one-parameter family of QFTs by changing the covariance between these two GFFs, Lie algebras arising only for some special values of this parameter, that is when the covariance matrix coincides with the Cartan matrix of some Lie algebra. More explicitly one can assume to be working with a pair of GFFs whose covariance matrix is given by

$$\mathbb{E}[X_i^g(x)X_j^g(y)] = A_{i,j}G_g(x,y),$$

but where $A = \begin{pmatrix} 2 & c \\ c & 2 \end{pmatrix}$ with $c \in (-2,2)$ would no longer be the Cartan matrix of some semisimple Lie algebra (except for the values $c = 0, -1$ corresponding to $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ and $\mathfrak{sl}_3$ Lie algebras). Interestingly these theories also enjoy conformal invariance thus define CFTs, but are not supposed to enjoy higher-spin symmetry. In this context it seems natural to wonder what is so specific about the theories defined via the $\mathfrak{sl}_3$ Lie algebra, and how can one see where W-symmetry does appear ? These questions are being investigated in a work in progress. More precisely it is expected that the existence of higher-spin currents that are *holomorphic* is granted if and only if the covariance matrix of the GFFs is given by the Green kernel times the Cartan matrix of the $\mathfrak{sl}_3$ Lie algebra.

5.1.2. *Local conformal structure of Toda Field Theories.* In LCFT, the Weyl anomaly (combined with diffeomorphism invariance of the theory) is in some sense equivalent to the existence of a holomorphic current: the stress-energy tensor $T(z)$. The expression of this tensor can be obtained by formally derivating the correlation function with respect to the metric

$$\langle T_{\mu,\nu}(z)V_{\alpha_1}(z_1)\cdots V_{\alpha_k}(z_N)\rangle_g := 4\pi\frac{\partial}{\partial g^{\mu,\nu}}\langle V_{\alpha_1}(z_1)\cdots V_{\alpha_k}(z_N)\rangle_g,$$

and then setting $T := T_{z,z} + \frac{c}{12}t$ where t is explicit and depends on the background metric g. The first Ward identity

$$\langle T(z_0)\prod_{k=1}^N V_{\alpha_k}(z_k)\rangle = \left(\sum_{k=1}^N \frac{\Delta_{\alpha_k}}{(z_0-z_k)^2} + \sum_{k=1}^N \frac{\partial_{z_k}}{z_0-z_k}\right)\langle\prod_{k=1}^N V_{\alpha_k}(z_k)\rangle \tag{5.1}$$

and the asymptotic behaviour of the stress-energy tensor $T(z) \sim \frac{1}{z^4}$ near infinity (which comes from the fact that we have conformally mapped the sphere to the plane) usually ensure conformal covariance of the model. In a similar way TCFTs feature higher-spin currents $\mathbf{W}^{(k)}(z)$ which should encode higher-spin symmetry via equations that take the same form as the Ward identity. As an application of our formalism it should be possible to

check that these identities hold (at least in the simplest $\mathfrak{sl}_3$ case) and study the properties of integrability provided by the W-symmetry. However the structure of W-algebras is much more complicated that the Virasoro one (for instance higher-spin tensors feature higher derivatives and the commutation relations of their modes are no longer linear) and the tools used to prove integrability of Liouville theory come with additional complications. However there is still some hope of actually solving these models: for instance it is predicted in the physics literature that one can find a differential equation for some four-point correlation functions, which would allow to derive the value of certain three-point correlation functions. See [10, Equation (14)] for a precise statement. These questions will be addressed in an upcoming series of work by Y. Huang and one of the author. The probabilistic framework should in particular allow to write down an explicit expression for the descendants of the primary fields $V_\alpha(x)$, expression which was left unidentified in the physics literature.

