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CAUCHY THEORY FOR THE WATER WAVES SYSTEM IN AN ANALYTIC FRAMEWORK

THOMAS ALAZARD, NICOLAS BURQ AND CLAUDE ZUILY

ABSTRACT. In this paper we consider the Cauchy problem for the gravity water-wave equations, in a domain with flat bottom and in arbitrary space dimension. We prove that if the data are of size ε in a space of analytic functions which have a holomorphic extension in a strip of size σ , then the solution exists up to a time of size C/ε in a space of analytic functions having at time t a holomorphic extension in a strip of size $\sigma - C'\varepsilon t$.

1. INTRODUCTION

The water-wave problem consists in describing, by means of the Euler equations, the dynamics of the free surface of a fluid. There are many different equations associated with this problem. Indeed, there are many different factors that dictate the dynamics of water-waves: The equations may be incompressible or not, irrotationnal or not, the fluid may have a fixed or moving bottom, and the restoring forces may be determined by gravitation or surface tension. The study of these equations has received a lot of attention during the last decades and there are now many cases in which the mathematical analysis is well developed. In particular, there are many recent results concerning the well-posedness of the water-waves equations in Sobolev spaces in large time, including global existence results (see [50, 51, 17, 5, 25, 22, 49, 53] for the equations without surface tension).

In addition to the analysis of the Cauchy problem, another line of research is the mathematical justification of the derivation of approximate equations describing water-waves dynamics in asymptotic regimes. The most famous examples are the equations introduced by Boussinesq and Korteweg-de Vries (see [45, 34, 44] and references there-in). Kano and Nishida [28, 27] gave, in the two dimensional case, the first justification of the Friedrichs expansion for the water-waves equations in terms of the shallowness parameter (by definition this is the ratio of the mean depth to the wavelength). In order to guarantee the existence of the solution for the full equations, they used an abstract Cauchy-Kowalevski theorem in a scale of Banach spaces, so that analyticity of the initial data is required (see also Kano [26] and Kano-Nishida [29]). These results have been extended to include initial data belonging to usual Sobolev spaces, by Craig [14], Iguchi [23, 24], Bona, Lannes and Saut [12] or Alvarez-Samaniego and Lannes [8] (see [34] for more references).

The study of various nonlinear partial differential equations in spaces of analytic functions has also received a great attention. We can mention the well-posedness results in analytic spaces by Kato and Masuda [30] which apply to many equations in fluid dynamics, the study of the Rayleigh-Taylor instability by Sulem-Sulem [47], the study of the Cauchy problem for the semi-linear one dimensional Schrödinger equations by Bona-Grujic and Kalisch [11], Selberg-D.O.da Silva [46], the work on the KdV equation by Hayashi [19], Tesfahun [48] and on the periodic BBM equation by Himonas-Petronilho [20], the work by Kucavica-Vicol [31] on the Euler equation, the work on quasilinear wave equations and other quasilinear systems by Alinhac and Métivier [7] and Kuksin-Nadirashvili [32], the work by Matsuyama and Ruzhansky [35] on the Kirchhoff equation, Gancedo-Granero-Belinchón-Scrobogna [16] for the Muskat problem and the one of Pierre [43] for the MHD equations. We should also mention the recent works by Mouhot-Villani [37], Bedrossian-Masmoudi-Mouhot [10] and by Grenier-Nguyen-Rodnianski [18] on the Landau damping for analytic and Gevrey data.

Inspired by the pioneering works of Kano-Nishida, our goal is to revisit the analysis of the water-problem with analytic data, using tools and methods that we developed previously to study the Cauchy problem with rough initial data. Our main result in this direction states that the solutions remain analytic for large time intervals.

Let us now state our problem more precisely. We are mainly interested in the study of the Cauchy problem for the gravity wave system, in any space dimension. There are many possible formulation for this problem. Here we use the classical Eulerian formulation and work with the so-called Craig-Sulem-Zakharov formulation, following [15, 52]. In this formulation, there are two unknowns: (i) the free surface elevation η and (ii) the trace ψ of the velocity potential on the free surface. These two unknowns depend on the time variable t and the horizontal space variable x . Motivated by possible applications to control theory ([1, 54]), we assume below that x belongs to the d dimensional torus $\mathbf{T}^d = (\mathbf{R}/2\pi\mathbf{Z})^d$, which means that the solutions are 2π -periodic in each variable x_j , $1 \leq j \leq d$.

We consider initial data in spaces of functions having a holomorphic extension to a fixed strip in the complex plane. Furthermore, in view of applications to control theory, we assume that the fluid domain has a flat bottom and consider a source term on the bottom which belongs merely to a classical Sobolev space. This problem can be written as follows. Given functions η_0, ψ_0 on \mathbf{T}^d , and b on $\mathbf{R} \times \mathbf{T}^d$, solve the system

$$(1.1) \quad \begin{cases} \partial_t \eta - G(\eta)(\psi, b) = 0, \\ \partial_t \psi + g\eta + \frac{1}{2} |\nabla_x \psi|^2 - \frac{1}{2} \frac{(\nabla_x \eta \cdot \nabla_x \psi + G(\eta)(\psi, b))^2}{1 + |\nabla_x \eta|^2} = 0, \\ \eta|_{t=0} = \eta_0, \quad \psi|_{t=0} = \psi_0. \end{cases}$$

Here $G(\eta)$ denotes the Dirichlet-Neuman operator, which is defined as follows. Given $h > 0$ and some fixed time t , let us introduce the fluid domain

$$\Omega(t) = \{(x, y) \in \mathbf{T}^d \times \mathbf{R} : -h < y < \eta(t, x)\}.$$

Then, define the potential $\phi = \phi(t, x, y)$ as the unique solution of the problem

$$(1.2) \quad \Delta\phi = 0 \quad \text{in } \Omega(t), \quad \phi|_{y=\eta(t,x)} = \psi(t, x), \quad \partial_y\phi|_{y=-h} = b(t, x).$$

Then the Dirichlet-Neumann operator is defined by

$$(1.3) \quad G(\eta)(\psi, b)(t, x) = \sqrt{1 + |\nabla_x \eta|^2} \partial_n \phi|_{y=\eta(t,x)} = (\partial_y \phi - \nabla_x \eta \cdot \nabla_x \phi)|_{y=\eta(t,x)}.$$

Recall that the equations (1.1) are derived from the following Euler equations in a set with moving boundary:

$$(1.4) \quad \begin{cases} \partial_t v + (v \cdot \nabla_{x,y})v = -\nabla_{x,y} P - gy & \text{in } \Omega = \{(t, x, y) : (x, y) \in \Omega(t)\}, \\ \operatorname{div}_{x,y} v = 0, & \text{in } \Omega, \\ \operatorname{curl}_{x,y} v = 0, & \text{in } \Omega, \\ P|_{y=\eta(t,x)} = 0, \end{cases}$$

where we set $v(t, x, y) = \nabla_{x,y} \phi(t, x, y)$ and $\psi(t, x) = \phi(t, x, \eta(t, x))$. We refer to [3] for the proof that from solutions of (1.1) one may define solutions of the Euler system (1.4). As in [27], we shall work in the spaces defined as follows. Given $d \geq 1$, $\sigma \geq 0$ and $s \geq 0$, we define

$$\mathcal{H}^{\sigma,s}(\mathbf{T}^d) = \left\{ u \in L^2(\mathbf{T}^d) : \|u\|_{\mathcal{H}^{\sigma,s}}^2 := \sum_{\xi \in \mathbf{Z}^d} e^{2\sigma|\xi|} \langle \xi \rangle^{2s} |\widehat{u}(\xi)|^2 < +\infty \right\}$$

with

$$\widehat{u}(\xi) = \int_{\mathbf{T}^d} e^{-ix \cdot \xi} u(x) dx, \quad \langle \xi \rangle = (1 + |\xi|^2)^{1/2}.$$

Several properties of these spaces are gathered in Appendix A.

Roughly speaking, the main result of this paper asserts that if the norms of the data η_0, ψ_0 in such spaces and the norm of b in some Sobolev space are of size $\varepsilon > 0$, then our system has a unique solution in these spaces up to the time c_*/ε for some $c_* > 0$. It is classical since the work of Kato and Masuda [30] that, for solutions with analytic initial data, the width of the strip of analyticity might decrease with time. The main novelty here is that we show that for small data of size ε , the decrease is at most linear in ε . To prove this result, we cannot rely on an abstract Cauchy-Kowalevski theorem (as the ones introduced by Nirenberg [38], Ovsjannikov [40], Nishida [39] or Baouendi-Goulaouic [9]; used by Ovsjannikov [41, 42] and Castro-Córdoba-Fefferman-Gancedo-Gómez-Serrano [13] to study the Cauchy problem for the water-waves equations). A key difference between our work and previous ones is that we shall use energy estimates in the spaces defined above, using the methods introduced in [6, 2, 4] to study the water-waves equations. To achieve these estimates we begin by a precise analysis of the Dirichlet-Neumann operator in the spaces of analytic functions. This requires a careful study of elliptic equations with variable coefficients, which is of independent interest.

1.1. The spaces of analytic functions and their characterizations. It is well known that functions in $\mathcal{H}^{\sigma,s}(\mathbf{T}^d)$ can be expressed as the traces on the real of functions which are holomorphic in a strip of the form

$$S_\sigma = \{z \in \mathbf{C}^d : \operatorname{Re} z \in \mathbf{T}^d, |\operatorname{Im} z| < \sigma\}.$$

More precisely, for $U \in \mathcal{Hol}(S_\sigma)$ and $|y| < \sigma$ we shall denote by U_y the function from \mathbf{T}^d to \mathbf{C} defined by $x \mapsto U(x + iy)$ (here $y \in \mathbf{R}^d$ and $|y|$ denotes its Euclidean norm). Then, for any $u \in \mathcal{H}^{\sigma,s}(\mathbf{T}^d)$, there exists $U \in \mathcal{Hol}(S_\sigma)$ such that $U_0 = u$ and

$$\sup_{|y| < \sigma} \|U_y\|_{H_x^s(\mathbf{T}^d)} \leq C \|u\|_{\mathcal{H}^{\sigma,s}}.$$

In Appendix A we prove a result which clarifies the converse.

Theorem 1.1. *Let $\sigma > 0$ and $s \geq 0$.*

(1) *Let $U \in \mathcal{Hol}(S_\sigma)$ be such that*

$$M_0 := \sup_{|y| < \sigma} \|U_y\|_{H_x^s(\mathbf{T}^d)} < +\infty$$

and set $u = U_0$. Then

(i) *If $d = 1$, then u belongs to $\mathcal{H}^{\sigma,s}(\mathbf{T}^d)$ and $\|u\|_{\mathcal{H}^{\sigma,s}} \leq 2M_0$.*

(ii) *If $d \geq 2$, then u belongs to $\mathcal{H}^{\delta,s}(\mathbf{T}^d)$ for any $\delta < \sigma$ and there exists a constant $C_\delta > 0$ such that $\|u\|_{\mathcal{H}^{\delta,s}} \leq C_\delta M_0$.*

(2) *Let $U \in \mathcal{Hol}(S_\sigma)$ be such that*

$$M_1 := \sup_{|y| < \sigma} \|U_y\|_{H_x^{s'}(\mathbf{T}^d)} < +\infty \quad \text{with} \quad s' > s + \frac{d-1}{4}.$$

Then the function $u = U_0$ belongs to $\mathcal{H}^{\sigma,s}(\mathbf{T}^d)$ and there exists a constant $C > 0$ such that $\|u\|_{\mathcal{H}^{\sigma,s}} \leq CM_1$.

Remark 1.2. (i) *In the case (ii), in general we do not have $u \in \mathcal{H}^{\sigma,s}(\mathbf{T}^d)$.*

For instance, if $s = 0$ and $d \geq 2$, a counterexample is provided by the function u such that $\hat{u}(\xi) = e^{-\sigma|\xi|} \langle \xi \rangle^{-\frac{\mu}{2}}$ where $\mu = d - \frac{1}{2} + \varepsilon$, $0 < \varepsilon < \frac{1}{10}$.

(ii) *If U_0 is radial, it suffices to assume in (2) that $\sup_{|y| < \sigma} \|U_y\|_{H_x^{s+\frac{d-1}{4}}(\mathbf{T}^d)}$ is finite.*

(iii) *All the properties of these spaces needed in this paper are gathered in the Appendix.*

(iv) *The same results hold with \mathbf{T}^d replaced by \mathbf{R}^d .*

1.2. Local in time well-posedness. Our first result states that, for data η_0, ψ_0 of size ε , the Cauchy-problem (1.1) has a unique solution in the space of analytic functions on a time interval of size 1.

