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▶ To cite this version:

Simon Allais, Tobias Soethe. Homologically visible closed geodesics on complete surfaces. Journal of Topology and Analysis, 2021, 16 (02), pp.181-203. 10.1142/S1793525321500692 . hal-03455611

HAL Id: hal-03455611
https://hal.science/hal-03455611v1

Submitted on 29 Nov 2021

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# HOMOLOGICALLY VISIBLE CLOSED GEODESICS ON COMPLETE SURFACES

SIMON ALLAIS AND TOBIAS SOETHE

Abstract. In this article, we give multiple situations when having one or two geometrically distinct closed geodesics on a complete Riemannian cylinder, a complete Möbius band or a complete Riemannian plane leads to having infinitely many geometrically distinct closed geodesics. In particular, we prove that any complete cylinder with isolated closed geodesics has zero, one or infinitely many homologically visible closed geodesics; this answers a question of Alberto Abbondandolo.

## 1. Introduction

The problem of the existence and multiplicity of closed geodesics plays an important role in both Riemannian geometry and dynamics. Going back to Hadamard and Poincaré [23, 27], it is still open for a large class of Riemannian manifolds. Given a complete Riemannian manifold $(M, g)$, a famous question is whether it possesses a closed geodesic for every Riemannian metric $g$. This is always true if $M$ is closed [10, 25, 17]. We can then ask whether the number of closed geodesics is infinite or not. It is known that every closed surface has infinitely many geometrically distinct closed geodesics [18, 5, 24]. However, this question is still open for spheres of higher dimension. In this article, we are interested in non-compact complete Riemannian surfaces for which even the existence of one closed geodesic fails in general: planes and cylinders (we also study the Möbius band that can have only one closed geodesic). For instance, the Euclidean plane does not possess any closed geodesic. Nevertheless, under specific geometric conditions, interesting results can be stated. In 1980, Bangert proved that any complete Riemannian cylinder, plane or Möbius band of finite area has infinitely many closed geodesics [4]. For the plane and the cylinder he proved the same result even under the weaker assumption of just the existence of a convex neighborhood of infinity. We will discuss this result in greater depth as it is used extensively in our proofs. The purpose of this article is to give simple conditions under which the existence of one or two distinct closed geodesics implies that a complete Riemannian cylinder, Möbius band or plane contains infinitely many geometrically distinct closed geodesics.

Let $S^1 := \mathbb{R}/\mathbb{Z}$ and let $M \simeq S^1 \times \mathbb{R}$ be a complete Riemannian cylinder. Let $\Lambda M$ be its loop space. Two loops $\alpha, \beta \in \Lambda M$ are said to be geometrically distinct if their images are distinct: $\alpha(S^1) \neq \beta(S^1)$. Throughout the article, by writing that two closed geodesics are distinct we will always mean that they are geometrically distinct. Given a ring $R$, a closed geodesic $\gamma \in \Lambda M$ is said to be homologically visible over $R$ if the local homology of the critical circle $S^1 \cdot \gamma \subset \Lambda M$ of the energy functional is non-zero over the coefficients ring $R$ (see Section 2 for precise definitions). With

*Date*: November 22, 2021.
2010 *Mathematics Subject Classification.* 53C22, 58E10.
*Key words and phrases.* closed geodesics, cylinder, min-max, Möbius band.

the exception of the Möbius band, every result is true over any coefficients ring $R$ (once fixed) so the ring $R$ will not be mentioned explicitly.

**Theorem 1.1.** *Let $M$ be a complete Riemannian cylinder where all closed geodesics are isolated and assume one of the following hypotheses:*

1. *there exists a contractible closed geodesic,*
2. *there exists a self-intersecting closed geodesic,*
3. *there exist two distinct closed geodesics that intersect,*
4. *there exists a closed geodesic of non-zero average index,*
5. *there exist two homologically visible closed geodesics.*

*Then $M$ contains infinitely many homologically visible closed geodesics intersecting some common compact set $K \subset M$ and at least one without self-intersection.*

Notice that according to Bott iteration theory, a closed geodesic $c$ has a non-zero average index if and only if some iterate $c^m$ has a non-zero index. The fact that hypothesis 5 implies that there exists infinitely many homologically visible closed geodesics proves a conjecture of Abbondandolo:

**Corollary 1.2.** *Any complete Riemannian cylinder where all closed geodesics are isolated has zero, one or infinitely many homologically visible closed geodesics.*

By essentially taking the double cover (see Section 7 for details), one can thus deduce the following counterpart of Theorem 1.1 when $M$ is a complete Möbius band.

**Corollary 1.3.** *Let $M$ be a complete Riemannian Möbius band where all closed geodesics are isolated and assume one of the following hypotheses:*

1. *there exists a contractible closed geodesic,*
2. *there exists a self-intersecting closed geodesic,*
3. *there exist two distinct closed geodesics that intersect,*
4. *there exists a closed geodesic of non-zero average index,*
5. *there exist two homologically visible closed geodesics over $\mathbb{Z}/2\mathbb{Z}$.*

*Then $M$ contains infinitely many closed geodesics intersecting some common compact set $K \subset M$ that are homologically visible over $\mathbb{Z}/2\mathbb{Z}$.*

According to Thorbergsson [29, Theorem 3.2], any complete Möbius band has at least one homologically visible closed geodesic without self-intersection (it is homologically visible as a local minimum of the energy, see Section 2 below).

**Corollary 1.4.** *Any complete Riemannian Möbius band where all closed geodesics are isolated has one or infinitely many homologically visible closed geodesics over $\mathbb{Z}/2\mathbb{Z}$.*

Similar results can also be obtained when $M \simeq \mathbb{R}^2$ is a complete plane:

**Theorem 1.5.** *Let $M$ be a complete Riemannian plane where all closed geodesics are isolated and assume one of the following hypotheses:*

1. *there exists a self-intersecting closed geodesic,*
2. *there exist two distinct closed geodesics that intersect,*
3. *there exists a closed geodesic of non-zero average index,*
4. *there exists a homologically visible closed geodesic.*

*Then $M$ contains infinitely many homologically visible closed geodesics intersecting some common compact set $K \subset M$ and at least one without self-intersection.*

**Corollary 1.6.** *Any complete Riemannian plane where all closed geodesics are isolated has zero or infinitely many homologically visible closed geodesics.*

It is easy to give counter-examples to Theorem 1.1 when none of the assumptions 1-5 holds by considering embedded cylinders of revolution

$$(\theta, z) \mapsto (r(z)\cos\theta, r(z)\sin\theta, z), \tag{1}$$

for well-chosen smooth maps $r : \mathbb{R} \to (0, +\infty)$. A complete cylinder may have no closed geodesic at all: take $r' > 0$. It can have an arbitrary large finite number $k \in \mathbb{N}$ of homologically invisible closed geodesics: take $r'(z) > 0$ for all $z \in \mathbb{R} \setminus \{z_1, \dots, z_k\}$ and $r'(z_i) = 0$. It can also have a unique visible closed geodesic: take $r' < 0$ on $(-\infty, 0)$, $r'(0) = 0$ and $r' > 0$ on $(0, +\infty)$ (one can as well add to this cylinder an arbitrary large finite number of homologically invisible closed geodesics the same way as before). By taking such an $r$ even and taking the quotient under the involution $(\theta, z) \mapsto (\theta + \pi, -z)$, one gets Möbius bands with only one homologically visible closed geodesic and as many homologically invisible closed geodesic as wanted. Remark that in our examples closed geodesics are without self-intersections and not contractible as implied by the theorem. Counter-examples where the theorem fails by lack of completeness can be found as well by choosing embedded cylinders of revolution restricting the domain of the embedding (1) to $(\theta, z) \in \mathbb{R}/2\pi\mathbb{Z} \times (a, b)$ for $a, b \in \mathbb{R}$. We could proceed as follows: take an even $r : [-1, 1] \to (0, +\infty)$ with $r' > 0$ on $[-1, 0)$ such that $z = 0$ is the only closed geodesic of the associated compact embedded cylinder. One can find such an $r$ by slightly modifying a Tannery surface: a sufficient condition is that the metric $g$ on the interior of the cylinder can be written as

$$g = [\alpha + h(\cos\rho)]^2 \, \mathrm{d}\rho^2 + \sin^2\rho \, \mathrm{d}\theta^2,$$

for a good choice of coordinates $(\rho, \theta) \in (0, \pi) \times S^1$, where $\alpha$ is irrational and $h : (-1, 1) \to (-\alpha, \alpha)$ is a smooth odd function (see for instance [9, Theorem 4.13]). Then extend $r$ to a smooth map $(-3, 1] \to (0, +\infty)$ with $r|_{(-3,-1)} < r(-1)$, $r' < 0$ on $(-3, -2)$ and $r' > 0$ on $(-2, -1)$. Then $z = -2$ and $z = 0$ are the only closed geodesic of the cylinder embedded by $r|_{(-3,1)}$ and are both visible.

In a similar way, we can give examples of complete planes with only an arbitrary finite number of homologically invisible closed geodesics by considering surfaces of revolution (1) parametrized by $\mathbb{R}/2\pi\mathbb{Z} \times [0, +\infty)$ with $r : [0, +\infty) \to [0, +\infty)$ being increasing and smooth on $(0, +\infty)$ with $r(0) = 0$ and $r'(z) \to +\infty$ when $z \to 0$ in a suitable way (*i.e.* so that the surface is smooth at the origin). Then, as above, we get homologically invisible closed geodesics on the inflexion points of $r$, and nowhere else.