5.2. **The semi-classical limit and Toda equations in W-geometry.** Let us comment here on the geometrical signification of TCFTs. Their path integral formulation rely on the action functional (1.1) that corresponds to the quantization (*i.e.* where we have introduced a coupling constant and considered an appropriate renormalization of the field and cosmological constants) of the action

$$S_T^c(\varphi, g) := \frac{1}{4\pi}\int_\Sigma \Big(\langle \partial_g\varphi(x), \partial_g\varphi(x)\rangle_g + 2R_g\langle \rho, \varphi(x)\rangle + 2\Lambda\sum_{i=1}^{n-1} e^{\langle e_i,\varphi(x)\rangle}\Big)\,\mathrm{v}_g(dx),$$

whose critical point is given by the solution of the Toda equation (1.2). Such a critical point exists and is unique as soon as Σ admits a metric for which R_g is negative and constant; when the surface has the topology of the sphere or the torus one may need the field to have certain logarithmic singularities in order for such a problem to admit a unique solution.

In the simplest case where $n-1=1$ the equation corresponds to the Liouville equation that describes (the conformal factor of) Riemannian metrics with constant negative curvature equal to $-\Lambda$ within the conformal class of (Σ, g). In general, such an interpretation remains possible but instead of working in the setup of conformal geometry the good framework to consider is the one of *W-geometry*. The interested reader may find more details on W-geometries in the work of Gervais and Matsuo [17]. One of its important features is that, in a way similar to the fact that W-algebras admit the Virasoro algebra as a subalgebra, W-geometries in some sense contain two-dimensional Riemannian geometry.

From the quantum theory viewpoint, since we have constructed TCFTs via a quantization of this classical model, it is natural to expect that when the coupling constant γ (which characterizes the level of randomness) is taken to zero—and under appropriate renormalizations—we recover the solution of the classical Toda equation (1.2) (when it exists and is unique). Such a result has been recently proved by H. Lacoin and the last two authors in [27] for LCFT; it should be reasonable to expect that the result extends for the general $\mathfrak{sl}_n$ TCFTs on the sphere.

6. Appendix

In the appendix, we gather rather general and classical results on Gaussian vectors: first comparison lemmas and then a statement of the Girsanov theorem.

Lemma 6.1. *Let F be some smooth function defined on $(\mathbb{R}^n)^d$ with at most polynomial growth at infinity for F as well as for its derivatives up to order 2. Assume that for $(x^1, \cdots, x^d) \in (\mathbb{R}^n)^d$ (where $x^i = (x^i_1, \cdots, x^i_n)$) the following inequalities hold:*

$$\text{for all } i \neq j \text{ and } k, k' \quad \frac{\partial^2 F}{\partial x^i_k \partial x^j_{k'}} \geqslant 0.$$

Let $X := (X^1, \cdots, X^d)$ and $\widetilde{X} := (\widetilde{X}^1, \cdots, \widetilde{X}^d)$ be two centered Gaussian vectors in $(\mathbb{R}^n)^d$ such that
1) for all $i \neq j$ and k, k'

$$\mathbb{E}[X^i_k X^j_{k'}] \leqslant \mathbb{E}[\widetilde{X}^i_k \widetilde{X}^j_{k'}].$$

2) for all i, X^i as the same law as $\widetilde{X}^i$.
Then the following inequality holds:

$$\mathbb{E}[F(X^1, \cdots, X^d)] \leqslant \mathbb{E}[F(\widetilde{X}^1, \cdots, \widetilde{X}^d)].$$

Proof. For $t \in [0, 1]$, we set $X_t = \sqrt{t} X + \sqrt{1-t}\widetilde{X}$, where X and $\widetilde{X}$ are independent, and