Definition 1.3. *Given a real number s and continuous time-dependent index $\sigma = \sigma(t) \geq 0$, we denote by $C^0([0, T], \mathcal{H}^{\sigma,s})$ the subspace of $C^0([0, T], H^s)$ which consists of those functions f such that*

$$F \in C^0([0, T], H^s),$$

where

$$F(t, \cdot) = e^{\sigma(t)|D_x|} f(t, \cdot).$$

Remark 1.4. *When $\sigma(t) = \sigma_0$, this definition coincides with the usual definition. In general, it is easy to show that for any $\sigma_0 \leq \inf_{t \in [0, T]} \sigma(t)$, we have*

$$C^0([0, T], \mathcal{H}^{\sigma,s}) \subset C^0([0, T], \mathcal{H}^{\sigma_0,s})$$

(with continuous embedding). In particular, since $\sigma(t) \geq 0$, we have

$$C^0([0, T], \mathcal{H}^{\sigma, s}) \subset C^0([0, T], H^s).$$

Theorem 1.5. *Let $d \geq 1, s > 2 + \frac{d}{2}, g > 0, h > 0$ and $0 < \lambda_0 < 1$. Then there exist positive constants ε_0, K, M such that for all $0 < \lambda \leq \lambda_0$, all $\varepsilon < \varepsilon_0$, for all $(\eta_0, \psi_0) \in \mathcal{H}^{\lambda h, s} \times \mathcal{H}^{\lambda h, s}$, and all $b \in L^\infty(\mathbf{R}, H^{s-1}(\mathbf{T}^d)) \cap L^2(\mathbf{R}, H^{s-\frac{1}{2}}(\mathbf{T}^d))$ satisfying*

$$\|b\|_{L^\infty(\mathbf{R}, H^{s-1}) \cap L^2(\mathbf{R}, H^{s-\frac{1}{2}})} + \|\eta_0\|_{\mathcal{H}^{\lambda h, s}} + \|\psi_0\|_{\mathcal{H}^{\lambda h, s}} \leq \varepsilon,$$

the Cauchy problem (1.1) has a unique solution

$$(1.5) \quad (\eta, \psi) \in C^0([0, T], \mathcal{H}^{\sigma, s} \times \mathcal{H}^{\sigma, s}) \cap L^2((0, T), \mathcal{H}^{\sigma, s+\frac{1}{2}} \times \mathcal{H}^{\sigma, s+\frac{1}{2}})$$

with

$$T = (\lambda h)/K \quad \text{and} \quad \sigma(t) = \lambda h - Kt,$$

such that

$$\sup_{t \in [0, T]} \|\eta(t)\|_{\mathcal{H}^{\sigma, s}}^2 + \sup_{t \in [0, T]} \|\psi(t)\|_{\mathcal{H}^{\sigma, s}}^2 + \int_0^T (\|\eta(t)\|_{\mathcal{H}^{\sigma, s+\frac{1}{2}}}^2 + \|\psi(t)\|_{\mathcal{H}^{\sigma, s+\frac{1}{2}}}^2) dt \leq M\varepsilon^2.$$

Remark 1.6. *This result complements the analysis by Kano and Nishida [27] and Kano [26] in which we allow a non-zero and non-analytic source term b .*

1.3. Well-posedness on large time intervals. Our main result improves Theorem 1.5 by showing that the solution exists and remains analytic on a large time interval whose size is proportional to the inverse of the size of the initial data. To state this result, we need to introduce two auxiliary functions. Following [4], we set

$$(1.6) \quad B = \frac{G(\eta)(\psi, b) + \nabla_x \eta \cdot \nabla_x \psi}{1 + |\nabla_x \eta|^2}, \quad V = \nabla_x \psi - B \nabla_x \eta.$$

They are the traces on the free surface of the Eulerian velocity field. Moreover, we shall set

$$N_s(b) = \|b\|_{L^\infty(\mathbf{R}, H^{s+\frac{1}{2}})} + \|\partial_t b\|_{L^\infty(\mathbf{R}, H^{s-\frac{1}{2}})} + \|b\|_{L^1(\mathbf{R}, H^{s+\frac{1}{2}})},$$

$$a(D_x)f = G(0)(f, 0) = |D_x| \tanh(h|D_x|)f.$$

Theorem 1.7. *Let $d \geq 1, g > 0, h > 0, 0 < \lambda_0 < 1$ and consider real-numbers $s > 3 + \frac{d}{2}$ and $s' \in [s-1, s)$. Then there exist positive constants ε_*, K_*, c_* , such that for all $\varepsilon \leq \varepsilon_*$, all $0 < \lambda \leq \lambda_0$, and all $(\eta_0, \psi_0) \in \mathcal{H}^{\lambda h, s+\frac{1}{2}} \times \mathcal{H}^{\lambda h, s+\frac{1}{2}}$, if*

$$(1.7) \quad N_s(b) + \|\eta_0\|_{\mathcal{H}^{\lambda h, s+\frac{1}{2}}} + \|a(D_x)^{\frac{1}{2}} \psi_0\|_{\mathcal{H}^{\lambda h, s}} + \|V_0\|_{\mathcal{H}^{\lambda h, s}} + \|B_0\|_{\mathcal{H}^{\lambda h, s}} \leq \varepsilon,$$

then the Cauchy problem (1.1) has a unique solution on the time interval $[0, \frac{c_*}{\varepsilon}]$ such that

$$(1.8) \quad (\eta, \psi, V, B) \in C^0\left(\left[0, \frac{c_*}{\varepsilon}\right], \mathcal{H}^{\sigma, s'+\frac{1}{2}} \times \mathcal{H}^{\sigma, s'+\frac{1}{2}} \times \mathcal{H}^{\sigma, s'} \times \mathcal{H}^{\sigma, s'}\right),$$

with $\sigma(t) = \lambda h - K_* \varepsilon t$.

- Remark 1.8.** (i) *One can assume without loss of generality that $\lambda h > K_* c_*$, so that $\sigma(t) > 0$ for all time t in $[0, c_*/\varepsilon]$.*
- (ii) *A loss in the radius of analyticity of size ε is optimal.*
- (iii) *With a little extra work one could prove that the above result holds with $s' = s$.*
- (iv) *As explained in the introduction some motivation for this work are future applications to control theory. Namely, we would like to understand the states reachable from initial rest $(\eta_0, \Psi_0, V_0, B_0) = (0, 0, 0, 0)$ by means of actions on the bottom (the function b). In some sense, this results shows that in this frame-work, the use of analytic regularity spaces is unavoidable. Indeed, subject to the control b in Sobolev spaces, the solution will remain analytic for large times.*

1.4. Organization of the paper. In the next three Sections (see Sections 2, 3 and 4), we prove auxiliary elliptic regularity results and apply them to study several different properties of the Dirichlet-Neumann operator $G(\eta)$ when η belongs to some analytic space. Theorem 1.5 is proved in Section 5, by performing a fixed point in a suitable family of analytic spaces. Theorem 1.7 is proved in Section 6. Here the analysis is more involved as we need to keep track of the linear terms, and we were not able to perform a fixed point. As a consequence, we have to use a compactness method and prove *a priori* estimates on a family of regularized systems in order to pass to the limit. In Appendix A we gathered several results concerning analytic spaces, including the proof of Theorem 1.1.

2. ELLIPTIC REGULARITY

All functions considered here will be real valued. We fix two real numbers s_0, h and a function $\eta = \eta(x)$ such that

$$s_0 > \frac{d}{2}, \quad h > 0, \quad \eta \in \mathcal{H}^{h, s_0+1}(\mathbf{T}^d), \quad \inf_{x \in \mathbf{R}^d} \eta(x) > -h.$$

Set

$$\begin{aligned} \Omega &= \{(x, y) : x \in \mathbf{T}^d, -h < y < \eta(x)\}, \\ \Sigma &= \{(x, y) : x \in \mathbf{T}^d, y = \eta(x)\}, \\ \Gamma &= \{(x, y) : x \in \mathbf{T}^d, y = -h\}. \end{aligned}$$

We denote by n the unit normal to Σ and by ∂_n the normal derivative:

$$n = \frac{1}{\sqrt{1 + |\nabla_x \eta|^2}} \begin{pmatrix} -\nabla \eta \\ 1 \end{pmatrix}, \quad \partial_n = \frac{1}{\sqrt{1 + |\nabla_x \eta|^2}} (\partial_y - \nabla_x \eta \cdot \nabla_x).$$

Given two functions $\psi = \psi(x)$ and $b = b(x)$, we consider the following elliptic problem:

$$(2.1) \quad \Delta_{x,y} u = 0 \text{ in } \Omega, \quad u|_{y=\eta} = \psi, \quad \partial_y u|_{y=-h} = b,$$

where $\Delta_{x,y} = \partial_y^2 + \Delta_x$. Hereafter, given a function $f = f(x, y)$, we use $f|_{y=\eta}$ as a short notation for the function $x \mapsto f(x, \eta(x))$. The goal of this section

is to obtain elliptic regularity results for the solutions of (2.1) in the spaces of analytic functions.

2.1. Preliminaries. For $h > 0$ we set $I_h = (-h, 0)$.

2.1.1. Straightening the free surface. We begin by making a change of variables to reduce the problem to a fixed domain of the form

$$\tilde{\Omega} = \mathbf{T}^d \times I_h = \{(x, z) : x \in \mathbf{T}^d, -h < z < 0\}.$$

This change of variables will take $\Delta_{x,y}$ to a strictly elliptic operator and the normal derivative ∂_n to a vector field which is transverse to the boundary $\{z = 0\}$.

We consider a simple change of variables of the form $(x, z) \mapsto (x, \rho(x, z))$. The simplest change of variables reads $(x, z) \mapsto (x, \frac{z+h}{h}\eta(x) + z)$. For technical reasons, we will consider another choice and introduce a *smoothing* change of variables (following Lannes [33]). This means that the function ρ is given by

$$\rho(x, z) := \frac{1}{h}(z+h)(e^{z|D_x|}\eta)(x) + z, \quad x \in \mathbf{T}^d, \quad -h \leq z \leq 0,$$

where $e^{z|D_x|}$ is the Fourier multiplier with symbol $e^{z|\xi|}$. Since $z \leq 0$, this is a smoothing operator, bounded from $H^\mu(\mathbf{T}^d)$ to $\mathcal{H}^{z,\mu}(\mathbf{T}^d)$ for any real number μ . Notice that $\rho(x, 0) = \eta(x)$ and $\rho(x, -h) = -h$. Since

$$\partial_z \rho(\cdot, z) - 1 = \frac{1}{h}e^{z|D_x|}\eta + \frac{1}{h}(z+h)e^{z|D_x|}|D_x|\eta,$$

for $s_0 > d/2$ the Sobolev embedding implies that for all $z \in I_h$ we have

$$\|\partial_z \rho(\cdot, z) - 1\|_{L^\infty(\mathbf{T}^d)} \lesssim \|\partial_z \rho(\cdot, z) - 1\|_{H^{s_0}(\mathbf{T}^d)} \lesssim \|\eta\|_{H^{s_0+1}(\mathbf{T}^d)}.$$

We refer the reader to Lemma A.15 in Appendix A for the proof of more general estimates. Therefore, if $\|\eta\|_{H^{s_0+1}} \leq \varepsilon_0$ with ε_0 small enough, the map $(x, z) \mapsto (x, \rho(x, z))$ is a diffeomorphism from $\tilde{\Omega}$ to Ω . By this change of variables, the derivatives ∂_y and ∇_x become, respectively,

$$\Lambda_1 = \frac{1}{\partial_z \rho} \partial_z, \quad \Lambda_2 = \nabla_x - \frac{\nabla_x \rho}{\partial_z \rho} \partial_z.$$

More precisely, set

$$\tilde{u}(x, z) = u(x, \rho(x, z)).$$

Then \tilde{u} solves

$$(2.2) \quad (\Lambda_1^2 + \Lambda_2^2) \tilde{u} = 0, \quad \text{in } \tilde{\Omega}, \quad \tilde{u}|_{z=0} = \psi, \quad (\partial_z \tilde{u})|_{z=-h} = (\partial_z \rho|_{z=-h})b.$$

Using the chain rule, one can expand $\Lambda_1^2 + \Lambda_2^2$ as follows:

$$\Lambda_1^2 + \Lambda_2^2 = \frac{1}{\partial_z \rho} \left(\alpha \partial_z^2 + \beta \Delta_x + \gamma \cdot \nabla_x \partial_z - \delta \partial_z \right),$$

where

$$\begin{cases} \alpha = \frac{1 + |\nabla_x \rho|^2}{\partial_z \rho}, & \beta = \partial_z \rho, & \gamma = -2\nabla_x \rho, \\ \delta = \frac{1 + |\nabla_x \rho|^2}{\partial_z \rho} \partial_z^2 \rho + \partial_z \rho \Delta_x \rho - 2\nabla_x \rho \cdot \nabla_x \partial_z \rho. \end{cases}$$

It will be useful to observe that $\Lambda_1^2 + \Lambda_2^2$ is a perturbation of $\Delta_{x,z} = \partial_z^2 + \Delta_x$, which can be written in divergence form. More precisely, by a direct computation, one can verify that

$$(2.3) \quad (\partial_z \rho)(\Lambda_1^2 + \Lambda_2^2) \tilde{u} = \partial_z \left(\frac{1 + |\nabla_x \rho|^2}{\partial_z \rho} \partial_z \tilde{u} - \nabla_x \rho \cdot \nabla_x \tilde{u} \right) + \operatorname{div}_x (\partial_z \rho \nabla_x \tilde{u} - \partial_z \tilde{u} \nabla_x \rho).$$

Consequently, it follows from (2.2) that

$$\Delta_{x,z} \tilde{u} + R \tilde{u} = 0 \text{ in } \tilde{\Omega},$$

where

$$(2.4) \quad R \tilde{u} = \partial_z \left(\frac{1 + |\nabla_x \rho|^2 - \partial_z \rho}{\partial_z \rho} \partial_z \tilde{u} - \nabla_x \rho \cdot \nabla_x \tilde{u} \right) + \operatorname{div}_x \left((\partial_z \rho - 1) \nabla_x \tilde{u} - \nabla_x \rho \partial_z \tilde{u} \right).$$

