On periodic points of Hamiltonian diffeomorphisms of CPd via generating functions

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ON PERIODIC POINTS OF HAMILTONIAN DIFFEOMORPHISMS OF $\mathbb{C}\mathrm{P}^d$ VIA GENERATING FUNCTIONS

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Abstract. Inspired by the techniques of Givental and Théret, we provide a proof with generating functions of a recent result of Ginzburg-Gürel concerning the periodic points of Hamiltonian diffeomorphisms of $\mathbb{C}\mathrm{P}^d$. For instance, we are able to prove that fixed points of pseudo-rotations are isolated as invariant sets or that a Hamiltonian diffeomorphism with a hyperbolic fixed point has infinitely many periodic points.

1. Introduction

Let $\mathbb{C}\mathrm{P}^d$ be the complex $2d$-dimensional space endowed with its classical symplectic structure ω, that is $\pi^*\omega = i^*\Omega$ where $\pi : \mathbb{S}^{2d+1} \to \mathbb{C}\mathrm{P}^d$ is the quotient map, $i : \mathbb{S}^{2d+1} \hookrightarrow \mathbb{C}^{d+1}$ is the inclusion map and $\Omega := \sum_j \mathrm{d}q_j \wedge \mathrm{d}p_j$ is the canonical symplectic form of $\mathbb{C}^{d+1} \simeq \mathbb{R}^{2(d+1)}$. We are interested in the study of Hamiltonian diffeomorphisms of $\mathbb{C}\mathrm{P}^d$, which are time-one maps of those vector fields X_t satisfying the Hamilton equations $X_t \lrcorner \omega = \mathrm{d}h_t$ for some smooth maps $h : [0,1] \times \mathbb{C}\mathrm{P}^d \to \mathbb{R}$ called Hamiltonian maps. Let $\mathrm{Ham}(\mathbb{C}\mathrm{P}^d)$ be the set of Hamiltonian diffeomorphisms of $\mathbb{C}\mathrm{P}^d$. In 1985, Fortune-Weinstein [4] proved that the Arnol'd conjecture holds for $\mathbb{C}\mathrm{P}^d$: any Hamiltonian diffeomorphism $\varphi \in \mathrm{Ham}(\mathbb{C}\mathrm{P}^d)$ has at least $d+1$ fixed points. Inspired by the work of Givental [9], Théret [17] used generating functions to reprove Fortune-Weinstein's theorem. Given $\varphi \in \mathrm{Ham}(\mathbb{C}\mathrm{P}^d)$, a k-periodic point $x \in \mathbb{C}\mathrm{P}^d$ of φ is by definition a fixed point of the k-iterated map φ^k. Contrary to aspherical symplectic manifolds like the $2d$-dimensional torus $\mathbb{T}^{2d}$ endowed with the canonical symplectic form, the Conley conjecture does not hold in $\mathbb{C}\mathrm{P}^d$: there exists Hamiltonian diffeomorphisms with only finitely many periodic points. For instance, one can take a rotation ρ of $\mathbb{C}\mathrm{P}^d$ defined by

$$\rho([z_1 : z_2 : \cdots : z_{d+1}]) := \left[e^{2i\pi a_1} z_1 : e^{2i\pi a_2} z_2 : \cdots : e^{2i\pi a_{d+1}} z_{d+1}\right],$$

with rationally independent coefficients $a_1, \ldots, a_{d+1} \in \mathbb{R}$. This is indeed a Hamiltonian diffeomorphism whose only periodic points are its fixed points: the projection of the canonical base of $\mathbb{C}^{d+1}$. Notice that this Hamiltonian diffeomorphism has the minimal number of periodic points. A Hamiltonian diffeomorphism of $\mathbb{C}\mathrm{P}^d$ which has exactly $d+1$ periodic points is called a *pseudo-rotation* of $\mathbb{C}\mathrm{P}^d$.

In the case $d = 1$, $\mathbb{C}\mathrm{P}^1 \simeq \mathbb{S}^2$ and Hamiltonian diffeomorphisms are the area preserving diffeomorphisms isotopic to identity. Franks [5, 6] proved that such area preserving homeomorphisms have either 2 or infinitely many periodic points.

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Therefore, the only Hamiltonian diffeomorphisms of $\mathbb{C}\mathrm{P}^1$ with finitely many periodic points are pseudo-rotations. In 1994, Hofer-Zehnder [11, p.263] conjectured a higher-dimensional generalization of this result: every Hamiltonian diffeomorphism of $\mathbb{C}\mathrm{P}^d$ has either $d+1$ or infinitely many periodic points (it was stated for more general symplectic manifolds). In this direction, a symplectic proof of Franks result (in the smooth setting) was provided by Collier *et al.* [2]. In 2019, Shelukhin [15] proved a version of Hofer-Zehnder conjecture: if a Hamiltonian diffeomorphism of a closed monotone symplectic manifold with semisimple quantum homology (*e.g.* $\mathbb{C}\mathrm{P}^d$) has a finite number of contractible periodic points then the sum of ranks of the local Floer homology groups at its contractible fixed points is equal to the total dimension of the homology of the manifold (that is $d+1$ for $\mathbb{C}\mathrm{P}^d$).

A compact invariant set $K \subset \mathbb{C}\mathrm{P}^d$ of a homeomorphism φ is said to be isolated if there exists a neighborhood U of K such that, for all $p \in U \setminus K$, $\varphi^k(p) \notin U$ for some $k \in \mathbb{Z}$. A fixed point of a Hamiltonian diffeomorphism is said to be homologically visible if its local Floer homology is non-trivial. The purpose of this article is to provide an elementary proof of the following theorem of Ginzburg-Gürel [8]:

Theorem 1.1. *Every Hamiltonian diffeomorphism of $\mathbb{C}\mathrm{P}^d$ which has a fixed point all of whose iterations are homologically visible that is isolated as an invariant set has infinitely many periodic points.*

As Ginzburg-Gürel already pointed out, Theorem 1.1 has two important corollaries. If x is a hyperbolic point then it is always isolated as an invariant set and the local cohomology of its iterations has rank 1.

Corollary 1.2. *Every Hamiltonian diffeomorphism of $\mathbb{C}\mathrm{P}^d$ with a hyperbolic fixed point has infinitely many periodic points.*

In fact, this theorem of Ginzburg-Gürel was originally proven in [7] in a more general setting, including some complex Grassmannians, $\mathbb{C}\mathrm{P}^d \times P^{2k}$ where P is symplectically aspherical and $k \leq d$, monotone products $\mathbb{C}\mathrm{P}^d \times \mathbb{C}\mathrm{P}^d$. We mention that the case of $\mathbb{C}\mathrm{P}^d \times \mathbb{T}^{2k}$, when $k \leq d$, can be deduced as well from our techniques.

In the special case of pseudo-rotations, every fixed point arises from a min-max principle and thus has a non-trivial local cohomology.

Corollary 1.3. *Each fixed point of a pseudo-rotation of $\mathbb{C}\mathrm{P}^d$ is not isolated as an invariant set.*

The original proof of Theorem 1.1 involves a non-trivial estimate on the energy of Floer trajectories leaving a periodic orbit called crossing energy theorem by Ginzburg-Gürel [7, Theorem 3.1] [8, Theorem 6.1] and is proved with a Gromov compactness like theorem on J-holomorphic curves. The second ingredient of the original proof is quantum homology, which is defined by means of Gromov-Witten invariants. Although we closely follow the original argument, our proof employs only elementary machinery: Morse theory and classical algebraic topology. Our main tool is generating functions, which are finite dimensional versions of the action functional for Hamiltonian diffeomorphisms of $\mathbb{R}^{2d}$. Inspired by Théret [17], we build a smooth map $\mathcal{T} : M \to \mathbb{R}$ defined on a finite dimensional manifold $M \subset \mathbb{R} \times \mathbb{C}\mathrm{P}^N$. There is a correspondence between critical points of $\mathcal{T}$ and capped fixed points of φ. With this map, the crossing energy theorem essentially boils down to elementary analysis. If $\zeta \in M$ is a critical point of $\mathcal{T}$ associated to a capped fixed point $\bar{z}$,

then $\mathrm{C}^*(\bar{z})$ is by definition the local cohomology of ζ with integral coefficients. In this setting, the q operator of quantum homology is mimicked by multiplication by $u^{d+1} \in H^*(M)$ where u is the generator of $H^2(\mathbb{C}\mathrm{P}^N)$ (notice that we have a morphism $H^*(\mathbb{C}\mathrm{P}^N) \to H^*(M)$, since $M \subset \mathbb{R} \times \mathbb{C}\mathrm{P}^N$) so that we can write the following identity when every object can be defined:

$$\mathrm{C}^*(\bar{z}\#A) = u^{d+1}\mathrm{C}^*(\bar{z}),$$

where A is the generator of $\pi_2(\mathbb{C}P^d) \simeq \mathbb{Z}$ satisfying $\langle [\omega], A\rangle = -\pi$ (see Proposition 5.11 for the precise statement and Section 5.7 for a further discussion about it).

Incidentally, we give a new composition formula for generating functions which is analogous to Chaperon's one [1] but works for the $\mathbb{C}$-linear identification of the diagonal $(z, Z) \mapsto (\frac{z+Z}{2}, i(z - Z))$. We also give an alternative way to study the projective join of a subspace of $\mathbb{C}\mathrm{P}^N$ with a projective subspace that does not involve equivariant cohomology.

Organization of the paper. In Section 2, we provide the background on generating functions. In Section 3, we provide the background on the Maslov index. In Section 4, we study the cohomological properties of the projective join needed to study the action of recapping on the cohomology of the sublevel sets of $\mathcal{T}$. In Section 5, we show how to use generating functions to provide a finite dimensional analogue of Floer cohomology in $\mathbb{C}\mathrm{P}^d$. In Section 6, we prove Theorem 1.1 and Corollaries 1.2 and 1.3, postponing the proof of the crossing energy theorem. In Section 7, we prove the crossing energy theorem in our setting.

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2. Generating functions

A generating function for Lagrangian submanifold of $T^*\mathbb{C}^n$ is a smooth function $F : \mathbb{C}^n \times \mathbb{C}^k \to \mathbb{R}$ such that 0 is a regular value of the $\mathbb{C}^k$-fiber derivative $\frac{\partial F}{\partial \xi}$. The space

$$\Sigma_F := \left\{ (q;\xi) \in \mathbb{C}^n \times \mathbb{C}^k \mid \frac{\partial F}{\partial \xi}(q;\xi) = 0 \right\} \tag{1}$$

is a smooth submanifold with dimension $2n$. Let $\iota_F : \Sigma_F \to T^*\mathbb{C}^n$ denotes the map $\iota_F(q;\xi) := (q, \partial_q F(q;\xi))$. Then ι_F is a Lagrangian immersion and we say that F generates the immersed Lagrangian submanifold $L := \iota_F(\Sigma_F)$.

A conical generating function of $\mathbb{C}^{2n} \simeq T^*\mathbb{C}^n$ is a C^1 map $F : \mathbb{C}^n \times \mathbb{C}^k \to \mathbb{R}$ such that

1. F is S^1-invariant and 2-homogeneous, that is
$$F(\lambda\zeta) = |\lambda|^2 F(\zeta), \quad \forall \lambda \in \mathbb{C}, \forall \zeta \in \mathbb{C}^n \times \mathbb{C}^k,$$
2. F is smooth in the neighborhood of $\Sigma_F \setminus 0$ where the subset $\Sigma_F \subset \mathbb{C}^n \times \mathbb{C}^k$ is still defined by (1)
3. 0 is a regular value of the fiber derivative $\partial_\xi F$ on $\mathbb{C}^n \times \mathbb{C}^k \setminus 0$.

The set Σ_F is $\mathbb{C}$-invariant and so is $\widetilde{L} := \iota_F(\Sigma_F)$. If $\pi : \mathbb{C}^{2n} \setminus 0 \to \mathbb{C}\mathrm{P}^{2n-1}$ denotes the quotient map, then $L := \pi(\widetilde{L})$ is a smooth immersed Lagrangian of $\mathbb{C}\mathrm{P}^{2n-1}$. We will say that $\widetilde{L}$ is a conical immersed Lagrangian.

A quadratic generating function $Q : \mathbb{C}^n \times \mathbb{C}^N \to \mathbb{R}$ is a generating function which is also a quadratic form. In this case, the induced Lagrangian $\iota_Q(\Sigma_Q)$ is a linear Lagrangian subspace of $T^*\mathbb{C}^n$. Notice that if $F : \mathbb{C}^n \times \mathbb{C}^k \to \mathbb{R}$ is a generating function of the Lagrangian $L \subset \mathbb{C}^{2n}$, then the quadratic form $\mathrm{d}^2F(x) : \mathbb{C}^n \times \mathbb{C}^k \to \mathbb{R}$, for $x \in \Sigma_F$, is a quadratic generating function of the tangent space $T_{\iota_F(x)}L \subset \mathbb{C}^{2n}$. The same is true if F is conical and $x \in \Sigma_F \setminus 0$. Moreover, $\mathbb{C}x \subset \ker \mathrm{d}^2F(x)$ in this case.

The existence of generating functions is well known for Lagrangians which are isotopic to the 0-section $\mathbb{C}^n \times \{0\}$ with a "suitably controlled" behavior at infinity (*e.g.* for a compactly supported isotopy or for a linear isotopy). In fact, we usually find a generating family of a whole isotopy $(L_t) := (\Phi_t(\mathbb{C}^n \times \{0\}))$, where (Φ_t) is a Hamiltonian flow, that is a continuous family of generating functions (F_t) with F_t generating L_t for all $t \in [0, 1]$. In Section 5, we give a construction of generating families for Hamiltonian flows.

There are strong uniqueness results relative to generating functions of linear Lagrangians or Lagrangians isotopic to the 0-section through compactly supported isotopies. Concerning quadratic generating functions, we will only need the following elementary result

Lemma 2.1 ([16, Prop. 35])**.** *For every quadratic generating function $Q : \mathbb{C}^n \times \mathbb{C}^k \to \mathbb{R}$ of the 0-section, there exists a linear fibered isomorphism A of $\mathbb{C}^n \times \mathbb{C}^k$ which is isotopic to the identity through linear fiberwise isomorphism such that $Q \circ A(q; \xi)$ does not depend on $q \in \mathbb{C}^n$. More precisely, if $Q(z) = \left\langle \widetilde{Q}z, z \right\rangle$ with*

$$\widetilde{Q} = \begin{bmatrix} a & b \\ {}^t b & c \end{bmatrix},$$

then c is invertible and $A(q; \xi) := (q; \xi - c^{-1\,t}bq)$ so that $Q \circ A(q; \xi) = {}^t\xi c \xi$.

Concerning the conical case, we will use the following

Lemma 2.2 ([17, Lemma 4.8])**.** *If $(F_t : \mathbb{C}^n \times \mathbb{C}^k \to \mathbb{R})$ is a smooth family of conical generating functions for the constant conical Lagrangian $L \subset T^*\mathbb{C}^n$, then there is a smooth isotopy (B_t) of conical fibered diffeomorphisms such that*

$$F_t \circ B_t = F_0, \quad \forall t.$$

Let $\mathrm{Ham}(\mathbb{C}^d)$ be the set of Hamiltonian diffeomorphisms of $\mathbb{C}^d \simeq T^*\mathbb{R}^d$. The map

$$\tau : \overline{\mathbb{C}^d} \times \mathbb{C}^d \to \mathbb{C}^{2d}, \quad \tau(z, Z) = \left(\frac{z + Z}{2}, i(z - Z) \right), \tag{2}$$

is a $\mathbb{C}$-linear symplectomorphism sending the diagonal $\{(z, z) \mid z \in \mathbb{C}^d\}$ to the 0-section of $\mathbb{C}^{2d}$. Let $\Phi \in \mathrm{Ham}(\mathbb{C}^d)$, the image of the graph $z \mapsto (z, \Phi(z))$ of Φ under τ is then a Lagrangian submanifold $L_\Phi \subset \mathbb{C}^{2d}$. A generating function of the Hamiltonian diffeomorphism Φ is a generating function of L_Φ. A generating family of a Hamiltonian flow (Φ_t) is a generating family of (L_{Φ_t}). Let Φ be a conical Hamiltonian diffeomorphism of $\mathbb{C}^d$, that is, a homeomorphism $\Phi : \mathbb{C}^d \to \mathbb{C}^d$ with $\Phi|_{\mathbb{C}^d \setminus 0} \in \mathrm{Ham}(\mathbb{C}^d \setminus 0)$ and which is $\mathbb{C}$-equivariant:

$$\Phi(\lambda z) = \lambda \Phi(z), \quad \forall \lambda \in \mathbb{C}, \forall z \in \mathbb{C}^d.$$

To simplify notation, we will write $\Phi \in \mathrm{Ham}_{\mathbb{C}}(\mathbb{C}^d)$ and say that Φ is a $\mathbb{C}$-equivariant Hamiltonian diffeomorphism. The last definition extends to Hamiltonian flows in

an obvious way. Then the induced subset $L_\Phi \subset \mathbb{C}^{2d}$ is a conical Lagrangian. A conical generating function of Φ (or simply a generating function of Φ) is a conical generating function of L_Φ. It extends to conical flows in an obvious way. As a consequence of the general case, if F is a generating function of $\Phi \in \mathrm{Ham}_{\mathbb{C}}(\mathbb{C}^d)$ and $(z;\xi) \neq 0$ is a critical point of F then z is a fixed point of Φ and $\mathrm{d}^2F(z;\xi)$ is a quadratic generating function of $\mathrm{d}\Phi(z)$. Moreover,

$$\dim\ker \mathrm{d}^2F(z;\xi) = \dim\ker(\mathrm{d}\Phi(z) - \mathrm{id}).$$

3. Maslov Index

3.1. **Maslov index of a path in $\mathbf{Sp}(2d)$.** Let $\Gamma = (\Gamma_t) : [0,1] \to \mathrm{Sp}(2d)$ be a continuous path in the space of symplectic matrices $\mathrm{Sp}(2d)$ of $\mathbb{R}^{2d} \simeq \mathbb{C}^d$. Then there exists a continuous family (Q_t) of quadratic generating functions such that, for $t \in [0,1]$, $Q_t : \mathbb{C}^N \to \mathbb{R}$ is generating Γ_t. The variation of index $\mathrm{ind}(Q_1) - \mathrm{ind}(Q_0) \in \mathbb{Z}$ is independent of the choice of (Q_t) and is called the Maslov index of Γ denoted

$$\mathrm{mas}((\Gamma_t)) := \mathrm{ind}(Q_1) - \mathrm{ind}(Q_0) \in \mathbb{Z}.$$

Other equivalent definitions of the Maslov index (which is sometimes also called Conley-Zehnder index) are available in the literature, see [14], [12] and references therein.

In order to state the general properties of mas, following Théret, in this section we will denote by $R \bullet S$ the concatenation of two paths $R = (R_t)$ and $S = (S_t)$ in $\mathrm{Sp}(2d)$ satisfying $R_1 = S_0$, that is $(R \bullet S)_t = R_{2t}$ for $t \in [0, 1/2]$ and $(R \bullet S)_t = S_{2t-1}$ for $t \in [1/2, 1]$. The path RS stands for the pointwise matrix product of two paths in $\mathrm{Sp}(2d)$ that is $(RS)_t = R_tS_t$ for all t. Given a path $R = (R_t)$ in $\mathrm{Sp}(2d)$, the path $R^{(-1)}$ will stand for the reverse path (R_{1-t}), whereas R^{-1} will stand for the path of inverses (R_t^{-1}). Identifying matrices with their canonical linear maps, for two square matrices A and B, $A \oplus B$ will stand for the square matrix

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

and given two paths $R = (R_t)$ and $S = (S_t)$ in $\mathrm{Sp}(2n)$ and $\mathrm{Sp}(2m)$ respectively, $(R \oplus S)_t := (R_t \oplus S_t)$ as a path in $\mathrm{Sp}(2(n+m))$. We recall the basic proprieties of the Maslov index (see for instance [16, Prop. 39 and 58]).

Proposition 3.1. *Let R be a path in $\mathrm{Sp}(2n)$,*

(1) if S is a path in $\mathrm{Sp}(2n)$ with $S_0 = R_1$, then $\mathrm{mas}(R \bullet S) = \mathrm{mas}(R) + \mathrm{mas}(S)$,
(2) the Maslov index of the reverse path is $\mathrm{mas}(R^{(-1)}) = -\mathrm{mas}(R)$,
(3) if S is a path in $\mathrm{Sp}(2m)$, then $\mathrm{mas}(R \oplus S) = \mathrm{mas}(R) + \mathrm{mas}(S)$,
(4) if $A \in \mathrm{Sp}(2d)$, then $\mathrm{mas}(ARA^{-1}) = \mathrm{mas}(R)$.
(5) if S is a path homotopic to R relative to endpoints, that is there exists a continuous family $s \mapsto R^s$ of paths in $\mathrm{Sp}(2n)$ with $R^0 = R$ and $R^1 = S$ such that $R_0^s \equiv R_0^0$ and $R_1^s \equiv R_1^0$, then $\mathrm{mas}(S) = \mathrm{mas}(R)$,
(6) if $S_t := \begin{bmatrix} \cos(2\pi t) & -\sin(2\pi t) \\ \sin(2\pi t) & \cos(2\pi t) \end{bmatrix} \in \mathrm{Sp}(2)$, $t \in [0,1]$, then $\mathrm{mas}(S) = -2$.

