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Freiheitssatz and phase transition for the density model of random groups

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Abstract

Magnus' Freiheitssatz [19] states that if a group is defined by a presentation with m generators and a single relator containing the last generating letter, then the first $m - 1$ letters freely generate a free subgroup. We study an analogue of this theorem in the Gromov density model of random groups [14], showing a phase transition phenomenon at density $d_r = \min\{\frac{1}{2}, 1 - \log_{2m-1}(2r - 1)\}$ with $1 \leq r \leq m - 1$: we prove that for a random group with m generators at density d , if $d < d_r$ then the first r letters freely generate a free subgroup; whereas if $d > d_r$ then the first r letters generate the whole group.

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1 Introduction

The *Freiheitssatz* (*freedom theorem* in German) is a fundamental theorem in combinatorial group theory. It was proposed by M. Dehn and proved by W. Magnus in his doctoral thesis [19] in 1930 (see [18] II.5). The theorem states that for a group presentation $G = \langle x_1, \dots, x_m | r \rangle$ where the single relator r is a cyclically reduced word, if x_m appears in r , then x_1, \dots, x_{m-1} freely generate a free subgroup of G .

Random groups are groups obtained by a probabilistic construction. Its first mentions, in terms of "generic property" for finitely presented groups, appear in the works of V. S. Guba [16] and M. Gromov [13] §0.2 in the late 1980s. The simplest model of random groups is the *few relator model* ([22] Definition 1). A few relator random group is defined by a group presentation $G_\ell = \langle x_1, \dots, x_m | r_1, \dots, r_k \rangle$ where the set of generators $X = \{x_1, \dots, x_m\}$ is fixed, and the relators r_1, \dots, r_k are chosen uniformly at random among all reduced words of X^\pm of length at most ℓ . The first well-known result of random groups ([13] §0.2) is that *asymptotically almost surely* (denoted by a.a.s., which means with probability converges to 1 when ℓ goes to infinity), a few relator random group G_ℓ is non-elementary hyperbolic.

For detailed surveys on random groups, see (in chronological order) [8] by E. Ghys, [22] by Y. Ollivier, [17] by I. Kapovich and P. Schupp and [6] by F. Bassino, C. Nicaud and P. Weil.

The density model of random groups

In 1993, Gromov introduced the *density model* of random groups in [14] 9.B. He considered a group presentation with a fixed set of m generators $X = \{x_1, \dots, x_m\}$ and $\lfloor (2m-1)^{d\ell} \rfloor$ randomly chosen relators, among the $2m(2m-1)^{\ell-1}$ reduced words of X^\pm of length ℓ . The parameter $d \in [0, 1]$ is called the *density*. Compare to the few relator model, the number of relators *grows exponentially* with the length ℓ . The main result of [14] 9.B is the *phase transition* at density one half: if $d > \frac{1}{2}$, then a.a.s. the group is trivial; if $d < \frac{1}{2}$, then a.a.s. the group is non-elementary hyperbolic.

In a 1996 paper [2], G. Arzhantseva and A. Ol'shanskii proved a few relator random group version of the Freiheitssatz: a.a.s. every $(m-1)$ -generated subgroup of a few relator random group G_ℓ is free. Arzhantseva proved several free subgroup properties subsequently for the few relator model in [3], [4] and [5]. Kapovich-Schupp [17] showed the existence of a small positive density $d(m)$ such that these results ([2], [3], [4] and [5]) can be generalized to a random group at any density $d < d(m)$. It was showed in [28] that the "every $(m-1)$ -generated subgroup is free" property [2] holds a.a.s. for a random group at any density $d < \frac{1}{120m^2 \ln(2m)}$.

In 2003, Gromov defined the general notion of random groups in [15] and proposed in Section 1.9 the following general problem: determining asymptotic invariants and phase transition phenomena for random groups. Since then, several variants of the phase transition phenomena have been discovered. For instance, A. Żuk [30] showed the freeness-property (T) phase transition for random triangular groups at density $1/3$ (see also [1] by Antoniuk-Łuczak-Świątkowski). Y. Ollivier proved in 2004 [21] the hyperbolicity-triviality transition for hyperbolic random groups, and in 2007 [23] the phase transition at density $1/5$ for Dehn's algorithm. In 2015 [12], D. Calegari and A. Walker showed that a random group at density $d < 1/2$ contains surface subgroups.

As we shall see, the main result of this paper is to highlight a new phase transition phenomenon, giving an analogue of the Freiheitssatz in the density model of

random groups. In particular, it partially answers Gromov's problem [15] 1.9 (iv): existence/nonexistence of non-free subgroups.

Main results

We say that a finite group presentation $G = \langle X | R \rangle$ satisfies the *Magnus Freiheitssatz property* if every subset of X of cardinality $|X| - 1$ freely generates a free subgroup of G . In particular, by Arzhantseva-Ol'shanskii's result [2], a few-relator random group G_ℓ has this property a.a.s.. We study the Magnus Freiheitssatz property in the density model of random groups.

Denote B_ℓ as the set of cyclically reduced words of $X^\pm = \{x_1^\pm, \dots, x_m^\pm\}$ of length at most ℓ . Let $G_\ell(m, d) = \langle X | R_\ell \rangle$ be a random group at density d . That is to say, R_ℓ is a permutation invariant random subset of B_ℓ with density d (see Section 3 or [28]). For example, it can be a uniform distribution on subsets of B_ℓ of cardinality $\lfloor |B_\ell| d \rfloor$, or a Bernoulli sampling on B_ℓ of parameter $|B_\ell|^{d-1}$.

For any $1 \leq r \leq m - 1$, let

$$d_r = \min \left\{ \frac{1}{2}, 1 - \log_{2m-1}(2r-1) \right\}.$$

Theorem 1 (Phase transition at density d_r , cf. Theorem 5.1).

1. If $d > d_r$, then a.a.s. x_1, \dots, x_r generate the whole group $G_\ell(m, d)$.
2. If $d < d_r$, then a.a.s. x_1, \dots, x_r freely generate a free subgroup of $G_\ell(m, d)$.

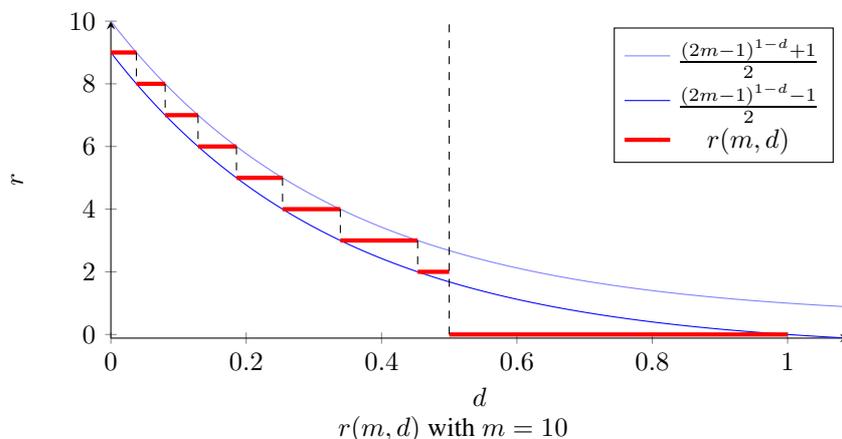
By symmetry, the set $\{x_1, \dots, x_r\}$ can be replaced by any subset X_r of X of cardinality r . In particular, if $0 \leq d < d_{m-1}$, then the group presentation $G_\ell(m, d) = \langle X | R_\ell \rangle$ has the Magnus Freiheitssatz property. More precisely for the first assertion, we prove that if $d > d_r$ then a.a.s. any generator x_i equals to a reduced word of X_r^\pm of length $\ell - 1$ in $G_\ell(m, d)$. Therefore, any relator $r_i \in R_\ell$ can be replaced by a reduced word r'_i of X_r^\pm of length at most $\ell(\ell - 1)$. Construct R'_ℓ by replacing every word of R_ℓ , we have the following result.

Corollary 2. *If $d_r < d < d_{r-1}$, then a.a.s. the group $G_\ell(m, d) = \langle X | R_\ell \rangle$ admits a presentation with r generators $\langle X_r | R'_\ell \rangle$ satisfying the Magnus Freiheitssatz property.*

Remark 3. We emphasize that R'_ℓ contains relators of lengths varying from ℓ to ℓ^2 . Such a presentation can not be studied using known methods in geometric or combinatorial group theory. Nevertheless, it gives us new examples of groups having the Magnus Freiheitssatz property.

Let $r = r(m, d)$ be the maximal number such that a.a.s. x_1, \dots, x_r freely generate a free subgroup of $G_\ell(m, d)$. By the phase transition at density $\frac{1}{2}$ [14], if $d > \frac{1}{2}$ then $r(m, d) = 0$. If $d \leq \frac{1}{2}$, by **Theorem 1**

$$\frac{(2m-1)^{1-d} - 1}{2} \leq r(m, d) \leq \frac{(2m-1)^{1-d} + 1}{2}.$$



As $r(m, d)$ is an integer, there is only one choice when d is not $1/2$ or one of d_r . Note that the value of $r(m, d)$ is not clear when $d \in \{d_1, \dots, d_{m-1}, 1/2\}$.

Our main theorem (Theorem 5.1) is a generalized version of **Theorem 1**. In the second assertion, we can replace the set $\{x_1, \dots, x_r\}$ by any set of r words of X^\pm of lengths at most $\frac{d_r-d}{5r}\ell$.