2.1.2. *The lifting of the trace.* Another standard approach consists in further transforming the problem by simplifying the Dirichlet boundary condition on $z = 0$. To do so, given a function $\underline{\psi} = \underline{\psi}(x, z)$ satisfying $\underline{\psi}(x, 0) = \psi(x)$, we shall set $v = \tilde{u} - \underline{\psi}$, solution to

$$\begin{aligned} (\Delta_{x,z} + R) v &= -(\Delta_{x,z} + R) \underline{\psi} \quad \text{in } \tilde{\Omega} = \mathbf{T}^d \times I_h, \\ v|_{z=0} &= 0, \quad (\partial_z v)|_{z=-h} = (\partial_z \rho|_{z=-h}) b - (\partial_z \underline{\psi}|_{z=-h}) b. \end{aligned}$$

Parallel to the choice of the coordinate ρ in the above paragraph, a convenient choice for $\underline{\psi}$ is to consider the solution of an elliptic problem, to gain some extra regularity inside the domain $\tilde{\Omega}$. Namely, we determine $\underline{\psi}$ by solving the problem

$$(2.5) \quad (\partial_z^2 + \Delta_x) \underline{\psi} = 0 \quad \text{in } \tilde{\Omega}, \quad \underline{\psi}|_{z=0} = \psi, \quad \partial_z \underline{\psi}|_{z=-h} = 0.$$

Notice that this problem can be explicitly solved using the Fourier transform in x . More precisely, we have

$$\underline{\psi}(x, z) = (2\pi)^{-d} \sum_{\xi \in \mathbf{Z}^d} e^{ix \cdot \xi} \widehat{\underline{\psi}}(\xi, z),$$

where

$$(2.6) \quad \widehat{\underline{\psi}}(\xi, z) = \frac{e^{z|\xi|}}{1 + e^{-2h|\xi|}} \widehat{\psi}(\xi) + \frac{e^{-2h|\xi| - z|\xi|}}{1 + e^{-2h|\xi|}} \widehat{\psi}(\xi).$$

Now, we set

$$(2.7) \quad G_0(0) \psi = \partial_z \underline{\psi}|_{z=0}.$$

This is the Dirichlet-Neumann operator associated to the problem (2.5). By using (2.6), we find that

$$G_0(0) = |D_x| \tanh(h|D_x|).$$

To shorten the notations, we shall set in the sequel,

$$(2.8) \quad a(D_x) = G_0(0).$$

By using the previous notation, we have the following result (see Lemma A.20 in Appendix A).

Lemma 2.1. *For all $\mu \in \mathbf{R}$, there exists a constant $C > 0$ such that for all $\sigma \geq 0$ and all ψ such that $a(D_x)^{\frac{1}{2}}\psi \in \mathcal{H}^{\sigma,\mu}$, there holds*

$$\begin{aligned} \|\nabla_{x,z}\underline{\psi}\|_{L^2(I_h, \mathcal{H}^{\sigma,\mu})} &\leq C \|a(D_x)^{\frac{1}{2}}\psi\|_{\mathcal{H}^{\sigma,\mu}}, \\ \|\partial_z^2 \underline{\psi}\|_{L^2(I_h, \mathcal{H}^{\sigma,\mu-1})} &\leq C \|a(D_x)^{\frac{1}{2}}\psi\|_{\mathcal{H}^{\sigma,\mu}}, \\ \|\nabla_{x,z}\underline{\psi}\|_{L^\infty(I_h, \mathcal{H}^{\sigma,\mu-\frac{1}{2}})} &\leq C \|a(D_x)^{\frac{1}{2}}\psi\|_{\mathcal{H}^{\sigma,\mu}}. \end{aligned}$$

Remark 2.2. *There exists a constant $C > 0$ such that for all $\sigma \geq 0$ and $\mu \in \mathbf{R}$ we have*

$$(2.9) \quad \||D_x|\psi\|_{\mathcal{H}^{\sigma,\mu-\frac{1}{2}}} + \|\nabla_x \psi\|_{\mathcal{H}^{\sigma,\mu-\frac{1}{2}}} \leq C \|a(D_x)^{\frac{1}{2}}\psi\|_{\mathcal{H}^{\sigma,\mu}}.$$

This follows from the inequality $|\xi| \leq C\langle \xi \rangle \tanh(h|\xi|)$ for all $\xi \in \mathbf{R}^d$.

2.2. Elliptic regularity in analytic spaces. In this paragraph, we specify the spaces in which we shall work to study the elliptic regularity theory. Recall the notation $I_h = (-h, 0)$ and consider the Dirichlet problem in a half-space:

$$(\partial_z^2 + \Delta_x)w = 0 \quad \text{in } \mathbf{T}^d \times I_h, \quad w|_{z=0} = \psi, \quad \partial_z w|_{z=-h} = \theta.$$

Then, by using a Fourier calculation analogous to (2.6), one verifies that if $\psi \in \mathcal{H}^{h,\mu}(\mathbf{T}^d)$ and $\theta \in H^{\mu-1}(\mathbf{T}^d)$, then

$$e^{(h+z)|D_x|}w \in C^0([-h, 0], H^\mu(\mathbf{T}^d)),$$

which is equivalent to

$$e^{z|D_x|}w \in C^0([-h, 0], \mathcal{H}^{h,\mu}(\mathbf{T}^d)).$$

Our aim is to obtain a similar result for solutions to the general problem with variable coefficients. However for the latter problem we will loose on the radius of analyticity. Namely, we will replace $e^{z|D_x|}$ (resp. $\mathcal{H}^{h,\mu}(\mathbf{T}^d)$) by $e^{\lambda z|D_x|}$ (resp. $\mathcal{H}^{\lambda h,\mu}(\mathbf{T}^d)$) for some $\lambda \in [0, 1)$. This leads us to introduce the following spaces.

Definition 2.3. *Let $\lambda \in [0, 1]$. For $\mu \in \mathbf{R}$, we introduce the spaces*

$$(2.10) \quad \begin{aligned} E^{\lambda,\mu} &= \{u : e^{\lambda z|D_x|}u \in C^0([-h, 0], \mathcal{H}^{\lambda h,\mu}(\mathbf{T}^d))\}, \\ F^{\lambda,\mu} &= \{u : e^{\lambda z|D_x|}u \in L^2(I_h, \mathcal{H}^{\lambda h,\mu}(\mathbf{T}^d))\}, \\ \mathcal{X}^{\lambda,\mu} &= E^{\lambda,\mu} \cap F^{\lambda,\mu+\frac{1}{2}}. \end{aligned}$$

Remark 2.4. *Lemma A.2 shows that $\nabla_{x,z}u \in F^{\sigma,\mu+\frac{1}{2}}$ and $D_{x,z}^\alpha u \in F^{\sigma,\mu-\frac{1}{2}}$ for $|\alpha| = 2$ imply that $\nabla_{x,z}u \in E^{\sigma,\mu}$.*

We are now in position to state our first two results concerning elliptic regularity in analytic Sobolev spaces.

Proposition 2.5. *Consider real numbers λ_0, s, μ such that*

$$0 \leq \lambda_0 < 1, \quad s > \frac{d}{2} + 1, \quad 0 \leq \mu \leq s - 1.$$

Then there exist two constants $\bar{\varepsilon} > 0$ and $C > 0$ such that for all $0 \leq \lambda \leq \lambda_0$, all $\eta \in \mathcal{H}^{\lambda h,s}(\mathbf{T}^d)$ satisfying $\|\eta\|_{\mathcal{H}^{\lambda h,s}} \leq \bar{\varepsilon}$, all $F \in F^{\lambda,\mu-1}$, all $\theta \in H^{\mu-\frac{1}{2}}(\mathbf{T}^d)$ and all $w \in L^2(I_h, H^{\mu+1}(\mathbf{T}^d))$ solutions of the problem

$$(2.11) \quad (\Delta_{x,z} + R)w = F \text{ in } \mathbf{T}^d \times I_h, \quad w|_{z=0} = 0, \quad (\partial_z w)|_{z=-h} = \theta,$$

the function w belongs to $F^{\lambda,\mu}$ and satisfies

$$(2.12) \quad \|\nabla_{x,z}w\|_{F^{\lambda,\mu}} \leq C \left(\|F\|_{F^{\lambda,\mu-1}} + \|\theta\|_{H^{\mu-\frac{1}{2}}(\mathbf{T}^d)} \right).$$

Remark 2.6. For our purposes the estimate (2.12) is interesting for λ close to 1.

Before proving this result, we pause to show how to deduce a variant of Proposition 2.5 with a non-vanishing trace on $z = 0$, assuming that the index μ is equal to $s - 1$.

Corollary 2.7. Consider two real numbers λ_0, s such that

$$0 \leq \lambda_0 < 1, \quad s > \frac{d}{2} + 1.$$

Then there exists two constants $\bar{\varepsilon} > 0$ and $C > 0$ such that for all $0 \leq \lambda \leq \lambda_0$, all $\eta \in \mathcal{H}^{\lambda h, s}(\mathbf{T}^d)$ satisfying $\|\eta\|_{\mathcal{H}^{\lambda h, s}} \leq \bar{\varepsilon}$, all $\psi \in \mathcal{H}^{\lambda h, s}(\mathbf{T}^d)$, all $F \in F^{\lambda, s-2}$, all $\theta \in H^{s-\frac{3}{2}}(\mathbf{T}^d)$ and all $\tilde{w} \in L^2(I_h, H^s(\mathbf{T}^d))$ solutions of the problem

$$(\Delta_{x,z} + R)\tilde{w} = F \text{ in } \mathbf{T}^d \times I_h, \quad \tilde{w}|_{z=0} = \psi, \quad (\partial_z \tilde{w})|_{z=-h} = \theta,$$

the function $\nabla_{x,z}\tilde{w}$ belongs to $F^{\lambda, s-1}$ and satisfies

$$(2.13) \quad \|\nabla_{x,z}\tilde{w}\|_{F^{\lambda, s-1}} \leq C \left(\|F\|_{F^{\lambda, s-2}} + \|a(D_x)^{\frac{1}{2}}\psi\|_{\mathcal{H}^{\lambda h, s-1}} + \|\theta\|_{H^{s-\frac{3}{2}}(\mathbf{T}^d)} \right).$$

Proof. Since by (2.5) we have, $\Delta_{x,z}\underline{\psi} = 0$, the function $w = \tilde{w} - \underline{\psi}$ satisfies

$$(\Delta_{x,z} + R)w = F - R\underline{\psi} \text{ in } \mathbf{T}^d \times I_h, \quad w|_{z=0} = 0, \quad (\partial_z w)|_{z=-h} = \theta.$$

Consequently, (2.13) follows from the estimate (2.12) given by Proposition 2.5, applied with $\mu = s - 1$, together with the estimate (A.28) below for the remainder $R\underline{\psi}$. \square

Proof of Proposition 2.5. As we have seen in the previous paragraph, one can use the Fourier transform to study the analytic regularity of the solutions to the linearized problem

$$\Delta_{x,z}w = F \text{ in } \mathbf{T}^d \times I_h, \quad w|_{z=0} = 0, \quad (\partial_z w)|_{z=-h} = \theta.$$

However, since the operator R is a differential operator with variable coefficients, to study the regularity of the solution to (2.11), we must proceed differently. We will use the multiplier method. More precisely, our strategy consists in conjugating the operator $\Delta_{x,z} + R$ by the weight $e^{\lambda(h+z)|D_x|}$. The trick here is that, when $\lambda < 1$, we obtain another coercive operator and then the desired estimate (2.12) will follow from an energy estimate. The proof thus consists in estimating the function $e^{\lambda(h+z)|D_x|}w$. To rigorously justify the computations, we shall truncate the symbol $e^{\lambda(h+z)|\xi|}$, using the following lemma.