Let (Φ_t) be a Hamiltonian flow on $\mathbb{C}^d$ starting at $\Phi_0 = \mathrm{id}$. If $z \in \mathbb{C}^d$ is a fixed-point of Φ_1, the Maslov index of z is set to be the Maslov index of the path $t \mapsto \mathrm{d}\Phi_t(z)$ in $\mathrm{Sp}(2d)$, that is

$$\mathrm{mas}(z, (\Phi_t)) := \mathrm{mas}((\mathrm{d}\Phi_t(z))).$$

Suppose that $F_t : \mathbb{C}^N \to \mathbb{C}$, $t \in [0,1]$, defines a continuous family of generating functions of (Φ_t). Let $\zeta_t \in \Sigma_{F_t} \subset \mathbb{C}^N$ be a continuous family associated to $\Phi_t(z)$. Then the continuous family of Hessians $Q_t := \mathrm{d}^2 F_t(\zeta_t)$ is a continuous family of quadratic generating functions of $\mathrm{d}\Phi_t(z)$, thus

$$\mathrm{mas}(z, (\Phi_t)) = \mathrm{ind}(\zeta_1, F_1) - \mathrm{ind}(\zeta_0, F_0),$$

where $\mathrm{ind}(\zeta, F) := \mathrm{ind}(\mathrm{d}^2 F(\zeta)) \in \mathbb{N}$ denotes the Morse index of F at the critical point ζ.

This definition is extended to every symplectic manifold M^{2d} as follows. Let (φ_t) be a Hamiltonian flow on M^{2d} starting at $\varphi_0 = \mathrm{id}$ and let $z \in M$ be a fixed point of φ_1 such that the loop $t \mapsto \varphi_t(z)$ is contractible. Let $D^2 := \{w \in \mathbb{C} \mid |w| \leq 1\}$ be the closed unit disk of $\mathbb{C}$. Since the loop is contractible, there exists a smooth map $u : D^2 \to M$ such that $u(e^{2i\pi t}) = \varphi_t(z)$. Then there exists a trivialization $D^2 \times \mathbb{C}^d \to u^*TM$, $(w, \zeta) \mapsto \xi(w)\zeta$ so that, for all $w \in D^2$, $\xi(w) : \mathbb{C}^d \to T_{u(w)}M$ is a symplectic map. Moreover, if we endow M with an almost complex structure, the trivialization can be made $\mathbb{C}$-linear. The set of every such trivialization is contractible, for a fixed choice of u (see [14, Lemma 5.1] for instance). Then $\gamma_t := \xi(e^{2i\pi t})^{-1}\mathrm{d}\varphi_t(z)\xi(1)$, $t \in [0,1]$, is a symplectic path in $\mathrm{Sp}(2d)$ and the Maslov index of z with respect to the *capping* u is set to be

$$\mathrm{mas}(z, u) := \mathrm{mas}((\gamma_t)).$$

It does not depend on the specific choice of trivialization, in fact it only depends on the homotopy class of u relative to the boundary ∂D^2. Thus, if $\pi_2(M) = 0$ any choice of u gives the same index.

3.2. **Maslov index of a $\mathbb{C}$-equivariant Hamiltonian diffeomorphism.** Let (Φ_t) be a $\mathbb{C}$-equivariant Hamiltonian flow on $\mathbb{C}^{d+1}$ lifting a Hamiltonian flow (φ_t) on $\mathbb{C}\mathrm{P}^d$. Let $Z_0 \in \mathbb{S}^{2d+1}$ be a fixed point of Φ_1 and denote by $Z_t := \Phi_t(Z_0)$, $t \in [0,1]$, the associated loop in $\mathbb{S}^{2d+1}$. Let $\pi : \mathbb{S}^{2d+1} \to \mathbb{C}\mathrm{P}^d$ be the quotient map. Let $z_t := \pi(Z_t)$ be the associated loop in $\mathbb{C}\mathrm{P}^d$ so that $z_t = \varphi_t(z_0)$. Let $U : D^2 \to \mathbb{S}^{2d+1}$ be any smooth capping of (Z_t), *i.e.* $Z_t = U(e^{2i\pi t})$. All such cappings are homotopic since $\pi_2(\mathbb{S}^{2d+1}) = 0$. We set $u := \pi \circ U$.

Proposition 3.2. *With the above notations,*

$$\mathrm{mas}(Z_0, (\Phi_t)) = \mathrm{mas}(z_0, u).$$

Proof. For all $t \in [0,1]$, let $\gamma_t := \mathrm{d}\varphi_t(z_0) : T_{z_0}\mathbb{C}\mathrm{P}^d \to T_{z_t}\mathbb{C}P^d$ and $\Gamma_t := \mathrm{d}\Phi_t(Z_0)$ which is a path in $\mathrm{Sp}(2(d+1))$. For all $w \in D^2$, let $\xi(w) : \mathbb{C}^d \to T_{u(w)}\mathbb{C}\mathrm{P}^d$ be a smooth family of $\mathbb{C}$-linear symplectic maps induced by u as explained above. Throughout the proof, if f denotes a map whose domain is D^2, then, for $t \in [0,1]$, $f_t := f(e^{2i\pi t})$. For all $t \in [0,1]$ let $\xi_t := \xi(e^{2i\pi t})$ and $\gamma'_t := \xi_t^{-1}\gamma_t\xi_0 \in \mathrm{Sp}(2d)$ so that

$$\mathrm{mas}(Z_0, (\Phi_t)) = \mathrm{mas}((\Gamma_t)) \quad \text{and} \quad \mathrm{mas}(z_0, u) = \mathrm{mas}((\gamma'_t)).$$

Notice that, for all $Z \in \mathbb{S}^{2d+1}$, the tangent space $T_{\pi(Z)}\mathbb{C}\mathrm{P}^d \simeq \mathbb{C}^{d+1}/\mathbb{C}Z$ is canonically isomorphic to $(\mathbb{C}Z)^\perp$ (given a $\mathbb{C}$-subspace $E \subset \mathbb{C}^{d+1}$, $E^\perp$ denotes its hermitian orthogonal subspace, which is also its Euclidean orthogonal subspace or its symplectic orthogonal subspace). Let $L(w) := (\mathbb{C}U(w))^\perp \to T_{u(w)}\mathbb{C}\mathrm{P}^d$, $w \in D^2$, be the induced continuous family of $\mathbb{C}$-linear symplectic maps. Let us define the following continuous family of endomorphism of $\mathbb{C}^{d+1}$ indexed by $w \in D^2$,

$$A(w) : \mathbb{C} \times \mathbb{C}^d \to \mathbb{C}U(w) \oplus (\mathbb{C}U(w))^\perp, \quad A(w)(\lambda, \zeta) = \lambda U(w) + L(w)^{-1}\xi(w)\zeta.$$

Since the linear maps $\lambda \mapsto \lambda U(w)$ and $L(w)^{-1}\xi(w)$ are symplectic maps and since both direct sums $\mathbb{C}\times\mathbb{C}^d$ and $\mathbb{C}U(w)\oplus(\mathbb{C}U(w))^\perp$ are symplectic-orthogonal sums, $A(w)\in \mathrm{Sp}(2(d+1))$.

Since Φ_t is a $\mathbb{C}$-equivariant diffeomorphism, The symplectic map $\mathrm{d}\Phi_t(Z_0)=\Gamma_t$ sends the orthogonal subspaces $\mathbb{C}Z_0$ and $(\mathbb{C}Z_0)^\perp$ respectively on $\mathbb{C}Z_t$ and $(\mathbb{C}Z_t)^\perp$ with

$$\Gamma_t(\lambda Z_0+\zeta)=\lambda Z_t+L_t^{-1}\gamma_t L_0\zeta,\quad \forall\lambda\in\mathbb{C},\forall\zeta\in(\mathbb{C}Z_0)^\perp,$$

where $L_t:=L(e^{2i\pi t}):(\mathbb{C}Z_t)^\perp\to T_{z_t}\mathbb{C}\mathrm{P}^d$. Thus $\Gamma'_t:=A_t^{-1}\Gamma_t A_0$ is the symplectic path $\Gamma'_t=I_2\oplus\gamma'_t$, so Proposition 3.1 (3) implies $\mathrm{mas}((\Gamma'_t))=\mathrm{mas}((\gamma'_t))$. Since $A_t=A(e^{2i\pi t})$ with $A:D^2\to\mathrm{Sp}(2(d+1))$ continuous, (Γ'_t) is homotopic to $(A_0^{-1}\Gamma_tA_0)$ relative to endpoints, thus $\mathrm{mas}((\Gamma'_t))=\mathrm{mas}((A_0^{-1}\Gamma_tA_0))=\mathrm{mas}((\Gamma_t))$, according to Proposition 3.1 (5) and (4). □

3.3. **Bott's iteration inequalities.** Let (Φ_t) be a Hamiltonian flow on $\mathbb{C}^d$ starting at $\Phi_0=\mathrm{id}$ and let $z\in\mathbb{C}^d$ be a fixed point. Even though $(\Phi_t(z))$ is a loop in $\mathbb{C}^d$, $\Gamma_t:=\mathrm{d}\Phi_t(z)$, $t\in\mathbb{R}_+$, defines only a path in $\mathrm{Sp}(2d)$, so that in general $\mathrm{mas}(\Gamma_{kt},t\in[0,1])\neq k\,\mathrm{mas}(\Gamma_t,t\in[0,1])$. Notice that the path $(\Gamma_t)_{t\in\mathbb{R}}$ only depends on $(\Gamma_t)_{t\in[0,1]}$ since $\Gamma_{t+k}=\Gamma_t\Gamma_1^k$ for $k\in\mathbb{N}$ and $t\geq 0$.

Theorem 3.3. *Let $\Gamma:=(\Gamma_t)_{t\geq0}$ be a continuous path in $\mathrm{Sp}(2d)$ such that $\Gamma_0=I_{2d}$ and $\Gamma_{t+k}=\Gamma_t\Gamma_1^k$ for all $k\in\mathbb{N}$ and $t>0$. Then the* average Maslov index

$$\overline{\mathrm{mas}}(\Gamma):=\lim_{k\to\infty}\frac{\mathrm{mas}(\Gamma_{kt},t\in[0,1])}{k}\in\mathbb{R}$$

is a well-defined real number and we have the iteration inequalities

$$k\,\overline{\mathrm{mas}}(\Gamma)-d\leq\mathrm{mas}(\Gamma_{kt},t\in[0,1]),$$
$$\mathrm{mas}(\Gamma_{kt},t\in[0,1])+\dim\ker(\Gamma_1^k-I_{2d})\leq k\,\overline{\mathrm{mas}}(\Gamma)+d.$$

We refer to [13, Theorem 3.6] for a more precise statement and a proof. Notice that, by definition, the average Maslov index is homogeneous:

$$\overline{\mathrm{mas}}((\Gamma_{kt}))=k\,\overline{\mathrm{mas}}((\Gamma_t)).$$

Let us denote by $\overline{\mathrm{mas}}(z,(\Phi_t))\in\mathbb{R}$ the average Maslov index of the fixed point z, that is

$$\overline{\mathrm{mas}}(z,(\Phi_t)):=\overline{\mathrm{mas}}(\mathrm{d}\Phi_t(z),t\geq0).$$

So that Theorem 3.3 gives for all $k\in\mathbb{N}$,

$$k\,\overline{\mathrm{mas}}(z,(\Phi_t))-d\leq\mathrm{mas}(z,(\Phi_{kt})),$$
$$\mathrm{mas}(z,(\Phi_{kt}))+\dim\ker(\mathrm{d}\Phi_k(z)-\mathrm{id})\leq k\,\overline{\mathrm{mas}}(z,(\Phi_t))+d.$$

This inequality can be extended to every symplectic manifold M^{2d} as follows. Let (φ_t) be a Hamiltonian flow on M^{2d} starting at $\varphi_0=\mathrm{id}$ and let $z\in M$ be a fixed point of φ_1 such that $(\varphi_t(z))$ is contractible. Let $u:D^2\to M$ be a capping of z and $\xi(w):\mathbb{C}^d\to T_{u(w)}M$, $w\in D^2$, be an induced trivialization. For $k\in\mathbb{N}^*$, let $u^k:D^2\to M$ be the smooth map $u^k(w):=u(w^k)$, $w\in D^2$. This map is the natural capping of z as a fixed point of the time-one map of the Hamiltonian flow (φ_{kt}) induced by (z,u). If $\bar z:=(z,u)$, it is often denoted by $\bar z^k=(z,u^k)$. An induced trivialization is $\xi^k(w):=\xi(w^k)$, so that $\gamma_t^{(k)}=\gamma_{kt}$, where $\gamma_t^{(k)}:=\xi^k(e^{2i\pi t})^{-1}\mathrm{d}\varphi_{kt}(z)\xi^k(1)$, and $\gamma_t:=$

$\xi(e^{2i\pi t})^{-1}\mathrm{d}\varphi_t(z)\xi(1)$, $t \geq 1$. Since $\mathrm{mas}(\bar{z}^k) := \mathrm{mas}(\gamma_t^{(k)}, t \in [0,1])$ with $\gamma_{t+k} = \gamma_t\gamma_1^k$ for all $k \in \mathbb{N}$ and $t \geq 0$, Theorem 3.3 gives for all $k \in \mathbb{N}$,

$$\begin{aligned} k\,\overline{\mathrm{mas}}(\bar{z}) - d \leq \mathrm{mas}(\bar{z}^k), \\ \mathrm{mas}(\bar{z}^k) + \dim\ker(\mathrm{d}\varphi_k(z) - \mathrm{id}) \leq k\,\overline{\mathrm{mas}}(\bar{z}) + d. \end{aligned} \tag{3}$$

where $\overline{\mathrm{mas}}(\bar{z}) := \overline{\mathrm{mas}}(\gamma_t, t \geq 0)$ is the average Maslov index of the capped fixed point $\bar{z} = (z, u)$. Let (Φ_t) be a $\mathbb{C}$-equivariant Hamiltonian flow of $\mathbb{C}^{d+1}$ with $\Phi_0 = \mathrm{id}$ which is the lift of a Hamiltonian flow (φ_t) of $\mathbb{C}\mathrm{P}^d$ with $\varphi_0 = \mathrm{id}$. Let $Z \in \mathbb{S}^{2d+1}$ be a fixed point of Φ_1 and $\bar{z} = (\pi(Z), u)$ be the capped fixed point of φ_1 associated to it, then

$$\mathrm{mas}(Z, (\Phi_{kt})) = \mathrm{mas}(\bar{z}^k), \quad \forall k \in \mathbb{N}.$$

Indeed, if $U : D^2 \to \mathbb{S}^{2d+1}$ is a capping of Z so that $u = \pi \circ U$, then U^k is a capping of Z relative to $(\Phi_{kt})_{t\in[0,1]}$ and $u^k = \pi \circ U^k$ (recall that $\mathrm{mas}(Z, (\Phi_{kt}))$ does not depend on the choice of capping since $\pi_2(\mathbb{S}^{2d+1}) = 0$). So, according to equation (3), for every fixed point $Z \in \mathbb{S}^{2d+1}$ of any $\mathbb{C}$-equivariant Hamiltonian flow (Φ_t) of $\mathbb{C}^{d+1}$ which is the lift of some Hamiltonian flow (φ_t) of $\mathbb{C}\mathrm{P}^d$, for all $k \in \mathbb{N}$,

$$\begin{aligned} k\,\overline{\mathrm{mas}}(Z, (\Phi_t)) - d \leq \mathrm{mas}(Z, (\Phi_{kt})), \\ \mathrm{mas}(Z, (\Phi_{kt})) + \dim\ker(\mathrm{d}\varphi_k(z) - \mathrm{id}) \leq k\,\overline{\mathrm{mas}}(Z, (\Phi_t)) + d, \end{aligned} \tag{4}$$

where $z := \pi(Z)$.

4. Projective join

In [9, Appendix], Givental studied the cohomology of projective joins by using S^1-equivariant cohomology. Here, we give an alternative way to study the special case of joins with a projective subspace.

Let $m, n \in \mathbb{N}$ and let $\pi : \mathbb{C}^{m+n+2} \setminus 0 \to \mathbb{C}\mathrm{P}^{m+n+1}$ be the quotient projection. We projectively embed $\mathbb{C}\mathrm{P}^m$ and $\mathbb{C}\mathrm{P}^n$ in $\mathbb{C}\mathrm{P}^{m+n+1}$ by identifying $\mathbb{C}\mathrm{P}^m$ with $\pi(\mathbb{C}^{m+1} \times 0 \setminus 0)$ and $\mathbb{C}\mathrm{P}^n$ with $\pi(0 \times \mathbb{C}^{n+1} \setminus 0)$ so that $\mathbb{C}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^m$ do not intersect. This is equivalent to considering two projective subspaces of respective $\mathbb{C}$-dimensions m and n in general position. Let $A \subset \mathbb{C}\mathrm{P}^m$ and $B \subset \mathbb{C}\mathrm{P}^n$. Then the projective join $A * B \subset \mathbb{C}\mathrm{P}^{m+n+1}$ is the union of every projective lines intersecting A and B. In other words, $A * B = A \cup B \cup \pi(\widetilde{A} \times \widetilde{B})$ where $\widetilde{A}$ and $\widetilde{B}$ are the lifts of A and B to $\mathbb{C}^{m+1} \setminus 0$ and $\mathbb{C}^{n+1} \setminus 0$ respectively. One can remark that $\mathbb{C}\mathrm{P}^m * \mathbb{C}\mathrm{P}^n = \mathbb{C}\mathrm{P}^{m+n+1}$ and that if $[a : b] \in \mathbb{C}\mathrm{P}^{m+n+1}$, with $a \in \mathbb{C}^{m+1}$ and $b \in \mathbb{C}^{n+1}$, does not belong to $\mathbb{C}\mathrm{P}^m$ nor to $\mathbb{C}\mathrm{P}^n$, then only one projective line intersecting $\mathbb{C}\mathrm{P}^m$ and $\mathbb{C}\mathrm{P}^n$ contains $[a : b]$, namely the line joining $[a : 0]$ to $[0 : b]$. Given $A \subset \mathbb{C}\mathrm{P}^m$, we denote by $p_A : A * \mathbb{C}\mathrm{P}^n \setminus \mathbb{C}\mathrm{P}^n \to A$ the projection $[a : b] \mapsto [a : 0]$.

In this paper, H^* will stand for the singular cohomology with integer coefficients. Given $A \subset \mathbb{C}\mathrm{P}^m$, let $T \subset A * \mathbb{C}\mathrm{P}^m$ be a tubular neighborhood of $\mathbb{C}\mathrm{P}^m$ such that $(A * \mathbb{C}\mathrm{P}^n, T)$ retracts on $(A * \mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)$. By excision $H^*(A * \mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n) \simeq H^*(A * \mathbb{C}\mathrm{P}^n \setminus \mathbb{C}\mathrm{P}^n, T \setminus \mathbb{C}\mathrm{P}^n)$. Using this identification, we define the cup-product $H^*(A * \mathbb{C}\mathrm{P}^n \setminus \mathbb{C}\mathrm{P}^n) \otimes H^*(A * \mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n) \to H^*(A * \mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)$ by the following commutative

diagram

$$\begin{array}{ccc} H^*(A*\mathbb{C}\mathrm{P}^n\setminus\mathbb{C}\mathrm{P}^n)\otimes H^*(A*\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n) & \xrightarrow{\smile} & H^*(A*\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n) \\ \downarrow\simeq & & \downarrow\simeq \\ H^*(A*\mathbb{C}\mathrm{P}^n\setminus\mathbb{C}\mathrm{P}^n)\otimes H^*(A*\mathbb{C}\mathrm{P}^n,T) & & H^*(A*\mathbb{C}\mathrm{P}^n,T) \\ \simeq\uparrow & & \simeq\uparrow \\ H^*(A*\mathbb{C}\mathrm{P}^n\setminus\mathbb{C}\mathrm{P}^n)\otimes H^*(A*\mathbb{C}\mathrm{P}^n\setminus\mathbb{C}\mathrm{P}^n,T\setminus\mathbb{C}\mathrm{P}^n) & \xrightarrow{\smile} & H^*(A*\mathbb{C}\mathrm{P}^n\setminus\mathbb{C}\mathrm{P}^n,T\setminus\mathbb{C}\mathrm{P}^n) \end{array},$$

where the vertical arrows are induced by inclusions and the bottom arrow is the usual cup-product. According to the long exact sequence of the couple $(\mathbb{C}\mathrm{P}^{m+n+1},\mathbb{C}\mathrm{P}^n)$, the map $H^{2(n+1)}(\mathbb{C}\mathrm{P}^{m+n+1},\mathbb{C}\mathrm{P}^n)\to H^{2(n+1)}(\mathbb{C}\mathrm{P}^{m+n+1})$ induced by the inclusion is an isomorphism (the dimension of $\mathbb{C}\mathrm{P}^n$ being $2n<2n+1$) so that we can see the class $u^{n+1}\in H^{2(n+1)}(\mathbb{C}\mathrm{P}^{m+n+1})$ in $H^{2(n+1)}(\mathbb{C}\mathrm{P}^{m+n+1},\mathbb{C}\mathrm{P}^n)$ *via* this identification. Given $A\subset\mathbb{C}\mathrm{P}^m$, let $t_A\in H^{2(n+1)}(A*\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n)$ be the image of $u^{n+1}\in H^{2(n+1)}(\mathbb{C}\mathrm{P}^{m+n+1},\mathbb{C}\mathrm{P}^n)$ induced by the inclusion and let $f_A:H^*(A)\to H^{*+2(n+1)}(A*\mathbb{C}\mathrm{P}^n)$ be the morphism given by $f_A(v):=p_A^*(v)\smile t_A$.