We say that $C_- \subset M$ (resp. $C_+$) is a neighborhood of $-\infty$ (resp. of $+\infty$) if $C_-$ contains $S^1 \times (-\infty, a)$ for some $a \in \mathbb{R}$ (resp. $S^1 \times (b, +\infty)$ for some $b \in \mathbb{R}$) for an arbitrarily fixed identification of $M$ with $S^1 \times \mathbb{R}$. In these terms, Bangert proved the following theorem (where for the notion of local convexity, we refer to Section 2.2):

**Theorem 1.7** ([4, Theorem 3, Remark 2])**.** *Let $M$ be a complete Riemannian cylinder where all closed geodesics are isolated and suppose there exist locally convex closed neighborhoods $C_-$ and $C_+$ with disjoint interior of $-\infty$ and $+\infty$ respectively such that the boundaries $\partial C_\pm$ are not totally geodesic. Then $M$ contains infinitely*

*many (homologically visible) closed geodesics intersecting a common compact set $K$ and at least one without self-intersections.*

In order to prove Theorem 1.1, we will extensively use a variation of the above theorem adapted to our problem. The desired applications require us to work with open neighbourhoods of infinity. This in turn necessitates a slighty different convexity property to be satisfied, which we call Property (C) (and which we introduce properly in Section 2.2, where it is discussed with its connections to local convexity). To avoid technicalities, we impose some slightly stronger assumptions (which will always be satisfied in our applications) to state the following modified version of Bangert's theorem:

**Theorem 1.8.** *Let $M$ be a complete Riemannian cylinder where all closed geodesics are isolated and suppose there exist disjoint open neighborhoods $C_-$ and $C_+$ of $-\infty$ and $+\infty$ respectively satisfying Property* (C) *and such that the boundaries $\partial C_\pm$ do not contain a simple closed geodesic. Then $M$ contains infinitely many homologically visible closed geodesics intersecting $K = M \setminus (C_- \cup C_+)$ and at least one without self-intersections.*

Our modified statement can be proven in the same way as the original theorem. However, since Bangert did not give the precise proof of Theorem 1.7 (but rather for its analogue in case of the plane), for the sake of completeness we give a comprehensive proof of Theorem 1.8 in the paper. The proof of Theorem 1.5 is quite similar and relies extensively on the analogous theorem of Bangert when $M$ is a plane where all closed geodesics are isolated: if there exists a locally convex open neighborhood of infinity $C \neq M$ with a boundary $\partial C$ which is not totally geodesic, $M$ contains infinitely many homologically visible closed geodesics [4, Theorem 3]. These two theorems were originally used by Bangert to prove that any complete Riemannian plane of finite area has infinitely many closed geodesics.

In fact, our results extend *verbatim* to the case where $M$ is a complete reversible Finsler manifold as we will essentially use variational properties of geodesics in our proof with no concern for geometric notion specific to Riemannian manifold. However, nothing can be said concerning the more general case of a complete (asymmetrical) Finsler manifold. The major issue is that, in the asymmetrical case, a closed subset of $M$ which is bounded by a geodesic is not locally convex (and neither does an open such set satisfy Property (C)). In this direction, we point out that the related question of whether or not infinitely many closed geodesics exist on every irreversible Finsler cylinder of finite area is still open [13, Question 2.3.2].

In order to put these results in perspective, we recall some known results concerning existence of closed geodesics on complete non-compact Riemannian manifolds. In 1978, Thorbergsson proved the existence of closed geodesics on a complete Riemannian manifold $M$ if it contains a convex compact set which is not homotopically trivial or if $M$ has a non-negative sectional curvature outside some compact set [29]. In the 1990s, Benci and Giannoni proved that any complete $d$-dimensional Riemannian manifold such that the limit superior of its sectional curvature at infinity is non-positive and the homology of its free loop space is non-trivial in some degree larger than $2d$ possesses a closed geodesic [7, 8]. In 2017, Asselle and Mazzucchelli showed the existence of infinitely many closed geodesics for complete $d$-dimensional Riemannian manifolds which have no close conjugate points at infinity and a free loop space with unbounded Betti numbers in degrees larger than $d$ [1]. They also

improved the result of Benci and Giannoni by replacing the asymptotic curvature assumption by an assumption on the conjugated points at infinity and by improving the bound on the homology of the free loop space. The existence of one closed geodesic in any complete Riemannian manifold of finite volume is a hard open problem (see for instance the following recent review of the subject [13]).

**Organization of the paper.** In Section 2, we fix notation and recall results of the variational theory of geodesics that we will need. In Section 3, we give a comprehensive proof of Theorem 1.8. In Section 4, we prove Theorem 1.1 when hypothesis 1, 2 or 3 is assumed. In Section 5, we prove Theorem 1.1 when hypothesis 4 is assumed. In Section 6, we prove the last case of Theorem 1.1. In Section 7, we prove Corollary 1.3. In Section 8, we prove Theorem 1.5.

**Acknowledgments.** This work started while the first author was visiting the Ruhr-Universität Bochum (Germany) in October 2019. The first author wishes to thank Alberto Abbondandolo and Stefan Suhr for the invitation, and the Ruhr-Universität Bochum for providing an excellent working environment. The first author is also grateful to his advisor Marco Mazzucchelli who introduced him to the result conjectured by Alberto Abbondandolo. To the latter the second author expresses his gratitude for his continuing help and advice. The second author is partially supported by the SFB/TRR 191 'Symplectic Structures in Geometry, Algebra and Dynamics', funded by the DFG (Projektnummer 281071066 – TRR 191). Both authors are thankful to the anonymous referee for his careful reading of the preprint and for providing important comments and suggestions.

## 2. Preliminaries

In this section, we recall some results of Riemannian geometry that we will use in our proofs and fix some notation. For the extension of these notions to the Finsler case, the reader may for instance look at [14, Section 2].

2.1. **The energy functional.** Given a complete Riemannian manifold with boundary $W$, we denote by $\Lambda W$ the space of $H^1$-maps $S^1 \to W$. In fact, if one wants to avoid analytic questions, we can always reduce our space to a finite-dimensional manifold of broken geodesics. For $\gamma \in \Lambda W$ and $m \in \mathbb{N}^*$, the iterated loop $\gamma^m \in \Lambda W$ is defined by $t \mapsto \gamma(mt)$. A geodesic is an immersed path $\gamma : \mathbb{R} \to W$ such that

$$\nabla_{\dot\gamma}\dot\gamma = 0,$$

where $\nabla$ denotes the Levi-Civita connection of the metric and $\dot\gamma$ stands for the derivative of $\gamma$. Therefore, in our convention, geodesics have constant speed. A closed geodesic is a geodesic $\gamma$ which is periodic: $\gamma(t+1) = \gamma(t)$ so that $\gamma \in \Lambda W$. Closed geodesics of $W$ are the critical points with non-zero critical value of the energy functional $E : \Lambda W \to [0, +\infty)$,

$$E(\gamma) := \int_{S^1} g_\gamma(\dot\gamma, \dot\gamma)\mathrm{d}t, \quad \forall \gamma \in \Lambda W.$$

The energy functional $E$ is $C^2$. If $W$ is a compact manifold (possibly with boundary), $E$ also satisfies the Palais-Smale condition and the $(-\nabla E)$-flow is defined for all time $t \geq 0$ (if $f$ is a real-valued on a Riemann-Hilbert manifold, $\nabla f$ denotes its gradient). We notice that every closed geodesic lies on a critical circle $S^1 \cdot \gamma$, where $S^1$ acts on $\Lambda W$ by $t \cdot \gamma := \gamma(t + \cdot)$. In our study we assume that $E$ has only isolated critical

circles (except for the constant loops which have zero value). Two closed geodesics $c_1$ and $c_2$ are said to be geometrically distinct if they do not have the same image in $W$.

2.2. **Finite-dimensional approximation of the loop space.** Morse's finite-dimensional approximation of the curve space over $W$, as presented by Bangert in [4] consists of the following data: an open set $\mathcal{O} \subset W$, an energy bound $\kappa > 0$ and a parameter $j \in \mathbb{N}$ satisfying $\frac{1}{j} < \frac{\varepsilon^2}{\kappa}$ where $\varepsilon > 0$ is smaller than the injectivity radius on $\mathcal{O}$. The positivity of $\varepsilon$ will be fulfilled if for instance $\mathcal{O}$ has compact closure, as will be the case in our considerations. The finite-dimensional approximation $\Omega = \Omega(\mathcal{O}, \kappa, j)$ is constructed as follows: it is the set of all curves $\gamma \in \Lambda W$ such that $E(\gamma) < \kappa$, $\gamma(i/j) \in \mathcal{O}$ and such that $\gamma|_{[i/j,(i+1)/j]}$ is a geodesic of length less than $\varepsilon$ for $0 \leq i \leq j-1$.

We call a closed subset $C$ of a manifold locally convex, if every point possesses a neighborhood $U \subset C$ such that any two sufficiently close points in $U$ can be joined by a unique geodesic which is entirely contained in $U$. We only define local convexity for closed sets, as open sets are always satisfying this property. Let $\Omega$ be a finite-dimensional approximation of $\Lambda W$ and $C \subset W$ a closed locally convex set with compact boundary such that $C \subset \mathcal{O}$. Then there exists an $\varepsilon > 0$ such that for any two points $p, q \in C$ with Riemannian distance $d(p,q) < \varepsilon$, if there exists a unique geodesic of length $= d(p,q)$ joining $p$ and $q$, it is contained entirely in $C$ [4, p. 85]. The negative gradient of the restriction of the energy functional to $\Omega$ is given by

$$-\nabla E|_\Omega(\gamma) = -2\Big(\dot\gamma_1(1/j) - \dot\gamma_2(1/j), \ldots, \dot\gamma_{j-1}((j-1)/j) - \dot\gamma_j((j-1)/j)\Big)$$

for $\gamma \in \Omega$, where $\gamma_i = \gamma|_{[(i-1)/j,i/j]}$ for $1 \leq i \leq j$ (see [20, p. 252]). Now from our choice of $j$ and Cauchy-Schwarz inequality, we get

$$d(\gamma((i-1)/j), \gamma(i/j))^2 \leq \frac{1}{j} E(\gamma|_{((i-1)/j,i/j)}) \leq \frac{\varepsilon^2}{\kappa}\kappa = \varepsilon^2 .$$

Consequently, by local convexity of $C$, the negative gradient flow of the finite-dimensional approximation of the energy functional respects $C$, i.e. $\gamma(S^1) \subset C$ implies $\phi_t(\gamma)(S^1) \subset C$ for all $t \geq 0$, where $\phi_t$ denotes the negative gradient flow (in finite-dimensional approximation).