Lemma 2.8. Given $\varepsilon > 0$, $\lambda \in [0, 1)$ and $z \in I_h$, define $q_\varepsilon(z, \cdot): \mathbf{R}^d \rightarrow \mathbf{R}$ by

$$q_\varepsilon(z, \xi) = \lambda \left(\frac{h|\xi|}{1 + \varepsilon|\xi|} + z|\xi| \right).$$

Then, for all $\xi, \zeta \in \mathbf{R}^d$ we have

$$\begin{aligned} q_\varepsilon(z, \xi) &\leq \lambda(h+z)|\xi|, & q_\varepsilon(-h, \xi) &= -\frac{\lambda h \varepsilon |\xi|^2}{1 + \varepsilon |\xi|} \leq 0, \\ q_\varepsilon(z, \xi) - q_\varepsilon(z, \zeta) &\leq \lambda h |\xi - \zeta|. \end{aligned}$$

Proof. The two first claims are obvious. Then set $\langle \varepsilon \xi \rangle = 1 + \varepsilon |\xi|$. We have

$$q_\varepsilon(z, \xi) - q_\varepsilon(z, \zeta) = \lambda(|\xi| - |\zeta|) \left(\frac{h}{\langle \varepsilon \xi \rangle \langle \varepsilon \zeta \rangle} + z \right).$$

If $|\xi| - |\zeta| \geq 0$, we use the fact that $\frac{h}{\langle \varepsilon \xi \rangle \langle \varepsilon \zeta \rangle} \leq h$, $z \leq 0$, together with the inequality $|\xi| - |\zeta| \leq |\xi - \zeta|$. If $|\xi| - |\zeta| \leq 0$ we use the fact that $\frac{h}{\langle \varepsilon \xi \rangle \langle \varepsilon \zeta \rangle} \geq 0$ and $0 \leq -z \leq h$. \square

Now we fix s, μ satisfying $s > d/2 + 1$ and $0 \leq \mu \leq s$ and set

$$\Lambda_\varepsilon(z) = e^{q_\varepsilon(z, D_x)} \langle D_x \rangle^\mu, \quad \langle D_x \rangle^\mu = (I - \Delta_x)^{\mu/2},$$

where q_ε is defined in Lemma 2.8. Given a function $f = f(x, z)$ defined for $x \in \mathbf{T}^d$ and $z \in I_h$, we define $\Lambda_\varepsilon f$ as usual by $(\Lambda_\varepsilon f)(\cdot, z) = \Lambda_\varepsilon(z) f(\cdot, z)$. Then we set

$$w_\varepsilon = \Lambda_\varepsilon w.$$

Notice that this definition is meaningful, since the symbol $e^{q_\varepsilon(x, \xi)}$ is bounded and w belongs to $L^2(I_h, H^1(\mathbf{T}^d))$. Our goal is to estimate the $H^1(\tilde{\Omega})$ norm of w_ε uniformly in ε , and this will imply the desired result by means of Fatou's lemma.

To form an equation on w_ε , we notice that, for any function $f = f(x, z)$,

$$(\partial_z - \lambda |D_x|) e^{q_\varepsilon(z, D_x)} f = e^{q_\varepsilon(z, D_x)} \partial_z f, \quad \nabla_x (e^{q_\varepsilon(z, D_x)} f) = e^{q_\varepsilon(z, D_x)} \nabla_x f.$$

Therefore, setting

$$(2.14) \quad P_\lambda = (\partial_z - \lambda |D_x|)^2 + \Delta_x,$$

we obtain

$$P_\lambda w_\varepsilon = \Lambda_\varepsilon (\partial_z^2 + \Delta_x) w.$$

Since v solves (2.11), we conclude that w_ε is solution of the problem

$$(2.15) \quad \begin{cases} P_\lambda w_\varepsilon = \Lambda_\varepsilon (-Rw + F), \\ w_\varepsilon|_{z=0} = 0, \quad (\partial_z w_\varepsilon - \lambda |D_x| w_\varepsilon)|_{z=-h} = e^{q_\varepsilon(-h, D_x)} \langle D_x \rangle^\mu \theta := \theta_\varepsilon. \end{cases}$$

Now, the rest of the proof is divided into three steps:

- First, we will prove that the operator P_λ is elliptic.
- The second step is elementary. We check that the contributions of the Cauchy data F and θ are estimated by the right-hand side of (2.12).
- In the third step, we prove a commutator estimate in analytic spaces and use it to deduce that the contribution of $e^{q_\varepsilon(z, D_x)} Rv$ can be absorbed by the elliptic regularity, under a smallness assumption on the coefficients in the operator R .

Step 1: The conjugated operator. We begin by studying the operator P_λ introduced in (2.14). We will see that it is an elliptic operator and prove some elementary elliptic estimates.

Recall that, by notation, $\tilde{\Omega} = \mathbf{T}^d \times I_h$. We denote by $\mathcal{H}_0^1(\tilde{\Omega})$ the subspace of $H^1(\tilde{\Omega})$ which consists of those functions whose trace on $z = 0$ vanishes, equipped with the $H^1(\tilde{\Omega})$ -norm. Poincaré's inequality applies in this setting and there is a positive constant $C_{\tilde{\Omega}}$ such that

$$(2.16) \quad \|u\|_{L^2(\tilde{\Omega})} \leq C_{\tilde{\Omega}} \|\nabla_{x,z} u\|_{L^2(\tilde{\Omega})}, \quad \forall u \in \mathcal{H}_0^1(\tilde{\Omega}).$$

Now, consider the bilinear form

$$\begin{aligned} a(u, v) &= (\partial_z u, \partial_z v)_{L^2(\tilde{\Omega})} + (1 - \lambda^2) (\nabla_x u, \nabla_x v)_{L^2(\tilde{\Omega})} \\ &\quad + \lambda (\partial_z u, |D_x|v)_{L^2(\tilde{\Omega})} - \lambda (|D_x|u, \partial_z v)_{L^2(\tilde{\Omega})}. \end{aligned}$$

This is a continuous bilinear form on $\mathcal{H}_0^1(\tilde{\Omega}) \times \mathcal{H}_0^1(\tilde{\Omega})$. Moreover, if $u \in \mathcal{H}_0^1(\tilde{\Omega}) \cap H^2(\tilde{\Omega})$, we can make the following computations:

$$\begin{aligned} (\partial_z u, \partial_z v)_{L^2(\tilde{\Omega})} &= -(\partial_z^2 u, v)_{L^2(\tilde{\Omega})} - \int_{\mathbf{T}^d} (\partial_z u) v|_{z=-h} dx, \\ (\nabla_x u, \nabla_x v)_{L^2(\tilde{\Omega})} &= -(\Delta_x u, v)_{L^2(\tilde{\Omega})}, \\ (\partial_z u, |D_x|v)_{L^2(\tilde{\Omega})} &= (|D_x| \partial_z u, v)_{L^2(\tilde{\Omega})}, \\ -(|D_x|u, \partial_z v)_{L^2(\tilde{\Omega})} &= (|D_x| \partial_z u, v)_{L^2(\tilde{\Omega})} + \int_{\mathbf{T}^d} (|D_x|u) v|_{z=-h} dx. \end{aligned}$$

It follows that

$$(2.17) \quad a(u, v) = (-P_\lambda u, v)_{L^2(\tilde{\Omega})} - \int_{\mathbf{T}^d} [(\partial_z u - \lambda |D_x|u)v]|_{z=-h} dx.$$

On the other hand, by using the assumption that $\lambda < 1$, and remembering that we are considering real-valued functions, we have

$$(2.18) \quad a(u, u) = \|\partial_z u\|_{L^2(\tilde{\Omega})}^2 + (1 - \lambda^2) \|\nabla_x u\|_{L^2(\tilde{\Omega})}^2 \geq C(1 - \lambda^2) \|u\|_{H^1(\tilde{\Omega})}^2,$$

where we used the classical Poincaré inequality (2.16). Here $C > 0$ is independent of λ .

With the notations in (2.15) consider the linear form on $\mathcal{H}_0^1(\tilde{\Omega})$,

$$L(f) = -\langle \Lambda_\varepsilon(-Rw + F), f \rangle - ((\theta_\varepsilon, f)|_{z=-h}),$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between $L_z^2(I_h, H^{-1}(\mathbf{T}^d))$ and $L_z^2(I_h, H^1(\mathbf{T}^d))$ and $((\cdot, \cdot))$ denotes the duality between $H^{-\frac{1}{2}}(\mathbf{T}^d)$ and $H^{\frac{1}{2}}(\mathbf{T}^d)$. We deduce from (2.17) that w_ε is solution of the problem,

$$(2.19) \quad a(w_\varepsilon, f) = L(f), \quad \forall f \in \mathcal{H}_0^1(\tilde{\Omega}).$$

Recall from (2.4) that Rw is given by,

$$Rv = \partial_z F_1 + \operatorname{div}_x F_2,$$

where

$$F_1 = \frac{1 + |\nabla_x \rho|^2 - \partial_z \rho}{\partial_z \rho} \partial_z w - \nabla_x \rho \cdot \nabla_x w, \quad F_2 = (\partial_z \rho - 1) \nabla_x w - \partial_z w \nabla_x \rho.$$

Parallel to the computations above we immediately verify that

$$\Lambda_\varepsilon \partial_z F_1 = (\partial_z - \lambda |D_x|)(\Lambda_\varepsilon F_1), \quad \Lambda_\varepsilon \operatorname{div}_x F_2 = \operatorname{div}_x (\Lambda_\varepsilon F_2),$$

so,

$$\Lambda_\varepsilon R w = (\partial_z - \lambda |D_x|)(\Lambda_\varepsilon F_1) + \operatorname{div}_x (\Lambda_\varepsilon F_2).$$

Integrating by parts with respect to z or x , we find that

$$\begin{aligned} \langle \Lambda_\varepsilon R w, f \rangle &= \langle \Lambda_\varepsilon F_1, (\partial_z - \lambda |D_x|) f \rangle - ((e^{q_\varepsilon(-h, D_x)} \langle D_x \rangle^\mu F_1|_{z=-h}, f|_{z=-h})) \\ &\quad + \langle \Lambda_\varepsilon F_2, \nabla_x f \rangle. \end{aligned}$$

The absolute value of the first and last term in the right-hand side above are estimated by means of the Cauchy-Schwarz inequality by

$$\left(\|\Lambda_\varepsilon F_1\|_{L^2(\tilde{\Omega})} + \|\Lambda_\varepsilon F_2\|_{L^2(\tilde{\Omega})} \right) \|\nabla_{x,z} f\|_{L^2(\tilde{\Omega})}.$$

To estimate the second term, we use the fact that $q_\varepsilon(-h, \xi) \leq 0$ and the trace theorem to write

$$\begin{aligned} &\left| ((e^{q_\varepsilon(-h, D_x)} \langle D_x \rangle^\mu F_1|_{z=-h}, f|_{z=-h})) \right| \\ &\leq \left\| e^{q_\varepsilon(-h, D_x)} \langle D_x \rangle^\mu F_1|_{z=-h} \right\|_{H^{-\frac{1}{2}}} \|f|_{z=-h}\|_{H^{\frac{1}{2}}(\mathbf{T}^d)} \\ &\leq \|\langle D_x \rangle^\mu F_1|_{z=-h}\|_{H^{-\frac{1}{2}}} \|f\|_{H^1(\tilde{\Omega})} \\ &\lesssim \|\langle D_x \rangle^\mu F_1|_{z=-h}\|_{H^{-\frac{1}{2}}} \|\nabla_{x,z} f\|_{L^2(\tilde{\Omega})}, \end{aligned}$$

where we used Poincaré's inequality. Similarly, we have

$$|(\theta_\varepsilon, f|_{z=-h})| \lesssim \|\theta_\varepsilon\|_{H^{-\frac{1}{2}}} \|\nabla_{x,z} f\|_{L^2(\tilde{\Omega})}.$$

Hence we conclude that

$$(2.20) \quad |L(f)| \leq (A + B + C + D) \|\nabla_{x,z} f\|_{L^2(\tilde{\Omega})},$$

where

$$\begin{aligned} A &= \|\Lambda_\varepsilon F\|_{L^2(I_h, H^{-1}(\mathbf{T}^d))}, \\ B &= \|\theta_\varepsilon\|_{H^{-\frac{1}{2}}}, \\ C &= \|\Lambda_\varepsilon F_1\|_{L^2(\tilde{\Omega})} + \|\Lambda_\varepsilon F_2\|_{L^2(\tilde{\Omega})}, \\ D &= \|\langle D_x \rangle^\mu F_1|_{z=-h}\|_{H^{-\frac{1}{2}}}. \end{aligned}$$

Thus, by combining (2.19) and (2.20) (applied with $f = w_\varepsilon$) together with the coercive inequality (2.18), we conclude that

$$(2.21) \quad \|\nabla_{x,z} w_\varepsilon\|_{L^2_z(I_h, L^2)} \leq C_\lambda (A + B + C + D),$$

where $C_\lambda = \frac{C}{1-\lambda}$.

Step 2: Estimates of A and B . Directly from Lemma 2.8 and (2.15) we deduce that

$$\|\theta_\varepsilon\|_{H^{-\frac{1}{2}}(\mathbf{T}^d)} \leq \|\theta\|_{H^{\mu-\frac{1}{2}}(\mathbf{T}^d)}.$$

Similarly, using the property $q_\varepsilon(z, \xi) \leq \lambda(h+z)|\xi|$, we obtain

$$\|\Lambda_\varepsilon F\|_{L^2_z(I_h, H^{-1}(\mathbf{T}^d))} \leq \|e^{\lambda z |D_x|} \langle D_x \rangle^\mu F\|_{L^2_z(I_h, \mathcal{H}^{\lambda h, -1})} \leq \|F\|_{F^{\lambda, \mu-1}}.$$

Then the terms A and B in (2.21) are estimated by the right-hand side of the desired inequality (2.12).

Step 3: estimate of C . We shall use the following lemma.