Proposition 4.1. *Let $A\subset\mathbb{C}\mathrm{P}^m$. One has the following isomorphisms:*

$$H^k(A*\mathbb{C}\mathrm{P}^n)\simeq\begin{cases}H^k(\mathbb{C}\mathrm{P}^n) & \text{for } k\leq 2n+1,\\ H^{k-2(n+1)}(A) & \text{for } k>2n+1,\end{cases}$$

where the isomorphisms $H^k(A*\mathbb{C}\mathrm{P}^n)\to H^k(\mathbb{C}\mathrm{P}^n)$ are induced by the inclusion and the isomorphisms $H^{k-2(n+1)}(A)\to H^k(A*\mathbb{C}\mathrm{P}^n)$ are given by f_A.

Proof. Let us consider the long exact sequence of the couple $(A*\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n)$:

$$\cdots\to H^*(A*\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n)\xrightarrow{j^*}H^*(A*\mathbb{C}\mathrm{P}^n)\xrightarrow{i^*}H^*(\mathbb{C}\mathrm{P}^n)\to\cdots \tag{5}$$

The inclusions of $A*\mathbb{C}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$ in $\mathbb{C}\mathrm{P}^{m+n+1}$ give the following commutative diagram:

$$\begin{array}{ccc} H^*(A*\mathbb{C}\mathrm{P}^n) & \xrightarrow{i^*} & H^*(\mathbb{C}\mathrm{P}^n) \\ \uparrow & \nearrow & \\ H^*(\mathbb{C}\mathrm{P}^{m+n+1}) & & \end{array}$$

where the diagonal arrow is onto (we recall that $\mathbb{C}\mathrm{P}^n$ is projectively embedded inside $\mathbb{C}\mathrm{P}^{m+n+1}$), thus i^* is onto. Hence the long exact sequence (5) can be reduced to the short exact sequence

$$0\to H^*(A*\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n)\xrightarrow{j^*}H^*(A*\mathbb{C}\mathrm{P}^n)\xrightarrow{i^*}H^*(\mathbb{C}\mathrm{P}^n)\to 0. \tag{6}$$

Let us consider $p_A:A*\mathbb{C}\mathrm{P}^n\setminus\mathbb{C}\mathrm{P}^n\to A$. This projection defines a complex vector bundle of dimension $n+1$. Indeed, let $E_A:=A*\mathbb{C}\mathrm{P}^n\setminus\mathbb{C}\mathrm{P}^n$ and $U_i\subset\mathbb{C}\mathrm{P}^{m+n+1}$ be the affine chart $\{[a_0:\cdots:a_m:z_0:\cdots:z_n]\mid a_i\neq 0\}$. Since the intersection of a projective line with the projective hyperplane $\mathbb{C}\mathrm{P}^{m+n+1}\setminus U_i$ is either a point or the projective line itself, we see that $p_A^{-1}(A\cap U_i)=E_A\cap U_i$. We then have the trivialization $E_A\cap U_i\simeq A\cap U_i\times\mathbb{C}^{n+1}$ given by $[a:z]\mapsto([a],z/a_i)$. Thus E_A is a fiber bundle, moreover this is the restriction of $E_{\mathbb{C}\mathrm{P}^m}$ to A. We can even say that $E_{\mathbb{C}\mathrm{P}^m}\simeq(\gamma_m^1)^{\oplus(n+1)}$ where γ_m^1 is the tautological fiber bundle of $\mathbb{C}\mathrm{P}^m$, by looking at the transition maps of the above trivialization charts (but this will not be relevant for us). Let us endow $\mathbb{C}\mathrm{P}^{m+n+1}$ with the Riemannian metric induced by the round

metric of $\mathbb{S}^{2(m+n)+3}$ and let $T \subset A*\mathbb{C}\mathrm{P}^n$ be the tubular neighborhood of $\mathbb{C}\mathrm{P}^n$ defined as the set of points at distance less than $r \in (0, \pi/2)$ of $\mathbb{C}\mathrm{P}^n$. Then the topological pair $(A * \mathbb{C}\mathrm{P}^n, T)$ retracts on $(A * \mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)$ so that the inclusion map induces an isomorphism $H^*(A*\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n) \simeq H^*(A*\mathbb{C}\mathrm{P}^n, T)$ in cohomology. Since the compact $\mathbb{C}\mathrm{P}^n$ is included in the interior of T, by excision $H^*(A * \mathbb{C}\mathrm{P}^n, T) \simeq H^*(E_A, T \cap E_A)$. In the trivialization charts, each fibers of $E_A \setminus T$ is a round ball of $\mathbb{C}^{n+1}$ so that $(E_A, E_A \setminus A)$ retracts on $(E_A, T \cap E_A)$. According to Thom isomorphism theorem,

$$H^{*-2(n+1)}(A) \simeq H^*(E_A, E_A \setminus A) \simeq H^*(A * \mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n),$$

where the isomorphism $H^{*-2(n+1)}(A) \to H^*(A * \mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)$ is given by the cup-product of the pull-back of the class by p_A with the Thom class $t'_A \in H^{2(n+1)}(A * \mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)$. Furthermore, since $H^k(\mathbb{C}\mathrm{P}^n)$ is zero when $k > 2n$ and $H^k(A*\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)$ is zero when $k < 2(n+1)$, the short exact sequence (6) obviously decomposes: $H^*(A * \mathbb{C}\mathrm{P}^n) \simeq H^*(A * \mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n) \oplus H^*(\mathbb{C}\mathrm{P}^n)$.

Since E_A is the restriction of $E_{\mathbb{C}\mathrm{P}^m}$, the Thom class t'_A is the image of the Thom class $t'_{\mathbb{C}\mathrm{P}^m}$ under the morphism induced by inclusion. Since j^* must be an isomorphism in degree $2(n+1)$ in the exact sequence (6) for $A = \mathbb{C}\mathrm{P}^m$, we must have $t'_{\mathbb{C}\mathrm{P}^m} = \pm u^{n+1}$ (recall that $\mathbb{C}\mathrm{P}^m * \mathbb{C}\mathrm{P}^n = \mathbb{C}\mathrm{P}^{m+n+1}$). In fact $t'_{\mathbb{C}\mathrm{P}^m} = u^{n+1} = t_{\mathbb{C}\mathrm{P}^m}$ as the orientation of a complex fiber $\simeq \mathbb{C}^{n+1}$ coincides with the orientation of a projective subspace of $\mathbb{C}$-dimension $n+1$ (they all come from the complex structure of $\mathbb{C}\mathrm{P}^{m+n+1}$). □

Following Givental, we define $\ell(A) \in \mathbb{N}$ for $A \subset \mathbb{C}\mathrm{P}^N$ as the rank of the morphism $H^*(\mathbb{C}\mathrm{P}^N) \to H^*(A)$ induced by the inclusion (*e.g.* $\ell(\mathbb{C}\mathrm{P}^n) = n+1$). This definition coincides with the equivariant cohomological index defined by Fadell and Rabinowitz [3] (in the special case of the free action of S^1 on $\mathbb{S}^{2N+1}$).

Corollary 4.2 ([9, Corollary A.2])**.** *Let $A \subset \mathbb{C}\mathrm{P}^m$, then $\ell(A*\mathbb{C}\mathrm{P}^n) = \ell(A) + n + 1$.*

Proof. Since $f_{\mathbb{C}\mathrm{P}^m}(u^k) = u^{n+1} \smile u^k$ for $0 \leq k \leq m$, we have the following commutative diagram:

$$\begin{array}{ccc} H^*(\mathbb{C}\mathrm{P}^{m+n+1}) & \xrightarrow{u^{n+1}\smile\cdot} & H^{*+2(n+1)}(\mathbb{C}\mathrm{P}^{m+n+1}) \\ \downarrow & & \downarrow \\ H^*(A) & \xrightarrow{f_A} & H^{*+2(n+1)}(A * \mathbb{C}\mathrm{P}^n) \end{array}$$

where the vertical arrows are induced by inclusions. For the grading $* = 2(\ell(A * \mathbb{C}\mathrm{P}^n) - n - 1)$, the map $u^{n+1} \smile \cdot$ is onto, so $\ell(A*\mathbb{C}\mathrm{P}^n) \leq \ell(A) + n + 1$. According to Proposition 5.2, the map f_A is an injection for the grading $* = 2\ell(A)$, so $\ell(A*\mathbb{C}\mathrm{P}^n) \geq \ell(A) + n + 1$. □

5. Generating functions of $\mathbb{C}$-equivariant Hamiltonian diffeomorphisms

5.1. **"Broken trajectories" and generating functions of $\mathbb{C}^d$.** Let $\Phi \in \mathrm{Ham}(\mathbb{C}^d)$ be a Hamiltonian diffeomorphism which can be decomposed as $\Phi = \sigma_n \circ \cdots \circ \sigma_1$, where every $\sigma_k \in \mathrm{Ham}(\mathbb{C}^d)$ is sufficiently C^1-close to id such that they admit generating functions $f_k : \mathbb{C}^d \to \mathbb{R}$ satisfying:

$$\forall z_k \in \mathbb{C}^d, \exists! w_k \in \mathbb{C}^d, \quad w_k = \frac{z_k + \sigma_k(z_k)}{2} \quad \text{and} \quad \nabla f_k(w_k) = i(z_k - \sigma_k(z_k)). \quad (7)$$

We call such generating functions without auxiliary variable elementary generating functions. We will say that the n-tuple $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$ is associated to the Hamiltonian flow (Φ_t) if there exists real numbers $0 = t_0 \leq t_1 \leq \cdots \leq t_n = 1$ such that $\sigma_k = \Phi_{t_k} \circ \Phi_{t_{k-1}}^{-1}$. A continuous family of such tuples $(\boldsymbol{\sigma}_s)$ will denote a family of tuples of the same size $n \geq 1$, $\boldsymbol{\sigma}_s =: (\sigma_{1,s}, \dots, \sigma_{n,s})$ such that the maps $s \mapsto \sigma_{k,s}$ are C^1-continuous. Every compactly supported Hamiltonian flow and every $\mathbb{C}$-equivariant Hamiltonian flow $(\Phi_s)_{s\in[0,1]}$ admits a continuous family of associated tuple $(\boldsymbol{\sigma}_s)$ that is $\boldsymbol{\sigma}_s$ is associated to Φ_s for all $s \in [0,1]$ (and the size can be taken as large as wanted).

Denote by $F_{\boldsymbol{\sigma}}$ the following function $(\mathbb{C}^d)^n \to \mathbb{R}$:

$$F_{\boldsymbol{\sigma}}(v_1, \dots, v_n) := \sum_{k=1}^{n} f_k\left(\frac{v_k + v_{k+1}}{2}\right) + \frac{1}{2}\left\langle v_k, i v_{k+1}\right\rangle, \tag{8}$$

with convention $v_{n+1} = v_1$. Let $A : (\mathbb{C}^d)^n \to (\mathbb{C}^d)^n$ denotes the linear map such that, for $\mathbf{v} = (v_1, \dots, v_n)$, $A(\mathbf{v}) = \mathbf{w}$ with $w_k = \frac{v_k + v_{k+1}}{2}$. Let $\psi : (\mathbb{C}^d)^n \to (\mathbb{C}^d)^n$ be the diffeomorphism $\psi(\mathbf{z}) = \mathbf{w}$ defined by (7).

Proposition 5.1. *Under the above hypothesis, we have*

$$\forall k, \forall \mathbf{v} \in (\mathbb{C}^d)^n, \quad \partial_{v_k} F_{\boldsymbol{\sigma}}(v_1, \dots, v_n) = i(z_k - \sigma_{k-1}(z_{k-1})),$$

where $\mathbf{z} := \psi^{-1} \circ A(\mathbf{v})$ *and* $z_0 := z_n$. *Moreover, if* n *is odd,* $F_{\boldsymbol{\sigma}}$ *is a generating function of* Φ *with* v_1 *as main variable.*

Proof. Let $F := F_{\boldsymbol{\sigma}}$. Given any n-tuple $\mathbf{v} \in (\mathbb{C}^d)^n$, we associate n-tuples $\mathbf{w}$ and $\mathbf{z}$ in $(\mathbb{C}^d)^n$ given by $\mathbf{w} = A(\mathbf{v})$ and $\psi(\mathbf{z}) = \mathbf{w}$. Then

$$\begin{aligned}
\partial_{v_k} F(\mathbf{v}) &= \frac{1}{2}\left(\nabla f_{k-1}\left(\frac{v_{k-1} + v_k}{2}\right) + \nabla f_k\left(\frac{v_k + v_{k+1}}{2}\right) + i(v_{k+1} - v_{k-1})\right) \\
&= \frac{1}{2}\left(\nabla f_{k-1}(w_{k-1}) + \nabla f_k(w_k)\right) + i(w_k - w_{k-1}) \\
&= i(z_k - \sigma_{k-1}(z_{k-1})).
\end{aligned}$$

where indices are seen in $\mathbb{Z}/n\mathbb{Z}$. Now suppose that n is odd, so that A is an isomorphism. If we denote by $\xi := (v_2, \dots, v_n)$ the auxiliary variables, we thus have $\partial_\xi F(\mathbf{v}) = 0$ if and only if $z_{k+1} = \sigma_k(z_k)$ for $1 \leq k \leq n-1$. Moreover, since $v_1 = \sum_k (-1)^{k+1} w_k$, if $\partial_\xi F(\mathbf{v}) = 0$ then

$$v_1 = \sum_{k=1}^{n} (-1)^{k+1} \frac{z_k + \sigma_k(z_k)}{2} = \frac{z_1 + \sigma_n(z_n)}{2},$$

as required (since $\sigma_n(z_n) = \Phi(z_1)$ recursively).

Finally we must show that $\partial_\xi F$ is transverse to 0. This is clear in the z-coordinates: the matrix

$$\mathrm{d}(\partial_\xi F)(\mathbf{v}) \cdot A^{-1} \cdot \mathrm{d}\psi(\mathbf{z}) = i \begin{bmatrix} -\mathrm{d}\sigma_1(z_1) & I_{2d} & & & \\ & -\mathrm{d}\sigma_2(z_2) & I_{2d} & & \\ & & \ddots & \ddots & \\ & & & -\mathrm{d}\sigma_n(z_n) & I_{2d} \end{bmatrix}$$

is invertible. □

This proposition provides a quantitative way to see how close a discrete trajectory $(z_1, \dots, z_n)$ given by $(v_1, \dots, v_n)$ is to a discrete trajectory of the dynamics $\sigma_n \circ \cdots \circ \sigma_1$.

If $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$ and $\boldsymbol{\delta} = (\delta_1, \dots, \delta_m)$, we write $(\boldsymbol{\sigma}, \boldsymbol{\delta}) = (\sigma_1, \dots, \sigma_n, \delta_1, \dots, \delta_m)$. We have the following decomposition formula:

$$\forall v_1, \dots, v_{n+m} \in \mathbb{C}^d, \quad F_{(\boldsymbol{\sigma},\boldsymbol{\delta})}(\mathbf{v}) = F_{(\boldsymbol{\sigma},\mathrm{id})}(v_1, \dots, v_{n+1}) + F_{(\boldsymbol{\delta},\mathrm{id})}(v_{n+1}, \dots, v_{n+m}, v_1). \tag{9}$$

The following proposition will be of special interest for us.

Proposition 5.2. *Let $\boldsymbol{\sigma}$ be m-tuple with m even and $\boldsymbol{\delta} := (U_1, \dots, U_n)$ be a n-tuple of unitary maps with n odd. Assume $U_n \circ \cdots \circ U_1 = \mathrm{id}$. Then the generating function $F_{(\boldsymbol{\sigma},\boldsymbol{\delta})}$ is equivalent to $F_{(\boldsymbol{\sigma},\mathrm{id})}$. More precisely, writing $\mathbf{v}^1 := (v_1, \dots, v_{n+1})$ and $\mathbf{v}^2 := (v_{n+2}, \dots, v_{m+n})$ there exists a $\mathbb{C}$-linear isomorphism $A_{\boldsymbol{\delta}} : (\mathbb{C}^d)^{m+n} \to (\mathbb{C}^d)^{m+n}$ of the form $A_{\boldsymbol{\delta}}(\mathbf{v}^1, \mathbf{v}^2) = (\mathbf{v}^1, \mathbf{v}^2 - A'_{\boldsymbol{\delta}}(\mathbf{v}^1))$ that does not depend on the choice of $\boldsymbol{\sigma}$ and a non-degenerate quadratic form Q of $(\mathbb{C}^d)^{n-1}$ such that*

$$F_{(\boldsymbol{\sigma},\boldsymbol{\delta})} \circ A_{\boldsymbol{\delta}}(\mathbf{v}^1, \mathbf{v}^2) = F_{(\boldsymbol{\sigma},\mathrm{id})}(\mathbf{v}^1) + Q(\mathbf{v}^2),$$

with $\mathrm{ind}(Q) = \mathrm{ind}(F_{(\boldsymbol{\delta},\mathrm{id})}) = \mathrm{ind}(F_{\boldsymbol{\delta}})$. *In fact*

$$Q(\mathbf{v}^2) := F_{\boldsymbol{\delta}}(0, \mathbf{v}^2).$$

In order to prove it, we will need the following lemma.

Lemma 5.3. *Let $n \in \mathbb{N}$ be odd. Let $U_1, \dots, U_n \in U(\mathbb{C}^d)$ be unitary maps generated by elementary quadratic generating functions and such that $U_n \circ \cdots \circ U_1 = \mathrm{id}$. Let $\boldsymbol{\delta} := (U_1, \dots, U_n)$. Then, writing $\mathbf{v}' := (v_2, \dots, v_n)$ there exists a $\mathbb{C}$-linear isomorphism $B_{\boldsymbol{\delta}} : (\mathbb{C}^d)^{n+1} \to (\mathbb{C}^d)^{n+1}$ of the form $B_{\boldsymbol{\delta}}(v_1, \mathbf{v}', v_{n+1}) = (v_1, \mathbf{v}' + B'_{\boldsymbol{\delta}}(v_1, v_{n+1}), v_{n+1})$ and a non-degenerate quadratic form $Q : (\mathbb{C}^d)^{n-1} \to \mathbb{R}$ such that*

$$\forall v_1, \dots v_{n+1} \in \mathbb{C}^d, \quad F_{(\boldsymbol{\delta},\mathrm{id})} \circ B_{\boldsymbol{\delta}}(v_1, \mathbf{v}', v_{n+1}) = Q(\mathbf{v}'),$$

with $\mathrm{ind}(Q) = \mathrm{ind}(F_{(\boldsymbol{\delta},\mathrm{id})}) = \mathrm{ind}(F_{\boldsymbol{\delta}})$. *In fact* $Q(\mathbf{v}') := F_{\boldsymbol{\delta}}(0, \mathbf{v}')$.

Proof. We will show that $F := F_{(\boldsymbol{\delta},\mathrm{id})}$ is a generating function of the 0-section of $(\mathbb{C}^d)^2$ with main variable $x := (v_1, v_{n+1})$ and auxiliary variable $\xi := \mathbf{v}'$. According to Proposition 5.1, $\partial_\xi F(\mathbf{v}) = 0$ implies $z_{k+1} = U_k z_k$ for $1 \leq k \leq n-1$ so that $z_n = U_{n-1} \cdots U_1 z_1 = U_n^{-1} z_1$. Since $n+1$ is even, one has

$$\mathrm{im}(A) = \left\{ \mathbf{w} \in (\mathbb{C}^d)^{n+1} \mid \sum_k (-1)^k w_k = 0 \right\}$$

so that if $\partial_\xi F(\mathbf{v}) = 0$ then

$$\sum_{k=1}^{n+1} (-1)^k \frac{z_k + U_k z_k}{2} = z_{n+1} - z_1 = 0.$$

Thus, for $\mathbf{v} \in (\mathbb{C}^d)^{n+1}$ such that $\partial_\xi F(\mathbf{v}) = 0$, according to Proposition 5.1,

$$\partial_x F(v) = \begin{pmatrix} i(z_1 - z_{n+1}) \\ i(z_{n+1} - U_n z_n) \end{pmatrix} = 0,$$

since we have seen that $U_n z_n = z_1$. We see that $\partial_\xi F$ is transverse to 0 easily in the z-coordinates.