We would like to use this property on slightly more general sets. Therefore we denote $B_r(p) := \{q \in W \mid d(p,q) < r\}$ and state the following property for a subset $C \subset W$.

> *There exists an $\varepsilon > 0$ such that if $p, q$ are elements in the same connected component of $B_\varepsilon(p) \cap \mathrm{int}(C)$, then whenever a unique geodesic connecting $p$ and $q$ exists, it is contained in that connected component.* (C)

Notice, that closed locally convex sets with compact boundary satisfy this property. Moreover, so do open sets whose boundary coincides with a broken geodesic with convex vertices.

The above discussion carries over and we have that the negative gradient flow of the finite-dimensional approximation of the energy functional respects sets that satisfy Property (C).

2.3. **Index of a closed geodesic.** The index of a closed geodesic $\gamma$ is the Morse index of $E$:

$$\mathrm{ind}(\gamma) := \mathrm{ind}(E, \gamma).$$

It is always finite. The behavior of this index under iteration $k \mapsto \mathrm{ind}(\gamma^k)$ was extensively studied by Bott in [12]. We simply recall that

$$\mathrm{ind}(\gamma^k) \geq k\,\overline{\mathrm{ind}}(\gamma) - \dim(W) + 1, \quad k \in \mathbb{N}, \tag{2}$$

where $\overline{\mathrm{ind}}(\gamma) \geq 0$ is the average index of $\gamma$ defined by

$$\overline{\mathrm{ind}}(\gamma) := \lim_{k\to\infty} \frac{\mathrm{ind}(\gamma^k)}{k}.$$

Let $p \in W$ and $\Omega_p W \subset \Lambda W$ be the set of loops based at $p$, that is $H^1$-paths $\gamma : [0,1] \to W$ such that $\gamma(0) = \gamma(1) = p$. Given a closed geodesic $\gamma \in \Lambda W$, we denote by $\mathrm{ind}_\Omega(\gamma) \in \mathbb{N}$ the Morse index

$$\mathrm{ind}_\Omega(\gamma) := \mathrm{ind}\left(E|_{\Omega_{\gamma(0)}W}, \gamma\right).$$

By inclusion, $\mathrm{ind}_\Omega(\gamma) \leq \mathrm{ind}(\gamma)$. In fact, we have the concavity inequality [3, Eq. (1.5)]:

$$\mathrm{ind}(\gamma) - \dim(W) + 1 \leq \mathrm{ind}_\Omega(\gamma) \leq \mathrm{ind}(\gamma). \tag{3}$$

A Jacobi field of the geodesic path $\gamma$ is a smooth map $J : \mathbb{R} \to \gamma^* TW$, satisfying

$$J(t) \in T_{\gamma(t)}W, \quad \forall t \in \mathbb{R} \quad \text{and} \quad \nabla^2_{\dot\gamma} J = \mathrm{R}(\dot\gamma, J)\dot\gamma,$$

where R denotes the Riemann tensor. Let $\mu(t)$ be the number of linearly independent Jacobi fields of $\gamma$ such that $J(0) = J(t) = 0$; the Morse index theorem states that

$$\mathrm{ind}_\Omega(\gamma) = \sum_{0<t<1} \mu(t). \tag{4}$$

The local homology of an isolated critical circle $S^1 \cdot \gamma$ over a ring $R$ is by definition

$$\mathrm{C}_*(S^1 \cdot \gamma; R) := H_*(\{E < E(\gamma)\} \cup S^1 \cdot \gamma, \{E < E(\gamma)\}; R),$$

where the set $\{E < E(\gamma)\}$ is $\{\delta \in \Lambda W \mid E(\delta) < E(\gamma)\}$, and $H_*$ denotes the singular homology. When the choice of the fixed ring $R$ is irrelevant, the symbol $R$ will omitted. According to the Gromoll-Meyer theory, local homology groups are finitely generated (see [21, remark following Lemma 1] for the case of an isolated critical point and [2, Proposition 3.1] for the reduction to this case). A closed geodesic is said to be homologically visible if $\mathrm{C}_*(S^1 \cdot \gamma) \neq 0$ and is said to be homologically invisible otherwise. Although this notion depends on the choice of coefficients ring $R$, the universal coefficients theorem implies that a closed geodesic is homologically invisible over every ring $R$ if and only if it is homologically invisible over $\mathbb{Z}$. By excision, for all neighborhood $U \subset \Lambda W$ of $S^1 \cdot \gamma$,

$$\mathrm{C}_*(S^1 \cdot \gamma) \simeq H_*\left(U \cap \left(\{E < E(\gamma)\} \cup S^1 \cdot \gamma\right), U \cap \{E < E(\gamma)\}\right). \tag{5}$$

Therefore, every local minimum of $E$ is homologically visible. We will be interested in the properties of the local homology $\mathrm{C}_*(S^1 \cdot \gamma)$ especially in the case where $\gamma$ is a closed geodesic of average index $\overline{\mathrm{ind}}(\gamma) = 0$ and whose image $\gamma(S^1)$ lies in the interior of $W$ ($\overline{\mathrm{ind}}(\gamma) = 0$ is equivalent to the fact that $\mathrm{ind}(\gamma^m)$ vanishes for all $m \geq 1$). Let $\gamma \in \Lambda W$ be such a closed geodesic. Given $m \in \mathbb{N}$, we denote by $\psi_m : \Lambda W \to \Lambda W$ the iteration map $\psi_m(\delta) := \delta^m$. According to a theorem of Gromoll-Meyer [22,

Theorem 3], the local homology $\mathrm{C}_d(S^1\cdot\gamma)$ is zero in degrees $d \geq 2\dim W$ and there exist infinitely many positive integers $m$ such that the induced map in homology

$$(\psi_m)_* : \mathrm{C}_*(S^1\cdot\gamma) \to \mathrm{C}_*(S^1\cdot\gamma^m) \tag{6}$$

is an isomorphism. On the other hand, a theorem of Bangert-Klingenberg [6, Corollary 1] states that there exists $m_0 \in \mathbb{N}$ above which for all $m \geq m_0$, there exists $e_m > m^2E(\gamma)$ such that the composition

$$\mathrm{C}_*(S^1\cdot\gamma) \xrightarrow{(\psi_m)_*} \mathrm{C}_*(S^1\cdot\gamma^m) \xrightarrow{\mathrm{inc}_*} H_*\left(\left\{E < e_m\right\}, \left\{E < m^2E(\gamma)\right\}\right) \tag{7}$$

is zero.

## 3. Proof of the adapted Bangert theorem

A closed geodesic $\gamma$ is a mountain pass if, for all neighborhoods $U \subset \Lambda M$ of $S^1\cdot\gamma$, $U \cap E^{-1}([0, E(\gamma))$ is not connected. For the proof of Theorem 1.8, we need the following statement, which tells us that isolated closed geodesics cannot remain mountain pass critical points of the energy functional when sufficiently iterated. A geometric proof is given by Bangert [4].

**Theorem 3.1** ([4, Theorem 2])**.** *Let $\gamma$ be an isolated closed geodesic on $M$, where $\dim M = 2$. Then there exists an integer $m_\gamma \in \mathbb{N}$ such that the following holds: For all integer $m \in \mathbb{N}$ with $m \geq m_\gamma$, there is a neighborhood $U$ of $S^1\cdot\gamma$ in $\Lambda M$ such that $U \cap E^{-1}([0, E(\gamma^m)))$ is connected.*

According to Gromoll-Meyer [22], given an isolated closed geodesic $\gamma$, there exists a connected neighborhood $U \subset \Lambda M$ of the critical circle $S^1\cdot\gamma$ such that

$$\mathrm{C}_*(S^1\cdot\gamma) \simeq H_*\left(U, U \cap E^{-1}\Big([0, E(\gamma))\Big)\right).$$

If $\gamma$ and all its iterates are homologically invisible, Theorem 3.1 is thus true for $m_\gamma = 1$.

*Proof of Theorem 1.8.* Assume there are only finitely many prime closed geodesics $\gamma_1, \ldots, \gamma_k$ which have homologically visible iterates and which intersect $W := M \setminus (C_- \cup C_+)$. We will now derive a contradiction from this assumption. We will define a suitable finite-dimensional approximation $\Omega = \Omega(\mathcal{O}, \kappa, j)$. Now as the statement of Theorem 3.1 remains true in a finite-dimensional approximation, we get that there exists $m_0 \in \mathbb{N}$ such that for all integers $m \geq m_0$ and for all $i \in \{1, \ldots, k\}$ the following holds:

i) There exists a neighborhood $U$ of $S^1\cdot\gamma_i^m$ in $\Omega$ such that $U \cap E^{-1}([0, E(\gamma_i^m)))$ is connected.