Outline of the paper

In Section 2, we first recall some essential tools in combinatorial group theory (Stallings graphs [27] and van Kampen diagrams [29]). We introduce *distortion van Kampen diagrams* to study the distortion of subgroups of a finitely presented group.

In order to give a concrete construction of random groups with density, we discuss probabilistic models of random subsets in Section 3 and recall the *intersection formula* by M. Gromov in [14]. Technical details are treated in [28].

Section 4 is dedicated to abstract van Kampen diagrams defined by Y. Ollivier in [21]. We apply his idea to distortion diagrams and define *abstract distortion diagrams*. The main technical lemma for our main theorem (Theorem 5.1) is to estimate the *number of fillings* of a given abstract distortion diagram (Lemma 4.12).

In the last section, we state a local result on distortion van Kampen diagrams (Lemma 5.2) and prove the main theorem by this lemma. The last subsection is then devoted to the proof of Lemma 5.2.

Acknowledgements. I would like to thank my supervisor, Thomas Delzant, for his patience and guidance, and for many interesting and helpful discussions on the subject.

2 Preliminaries on group theory

In this section, we fix a finite group presentation $G = \langle X | R \rangle$ where X is the set of generators and R is the set of relators. A word u in the alphabet X^\pm is called *reduced* if it has no sub-words of type xx^{-1} or $x^{-1}x$ for any $x \in X$. If u and v are words that represent the same element in G , we denote $u =_G v$.

We consider oriented combinatorial graphs and 2-complexes as defined in Chapter III.2. of Lyndon and Schupp [18].

2.1 Stallings graphs (graphs generating subgroups)

A *graph* is a pair $\Gamma = (V, E)$ where V is the set of *vertices* (also called points) and E is the set of (oriented) *edges*. Every edge $e \in E$ has a starting point $\alpha(e) \in V$, an ending point $\omega(e) \in V$ and an inverse edge $e^{-1} \in E$, satisfying $\alpha(e^{-1}) = \omega(e)$, $\omega(e^{-1}) = \alpha(e)$ and $(e^{-1})^{-1} = e$. The vertices $\alpha(e)$ and $\omega(e)$ are called the endpoints of e . An *undirected edge* is a pair of inverse edges $\{e, e^{-1}\}$. The *size* $|\Gamma|$ of a graph is the number of its undirected edges. The *rank* $\text{rk}(\Gamma)$ is the rank of its fundamental free group, which equals to $|\Gamma| - |V| + 1$ by Euler's characteristic.

A *path* on a graph Γ is a non-empty finite sequence of edges $p = e_1 \dots e_k$ such that $\omega(e_i) = \alpha(e_{i+1})$ for $i \in \{1, \dots, k-1\}$. The starting point and the ending point of the path p are defined by $\alpha(p) = \alpha(e_1)$ and $\omega(p) = \omega(e_k)$. The inverse of p is the path $p^{-1} = e_k^{-1} \dots e_1^{-1}$. A path is called *reduced* if there is no subsequence of the form ee^{-1} . A *loop* is a path whose starting point and ending point coincide. In this case $\alpha(p) = \omega(p)$ is called the starting point of the loop. A loop $p = e_1 \dots e_k$ is *cyclically reduced* if it is a reduced path with $e_k \neq e_1^{-1}$.

An *arc* of a graph Γ is a reduced path passing only by vertices of degree 2, except possibly for its endpoints. A *maximal arc* is an arc that can not be extended to another arc. Note that the endpoints of a maximal arc are not of degree 2. The following elementary fact for finite connected graphs can be deduced by Euler's characteristic.

Lemma 2.1. *Let Γ be a finite connected graph of rank $r \geq 1$ with no vertices of degree 1.*

1. *The number of vertices of degree at least 3 is bounded by $2(r-1)$.*
2. *The number of maximal arcs of Γ is bounded by $3(r-1)$.*

Lemma 2.2. *The number of topological types of finite connected graphs of rank at most r with no vertices of degree 1 is bounded by $(2r)^{6r}$.*

Proof. If $r = 1$ then the only topological type is a simple cycle. If $r \geq 2$, we may draw $a \leq 3(r-1)$ arcs on a set of $v \leq 2(r-1)$ vertices. There are at most $(v^2)^a \leq (2r)^{6r}$ ways. \square

A *labeled graph* (with respect to the alphabet X) is a graph $\Gamma = (V, E)$ with a labelling function on edges by generators $\varphi : E \rightarrow X^\pm$, satisfying $\varphi(e^{-1}) = \varphi(e)^{-1}$. We denote briefly $\Gamma = (V, E, \varphi)$. The labeling function φ extends naturally on the paths of Γ . If $p = e_1 \dots e_k$ is a path of Γ , then the word $\varphi(p) = \varphi(e_1) \dots \varphi(e_k)$ is called the *labeling word* of p . We say that a word u is *readable* on a labeled graph Γ if there exists a path p of Γ whose labeling word is u .

Labeled graphs are considered by Stallings [27] to represent subgroups of a free group. Let $\Gamma = (V, E, \varphi)$ be a finite connected labeled graph. Labeling words of the loops starting at a vertex $o \in V$ form a subgroup H of $G = \langle X | R \rangle$, which is the image of the fundamental group $\pi_1(\Gamma, o)$ by the group homomorphism induced by φ . If H is a conjugate of the subgroup $\varphi(\pi_1(\Gamma, o))$ in G for some $o \in V$, we say that H is *generated* by the labeled graph Γ .

Conversely, any finitely generated subgroup H can be generated by a labeled graph. One can choose a system of generators h_1, \dots, h_r of H , and label them on the wedge of r simple cycles of lengths $|h_1|, \dots, |h_r|$. A labeled graph is *reduced* if it has no pair of edges with the same label and starting point, and, it has no vertices of degree 1. By doing reductions on the construction above (see [2] and [20]), if H is a subgroup of rank r , then there is a *reduced* labeled graph of rank r that generates H .

2.2 Van Kampen diagrams

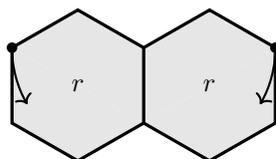
We consider van Kampen diagrams defined by Lyndon and Schupp in [18] Chapter III.9. A 2-complex is a triplet $W = (V, E, F)$, where (V, E) is a graph and F is the set of (oriented) faces. Every face $f \in F$ has a boundary ∂f , which is a cyclically reduced loop of (V, E) , and an inverse face $f^{-1} \in F$ satisfying $\partial(f^{-1}) = (\partial f)^{-1}$ and $(f^{-1})^{-1} = f$. An *undirected face* is a pair of inverse faces $\{f, f^{-1}\}$. The size $|W|$ is the number of undirected faces.

Note that our definition is slightly more precise than [18]: Every face $f \in F$ has a starting point and an orientation given by ∂f . If $\partial f = e_1 \dots e_k$, we say that e_i is *attached* to f and is the i -th boundary edge of f for $1 \leq i \leq k$. In this case, we say that $\{e_i, e_i^{-1}\}$ is attached to $\{f, f^{-1}\}$. An edge is called *isolated* if it is not attached to any face.

A *van Kampen diagram* (with respect to $G = \langle X|R \rangle$) is a finite, planar (embedded in \mathbb{R}^2) and simply connected 2-complex $D = (V, E, F)$ with two compatible labeling functions, on edges by generators $\varphi_1 : E \rightarrow X^\pm$ and on faces by relators $\varphi_2 : F \rightarrow R^\pm$. Compatible means that (V, E, φ_1) is a labeled graph, $\varphi_2(f^{-1}) = \varphi_2(f)^{-1}$ and $\varphi_1(\partial f) = \varphi_2(f)$. Note that if a diagram D has no isolated edges (for example, a disk), then φ_1 is determined by φ_2 . We denote briefly $D = (V, E, F, \varphi_1, \varphi_2)$.

According to [11] p.159, a van Kampen diagram is either a disk or a concatenation of disks and segments. The boundary ∂D is the boundary of $\mathbb{R}^2 \setminus D$, which is a sub-graph of its underlying graph (V, E) . A *boundary path* is a path on ∂D defined in a natural way in [18] p.150. A *boundary word* of D is then the labeling word of a boundary path, unique up to cyclic conjugations and inversions. The *boundary length* of D is the length of a boundary path, denoted $|\partial D|$.

Let $D = (V, E, F, \varphi_1, \varphi_2)$ be a van Kampen diagram. A pair of faces $f, f' \in F$ is *reducible* if they have the same label and there is a common edge on their boundaries at the same position. A van Kampen diagram is called *reduced* if there is no reducible pair of faces.



a reducible pair of faces

In 1933, E. van Kampen showed in [29] that a word u of X^\pm is trivial in a finitely presented group $G = \langle X|R \rangle$ if and only if it is a boundary word of a van Kampen diagram of G . In [24] §11.6, A. Ol'shanskii improved this result to *reduced* diagrams.

Lemma 2.3 (Van Kampen's lemma, Ol'shanskii's version). *A word w of X^\pm is trivial in $G = \langle X|R \rangle$ if and only if it is a boundary word of a **reduced** van Kampen diagram.*

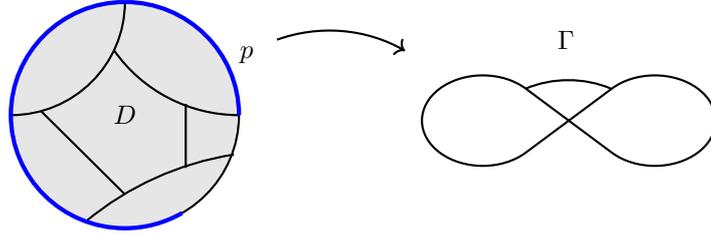
2.3 Distortion van Kampen diagrams

Let $G = \langle X|R \rangle$ be a group presentation. For any word u of X^\pm , we denote $|u|$ its word length and $\|u\|_G$ the distance between the endpoints of its image in the Cayley graph $\text{Cay}(G, X)$. Let Γ be a finite, connected and reduced labeled graph. Its universal

covering $\tilde{\Gamma}$ is an infinite, connected and reduced labeled tree, with a natural label-preserving map $\tilde{\Gamma} \rightarrow \text{Cay}(G, X)$.