Lemma 2.9. *Consider two real numbers*

$$\nu > \frac{d}{2}, \quad 0 \leq \mu \leq \nu.$$

Given two functions $r, u: \mathbf{T}^d \times I_h \rightarrow \mathbf{R}$, set

$$U(\cdot, z) = e^{q_\varepsilon(z, D_x)} \langle D_x \rangle^\mu (r(\cdot, z) u(\cdot, z)).$$

Then there exists a constant $C = C(d, \nu, \mu, h) > 0$ such that

$$\|U\|_{L_z^2(I_h, L^2(\mathbf{T}^d))} \leq C \|r\|_{L_z^\infty(I_h, \mathcal{H}^{\lambda h, \nu})} \|e^{q_\varepsilon(z, D_x)} \langle D_x \rangle^\mu u\|_{L_z^2(I_h, L^2(\mathbf{T}^d))}.$$

Proof. By Lemma 2.8 we have

$$e^{q_\varepsilon(z, \xi)} = e^{q_\varepsilon(z, \xi) - q_\varepsilon(z, \zeta)} e^{q_\varepsilon(z, \zeta)} \leq e^{\lambda h |\xi - \zeta|} e^{q_\varepsilon(z, \zeta)}.$$

So writing $\widehat{U}(\xi, z)$ under the form

$$\frac{1}{(2\pi)^d} \int \frac{\langle \xi \rangle^\mu}{\langle \xi - \zeta \rangle^\nu \langle \zeta \rangle^\mu} e^{q_\varepsilon(z, \xi) - q_\varepsilon(z, \zeta)} \langle \xi - \zeta \rangle^\nu \widehat{r}(\xi - \zeta, z) e^{q_\varepsilon(z, \zeta)} \langle \zeta \rangle^\mu \widehat{u}(\zeta, z) d\zeta,$$

we find that

$$|\widehat{U}(\xi, z)| \leq \frac{1}{(2\pi)^d} \int F(\xi, \zeta) f_1(\xi - \zeta, z) f_2(\zeta, z) d\zeta,$$

where

$$F(\xi, \zeta) = \langle \xi \rangle^\mu \langle \xi - \zeta \rangle^{-\nu} \langle \zeta \rangle^{-\mu},$$

and

$$f_1(\theta, z) = e^{\lambda h |\theta|} \langle \theta \rangle^\nu |\widehat{r}(\theta, z)|, \quad f_2(\zeta, z) = e^{q_\varepsilon(z, \zeta)} \langle \zeta \rangle^\mu |\widehat{u}(\zeta, z)|.$$

Now, we are exactly in the setting introduced by Hörmander to study the continuity of the product in Sobolev spaces (see Lemma A.3 in Appendix A and take $s_3 = \mu$, $s_1 = \nu$ and $s_2 = \mu$). We infer that, for any fixed $z \in I_h$,

$$\|U(\cdot, z)\|_{L^2}^2 \leq C \|f_1(\cdot, z)\|_{L^2}^2 \|f_2(\cdot, z)\|_{L^2}^2.$$

Using the Plancherel's identity and then integrating in z , we obtain the desired result. This completes the proof of Lemma 2.9. \square

We are now in position to estimate $\|\Lambda_\varepsilon F_1\|_{L^2(\widetilde{\Omega})} + \|\Lambda_\varepsilon F_2\|_{L^2(\widetilde{\Omega})}$. Remembering that

$$F_1 = \frac{1 + |\nabla_x \rho|^2 - \partial_z \rho}{\partial_z \rho} \partial_z w - \nabla_x \rho \cdot \nabla_x w, \quad F_2 = (\partial_z \rho - 1) \nabla_x w - \partial_z w \nabla_x \rho,$$

we see that it is sufficient to estimate the $L^2(\widetilde{\Omega})$ -norm of a term of the form $\Lambda_\varepsilon(\alpha\beta)$, where the factors of the product are

$$\alpha \in \left\{ \frac{1 + |\nabla_x \rho|^2 - \partial_z \rho}{\partial_z \rho}, \nabla_x \rho, \partial_z \rho - 1 \right\}, \quad \beta \in \{\nabla_x w, \partial_z w\}.$$

Since $s > d/2 + 1$ and $0 \leq \mu \leq s - 1$ by assumptions, we can apply Lemma 2.9 with $\nu = s - 1$, to obtain

$$\|\Lambda_\varepsilon(\alpha\beta)\|_{L^2(\tilde{\Omega})} \lesssim \|\alpha\|_{L^\infty(I_h, \mathcal{H}^{\lambda h, s-1})} \|\Lambda_\varepsilon\beta\|_{L^2(\tilde{\Omega})}.$$

Now we claim that there exists $C > 0$ such that

$$(2.22) \quad \|\Lambda_\varepsilon\beta\|_{L^2(\tilde{\Omega})} \leq C \|\nabla_{x,z}w_\varepsilon\|_{L^2(\tilde{\Omega})},$$

$$(2.23) \quad \|\alpha\|_{L^\infty(I_h, \mathcal{H}^{\lambda h, s-1})} \leq C\bar{\varepsilon} \quad \text{provided} \quad \|\eta\|_{\mathcal{H}^{\lambda h, s}} \leq \bar{\varepsilon}.$$

The proof of (2.22) is straightforward; indeed

$$\Lambda_\varepsilon\partial_{x_j}w = \partial_{x_j}\Lambda_\varepsilon w = \partial_{x_j}w_\varepsilon, \quad \Lambda_\varepsilon\partial_z w = (\partial_z - \lambda|D_x|)\Lambda_\varepsilon w = (\partial_z - \lambda|D_x|)w_\varepsilon.$$

The second estimate follows from Lemma A.17. Now, by using (2.23) and (2.22), we obtain that, if $\|\eta\|_{\mathcal{H}^{\lambda h, s}} \leq \bar{\varepsilon}$, then we have

$$\|\Lambda_\varepsilon F_1\|_{L^2(\tilde{\Omega})} + \|\Lambda_\varepsilon F_2\|_{L^2(\tilde{\Omega})} \lesssim \bar{\varepsilon} \|\nabla_{x,z}w_\varepsilon\|_{L^2(\tilde{\Omega})}.$$

Step 4: estimate of D . It remains only to estimate

$$D = \|\langle D_x \rangle^\mu F_1|_{z=-h}\|_{H^{-\frac{1}{2}}},$$

where recall that

$$F_1 = \frac{1 + |\nabla_x \rho|^2 - \partial_z \rho}{\partial_z \rho} \partial_z w - \nabla_x \rho \cdot \nabla_x w.$$

By definition of ρ (see (A.19)) we have

$$\partial_z \rho|_{z=-h} = 1 + \frac{1}{h} e^{-h|D_x|} \eta(x), \quad \nabla_x \rho|_{z=-h} = 0.$$

Since $\partial_z v|_{z=-h} = \theta$ we obtain

$$F_1|_{z=-h} = -\frac{e^{-h|D_x|}\eta}{h + e^{-h|D_x|}\eta} \theta.$$

Then $D \leq C \|\theta\|_{H^{\mu-\frac{1}{2}}}$, where C depends only on h .

Step 5: End of the proof. By plugging the previous estimates in (2.21), we conclude that

$$\|\nabla_{x,z}w_\varepsilon\|_{L_z^2(I_h, L^2)} \leq C_\lambda (\|\theta\|_{H^{\mu-\frac{1}{2}}(\mathbf{T}^d)} + \|F\|_{F^{\lambda, \mu-1}}) + C'_\lambda \bar{\varepsilon} \|\nabla_{x,z}w_\varepsilon\|_{L_z^2(I_h, L^2)}.$$

Now, taking $\bar{\varepsilon}$ such that $C'_\lambda \bar{\varepsilon} < 1$, we obtain the uniform estimate

$$\sup_{\varepsilon \in (0, \bar{\varepsilon}]} \|\nabla_{x,z}w_\varepsilon\|_{L_z^2(I_h, L^2)} \leq C''_\lambda (\|\theta\|_{H^{\mu-\frac{1}{2}}(\mathbf{T}^d)} + \|F\|_{F^{\lambda, \mu-1}}).$$

It follows from Fatou's lemma that

$$\|\nabla_{x,z}w\|_{L_z^2(I_h, L^2)} \leq C''_\lambda (\|\theta\|_{H^{\mu-\frac{1}{2}}(\mathbf{T}^d)} + \|F\|_{F^{\lambda, \mu-1}}).$$

This completes the proof of Proposition 2.5. \square

2.3. Sharp elliptic estimates. We consider again the problem,

$$(2.24) \quad (\Delta_{x,z} + R)\tilde{w} = F \text{ in } \mathbf{T}^d \times I_h, \quad \tilde{w}|_{z=0} = \psi, \quad \partial_z \tilde{w}|_{z=-h} = \theta,$$

and our purpose is to refine the result proved in Corollary 2.7.

Let us recall some elliptic estimates proved in [4, 6] for this problem:

$$\|\nabla_{x,z} \tilde{w}\|_{L^2(I_h; H^s)} \leq \mathcal{F} \left(\|\eta\|_{H^{s+\frac{1}{2}}} \right) \left(\|F\|_{L^2_z(I_h; H^{s-1})} + \|\psi\|_{H^{s+\frac{1}{2}}} + \|\theta\|_{H^{s-\frac{1}{2}}} \right)$$

for some non-decreasing function $\mathcal{F}: \mathbf{R}_+ \rightarrow \mathbf{R}_+$. Here we will prove an analogue of this estimate in the analytic setting. As in the previous paragraph, we will make a smallness assumption on η . But we will only assume that the $\mathcal{H}^{\lambda, s}$ -norm of η is small, and not the $\mathcal{H}^{\lambda, s+\frac{1}{2}}$ -norm. Also, for later purposes, we will prove a sharp estimate which is tame in the sense that the $\mathcal{H}^{\lambda, s+\frac{1}{2}}$ -norm of η will multiply only the $\mathcal{H}^{\lambda, s}$ -norm of ψ .

Theorem 2.10. *Consider two real numbers λ_0, s such that*

$$0 \leq \lambda_0 < 1, \quad s > \frac{d}{2} + 2.$$

Then there exist two constants $\bar{\varepsilon} > 0$ and $C > 0$ such that for all $0 \leq \lambda \leq \lambda_0$, all $\eta \in \mathcal{H}^{\lambda, s+\frac{1}{2}}(\mathbf{T}^d)$ satisfying $\|\eta\|_{\mathcal{H}^{\lambda, s}} \leq \bar{\varepsilon}$, all $\psi \in \mathcal{H}^{\lambda, s+\frac{1}{2}}(\mathbf{T}^d)$, all $\theta \in H^{s-\frac{1}{2}}(\mathbf{T}^d)$ and all solutions \tilde{w} of problem (2.24), if we set

$$I_1 = \|\nabla_{x,z} \tilde{w}\|_{F^{\lambda, s}}, \quad I_2 = \|\partial_z^2 \tilde{w}\|_{F^{\lambda, s-1}}, \quad I_3 = \|\nabla_{x,z} \tilde{w}\|_{E^{\lambda, s-\frac{1}{2}}},$$

then

$$(2.25) \quad \sum_{k=1}^3 I_k \leq C \left(\|F\|_{F^{\lambda, s-1}} + \|a(D_x)^{\frac{1}{2}} \psi\|_{\mathcal{H}^{\lambda, s}} + \|\theta\|_{H^{s-\frac{1}{2}}} \right) \\ + C \|\eta\|_{\mathcal{H}^{\lambda, s+\frac{1}{2}}} \left(\|F\|_{F^{\lambda, s-2}} + \|a(D_x)^{\frac{1}{2}} \psi\|_{\mathcal{H}^{\lambda, s-\frac{1}{2}}} + \|\theta\|_{H^{s-\frac{3}{2}}} \right),$$

where recall that $a(D_x) = G_0(0)$ as introduced in (2.8).

Remark 2.11. *Compared to the statement of Corollary 2.7 we notice that, under the same smallness assumption on η , we are considering smoother data ψ, F and θ as well as smoother solutions. Indeed, now θ and F belongs to $H^{s-1/2}(\mathbf{T}^d)$ and $F^{\lambda, s-1}$ respectively, while we considered before the case where they belong to $H^{\mu-1/2}(\mathbf{T}^d)$ and $F^{\lambda, \mu-1}$ for some $\mu \leq s-1$. We will reduce the analysis to the previous case by considering the equation satisfied by $\nabla_x \tilde{w}$.*

The rest of this section is devoted to the proof of this result.

Proof of Theorem 2.10. We divide the proof into three steps.

Step 1. Notice that by interpolation (see Lemma A.12, (iv) with $\mu = s - \frac{1}{2}$), we have

$$(2.26) \quad I_3 \leq C(I_1 + I_2).$$

Therefore, it is enough to estimate I_1 and I_2 .