Set $A := \max\{E(\gamma_i^{m_0}) \,|\, i \in \{1, \ldots, k\}\}$, and notice that $A$ is larger than the energy of a closed geodesic of mountain pass type. We fix an identification of $\pi_1(M)$ with $\mathbb{Z}$ and denote by $[\gamma] \in \mathbb{Z}$ the class of a loop $\gamma \in \Lambda M$. We define the following sets of curves:

$$P_n^{\pm} := \{\gamma \in \Omega \,|\, \gamma(S^1) \subset C_{\pm},\ [\gamma] = n\}.$$

In the following for each $U, V \subset M$, we will denote

$$\mathrm{dist}(U, V) := \inf_{x\in U,\, y\in V} d(x, y).$$

Choose $\delta > 0$. Then there exists an $n \in \mathbb{N}$ such that for any curve $\gamma \in P_n^{\pm}$ satisfying $\operatorname{dist}(\gamma(S^1), W) < \delta$ it holds that $E(\gamma) \geq A$. We can now say how exactly the finite-dimensional approximation has to be chosen:

- Choose a $\kappa > 0$ large enough such that there exists a homotopy $h : [0,1] \to E^{-1}([0,\kappa))$ in $\Omega$ from $h_0 \in P_n^-$ to $h_1 \in P_n^+$ with
$$\operatorname{dist}\Big(h_t(S^1), W\Big) < \delta, \quad \forall t \in [0,1].$$
- Set $\mathcal{O} := \{p \in M \mid \operatorname{dist}(p, W) < R\}$, where $R > 2\kappa^{\frac{1}{2}} + \delta$ such that $\mathcal{O}$ contains $\gamma_1, \dots, \gamma_k$.
- Choose $j$ such that the $(-\nabla E)$-flow of the finite-dimensional approximation respects $C_\pm$, as described above.

A technical issue is given by the fact that the gradient flow of $-\nabla E$ may not be defined for all times as the sublevel sets of $E|_\Omega$ are not compact. Ultimately we are only going to be interested in curves intersecting the compact set $W$, *i.e.* the subset
$$K := \{\gamma \in \Omega \,|\, \gamma(S^1) \cap W \neq \varnothing\}$$
of $\Omega$. We introduce a smooth function $g : \Omega \to [0,1]$ with the property that
$$\begin{cases} g(\gamma) = 1 & \text{if } \operatorname{dist}(\gamma(S^1), W) \leq \frac{1}{2}\kappa^{\frac{1}{2}}, \\ g(\gamma) = 0 & \text{if } \operatorname{dist}(\gamma(S^1), W) > \frac{3}{2}\kappa^{\frac{1}{2}}. \end{cases}$$
Then the flow $\phi_t$ of $-g\nabla E$ is defined for all times $t \geq 0$ and coincides with the negative gradient flow for curves in $K$. Two crucial observations about the set $K$ are the following: firstly, for all $\bar{\kappa} < \kappa$ the set $K \cap E^{-1}([0,\bar{\kappa}])$ is compact. Secondly, if $\phi_t(\gamma) \in K$ for some $\gamma \in \Omega$ and some time $t \geq 0$, we already have $\gamma \in K$ as the flow $\phi_t$ respects the sets $C_\pm$ (as those satisfy Property (C)). From this it follows:

ii) Let $0 < \kappa_0 < \kappa_0 + \varepsilon < \kappa$. Let $V$ denote a neighborhood of the closed geodesics in $K$ of energy $\kappa_0$. Suppose there is no closed geodesic in $K \cap E^{-1}((\kappa_0, \kappa_0 + \varepsilon])$. Then there exists a time $\tau > 0$, such that
$$\phi_\tau\left(E^{-1}\left([0, \kappa_0 + \varepsilon]\right)\right) \cap K \subset E^{-1}([0, \kappa_0)) \cup V.$$

This is just the deformation lemma; for a proof see for instance [28, Lemma 3.4]. We are now set to complete the proof of the theorem. Define the set of homotopies
$$\Pi := \left\{\beta : [0,1] \to \Omega \text{ continuous} \,|\, \beta_0 \in P_n^-,\, \beta_1 \in P_n^+\right\}.$$
Note that $\Pi$ is not empty, as $h \in \Pi$. Furthermore, $\phi_t \circ \beta \in \Pi$ for all $\beta \in \Pi$ and all $t \geq 0$ as the flow respects the sets $C_\pm$ and therefore $\phi_t(\beta_0) \in P_n^-$ and $\phi_t(\beta_1) \in P_n^+$ for all $t \geq 0$. Define now
$$\kappa_0 := \inf_{\beta \in \Pi} \max_{\substack{t \in [0,1] \\ \beta_t \in K}} E(\beta_t)\,.$$

By definition of $\kappa$, one has $\kappa_0 < \kappa$. For every $\beta \in \Pi$ for time $t_0 := \min\{t \in [0,1] \,|\, \beta_t \notin P_n^-\}$ it holds that $\beta_{t_0} \in K$ and $E(\beta_{t_0}) \geq A$ (as $E$ and $\beta$ are continuous and there exists a sequence $(t_k) \nearrow t_0$ such that $\beta_{t_k} \in P_n^-$ and $\operatorname{dist}(\beta_{t_k}, W) < \delta$). Consequently, we get $\kappa_0 \geq A$. Since $\kappa_0 < \kappa$, for $\varepsilon > 0$ small enough, the subset $K \cap E^{-1}([0, \kappa_0 + \varepsilon])$ is compact and there are only finitely many $S^1$-orbits of closed geodesics inside (we assumed every orbit to be isolated). Let $\{S^1 \cdot d_j\}_{1 \leq j \leq l}$ denote the critical circles of energy $\kappa_0$ in $K$. By definition of $A$ and by using i) when $d_j$ is a power of some $\gamma_i$ (otherwise this is true according to the remark just before the

beginning of the proof), there exist disjoint neighborhoods $U_j$ of the $S^1 \cdot d_j$'s such that $U_j \cap E^{-1}([0, \kappa_0))$ is connected for all $j$.
We now want to argue that we can choose $U_j \subset K$. We first argue, that $d_j \in \partial K$ implies $d_j(S^1) \subset \partial C_\pm$. To that end suppose that $d_j \in \partial K$ but it is not contained in $\partial C_\pm$. Then $d_j|_{[-a,0)}$ is contained in $C_\pm$ for some $a \in (0, \epsilon/2)$ (where this $\epsilon$ denotes the constant coming from Property (C)) and $d_j(0) \in \partial C_\pm$. There are two cases: Either $d_j(b_n) \in C_\pm$ for some infinitesimal sequence $b_n > 0$ or $d_j|_{[0,b]}$ is contained in $\partial C_\pm$ for some $b \in (0, \epsilon/2)$.
In the first case: $d_j(b_n)$ and $d_j(-a)$ belong to the same connected component of $C_\pm \cap B_\epsilon(\delta(-a))$ since there is a curve arbitrarily close to $d_j$ which is contained in $C_\pm$ (as $d_j \in \partial K$). However, the shortest geodesic connecting $d_j(b_n)$ and $d_j(-a)$ is not contained in $C_\pm$ which contradicts Property C.
In the second case: let $\gamma$ be a short geodesic segment which is transverse to $d_j$ passing through $d_j(-a)$ and contained in $C_\pm$. Then since $d_j$ is in $\partial K$ there is a curve completely contained in $C_\pm$ connecting a point $x$ arbitrarily close to $d_j(b)$ and a point on $\gamma$. This means that $x$ is in the same connected component of $C_\pm \cap B_\epsilon(x)$ as are all the points of $\gamma$. If $x$ is close enough to $d_j(b)$ however, one can find a short geodesic segment from $x$ to a point on $\gamma$ passing through $d_j(0)$. Therefore this geodesic segment is not contained in $C_\pm$ thus contradicting Property C.
If $d_j$ is an iterate of a simple closed geodesic, by the above it cannot be contained in $\partial K$ since by assumption $\partial C_\pm$ do not contain simple closed geodesics. Conversely, if it is not an iterate, it has a (transverse) self-intersection. Then all the curves close to $d_j$ will also intersect $d_j$ and therefore $\partial C_\pm$. Consequently, in this case $U_j$ is also contained in $K$.
Now because there are only finitely many closed geodesics in $K \cap E^{-1}([0, \kappa_0 + \varepsilon])$ for $\varepsilon > 0$ small enough, one can take $\varepsilon > 0$ such that there is no closed geodesic in $K \cap E^{-1}((\kappa_0, \kappa_0 + \varepsilon])$. By the definition of $\kappa_0$ there exists a homotopy $\beta \in \Pi$ satisfying $E(\beta_t) \leq \kappa_0 + \varepsilon$ for all $t \in [0, 1]$ such that $\beta_t \in K$. Choose neighborhoods $V_j$ of $S^1 \cdot d_j$ such that $\overline{V_j} \subset \mathrm{int}(U_j)$ and use property ii) on the neighborhood $V := \bigcup_{k=1}^l V_j$ of closed geodesics of energy $\kappa_0$ in $K$ to obtain a $\tau > 0$ with the property that for the homotopy $\phi_\tau \circ \beta$ we have that $(\phi_\tau \circ \beta)_t \in K$ implies $E((\phi_\tau \circ \beta)_t) < \kappa_0$ or $(\phi_\tau \circ \beta)_t \in V$. Now $(\phi_\tau \circ \beta)^{-1}(V) = \bigcup_{r=1}^m (t_r, t'_r)$ and by our choice of the $V_j$ we have $(\phi_\tau \circ \beta)([t_r, t'_r]) \subset U_j$ and for the endpoints $(\phi_\tau \circ \beta)_{t_r}, (\phi_\tau \circ \beta)_{t'_r} \in U_j \cap E^{-1}([0, \kappa_0))$ for some $j \in \{1, \ldots, l\}$ (which is why we applied ii) only to $V$ and not to $\bigcup_{j=1}^l U_j$ directly). Now, by using i) if $d_j$ a power of some $\gamma_i$ (otherwise it is true by the remark just before the beginning of the proof), we know that $U_j \cap E^{-1}([0, \kappa_0))$ is connected and consequently we can replace $(\phi_\tau \circ \beta)|_{[t_r, t'_r]}$ by a path in $E^{-1}([0, \kappa_0))$ with the same endpoints. After $m$ steps we obtain a homotopy $\hat{\beta} : [0, 1] \to \Omega$ such that $E(\hat{\beta}_t) < \kappa_0$ when $\hat{\beta}_t \in K$. Since $(\phi_\tau \circ \beta)_0, (\phi_\tau \circ \beta)_1 \notin K$ it follows that $(\phi_\tau \circ \beta)_0, (\phi_\tau \circ \beta)_1 \notin \bigcup_{j=1}^l U_j$ and therefore $\hat{\beta}_0 \in P_n^-$, $\hat{\beta}_1 \in P_n^+$, hence $\hat{\beta} \in \Pi$. This contradicts the minimality of $\kappa_0$.