If the map $\tilde{\Gamma} \rightarrow \text{Cay}(G, X)$ is a λ -quasi isometric embedding with some $\lambda \geq 1$ (in other words, every reduced word u readable on Γ satisfies $|u| \leq \lambda \|u\|_G$), then any subgroup generated by Γ is a free group. To study this question, we introduce *distortion van Kampen diagrams*.

Definition 2.4 (Distortion diagram). *A distortion van Kampen diagram of (G, Γ) is a pair (D, p) where D is a van Kampen diagram of G and p is a cyclic sub-path ∂D whose labeling word is readable on Γ .*



Lemma 2.5. *Let $\lambda \geq 1$. If every **disk-like** and **reduced** distortion van Kampen diagram (D, p) of (G, Γ) satisfies*

$$|p| \leq \frac{\lambda}{1 + \lambda} |\partial D|, \quad (\star)$$

then the map $\tilde{\Gamma} \rightarrow \text{Cay}(G, X)$ is a λ -quasi isometric embedding.

In particular, any subgroup generated by Γ is free.

Proof. Let u be a reduced word that is readable on Γ . Let v be a shortest word (whose image is a geodesic in G) such that $uv =_G 1$. We shall check that $|u| \leq \lambda |v|$.

By van Kampen's lemma (Lemma 2.3), there exists a reduced van Kampen diagram D whose boundary word is uv . If D is disk-like, then by the hypothesis (\star) we have $|u| \leq \frac{\lambda}{1 + \lambda} (|u| + |v|)$, which gives $|u| \leq \lambda |v|$.

Otherwise, we decompose D into disks and segments D_1, \dots, D_k (as in [11] p.159). The path of v does not intersect itself because it is a geodesic in G . The path of u on D does not intersect itself. If it did, as u is reduced, there would be a disk-like sub-diagram whose boundary word is readable on Γ , which is impossible because of (\star) .

Hence, for any $1 \leq i \leq k$, there are exactly two vertices on ∂D_i separating u and v , which are the only possible vertices of degree not equal to 2. The boundary word of D_i is written as $u_i v_i$ where u_i is a subword of u and v_i is a subword of v . If D_i is a segment, then it is read once by u and once by v with opposite directions, so $|u_i| = |v_i| \leq \lambda |v_i|$. If D_i is a disk, then $|u_i| \leq \lambda |v_i|$ by (\star) . We conclude that

$$|u| = \sum_{i=1}^k |u_i| \leq \sum_{i=1}^k \lambda |v_i| = \lambda |v|.$$

□

2.4 Hyperbolic groups

In this subsection we recall several facts of hyperbolic groups defined by M. Gromov in [13]. Let $G = \langle X | R \rangle$ be a finite group presentation. The Cayley graph $\text{Cay}(G, X)$

with the usual length metric is δ -hyperbolic if each side of any geodesic triangle is δ -close to the two other sides ([10] Chapter 1). In this case, G is called a *hyperbolic group*.

We start by a criterion of hyperbolicity in [13] Chapter 2.3. See also [26] by H. Short and [10] Chapter 6. For a precise estimation of hyperbolicity constants, see [9] Lemma 3.11 and Lemma 3.12 by C. Champetier.

Theorem 2.6 (Isoperimetric inequality). *Let ℓ be the longest relator length in R . The group $G = \langle X|R \rangle$ is hyperbolic if and only if there exists a real number $\beta > 0$ such that every reduced van Kampen diagram D satisfies the following isoperimetric inequality :*

$$|\partial D| \geq \beta \ell |D|.$$

In this case, the Cayley graph $\text{Cay}(G, X)$ is δ -hyperbolic with

$$\delta = \frac{4\ell}{\beta}.$$

The local-global principle of hyperbolicity is due to M. Gromov in [13]. For other proofs, see [7] Chapter 8 by B. H. Bowditch or [25] by P. Papasoglu. We state here a sharpened version by Y. Ollivier in [23] Proposition 8.

Theorem 2.7. (*Local-global principal of hyperbolicity*) *For any $\alpha > 0$ and $\varepsilon > 0$, there exists an integer $K = K(\alpha, \varepsilon)$ such that, if every reduced disk-like diagram D with $|D| \leq K$ satisfies*

$$|\partial D| \geq \alpha \ell |D|,$$

then every reduced diagram D satisfies

$$|\partial D| \geq (\alpha - \varepsilon) \ell |D|.$$

Recall that a path p in $\text{Cay}(X, R)$ is a λ -quasi-geodesic if every sub-path u of p satisfies $|u| \leq \lambda \|u\|_G$. It is a L -local λ -quasi geodesic if such an inequality is satisfied by every sub-path of length at most L . Here is the local-global principle for quasi-geodesics in hyperbolic groups, stated by Gromov in [13] 7.2.A and 7.2.B. See [10] Chapter 3 for a proof.

Theorem 2.8. *Let $G = \langle X|R \rangle$ be a group presentation such that $\text{Cay}(G, X)$ is δ -hyperbolic. Let $\lambda \geq 1$, then*

1. *Every λ -quasi-geodesic is $100\delta(1 + \log \lambda)$ close to any geodesic joining its endpoints.*
2. *Every $1000\lambda\delta$ -local λ -quasi-geodesic is a (global) 2λ -quasi-geodesic.*

3 Random subsets and random groups

In this section, we recall the definition of *random groups with density* by M. Gromov in [14]. Proofs of Proposition 3.2, Proposition 3.3, Theorem 3.4 and Theorem 3.5 are in [28].

3.1 Densable sequences of random subsets

A *random subset* A of a finite set E is a $\mathcal{P}(E)$ -valued random variable, where $\mathcal{P}(E)$ is the set of subsets of E . We say that A is *permutation invariant* if $\Pr(A = a) = \Pr(A = \sigma(a))$ for any permutation σ of E and any subset a of E .

In this subsection, we consider a sequence of finite sets $\mathbf{E} = (E_\ell)_{\ell \in \mathbb{N}}$ with $|E_\ell| \xrightarrow{\ell \rightarrow \infty} \infty$. Let (Q_ℓ) be a sequence of events. We say that the event Q_ℓ holds *asymptotically almost surely* if $\Pr(Q_\ell) \xrightarrow{\ell \rightarrow \infty} 1$. We denote briefly a.a.s. Q_ℓ . Note that the intersection of a finite number of events that hold a.a.s. is an event that holds a.a.s.. In addition, we have the following proposition.

Proposition 3.1. *Let $\mathbf{Q} = (Q_\ell)$, $\mathbf{R} = (R_\ell)$ be sequences of events. If a.a.s. Q_ℓ and a.a.s. " R_ℓ under the condition Q_ℓ ", then a.a.s. R_ℓ .*

Proof. Denote by $\overline{Q_\ell}$ the complement of Q_ℓ . By the two hypotheses, $\Pr(Q_\ell) \rightarrow 1$ and $\Pr(R_\ell|Q_\ell) \rightarrow 1$. Either $\overline{Q_\ell}$ is empty and $\Pr(R_\ell) = \Pr(R_\ell|Q_\ell) \rightarrow 1$, or by the formula of total probability

$$\Pr(R_\ell) = \Pr(Q_\ell)\Pr(R_\ell|Q_\ell) + \Pr(\overline{Q_\ell})\Pr(R_\ell|\overline{Q_\ell}) \xrightarrow{\ell \rightarrow \infty} 1.$$

□

Let $d \in \{-\infty\} \cup [0, 1]$. A sequence of random subsets $\mathbf{A} = (A_\ell)$ of $\mathbf{E} = (E_\ell)$ is *densable with density d* if the sequence of real-valued random variables

$$\log_{|E_\ell|}(|A_\ell|)$$

converges in probability (or in distribution) to the constant d . We denote

$$\text{dens } \mathbf{A} = d.$$

By definition, $\text{dens } \mathbf{A} = d$ if and only if

$$\forall \varepsilon > 0 \text{ a.a.s. } |E_\ell|^{d-\varepsilon} \leq |A_\ell| \leq |E_\ell|^{d+\varepsilon}.$$

In particular, $\text{dens } \mathbf{A} = -\infty$ if and only if a.a.s. $A_\ell = \emptyset$; $\text{dens } \mathbf{A} = 0$ if and only if a.a.s. $A_\ell \neq \emptyset$ and $|A_\ell|$ is sub-exponential.

Here is the main example of a densable sequence of permutation invariant random subsets. The proofs of Theorem 3.4 and Theorem 3.5 are much simpler in this model (see [28]).

Proposition 3.2 (Bernoulli density model, [28] Proposition 1.12). *Let $0 < d \leq 1$. Let A_ℓ be a sequence of random subsets of E_ℓ such that every element $e \in E_\ell$ is taken independently with probability $p_\ell = |E_\ell|^{d-1}$. Then $\mathbf{A} = (A_\ell)$ is a densable sequence of permutation invariant random subsets with density d .*

Note that in the case $d = 0$, the Bernoulli model is not densable. If A_ℓ is a Bernoulli sequence with density $d > 0$, then for any distinct elements e_1, \dots, e_k in E_ℓ , we have $\Pr(e_1, \dots, e_k \in A_\ell) = p_\ell^k = |E_\ell|^{k(d-1)}$ by independence. This property is, in general, not true for an arbitrary densable sequence of permutation invariant random subsets. Nevertheless, it can be approached asymptotically.