The lemma is now a direct application of Lemma 2.1 In our case, it gives

$$F_{(\boldsymbol{\delta},\mathrm{id})}(v_1, \mathbf{v}' + B'_{\boldsymbol{\delta}}(v_1, v_{n+1}), v_{n+1}) = Q(\mathbf{v}')$$

where

$$Q(\mathbf{v}') = F_{(\boldsymbol{\delta},\mathrm{id})}(0, \mathbf{v}', 0) = F_{\boldsymbol{\delta}}(0, \mathbf{v}').$$

Since the map $B_{\boldsymbol{\delta}}(\mathbf{v}) := (v_1, \mathbf{v}' + B'_{\boldsymbol{\delta}}(v_1, v_{n+1}), v_{n+1})$ is a linear isomorphism, $\mathrm{ind}(Q) = \mathrm{ind}(F_{(\boldsymbol{\delta},\mathrm{id})})$. In fact $\mathrm{ind}(Q) = \mathrm{ind}(F_{\boldsymbol{\delta}})$, as can be seen by applying Lemma 2.1, this time to the quadratic generating function of the 0-section $F_{\boldsymbol{\delta}}$ with v_1 as main variable. □

Proof of Proposition 5.2. This is a direct application of Lemma 5.3 to the function $F_{(\boldsymbol{\delta},\mathrm{id})}$ together with the decomposition formula (9). □

The following lemma will be useful to relate critical points of Hamiltonian diffeomorphisms with a common factor.

Lemma 5.4. *Let $\boldsymbol{\sigma}$, $\boldsymbol{\delta}$ and $\boldsymbol{\delta}'$ be respectively an m-tuple and two n-tuples of small Hamiltonians diffeomorphism of $\mathbb{C}^d$ as above. Let ψ and ψ' be the diffeomorphisms $\mathbf{z} \mapsto \mathbf{w}$ of $(\mathbb{C}^d)^{m+n+1}$ defined by (7) for the tuples $(\boldsymbol{\sigma}, \boldsymbol{\delta}, \mathrm{id})$ and $(\boldsymbol{\sigma}, \boldsymbol{\delta}', \mathrm{id})$ respectively. Let $A : \mathbf{v} \mapsto \mathbf{w}$ be the linear map of $(\mathbb{C}^d)^{m+n+1}$ defined as above. Then for all $\mathbf{z}^1 \in (\mathbb{C}^d)^m$, $\mathbf{z}^2, \mathbf{z}^3 \in (\mathbb{C}^d)^n$ and $z_{m+n+1} \in \mathbb{C}^d$, we have $(v_1, \dots, v_m) = (v'_1, \dots, v'_m)$ where $\psi(\mathbf{z}^1, \mathbf{z}^2, z_{n+m+1}) = A(\mathbf{v})$ and $\psi'(\mathbf{z}^1, \mathbf{z}^3, z_{n+m+1}) = A(\mathbf{v}')$.*

Proof. Under the above hypothesis,

$$\frac{v_{m+n+1} + v_1}{2} = z_{m+n+1} = \frac{v'_{m+n+1} + v'_1}{2} \quad \text{and} \quad \frac{v_k + v_{k+1}}{2} = \frac{z^1_k + \sigma_k(z^1_k)}{2} = \frac{v'_k + v'_{k+1}}{2},$$

where $k \in \{1, \dots, n\}$. So that, with matrices,

$$A\mathbf{v}' = \begin{bmatrix} I_{2md} & & \\ & * & \\ & & I_{2d} \end{bmatrix} A\mathbf{v}.$$

The conclusion then follows from a direct computation. □

5.2. **Generating family of the S^1-action.** In this section we follow Théret [17] and study generating families of the unitary (Hamiltonian) flow $g : t \mapsto e^{-2i\pi t}$ of $\mathbb{C}^{d+1}$ For $|t| < 1/2$, the Hamiltonian diffeomorphism $g_t(z) := e^{-2i\pi t}z$ admits the elementary quadratic generating function

$$q_t(w) := -\tan(\pi t)\|w\|^2, \quad \forall w \in \mathbb{C}^{d+1}.$$

Let $\boldsymbol{\delta}_t$ be the m-tuple $(g_{t/(m-1)}, \dots, g_{t/(m-1)}, \mathrm{id})$ with $m \geq 5$ odd such that $F_{\boldsymbol{\delta}_t}$ generates g_t for $t \in (-\varepsilon, 1+\varepsilon)$ where $\varepsilon > 0$ is arbitrarily fixed (we have only put a final "id" in $\boldsymbol{\delta}_t$ in order for us to apply Lemma 5.4 in a further section without trouble).

Lemma 5.5. *With the above notation,*

$$\mathrm{ind}(F_{\boldsymbol{\delta}_1}) - \mathrm{ind}(F_{\boldsymbol{\delta}_0}) = 2(d+1).$$

Proof. According to Proposition 3.1 (3) and (6), $\mathrm{mas}(z, (g_t)) = 2(d+1)$ for all $z \in \mathbb{C}^{d+1}$. Since $F_{\boldsymbol{\delta}_t}$ is a generating function of g_t, the result follows by definition of the Maslov index. □

Lemma 5.6 (compare with [17, Lemma 4.4])**.** *Let $\boldsymbol{\sigma}$ be a m'-tuple, with m' even, such that $F_t := F_{(\boldsymbol{\sigma},\boldsymbol{\delta}_t)} : (\mathbb{C}^{d+1})^{m'+m} \to \mathbb{R}$ is a smooth family of conical generating functions. Then*

(i) $\partial_t F_t(\mathbf{v}) \leq 0$, $\forall \mathbf{v} \in (\mathbb{C}^{d+1})^{m'+m}$,
(ii) $\partial_t F_t(\mathbf{v}) < 0$, $\forall \mathbf{v} \in \Sigma_{F_t} \setminus 0$.

Proof. The first property is a direct consequence of the definitions and the fact that $\partial_t(\tan(\pi t/m)) > 0$. Let $\mathbf{v} = (v_1, \dots, v_{m+m'}) \in (\mathbb{C}^{d+1})^{m+m'}$ be such that $\partial_t F_t(\mathbf{v}) = 0$. Then, for $m'+1 \leq k < m'+m$, $w_k := \frac{v_k + v_{k+1}}{2} = 0$ thus $z_k = 0$ where the family $\mathbf{z} = (z_k)$ is associated to the family $\mathbf{w} = (w_k)$ *via* (7) as usual. Thus if $\mathbf{v} \in \Sigma_{F_t}$, z_k must be 0 for all k for the sequence $(z_1, \dots, z_{m'+m})$ to be the discrete dynamics of conical diffeomorphisms, hence $\mathbf{w} = 0$ and $\mathbf{v} = 0$. □

5.3. **A discrete variational principle for $\mathbb{C}$-equivariant Hamiltonian diffeomorphisms.** Let (φ_t) be the Hamiltonian flow of $\mathbb{C}\mathrm{P}^d$ associated to the Hamiltonian map $h : [0,1] \times \mathbb{C}\mathrm{P}^d \to \mathbb{R}$. Let $\tilde{h} : [0,1] \times \mathbb{S}^{2d+1} \to \mathbb{R}$ be the S^1-invariant lift of h defined by $\tilde{h}_t := h_t \circ \pi$ where $\pi : \mathbb{S}^{2d+1} \to \mathbb{C}\mathrm{P}^d$ is the quotient map $\pi(z) := [z]$. Let $H : [0,1] \times \mathbb{C}^{d+1} \to \mathbb{R}$ be the 2-homogeneous Hamiltonian map such that $H_t(\lambda x) := \lambda^2 \tilde{h}_t(x)$ for all $x \in \mathbb{S}^{2d+1}$. It defines a $\mathbb{C}$-equivariant symplectic flow (Φ_t) stabilizing the Euclidean sphere $\mathbb{S}^{2d+1}$ and such that

$$\pi \circ \Phi_t|_{\mathbb{S}^{2d+1}} = \varphi_t \circ \pi, \quad \forall t \in [0,1]. \tag{10}$$

This flow (Φ_t) is uniquely defined by the choice of Hamiltonian map (h_t) of (φ_t). In fact, if (Φ'_t) is a $\mathbb{C}$-equivariant Hamiltonian flow stabilizing the sphere and such that (10), then $\Phi'_t = e^{i\theta(t)}\Phi_t$ which boils down to a change of equivalent Hamiltonian map (h'_t) for (φ_t). We will usually write $\varphi := \varphi_1 \in \mathrm{Ham}(\mathbb{C}\mathrm{P}^d)$ and $\Phi := \Phi_1 \in \mathrm{Ham}_{\mathbb{C}}(\mathbb{C}^{d+1})$. Given a choice of Hamiltonian map (h_t), the *action* $a(x) \in \mathbb{R}/\mathbb{Z}$ of a fixed point $x \in \mathbb{C}\mathrm{P}^d$ is defined by

$$a(x) := -\frac{1}{\pi}\left(\int_D \omega + \int_0^1 h_t \circ \varphi_t(x)\mathrm{d}t\right) \in \mathbb{R}/\mathbb{Z},$$

where $D \subset \mathbb{C}\mathrm{P}^d$ is a 2-disc filling the contractible loop $\gamma := (\varphi_t)_{t\in[0,1]}$, that is $\partial D = \gamma$ (the $1/\pi$ factor is a standard renormalization to simplify notations). Fixed points $x \in \mathbb{C}\mathrm{P}^d$ of action $a \in \mathbb{R}/\mathbb{Z}$ are in one-to-one correspondence with $\mathbb{C}$-lines $\mathbb{C}X \subset \mathbb{C}^{d+1}$ such that $\Phi_1(X) = e^{2i\pi a}X$, $X \in \mathbb{C}^{d+1} \setminus 0$ (see [17, Prop. 5.8]). Since the action only depends on the choice of lift (Φ_t), when such a lift is given, we will simply call it the action of (φ_t) or the action of φ.

Following Théret, we now define a map $\mathcal{T} : M \to \mathbb{R}$ that provides a variational principle for fixed points of (φ_t). Let $\varepsilon > 0$, let $(\boldsymbol{\delta}_t)$ be one of the families of odd tuples associated to (g_t) defined in Section 5.2 for $t \in (-\varepsilon, 1+\varepsilon)$ and let $(\boldsymbol{\sigma}_s)$ be an even continuous family of tuples associated to (Φ_s) of the form $(\boldsymbol{\sigma}'_s, \boldsymbol{\delta}_0)$. Then $F_{s,t} := F_{(\boldsymbol{\sigma}_s, \boldsymbol{\delta}_t)} : \mathbb{C}^{N+1} \to \mathbb{R}$ gives us a family of conical functions generating $e^{-2i\pi t}\Phi_s$. In order to simplify notation, let $F_t := F_{1,t}$ be the family of conical functions generating $e^{-2i\pi t}\Phi$, $t \in (-\varepsilon, 1+\varepsilon)$. Let $\tilde{f} : (-\varepsilon, 1+\varepsilon) \times \mathbb{S}^{2N+1} \to \mathbb{R}$ be the S^1-invariant function $\tilde{f}(t,\zeta) := F_t(\zeta)$ for $|\zeta| = 1$ and $f : (-\varepsilon, 1+\varepsilon) \times \mathbb{C}\mathrm{P}^N \to \mathbb{R}$ be the induced function. Then there is a one-to-one correspondence between fixed points of φ of action $\bar{t} \in \mathbb{R}/\mathbb{Z}$ and critical points of $f(t,\cdot)$ with value 0 for any $t \in (-\varepsilon, 1+\varepsilon)$.

According to property (ii) of Lemma 5.6, the differential $\mathrm{d}\tilde{f} = \partial_t(F_t)\mathrm{d}t + \mathrm{d}F_t$ never vanishes on $\mathbb{C}^{N+1} \setminus 0$ so 0 is a regular value of f. Let $I := (-\varepsilon, 1+\varepsilon)$. Let $M := \{(t,\zeta) \in I \times \mathbb{C}\mathrm{P}^N \mid f(t,\zeta) = 0\}$ and $\mathcal{T} : M \to I$ be the projection onto the first factor. Fixed points of action $\bar{t} \in \mathbb{R}/\mathbb{Z}$ are in one-to-one correspondence with critical points of $\mathcal{T}$ with value t: more precisely $\mathrm{d}\mathcal{T}(t,\zeta) = 0 \Leftrightarrow \mathrm{d}_\zeta f(t,\zeta) = 0$. Moreover, if $(t,\zeta) \in M$ is a critical point of $\mathcal{T}$, then the Hessian $\mathrm{d}^2\mathcal{T}(t,\zeta)$ is

equivalent as a quadratic form to $\mathrm{d}^2_{\zeta,\zeta} f(t,\zeta)$ which is equivalent to $\mathrm{d}^2 F_t(\tilde{\zeta})$ restricted to a complement of the $\mathbb{C}$-line induced by $\tilde{\zeta} \in \mathbb{S}^{2N+1}$, where $\tilde{\zeta}$ is a lift of $\zeta \in \mathbb{C}\mathrm{P}^N$ (because F_t is conical). Since this line $\mathbb{C}\tilde{\zeta}$ is included in $\ker \mathrm{d}^2 F_t(\tilde{\zeta})$, critical points $(t,\zeta) \in M$ and $\tilde{\zeta} \in \mathbb{C}^{N+1}$ share the same index. Moreover, if $z \in \mathbb{C}\mathrm{P}^d$ and $Z \in \mathbb{C}^{d+1}$ are fixed points associated to $\zeta \in \mathbb{C}\mathrm{P}^N$ and $\tilde{\zeta} \in \mathbb{C}^{N+1}$ respectively, since

$$\dim \ker \mathrm{d}^2 F_t(\tilde{\zeta}) = \dim \ker(e^{-2i\pi t}\mathrm{d}\Phi(Z) - \mathrm{id})$$

one has

$$\dim \ker \mathrm{d}^2_{\zeta,\zeta} f(t,\zeta) = \dim \ker(\mathrm{d}\varphi(z) - \mathrm{id}) =: \nu(z). \tag{11}$$

5.4. **Cohomology of sublevel sets of $\mathcal{T}$.** We recall that H^* denotes the singular cohomology with integral coefficients. Let $p : I \times \mathbb{C}\mathrm{P}^N \to \mathbb{C}\mathrm{P}^N$ be the projection on the second space and $i : M \hookrightarrow I \times \mathbb{C}\mathrm{P}^N$ be the inclusion map. Let $\widehat{F}_t : \mathbb{C}\mathrm{P}^N \to \mathbb{R}$ be the C^1 map induced by $F_t|_{\mathbb{S}^{2N+1}}$. According to Lemma 5.6, if $s \leq t$, then $\widehat{F}_t \leq \widehat{F}_s$ so that $\left\{\widehat{F}_s \leq 0\right\} \subset \left\{\widehat{F}_t \leq 0\right\}$. Thus the subspace

$$A_t := \left\{(s,\zeta) \in (-\varepsilon, t] \times \mathbb{C}\mathrm{P}^N \mid \widehat{F}_s(\zeta) \leq 0\right\}$$

retracts on $t \times \left\{\widehat{F}_t \leq 0\right\}$, hence p induces an isomorphism $H^*\left(\left\{\widehat{F}_t \leq 0\right\}\right) \to H^*(A_t)$ for all $t \in I$ and thus induces isomorphisms

$$p^* : H^*\left(\left\{\widehat{F}_b \leq 0\right\}, \left\{\widehat{F}_a \leq 0\right\}\right) \to H^*(A_b, A_a), \tag{12}$$

for $a \leq b$ in I. Let $a \leq b$ in I and $e > 0$ such that $a - e \in I$, the subspace A_b retracts on $\{\mathcal{T} \leq b\} \cup A_{a-e}$ by $(t,\zeta) \mapsto (s,\zeta)$ where s is the maximal $r \in (a-e, t]$ satisfying $\widehat{F}_r(\zeta) = 0$ or $s = a - e$ if such a max does not exist. By excision, we then have that i induces an isomorphism

$$i^* : H^*(A_b, A_a) \to H^*(\{\mathcal{T} \leq b\}, \{\mathcal{T} \leq a\}), \tag{13}$$

for all $a \leq b$ in I. Putting (12) and (13) together, we get the following

Lemma 5.7. *For all $a \leq b$ in I, the composition $p \circ i$ induces an isomorphism in cohomology*

$$H^*\left(\left\{\widehat{F}_b \leq 0\right\}, \left\{\widehat{F}_a \leq 0\right\}\right) \simeq H^*(\{\mathcal{T} \leq b\}, \{\mathcal{T} \leq a\}).$$

The above statement still holds when the inequalities on one or both sides of the topological pairs are replaced by strict inequalities. It also extends to local cohomology, the precise statement being given in the next section.

5.5. **Local cohomology of a fixed point.** Let $z \in \mathbb{C}\mathrm{P}^d$ be a fixed point of φ with $\Phi(Z) = e^{2i\pi t} Z$ where $Z \in \mathbb{S}^{2d+1}$ is a lift of z. We assume that z is an isolated fixed point and that its action value is isolated among the action values of φ (it is always the case when φ has a finite number of fixed point). We denote by $\mathrm{C}^*(z,t)$ the local cohomology of $\mathcal{T}$ at the critical point (t,ζ) corresponding to z, *i.e.*

$$\mathrm{C}^*(z,t) = H^*(\{\mathcal{T} \leq t\}, \{\mathcal{T} \leq t\} \setminus (t,\zeta)).$$

This group depends only on the germ of $\mathcal{T}$ at (t,ζ). Namely, for all neighborhoods $U \subset M$ of (t,ζ),

$$\mathrm{C}^*(z,t) \simeq H^*(U \cap \{\mathcal{T} \leq t\}, U \cap \{\mathcal{T} \leq t\} \setminus (t,\zeta)).$$

By an argument similar to the proof of Lemma 5.7, the map $p \circ i : M \to \mathbb{C}\mathrm{P}^N$ induces an isomorphism

$$\mathrm{C}^*(z,t) \simeq H^*\left(\left\{\widehat{F_t} \leq 0\right\}, \left\{\widehat{F_t} \leq 0\right\} \setminus \zeta\right).$$

Thus $\mathrm{C}^*(z,t)$ is isomorphic to the local cohomology of $\widehat{F_t}$ at the point ζ which we denote by $\mathrm{C}^*(\widehat{F_t};\zeta)$. The support of a cohomology group C^* is defined by

$$\operatorname{supp} C^* := \{k \in \mathbb{Z} \mid C^k \neq 0\} \subset \mathbb{Z}.$$

A classical result due to Gromoll-Meyer [10, remark following Lemma 1] implies that for any smooth function $f : M \to \mathbb{R}$ and any isolated critical point $x \in M$,

$$\operatorname{supp} \mathrm{C}^*(f;x) \subset \left[\operatorname{ind}(x,f), \operatorname{ind}(x,f) + \dim \ker \mathrm{d}^2 f(x)\right].$$

According to (11), we thus have

$$\operatorname{supp} \mathrm{C}^*(z,t) \subset \left[\operatorname{ind}(\tilde{\zeta}, F_t), \operatorname{ind}(\tilde{\zeta}, F_t) + \nu(z)\right], \tag{14}$$

where $\tilde{\zeta} \in \mathbb{C}^{N+1}$ is a lift of ζ.

We want to study the relationship between the local cohomology groups $\mathrm{C}^*(z,t)$ and $\mathrm{C}^*(z,t+1)$ when z is a fixed point of action $t \in (-\varepsilon, \varepsilon)$. Let us assume that $t = 0$ and let $(j, \zeta_j) \in M$, $j = 0, 1$, be the critical points associated to z. Let $(\mathbf{u}^j, \mathbf{v}^j) \in (\mathbb{C}^{d+1})^{n+1} \times (\mathbb{C}^{d+1})^{m-1}$ be lifts of ζ_j. According to Lemma 5.4, one can take $\mathbf{u}^0 = \mathbf{u}^1$. According to proposition 5.2, there exist $\mathbb{C}$-linear maps $A_j : (\mathbb{C}^{d+1})^{m+n} \to (\mathbb{C}^{d+1})^{m-1}$ such that $F_j(\mathbf{u}, A_j(\mathbf{u},\mathbf{v})) = g(\mathbf{u}) + Q_j(\mathbf{v})$ where $g = F_{(\boldsymbol{\sigma},\mathrm{id})}$, $Q_j(\mathbf{v}) = F_{\boldsymbol{\delta}_j}(0,\mathbf{v})$ and the linear maps have the form $A_j(\mathbf{u},\mathbf{v}) = \mathbf{v} + B_j(\mathbf{u})$. Since the $(\mathbf{u}^0, \mathbf{v}^j)$'s are critical points and the Q_j's are non-degenerate $A_j(\mathbf{u}^0, 0) = \mathbf{v}^j$.