Step 2. We now prove that

$$(2.27) \quad \begin{aligned} I_2 \leq & C(I_1 + \|F\|_{F^{\lambda, s-1}} \\ & + \|\eta\|_{\mathcal{H}^{\lambda h, s+\frac{1}{2}}} (\|F\|_{F^{\lambda, s-2}} + \|a(D_x)^{\frac{1}{2}}\psi\|_{\mathcal{H}^{\lambda h, s-\frac{1}{2}}} + \|\theta\|_{H^{s-\frac{3}{2}}(\mathbb{T}^d)})). \end{aligned}$$

Since $(\Delta_{x,z} + R)w = F$, we have $\partial_z^2 w = F - \Delta_x w - R w$. Now, we use the technical estimate (A.22) in Appendix A to handle the contribution of $R w$. It follows that

$$\|\partial_z^2 \tilde{w}\|_{F^{\lambda, s-2}} \leq \|F\|_{F^{\lambda, s-2}} + \|\nabla_x \tilde{w}\|_{F^{\lambda, s-1}} + C\bar{\varepsilon}(\|\nabla_{x,z} \tilde{w}\|_{F^{\lambda, s-1}} + \|\partial_z^2 \tilde{w}\|_{F^{\lambda, s-2}}).$$

Taking $\bar{\varepsilon}$ such that $C\bar{\varepsilon} \leq \frac{1}{2}$, we deduce that

$$(2.28) \quad \|\partial_z^2 \tilde{w}\|_{F^{\lambda, s-2}} \leq C(\|F\|_{F^{\lambda, s-2}} + \|\nabla_{x,z} \tilde{w}\|_{F^{\lambda, s-1}}).$$

By the same way, using (A.24) below, we get

$$\begin{aligned} \|\partial_z^2 \tilde{w}\|_{F^{\lambda, s-1}} \leq & \|F\|_{F^{\lambda, s-1}} + \|\nabla_x \tilde{w}\|_{F^{\lambda, s}} + C\bar{\varepsilon}(\|\nabla_x \tilde{w}\|_{F^{\lambda h, s}} + \|\partial_z^2 \tilde{w}\|_{F^{\lambda, s-1}}) \\ & + C(\|\partial_z \tilde{w}\|_{F^{\lambda h, s-1}} + \|\eta\|_{\mathcal{H}^{\lambda h, s+\frac{1}{2}}} (\|\partial_z^2 \tilde{w}\|_{F^{\lambda, s-\frac{5}{2}}} + \|\partial_z \tilde{w}\|_{F^{\lambda, s-\frac{3}{2}}}). \end{aligned}$$

As before, taking $\bar{\varepsilon}$ so small that $C\bar{\varepsilon} \leq \frac{1}{2}$, we deduce that

$$\begin{aligned} \|\partial_z^2 \tilde{w}\|_{F^{\lambda, s-1}} \leq & C(\|F\|_{F^{\lambda, s-1}} + \|\nabla_{x,z} \tilde{w}\|_{F^{\lambda, s}} \\ & + \|\eta\|_{\mathcal{H}^{\lambda h, s+\frac{1}{2}}} (\|\partial_z^2 \tilde{w}\|_{F^{\lambda, s-\frac{5}{2}}} + \|\partial_z \tilde{w}\|_{F^{\lambda, s-\frac{3}{2}}}). \end{aligned}$$

Now, since $s - \frac{5}{2} \leq s - 2$ and $s - \frac{3}{2} \leq s - 1$, we can use (2.28). It follows that

$$(2.29) \quad \begin{aligned} \|\partial_z^2 \tilde{w}\|_{F^{\lambda, s-1}} \leq & C(\|F\|_{F^{\lambda, s-1}} + \|\nabla_{x,z} \tilde{w}\|_{F^{\lambda, s}} \\ & + \|\eta\|_{\mathcal{H}^{\lambda h, s+\frac{1}{2}}} (\|F\|_{F^{\lambda, s-2}} + \|\nabla_{x,z} \tilde{w}\|_{F^{\lambda, s-1}})). \end{aligned}$$

Now by Corollary 2.7 we have

$$(2.30) \quad \|\nabla_{x,z} \tilde{w}\|_{F^{\lambda, s-1}} \leq C\left(\|F\|_{F^{\lambda, s-2}} + \|a(D_x)^{\frac{1}{2}}\psi\|_{\mathcal{H}^{\lambda h, s-\frac{1}{2}}} + \|\theta\|_{H^{s-\frac{3}{2}}(\mathbb{T}^d)}\right).$$

Since $\|\nabla_{x,z} \tilde{w}\|_{F^{\lambda, s}} = I_1$ by definition, we obtain the desired estimate (2.27) by plugging the previous inequality in (2.29).

Step 3. We are left with the estimate of I_1 . We shall prove that

$$(2.31) \quad \begin{aligned} I_1 \leq & C\left(\|F\|_{F^{\lambda, s-1}} + \|a(D_x)^{\frac{1}{2}}\psi\|_{\mathcal{H}^{\lambda h, s}} + \|\theta\|_{H^{s-\frac{1}{2}}}\right) \\ & + C\|\eta\|_{\mathcal{H}^{\lambda h, s+\frac{1}{2}}}\left(\|F\|_{F^{\lambda, s-2}} + \|a(D_x)^{\frac{1}{2}}\psi\|_{\mathcal{H}^{\lambda h, s-\frac{1}{2}}} + \|\theta\|_{H^{s-\frac{3}{2}}}\right) \\ & + C\|\eta\|_{\mathcal{H}^{\lambda h, s}}(I_1 + I_2). \end{aligned}$$

This will conclude the proof of desired estimate (2.25); by taking an appropriate linear combination of (2.26), (2.27) and (2.31) and then taking $\bar{\varepsilon} \geq \|\eta\|_{\mathcal{H}^{\lambda h, s}}$ small enough to absorb the contribution of $I_1 + I_2$ in the right-hand side of (2.31). Notice that

$$(2.32) \quad I_1 = \|\nabla_{x,z} \tilde{w}\|_{F^{\lambda, s}} \leq \|\nabla_{x,z} \tilde{w}\|_{F^{\lambda, s-1}} + \|\nabla_{x,z} \nabla_x \tilde{w}\|_{F^{\lambda, s-1}}.$$

We will prove that these two terms are bounded by the right-hand side of (2.31). The first term $\|\nabla_{x,z}\tilde{w}\|_{F^{\lambda,s-1}}$ has been estimated in Corollary 2.7. By (2.13) we have

$$\|\nabla_{x,z}\tilde{w}\|_{F^{\lambda,s-1}} \leq C \left(\|F\|_{F^{\lambda,s-2}} + \|a(D_x)^{\frac{1}{2}}\psi\|_{\mathcal{H}^{\lambda h,s-1}} + \|\theta\|_{H^{s-\frac{3}{2}}(\mathbf{T}^d)} \right),$$

which immediately implies that it is bounded by the right-hand side of (2.31). Hence, it remains only to estimate $\|\nabla_{x,z}\nabla_x\tilde{w}\|_{F^{\lambda,s-1}}$. We will estimate the $F^{\lambda,s-1}$ -norm of $\nabla_{x,z}\partial_j\tilde{w}$ for $1 \leq j \leq d$. Notice that $\partial_j\tilde{w}$ satisfies

$$\begin{cases} (\Delta_{x,z} + R)\partial_j\tilde{w} = -[\partial_j, R]w + \partial_j F & \text{in } \mathbf{T}^d \times I_h, \\ \tilde{w}|_{z=0} = \partial_j\psi, \quad \partial_z\tilde{w}|_{z=-h} = \partial_j\theta, \end{cases}$$

where $[\partial_j, R]\tilde{w} = \partial_j(R\tilde{w}) - R\partial_j\tilde{w}$. It follows from Corollary 2.7 that

$$(2.33) \quad \|\nabla_{x,z}\partial_j\tilde{w}\|_{F^{\lambda,s-1}} \leq C \left(\|[\partial_j, R]\tilde{w}\|_{F^{\lambda,s-2}} + \|\partial_j F\|_{F^{\lambda,s-2}} + \|a(D_x)^{\frac{1}{2}}\partial_j\psi\|_{\mathcal{H}^{\lambda h,s-1}} + \|\partial_j\theta\|_{H^{s-\frac{3}{2}}} \right).$$

The key point is to estimate the commutator $[\partial_j, R]w$. We claim that

$$(2.34) \quad \|[\partial_j, R]\tilde{w}\|_{F^{\lambda,s-2}} \leq C\|\eta\|_{\mathcal{H}^{\lambda h,s}}\|\nabla_{x,z}^2\tilde{w}\|_{F^{\lambda,s-1}} + C\|\eta\|_{\mathcal{H}^{\lambda h,s+\frac{1}{2}}}\|\nabla_{x,z}\tilde{w}\|_{F^{\lambda,s-1}}.$$

Let us assume this claim and conclude the proof. By combining (2.33), (2.34) and (2.30), we have

$$\begin{aligned} \|\nabla_{x,z}\partial_j\tilde{w}\|_{F^{\lambda,s-1}} &\leq C \left(\|F\|_{F^{\lambda,s-1}} + \|a(D_x)^{\frac{1}{2}}\psi\|_{\mathcal{H}^{\lambda h,s}} + \|\theta\|_{H^{s-\frac{1}{2}}} \right) \\ &\quad + C\|\eta\|_{\mathcal{H}^{\lambda h,s+\frac{1}{2}}} \left(\|F\|_{F^{\lambda,s-2}} + \|a(D_x)^{\frac{1}{2}}\psi\|_{\mathcal{H}^{\lambda h,s-\frac{1}{2}}} + \|\theta\|_{H^{s-\frac{3}{2}}} \right) \\ &\quad + C\|\eta\|_{\mathcal{H}^{\lambda h,s}}\|\nabla_{x,z}^2w\|_{F^{\lambda,s-1}}. \end{aligned}$$

By summing all these estimates for $1 \leq j \leq d$, this will give the required estimate for

$$\|\nabla_{x,z}\nabla_x w\|_{F^{\lambda,s-1}},$$

which will conclude the proof of the theorem as explained after (2.31).

We now have to prove the claim (2.34). Recall that

$$(2.35) \quad R = a\partial_z^2 + b\Delta_x + c \cdot \nabla_x \partial_z - d\partial_z,$$

where

$$\begin{cases} a = \frac{1 + |\nabla_x \rho|^2}{\partial_z \rho} - 1, & b = \partial_z \rho - 1, & c = -2\nabla_x \rho, \\ d = \frac{1 + |\nabla_x \rho|^2}{\partial_z \rho} \partial_z^2 \rho + \partial_z \rho \Delta_x \rho - 2\nabla_x \rho \cdot \nabla_x \partial_z \rho. \end{cases}$$

We have

$$[\partial_j, R]w = (\partial_j a)\partial_z^2 w + (\partial_j b)\Delta_x w + (\partial_j c) \cdot \nabla_x \partial_z w - (\partial_j d)\partial_z w.$$

Then we use statement (i) in Lemma A.12 (applied with $s_1 = s_3 = s_1 = s - 2 > d/2$) to write that

$$\begin{aligned} \|(\partial_j a) \partial_z^2 \tilde{w}\|_{F^{\lambda, s-2}} &\lesssim \|\partial_j a\|_{E^{\lambda, s-2}} \|\partial_z^2 \tilde{w}\|_{F^{\lambda, s-2}}, \\ \|(\partial_j b) \Delta_x \tilde{w}\|_{F^{\lambda, s-2}} &\lesssim \|\partial_j b\|_{E^{\lambda, s-2}} \|\Delta_x \tilde{w}\|_{F^{\lambda, s-2}}, \\ \|(\partial_j c) \cdot \nabla_x \partial_z \tilde{w}\|_{F^{\lambda, s-2}} &\lesssim \|\partial_j c\|_{E^{\lambda, s-2}} \|\nabla_x \partial_z \tilde{w}\|_{F^{\lambda, s-2}}. \end{aligned}$$

By lemma A.17 we have

$$\|a\|_{E^{\lambda, s-1}} + \|b\|_{E^{\lambda, s-1}} + \|c\|_{E^{\lambda, s-1}} \lesssim \|\eta\|_{\mathcal{H}^{\lambda h, s}}.$$

Therefore,

$$\begin{aligned} &\|(\partial_j a) \partial_z^2 \tilde{w}\|_{F^{\lambda, s-2}} + \|(\partial_j b) \Delta_x \tilde{w}\|_{F^{\lambda, s-2}} + \|(\partial_j c) \partial_z \tilde{w}\|_{F^{\lambda, s-2}} \\ &\lesssim \|\eta\|_{\mathcal{H}^{\lambda h, s}} \|\nabla_{x, z}^2 \tilde{w}\|_{F^{\lambda, s-2}}. \end{aligned}$$

It remains to estimate the term $(\partial_j d) \partial_z \tilde{w}$. To do so, again, we begin by applying the product rule given by (ii) in Lemma A.12 (applied with s replaced by $s - 2 > d/2$) to write that

$$\|(\partial_j d) \partial_z \tilde{w}\|_{F^{\lambda, s-2}} \lesssim \|\partial_j d\|_{F^{\lambda, s-2}} \|\partial_z \tilde{w}\|_{E^{\lambda, s-2}}.$$

Then, by Lemma A.17 we have

$$\|\partial_j d\|_{F^{\lambda, s-2}} \leq \|d\|_{F^{\lambda, s-1}} \lesssim \|\eta\|_{\mathcal{H}^{\lambda h, s + \frac{1}{2}}}.$$

By combining the previous estimates, we see that, to complete the proof of the claim (2.34), it remains to estimate $\|\partial_z \tilde{w}\|_{E^{\lambda, s-2}}$ and $\|\nabla_{x, z}^2 \tilde{w}\|_{F^{\lambda, s-2}}$ in terms of $\|\nabla_{x, z} \tilde{w}\|_{F^{\lambda, s-1}}$. Since we will need to prove a similar result later on, we pause here to prove a general result.