Proposition 3.3 (Similar to [28] Lemma 3.10). *Let $\mathbf{A} = (A_\ell)$ be a densable sequence of permutation invariant random subsets of $\mathbf{E} = (E_\ell)$ with density d . Let $\varepsilon > 0$. Denote Q_ℓ the event $|E_\ell|^{d-\varepsilon} \leq |A_\ell| \leq |E_\ell|^{d+\varepsilon}$ (we have a.a.s. Q_ℓ by definition). Let e_1, \dots, e_k be distinct elements in E_ℓ . For ℓ large enough,*

$$|E_\ell|^{k(d-1-2\varepsilon)} \leq \Pr(e_1, \dots, e_k \in A_\ell | Q_\ell) \leq |E_\ell|^{k(d-1+2\varepsilon)}. \quad \square$$

3.2 The intersection formula

We recall here the *intersection formula* for random subsets. See [14] for the original version by M. Gromov, and [28] Section 2 for a proof.

Theorem 3.4 (The intersection formula). *Let $\mathbf{A} = (A_\ell)$, $\mathbf{B} = (B_\ell)$ be independent densable sequences of permutation invariant random subsets.*

1. *If $\text{dens } \mathbf{A} + \text{dens } \mathbf{B} < 1$, then a.a.s. $A_\ell \cap B_\ell = \emptyset$.*
2. *If $\text{dens } \mathbf{A} + \text{dens } \mathbf{B} > 1$, then $\mathbf{A} \cap \mathbf{B} := (A_\ell \cap B_\ell)$ is a densable sequence of permutation invariant random subset and*

$$\text{dens}(\mathbf{A} \cap \mathbf{B}) = \text{dens } \mathbf{A} + \text{dens } \mathbf{B} - 1.$$

In particular, a.a.s. $A_\ell \cap B_\ell \neq \emptyset$. \square

A *fixed* subset can be regarded as a constant random subset. The density of a sequence of fixed subsets can be defined by the same way. Note that a sequence of subsets $\mathbf{F} = (F_\ell)$ of $\mathbf{E} = (E_\ell)$ is densable with density d if and only if

$$|F_\ell| = |E_\ell|^{d+o(1)}.$$

We consider also the intersection between a sequence of random subsets and a sequence of fixed subsets. See [28] Section 3 for a proof.

Theorem 3.5 ([28] Theorem 3.7). *Let $\mathbf{A} = (A_\ell)$ be a densable sequence of permutation invariant random subsets of \mathbf{E} . Let $\mathbf{F} = (F_\ell)$ be a densable sequence of fixed subsets.*

1. *If $\text{dens } \mathbf{A} + \text{dens } \mathbf{F} < 1$, then a.a.s. $A_\ell \cap F_\ell = \emptyset$.*
2. *If $\text{dens } \mathbf{A} + \text{dens } \mathbf{F} > 1$, then the sequence $\mathbf{A} \cap \mathbf{F}$ is densable in \mathbf{E} , with density*

$$\text{dens } \mathbf{A} + \text{dens } \mathbf{F} - 1.$$

In addition, $\mathbf{A} \cap \mathbf{F}$ is densable and permutation invariant in \mathbf{F} , with density

$$\frac{\text{dens } \mathbf{A} + \text{dens } \mathbf{F} - 1}{\text{dens } \mathbf{F}}.$$

\square

3.3 The density model of random groups

Fix an alphabet $X = \{x_1, \dots, x_m\}$ as generators of group presentations. Let B_ℓ be the set of cyclically reduced words on $X^\pm = \{x_1^\pm, \dots, x_m^\pm\}$ of lengths at most ℓ . Note that

$$|B_\ell| = (2m - 1)^{\ell + o(\ell)}.$$

We consider a sequence of random groups $\mathbf{G}(m, d) = (G_\ell(m, d))$ defined by random presentations $G_\ell(m, d) := \langle X | R_\ell \rangle$ where $\mathbf{R} = (R_\ell)$ is a dense sequence of permutation invariant random subsets of $\mathbf{B} = (B_\ell)$ with density d . Such a sequence is called a *sequence of random groups at density d* .

The number of relators $|R_\ell|$ is a real-valued random variable and is concentrated to $(2m - 1)^{d\ell}$. More precisely, for any $\varepsilon > 0$ a.s.

$$(2m - 1)^{d\ell - \varepsilon\ell} \leq |R_\ell| \leq (2m - 1)^{d\ell + \varepsilon\ell}.$$

We are interested in asymptotic behaviors of a sequence of random groups. In his book [14], Gromov observed that there is a *phase transition* at density $1/2$.

Theorem 3.6 (Phase transition at density $1/2$). *Let $\mathbf{G}(m, d) = (G_\ell(m, d)) = (\langle X | R_\ell \rangle)$ be a sequence of random groups at density d .*

1. *If $d > 1/2$, then a.s. $G_\ell(m, d)$ is a trivial group.*
2. *If $d < 1/2$, then a.s. $G_\ell(m, d)$ is a hyperbolic group, and the Cayley graph $\text{Cay}(G_\ell, X)$ is δ -hyperbolic with $\delta = \frac{4\ell}{1-2d}$.*

In addition, for any $s > 0$, a.s. every reduced van Kampen diagram D of $G_\ell(m, d)$ satisfies the isoperimetric inequality

$$|\partial D| \geq (1 - 2d - s)\ell|D|.$$

The proof of our main theorem (Theorem 5.1) is very similar to Ollivier's proof [22] for this theorem. We give here a proof for the first assertion and an idea of proof for the second assertion.

Proof of Theorem 3.6.1. Let S_ℓ be the set of cyclically reduced words of length exactly ℓ . The sequence $(S_{\ell-1})$ is a fixed sequence of subsets of $\mathbf{B} = (B_\ell)$ of density 1. By the intersection formula (Theorem 3.5), the two sequences $(x_1 R_\ell \cap x_1 S_{\ell-1})$ and $(R_\ell \cap x_1 S_{\ell-1})$ are both sequences of random subsets of $(x_1 S_{\ell-1})$ with density d . By the intersection formula between random subsets (Theorem 3.4), their intersection is a sequence of random subsets with density $(2d - 1) > 0$, which is a.s. not empty. Thus, a.s. there exists a word $w \in S_{\ell-1}$ such that $w \in R_\ell$ and $x_1 w \in R_\ell$, so a.s. $x_1 = 1$ in G_ℓ by canceling w .

The argument works for every generator $x_i \in X$. By intersecting a finite number of a.s. events, a.s. G_ℓ is isomorphic to the trivial group. \square

By Theorem 2.6 and Theorem 2.7, to prove Theorem 3.6.2, it is sufficient to find a local isoperimetric inequality. See [22] for a proof by Y. Ollivier.

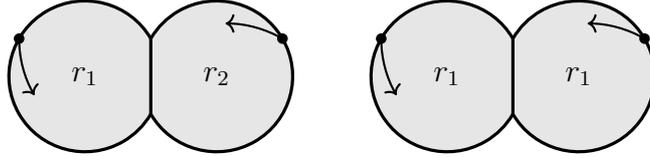
Lemma 3.7 (Local isoperimetric inequality). *Let $s > 0$. If $d < 1/2$, then for $K = K(1 - 2d - \frac{s}{2}, \frac{s}{2})$ provided by Theorem 2.7, a.s. any reduced disc-like diagram D of $G_\ell(m, d)$ with at most K faces satisfies the isoperimetric inequality*

$$|\partial D| \geq \left(1 - 2d - \frac{s}{2}\right) \ell |D|.$$

\square

4 Abstract diagrams

Two van Kampen diagrams having the same underlying 2-complex may not use the same number of relators. For example, to check that if a group satisfies the $C'(\lambda)$ small cancellation condition ([28] Theorem 4.3), we have to consider the following two types of diagrams. By Proposition 3.3, they have different probabilities to be a diagram of a given random group.



Y. Ollivier introduced *abstract van Kampen diagrams* ([21] p.10) to surround this problem.

4.1 Abstract van Kampen diagrams

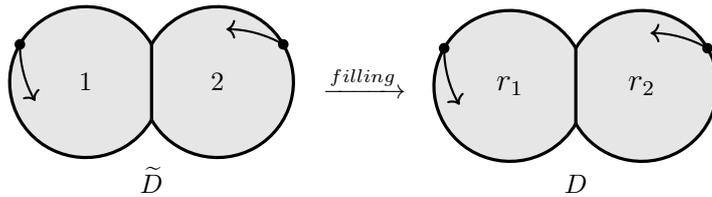
Definition 4.1 (Abstract diagram, Ollivier [22]). *An abstract van Kampen diagram \tilde{D} is a finite, planar and simply-connected 2-complex (V, E, F) with a labeling function on faces by integer numbers $\tilde{\varphi}_2 : F \rightarrow \{1, 1^-, 2, 2^-, \dots, k, k^-\}$ satisfying $\tilde{\varphi}_2(f^{-1}) = \tilde{\varphi}_2(f)^-$. We denote $\tilde{D} = (V, E, F, \tilde{\varphi}_2)$,*

By convention $(i^-)^- = i$ for any $1 \leq i \leq k$. The numbers $\{1, \dots, k\}$ are called *abstract relators* of \tilde{D} .