Lemma 5.8. *Let $C : (\mathbb{C}^{d+1})^{n+1} \to (\mathbb{C}^{d+1})^{m-1}$ be the linear map such that the following diagram commutes:*

$$\begin{array}{ccc} \{g \leq 0\} \times \{Q_1 \leq 0\} & \xrightarrow{(\mathbf{u},\mathbf{v}) \mapsto (\mathbf{u}, A_1(\mathbf{u},\mathbf{v}))} & \{F_1 \leq 0\} \\ {\scriptstyle (\mathbf{u},\mathbf{v}) \mapsto (\mathbf{u}, \mathbf{v} + C(\mathbf{u}))}\uparrow & & \uparrow \\ \{g \leq 0\} \times \{Q_0 \leq 0\} & \xrightarrow{(\mathbf{u},\mathbf{v}) \mapsto (\mathbf{u}, A_0(\mathbf{u},\mathbf{v}))} & \{F_0 \leq 0\} \end{array} \tag{15}$$

The set $\{Q_0 \leq 0\}$ is included in $\{Q_1 \leq 0\}$ and the left hand vertical arrow is homotopic to the inclusion map $\{g \leq 0\} \times \{Q_0 \leq 0\} \hookrightarrow \{g \leq 0\} \times \{Q_1 \leq 0\}$.

Proof. According to Lemma 5.6, $Q_1 \leq Q_0$ so that we have the inclusion of their sublevel sets. We now prove that the map $(\mathbf{u},\mathbf{v}) \mapsto (\mathbf{u}, \mathbf{v} + sC(\mathbf{u}))$ is well-defined from $\{g \leq 0\} \times \{Q_0 \leq 0\}$ to $\{g \leq 0\} \times \{Q_1 \leq 0\}$ for $s \in [0,1]$ in order to conclude. By the above inclusion, this is true for $s = 0$. For all $(\mathbf{u},\mathbf{v}) \in \{g \leq 0\} \times \{Q_0 \leq 0\}$, we have $Q_1(\mathbf{v} + C(\mathbf{u})) \leq 0$ by definition of C. Thus for $s \in (0,1]$, since $(\mathbf{u}, \mathbf{v}/s)$ is also in $\{g \leq 0\} \times \{Q_0 \leq 0\}$, one has

$$Q_1(\mathbf{v}/s + C(\mathbf{u})) = s^{-2} Q_1(\mathbf{v} + sC(\mathbf{u})) \leq 0.$$

Hence the result follows for $s \neq 0$. □

Lemma 5.9. *In the diagram (15), the sublevel sets $\{Q_j \leq 0\}$ retract on the maximal negative $\mathbb{C}$-subspaces of the Q_j's through $\mathbb{C}$-linear maps. By taking the projection of*

these spaces on $\mathbb{C}\mathrm{P}^N$*, we get the following commutative diagram in cohomology:*

$$\begin{array}{ccc} H^*(X * \mathbb{C}\mathrm{P}^{k+d+1}) & \xleftarrow[\simeq]{} & H^*\left(\left\{\widehat{F}_1 \leq 0\right\}\right) \\ \downarrow & & \downarrow \\ H^*(X * \mathbb{C}\mathrm{P}^{k}) & \xleftarrow[\simeq]{} & H^*\left(\left\{\widehat{F}_0 \leq 0\right\}\right) \end{array}$$

where $X = \{\widehat{g} \leq 0\}$*,* $\mathbb{C}\mathrm{P}^k \subset \mathbb{C}\mathrm{P}^{k+d+1}$ *are projectively embedded subspaces of* $\mathbb{C}\mathrm{P}^N$ *(the projections of the negative spaces of the* Q_j*'s), vertical maps are induced by inclusions and the horizontal maps are isomorphisms induced by* $[\mathbf{u} : \mathbf{v}] \mapsto [\mathbf{u} : A_j(\mathbf{u}, \mathbf{v})]$*. The same is true when we replace* $\{\widehat{F}_j \leq 0\}$ *by* $\{\widehat{F}_j \leq 0\} \setminus [\mathbf{u}^0 : \mathbf{v}^j]$ *and* $X * \mathbb{C}\mathrm{P}^{k+j(d+1)}$ *by* $X * \mathbb{C}\mathrm{P}^{k+j(d+1)} \setminus [\mathbf{u}^0 : 0]$*.*

Proof. The variation of dimension between the maximal negative subspaces of Q_1 and Q_0 is $2(d+1)$ since

$$\mathrm{ind}(Q_1) - \mathrm{ind}(Q_0) = \mathrm{ind}(F_{\boldsymbol{\delta}_1}) - \mathrm{ind}(F_{\boldsymbol{\delta}_0}) = 2(d+1),$$

according to Proposition 5.2 together with Lemma 5.5. The commutativity of the diagram in the statement is a direct consequence of Lemma 5.8. According to the five lemma applied to long exact sequences of pairs, in order to complete the proof, we only need to show that the following morphisms induced by $[\mathbf{u} : \mathbf{v}] \mapsto [\mathbf{u} : A_j(\mathbf{u}, \mathbf{v})]$ are isomorphisms:

$$\begin{aligned} H^*\left(\left\{\widehat{F}_j \leq 0\right\}\right) &\to H^*\left(X * \mathbb{C}\mathrm{P}^{k+j(d+1)}\right), \\ \mathrm{C}^*\left(\widehat{F}_j; [\mathbf{u}^0 : \mathbf{v}^j]\right) &\to H^*\left(X * \mathbb{C}\mathrm{P}^{k+j(d+1)}, X * \mathbb{C}\mathrm{P}^{k+j(d+1)} \setminus [\mathbf{u}^0 : 0]\right), \end{aligned} \tag{16}$$

for $j \in \{0, 1\}$.

In order to show that the first morphism of (16) is an isomorphism, we need to deform the zero sublevel set of $\widehat{F}_j$ into a certain non-singular sublevel set. Let $\eta \in (0, 1/2)$ be a small number. For $j \in \{0, 1\}$ and $t \in [j - \eta, j + \eta]$, let $f_t : (\mathbb{C}^{d+1})^{m+n} \to \mathbb{R}$ be the S^1-invariant and 2-homogeneous map $f_t := F_{(\boldsymbol{\sigma}', \boldsymbol{\delta}_{t-j}, \boldsymbol{\delta}_j)}$, where $\boldsymbol{\sigma}'$ is the tuple satisfying $\boldsymbol{\sigma} = (\boldsymbol{\sigma}', \boldsymbol{\delta}_0)$. In particular, $f_j = F_j$. In a way similar to Section 5.3, we define a C^1-map $\mathcal{T}' : M' \to (-\eta, \eta) \cup (1 - \eta, 1 + \eta)$ as the retriction of the projection $\mathbb{R} \times \mathbb{C}\mathrm{P}^N \to \mathbb{C}\mathrm{P}^N$ to the submanifold M' of those (t, ζ) satisfying $\widehat{f}_t(\zeta) = 0$. By the same argument used in Section 5.4, the statement of Lemma 5.7 is still true if one replaces the F_t's with f_t's, $\mathcal{T}$ with $\mathcal{T}'$ and I with $(-\eta, \eta)$ or $(1 - \eta, 1 + \eta)$. By applying Morse's deformation lemma to $\mathcal{T}'$, we thus get that for $\eta' \in (0, \eta)$ such that $(0, \eta']$ does not contain any action value of φ the inclusion morphism

$$H^*\left(\left\{\widehat{f}_{j+\eta'} \leq 0\right\}\right) \to H^*\left(\left\{\widehat{f}_j \leq 0\right\}\right) = H^*\left(\left\{\widehat{F}_j \leq 0\right\}\right)$$

is an isomorphim. By our choice of f_t and by construction of A_j,

$$f_{t+j}(\mathbf{u}, A_j(\mathbf{u}, \mathbf{v})) = g_t(\mathbf{u}) + Q_j(\mathbf{v}),$$

where $t \in (-\eta, \eta)$ and $g_t := F_{(\boldsymbol{\sigma}', \boldsymbol{\delta}_t, \mathrm{id})}$. In particular, $g_0 = g$ and g_t is non-decreasing with t. By the same reasoning as above, Morse's deformation lemma implies that the inclusion morphism

$$H^*\left(\{\widehat{g}_{\eta'} \leq 0\}\right) \to H^*\left(\{\widehat{g}_0 \leq 0\}\right) = H^*\left(\{\widehat{g} \leq 0\}\right)$$

is an isomorphism. By Proposition 4.1, it implies that the inclusion morphisms

$$H^*\left(\{\widehat{g}_{\eta'} \leq 0\} * \left\{\widehat{Q}_j \leq 0\right\}\right) \to H^*\left(\{\widehat{g} \leq 0\} * \left\{\widehat{Q}_j \leq 0\right\}\right)$$

are isomorphisms for $j \in \{0,1\}$. Now, proving that the first morphism of (16) is an isomorphism is equivalent to proving that the inclusion morphism

$$H^*\left(\left\{\widehat{g_{\eta'} \oplus Q_j} \leq 0\right\}\right) \to H^*\left(\{\widehat{g}_{\eta'} \leq 0\} * \left\{\widehat{Q}_j \leq 0\right\}\right)$$

is an isomorphism. It boils down to the fact that $\left\{\widehat{g_{\eta'} \oplus Q_j} \leq 0\right\}$ retracts on $\{\widehat{g}_{\eta'} \leq 0\} * \left\{\widehat{Q_j} \leq 0\right\}$, which is proved by Givental in [9, Proposition B.1] (here, we need that $g_{\eta'}$ does not have any $\mathbb{C}$-line of critical points, contrary to g).

Let us prove that the second morphism of (16) is also an isomorphism. Since $\mathbf{u}^0 \neq 0$, one can assume that $\mathbf{u}^0 = (1, \mathbf{u}^1)$. By excision to affine neighborhood $\{[\mathbf{u} : \mathbf{v}] \mid u_1 \neq 0\}$, the inclusion morphism

$$\mathrm{C}^*\left(\widehat{F}_j; [\mathbf{u}^0 : \mathbf{v}^j]\right) \to \mathrm{C}^*\left(F_j(1,\cdot); (\mathbf{u}^1, \mathbf{v}^j)\right)$$

is an isomorphism. The image of the local cohomology of $F_j(1,\cdot)$ at $(\mathbf{u}^1, \mathbf{v}^j)$ under $(\mathbf{u}, \mathbf{v}) \mapsto (\mathbf{u}, A_j(1, \mathbf{u}, \mathbf{v}))$ is the local cohomology of $g(1,\cdot) \oplus Q_j$ at $(\mathbf{u}^1, 0)$, which is isomorphic to

$$H^*\left(\{g(1,\cdot) \leq 0\} \times \{Q_j \leq 0\}, \{g(1,\cdot) \leq 0\} \times \{Q_j \leq 0\} \setminus (\mathbf{u}^1, 0)\right),$$

under the inclusion morphism, according to the shifting theorem of Gromoll-Meyer [10, §3]. By excision to the same affine neighborhood as above applied to the target space of the second morphism of (16), we finally get that this morphism is an isomorphism. □

According to this lemma, regarding cohomology of sublevel sets, one can essentially assume that $\{\widehat{F}_0 \leq 0\} = X * \mathbb{C}\mathrm{P}^k$ and $\{\widehat{F}_1 \leq 0\} = X * \mathbb{C}\mathrm{P}^{k+d+1}$.

Proposition 5.10. *Let $z \in \mathbb{C}\mathrm{P}^d$ be a fixed point of φ with action $t \in (-\varepsilon, \varepsilon)$, then*

$$\mathrm{C}^*(z, t+1) \simeq \mathrm{C}^{*-2(d+1)}(z, t).$$

Proof. Let us first assume that $t = 0$. Since we are in the hypothesis of Lemma 5.9, we keep the same notation with $j \in \{0,1\}$. The coordinate changes $[\mathbf{u} : \mathbf{v}] \mapsto [\mathbf{u} : A_j(\mathbf{u}, \mathbf{v})]$ induce isomorphisms in local cohomologies:

$$\mathrm{C}^*(z, j) \simeq H^*\left(X * \mathbb{C}\mathrm{P}^{k+j(d+1)}, X * \mathbb{C}\mathrm{P}^{k+j(d+1)} \setminus [\mathbf{u}^0 : 0]\right). \tag{17}$$

We recall that, according to the proof of Proposition 4.1, $X * \mathbb{C}\mathrm{P}^{k+j(d+1)} \setminus \mathbb{C}\mathrm{P}^{k+j(d+1)}$ is a complex fiber bundle with fibers $\mathbb{C}^{k+1+j(d+1)}$. Let $U \subset X$ be an open set of trivialization containing $[\mathbf{u}^0] \in X$. Then by excision,

$$\mathrm{C}^*(z, j) \simeq H^*\left(U \times \mathbb{C}^{k+1+j(d+1)}, U \times \mathbb{C}^{k+1+j(d+1)} \setminus ([\mathbf{u}^0], 0)\right),$$

and Künneth formula gives

$$\mathrm{C}^*(z, j) \simeq H^{*-2(k+1+j(d+1))}\left(U, U \setminus [\mathbf{u}^0]\right), \tag{18}$$

which concludes the case $t = 0$.

Let us assume that $t \in (-\varepsilon, \varepsilon)$. We recall that $\boldsymbol{\sigma}_1 = (\boldsymbol{\sigma}'_1, \boldsymbol{\delta}_0)$, thus $s \mapsto (\boldsymbol{\sigma}'_1, \boldsymbol{\delta}_{st}, \boldsymbol{\delta}_{(1-s)t})$ is a continuous family of tuples associated to the constant flow $e^{-2i\pi t}\Phi$ starting at $(\boldsymbol{\sigma}, \boldsymbol{\delta}_t)$. Therefore, according to Lemma 2.2, there exists an isotopy (B_s) with $B_0 = \mathrm{id}$ and $F_{(\boldsymbol{\sigma}'_1, \boldsymbol{\delta}_t, \boldsymbol{\delta}_0)} \circ B_1 = F_t$. Let $G_s := F_{(\boldsymbol{\sigma}'_1, \boldsymbol{\delta}_t, \boldsymbol{\delta}_s)}$, then we have

$G_0 \circ B_1 = F_t$ and, by the same way, we have $G_1 \circ B'_1 = F_{t+1}$ for another isotopy (B'_s). The proof follows along the same lines as before replacing (F_s) by (G_s). □

We want to elaborate on this local statement when the subgroup $\mathrm{C}^*(z,t)$ "persists in the action window $(t,t+1]$". Let $z \in \mathbb{C}\mathrm{P}^d$ be a fixed point of φ with action $t \in (-\varepsilon,\varepsilon)$. By excision, we have an isomorphism

$$H^*(\{\mathcal{T} \leq t\},\{\mathcal{T} < t\}) \simeq \bigoplus_i \mathrm{C}^*(z_i,t), \tag{19}$$

where $z_1 = z, z_2, z_3, \ldots$ is the finite family of fixed points of φ with action t. We will make use of the following maps induced by inclusions:

$$j_1^* : H^*(\{\mathcal{T} \leq t+1\},\{\mathcal{T} < t\}) \to H^*(\{\mathcal{T} \leq t\},\{\mathcal{T} < t\})$$

and

$$j_2^* : H^*(\{\mathcal{T} \leq t+1\},\{\mathcal{T} < t+1\}) \to H^*(\{\mathcal{T} \leq t+1\},\{\mathcal{T} < t\}).$$

We recall that $u^{d+1} \in H^{2(d+1)}(\mathbb{C}\mathrm{P}^N)$ acts on these relative cohomology groups by the cup-product, identifying $H^*(\mathbb{C}\mathrm{P}^N)$ with $H^*(I \times \mathbb{C}\mathrm{P}^N)$ *via* the projection $I \times \mathbb{C}\mathrm{P}^N \to \mathbb{C}\mathrm{P}^N$.

Proposition 5.11. *Let $z \in \mathbb{C}\mathrm{P}^d$ be a fixed point of φ with zero action such that there exists a subgroup $G(z)$ of $H^*(\{\mathcal{T} \leq 1\},\{\mathcal{T} < 0\})$ whose image $j_1^*G(z)$ is $\mathrm{C}^*(z,0)$ under the identification (19). Then $\mathrm{C}^*(z,1)$ trivially intersects $\ker j_2^*$ and we have the isomorphism*

$$\mathrm{C}^*(z,1) \simeq j_2^*\mathrm{C}^*(z,1) = u^{d+1}G(z).$$

Proof. In order to simplify notations, let $M^{\leq j} := \{\mathcal{T} \leq j\}$ and $M^{<j} := M^{\leq j} \setminus [\mathbf{u}^0 : \mathbf{v}^j]$ where $j \in \{0;1\}$. The proposition is then a direct consequence of the commutativity of the following diagram:

$$\begin{array}{ccc} H^{*-2(d+1)}(M^{\leq 1}, M^{<0}) & \xrightarrow{u^{d+1}\smile .} & H^*(M^{\leq 1}, M^{<0}) \\ \downarrow & & \uparrow \\ H^{*-2(d+1)}(M^{\leq 0}, M^{<0}) & \xrightarrow{\simeq} & H^*(M^{\leq 1}, M^{<1}) \end{array}$$

where the vertical arrows are induced by inclusion and the bottom one is constructed as follows. By definition $H^*(M^{\leq j}, M^{<j}) = \mathrm{C}^*(z,j)$. In the proof of Proposition 5.10, we have seen that excision gives (17) after identifying $M^{\leq j}$ with $X * \mathbb{C}\mathrm{P}^{k+j(d+1)}$ and taking a trivialization neighborhood $U \subset X$ of the induced covering. At last, we had an isomorphism (18) between local cohomologies $\mathrm{C}^*(z,j)$ and $H^*(U, U \setminus [\mathbf{u}^0])$ with a shift in degrees. This last isomorphism is in fact induced by the cup-product of an element of $H^*(U, U \setminus [\mathbf{u}^0])$ with the restriction of u^{k+d+1} since the Thom class of the fiber bundle covering X is also a restriction of u^{k+d+1} according to Proposition 4.1. Thus, the cup-product by u^{d+1} makes explicit the isomorphism $H^{*-2(d+1)}(M^{\leq 0}, M^{<0}) \xrightarrow{\simeq} H^*(M^{\leq 1}, M^{<1})$ (after applying a suitable excision), so that the commutativity of the diagram follows. □

5.6. **Iteration properties of $\mathcal{T}$.** Given a n-tuple $\boldsymbol{\sigma}$ and a integer $m \geq 0$, we denote by $\boldsymbol{\sigma}^m$ the mn-tuple $(\boldsymbol{\sigma}, \ldots, \boldsymbol{\sigma})$. For $m \geq 1$, let $(\boldsymbol{\sigma}_s^{(m)})$ be the family of tuples of (Φ_{ms}) defined by

$$\boldsymbol{\sigma}^{(m)}_{k/m+s} = \left(\boldsymbol{\sigma}_1^k, \boldsymbol{\sigma}_s, \boldsymbol{\sigma}_0^{m-k-1}\right), \quad \forall s \in [0,1/m], 0 \leq k \leq m-1.$$

Let $F^m_{s,t} := F_{(\boldsymbol{\sigma}^{(m)}_s, \boldsymbol{\delta}_t)}$ be the induced generating family of $e^{-2i\pi t}\Phi_{ms}$. We denote by

$$M_m := \left\{(t,[\zeta]) \in (-\varepsilon, 1+\varepsilon) \times \mathbb{C}\mathrm{P}^{N(m)} \mid F^m_{1,t}(\zeta) = 0\right\}, \quad \mathcal{T}^m : M_m \to (-\varepsilon, 1+\varepsilon)$$

the discrete action associated to Φ^m *via* the generating family $F^m_{1,t} : \mathbb{C}^{N(m)+1} \to \mathbb{R}$.

According to Proposition 5.10, in order to study the local cohomology groups of $\mathcal{T}$ and $\mathcal{T}^m$, it is enough to consider points of value in $[0,1)$. Let $y \in \mathbb{C}\mathrm{P}^d$ be a fixed point of φ and $t(y) \in [0,1)$ be uniquely defined by $\Phi(\tilde{y}) = e^{2i\pi t(y)}\tilde{y}$, where $\pi(\tilde{y}) = y$. We define the *index* of the fixed point y by $\mathrm{ind}(y) := \mathrm{ind}(\zeta, F_{1,t(y)})$. We extend these definitions to iterated diffeomorphisms φ^m in the following way: if $y \in \mathbb{C}\mathrm{P}^d$ is a fixed point of φ, $t(y^m)$ denotes the only $t \in [0,1)$ such that $\Phi^m(\tilde{y}) = e^{2i\pi t}\tilde{y}$, hence satisfies

$$t(y^m) = mt(y) - \lfloor mt(y) \rfloor.$$

The same way, the m-iterated index of y designates the integer

$$\mathrm{ind}(y^m) := \mathrm{ind}\left(\zeta, F^m_{1,t}\right),$$

for some critical point ζ associated to the fixed point $\tilde{y}$ of the diffeomorphism $e^{-2i\pi t(y^m)}\Phi^m$.