A closed geodesic without self-intersections can be constructed like in [4, Theorem 1]. One should note that this requires a curve shortening process which takes simple curves into simple curves. Alternatively to the one by Lusternik and Schnirelmann quoted by Bangert, one could also use the more recent Grayson-Oaks curve shortening [19, 26]. □

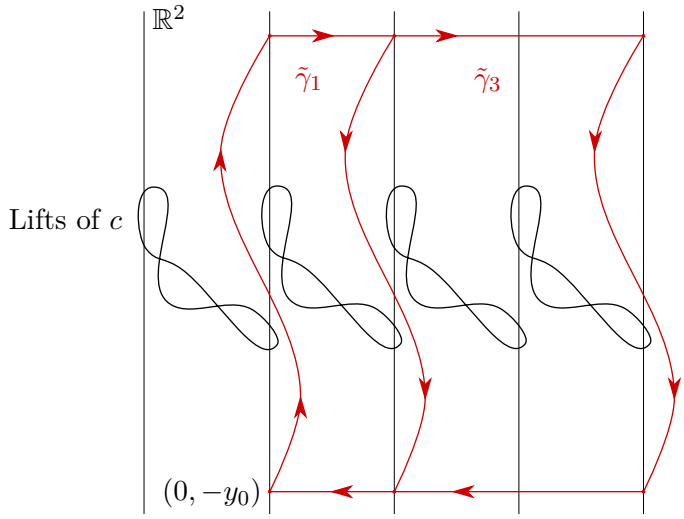


FIGURE 1. The family of loops $(\tilde{\gamma}_n)$

## 4. Contractible and intersecting closed geodesics

Here $M$ still denotes a complete Riemannian cylinder. We assume that there exists a contractible closed geodesic $c \in \Lambda M$. Let us consider the unbounded components of $M \setminus c(S^1)$. Since $c(S^1)$ is bounded, there are at most two distinct unbounded components. If there are two distinct unbounded components $C_-$ and $C_+$, one can assume that $C_-$ is a neighborhood of $-\infty$ and $C_+$ is a neighborhood of $+\infty$. By $C_\pm$ we will mean any of these two neighborhoods. Then $\partial C_\pm$ is a broken geodesic with angles strictly less than $\pi$ inside $C_\pm$ since $c$ is a closed geodesic (see Figure 2 for an instance of $\partial C_+$). Hence $C_\pm$ is satisfies Property (C). Moreover if the boundary were a simple closed geodesic, then $\partial C_\pm$ would be parametrised by $c$ which is impossible for $c$ is contractible. We can thus apply Theorem 1.8 in this case.

We now assume that $M \setminus c(S^1)$ has only one unbounded component $C$. Let us identify $M$ with $S^1 \times \mathbb{R}$ in the remainder of this proof in order to fix the notation. Let $\pi : \mathbb{R}^2 \to S^1 \times \mathbb{R}$ be the universal cover of $S^1 \times \mathbb{R}$. By compactness of $c(S^1)$, there exists $A > 0$ such that $c(S^1) \subset S^1 \times (-A, A)$. Let $y_0 > A$, since $S^1 \times (-\infty, -A)$ and $S^1 \times (A, +\infty)$ belong to the same component of $M \setminus c(S^1)$, there exists a smooth path $\alpha : [0,1] \to M \setminus c(S^1)$ such that $\alpha(0) = (0, -y_0)$ and $\alpha(1) = (0, y_0)$. Let $\beta_0$ be the smooth lift of $\alpha$ in $\mathbb{R}^2$ such that $\beta_0(0) = (0, -y_0)$ and $\beta_0(1) = (n_0, y_0)$ for some $n_0 \in \mathbb{Z}$ that we can take equal to $n_0 = 0$ by chaining $\alpha$ with $t \mapsto (tn_0 \bmod 1, y_0)$. Let $\delta_{n,\pm} : [0,1] \to \mathbb{R}^2$ be the path $t \mapsto (nt, \pm y_0)$ and $\beta_n : [0,1] \to \mathbb{R}^2$ be the family of lifts $\beta_n := (n, 0) + \beta_0$, $n \in \mathbb{N}$. We define the family of loops $\tilde{\gamma}_n \in \Lambda\mathbb{R}^2$ by

$$\tilde{\gamma}_n := \beta_0 \cdot \delta_{n,+} \cdot \beta_n^{-1} \cdot \delta_{n,-}^{-1}.$$

They project to $\gamma_n := \pi \circ \tilde{\gamma}_n$ in $M \setminus c(S^1)$. Let $q_0 \in \mathbb{R}^2$ be a lift of some point of $c(S^1)$ and define $q_n := q_0 + (n, 0)$. Then the first homology group $H_1(\mathbb{R}^2 \setminus \{q_n\}_{n\in\mathbb{Z}})$ is the free abelian group with generators $(g_n)_{n\in\mathbb{Z}}$, and by construction the class of $\tilde{\gamma}_n$ is $g_1 + g_2 + \cdots + g_n$. The covering transformations of $\mathbb{R}^2 \setminus \{q_n\}_{n\in\mathbb{Z}} \to S^1 \times \mathbb{R} \setminus \pi(q_0)$, which form a group isomorphic to $\mathbb{Z}$, act on the first homology group by $k \cdot g_i = g_{i+k}$. Therefore, for natural integers $n \neq m$ and integers $k, l \in \mathbb{Z}^*$, the fact that

$$k(g_{1+a} + g_{2+a} + \cdots + g_{n+a}) \neq l(g_{1+b} + g_{2+b} + \cdots + g_{m+b}), \quad \forall a, b \in \mathbb{Z},$$

implies that the iterated loops $\gamma_n^k$ and $\gamma_m^l$ are not freely homotopic in $M \setminus \pi(q_0)$ and hence in the unbounded component $C$ of $M \setminus c(S^1)$. For $\gamma \in \Lambda C$, let us denote by $[\gamma]$ the free homotopy class of $\gamma$. For $m \geq 2$, let us consider the infimum

$$e_m := \inf_{\substack{\gamma \in \Lambda C \\ [\gamma]=[\gamma_m]}} E(\gamma).$$

Let $K \subset M$ be a compact set that contains $c(S^1)$ such that $M \setminus K$ has two distinct unbounded component. Since any $\gamma \in \Lambda C$ that is freely homotopic to $\gamma_m$ must intersect $K$, one can restrict the domain of the infimum to those $\gamma$ which image is inside the compact set $L \subset M$ of points that are at distance at most $\sqrt{e_m}$ of $K$ (which is compact by completeness of the metric on $M$). Indeed, if $\gamma$ were a loop of length $\geq 2\sqrt{e_m}$ then $E(\gamma) \geq 4e_m$ by Cauchy-Schwarz inequality. By compactness of $L$, we can use a finite-dimensional approximation to get a closed geodesic $c_m$ on $C \cup c(S^1)$ with $E(c_m) = e_m$ that is a limit of broken geodesics on $C$ freely homotopic to $\gamma_m$. By uniqueness of the Cauchy problem, if $c_m$ intersect $c(S^1)$, the closed geodesic must be a power of $c$ (up to a translation of the parametrisation). This is impossible since the powers of $c$ are not in the closure of $\{\gamma \in \Lambda C \mid [\gamma] = [\gamma_m]\}$ for $m \geq 2$ (such $\gamma$'s must intersect every line joining both ends $\pm\infty$ of $M$). Therefore, the above infimum is reached by the closed geodesic $c_m \in \Lambda C$. We thus get a family of closed geodesics $(c_m)$ such that $[c_m^k] \neq [c_n^l]$ for all $k, l \in \mathbb{Z}^*$ and $m \neq n$. Therefore the closed geodesics $(c_m)$ are geometrically distinct. They all intersect the compact set $K$. As a local minimum, every $c_m$ is homologically visible. A closed geodesic without self-intersection can be found be choosing a simple closed curve in $C$ which is close to $\partial C$ and by applying Grayson-Oaks curve shortening [19, 26] to this curve.

Now that Theorem 1.1 is proved under hypothesis 1, in order to prove it when there is one self-intersecting closed geodesic $c$ or two intersecting ones $c_1$ and $c_2$, one can assume that these geodesics are not contractible. Therefore, in both respective cases, $M \setminus c(S^1)$ or $M \setminus (c_1(S^1) \cup c_2(S^1))$ has exactly two unbounded connected components $C_-$ and $C_+$, which satisfy Property (C) by construction. The intersection hypothesis then implies that none of the boundaries $\partial C_\pm$ is a simple closed geodesic. Hence the conclusion follows by applying Theorem 1.8.