Similarly to a van Kampen diagram, a pair of faces $f, f' \in F$ is *reducible* if they have the same label, and they share an edge at the same position of their boundaries. An abstract diagram is called *reduced* if there is no reducible pair of faces.

Let $D = (V, E, F, \varphi_1, \varphi_2)$ be a van Kampen diagram of a group presentation $G = \langle X | R \rangle$. Let $\{r_1, \dots, r_k\} \subset R$ be the set of relators used in D . Define $\tilde{\varphi}_2 : F \rightarrow \{1, 1^-, \dots, k, k^-\}$ by $\tilde{\varphi}_2(f) = i$ if $\varphi_2(f) = r_i$. We obtain an abstract diagram $\tilde{D} = (V, E, F, \tilde{\varphi}_2)$ with k abstract relators, called an *underlying abstract diagram* of D .

An abstract diagram \tilde{D} is *fillable* by a group presentation $G = \langle X | R \rangle$ (or by a set of relators R) if there exists a van Kampen diagram D of G , called a *filled diagram* of \tilde{D} , whose underlying abstract diagram is \tilde{D} . That is to say, there exists k different relators $r_1, \dots, r_k \in R$ such that the construction $\varphi_2(f) = r_i$ if and only if $\tilde{\varphi}_2(f) = i$ gives a diagram $D = (V, E, F, \varphi_1, \varphi_2)$ of G . The k -tuple (r_1, \dots, r_k) is called a *filling* of \tilde{D} . As we picked different relators, \tilde{D} is reduced if and only if a filled diagram D is reduced.

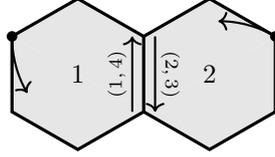


We assume that faces with the same label of \tilde{D} have the same boundary length, otherwise \tilde{D} would never be fillable. Denote ℓ_i the length of the abstract relator i for $1 \leq i \leq k$. Let $\ell = \max\{\ell_1, \dots, \ell_k\}$ be the maximal boundary length of faces of \tilde{D} .

Notation. The pairs of integers $(i, 1), \dots, (i, \ell_i)$ are called abstract letters of i .

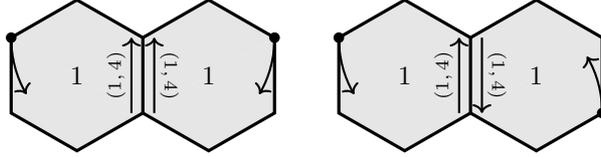
The set of abstract letters of \tilde{D} , denoted \tilde{X} , is then a subset of $\{1, \dots, k\} \times \{1, \dots, \ell\}$, endowed with the lexicographic order.

We decorate undirected edges of \tilde{D} by abstract letters and directions. Let $f \in F$ labeled by i and let $e \in E$ at the j -th position of ∂f . The edge $\{e, e^{-1}\}$ is decorated, on the side of $\{f, f^{-1}\}$, by an arrow indicating the direction of e and the abstract letter (i, j) . This decoration on $\{e, e^{-1}\}$ is called the *decoration from f at the position j* . The number of decorations on an edge $\{e, e^{-1}\}$ is the number of its adjacent faces $\{f, f^{-1}\}$ with multiplicity (0, 1 or 2 when \tilde{D} is planar).



For any filling (r_1, \dots, r_k) of \tilde{D} , we construct the *canonical function* $\phi : \tilde{X} \rightarrow X^\pm$ such that $r_i = \phi(i, 1) \dots \phi(i, \ell_i)$ for any $1 \leq i \leq k$. If an edge $\{e, e^{-1}\}$ is decorated by two abstract letters $(i, j), (i', j')$, then $\phi(i', j') = \phi(i, j)$ if they have the same direction, or $\phi(i', j') = \phi(i, j)^{-1}$ if they have opposite directions. For example, the diagram above deduce $\phi(1, 4) = \phi(2, 3)^{-1}$.

Note that if \tilde{D} is reduced, then by definition an abstract letter can not be decorated twice on an edge with the same direction. If \tilde{D} is fillable (by the set of all relators), then an abstract letter (i, j) can not be decorated twice on an undirected edge with opposite directions, otherwise we have $\phi(i, j) = \phi(i, j)^{-1}$ in the set of generators X .



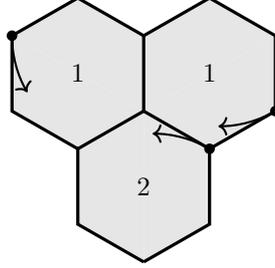
In the following we assume that \tilde{D} is fillable and reduced, so that the abstract letters decorated on an edge $\{e, e^{-1}\}$ are all different. In particular, there exists a unique face $\{f, f^{-1}\}$ (at a unique position) from which the decoration is (lexicographically) minimal. Whence the following two definitions.

Definition 4.2 (Preferred face of an edge). Let $\{e, e^{-1}\}$ be an edge of \tilde{D} . Let $\{f, f^{-1}\}$ be the adjacent face of $\{e, e^{-1}\}$ from which the decoration is minimal. Then $\{f, f^{-1}\}$ is called the **preferred face** of $\{e, e^{-1}\}$.

Definition 4.3 (free-to-fill). An abstract letter (i, j) of \tilde{D} is **free-to-fill** if, for any edge $\{e, e^{-1}\}$ decorated by (i, j) , it is the minimal decoration on $\{e, e^{-1}\}$.

Note that an abstract letter (i, j) is free-to-fill if and only if every face f labeled by i is the preferred face of its j -th boundary edge. In other words, if (i, j) is *not free-to-fill*, then there exists an edge $\{e, e^{-1}\}$ decorated by (i, j) that has another decoration $(i', j') < (i, j)$.

For example, in the abstract diagram below, $(1, 4)$, $(2, 1)$ and $(2, 2)$ are not free-to-fill. The other abstract letters are free-to-fill.



Denote $F^+ = \{f \in F \mid \tilde{\varphi}_2(f) \in \{1, \dots, k\}\}$. It gives a preferred orientation for each undirected face of $\tilde{D} = (V, E, F, \tilde{\varphi}_2)$. Let \overline{E} be the set of undirected edges of \tilde{D} .

Lemma 4.4. *Let \tilde{D} be a reduced fillable abstract diagram without isolated edges. For every face $f \in F^+$, let \overline{E}_f be the set of edges $\{e, e^{-1}\}$ on the boundary of $\{f, f^{-1}\}$ such that $\{f, f^{-1}\}$ is the preferred face of $\{e, e^{-1}\}$. Then*

$$\overline{E} = \bigsqcup_{f \in F^+} \overline{E}_f.$$

Proof. For every edge $\{e, e^{-1}\}$ there exists a unique face $f \in F^+$ such that $\{e, e^{-1}\} \in \overline{E}_f$. Hence the sets \overline{E}_f with $f \in F^+$ are pairwise disjoint. Their reunion is the set of edges because every edge is adjacent to at least one face. \square

4.2 Abstract distortion van Kampen diagrams

We generalize the idea of abstract diagrams to distortion van Kampen diagrams.

Definition 4.5 (Abstract distortion diagram). *An abstract distortion van Kampen diagram is a pair (\tilde{D}, p) where \tilde{D} is an abstract diagram and p is a path on $\partial\tilde{D}$.*

Let $G = \langle X \mid R \rangle$ be a group presentation and let Γ be a labeled graph. An abstract distortion diagram (\tilde{D}, p) is *fillable* by the pair (G, Γ) (or by the pair (R, Γ)) if there exists a filled diagram D of \tilde{D} such that (D, p) is a distortion diagram of (G, Γ) . The distortion diagram (D, p) is called a *filled distortion diagram* of (\tilde{D}, p) .

In the following, an abstract distortion diagram (\tilde{D}, p) is reduced, fillable and without isolated edges. Recall that $\tilde{X} \subset \{1, \dots, k\} \times \{1, \dots, \ell\}$ is the set of abstract letters. Let \overline{p} be the set of undirected edges given by p . In an abstract distortion diagram we distinguish between two types of free-to-fill abstract letters: those that decorate an edge of \overline{p} and those that do not.

Definition 4.6. *Let (i, j) be an abstract letter of (\tilde{D}, p) .*

- (i) (i, j) is **free-to-fill** if it is free-to-fill for the abstract diagram \tilde{D} and it does not decorate any edge of \overline{p} .
- (ii) (i, j) is **semi-free-to-fill** if it is free-to-fill for the abstract diagram \tilde{D} and it decorates an edge of \overline{p} .

(iii) Otherwise (i, j) is not free-to-fill.

Notation. Let i be an abstract relator of \tilde{D} . We denote α_i the number of faces labeled by i , η_i the number of free-to-fill abstract letters of i , and η'_i the number of semi-free-to-fill abstract letters of i .

Note that $\ell_i - \eta_i - \eta'_i$ is the number of non free-to-fill edges.

Lemma 4.7. Recall that \overline{E}_f is the set of edges on the boundary of f that prefers $\{f, f^{-1}\}$. Let i be an abstract relator. For any face $f \in F$ with $\tilde{\varphi}_2(f) = i$, we have

$$\eta'_i \leq |\overline{E}_f \cap \overline{p}| \quad \text{and} \quad \eta_i \leq |\overline{E}_f| - |\overline{E}_f \cap \overline{p}|.$$

Proof. Let $\{e, e^{-1}\}$ be the edge at the j -th position of ∂f . It is decorated by (i, j) . If $\{f, f^{-1}\}$ is not preferred by $\{e, e^{-1}\}$, then (i, j) is not free-to-fill because there is a smaller decoration on $\{e, e^{-1}\}$.