According to (14),

$$\operatorname{supp} \mathrm{C}^*(y^m, t(y^m)) \subset \left[\mathrm{ind}(y^m), \mathrm{ind}(y^m) + \nu(y^m)\right].$$

By definition of the Maslov index,

$$\mathrm{ind}(y^m) = \mathrm{mas}\left(\tilde{y}, \left(e^{-2i\pi t(y^m)s}\Phi_{ms}\right)\right) + i(m),$$

where $i(m) := \mathrm{ind}(F^m_{0,0})$ only depends on $m \in \mathbb{N}^*$. Thus, according to Bott's iteration inequalities (4),

$$\operatorname{supp} \mathrm{C}^*(y^m, t(y^m)) \subset \left[\overline{\mathrm{ind}}(y^m) - d, \overline{\mathrm{ind}}(y^m) + d\right], \tag{20}$$

where $\overline{\mathrm{ind}}(y^m) := i(m) + \overline{\mathrm{mas}}(\tilde{y}, (e^{-2i\pi t(y^m)s}\Phi_{ms}))$.

Lemma 5.12. *Let $y \in \mathbb{C}\mathrm{P}^d$ be a fixed point of φ, then*

$$\overline{\mathrm{ind}}(y^m) = m\,\overline{\mathrm{mas}}\left(\tilde{y}, \left(e^{-2i\pi t(y)s}\Phi_s\right)\right) - 2(d+1)\lfloor mt(y)\rfloor + i(m), \quad \forall m \in \mathbb{N}^*.$$

Proof. Let $y \in \mathbb{C}\mathrm{P}^d$ be fixed by φ, $m \in \mathbb{N}^*$ and $k \in \mathbb{N}^*$. In $\mathrm{Sp}(2(d+1))$, the path $s \mapsto \mathrm{d}\left(e^{-2i\pi kmt(y)s}\Phi_{kms}\right)(\tilde{y})$ is homotopic relative to endpoints to the concatenation of the path $s \mapsto \mathrm{d}\left(e^{-2i\pi kt(y^m)s}\Phi_{kms}\right)(\tilde{y})$ and the loop $\Gamma : s \mapsto e^{-2i\pi k\lfloor mt(y)\rfloor s}$, thus Proposition 3.1 (1) and (5) implies

$$\mathrm{mas}\left(\tilde{y}, \left(e^{-2i\pi kmt(y)s}\Phi_{kms}\right)\right) = \mathrm{mas}\left(\tilde{y}, \left(e^{-2i\pi kt(y^m)s}\Phi_{kms}\right)\right) + \mathrm{mas}(\Gamma),$$

According to Proposition 3.1, $\mathrm{mas}(\Gamma) = 2(d+1)k\lfloor mt(y)\rfloor$, thus, dividing by k and letting $k \to \infty$, we get

$$\overline{\mathrm{mas}}\left(\tilde{y}, \left(e^{-2i\pi mt(y)s}\Phi_{ms}\right)\right) = \overline{\mathrm{mas}}\left(\tilde{y}, \left(e^{-2i\pi t(y^m)s}\Phi_{ms}\right)\right) + 2(d+1)\lfloor mt(y)\rfloor.$$

□

5.7. **Remarks on the parallel with Floer homology.** Our construction of the cohomology groups of the sublevel sets depends *a priori* on the choice of the tuple $\boldsymbol{\sigma}$ of the Hamiltonian diffeomorphism Φ. We discuss uniqueness properties of this group and links with Floer homology without complete proofs, as it will not be necessary for this paper. Given a $\mathbb{C}$-equivariant flow (Φ_s) and real numbers $a < b$ not in the action spectrum of φ, we can define $F_{s,t}$ generating $e^{-2i\pi t}\Phi_s$ for $s \in [0,1]$ and $t \in I$, where I is an interval containing some $a' < a$ and $b' > b$, the same way as before. Let $i_0 := \mathrm{ind}(F_{0,0})$, we define the cohomology group

$$G^*_{(a,b)}((\Phi_s)) := H^{*-i_0}(\{\mathcal{T} \leq b\}, \{\mathcal{T} \leq a\}).$$

By using Lemma 2.2 together with Lemma 5.3, one can show that this group does not depend on the specific choice of $(\boldsymbol{\sigma}_s)$ associated to the Hamiltonian flow (Φ_s). Moreover, one can show that it only depends on the homotopy class of Hamiltonian path (Φ_s) relative to endpoints the same way Théret showed it for its rotation numbers in [17, Prop. 5.7]. We can go a little further if we normalize the Hamiltonian map (h_s) of the flow (φ_s) so that for instance

$$\int_{\mathbb{C}\mathrm{P}^d} h_s \omega^d = 0, \quad \forall s \in [0,1].$$

Then (Φ_s) is uniquely determined by (φ_s) and the group $G^*_{(a,b)}((\Phi_s))$ only depends on the homotopy class of (φ_s) relative to endpoints, that is the choice of a lift $\widetilde{\varphi} \in \widetilde{\mathrm{Ham}}(\mathbb{C}\mathrm{P}^d)$ of φ to the universal cover of $\mathrm{Ham}(\mathbb{C}\mathrm{P}^d)$.

When there are finitely many fixed points, the building blocks of $G^*_{(a,b)}$ are the local cohomology groups $\mathrm{C}^{*-i_0}(z,t)$ for fixed points $z \in \mathbb{C}\mathrm{P}^d$ of φ and $t \in (a,b)$. Giving a couple (z,t) is equivalent to giving a capped orbit $\bar{z} = (z, u_t)$ where u_t is the capping naturally induced by the lift $e^{-2i\pi t}\Phi$, as seen in Section 3.2. We can remark that both $\mathrm{C}^*(\bar{z}) := \mathrm{C}^{*-i_0}(z,t)$ and local Floer cohomology $HF^*(\bar{z})$ have their support in $[\overline{\mathrm{mas}}(\bar{z}) - d, \overline{\mathrm{mas}}(\bar{z}) + d]$ and that they are equal when $\bar{z}$ is non-degenerate. One can prove that they are isomorphic by using the isomorphism between cohomology of generating functions of Hamiltonian diffeomorphisms of $\mathbb{C}^{d+1}$ and their Floer cohomology in [18].

Proposition 5.11 essentially asserts that for all capped orbit $\bar{z}$ of φ,

$$\mathrm{C}^*(A_0 \# \bar{z}) = u^{d+1}\mathrm{C}^*(\bar{z}),$$

where A_0 is the generator of $\pi_2(\mathbb{C}\mathrm{P}^d) \simeq \mathbb{Z}$ satisfying $\langle [\omega], A_0 \rangle = -\pi$. The reader familiar to quantum homology can interpret this relation as the fact that the class u^{d+1} of $H^*(\mathbb{C}\mathrm{P}^\infty)$ acts on the group $G^*_{(a,b)}$ the same way as the operator q of the quantum homology of $\mathbb{C}\mathrm{P}^d$ acts on the group $HF_*^{(a,b)}$ (see [8, Sect. 2] but beware signs of ω and action are opposite to our convention). Moreover, the relation

$$[\mathrm{pt}] * [\mathbb{C}\mathrm{P}^{d-1}] = \mathrm{q}[\mathbb{C}\mathrm{P}^d]$$

in the quantum homology of $\mathbb{C}\mathrm{P}^d$ can be interpreted as

$$u^d \smile u = u^{d+1} \smile 1,$$

seeing u^d, u and 1 respectively as the Poincaré duals of the homology classes $[\mathrm{pt}]$, $[\mathbb{C}\mathrm{P}^{d-1}]$ and $[\mathbb{C}\mathrm{P}^d]$ in $H_*(\mathbb{C}\mathrm{P}^d)$. This relation is fundamental in the original proof of Ginzburg-Gürel and thus explains the fundamental role of the subordinated classes 1, u and u^{d+1} in our proof.

6. Proof of Theorem 1.1 and its corollaries

In this section, we prove Theorem 1.1, postponing the proof of the crossing energy theorem to Section 7. We then provide the proofs of Corollaries 1.2 and 1.3 sketched in the introduction.

6.1. **Preliminaries.** Let $\varphi \in \mathrm{Ham}(\mathbb{C}\mathrm{P}^d)$ be the time-one map of a Hamiltonian flow (φ_s) with a fixed point $x \in \mathbb{C}\mathrm{P}^d$ which is isolated as an invariant set. Moreover, let (Φ_s) be a $\mathbb{C}$-equivariant Hamiltonian flow lifting (φ_s) and let us suppose that $t(x) = 0$ and that the local cohomology groups of x associated to the iterations of φ are all non-zero. We will prove Theorem 1.1 by contradiction: let us assume that φ has only finitely many periodic points so that $\mathcal{T}^m$ has only isolated critical points in a finite number for all $m \in \mathbb{N}^*$. In our construction of $\mathcal{T}$, we take $\varepsilon < 1/2$ so that any fixed point of φ have at most 2 associated critical points. Taking an iteration, we might suppose that any periodic point is a fixed point of φ. For all $m \in \mathbb{N}^*$, let $(j, \zeta_j^m) \in M_m$, $j \in \{0, 1\}$, be the critical points of $\mathcal{T}$ associated to x. In Section 7, we prove the crossing energy theorem which applies to our point x, isolated as an invariant set, in the following way:

Theorem 6.1. *There exist $c_\infty > 0$, families of open neighborhoods $V_j^m, W_j^m \subset M_m$ of (j, ζ_j^m) with $W_j^m \subset V_j^m$ which do not intersect $\mathrm{Crit}(\mathcal{T}^m) \setminus (j, \zeta_j^m)$ and an adapted pseudo-gradient X_m of $\mathcal{T}^m$, such that any gradient or reversed-gradient flow line $\mathbf{u} : \mathbb{R} \to M_m$, $\dot{\mathbf{u}} = \pm X_m(\mathbf{u})$, with $\mathbf{u}(s) \notin V_j^m$ and $\mathbf{u}(t) \in W_j^m$ for some $m \in \mathbb{N}^*$ and $j \in \{0, 1\}$ satisfies*

$$|\mathcal{T}^m(\mathbf{u}(s)) - \mathcal{T}^m(\mathbf{u}(t))| > c_\infty.$$

Let $c_\infty > 0$ be given by the above result. Without loss of generality, we suppose that

$$0 < c_\infty < \frac{1}{2(d+1)}. \tag{21}$$

6.2. **Augmented action.** By the analogy with Ginzburg-Gürel augmented action, for any fixed point $y \in \mathbb{C}\mathrm{P}^d$ we define

$$\tilde{a}(y^m) := mt(y) - \frac{1}{2(d+1)}\,\overline{\mathrm{mas}}(\tilde{y}, (e^{-2i\pi mst(y)}\Phi_{ms})) = m\tilde{a}(y).$$

According to Lemma 5.12,

$$\begin{aligned}\overline{\mathrm{ind}}(y^m) - \overline{\mathrm{ind}}(x^m) &= m\left(\overline{\mathrm{mas}}\left(\tilde{y}, \left(e^{-2i\pi t(y)s}\Phi_s\right)\right) - \overline{\mathrm{mas}}\left(\tilde{x}, \left(e^{-2i\pi t(x)s}\Phi_s\right)\right)\right) \\ &\quad - 2(d+1)\lfloor mt(y)\rfloor \\ &= 2m(d+1)\left(\tilde{a}(x) - \tilde{a}(y)\right) + 2(d+1)t(y^m).\end{aligned} \tag{22}$$

By Dirichlet's lemma, one can find $m \in \mathbb{N}^*$ such that, for all fixed points y, the fraction part of each $mt(y)$, which is $t(y^m)$, satisfies

$$t(y^m) \in [0, c_\infty) \cup (1 - c_\infty, 1) \tag{23}$$

with m taken sufficiently large so that

$$|\tilde{a}(y) - \tilde{a}(x)| = 0 \quad \text{or} \quad m|\tilde{a}(y) - \tilde{a}(x)| > 3. \tag{24}$$

Thus equation (22) together with assumption (21) implies the following lemma.

Lemma 6.2. *Let $m \in \mathbb{N}^*$ be so that (23) and (24) are satisfied. Given any fixed point $y \in \mathbb{C}\mathrm{P}^d$, we have:*

- $\left|\overline{\mathrm{ind}}(y^m) - \overline{\mathrm{ind}}(x^m)\right| \leq 2d+1$ *implies* $t(y^m) < c_\infty$,
- $\left|\overline{\mathrm{ind}}(y^m) - \left(\overline{\mathrm{ind}}(x^m) + 2(d+1)\right)\right| \leq 2d+1$ *implies* $t(y^m) > 1 - c_\infty$.

Given two subsets $A, B \subset \mathbb{R}$, we denote the smallest distance among their points by

$$\mathrm{dist}(A,B) := \inf\{|a-b| \mid a \in A, b \in B\} \in [0,+\infty].$$

According to Proposition 5.10 and equation (20), Lemma 6.2 implies

Corollary 6.3. *Let $m \in \mathbb{N}^*$ be so that (23) and (24) are satisfied. Given any fixed point $y \in \mathbb{C}\mathrm{P}^d$ of φ^m of action $\bar{t}$ with $t \in (-\varepsilon, 1+\varepsilon)$, if*

$$\mathrm{dist}(\mathrm{supp}\,\mathrm{C}^*(y,t), \mathrm{supp}\,\mathrm{C}^*(x,j)) \leq 1,$$

then $|t-j| < c_\infty$, for $j \in \{0,1\}$.

6.3. **Subordinated min-max.** By taking the m-th iteration of $\varphi \in \mathbb{C}\mathrm{P}^d$, we can suppose that φ satisfies Corollary 6.3 for $m=1$. Let $G_t : M \to M$ be the gradient flow associated to the pseudo-gradient of Theorem 6.1 at time $t \in \mathbb{R}$. In order to simplify notation in this section, given any subset $U \subset M$ and any $b \in I$, we set $U^{\leq b} := U \cap \{\mathcal{T} \leq b\}$ and $U^{<b} := U \cap \{\mathcal{T} < b\}$, and we denote by $\mathrm{C}^*(y)$ the local homology associated to a critical point $y \in M$. For all critical points $y \in M$, let us define a specific flow-out $U(y)$, that is an open neighborhood of y which is invariant under G_t for all $t \geq 0$. We take a small neighborhood B of y, then we set

$$U'(y) := \bigcup_{t \geq 0} G_t(B),$$

let $\{z_j\}$ be the family of critical points in the closure of $U'(y)$, we then define $U(y) := U'(y) \cup \{z_j\}$ which is in fact open if one takes B small enough. Let $z_0 := (0, \zeta_0)$ and $z_1 := (1, \zeta_1)$ be the two critical points associated to the fixed point $x \in \mathbb{C}\mathrm{P}^d$. Applying Theorem 6.1 together with Corollary 6.3, we choose B small enough such that $z_j \in U(y)$ implies that

$$\mathrm{dist}(\mathrm{supp}\,\mathrm{C}^*(y), \mathrm{supp}\,\mathrm{C}^*(z_j)) > 1, \tag{25}$$

and in the case where $y = z_j$, we do the same so that this last equation holds also for critical points $y \in U(z_j)$. We also ask that if some critical point y is in $U(y')$, then $U(y) \subset U(y')$ (this could be achieved by induction on the critical values, starting by defining the flow-out for the critical points of largest value). We first prove that the local cohomology $\mathrm{C}^*(z_0)$ "persists in the action window $[0, 1+\varepsilon)$". Let $v_0 \in \mathrm{C}^*(z_0)$ be a non-zero class, which exists by hypothesis.

Lemma 6.4. *For all $b \in [0, 1+\varepsilon)$, there exists a class $v \in H^*(M^{\leq b}, M^{<0})$ such that its image under the morphism induced by the inclusion $H^*(M^{\leq b}, M^{<0}) \to H^*(M^{\leq 0}, M^{<0})$ is v_0. Moreover, given one of the above flow-outs $U = U(y)$,*

$$v \notin \ker\left(H^*(M^{\leq b}, M^{<0}) \to H^*(U^{\leq b}, U^{<0})\right)$$

if and only if $z_0 \in U$, where the morphism is induced by inclusion.

Proof. According to Morse deformation lemma, if the lemma is true for b and $(b,c] \subset I$ does not contain any critical value, then the lemma is also true for c. Since there is a finite number of critical values, we can thus prove this lemma inductively on the critical value $b \geq 0$. We start with the case $b = 0$. As we have seen, by excision $\mathrm{C}^*(z_0) \subset H^*(M^{\leq 0}, M^{<0})$ and taking $v = v_0$ under this injection is enough.

Let us assume that $b > 0$ is a critical value and that the lemma is true on $[0, b)$. Let (y_k) be the family of critical points of value b and $U_k := U(y_k)$ be their associated flows-out. We will work with the following commutative diagram:

$$\begin{array}{ccccc} H^*(M^{\leq b}, M^{<b}) & \xrightarrow{j^*} & H^*(M^{\leq b}, M^{<0}) & \xrightarrow{i^*} & H^*(M^{<b}, M^{<0}) \\ \downarrow & & \downarrow & & \downarrow \\ H^*(U_k^{\leq b}, U_k^{<b}) & \xrightarrow{j_k^*} & H^*(U_k^{\leq b}, U_k^{<0}) & \xrightarrow{i_k^*} & H^*(U_k^{<b}, U_k^{<0}) \end{array} \tag{26}$$

where every arrow is induced by inclusion. By Morse deformation lemma $M^{<b}$ and $U^{<b}$ on the right hand side of the diagram can be replaced by $M^{\leq c}$ and $U^{\leq c}$ for some $c < b$ close enough. By induction, there exists $v' \in H^*(M^{<b}, M^{<0})$ satisfying the lemma (with symbol $\leq b$ replaced by $< b$). Let us first show that v' is in the image of i^*. According to the long exact sequence of the triple $(M^{\leq b}, M^{<b}, M^{<0})$, it boils down to showing that $\partial^* v' = 0$ where ∂^* is the coboundary map. By contradiction let us assume that $\partial^* v' \neq 0$. By excision, we recall that

$$H^*(M^{\leq b}, M^{<b}) \simeq \bigoplus_k \mathrm{C}^*(y_k),$$

thus if $\partial^* v' \neq 0$ then $\partial_k^* v' \neq 0$ for some k, where ∂_k^* is the composition of the coboundary with the projection on $\mathrm{C}^*(y_k)$ with respect to the above direct sum. Identifying $H^*(U_k^{\leq b}, U_k^{<b})$ with $\mathrm{C}^*(y_k)$ by excision, one has the following commutative diagram:

$$\begin{array}{ccc} H^*(M^{<b}, M^{<0}) & \xrightarrow{\partial^*} & H^{*+1}(M^{\leq b}, M^{<b}) \\ \downarrow & \searrow^{\partial_k^*} & \downarrow \\ H^*(U_k^{<b}, U_k^{<0}) & \longrightarrow & \mathrm{C}^{*+1}(y_k) \end{array}$$

where the vertical arrows are induced by inclusions and the horizontal are coboundary maps. Thus we see that v' is not in the kernel of the left hand side arrow, so that by induction hypothesis $z_0 \in U_k$. But according to equation (25), if ℓ is the degree of v' (which maps to $v_0 \in \mathrm{C}^\ell(z_0)$), then $\mathrm{C}^{\ell+1}(y_k) = 0$, a contradiction.

Hence $\partial^* v' = 0$ and there exists $v'' \in H^*(M^{\leq b}, M^{<0})$ such that $i^* v'' = v'$. This v'' maps to v_0 as required but does not satisfy the second conclusion of the lemma *a priori*. We now explain how to build v in the inverse image of v'. For a fixed k, let $v_k'' \in H^*(U_k^{\leq b}, U_k^{<0})$ be the image of v'' under the vertical arrow of (26). For v'' to satisfy the conclusions of the lemma, we need v_k'' to be zero if and only if the image of $v' = i^* v''$ under its vertical arrow is zero. If $i_k^* v_k'' = 0$, then there exists $w_k' \in H^*(U_k^{\leq b}, U_k^{<b})$ such that $j_k^* w_k' = v_k''$. We recall that the left hand side arrow is equivalent to the projection

$$\bigoplus_\ell \mathrm{C}^*(y_\ell) \to \mathrm{C}^*(y_k),$$

let $w_k \in H^*(M^{\leq b}, M^{<b})$ be then the image of w_k' under the inclusion $\mathrm{C}^*(y_k) \subset H^*(M^{\leq b}, M^{<b})$. We finally set

$$v := v'' - \sum_k j^*(w_k) \in H^*(M^{\leq b}, M^{<0})$$

to be the wanted solution.