## 5. Closed geodesic of non-zero average index

We assume that there exists a closed geodesic $c \in \Lambda M$ of average index $\overline{\mathrm{ind}}(c) > 0$. If $c$ is contractible or self-intersecting, we already know that there are infinitely many homologically visible closed geodesics. Let us assume that $c$ is an embedded curve generating $\pi_1(M) \simeq \mathbb{Z}$. By a slight abuse of notation, we identify the loop $c : S^1 \to M$ with its lift $\mathbb{R} \to M$.

**Lemma 5.1.** *There exist a non-zero Jacobi field $J : \mathbb{R} \to c^*TM$ of $c$ and $\delta > 0$ such that $J(s) \neq 0$ for all $s \in (0, \delta)$ and $J(0) = J(\delta) = 0$.*

*Proof.* Since $\overline{\mathrm{ind}}(c) > 0$, Bott iteration inequality (2) and the concavity bound (3) imply that there exists $k \in \mathbb{N}^*$ such that

$$\mathrm{ind}_\Omega(c^k) \geq 1.$$

Let us fix such a $k \geq 1$. The conclusion is now a direct application of the Morse index theorem (4) to the geodesic path $c^k$. □

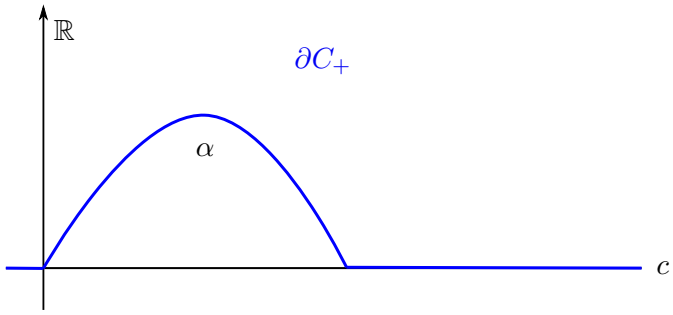


FIGURE 2. Construction of the neighborhood $C_+$ ($\alpha$ could also have self-intersections)

In order to fix notation, let us identify the image of the loop $c$ to $S^1 \times \{0\}$, with $c(s) = (s, 0)$ for $s \in S^1$, so that $M \setminus c(S^1)$ is the disjoint union of the neighborhood $S^1 \times (-\infty, 0)$ of $-\infty$ and the neighborhood $S^1 \times (0, +\infty)$ of $+\infty$ (we only need this identification to be a homeomorphism). Let $J : \mathbb{R} \to c^*TM$ and $\delta > 0$ be the Jacobi field and the positive number given by Lemma 5.1. Let $\varepsilon > 0$ and $I := (-\varepsilon, \delta + \varepsilon)$. Since there exists a smooth family $(\beta_s)_{s\in(-1,1)}$ of geodesic paths $I \to M$ such that $J|_I = \frac{\partial \beta_s}{\partial s}|_{s=0}$, it implies that there exists a geodesic path $\alpha : [0, 1] \to S^1 \times [0, +\infty)$ intersecting $c$ (transversally) only at its endpoints.

By construction, the unbounded component $C_+$ of $S^1 \times (0, +\infty) \setminus \alpha([0, 1])$ has a boundary which is a broken geodesic with angles strictly less than $\pi$. By symmetry, we get two disjoint neighborhoods of $+\infty$ and $-\infty$ respectively which are locally convex and whose boundaries are not totally geodesic, we can thus apply Theorem 1.8.

## 6. Two homologically visible closed geodesics

Here $M$ denotes a complete Riemannian cylinder. We fix an identification of $\pi_1(M)$ with $\mathbb{Z}$ and denote by $[\gamma] \in \mathbb{Z}$ the class of a loop $\gamma \in \Lambda M$. We assume that there exist two geometrically distinct and homologically visible closed geodesics. We suppose by contradiction that for any compact set $K \subset M$ only a finite number of geometrically distinct homologically visible closed geodesics intersect $K$. By the previous cases of Theorem 1.1, every prime closed geodesic of $M$ must be embedded, non-contractible, without intersections with another closed geodesic, and of zero average index. Thus the images of closed geodesics of $M \simeq S^1 \times \mathbb{R}$ with a homologically visible iterate are naturally ordered by their smallest intersection with $* \times \mathbb{R}$ where $*$ denotes any point of $S^1$. The order is independent of the choice of $* \in S^1$. We will say that two closed geodesics are consecutive if they are so with respect to this order. Since only a finite number of geometrically distinct homologically visible closed geodesic intersect a given compact set, one can talk about the next and the previous one with respect to this order.

**Lemma 6.1.** *There exist two closed embedded geodesics $c_1$ and $c_2$ of $M$ with degree $[c_1] = [c_2] = 1$ bounding a compact locally convex cylinder $C \simeq S^1 \times [0, 1]$ such that*

*(1) $c_1$ is a local minimum of $E|_{\Lambda C}$,*
*(2) $c_2$ is not a local minimum of $E|_{\Lambda C}$,*
*(3) $c_1$ and $c_2$ are the only closed geodesics of $M$ inside $C$ that have homologically visible iterates.*

*Proof.* We first show that two consecutive closed geodesics among closed geodesics that possess homologically visible iterates cannot be both local minima of $E|_{\Lambda C'}$ if $C'$ is the compact cylinder that they bound. By contradiction, let us assume so and let us call $\gamma_0$ and $\gamma_1$ these two geodesics. Up to a change of parametrization, one can assume that $[\gamma_0] = [\gamma_1]$ and thus that these two geodesics are homotopic in $\Lambda C'$. Let

$$\Pi := \{h : [0,1] \to \Lambda C' \text{ continuous} \mid h(0) = \gamma_0 \text{ and } h(1) = \gamma_1\}$$

denote the set of homotopy of loops in $C'$ starting at $\gamma_0$ and ending at $\gamma_1$. We consider the following min-max:

$$\tau = \inf_{h \in \Pi} \max E \circ h.$$

By compactness of $C'$, $E|_{\Lambda C'}$ satisfies Palais-Smale (alternatively, one can work in the compact finite-dimensional manifold of $k$-broken-geodesics of energy $\leq c+\varepsilon$ for a large $k \in \mathbb{N}$ and $\varepsilon > 0$). Let $e := \max(E(\gamma_0), E(\gamma_1))$. Since the critical orbits $S^1 \cdot \gamma_0$ and $S^1 \cdot \gamma_1$ are isolated local minima of $E|_{\Lambda C'}$ that satisfies Palais-Smale, $\tau > e$. By local convexity of $C'$, the $(-\nabla E)$-flow preserves $\Lambda C'$. By the minimax principle, $\tau$ is thus a critical value of $E|_{\Lambda C'}$ and there exists a homologically visible closed geodesic $\gamma \in \Lambda C'$ of energy $\tau$. Hence $\gamma_0$ and $\gamma_1$ are not consecutive, a contradiction.

By a similar argument, we show that one out of two consecutive closed geodesics among those that possess homologically visible iterates is a local minimum of $E|_{\Lambda C'}$. Indeed, otherwise one has that

$$\inf_{\substack{\gamma \in \Lambda C' \\ [\gamma]=1}} E(\gamma) < \min(E(\gamma_0), E(\gamma_1)),$$

and this infimum is reached for some closed geodesic in $C'$ by compactness and local convexity of $C'$ (and this is not a point since $E(\gamma) \geq (2r)^2$ for all $\gamma \in \Lambda C'$ of degree $[\gamma] = 1$ where $r > 0$ denotes the injectivity radius of the compact Riemannian manifold with boundary $C'$). This new closed geodesic is a local minimum of $E$ by definition and thus homologically visible.

The requirements of the lemma are thus fulfilled by taking any two consecutive closed geodesics among those with homologically visible iterates. □

*Proof of Theorem 1.1.* Let $c_1$ and $c_2$ be closed geodesics of $M$ satisfying Lemma 6.1. We will reach a contradiction by finding a homologically visible geodesic which is not $c_1$ or $c_2$ and arbitrarily close to $C$.

Let $x \in \mathrm{Int}(C)$ be outside the image of the isolated set of closed geodesics and let $\gamma_1 \in \Lambda C$ be the loop of degree $[\gamma_1] = 1$ based at $x$ of minimal length. It exists by local convexity and compactness of $C$. The loop $\gamma_1$ is not a periodic geodesic (this is a geodesic as a path $[0,1] \to C$ but not as a loop $S^1 \to C$) by our specific choice of $x$. This loop lies inside $\mathrm{Int}(C)$ so that either the connected component of $C \setminus \gamma_1(S^1)$ containing $c_1$ or the connected component containing $c_2$ is locally convex – depending on the angle of $\gamma_1$ at $\gamma_1(0) = \gamma_1(1) = x$. If the connected component containing $c_2$ were locally convex, then the infimum of $E$ among loops of degree one lying inside the locally convex compact cylinder bounded by $\gamma_1$ and $c_2$ would give a closed geodesic loop $\neq c_1$ which would be a local minimum. Thus the connected component of $C \setminus \gamma_1(S^1)$ containing $c_1$ is a locally convex compact cylinder. Hence the unbounded component of $M \setminus \gamma_1(S^1)$ containing $c_1$ is a locally convex neighborhood of $-\infty$ which is not totally geodesic since $\gamma_1$ is not a closed geodesic.