Lemma 2.12. *Consider two real numbers*

$$s > \frac{d}{2} + 2, \quad \lambda \in [0, 1).$$

Then there exist two constants $\bar{\varepsilon} > 0$ and $C > 0$ such that for all $\eta \in \mathcal{H}^{\lambda h, s}(\mathbf{T}^d)$ satisfying $\|\eta\|_{\mathcal{H}^{\lambda h, s}} \leq \bar{\varepsilon}$, if v satisfies $\Delta_{x, z} v + Rv = f$, then

$$(2.36) \quad \|\nabla_{x, z} v\|_{E^{\lambda, s - \frac{3}{2}}} + \|\partial_z^2 v\|_{F^{\lambda, s-2}} \leq C \|\nabla_{x, z} v\|_{F^{\lambda, s-1}} + C \|f\|_{F^{\lambda, s-2}}.$$

Proof. By interpolation (see statement (iv) in Lemma A.12), we have

$$\|\partial_z v\|_{E^{\lambda, s - \frac{3}{2}}} \lesssim \|\partial_z^2 v\|_{F^{\lambda, s-2}} + \|\partial_z v\|_{F^{\lambda, s-1}}.$$

Therefore, it is sufficient to prove that $\|\partial_z^2 v\|_{F^{\lambda, s-2}}$ is estimated by the right-hand side of (2.36). To do so, we repeat the arguments used in Step 2. Namely, we write $\partial_z^2 v = -\Delta_x v - Rv + f$, to infer that

$$\|\partial_z^2 v\|_{F^{\lambda, s-2}} \leq \|\nabla_x v\|_{F^{\lambda, s-1}} + \|Rv\|_{F^{\lambda, s-2}} + \|f\|_{F^{\lambda, s-2}}.$$

Then we use the estimate (A.22) to estimate the contribution of Rv , which implies that

$$\|\partial_z^2 v\|_{F^{\lambda, s-2}} \leq \|\nabla_x v\|_{F^{\lambda, s-1}} + C\bar{\varepsilon} (\|\nabla_{x, z} v\|_{F^{\lambda, s-1}} + \|\partial_z^2 v\|_{F^{\lambda, s-2}}) + \|f\|_{F^{\lambda, s-2}}.$$

Then we conclude the proof by taking $\bar{\varepsilon}$ so small that $C\bar{\varepsilon} \leq 1/2$. \square

By applying the previous lemma to $(v, f) = (\tilde{w}, F)$, we complete the proof of the claim (2.34), which in turn concludes the proof of Theorem 2.10. \square

We consider eventually the problem

$$(2.37) \quad (\Delta_{x,z} + R)\tilde{u} = 0 \text{ in } \mathbf{T}^d \times I_h, \quad \tilde{u}|_{z=0} = \psi, \quad \partial_z \tilde{u}|_{z=-h} = (\partial_z \rho|_{z=-h})b.$$

Corollary 2.13. *Consider two real numbers λ_0, s such that*

$$0 \leq \lambda_0 < 1, \quad s > \frac{d}{2} + 2.$$

Then there exist two constants $\bar{\varepsilon} > 0$ and $C > 0$ such that for all $0 \leq \lambda \leq \lambda_0$, all $\eta \in \mathcal{H}^{\lambda h, s + \frac{1}{2}}(\mathbf{T}^d)$ satisfying $\|\eta\|_{\mathcal{H}^{\lambda h, s}} \leq \bar{\varepsilon}$, all $\psi \in \mathcal{H}^{\lambda h, s + \frac{1}{2}}(\mathbf{T}^d)$, all $b \in H^{s - \frac{1}{2}}(\mathbf{T}^d)$ and all solutions \tilde{u} of problem (2.37), if we set

$$I_1 = \|\nabla_{x,z} \tilde{u}\|_{F^{\lambda, s}}, \quad I_2 = \|\partial_z^2 \tilde{u}\|_{F^{\lambda, s-1}}, \quad I_3 = \|\nabla_{x,z} \tilde{u}\|_{E^{\lambda, s - \frac{1}{2}}},$$

then

$$\begin{aligned} \sum_{k=1}^3 I_k \leq C & \left(\|a(D_x)^{\frac{1}{2}} \psi\|_{\mathcal{H}^{\lambda h, s}} + \|b\|_{H^{s - \frac{1}{2}}} \right. \\ & \left. + \|\eta\|_{\mathcal{H}^{\lambda h, s + \frac{1}{2}}} \left(\|a(D_x)^{\frac{1}{2}} \psi\|_{\mathcal{H}^{\lambda h, s - \frac{1}{2}}} + \|b\|_{H^{s - \frac{3}{2}}} \right) \right), \end{aligned}$$

where we recall that $a(D_x) = G_0(0)$ is introduced in (2.8).

Proof. This follows from Theorem 2.10 and the fact that, since

$$\partial_z \rho|_{z=-h} = 1 + \frac{1}{h} e^{-h|D_x|} \eta$$

with $\|\eta\|_{\mathcal{H}^{\lambda h, s}} \leq 1$, we have, for $\mu = s - \frac{1}{2}$, $\mu = s - \frac{3}{2}$,

$$\|(\partial_z \rho|_{z=-h})b\|_{H^\mu} \leq C \|b\|_{H^\mu}.$$

This completes the proof of Corollary 2.13. \square

3. THE DIRICHLET NEUMANN OPERATOR

Given functions ψ, b we consider the problem,

$$(3.1) \quad \Delta_{x,y} u = 0 \text{ in } \Omega, \quad u|_{y=\eta} = \psi, \quad \partial_y u|_{y=-h} = b.$$

We set

$$G(\eta)(\psi, b) = \sqrt{1 + |\nabla_x \eta|^2} \partial_n u|_{y=\eta} = (\partial_y u - \nabla_x \eta \cdot \nabla_x u)|_{y=\eta}.$$

This is the Dirichlet-Neumann operator associated to problem (3.1). Notice that using the notations in (2.7) we have

$$G_0(0)\psi = G(0)(\psi, 0) = a(D_x)\psi = |D_x| \tanh(h|D_x|)\psi.$$

We have then the following result.

Theorem 3.1. *Consider a real number $s > 2 + d/2$. For all $0 \leq \lambda_0 < 1$, there exist $\bar{\varepsilon} > 0$ and $C > 0$ such that for all $0 \leq \lambda \leq \lambda_0$, for all $\eta \in$*

$\mathcal{H}^{\lambda h, s + \frac{1}{2}}(\mathbf{T}^d)$ satisfying $\|\eta\|_{\mathcal{H}^{\lambda h, s}} \leq \bar{\varepsilon}$, all ψ such that $a(D_x)^{\frac{1}{2}}\psi \in \mathcal{H}^{\lambda h, s}(\mathbf{T}^d)$, all $b \in H^{s - \frac{1}{2}}(\mathbf{T}^d)$ we have

$$(3.2) \quad \|G(\eta)(\psi, b)\|_{\mathcal{H}^{\lambda h, s - \frac{1}{2}}} \leq C(\|a(D_x)^{\frac{1}{2}}\psi\|_{\mathcal{H}^{\lambda h, s}} + \|b\|_{H^{s - \frac{1}{2}}}) \\ + \|\eta\|_{\mathcal{H}^{\lambda h, s + \frac{1}{2}}}(\|a(D_x)^{\frac{1}{2}}\psi\|_{\mathcal{H}^{\lambda h, s - \frac{1}{2}}} + \|b\|_{H^{s - \frac{3}{2}}}).$$

Remark 3.2. (i) We insist on the fact that the constants $\bar{\varepsilon}$ and C in the above Theorem depend on λ_0 but not on λ as soon as $0 \leq \lambda \leq \lambda_0$.

(ii) Assume that $s > 2 + \delta + \frac{d}{2}$, $b \in H^{s - \frac{1}{2}}(\mathbf{T}^d)$ and for $j = 1, 2$,

$$\eta_j \in \mathcal{H}^{\lambda h, s + \frac{1}{2}}(\mathbf{T}^d) \text{ with } \|\eta_j\|_{\mathcal{H}^{\lambda h, s}} \leq \bar{\varepsilon}, \quad a(D_x)^{\frac{1}{2}}\psi_j \in \mathcal{H}^{\lambda h, s}(\mathbf{T}^d).$$

Then we may apply the above Theorem with $s' = s - \delta$, and we obtain an estimate of $\|G(\eta)(\psi, b)\|_{\mathcal{H}^{\lambda h, s - \frac{1}{2} - \delta}}$ by the right hand side of (3.2) for s replaced by $s - \delta$.

Proof of Theorem 3.1. We use the notations introduced in Section 2.1. In the variable $(x, z) \in \mathbf{T}^d \times (-h, 0)$, we have

$$G(\eta)(\psi, b) = \left(\frac{1 + |\nabla_x \rho|^2}{\partial_z \rho} \partial_z \tilde{u} - \nabla_x \rho \cdot \nabla_x \tilde{u} \right) \Big|_{z=0},$$

where $\tilde{u} = u(x, \rho(x, z))$ satisfies the following elliptic boundary value problem:

$$(\Delta_{x,z} + R)\tilde{u} = 0, \quad \tilde{u}|_{z=0} = \psi, \quad (\partial_z \tilde{u})|_{z=-h} = (\partial_z \rho|_{z=-h})b.$$

Let us introduce the function

$$U = \frac{1 + |\nabla_x \rho|^2}{\partial_z \rho} \partial_z \tilde{u} - \nabla_x \rho \cdot \nabla_x \tilde{u},$$

and set $I_h = [-h, 0]$. Since $G(\eta)(\psi, b) = U|_{z=0}$, by definition of the spaces $E^{\lambda, \mu}$, we have

$$\|G(\eta)(\psi, b)\|_{\mathcal{H}^{\lambda h, s - \frac{1}{2}}} \leq \|U\|_{E^{\lambda, s - \frac{1}{2}}}.$$

Now, we use an interpolation argument (see statement (iv) in Lemma A.12) to infer that

$$(3.3) \quad \|G(\eta)(\psi, b)\|_{\mathcal{H}^{\lambda h, s - \frac{1}{2}}} \leq \|U\|_{E^{\lambda, s - \frac{1}{2}}} \leq C \|\partial_z U\|_{F^{\lambda, s - 1}} + C \|U\|_{F^{\lambda h, s}}.$$

The rest of the proof consists in estimating U and $\partial_z U$ in terms of $\nabla_{x,z} \tilde{u}$, so that the required estimate (3.2) will be a consequence of Corollary 2.7 and Theorem 2.10.

Lemma 3.3. For all $s > 2 + d/2$ and all $0 \leq \lambda < 1$, there exist $\bar{\varepsilon} > 0$ and $C > 0$ such that for all $\eta \in \mathcal{H}^{\lambda h, s + \frac{1}{2}}$ satisfying $\|\eta\|_{\mathcal{H}^{\lambda h, s}} \leq \bar{\varepsilon}$, there holds

$$(3.4) \quad \|U\|_{F^{\lambda, s}} \leq C(\|\nabla_{x,z} \tilde{u}\|_{F^{\lambda, s}} + \|\eta\|_{\mathcal{H}^{\lambda h, s + \frac{1}{2}}}) \|\nabla_{x,z} \tilde{u}\|_{F^{\lambda, s - 1}},$$

and

$$(3.5) \quad \|\partial_z U\|_{F^{\lambda, s - 1}} \leq C(\|\nabla_{x,z} \tilde{u}\|_{F^{\lambda, s}} + \|\eta\|_{\mathcal{H}^{\lambda h, s + \frac{1}{2}}}) \|\nabla_{x,z} \tilde{u}\|_{F^{\lambda, s - 1}}.$$

Proof. Write U under the form

$$U = (1 + a)\partial_z \tilde{u} - \nabla_x \rho \cdot \nabla_x \tilde{u},$$

where a is as defined in (2.35). Then, the tame product estimate (A.17) below (applied with $s_0 = s - 2$) implies that

$$\begin{aligned} \|U\|_{F^{\lambda,s}} &\lesssim (1 + \|a\|_{E^{\lambda,s-2}}) \|\partial_z \tilde{u}\|_{F^{\lambda h,s}} + \|\nabla_x \rho\|_{E^{\lambda,s-2}} \|\nabla_x \tilde{u}\|_{F^{\lambda,s}} \\ &\quad + \|a\|_{F^{\lambda h,s}} \|\partial_z \tilde{u}\|_{E^{\lambda,s-2}} + \|\nabla_x \rho\|_{F^{\lambda,s}} \|\nabla_x \tilde{u}\|_{E^{\lambda,s-2}}. \end{aligned}$$

The contribution of ρ is estimated by means of Lemma A.15 and the one of a is estimated by Lemma A.17 (which implies that a belongs to the space \mathcal{E}_1). Consequently, provided $\|\eta\|_{\mathcal{H}^{\lambda h,s}} \leq \bar{\varepsilon} \leq 1$, we have

$$\begin{aligned} \|a\|_{E^{\lambda,s-2}} + \|\nabla_x \rho\|_{E^{\lambda,s-2}} &\leq C \|\eta\|_{\mathcal{H}^{\lambda h,s}} \leq C, \\ \|a\|_{F^{\lambda,s}} + \|\nabla_x \rho\|_{F^{\lambda,s}} &\leq C \|\eta\|_{\mathcal{H}^{\lambda h,s+\frac{1}{2}}}. \end{aligned}$$

By combining the previous estimates, we obtain that

$$(3.6) \quad \|U\|_{F^{\lambda,s}} \leq C \|\nabla_{x,z} \tilde{u}\|_{F^{\lambda,s}} + \|\eta\|_{\mathcal{H}^{\lambda h,s+\frac{1}{2}}} \|\nabla_{x,z} \tilde{u}\|_{E^{\lambda,s-2}},$$

Now, since $\Delta_{x,z} \tilde{u} + R\tilde{u} = 0$, it follows from Lemma 2.12 that

$$(3.7) \quad \|\nabla_{x,z} \tilde{u}\|_{E^{\lambda,s-2}} \leq C \|\nabla_{x,z} \tilde{u}\|_{F^{\lambda,s-1}}.$$

By plugging this bound in (3.6), we conclude the proof of the estimate (3.4).