The conclusion is true for the $U = U_k$ with this choice of v, by construction. Let U be the flow-out of some critical point. If U does not contain any of the y_k, by the Morse deformation lemma, $U^{\leq b}$ retracts on $U^{<b}$ so that the conclusion follows by induction. Otherwise, let (y_{k_q}) be the sub-family of (y_k) included in U, so that $U_{k_q} \subset U$ by construction of our flows-out. If $z_0 \in U$, then by hypothesis, v' is not in the kernel of $H^*(M^{<b}, M^{<0}) \to H^*(U^{<b}, U^{<0})$ and, as $i^*v = v'$, v is neither in the kernel of $H^*(M^{\leq b}, M^{<0}) \to H^*(U^{\leq b}, U^{<0})$. Conversely, if v is not in the above kernel, either v' is not in the kernel of its restriction to U, in which case $z_0 \in U$ by induction, or its image $v_U \in H^*(U^{\leq b}, U^{<0})$ under the above map has the form $j_U^* w_U$ where $j_U^* : H^*(U^{\leq b}, U^{<b}) \to H^*(U^{\leq b}, U^{<0})$. Now by excision $H^*(U^{\leq b}, U^{<b})$ is isomorphic to the direct sum of the $H^*(U_{k_q}^{\leq b}, U_{k_q}^{<b})$'s. Thus, there is a k_q such that w_U projects on $w_{k_q} \neq 0$. The commutativity of the left hand square of (26) for $k = k_q$ together with the construction of v yields a contradiction. □

Corollary 6.5. *There exists a subgroup $G \subset H^*(M^{\leq 1}, M^{<0})$ whose image under the map $H^*(M^{\leq 1}, M^{<0}) \to H^*(M^{\leq 0}, M^{<0})$ is $\mathrm{C}^*(z_0)$ and such that its image under $H^*(M^{\leq 1}, M^{<0}) \to H^*(U^{\leq 1}, U^{<0})$ is non-zero if and only if $z_0 \in U$, where $U := U(y)$ is a flow-out.*

Proof. Take G to be the subgroup of $H^*(M^{\leq 1}, M^{<0})$ generated by every v given by Lemma 6.4 for $b = 1$ and each $v_0 \in \mathrm{C}^*(z_0) \setminus 0$. □

By using (25) for z_1, one can prove dually the following lemma.

Lemma 6.6. *The subgroup $\mathrm{C}^*(z_1) \subset H^*(M^{\leq 1}, M^{<1})$ trivially intersects the kernel of the map*

$$H^*(M^{\leq 1}, M^{<1}) \to H^*(U^{\leq 1}, U^{<0})$$

induced by inclusion, where $U := U(z_1)$ is the flow-out of z_1.

Proof of Theorem 1.1. Let $v \in H^*(M^{\leq 1}, M^{<0})$ be the class given by Lemma 6.4 for $b = 1$. Thus, applying Proposition 5.11 to Corollary 6.5, there exists a class $w \in \mathrm{C}^*(z_1)$ that maps to $u^{d+1}v \in H^*(M^{\leq 1}, M^{<0})$. Considering the restriction map $H^*(M^{\leq 1}, M^{<0}) \to H^*(U^{\leq 1}, U^{<0})$, where $U := U(z_1)$ is the flow-out of z_1, the image of $u^{d+1}v$ is non-zero by Lemma 6.6, thus the image v' of v is non-zero. Therefore, Lemma 6.4 implies that $z_0 \in U(z_1)$. For $t \in [0, 1]$, let i_t^* be the map induced by inclusion

$$i_t^* : H^*(U^{\leq 1}, U^{<0}) \to H^*(U^{\leq t}, U^{<0}),$$

and, for $0 \leq k \leq d + 1$, let $\tau_k \in [0, 1]$ be the family of min-max values

$$\tau_k := \inf \left\{ t \geq 0 \mid u^k v' \notin \ker i_t^* \right\}. \tag{27}$$

As we have just seen, $u^{d+1}v' \neq 0$ so that $u^k v' \neq 0$ for $k \leq d + 1$. Lemma 6.4 implies that $\tau_0 = 0$. By the long exact sequence of the triple $(U^{\leq 1}, U^{<1}, U^{<0})$, the image of $u^{d+1}v'$ under $H^*(U^{\leq 1}, U^{<0}) \to H^*(U^{<1}, U^{<0})$ is zero, thus $\tau_{d+1} = 1$. By Lyusternik-Schnirelmann theory, τ_1 is a critical value of $\mathcal{T}|_U$ and $\tau_0 < \tau_1 < \tau_{d+1}$ since $\mathcal{T}$ has a finite number of critical values (we recall that $d \geq 1$). Let (y_j) be the family of critical points of value τ_1 in U. According to Theorem 6.1, if the flow-out U is taken small enough, $\tau_1 \leq 1 - c_\infty$, thus $\tau_1 < c_\infty$ since there are no critical points with value in $[c_\infty, 1 - c_\infty]$. Since $H^*(U^{\leq \tau_1}, U^{<\tau_1})$ decomposes in the direct sum of the local cohomologies of the y_j's, we find by similar arguments as before that there exists a j such that the image of uv' on $H^*(U_j^{\leq \tau_1}, U_j^{<0})$ is non-zero. For this j, the

image of v' under the same map is thus also non-zero, hence $z_0 \in U_j$ by Lemma 6.4. But according to Theorem 6.1, for U_j taken small enough in our proofs, one must have $\tau_1 \geq c_\infty$, a contradiction. □

6.4. **Corollaries.**

Proof of Corollary 1.2. Let $x \in \mathbb{C}\mathrm{P}^d$ be a hyperbolic fixed point of $\varphi \in \mathrm{Ham}(\mathbb{C}\mathrm{P}^d)$ and $\Phi \in \mathrm{Ham}_{\mathbb{C}}(\mathbb{C}^{d+1})$ be a lift of φ. According to Theorem 1.1, it is enough to prove that the local cohomology group $\mathrm{C}^*(x, t(x^k))$ is non zero for all iteration $k \in \mathbb{N}^*$. In Section 5.5, we have seen that $\mathrm{C}^*(x, t(x^k)) \simeq \mathrm{C}^*(\widehat{F}_{t(x^k)}; \zeta_k)$ where $\zeta_k \in \mathbb{C}\mathrm{P}^{N(k)}$ is the critical point of the map $\widehat{F}_{t(x^k)} : \mathbb{C}\mathrm{P}^{N(k)} \to \mathbb{R}$ induced by the generating function $F_{t(x^k)}$ of $e^{-2i\pi t(x^k)}\Phi \in \mathrm{Ham}_{\mathbb{C}}(\mathbb{C}^{d+1})$. Since x is hyperbolic, $\dim \ker(\mathrm{d}\varphi(x)^k - \mathrm{id}) = 0$ for all $k \in \mathbb{N}^*$, thus $\mathrm{d}^2\widehat{F}_{t(x^k)}$ is non-degenerate according to (11) and $\mathrm{C}^*(x, t(x^k))$ has rank 1. □

Proof of Corollary 1.3. Let $\varphi \in \mathrm{Ham}(\mathbb{C}\mathrm{P}^d)$ be a pseudo-rotation of $\mathbb{C}\mathrm{P}^d$ with fixed points $x_1, \dots, x_{d+1} \in \mathbb{C}\mathrm{P}^d$ and a lift $\Phi \in \mathrm{Ham}_{\mathbb{C}}(\mathbb{C}^{d+1})$. According to Theorem 1.1, it is enough to prove that the local cohomology groups $\mathrm{C}^*(x_j, t(x_j^k))$ are non-zero for all j and all $k \in \mathbb{N}^*$. This is a consequence of the following fact due to Théret [17]: given any $\varphi \in \mathrm{Ham}(\mathbb{C}\mathrm{P}^d)$, there always exists some integer $k \in \mathbb{N}$ such that the classes $u^k, u^{k+1}, \dots, u^{k+d} \in H^*(\mathbb{C}\mathrm{P}^N)$ are not in the kernel of the map induced by the projection $I \times \mathbb{C}\mathrm{P}^N \to \mathbb{C}\mathrm{P}^N$

$$H^*(\mathbb{C}\mathrm{P}^N) \to H^*(\{\mathcal{T} \leq 1 + \varepsilon'\}, \{\mathcal{T} \leq -\varepsilon'\})$$

with associated min-max values in $[0, 1)$ (that is the values τ_k of (27) where the $u^k v'$'s are replaced by the u^{k+j}'s). We briefly recall the proof in our setting.

Given $t \in [0, 1]$, let $\ell(t) \in \mathbb{N}$ be the integer

$$\ell(t) := \max\left\{k \in \mathbb{N} \mid u^k \notin \ker\left(H^*(\mathbb{C}\mathrm{P}^N) \to H^*(\{\mathcal{T} \leq t\})\right)\right\}.$$

We can show that $\ell(t+1) = \ell(t) + d + 1$ by combining Corollary 4.2 and Lemma 5.9, hence the statement for $k := \ell(0) + 1$.

In the case where $\varphi \in \mathrm{Ham}(\mathbb{C}\mathrm{P}^d)$ has isolated fixed points, the min-max values of the u^{k+j}'s must be different by Lyusternik-Schnirelmann theory. Thus any $\varphi \in \mathrm{Ham}(\mathbb{C}\mathrm{P}^d)$ with isolated fixed points has at least $d + 1$ fixed points with non-zero associated local cohomology. Hence the conclusion holds, since every iteration of a pseudo-rotation have only $d + 1$ fixed points. □

7. Ginzburg-Gürel crossing Theorem for generating functions

In this section, we prove the analogue of Ginzburg-Gürel crossing theorem for generating functions. Since the proof in $\mathbb{C}\mathrm{P}^d$ is essentially the same as the one in $\mathbb{C}^d$ with some technical changes which could make it less transparent to the reader, we first provide the argument for $\mathbb{C}^d$, even though the $\mathbb{C}^d$ setting will not be employed in this paper.

7.1. **Crossing energy theorem in $\mathbb{C}^d$.** If $\boldsymbol{\sigma} := (\sigma, \dots, \sigma)$ is a tuple of even size associated to Φ, then $\boldsymbol{\sigma}^m$ is a tuple of even size of the iterated diffeomorphism Φ^m. Given any $x \in \mathbb{C}^d$, let $B_r^{2d}(x) := \{z \in \mathbb{C}^d \mid |z - x| < r\}$ or simply $B_r(x)$. We will denote by A_m the linear isomorphism of $(\mathbb{C}^d)^{mn+1}$ defined by $A_m(\mathbf{v}) := \mathbf{w}$ where $w_k = \frac{v_k + v_{k+1}}{2}$. Throughout this section, we will study the generating functions $F_{(\boldsymbol{\sigma}, \mathrm{id})}$

of Φ^m with a linear change of coordinates: let $F^m(w) := F_{(\boldsymbol{\sigma},\mathrm{id})} \circ A_m^{-1}(w)$. Given a tuple $\boldsymbol{\delta} := (\delta_1, \dots, \delta_m)$, with $\delta_k \in \mathrm{Ham}(\mathbb{C}^d)$ for all k, $x \in \mathbb{C}^d$ and a radius $r > 0$, we denote by $B_r(x, \boldsymbol{\delta}) \subset (\mathbb{C}^d)^m$ the open set

$$B_r\left(\frac{x + \delta_1(x)}{2}\right) \times B_r\left(\frac{\delta_1 + \delta_2 \circ \delta_1(x)}{2}\right) \times \cdots \\ \cdots \times B_r\left(\frac{\delta_{m-2} \circ \cdots \circ \delta_1(x) + \delta_{m-1} \circ \cdots \circ \delta_1(x)}{2}\right),$$

that is $B_r(x, \boldsymbol{\delta}) = \prod_j B_r(w_j)$ where the m-tuple $\mathbf{w}$ is associated to the discrete trajectory $(x, \delta_1(x), \dots, \delta_{m-1} \circ \cdots \circ \delta_1(x))$ of the discrete dynamics of $\boldsymbol{\delta}$.

Lemma 7.1. *Let $\boldsymbol{\sigma} := (\sigma_1, \dots, \sigma_n)$ be such an n-tuple and $x \in \mathbb{C}^d$ be a fixed point of $\sigma_n \circ \cdots \circ \sigma_1$. Suppose there exists a sequence $(m_j)_{j\geq 0}$ such that there exists a sequence $(\mathbf{w}^j)_{j\geq 0}$ with $\mathbf{w}^j \in B_r(x, (\boldsymbol{\sigma}^{m_j}, \mathrm{id})) \setminus B_{r/2}(x, (\boldsymbol{\sigma}^{m_j}, \mathrm{id}))$ satisfying,*

$$\left|\nabla F^{m_j}\left(\mathbf{w}^j\right)\right|^2 = \sum_{k=1}^{m_j n+1} \left|\partial_{w_k} F^{m_j}\left(\mathbf{w}^j\right)\right|^2 \xrightarrow{j\to\infty} 0. \tag{28}$$

Let $(a_1, \dots, a_n) \in (\mathbb{C}^d)^n$ be such that $B_r(x, \boldsymbol{\sigma}) = B_r(a_1) \times \cdots \times B_r(a_n)$. Then, there exists a sequence $(z_j)_{j\in\mathbb{Z}} \in (\mathbb{C}^d)^{\mathbb{Z}}$ and some integer $1 \leq q \leq n$ such that $z_{j+1} = \sigma_j(z_j)$ with

$$\begin{cases} \frac{z_j + \sigma_j(z_j)}{2} \in \overline{B_r^{2d}(a_{j \bmod n})} & \text{for all } j \in \mathbb{Z}, \\ \frac{z_q + \sigma_q(z_q)}{2} \notin B_{r/2}^{2d}(a_q) \text{ or } z_q = z_1 \notin B_{r/2}^{2d}(x). \end{cases} \tag{29}$$

Remark that Proposition 5.1 and (28) imply

$$|z_k^j - \sigma_{k-1}(z_{k-1}^j)| \xrightarrow{j\to\infty} 0 \text{ for } 1 < k \leq m_j n + 1 \quad \text{and} \quad |z_1^j - z_{m_j n+1}^j| \xrightarrow{j\to\infty} 0,$$

where $\mathbf{z}^j$ is the discrete trajectory associated to $\mathbf{w}^j$ *via* relations (7). Indeed, $\partial_{v_k} F^{m_j} = \frac{1}{2}(\partial_{w_k} F^{m_j} + \partial_{w_{k-1}} F^{m_j})$. Thus, the proof essentially consists in an elementary application of the Cantor's diagonal argument to ultimately get a discrete trajectory of the dynamic $\sigma_n \circ \cdots \circ \sigma_1$ whose special property (29) comes from the domain of the $\mathbf{w}^j$'s.

Proof. We first prove the case where $(m_j)_{j\geq 0}$ admits a bounded subsequence for a better understanding of the general case. Taking an extracted subsequence, we might suppose that $m_j \equiv m \in \mathbb{N}^*$. Then by relative compactness, we might suppose that $\mathbf{w}^j \to \mathbf{w} \in \overline{B_r(x, (\boldsymbol{\sigma}^m, 0))} \setminus B_{r/2}(x, (\boldsymbol{\sigma}^m, 0))$. Let us define $(z'_j)_{1\leq j\leq mn+1} \in (\mathbb{C}^d)^{mn+1}$ by the relations (7) for $\mathbf{w}$. According to Proposition 5.1, since $\nabla F^m(A_0^{-1}\mathbf{v}) = 0$, one has $z'_{j+1} = \sigma_{j \bmod n}(z'_j)$ for $1 \leq j \leq mn$ and $z'_1 = z'_{mn+1}$. Since $w \notin B_{r/2}(x, (\boldsymbol{\sigma}^m, 0))$, there is some integer $1 \leq q' \leq mn + 1$ such that $w_{q'} \notin B_{r/2}^{2d}(a_{q' \bmod n})$ if $q' \neq mn + 1$ or $w_{q'} \notin B_{r/2}^{2d}(x)$ otherwise. If $q' \neq mn + 1$, let $k \in \mathbb{N}$ be such that $kn + 1 \leq q' < (k + 1)n + 1$ and let $1 \leq q \leq n$ be the integer $q = q' - kn$. The desired sequence $(z_j)_{j\in\mathbb{Z}}$ is then the mn-periodic sequence such that

$$z_j = z'_{j+kn}, \quad -kn + 1 \leq j \leq (m - k)n.$$

In this case,

$$\frac{z_q + \sigma_q(z_q)}{2} = w_{q'} \notin B_{r/2}^{2d}(a_q).$$

If $q' = mn+1$, then the wanted sequence is the mn-periodic sequence such that $z_j = z'_j$ for $1 \leq j \leq mn$ with $q = 1$ and in this case

$$z_q = z_1 = w_{mn+1} \notin B^{2d}_{r/2}(x).$$

Now suppose that $(m_j)_{j\geq 0}$ admits no bounded infinite subsequence. Taking an extracted subsequence, we might suppose that (m_j) is increasing. For all $j \in \mathbb{N}$, let $1 \leq q'_j < m_j n + 1$ be such that $w^j_{q'_j} \notin B_{r/2}(a_{q'_j \bmod n})$ or $q'_j = m_j n + 1$ if such an integer does not exist. Similarly to the bounded case, we first suppose that we can take an extracted subsequence $q'_j \neq m_j n + 1$ for all $j \geq 0$. Let $k_j \in \mathbb{N}$ be such that $k_j n + 1 \leq q'_j < (k_j + 1)n + 1$ and let $1 \leq q_j \leq n$ be the integer $q_j = q'_j - k_j n$. Taking an extracting subsequence, we might suppose $q_j \equiv q$. For all $j \geq 0$, let $(\mathbf{w}'^j) \in (\mathbb{C}^d)^{\mathbb{Z}}$ be the $m_j n$-periodic sequence such that $w'^j_k = w^j_{k+k_j n}$ for $-k_j n + 1 \leq j \leq (m_j - k_j)n$.

Let $M \in \mathbb{N}^*$ and let us consider the sequence $(\mathbf{w}^{M,j})_{j\geq 0}$ in $(\mathbb{C}^d)^{2M+1}$ defined by restriction: for all $j \geq 0$, $(w^{M,j}_k)_{-M-1\leq k\leq M} := (w'^j_k)_{-M-1\leq k\leq M}$. Now, we can extract a subsequence $j^M_p \to \infty$, such that $(\mathbf{w}^{M,j^M_p})_p$ converges. Since $\partial_{w_{k+k_{j^n}}} F^{m_j}(\mathbf{w}^j) \to 0$ for all $-M - 1 \leq k \leq M$, the associated $(2M + 1)$-tuple $(z_j)_{-M\leq j\leq M}$, now satisfies $z_{j+1} = \sigma_j(z_j)$ for $-M \leq j \leq M - 1$. By a diagonal extraction associated to subsequences $(j^M_p)_p$ as M goes to infinity, we extend our $(2M + 1)$-tuples (z_j) to a sequence in $\mathbb{Z}$ with the wanted properties. In particular $\frac{z_q+\sigma_q(z_q)}{2} \notin B^{2d}_{r/2}(a_q)$.