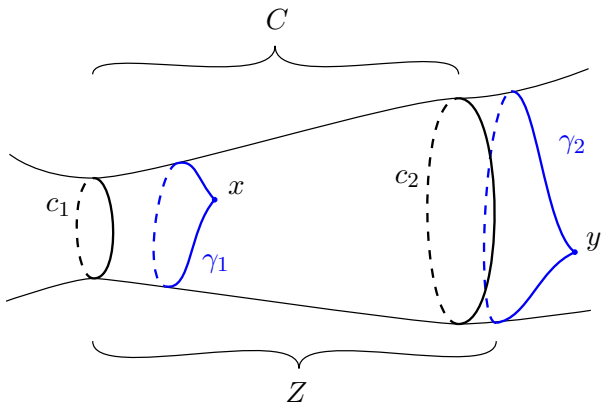


FIGURE 3. Construction of cylinder $Z$

Let $c_3$ be the homologically visible closed geodesic succeeding $c_2$ if it exists. Let $C'$ be either the compact cylinder that $c_2$ and $c_3$ bound or the infinite cylinder $\simeq S^1 \times [0, +\infty)$ with boundary $c_2$ and ending at $+\infty$, depending on the existence of $c_3$ (so that $C \cap C' = c_2(S^1)$ in both cases). Let $y \in \mathrm{Int}(C')$ be outside the image of any closed geodesic and let $\gamma_2 \in \Lambda C'$ be a loop of degree $[\gamma_2] = 1$ based at $y$ of minimal length. Since $C'$ is complete and locally convex, $\gamma_2$ exists. It cannot be a closed geodesic by our specific choice of $y$. One of the two unbounded components of $M \setminus \gamma_2(S^1)$ is thus locally convex, depending on the angle of $\gamma_2$ at $\gamma_2(0) = \gamma_2(1) = y$. If the neighborhood of $+\infty$ was the locally convex one, by Theorem 1.8 applied to the locally convex neighborhood of $-\infty$ defined above with $\gamma_1$ and this neighborhood of $+\infty$, there would be infinitely many homologically visible and geometrically distinct closed geodesics intersecting some compact set of $M$. Thus the neighborhood of $-\infty$ is the locally convex unbounded component of $M \setminus \gamma_2(S^1)$. Restricting this neighborhood to the compact cylinder $C \cup C'$, one gets a compact locally convex cylinder $Z$ intersecting only two geodesics $c_1$ and $c_2$ that possess homologically visible iterates, moreover $c_1(S^1) \subset \partial Z$ and $c_2(S^1) \subset \mathrm{Int}(Z)$.

Let $k \in \mathbb{N}^*$ be such that $\mathrm{C}_*(S^1 \cdot c_2^k) \neq 0$. Let $\Lambda_h \subset \Lambda Z$ be the connected component of loops $\gamma \in \Lambda Z$ of degree $[\gamma] = h$. For all $m \in \mathbb{N}^*$, let $\psi_m : \Lambda_k \to \Lambda_{km}$ be the iteration map $\psi_m(\gamma) := \gamma^m$. According to the Bangert-Klingenberg theorem (7), there exists $m_0 \in \mathbb{N}$ above which for all $m \geq m_0$ there exists $e_m > m^2 E(c_2^k)$ such that the composition

$$\mathrm{C}_*(S^1 \cdot c_2^k) \xrightarrow{(\psi_m)_*} \mathrm{C}_*(S^1 \cdot c_2^{km}) \xrightarrow{\mathrm{inc}_*} H_*\left(\left\{E|_{\Lambda_{km}} < e_m\right\}, \left\{E|_{\Lambda_{km}} < m^2 E(c_2^k)\right\}\right)$$

is zero. According to the Gromoll-Meyer theorem (6), since $\overline{\mathrm{ind}}(c_2^k) = k\,\overline{\mathrm{ind}}(c_2) = 0$, there exist infinitely many $m$ such that

$$(\psi_m)_* : \mathrm{C}_*(S^1 \cdot c_2^k) \to \mathrm{C}_*(S^1 \cdot c_2^{km})$$

is an isomorphism. Let $m \geq m_0$ be such an integer, then the inclusion induces a zero map

$$\mathrm{C}_*(S^1 \cdot c_2^{km}) \xrightarrow{\mathrm{inc}_*} H_*\left(\left\{E|_{\Lambda_{km}} < e_m\right\}, \left\{E|_{\Lambda_{km}} < m^2 E(c_2^k)\right\}\right),$$

which contradicts the fact that $c_2^{km}$ is the homologically visible critical point of $E|_{\Lambda_{km}}$ of maximal value. Critical points of $E|_{\Lambda_{km}}$ are closed geodesics of $Z$ of degree $km$. Thus $S^1 \cdot c_1^{km}$ and $S^1 \cdot c_2^{km}$ are the only homologically visible critical circles of

$E|_{\Lambda_{km}}$ (and $E(c_2^{km}) > E(c_1^{km})$ since $c_1$ is the only local minimum in $C$). Since $Z$ is locally convex, the $(-\nabla E)$-flow preserves $\Lambda Z$. Moreover $Z$ is compact and has only isolated closed geodesics, we can thus apply Morse theoretical arguments since $E|_{\Lambda_{km}}$ has isolated critical circles and satisfies Palais-Smale or, alternatively, one can restrict $E$ to the finite-dimensional subspace of $j$-broken-geodesics of $\Lambda_{km}$ of energy less than $e_m + \varepsilon$ for some large $j \in \mathbb{N}$ and some $\varepsilon > 0$. Thus, if $S^1 \cdot c_2^{km}$ were the only homologically visible critical circle of energy $\geq m^2 E(c_2^k)$, Morse deformation lemma would imply $\mathrm{inc}_*$ to be an isomorphism. □

## 7. The case of the Möbius band

Assuming Theorem 1.1 concerning complete cylinders, we deduce Corollary 1.3.

Let $(M, g)$ be a complete Möbius band and let us denote by $\pi : \widetilde{M} \to M$ its connected double cover. Hence, $(\widetilde{M}, \tilde{g})$ with $\tilde{g} := \pi^* g$ is a complete cylinder. Let us denote by $E : \Lambda M \to \mathbb{R}$ and $\widetilde{E} : \Lambda \widetilde{M} \to \mathbb{R}$ the respective energy functionals of $M$ and $\widetilde{M}$. Any closed geodesic of $M$ is covered by one or two closed geodesics of $\widetilde{M}$. The proof would be obvious if the homological visibility of one of the iterates of the geodesic on $M$ were equivalent to the homological visibility of one of the iterates of the covering geodesics. However, it is not clear whether this is the case when only one closed geodesic covers the closed geodesic on $M$. We will see that the Smith inequality (8) will yield equivalence over the field $R = \mathbb{Z}/2\mathbb{Z}$.

In the statements of the following two lemmas, we use the above notation $M$, $\widetilde{M}$, $E$, $\widetilde{E}$ where $\pi : \widetilde{M} \to M$ denotes any Riemannian cover of some Riemannian manifold $M$.

**Lemma 7.1.** *Let $\tilde{c} \in \Lambda \widetilde{M}$ be a closed geodesic and let $c := \pi \circ \tilde{c}$. Then the map $\pi_\sharp : \Lambda \widetilde{M} \to \Lambda M$, $\tilde{\gamma} \mapsto \pi \circ \tilde{\gamma}$, induces an isomorphism*

$$\mathrm{C}_*(S^1 \cdot \tilde{c}) \xrightarrow{\simeq} \mathrm{C}_*(S^1 \cdot c).$$

*Moreover,* $\mathrm{ind}(\tilde{c}) = \mathrm{ind}(c)$.

*Proof.* Since $\pi$ is a covering map, the map $\pi_\sharp$ is a diffeomorphism in a small neighborhood $\widetilde{U}$ of $S^1 \cdot \tilde{c}$ by the uniqueness of the lift to $\widetilde{U}$ of a loop belonging to the neighborhood $U := \pi_\sharp(\widetilde{U})$ of $S^1 \cdot c$. Since $\widetilde{E} = E \circ \pi_\sharp$, the Morse indices $\mathrm{ind}(c)$ and $\mathrm{ind}(\tilde{c})$ are equal. The conclusion now follows from the local property (5) of the local homologies of $S^1 \cdot \tilde{c}$ and $S^1 \cdot c$. □

Given a group $G$ acting on a space $X$, let $X^G \subset X$ be the set of fixed points of $G$. According to the Smith inequality,

$$\dim H_*\Big(X; \mathbb{Z}/p\mathbb{Z}\Big) \geq \dim H_*\Big(X^{\mathbb{Z}/p\mathbb{Z}}; \mathbb{Z}/p\mathbb{Z}\Big), \tag{8}$$

where $X$ is a locally compact space or pair such that $H_*(X; \mathbb{Z}/p\mathbb{Z})$ is finitely generated, a space on which acts the group $\mathbb{Z}/p\mathbb{Z}$ with $p$ prime (see for instance [11, Chapter IV, §4.1]). Here $\dim H_*$ denotes the total dimension $\sum_k \dim H_k$. The following lemma is a counterpart of a result of Çineli-Ginzburg relating the local homologies of a Hamiltonian orbit and its $p$-iterate [15].

**Lemma 7.2.** *For every isolated closed geodesic $c \in \Lambda M$ and every prime number $p$,*

$$\dim \mathrm{C}_*(S^1 \cdot c^p; \mathbb{Z}/p\mathbb{Z}) \geq \dim \mathrm{C}_*(S^1 \cdot c; \mathbb{Z}/p\mathbb{Z}).$$

Let us notice that this last inequality is an equality when $p$ is large enough according to Gromoll-Meyer theory [22, Theorem 3] (the coefficients field $\mathbb{Z}/p\mathbb{Z}$ can be replaced by any ring $R$ in this case).