$$G(t) = \mathbb{E}[F(X_t)].$$

By using Gaussian integration by parts, we get the following relation

$$\begin{aligned}
G'(t) &= \frac{1}{2} \sum_{i=1}^d \sum_{k=1}^n \mathbb{E}\left[\frac{\partial F}{\partial x^i_k}(X_t)(\frac{1}{\sqrt{t}} X^i_k - \frac{1}{\sqrt{1-t}} \widetilde{X}^i_k)\right] \\
&= \frac{1}{2} \sum_{i=1}^d \sum_{k=1}^n \sum_{i'=1}^d \sum_{k'=1}^n \mathbb{E}\left[\frac{\partial^2 F}{\partial x^i_k \partial x^{i'}_{k'}}(X_t)\right] \mathbb{E}\left[(\sqrt{t} X^{i'}_{k'} + \sqrt{1-t}\widetilde{X}^{i'}_{k'})(\frac{1}{\sqrt{t}} X^i_k - \frac{1}{\sqrt{1-t}} \widetilde{X}^i_k)\right] \\
&= \frac{1}{2} \sum_{i=1}^d \sum_{k=1}^n \sum_{i' \neq i}^d \sum_{k'=1}^n \mathbb{E}\left[\frac{\partial^2 F}{\partial x^i_k \partial x^{i'}_{k'}}(X_t)\right] \left(\mathbb{E}[X^{i'}_{k'} X^i_k] - \mathbb{E}[\widetilde{X}^{i'}_{k'} \widetilde{X}^i_k]\right) \\
&\leqslant 0.
\end{aligned}$$

Therefore $G(1) \leqslant G(0)$. □

Corollary 6.2. *Let G be some smooth function defined on $(\mathbb{R}_+)^d$ with at most polynomial growth at infinity for G as well as for its derivatives up to order 2 and consider a partition $P_1, \ldots, P_m$ of the set $\{1, \ldots, d\}$. Assume that for $(x_1, \cdots, x_d) \in \mathbb{R}^d_+$, the following inequality holds for all $r, r' \in \{1, \ldots, m\}$ with $r \neq r'$, all $i \in P_r$ and all $j \in P_{r'}$*

$$\frac{\partial^2 G}{\partial x_i \partial x_j} \geqslant 0.$$

Further assume that $X^1, \cdots, X^d$ is a family of continuous centered Gaussian fields respectively defined over domains $D_i \subset \mathbb{R}^n$ (for $i = 1, \ldots, d$) such that for all $r, r' \in \{1, \ldots, m\}$ with $r \neq r'$, all $i \in P_r$ and all $j \in P_{r'}$

$$\forall x \in D_i, \forall x' \in D_j, \quad \mathbb{E}[X^i(x) X^j(x')] \leqslant 0.$$

Let $\widetilde{X} = (\widetilde{X}^1, \cdots, \widetilde{X}^d)$ be another family of continuous centered Gaussian fields such that:
1) for all $r = 1, \dots, m$, $(\widetilde{X}^i)_{i \in P_r}$ has same distribution as $(X^i)_{i \in P_r}$.
2) the families $(\widetilde{X}^i)_{i \in P_1}, \dots, (\widetilde{X}^i)_{i \in P_m}$ are independent.

Eventually, let $f_1, \dots, f_d$ be a family of positive functions each of which respectively defined on D_i. For $i = 1, \dots, d$, we set

$$M^i := \int_{D_i} e^{X^i(x) - \frac{1}{2}\mathbb{E}[X^i(x)^2]} f_i(x)\, dx \quad \text{and} \quad \widetilde{M}^i := \int_{D_i} e^{\widetilde{X}^i(x) - \frac{1}{2}\mathbb{E}[X^i(x)^2]} f_i(x)\, dx.$$

Then the following inequality holds

$$\mathbb{E}[G(M^1, \cdots, M^d)] \leqslant \mathbb{E}[G(\widetilde{M}^1, \cdots, \widetilde{M}^d)].$$

Proof. Up to a discretization of the fields, it suffices to apply Lemma 6.1 with

$$F(x^1, \dots, x^d) := G\left(\sum_{k_1} p^1_{k_1} e^{\gamma X^1_{k_1} - \frac{\gamma^2}{2}\mathbb{E}[(X^1_{k_1})^2]}, \cdots, \sum_{k_d} p^d_{k_d} e^{\gamma X^d_{k_d} - \frac{\gamma^2}{2}\mathbb{E}[(X^d_{k_d})^2]} \right)$$

for some nonnegative numbers $p^i_{k_i}$ obtained by discretizing f_i over D_i. □