Thus, if $\{e, e^{-1}\} \in \overline{E}_f \cap \overline{p}$ then (i, j) is semi-free-to-fill, which gives the first inequality. Similarly, if $\{e, e^{-1}\} \in \overline{E}_f \setminus \overline{p}$, then (i, j) is free-to-fill and we have the second inequality. \square

Lemma 4.8. Recall that \overline{E} is the set of undirected edges. The following two inequalities hold.

$$\sum_{i=1}^k \alpha_i \eta'_i \leq |\overline{p}|, \quad \sum_{i=1}^k \alpha_i \eta_i \leq |\overline{E}| - |\overline{p}|.$$

Proof. By Lemma 4.7, for every $1 \leq i \leq k$

$$\alpha_i \eta'_i \leq \sum_{f \in F, \tilde{\varphi}_2(f)=i} |\overline{E}_f \cap \overline{p}|.$$

Apply Lemma 4.4,

$$\sum_{i=1}^k \alpha_i \eta'_i \leq \sum_{f \in F^+} |\overline{E}_f \cap \overline{p}| \leq |\overline{p}|.$$

We get the second inequality by replacing η'_i by η_i and $|\overline{p}|$ by $|\overline{E} \setminus \overline{p}|$. \square

4.3 The number of fillings of an abstract distortion diagram

Recall that B_ℓ is the set of cyclically reduced words on $X^\pm = \{x_1^\pm, \dots, x_m^\pm\}$ of length at most ℓ . Let Γ be a graph labeled by X with $\text{rk}(\Gamma) \leq r$.

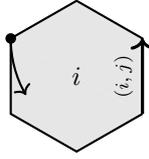
Let (\tilde{D}, p) be an abstract distortion diagram with k abstract relators. Assume that \tilde{D} is reduced, fillable and has no isolated edges. Let ℓ be the longest boundary length of faces of \tilde{D} .

Denote $N_\ell(\tilde{D}, p, \Gamma)$ the set of fillings (r_1, \dots, r_k) of (\tilde{D}, p) by (B_ℓ, Γ) . In this subsection, we give an upper bound of the number of fillings $|N_\ell(\tilde{D}, p, \Gamma)|$.

Lemma 4.9. The number of reduced words u of length L that is readable on Γ is at most $2|\Gamma|(2r-1)^L$.

Proof. We estimate the number of paths p on Γ whose labeling word can be reduced. Take an oriented edge of Γ as the first edge of p , there are $2|\Gamma|$ choices. Every vertex is of degree at most $2r$ because $\text{rk}(\Gamma) \leq r$. As p is reduced, every time we take the next edge, there are at most $(2r - 1)$ choices. Hence there are at most $2|\Gamma|(2r - 1)^L$ paths. \square

A vertex of (\tilde{D}, p) is called *distinguished* if it is either of degree at least 3, or the starting point of a face, or an endpoint of p . Let i be an abstract letter of (\tilde{D}, p) . It can be regarded as a 2-complex with two inverse faces $\{i, i^-\}$ and $2\ell_i$ edges $(i, 1), \dots, (i, \ell_i)$ with their inverses, such that $\partial i = (i, 1) \dots (i, \ell_i)$.



A vertex of ∂i is *marked* if there exists a face f of \tilde{D} labeled by i such that the corresponding vertex is distinguished. Note that the starting point of ∂i is marked. Marked vertices divide the loop ∂i into segments, called *elementary segments*.

Consequently, an elementary segment is a sequence of abstract letters $(i, j)(i, j + 1) \dots (i, j + t)$ such that, if a path $e_j \dots e_{j+t}$ on \tilde{D} is decorated by $(i, j) \dots (i, j + t)$, then it passes by no distinguished points except for its endpoints.

Lemma 4.10. *Let $(i, j) \dots (i, j + t)$ be an elementary segment. The abstract letters $(i, j), \dots, (i, j + t)$ are either all free-to-fill, or all semi-free-to-fill, or all not free-to-fill.*

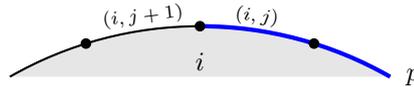
Proof. We shall check that if the vertex between two consecutive abstract letters (i, j) and $(i, j + 1)$ is not marked, then they are of the same type.

Recall that if an edge $\{e_1, e_1^{-1}\}$ is decorated by (i, j) from the face $\{f, f^{-1}\}$, then there is an edge $\{e_2, e_2^{-1}\}$ next to $\{e_1, e_1^{-1}\}$, decorated by $(i, j + 1)$ from the same face $\{f, f^{-1}\}$. Assume that the vertex between (i, j) and $(i, j + 1)$ is not marked so that the vertex between $\{e_1, e_1^{-1}\}$ and $\{e_2, e_2^{-1}\}$ is not distinguished.

We suppose by contradiction that (i, j) and $(i, j + 1)$ are not of the same type. There are $3^2 - 3 = 6$ cases, grouped into three cases.

case 1. (i, j) is semi-free-to-fill and $(i, j + 1)$ is free-to-fill, or inversely:

Recall that if (i, j) is semi-free-to-fill in the abstract distortion diagram (\tilde{D}, p) , then it decorates an undirected edge $\{e_1, e_1^{-1}\}$ on p . As $(i, j + 1)$ is free-to-fill, the edge $\{e_2, e_2^{-1}\}$ decorated by $(i, j + 1)$ from the same face is not on p . So the vertex between $\{e_1, e_1^{-1}\}$ and $\{e_2, e_2^{-1}\}$ is distinguished, contradiction.

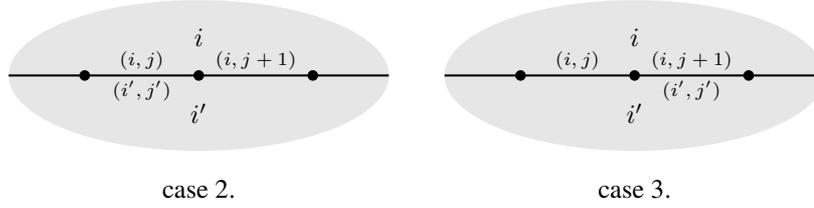


case 1.

case 2. (i, j) is not free-to-fill, and $(i, j + 1)$ is free-to-fill or semi-free-to-fill:

By definition, there is an edge $\{e_1, e_1^{-1}\}$ decorated by (i, j) having a smaller decoration $(i', j') < (i, j)$. Let $\{f, f^{-1}\}, \{f', f'^{-1}\}$ be the faces attached by $\{e_1, e_1^{-1}\}$ such that f is labeled by i and f' is labeled by i' .

Let $\{e_2, e_2^{-1}\}$ be the edge next to $\{e_1, e_1^{-1}\}$, decorated by $(i, j+1)$ from the face $\{f, f^{-1}\}$. As the vertex between e_1 and e_2 is not distinguished, $\{e_2, e_2^{-1}\}$ is attached to the face $\{f', f'^{-1}\}$. It is then decorated by $(i', j'+1)$ or $(i', j'-1)$ from $\{f', f'^{-1}\}$. Because $(i, j+1)$ is free-to-fill, we have $(i, j+1) < (i', j'+1)$ or $(i, j+1) < (i', j'-1)$. Both are impossible because $(i, j) > (i', j')$.



case 3. (i, j) is free-to-fill or semi-free-to-fill, and $(i, j+1)$ is not free-to-fill:

By the same argument, there exists an abstract letter $(i', j') < (i, j+1)$ such that $(i, j) < (i', j'+1)$ or $(i, j) < (i', j'-1)$. The second one is obviously impossible. If the first one held, then $(i', j') < (i, j+1) < (i', j'+2)$, so $(i', j') = (i, j)$, and there was an edge decorated by (i, j) and $(i, j+1)$ with opposite directions. The canonical function $\phi : \tilde{X} \rightarrow X$ gives $\phi(i, j+1) = \phi(i, j)^{-1}$, which is impossible because $r_i = \phi(i, 1) \dots \phi(i, \ell_i)$ should be a reduced word.

□

Lemma 4.11. *Let (\tilde{D}, p) be an abstract distortion diagram with no isolated edges.*

(i) *The number of distinguished vertices of (\tilde{D}, p) is at most $3|\tilde{D}|$.*

(ii) *The number of elementary segments of an abstract letter i is at most $3|\tilde{D}|^2$.*

Proof. The underlying 1-complex of \tilde{D} is a graph of rank $|\tilde{D}|$ without isolated edges. By Lemma 2.1 there are at most $2(|\tilde{D}| - 1)$ vertices of degree ≥ 3 . We add $k \leq |\tilde{D}|$ starting points and 2 endpoints of p , there are at most $3|\tilde{D}|$ distinguished vertices on (\tilde{D}, p) .

The number of faces of \tilde{D} labeled by i is at most $|\tilde{D}|$. Every face brings at most $3|\tilde{D}|$ marked vertices to ∂i , so there are at most $3|\tilde{D}|^2$ marked vertices on ∂i . □

Lemma 4.12. *Let (\tilde{D}, p) be a reduced abstract distortion diagram with no isolated edges and with k abstract relators. Let Γ be a labeled graph with $\text{rk}(\Gamma) \leq r$. Recall that η_i is the number of free-to-fill abstract letters of i and η'_i is the number of semi-free-to-fill abstract letters of i .*

$$|N_\ell(\tilde{D}, p, \Gamma)| \leq \left(\frac{2m}{2m-1} \right)^k (2|\Gamma|)^{3|\tilde{D}|^2 k} (2m-1)^{\sum_{i=1}^k \eta_i} (2r-1)^{\sum_{i=1}^k \eta'_i}.$$

Proof. We fill abstract letters of \tilde{D} in lexicographic order. We shall prove that if the abstract relators $1, \dots, i-1$ are filled, then there are at most

$$\left(\frac{2m}{2m-1}\right) (2|\Gamma|)^{3|\tilde{D}|^2} (2m-1)^{\eta_i} (2r-1)^{\eta'_i}$$

ways to fill the i -th abstract relator.