*Proof.* Since the local homology of $S^1 \cdot c$ only depends on a small neighborhood of $S^1 \cdot c$ (local property (5)), one can assume that $M$ is a closed manifold. Let $X \subset \Lambda M$ be the topological pair

$$X := \Big(\{E < E(c^p)\} \cup S^1 \cdot c^p, \{E < E(c^p)\}\Big).$$

This pair retracts on a locally compact pair by using a finite-dimensional approximation. According to the Gromoll-Meyer theory, the homology group $H_*(X; \mathbb{Z}/p\mathbb{Z}) = \mathrm{C}_*(S^1 \cdot c; \mathbb{Z}/p\mathbb{Z})$ is finitely generated (see Section 2). By viewing $\mathbb{Z}/p\mathbb{Z}$ as the subgroup of $p$-th roots of unity, $\mathbb{Z}/p\mathbb{Z} \subset S^1$ acts on $\Lambda M$. This action preserves the sublevel sets of $E$ so it preserves $X$ and $\gamma \mapsto \gamma^p$ induces a homeomorphism

$$\Big(\{E < E(c)\} \cup S^1 \cdot c, \{E < E(c)\}\Big) \xrightarrow{\simeq} X^{\mathbb{Z}/p\mathbb{Z}}.$$

This is now a direct consequence of the Smith inequality (8). □

*Proof of Corollary 1.3.* Let $\pi : \widetilde{M} \to M$ be the connected double cover of the complete Möbius band $M$. Let us identify $H_1(M; \mathbb{Z})$ and $H_1(\widetilde{M}; \mathbb{Z})$ with $\mathbb{Z}$, so that the induced morphism $\pi_* : H_1(\widetilde{M}; \mathbb{Z}) \to H_1(M; \mathbb{Z})$ is the multiplication by 2. Given a closed geodesic $\gamma \in \Lambda M$, we denote by $[\gamma] \in \mathbb{Z}$ its homology class. By the lifting property of covers, there exists $\tilde{\gamma} \in \Lambda\widetilde{M}$ such that $\gamma = \pi \circ \tilde{\gamma}$ if and only if $[\gamma]$ is even (we recall that $\pi_1(M) \simeq H_1(M; \mathbb{Z})$ for $M$ and for $\widetilde{M}$ as well).

A contractible closed geodesic of $M$ is covered by contractible closed geodesics of $\widetilde{M}$ thus hypothesis 1 on $M$ implies hypothesis 1 on $\widetilde{M}$. A self-intersecting closed geodesic of $M$ is either covered by a self-intersecting closed geodesic or two intersecting closed geodesics of $\widetilde{M}$ thus hypothesis 2 on $M$ either implies hypothesis 2 or hypothesis 3 on $\widetilde{M}$. Two intersecting closed geodesics on $M$ admit intersecting lifts on $\widetilde{M}$ thus hypothesis 3 on $M$ implies hypothesis 3 on $\widetilde{M}$. If hypothesis 4 is satisfied on $M$, let $c \in \Lambda M$ be a closed geodesic with $\overline{\mathrm{ind}}(c) > 0$. Now $[c^2] = 2[c]$ is even so there exists a closed geodesic $\tilde{\gamma} \in \Lambda\widetilde{M}$ such that $c^2 = \pi \circ \tilde{\gamma}$. According to Lemma 7.1, $\overline{\mathrm{ind}}(\tilde{\gamma}) = \overline{\mathrm{ind}}(c^2) = 2\,\overline{\mathrm{ind}}(c) > 0$ and therefore hypothesis 4 is also satisfied on $\widetilde{M}$. Finally, if hypothesis 5 is satisfied on $M$ and $c_1, c_2 \in \Lambda M$ denote the two closed geodesics that are homologically visible over $\mathbb{Z}/2\mathbb{Z}$, $c_1^2$ and $c_2^2$ are also homologically visible over $\mathbb{Z}/2\mathbb{Z}$ by Lemma 7.2 and one can apply Lemma 7.1 as before to get that hypothesis 5 is satisfied on $\widetilde{M}$ over $\mathbb{Z}/2\mathbb{Z}$.

According to Theorem 1.1, in any of the above cases $\widetilde{M}$ contains infinitely many closed geodesics intersecting some common compact set $\widetilde{K}$ that are homologically visible over $\mathbb{Z}/2\mathbb{Z}$. By Lemma 7.1, the projection of these closed geodesics gives infinitely many closed geodesics intersecting the compact set $\pi(\widetilde{K})$ that are homologically visible over $\mathbb{Z}/2\mathbb{Z}$. □

## 8. The case of the plane

Let $M \simeq \mathbb{R}^2$ be a complete Riemannian plane with isolated closed geodesics. Using what we have seen in the previous sections, we now give the proof of Theorem 1.5.

*Proof of Theorem 1.5.* When hypothesis 1, 2 or 3 is assumed, the conclusion follows from the same argument as in the case of the cylinder: by construction of an open neighborhood $C \neq M$ of infinity. More precisely, this neighborhood $C$ is the unbounded component of $M \setminus c(S^1)$ or $M \setminus (c_1(S^1) \cup c_2(S^1))$ if $c$ is self-intersecting or $c_1$ and $c_2$ are intersecting closed geodesics. In the case when there exists a closed geodesic $c$ of non-zero average index, $C$ is constructed by "integrating a Jacobi field" along $c$ as was done in Section 5.

Now, let us assume that all the closed geodesics of $M$ are without self-intersection, with zero average index and do not intersect any other closed geodesic. Moreover, let us assume that only finitely many (geometrically distinct) closed geodesics intersect any given compact set among homologically visible closed geodesics. Let us show by contradiction that it cannot occur whenever $M$ possesses at least one homologically visible closed geodesic. Let $c$ be a simple closed geodesic that has a homologically visible iterate and such that there is not any homologically visible closed geodesic inside the disk $D$ bounded by $c$. Let $G = \bigcup_\gamma \gamma(S^1) \subset M$ be the union of the images of the closed geodesics $\gamma$ of $M$. Let $U$ be the connected component of $M \setminus (D \cup G)$ that contains $c(S^1)$ in its boundary. Since $U$ contains loops that are not contractible in $\mathbb{R}^2 \setminus D$ (by taking loops close to the boundary $c(S^1)$), $U$ is not simply connected. Let $y \in U$ and let $\gamma \in \Lambda\overline{U}$ be a loop minimizing the length among the loop of $\overline{U}$ based at $y$ that are freely homotopic to $c$ (it exists since $\overline{U}$ is complete). Since $\partial U$ is a disjoint union of closed geodesics, $\gamma$ lies in the interior of $U$ and is a geodesic path. Depending on the angle that $\gamma$ makes at $y$, either the unbounded component of $M \setminus \gamma(S^1)$ is locally convex and not totally geodesic or the bounded component containing $c$ is locally convex. In the first case, one can apply Bangert's theorem to get a contradiction.

Let us now apply an argument similar to the one given in [6, Theorem 3] in order to conclude the proof. We can thus assume that $c$ lies in the interior of a compact and locally convex subset $K \subset M$ and that some powers of $c$ are the only homologically visible closed geodesics of $K$. Since $\overline{\mathrm{ind}}(c) = 0$, the local homology groups $\mathrm{C}_d(S^1 \cdot c^m)$ are trivial in degrees $d \geq 4$ for all $m \in \mathbb{N}$. Let $d \in \{0, 1, 2, 3\}$ be the maximal degree such that $\mathrm{C}_d(S^1 \cdot c^m) \neq 0$ for some $m \in \mathbb{N}^*$. Let $k \in \mathbb{N}^*$ be such that $\mathrm{C}_d(S^1 \cdot c^k) \neq 0$. According to Gromoll-Meyer theory, there exist infinitely many $m \in \mathbb{N}^*$ such that the map induced by the iteration map

$$(\psi_m)_* : \mathrm{C}_*(S^1 \cdot c^k) \to \mathrm{C}_*(S^1 \cdot c^{km})$$

is an isomorphism. As above, according to the Bangert-Klingenberg theorem (7), there exists $m_0 \in \mathbb{N}^*$ such that, for all such $m \in \mathbb{N}^*$ greater than $m_0$, the inclusion of sublevel sets of $E|_{\Lambda K}$

$$\mathrm{C}_*(S^1 \cdot c^{km}) \xrightarrow{\mathrm{inc}_*} H_*\left(\left\{E|_{\Lambda K} < e_m\right\}, \left\{E|_{\Lambda K} < m^2 E(c^k)\right\}\right)$$

induces the zero map, for some $e_m > m^2 E(c^k)$. Thus, for such an $m$, the long exact sequence of the triple

$$\left(\left\{E|_{\Lambda K} < e_m\right\}, \left\{E|_{\Lambda K} < m^2 E(c^k)\right\} \cup S^1 \cdot c^{km}, \left\{E|_{\Lambda K} < m^2 E(c^k)\right\}\right)$$

implies that

$$H_{d+1}\left(\left\{E|_{\Lambda K} < e_m\right\}, \left\{E|_{\Lambda K} < m^2 E(c^k)\right\} \cup S^1 \cdot c^{km}\right) \neq 0.$$

Therefore, by the Morse deformation lemma applied to the $C^2$ function $E|_{\Lambda K}$ which satisfies the Palais-Smale condition and whose anti-gradient flow preserves $\Lambda K$ (by compactness and local convexity of $K$), there must be a closed geodesic $\gamma \in \Lambda K$ such that $\mathrm{C}_{d+1}(S^1 \cdot \gamma) \neq 0$ (see for instance [16, Theorems 4.2 and 4.3 p. 35-36] where one can replace isolated critical points by isolated critical $S^1$-orbits *verbatim*). By maximality of $d$, $\gamma$ and $c$ are geometrically distinct. But $c$ is the only homologically visible closed geodesic of $K$, a contradiction. □

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