After these two comparison lemmas we recall the statement of the Girsanov theorem. It can be adapted in a way similar to the above proof via a regularization procedure in order to fit to the GFF we have considered throughout the present document:

Theorem 6.3 (Girsanov theorem)**.** *Let D be a subdomain of $\mathbb{C}$ and*

$$(\boldsymbol{X}(x))_{x \in D} := (X_1(x), \cdots, X_{n-1}(x))_{x \in D}$$

be a family of smooth centered Gaussian field; also consider Z any Gaussian variable belonging to the L^2 closure of the subspace spanned by $(\boldsymbol{X}(x))_{x \in D}$. Then, for any bounded functional F over the space of continuous functions one has that

$$\mathbb{E}\left[e^{Z - \frac{\mathbb{E}\left[Z^2\right]}{2}} F(\boldsymbol{X}(x))_{x \in D} \right] = \mathbb{E}\left[F\left(\boldsymbol{X}(x) + \mathbb{E}\left[Z \boldsymbol{X}(x)\right]\right)_{x \in D} \right].$$

References

[1] T. Arakawa. Representation theory of W-algebras and Higgs branch conjecture. In *Proc. Int. Cong. of Math.*, volume 2, pages 1263–1282. World Scientific, 2018.

[2] A.A. Belavin, A.M. Polyakov, and A.B. Zamolodchikov. Infinite conformal symmetry in two-dimensional quantum field theory. *Nuclear Physics B*, 241(2):333 – 380, 1984.

[3] R. Borcherds. Vertex algebras, Kac-Moody algebras, and the Monster. In *Proceedings of the National Academy of Sciences of the United States of America*, volume 83, pages 3068–3071, 1986.

[4] F. David, A. Kupiainen, R. Rhodes, and V. Vargas. Liouville Quantum Gravity on the Riemann Sphere. *Communications in Mathematical Physics*, 342:869, 2016.

[5] F. David, R. Rhodes, and V. Vargas. Liouville quantum gravity on complex tori. *Journal of Mathematical Physics*, 57(2):022302, 2016.

[6] J. Ding, J. Dubédat, A. Dunlap, and H. Falconet. Tightness of Liouville first passage percolation for $\gamma \in (0, 2)$. *Publications mathématiques de l'IHÉS*, 132:353–403, 2020.

[7] J. Dubédat. SLE and the Free Field: partition functions and couplings. *Journal of the AMS*, 22 (4):995–1054, 2009.

[8] J. Dubédat, H. Falconet, E. Gwynne, J. Pfeffer, and X. Sun. Weak LQG metrics and Liouville first passage percolation. *Probability Theory and Related Fields*, 178:369–436, 2020.

[9] B. Duplantier and S. Miller, J. Sheffield. Liouville quantum gravity as a mating of trees. *arXiv preprint*, arXiv:1409.7055, 2014.

[10] V.A. Fateev and A.V. Litvinov. On differential equation on four-point correlation function in the Conformal Toda Field Theory. *JETP Lett.*, 81:594–598, 2005.

[11] V.A. Fateev and Sergei L. Lukyanov. The Models of Two-Dimensional Conformal Quantum Field Theory with Z(n) Symmetry. *Int. J. Mod. Phys. A*, 3:507, 1988.

[12] V.A. Fateev and A.B. Zamolodchikov. Parafermionic Currents in the Two-Dimensional Conformal Quantum Field Theory and Selfdual Critical Points in Z(n) Invariant Statistical Systems. *Sov. Phys. JETP*, 62:215–225, 1985.

[13] V.A. Fateev and A.B. Zamolodchikov. Conformal quantum field theory models in two dimensions having Z3 symmetry. *Nuclear Physics B*, 280:644 – 660, 1987.