By Lemma 4.10, we fill elementary segments of i in order. Let u be an elementary segment of i . If u is free-to-fill, then there are at most $(2m-1)^{|u|}$ ways to fill u , or at most $2m(2m-1)^{|u|-1}$ ways if u is the first segment of i . If u is semi-free-to-fill, then there are at most $2|\Gamma|(2r-1)^{|u|}$ ways to fill u by lemma 4.9. If u is not free-to-fill there is only one choice.

The sum of the lengths of free-to-fill segments is η_i , and the sum of the lengths of semi-free-to-fill segments is η'_i . As the number of semi-free-to-fill segments is at most $3|\tilde{D}|^2$ (Lemma 4.11), there are at most $2m(2m-1)^{\eta_i-1}(2|\Gamma|)^{3|\tilde{D}|^2}(2r-1)^{\eta'_i}$ ways to fill the abstract relator i . \square

5 Freiheitssatz for random groups

Recall that B_ℓ is the set of cyclically reduced words of $X^\pm = \{x_1^\pm, \dots, x_m^\pm\}$, and that $|B_\ell| = (2m-1)^{\ell+o(\ell)}$. The set of cyclically reduced words on $X_r^\pm = \{x_1^\pm, \dots, x_r^\pm\}$ of length at most ℓ is of cardinality $(2r-1)^{\ell+o(\ell)}$. Its density in B_ℓ is

$$c_r = \log_{2m-1}(2r-1).$$

In this section, we prove that there is a phase transition at density

$$d_r = \min\left\{\frac{1}{2}, 1 - c_r\right\}.$$

5.1 Statement of the theorem

Theorem 5.1 (Phase transition at density d_r). *Let $\mathbf{G}(m, d) = (G_\ell(m, d))$ be a sequence of random groups at density d .*

1. *If $d > d_r$, then a.s., $G_\ell(m, d)$ is generated by x_1, \dots, x_r (or by any subset of X of cardinality r).*
2. *If $d < d_r$, then a.s., for every reduced labeled graph Γ with $\text{rk}(\Gamma) \leq r$ and $|\Gamma| \leq \frac{d_r-d}{5}\ell$, the canonical map $\tilde{\Gamma} \rightarrow \text{Cay}(G_\ell, X)$ is a $\frac{10}{d_r-d}$ -quasi-isometric embedding.*

In particular, a.s. every subgroup of $G_\ell(m, d)$ generated by a reduced labeled graph Γ with $\text{rk}(\Gamma) \leq r$ and $|\Gamma| \leq \frac{d_r-d}{5}\ell$ is a free group of rank r .

In particular, a.s. x_1, \dots, x_r freely generate a free subgroup of $G_\ell(m, d)$.

Proof of theorem 5.1.1. We assume that $d < 1/2$, otherwise a.s. $G_\ell(m, d)$ is trivial by Theorem 3.6. Recall that $X = \{x_1, \dots, x_m\}$ and $X_r = \{x_1, \dots, x_r\}$.

Let A_ℓ be the set of words of type $x_{r+1}w$ where w is a cyclically reduced word of X_r^\pm of length $\ell-1$. The density of (A_ℓ) in (B_ℓ) is c_r . By hypothesis $c_r + d > 1$.

Apply the intersection formula (Theorem 3.4), a.a.s. the intersection $R_\ell \cap A_\ell$ is not empty. Hence a.a.s. there exists a cyclically reduced word w_{r+1} of X_r^\pm such that $x_{r+1}w_{r+1} \in R_\ell$, which implies $x_{r+1} =_{G_\ell} w_{r+1}$.

Apply the same argument to the other generators x_{r+2}, \dots, x_m . A.a.s. there are cyclically words w_{r+1}, \dots, w_m of X_r^\pm such that $x_i =_{G_\ell} w_i$ for any $r+1 \leq i \leq m$. Hence a.a.s. every word of X^\pm equals to a word of X_r^\pm in G_ℓ . \square

The proof of the second assertion is similar to the proof of Theorem 3.6.2. We work first on a local result.

Lemma 5.2 (Local diagrams). *Let $K > 0$. Let $(G_\ell) = (G_\ell(m, d))$ be a sequence of random groups with $d < d_r$. Then a.a.s. for every reduced labeled graph Γ with $\text{rk}(\Gamma) \leq r$ and $|\Gamma| \leq \frac{d_r-d}{5}\ell$, every **disc-like reduced distortion diagram** (D, p) of (G_ℓ, Γ) with $|D| \leq K$ satisfies*

$$|p| \leq \left(1 - \frac{d_r - d}{5}\right) |\partial D|. \quad (\star)$$

The proof of this lemma is in the next subsection.

Proof of Theorem 5.1.2 by Lemma 5.2.

By Lemma 3.1 and Theorem 3.6, we can assume that every diagram D of $G_\ell(m, d)$ satisfies $|\partial D| \geq (1-2d)/2|D|\ell$ and that $G_\ell(m, d)$ is δ -hyperbolic with $\delta = \frac{4\ell}{1-2d}$.

Let Γ be a reduced labeled graph with $|\Gamma| \leq \frac{d_r-d}{5}\ell$ and $\text{rk}(\Gamma) \leq r$. Let $\lambda = \frac{5}{d_r-d}$. By the local-global principle of quasi-geodesics (Theorem 2.8), in order to prove that $\tilde{\Gamma} \rightarrow \text{Cay}(X, G_\ell)$ is a (global) 2λ -quasi-isometric embedding, we prove that every reduced word u readable on Γ is a $\frac{4000\lambda\ell}{1-2d}$ -local λ -quasi-geodesic.

Let u be a reduced word read on Γ with $|u| \leq \frac{4000\lambda\ell}{1-2d}$. Let v be a geodesic in G_ℓ joining endpoints of the image of u in G_ℓ . We shall prove that $|u| \leq \lambda|v|$. By van Kampen's lemma (Lemma 2.3) there exists a diagram D of G_ℓ whose boundary word is uv . By the isoperimetric inequality (Theorem 3.6.2),

$$|D| \leq \frac{2|\partial D|}{(1-2d)\ell} \leq \frac{4|u|}{(1-2d)\ell} \leq \frac{50000\lambda}{(1-2d)^2}.$$

Apply Lemma 5.2 with $K = \frac{50000\lambda}{(1-2d)^2} = \frac{250000}{(1-2d)^2(d_r-d)}$. If D is disk-like, then by (\star) we have

$$|u| \leq \left(1 - \frac{d_r - d}{5}\right) (|u| + |v|) \leq \frac{\lambda}{1 + \lambda} (|u| + |v|),$$

which implies $|u| \leq \lambda|v|$.

Otherwise we decompose D into discs and segments. By the same argument of Lemma 2.5, because every disc-like sub-diagram is a distortion diagram satisfying (\star) , we still have $|u| \leq \lambda|v|$. \square

5.2 Proof of Lemma 5.2

Let $(G_\ell) = (G_\ell(m, d)) = ((X|R_\ell))$ be a sequence of random groups with density $d < d_r$. To prove Proposition 5.2, we work first on the fillability of an abstract distortion diagram. Denote

$$\varepsilon_d = \frac{d_r - d}{5}.$$

Let Q_ℓ be the a.a.s. event $(2m - 1)^{(d - \varepsilon_d)\ell} \leq |R_\ell| \leq (2m - 1)^{(d + \varepsilon_d)\ell}$.

Lemma 5.3 (Fillability of an abstract distortion diagram). *Let $K > 0$. Let Γ be a reduced labeled graph with $\text{rk}(\Gamma) \leq r$ and $|\Gamma| \leq \varepsilon_d \ell$. Let (\tilde{D}, p) be a disc-like abstract distortion diagram with $|\tilde{D}| \leq K$ that satisfies*

$$|p| > (1 - \varepsilon_d) |\partial \tilde{D}|.$$

Then for ℓ large enough,

$$\mathbf{P} = \Pr \left((\tilde{D}, p) \text{ is fillable by } (G_\ell, \Gamma) \mid Q_\ell \right) \leq \ell^{10K^3} (2m - 1)^{-2\varepsilon_d \ell}.$$

Proof. We shall prove the lemma in four steps. We omit "for ℓ large enough" in every step. Recall that α_i is the number of faces labeled by i , η_i the number of free-to-fill abstract letters of i , and η'_i the number of semi-free-to-fill abstract letters of i .

$$\textbf{Step 1: } \log_{2m-1} \mathbf{P} \leq \sum_{i=1}^k (\eta_i + c_r \eta'_i + (d - 1 + 2\varepsilon_d)\ell) + 10K^3 \log_{2m-1} \ell. \quad (1)$$

According to Proposition 3.3, if (r_1, \dots, r_k) is a filling of \tilde{D} by B_ℓ , then for ℓ large enough $\Pr(r_1, \dots, r_k \in R_\ell \mid Q_\ell) \leq (2m - 1)^{k(d - 1 + 2\varepsilon_d)\ell}$. Recall that $N_\ell(\tilde{D}, p, \Gamma)$ is the set of fillings of (\tilde{D}, p) by (B_ℓ, Γ) .