If one cannot extract a subsequence such that $q'_j \neq m_j n + 1$, we can extract a subsequence such that $q'_j \equiv m_j n + 1$. Then take $q = 1$ and define $(\mathbf{w}'^j)$ to be the $m_j n$-periodic sequence such that $w'^j_k = w^j_k$ for $1 \leq k \leq m_j n$. By the same way as above, one gets the wanted (z_j) by a diagonal extraction and in this case $z_q = z_1 \notin B^{2d}_{r/2}(x)$. □

Theorem 7.2. *Let $\Phi \in \mathrm{Ham}(\mathbb{C}^d)$ admitting C^1-small n-tuples $\boldsymbol{\sigma}$. Suppose that $x \in \mathbb{C}^d$ is a fixed point of Φ which is isolated as an invariant set. Then for every sufficiently small $r > 0$, there exists $c_\infty > 0$ and a n-tuple $\boldsymbol{\sigma}$ associated to Φ, with n even, such that for all $m \geq 1$, any gradient or reverse-gradient flow line $\mathbf{u} : \mathbb{R} \to (\mathbb{C}^d)^{mn+1}$, $\dot{\mathbf{u}} = \pm\nabla F^m(\mathbf{u})$, with $\mathbf{u}(0) \in \partial B_r(x, (\boldsymbol{\sigma}^m, \mathrm{id}))$ and $\mathbf{u}(\tau) \in B_{r/2}(x, (\boldsymbol{\sigma}^m, \mathrm{id}))$ for some $\tau \in \mathbb{R}$ satisfies*

$$|F^m(\mathbf{u}(0)) - F^m(\mathbf{u}(\tau))| > c_\infty.$$

Proof. Since x is isolated as an invariant set, there exists some $R > 0$ such that for all $z \in B^{2d}_R(x) \setminus \{x\}$, there exists $k \in \mathbb{Z}$ such that $\Phi^k(z) \notin B^{2d}_R(x)$. Fix such an $R > 0$ and choose an even tuple $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ such that $|z - \sigma_j(z)| < R/8$ for all $z \in B^{2d}_R(x)$. Let $m \geq 1$ and $\mathbf{u} : \mathbb{R}_+ \to (\mathbb{C}^d)^{mn+1}$ be as in the statement of the theorem, we may suppose that $\mathbf{u}$ takes its values in $B_{R/2}(x, (\boldsymbol{\sigma}^m, \mathrm{id}))$. Let $\tau > 0$ be such that $\mathbf{u}(\tau) \in \partial B_{R/4}(x, (\boldsymbol{\sigma}^m, \mathrm{id}))$. In order to prove the theorem, it is enough to show that there exists $c_\infty > 0$ independent of $m \geq 1$ and $\mathbf{u}$ satisfying

$$|F^m(\mathbf{u}(0)) - F^m(\mathbf{u}(\tau))| > c_\infty.$$

By contradiction, suppose there exists a sequence $(m_j)_{j\geq 0}$ and a sequence of gradient or reverse-gradient flow lines $\mathbf{u}^j : [0, \tau_j] \to B_{R/2}(x, (\boldsymbol{\sigma}^{m_j}, \mathrm{id}))$, $\dot{\mathbf{u}}^j = \pm\nabla F^{m_j}(\mathbf{u}^j)$, with $\mathbf{u}^j(0) \in \partial B_{R/2}(x, (\boldsymbol{\sigma}^{m_j}, \mathrm{id}))$ and $\mathbf{u}^j(\tau_j) \in \partial B_{R/4}(x, (\boldsymbol{\sigma}^{m_j}, \mathrm{id}))$ such that

$$\left|F^{m_j}(\mathbf{u}^j(0)) - F^{m_j}(\mathbf{u}^j(\tau_j))\right| \to 0.$$

For some $1 \leq k_j \leq m_j n + 1$, one has $|u^j_{k_j}(0) - u^j_{k_j}(\tau_j)| \geq R/4$ so

$$R/4 \leq \int_0^{\tau_j} |\dot{u}^j_{k_j}(s)|\mathrm{d}s \leq \int_0^{\tau_j} |\dot{\mathbf{u}}^j(s)|\mathrm{d}s = \int_0^{\tau_j} |\nabla F^{m_j}(\mathbf{u}^j(s))|\mathrm{d}s,$$

but

$$\left(\int_0^{\tau_j} |\nabla F^{m_j}(\mathbf{u}^j(s))|\mathrm{d}s\right)^2 \leq \tau_j \int_0^{\tau_j} |\nabla F^{m_j}(\mathbf{u}^j(s))|^2\mathrm{d}s \\ = \tau_j \left(F^{m_j}(\mathbf{u}^j(0)) - F^{m_j}(\mathbf{u}^j(\tau_j))\right),$$

thus $\tau_j \to +\infty$. Combined with $\int_0^{\tau_j} |\nabla F^{m_j}(\mathbf{u}(s))|^2\mathrm{d}s \to 0$, it implies that there exists a sequence $(s_j)_{j\geq 0}$ with $s_j \in [0, \tau_j]$, such that the sequence $(\mathbf{u}^j(s_j)))_{j\geq 0}$ satisfies the hypothesis of Lemma 7.1 with $r = R/2$.

Therefore, according to Lemma 7.1, there exists a sequence $(z_j)_{j\in\mathbb{Z}} \in (\mathbb{C}^d)^{\mathbb{Z}}$ and some integer $1 \leq q \leq n$, such that $|z_j + \sigma_j(z_j) - 2x| \leq R$ which implies that $|z_j - x| \leq R/2 + R/16 < R$ by the specific choice of σ_j, $|z_q - x| > R/4$ and $z_{j+1} = \sigma_j(z_j)$. Thus, for all $k \in \mathbb{Z}$, $\Phi^k(z_1) = z_{kn+1} \in B^{2d}_R(x)$ with $z_1 \neq x$ since

$$\sigma_{q-1} \circ \cdots \circ \sigma_1(z_1) = z_q \neq x = \sigma_{q-1} \circ \cdots \circ \sigma_1(x),$$

a contradiction. □

7.2. **Crossing energy theorem in $\mathbb{C}\mathbf{P}^d$.** We employ the notation of Section 6. We recall that $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_{n_1})$ is a specific n_1-tuple, with n_1 even, associated to Φ, $\boldsymbol{\delta}_t = (g_{t/(n_2-1)}, \ldots, g_{t/(n_2-1)}, \mathrm{id})$ is a n_2-tuple, with n_2 odd, associated to $e^{-2i\pi t}$, $F^m_{1,t} = F_{(\boldsymbol{\sigma}^m, \boldsymbol{\delta}_t)}$ is a conical generating function of the conical Hamiltonian diffeomorphism $e^{-2i\pi t}\Phi^m$, $M_m := \{(t, [z]) \in I \times \mathbb{C}\mathrm{P}^{N(m)} \mid F^m_t(z) = 0\}$ is the domain of the projection map $\mathcal{T}^m : M_m \to I$ with $N(m) = (d+1)(n_1 m + n_2) - 1$. Similarly to the $\mathbb{C}^d$-case, we apply a linear change of coordinates and study the function $F^m_t := F^m_{1,t} \circ A_m^{-1}$, and by a slight abuse of notation we still denote by M_m and $\mathcal{T}^m$ the domains and functions seen in the induced projective chart.

The proof of the crossing energy theorem in $\mathbb{C}\mathrm{P}^d$ follows the same lines as the $\mathbb{C}^d$ case. First, we need an analogue to Lemma 7.1. We have to define a neighborhood of $\mathbb{C}\mathrm{P}^{N(m)}$ similar to $B(x, (\boldsymbol{\sigma}^m, \mathrm{id}))$ in the $\mathbb{C}^d$ case. Let $B_1 \subset \mathbb{C}^{d+1}$ be the unit Euclidean ball centered at the origin, so that, for $k \in \mathbb{N}^*$, $\partial(B^k_1) \subset (\mathbb{C}^{d+1})^k$ denotes the sphere

$$\partial(B^k_1) = \bigcup_{1\leq j\leq k} B^{j-1}_1 \times \mathbb{S}^{2d+1} \times B^{k-j}_1.$$

Let $\pi_m : \partial(B^{mn_1+n_2}_1) \to \mathbb{C}\mathrm{P}^{N(m)}$ be the quotient map by the diagonal action of S^1. We define now a S^1-equivariant neighborhood in the sphere $\partial(B^{mn_1+n_2}_1)$ of the normalized w-coordinates of some point $x \in \mathbb{C}^{d+1} \setminus 0$ relative to F^m_t. Let $\mathbf{a} = (a_1, \ldots, a_{mn_1+n_2}) \in (\mathbb{C}^{d+1})^{mn_1+n_2}$ be the w-coordinates of x, that is $B_r(x, (\boldsymbol{\sigma}^m, \boldsymbol{\delta}_t)) = \prod_j B_r(a_j)$. Let $\lambda > 0$ such that $\lambda\mathbf{a} \in \partial(B^{mn_1+n_2}_1)$. For $r > 1$ we define

$$U_r(x, m, t) := S^1 \cdot B_r(\lambda x, (\boldsymbol{\sigma}^m, \boldsymbol{\delta}_t)) \cap \partial(B^{mn_1+n_2}_1),$$

where $S^1 \cdot E := \{\mu z \mid z \in E,\ \mu \in S^1\}$ for any subset $E \subset (\mathbb{C}^d)^{mn_1+n_2}$. Let $V_r(x, m, t) \subset \mathbb{C}\mathrm{P}^{N(m)}$ be the projection of this neighborhood on $\mathbb{C}\mathrm{P}^{N(m)}$.

Lemma 7.3. *Let $x \in \mathbb{C}^{d+1}\backslash 0$ be a fixed point of Φ. Suppose there exists an increasing sequence of positive integers $(m_j)_{j\geq 0}$ such that there exist a sequence $(t_j)_{j\geq 0}$ in I satisfying $(t_j) \to t \in \{0, 1\}$ and a sequence $(\mathbf{w}^j)_{j\geq 0}$ with $\mathbf{w}^j \in U_r(x, m_j, t)\backslash U_{r/2}(x, m_j, t)$*

satisfying,

$$\left|\nabla F_{t_j}^{m_j}\left(\mathbf{w}^j\right)\right|^2 = \sum_{k=1}^{m_j n_1+n_2} \left|\partial_{w_k} F_{t_j}^{m_j}\left(\mathbf{w}^j\right)\right|^2 \xrightarrow{j\to\infty} 0.$$

Let $\mathbf{a} := (a_1, \dots, a_{n_1}) \in (\mathbb{C}^{d+1})^{n_1}$ *be such that* $B_r(x, \boldsymbol{\sigma}) = \prod_j B_r(a_j)$ *and* $\tilde{\mathbf{a}} := (\tilde{a}_1, \dots, \tilde{a}_{n_2}) \in (\mathbb{C}^{d+1})^{n_2}$ *be the* n_2*-tuple*

$$\tilde{a}_k := \frac{g_{(k-1)t/(n_2-1)}(x) + g_{kt/(n_2-1)}(x)}{2} \quad \textit{for} \quad 1 \le k \le n_2 - 1 \quad \textit{and} \quad \tilde{a}_{n_2} = x.$$

Then, there exists a possibly infinite integer $\kappa \in \mathbb{Z} \cup \{+\infty\}$ *such that there exists a sequence* $(b_j) \in (\mathbb{C}^{d+1})^{\mathbb{Z}}$ *defined by*

$$b_j := \begin{cases} a_{j \bmod n_1} & \textit{if } j \le \kappa n_1, \\ \tilde{a}_{j-\kappa n_1} & \textit{if } \kappa n_1 + 1 \le j \le \kappa n_1 + n_2, \\ a_{j-n_2 \bmod n_1} & \textit{if } j \ge \kappa n_1 + n_2 + 1, \end{cases}$$

a sequence $(z_j)_{j\in\mathbb{Z}} \in (\mathbb{C}^{d+1} \setminus 0)^{\mathbb{Z}}$ *satisfying*

$$z_{j+1} = \begin{cases} \sigma_{j \bmod n_1}(z_j) & \textit{if } j \le \kappa n_1, \\ g_{t/n_2}(z_j) & \textit{if } \kappa n_1 + 1 \le j \le \kappa n_1 + n_2, \\ \sigma_{j-n_2 \bmod n_1}(z_j) & \textit{if } j \ge \kappa n_1 + n_2 + 1, \end{cases}$$

and some integer $1 \le q \le n_1 + n_2$ *such that*

$$\begin{cases} \frac{z_j + z_{j+1}}{2} \in \mathbb{C} \cdot \overline{B_r^{2d}(b_j)} & \textit{for all } j \in \mathbb{Z}, \\ \frac{z_q + z_{q+1}}{2} \notin \mathbb{C} \cdot B_{r/2}^{2d}(b_q) \end{cases}$$

Proof. The proof follows along the same lines as Lemma 7.1 with just additional calligraphic difficulties, we will only underline the key changes.

Let $x \in \mathbb{C}^{d+1} \setminus 0$, $\mathbf{a} \in (\mathbb{C}^{d+1})^{n_1}$ and $\tilde{\mathbf{a}} \in (\mathbb{C}^{d+1})^{n_2}$ satisfying the assumptions of the lemma. Let $\lambda > 0$ be such that $(\lambda\mathbf{a}, \lambda\tilde{\mathbf{a}}) \in \partial(B_1^{n_1+n_2})$ (it exists since $x \neq 0$), then

$$U_r(x, m, t) = S^1 \cdot \left[\left(\prod_{k=1}^{n_1} B_r^{2(d+1)}(\lambda a_k)\right)^m \times \prod_{k=1}^{n_2} B_r^{2(d+1)}(\lambda \tilde{a}_k)\right] \cap \partial(B_1^{mn_1+n_2}).$$

Let $(\mathbf{w}^j)$ be satisfying the assumptions of the lemma. By S^1-invariance of the function $|\partial F_{t_j}^{m_j}|$ and the neighborhood $U_r(x, m_j, t)$, we can suppose that

$$\mathbf{w}^j \in \left(\prod_{k=1}^{n_1} B_r^{2(d+1)}(\lambda a_k)\right)^{m_j} \times \prod_{k=1}^{n_2} B_r^{2(d+1)}(\lambda \tilde{a}_k).$$

The result follows from Cantor's diagonal argument applied to the sequence $(\mathbf{w}^j/\lambda)$ in the same way as in the proof of Lemma 7.1. □

In order to state the crossing energy theorem in $\mathbb{C}\mathrm{P}^d$, we will need to define a "good" pseudo-gradient X_m for the function $\mathcal{T}^m$. For technical reasons, the projection $\pi_m : \partial(B_1^{mn_1+n_2}) \to \mathbb{C}\mathrm{P}^{N(m)}$ is the most natural for our problem. However the sphere $\partial(B_1^{mn_1+n_2})$ is not smooth, we thus introduce a smooth S^1-invariant sphere $\Sigma_m \subset (\mathbb{C}^{d+1})^{mn_1+n_2}$:

$$\Sigma_m := \left\{\mathbf{z} \in (\mathbb{C}^{d+1})^{mn_1+n_2} \mid \sum_{k=1}^{mn_1+n_2} |z_k|^{p_m} = 1\right\},$$

where $p_m \geq 2$ is chosen such that,

$$\forall \mathbf{z} \in \Sigma_m, \exists \lambda \in [1,2], \quad \lambda \mathbf{z} \in \partial(B_1^{mn_1+n_2}),$$

(necessarily $(p_m) \to \infty$). We endow $\mathbb{C}\mathrm{P}^{N(m)}$ with the Riemannian metric induced by the S^1-invariant projection $\pi'_m : \Sigma_m \to \mathbb{C}\mathrm{P}^{N(m)}$. Since

$$\operatorname{dist}(\partial U_r(x,m,t), U_{r/2}(x,m,t)) \geq r/2,$$

the condition on p_m implies that

$$\operatorname{dist}(\partial V_r(x,m,t), V_{r/2}(x,m,t)) \geq r/4. \tag{30}$$

Let $f^m : I \times \mathbb{C}\mathrm{P}^{N(m)} \to \mathbb{R}$ be the C^1 function satisfying $f^m(t, \pi'_m(\mathbf{z})) = F_t^m(\mathbf{z})$ for all $\mathbf{z} \in \Sigma_m$, so that $M_m = \{(t,\zeta) \in I \times \mathbb{C}\mathrm{P}^{N(m)} \mid f^m(t,\zeta) = 0\}$. The pseudo-gradient X_m of $\mathcal{T}^m$ is defined by

$$X_m(t,\zeta) := \partial_t f^m(t,\zeta) \nabla f^m(t,\zeta) - |\nabla f^m(t,\zeta)|^2 \frac{\partial}{\partial t}$$

We have $\left\langle X_m, -\frac{\partial}{\partial t} \right\rangle \geq 0$ with equality if and only if $\nabla f^m = 0$, that is to say $\mathrm{d}\mathcal{T}^m = 0$.

Theorem 7.4. *Let $\Phi \in \mathrm{Ham}_{\mathbb{C}}(\mathbb{C}^{d+1})$ be a lift of $\varphi \in \mathrm{Ham}(\mathbb{C}\mathrm{P}^d)$. Suppose that $x \in \mathbb{C}^{d+1} \setminus 0$ is a fixed point of Φ such that $[x] \in \mathbb{C}\mathrm{P}^d$ is isolated as an invariant set of φ. Then for every sufficiently small $r > 0$, there exists $c_\infty > 0$ and a tuple $\boldsymbol{\sigma}$ associated to Φ such that for all $m \geq 1$, if $(t, \zeta_t^m) \in M_m$ denotes the critical point of $\mathcal{T}^m$ with critical value $t \in \{0,1\}$ associated to x, any gradient flow line $\mathbf{u} : \mathbb{R} \to M_m$, $\dot{\mathbf{u}} = \pm X_m(\mathbf{u})$, with $\mathbf{u}(0) \in I \times \partial V_r(x,m,t)$ and $\mathbf{u}(\tau) \in I \times V_{r/2}(x,m,t)$ for some $\tau \in \mathbb{R}$ satisfies*

$$|\mathcal{T}^m(\mathbf{u}(0)) - \mathcal{T}^m(\mathbf{u}(\tau))| > c_\infty.$$

The pseudo-gradient X_m can be replaced by a pseudo-gradient C^0-close to it, e.g. *a Morse-Smale adapted pseudo-gradient if $\mathcal{T}^m$ is a Morse function.*

Proof. We follow the steps of the proof of Theorem 7.2. By contradiction, suppose that there exists a sequence $(m_j)_{j\geq 0}$ and a sequence of pseudo-gradient flow line $\mathbf{u}^j : [0, \tau_j] \to U_r(x, m_j, t)$, $\dot{\mathbf{u}}^j = \pm X_{m_j}(\mathbf{u}^j)$ with $\mathbf{u}^j(0) \in \partial V_r(x,m_j,t)$ and $\mathbf{u}^j(\tau_j) \in V_{r/2}(x,m_j,t)$ such that

$$\left|\mathcal{T}^{m_j}(\mathbf{u}^j(0)) - \mathcal{T}^{m_j}(\mathbf{u}^j(\tau_j))\right| \xrightarrow{j\to+\infty} 0 \quad \text{and} \quad \mathcal{T}^{m_j}(\mathbf{u}^j(0)) \xrightarrow{j\to+\infty} 0.$$

First we must show that $\tau_j \not\to 0$. Let $p_2 : I \times \mathbb{C}\mathrm{P}^N \to \mathbb{C}\mathrm{P}^N$ be the projection onto the second factor, then (30) implies that

$$\frac{r}{4} \leq \int_0^{\tau_j} |\mathrm{d}p_2 \cdot \dot{\mathbf{u}}^j| \mathrm{d}s,$$

so

$$\left(\frac{r}{4}\right)^2 \leq \tau_j \int_0^{\tau_j} |\mathrm{d}p_2 \cdot X_{m_j}(\mathbf{u}^j)|^2 \mathrm{d}s = \tau_j \int_0^{\tau_j} (\partial_t f^{m_j}(\mathbf{u}^j))^2 |\nabla_\zeta f^{m_j}(\mathbf{u}^j)|^2 \mathrm{d}s.$$

Remark that there exists some $C > 0$ independent of m (it only depends on $(\boldsymbol{\delta}_t)$) such that $0 \leq -\partial_t f^m < C$, thus

$$\int_0^{\tau_j} |\mathrm{d}p_2 \cdot X_{m_j}(\mathbf{u}^j)|^2 \mathrm{d}s \leq C^2 \int_0^{\tau_j} |\nabla_\zeta f^{m_j}(\mathbf{u}^j)|^2 \mathrm{d}s.$$

This last term goes to 0 since

$$\left|\mathcal{T}^{m_j}(\mathbf{u}^j(0)) - \mathcal{T}^{m_j}(\mathbf{u}^j(\tau_j))\right| = \int_0^{\tau_j} \left\langle -\frac{\partial}{\partial t}, X_{m_j}(\mathbf{u}) \right\rangle \mathrm{d}s = \int_0^{\tau_j} |\nabla_\zeta f^{m_j}(\mathbf{u}^j)|^2 \mathrm{d}s.$$

Therefore, $\tau_j \to +\infty$ and thus there exists a sequence $(s_j)_{j\geq 0}$ in $I \times V_r(x,m,t) \setminus I \times V_{r/2}(x,m,t)$ such that $|\nabla_\zeta f^{m_j}(s_j)| \to 0$.

Let $(t^j; \lambda_j \mathbf{w}^j) \in I \times U_r(x, m_j, t)$ be lifted from s_j with $\mathbf{w}^j \in \Sigma_{m_j}$ and $\lambda_j \in [1,2]$ such that $\lambda_j \mathbf{w}^j \in \partial(B_1^{m_j n_1 + n_2})$ (which exists by definition of Σ_{m_j}). Since $t^j = \mathcal{T}^{m_j}(s_j)$, one has $t^j \to t$. Since $|\nabla_\zeta f^{m_j}(s_j)| \to 0$, the norm of the orthogonal projection of $\nabla F_{t^j}(\mathbf{w}^j) \in \mathbb{C}^{N(m_j)+1}$ on the sphere Σ_{m_j} goes to zero as $j \to \infty$. The radial component is $\langle \mathbf{w}^j, \nabla F_{t^j}(\mathbf{w}^j) \rangle = 2F_{t^j}(\mathbf{w}^j) = 0$, hence $|\nabla F_{t^j}^{m_j}(\mathbf{w}^j)| \to 0$. Since $\lambda_j \in [1/2, 1]$, the homogeneity of $F_{t^j}^{m_j}$ implies that

$$\left|\nabla F_{t^j}^{m_j}(\lambda_j \mathbf{w}^j)\right| \xrightarrow{j\to\infty} 0.$$

We can thus apply Lemma 7.3 to the sequences (m_j), $(\lambda_j \mathbf{w}^j)$ and the fixed point $x \in \mathbb{C}^{d+1}$. We then find a sequence $(z_j)_{j\in\mathbb{Z}}$ in $\mathbb{C}^{d+1}$ such that $\varphi^k([z_0])$ stays close to $[x]$ for all $k \in \mathbb{Z}$ with $[z_0] \neq [x]$. □

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