[14] L. Fehér, L. O'Raifeartaigh, P. Ruelle, I. Tsutsui, and A. Wipf. On Hamiltonian reductions of the Wess-Zumino-Novikov-Witten theories. *Physics Reports*, 222(1):1 – 64, 1992.

[15] I. Frenkel, J. Lepowsky, and A. Meurman. *Vertex Operator Algebras and the Monster*. Academic Press, 1989.

[16] H. Freudenthal and H. de Vries. *Linear lie groups*. Pure and Applied Mathematics. Academic Press, 1969.

[17] J.-L. Gervais and Y. Matsuo. W geometries. *Phys. Lett. B*, 274:309–316, 1992.

[18] C. Guillarmou, A. Kupiainen, R. Rhodes, and V. Vargas. Conformal bootstrap in Liouville Theory. *arXiv preprint*, arXiv:2005.11530, 2020.

[19] C. Guillarmou, R. Rhodes, and V. Vargas. Polyakov's formulation of $2d$ bosonic string theory. *Publications Mathématiques de l'IHÉS*, 130:111–185, 2016.

[20] E. Gwynne and J. Miller. Existence and uniqueness of the Liouville quantum gravity metric for $\gamma \in (0, 2)$. *Inventiones mathematicae*, 223:213–333, 2021.

[21] Y. Huang, R. Rhodes, and V. Vargas. Liouville quantum gravity on the unit disk. *Ann. Inst. H. Poincaré Probab. Statist.*, 54(3):1694–1730, 08 2018.

[22] J. Humphreys. *Introduction to Lie Algebras and Representation Theory*. Springer, Berlin, 1972.

[23] Michio Jimbo, Tetsuji Miwa, and Masato Okado. Solvable lattice models related to the vector representation of classical simple Lie algebras. *Comm. Math. Phys.*, 116(3):507–525, 1988.

[24] J.-P. Kahane. Sur le chaos multiplicatif. *Annales des sciences mathématiques du Québec*, 1985.

[25] A. Kupiainen, R. Rhodes, and V. Vargas. Local Conformal Structure of Liouville Quantum Gravity. *Communications in Mathematical Physics*, 2018.

[26] A. Kupiainen, R. Rhodes, and V. Vargas. Integrability of Liouville theory: proof of the DOZZ formula. *Annals of Mathematics*, 191(1):81–166, 2020.

[27] H. Lacoin, R. Rhodes, and V. Vargas. The semiclassical limit of Liouville conformal field theory. *arXiv preprint*, arXiv:1903.08883, 2019.

[28] J.-F. Le Gall. Uniqueness and universality of the Brownian map. *Ann. Probab.*, 41(4):2880–2960, 07 2013.

[29] G. Miermont. The Brownian map is the scaling limit of uniform random plane quadrangulations. *Acta Math.*, 210(2):319–401, 2013.

[30] Seiberg N. Notes on Quantum Liouville Theory and Quantum Gravity. *Progress of Theoretical Physics Supplement*, 102:319–349, 03 1990.

[31] B Osgood, R Phillips, and P Sarnak. Extremals of determinants of Laplacians. *Journal of Functional Analysis*, 80(1):148 – 211, 1988.

[32] A. Polyakov. Quantum Geometry of bosonic strings. *Physics Letters B*, 103:207:210, 1981.

[33] R. Rhodes and V. Vargas. Gaussian multiplicative chaos and applications: A review. *Probab. Surveys*, 11:315–392, 2014.

[34] O. Schramm. Scaling limits of loop-erased random walks and uniform spanning trees [mr1776084]. In *Selected works of Oded Schramm. Volume 1, 2*, Sel. Works Probab. Stat., pages 791–858. Springer, New York, 2011.

[35] S. Sheffield. Gaussian free field for mathematicians. *Probability theory and related fields*, 139:521, 2007.

[36] A. B. Zamolodchikov. Infinite additional symmetries in two-dimensional conformal quantum field theory. *Theoretical and Mathematical Physics*, 65(3):1205–1213, December 1985.

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