Apply Lemma 4.12 with $|\Gamma| \leq \varepsilon_d \ell$ and $|\tilde{D}| \leq K$,

$$\begin{aligned} & \Pr \left((\tilde{D}, p) \text{ is fillable by } (G_\ell, \Gamma) \mid Q_\ell \right) \\ & \leq \sum_{(r_1, \dots, r_k) \in N_\ell(\tilde{D}, p, \Gamma)} \Pr(r_1, \dots, r_k \in R_\ell \mid Q_\ell) \\ & \leq |N_\ell(\tilde{D}, p, \Gamma)| (2m - 1)^{k(d - 1 + 2\varepsilon_d)\ell} \\ & \leq \ell^{10K^3} (2m - 1)^{\sum_{i=1}^k \eta_i} (2r - 1)^{\sum_{i=1}^k \eta'_i} (2m - 1)^{k(d - 1 + 2\varepsilon_d)\ell}. \end{aligned}$$

Hence the inequality (1) by applying \log_{2m-1} .

$$\textbf{Step 2: } |\tilde{D}| (\log_{2m-1} \mathbf{P} - 10K^3 \log_{2m-1} \ell) \leq \sum_{i=1}^k \alpha_i (\eta_i + c_r \eta'_i + (d - 1 + 2\varepsilon_d)\ell). \quad (2)$$

Let \tilde{D}_i be the sub-diagram of \tilde{D} consisting of the faces labeled by the first i abstract relators $1^\pm, \dots, i^\pm$ and the edges attached to them. Apply (1) to \tilde{D}_i , and denote \mathbf{P}_i the probability obtained. We have

$$\log_{2m-1} \mathbf{P} \leq \log_{2m-1} \mathbf{P}_i \leq \sum_{s=1}^i (\eta_s + c_r \eta'_s + (d - 1 + 2\varepsilon_d)\ell) + 10K^3 \log_{2m-1} \ell.$$

Without loss of generality, we assume $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_k$. Note that $\log_{2m-1} \mathbf{P}$ is negative and that $\alpha_1 \leq |\tilde{D}|$. By Abel's summation formula, with convention $\alpha_{k+1} = 0$,

$$\begin{aligned}
& \sum_{i=1}^k \alpha_i (c_r \eta'_i + \eta_i + (d-1+2\varepsilon_d)\ell) \\
&= \sum_{i=1}^k \left[(\alpha_i - \alpha_{i+1}) \sum_{s=1}^i (c_r \eta'_s + \eta_s + (d-1+2\varepsilon_d)\ell) \right] \\
&\geq \sum_{i=1}^k [(\alpha_i - \alpha_{i+1})(\log_{2m-1} \mathbf{P} - 10K^3 \log_{2m-1} \ell)] \\
&\geq \alpha_1 (\log_{2m-1} \mathbf{P} - 10K^3 \log_{2m-1} \ell) \\
&\geq |\tilde{D}| (\log_{2m-1} \mathbf{P} - 10K^3 \log_{2m-1} \ell).
\end{aligned}$$

Step 3: $\log_{2m-1} \mathbf{P} \leq \left(d - \frac{1}{2} + 2\varepsilon_d\right) \ell + \left(c_r - \frac{1}{2} + \varepsilon_d\right) \frac{|\partial \tilde{D}|}{|\tilde{D}|} + 10K^3 \log_{2m-1} \ell.$ (3)

Let $\varepsilon'_d > 0$ such that $|\bar{p}| = (1 - \varepsilon'_d)|\partial \tilde{D}|$. By hypothesis $\varepsilon'_d < \varepsilon_d$. Because \tilde{D} is disc-like and the boundary length of every face is $\leq \ell$, the number of undirected edges $|\bar{E}|$ is less than $\frac{|\tilde{D}|\ell - |\partial \tilde{D}|}{2} + |\partial \tilde{D}|$. Apply Lemma 4.8, we get

$$\begin{aligned}
& \sum_{i=1}^k \alpha_i \eta'_i \leq |\bar{p}| = (1 - \varepsilon'_d)|\partial \tilde{D}|, \\
& \sum_{i=1}^k \alpha_i \eta_i \leq |\bar{E}| - |\bar{p}| \leq \frac{|\tilde{D}|\ell}{2} + \left(\varepsilon'_d - \frac{1}{2}\right) |\partial \tilde{D}|.
\end{aligned}$$

Note that $\sum_{i=1}^k \alpha_i = |\tilde{D}|$. So we have

$$\begin{aligned}
& \sum_{i=1}^k \alpha_i (c_r \eta'_i + \eta_i + (d-1+2\varepsilon_d)\ell) \\
&\leq c_r (1 - \varepsilon'_d) |\partial \tilde{D}| + \frac{|\tilde{D}|\ell}{2} + \left(\varepsilon'_d - \frac{1}{2}\right) |\partial \tilde{D}| + (d-1+2\varepsilon_d) |\tilde{D}|\ell \\
&\leq \left(d - \frac{1}{2} + 2\varepsilon_d\right) |\tilde{D}|\ell + \left(c_r - \frac{1}{2} + \varepsilon_d\right) |\partial \tilde{D}|.
\end{aligned}$$

Combine with (2) we get the inequality (3)

Step 4: $\left(d - \frac{1}{2} + 2\varepsilon_d\right) \ell + \left(c_r - \frac{1}{2} + \varepsilon_d\right) \frac{|\partial \tilde{D}|}{|\tilde{D}|} \leq -2\varepsilon_d \ell.$ (4)

Recall that $c_r = \log_{2m-1}(2r-1)$. Note that $|\partial \tilde{D}| \leq \ell |\tilde{D}|$ and that $d = d_r - 5\varepsilon_d$. There are two cases:

(a) If $c_r \geq \frac{1}{2}$, then $d = 1 - c_r - 5\varepsilon_d$ and $c_r - \frac{1}{2} + \varepsilon_d \geq 0$, so

$$\begin{aligned} & \left(d - \frac{1}{2} + 2\varepsilon_d\right) |\tilde{D}|\ell + \left(c_r - \frac{1}{2} + \varepsilon_d\right) |\partial\tilde{D}| \\ & \leq (d + c_r - 1 + 3\varepsilon_d) |\tilde{D}|\ell \\ & \leq -2\varepsilon_d |\tilde{D}|\ell \end{aligned}$$

(b) If $c_r < \frac{1}{2}$, then $d = \frac{1}{2} - 5\varepsilon_d$ and $c_r - \frac{1}{2} + \varepsilon_d \leq \varepsilon_d$, so

$$\begin{aligned} & \left(d - \frac{1}{2} + 2\varepsilon_d\right) |\tilde{D}|\ell + \left(c_r - \frac{1}{2} + \varepsilon_d\right) |\partial\tilde{D}| \\ & \leq \left(d - \frac{1}{2} + 3\varepsilon_d\right) |\tilde{D}|\ell \\ & \leq -2\varepsilon_d |\tilde{D}|\ell \end{aligned}$$

By (3) and (4), for ℓ large enough $\log_{2m-1}(\mathbf{P}) \leq -2\varepsilon_d \ell + 10K^3 \log_{2m-1} \ell$. \square

By Lemma 2.1 and Lemma 2.2, we have the following two results.

Lemma 5.4. *For ℓ large enough, the number of reduced labeled connected graphs Γ (with respect to X) with $|\Gamma| \leq \varepsilon_d \ell$ and $\text{rk}(\Gamma) \leq r$ is bounded by*

$$(2m-1)^{\varepsilon_d \ell} \ell^{3r}.$$

Lemma 5.5. *For ℓ large enough, the number of disc-like abstract distortion diagrams (\tilde{D}, p) with $|\tilde{D}| \leq K$ and $|\partial f| \leq \ell$ for all faces $f \in F$ is bounded by*

$$\ell^{5K}.$$

Proof of Lemma 5.2. Recall that $\varepsilon_d = \frac{d_r - d}{5}$.

We shall prove that a.s. for every reduced labeled graph Γ with $\text{rk}(\Gamma) \leq r$ and $|\Gamma| \leq \varepsilon_d \ell$, every reduced distortion diagram (D, p) of (G_ℓ, Γ) with $|D| \leq K$ satisfies $|p| \leq (1 - \varepsilon_d) |\partial D|$.

Apply Lemma 5.3, Lemma 5.4 and Lemma 5.5. The probability that there exists a reduced labeled graph Γ with $\text{rk}(\Gamma) \leq r$, $|\Gamma| \leq \varepsilon_d \ell$ and there exists a disc-like reduced abstract distortion diagram (\tilde{D}, p) with $|\tilde{D}| \leq K$, $|p| > (1 - \varepsilon_d) |\partial\tilde{D}|$ such that (\tilde{D}, p) is fillable by (G_ℓ, Γ) is bounded by

$$(2m-1)^{\varepsilon_d \ell} \ell^{3r} \times \ell^{5K} \times \ell^{10K^3} (2m-1)^{-2\varepsilon_d \ell} = (2m-1)^{-\varepsilon_d \ell + O(\log \ell)}.$$

So the probability that there exists a reduced labeled graph Γ with $\text{rk}(\Gamma) \leq r$, $|\Gamma| \leq \varepsilon_d \ell$ and there exists a disc-like reduced distortion diagram (D, p) of (G_ℓ, Γ) with $|D| \leq K$ that satisfies $|p| > (1 - \varepsilon_d) |\partial\tilde{D}|$ is bounded by

$$(2m-1)^{-\varepsilon_d \ell + O(\log \ell)},$$

which goes to 0 when ℓ goes to infinity. \square

This completes the proof of Theorem 5.1.

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