We now estimate $\partial_z U$. To do so, we exploit the fact that the equation

$$\Delta_{x,z} \tilde{u} + R\tilde{u} = 0$$

can be written in divergence form, as we have seen in (2.3). More precisely, we have

$$(3.8) \quad \partial_z \left(\frac{1 + |\nabla_x \rho|^2}{\partial_z \rho} \partial_z \tilde{u} - \nabla_x \rho \cdot \nabla_x \tilde{u} \right) + \operatorname{div}_x (\partial_z \rho \nabla_x \tilde{u} - \partial_z \tilde{u} \nabla_x \rho) = 0.$$

This immediately implies that

$$\|\partial_z U\|_{F^{\lambda,s-1}} \leq \|\partial_z \rho \nabla_x \tilde{u} - \partial_z \tilde{u} \nabla_x \rho\|_{F^{\lambda,s}}.$$

Now, as above, we apply the same product estimate (A.17) below to infer that

$$\|\partial_z U\|_{F^{\lambda,s-1}} \lesssim \|\nabla_{x,z} \rho\|_{E^{\lambda,s-2}} \|\nabla_{x,z} \tilde{u}\|_{F^{\lambda,s}} + \|\nabla_{x,z} \rho\|_{F^{\lambda,s}} \|\nabla_{x,z} \tilde{u}\|_{E^{\lambda,s-2}}.$$

Then we use Lemma A.15 to obtain

$$\|\partial_z U\|_{F^{\lambda,s-1}} \leq C \|\nabla_{x,z} \tilde{u}\|_{F^{\lambda,s}} + \|\eta\|_{\mathcal{H}^{\lambda,s+\frac{1}{2}}} \|\nabla_{x,z} \tilde{u}\|_{E^{\lambda,s-2}},$$

and hence, the desired estimate (3.5) follows from (3.7). \square

In view of (3.3) and the previous lemma, the estimate (3.2) follows directly from Corollary 2.7 and Theorem 2.10. \square

In the following result we shall prove that, in a certain sense, the Dirichlet-Neuman operator is Lipschitz with respect to (ψ, η) . Let us introduce some notations. If (θ_1, θ_2) is a given couple of functions and $t \in \mathbf{R}$, we shall set

$$\theta = \theta_1 - \theta_2, \quad \|(\theta_1, \theta_2)\|_{\mathcal{H}^{\lambda h,t}} = \sum_{j=1}^2 \|\theta_j\|_{\mathcal{H}^{\lambda h,t}},$$

and we shall use these notations if $\theta = \eta$ or $a(D_x)^{\frac{1}{2}}\psi$. Moreover, we set

$$(3.9) \quad \begin{cases} \mathcal{G}_j = G(\eta_j)(\psi_j, b), j = 1, 2, \\ \lambda_1 = \|(\eta_1, \eta_2)\|_{\mathcal{H}^{\lambda h, s + \frac{1}{2}}} \left(\| (a(D_x)^{\frac{1}{2}}\psi_1, a(D_x)^{\frac{1}{2}}\psi_2) \|_{\mathcal{H}^{\lambda h, s - \frac{1}{2}}} + \|b\|_{H^{s - \frac{3}{2}}} \right) \\ \quad + \| (a(D_x)^{\frac{1}{2}}\psi_1, a(D_x)^{\frac{1}{2}}\psi_2) \|_{\mathcal{H}^{\lambda h, s}} + \|b\|_{H^{s - \frac{1}{2}}}, \\ \lambda_2 = \| (a(D_x)^{\frac{1}{2}}\psi_1, a(D_x)^{\frac{1}{2}}\psi_2) \|_{\mathcal{H}^{\lambda h, s - 1}} + \|b\|_{H^{s - \frac{3}{2}}}, \\ \lambda_3 = \|(\eta_1, \eta_2)\|_{\mathcal{H}^{\lambda h, s + \frac{1}{2}}}. \end{cases}$$

Then we have the following result.

Theorem 3.4. *For all $s > d/2 + 2$ and all $0 \leq \lambda_0 < 1$, there exist $C > 0$ and $\bar{\varepsilon} > 0$ such that for all $0 \leq \lambda \leq \lambda_0$, for all $\eta_j \in \mathcal{H}^{\lambda h, s + \frac{1}{2}}(\mathbf{T}^d)$ satisfying $\|\eta_j\|_{\mathcal{H}^{\lambda h, s}} \leq \bar{\varepsilon}$, all ψ_j such that $a(D_x)^{\frac{1}{2}}\psi_j \in \mathcal{H}^{\lambda h, s}(\mathbf{T}^d)$, $j = 1, 2$ and all $b \in H^{s - \frac{1}{2}}(\mathbf{T}^d)$ we have*

$$\begin{aligned} & \|\mathcal{G}_1 - \mathcal{G}_2\|_{\mathcal{H}^{\lambda h, s - \frac{1}{2}}} \\ & \leq C \left(\lambda_1 \|\eta_1 - \eta_2\|_{\mathcal{H}^{\lambda h, s}} + \lambda_2 \|\eta_1 - \eta_2\|_{\mathcal{H}^{\lambda h, s + \frac{1}{2}}} \right. \\ & \quad \left. + \lambda_3 \|a(D_x)^{\frac{1}{2}}(\psi_1 - \psi_2)\|_{\mathcal{H}^{\lambda h, s - \frac{1}{2}}} + \|a(D_x)^{\frac{1}{2}}(\psi_1 - \psi_2)\|_{\mathcal{H}^{\lambda h, s}} \right). \end{aligned}$$

Remark 3.5. (1) *The constants $\bar{\varepsilon}$ and C in the above Theorem depend on λ_0 but not on λ for $0 \leq \lambda \leq \lambda_0$.*

(2) *Assume that $s > 2 + \delta + \frac{d}{2}$, $\delta > 0$, $b \in H^{s - \frac{1}{2}}(\mathbf{T}^d)$ and for $j = 1, 2$,*

$$\eta_j \in \mathcal{H}^{\lambda h, s + \frac{1}{2}}(\mathbf{T}^d) \text{ with } \|\eta_j\|_{\mathcal{H}^{\lambda h, s}} \leq \bar{\varepsilon}, \quad a(D_x)^{\frac{1}{2}}\psi_j \in \mathcal{H}^{\lambda h, s}(\mathbf{T}^d).$$

Then we may apply the above Theorem to $s' = s - \delta$ and we get an estimate of the term, $\|\mathcal{G}_1 - \mathcal{G}_2\|_{\mathcal{H}^{\lambda h, s - \frac{1}{2} - \delta}}$ by the right hand side for s replaced by $s - \delta$.

Proof of Theorem 3.4. Introduce the functions

$$U_j = \frac{1 + |\nabla_x \rho_j|^2}{\partial_z \rho_j} \partial_z \tilde{u}_j - \nabla_x \rho_j \cdot \nabla_x \tilde{u}_j$$

for $j = 1, 2$, where

$$(3.10) \quad (\Delta_{x,z} + R_j)\tilde{u}_j = 0, \quad \tilde{u}_j|_{z=0} = \psi_j, \quad (\partial_z \tilde{u}_j)|_{z=-h} = (\partial_z \rho_j|_{z=-h})b.$$

Then, by definition we have

$$\mathcal{G}_1 - \mathcal{G}_2 = (U_1 - U_2)|_{z=0}.$$

As in (3.3), we have

$$\begin{aligned} \|\mathcal{G}_1 - \mathcal{G}_2\|_{\mathcal{H}^{\lambda h, s - \frac{1}{2}}} & \leq \|U_1 - U_2\|_{E^{\lambda, s - \frac{1}{2}}} \\ & \leq C(\|U_1 - U_2\|_{F^{\lambda, s}} + \|\partial_z(U_1 - U_2)\|_{F^{\lambda, s - 1}}). \end{aligned}$$

Now according to (3.8), our equation on \tilde{u}_j reads

$$\partial_z U_j + \operatorname{div}_x V_j = 0$$

with

$$(3.11) \quad U_j = \frac{1 + |\nabla_x \rho_j|^2}{\partial_z \rho_j} \partial_z \tilde{u}_j - \nabla_x \rho_j \cdot \nabla_x \tilde{u}_j, \quad V_j = \partial_z \rho_j \nabla_x \tilde{u}_j - \partial_z \tilde{u}_j \nabla_x \rho_j,$$

It follows that

$$(3.12) \quad \|\mathcal{G}_1 - \mathcal{G}_2\|_{\mathcal{H}^{\lambda, s - \frac{1}{2}}} \leq C(\|U_1 - U_2\|_{F^{\lambda, s}} + \|V_1 - V_2\|_{F^{\lambda, s}}).$$

Recall that

$$\partial_z \rho_j = 1 + q_j \quad \text{and} \quad \nabla_x \rho_j = \frac{1}{h}(z + h)e^{z|D_x|} \nabla_x \eta_j,$$

where

$$\begin{cases} q_j = \frac{1}{h} e^{z|D_x|} \eta_j + \frac{1}{h} (z + h) e^{z|D_x|} |D_x| \eta_j, \\ \partial_z \rho_j|_{z=-h} = 1 + \frac{1}{h} e^{-h|D_x|} \eta_j, \\ \frac{1 + |\nabla_x \rho_j|^2}{\partial_z \rho_j} = 1 + A_j, \quad A_j = -f(q_j) - |\nabla_x \rho_j|^2 f(q_j) + |\nabla_x \rho_j|^2, \end{cases}$$

with

$$f(q_j) = \frac{q_j}{1 + q_j}.$$

With these notations and using (3.11) we can write

$$(3.13) \quad \begin{cases} \tilde{u} = \tilde{u}_1 - \tilde{u}_2, \\ U_1 - U_2 = (A_1 - A_2) \partial_z \tilde{u}_2 - (\nabla_x \rho_1 - \nabla_x \rho_2) \cdot \nabla_x \tilde{u}_2 \\ \quad + (1 + A_1) \partial_z \tilde{u} - \nabla_x \rho_1 \cdot \nabla_x \tilde{u}, \\ V_1 - V_2 = (\partial_z \rho_1 - \partial_z \rho_2) \nabla_x \tilde{u}_2 \\ \quad - (\nabla_x \rho_1 - \nabla_x \rho_2) \partial_z \tilde{u}_2 + \partial_z \rho_1 \nabla_x \tilde{u} - \nabla_x \rho_1 \partial_z \tilde{u}. \end{cases}$$

The first two terms in the right hand side of (3.13) are of the form

$$(p_1 - p_2) \nabla_{x,z} \tilde{u}_2 \quad \text{with } p \in \mathcal{F}_1$$

by Lemma A.19, where \mathcal{F}_1 has been defined in Definition A.18. They are estimated as follows. Using Lemma A.12 (iii) with $s_0 = s - \frac{3}{2}$, we can write

$$\begin{aligned} \|(p_1 - p_2) \nabla_{x,z} \tilde{u}_2\|_{F^{\lambda, s}} &\lesssim \|p_1 - p_2\|_{E^{\lambda, s - \frac{3}{2}}} \|\nabla_{x,z} \tilde{u}_2\|_{F^{\lambda, s}} \\ &\quad + \|p_1 - p_2\|_{F^{\lambda, s}} \|\nabla_{x,z} \tilde{u}_2\|_{E^{\lambda, s - \frac{3}{2}}}, \end{aligned}$$

and hence,

$$\begin{aligned} \|(p_1 - p_2) \nabla_{x,z} \tilde{u}_2\|_{F^{\lambda, s}} &\lesssim \|p_1 - p_2\|_{L^\infty(I_h, \mathcal{H}^{\lambda, s - \frac{3}{2}})} \|\nabla_{x,z} \tilde{u}_2\|_{F^{\lambda, s}} \\ &\quad + \|p_1 - p_2\|_{L^2(I_h, \mathcal{H}^{\lambda, s})} \|\nabla_{x,z} \tilde{u}_2\|_{E^{\lambda, s - \frac{3}{2}}}. \end{aligned}$$

Using the definition of \mathcal{F}_1 , (3.10), Corollary 2.13, Lemma 2.12, Corollary 2.7 and the notation in (3.9), we obtain

$$(3.14) \quad \|(p_1 - p_2) \nabla_{x,z} \tilde{u}_2\|_{F^{\lambda, s}} \leq \lambda_1 \|\eta_1 - \eta_2\|_{\mathcal{H}^{\lambda, s}} + \lambda_2 \|\eta_1 - \eta_2\|_{\mathcal{H}^{\lambda, s + \frac{1}{2}}}.$$

The last two terms in the right hand side of (3.13) are of the form

$$q \nabla_{x,z} (\tilde{u}_1 - \tilde{u}_2) = q \nabla_{x,z} \tilde{u}, \quad \text{with } q = 1 \text{ or } q \in \mathcal{E}_1,$$

where \mathcal{E}_1 has been defined in Definition A.16. They are estimated as follows. By Lemma A.12 with $s_0 = s - \frac{3}{2}$ we can write

$$\|q\nabla_{x,z}\tilde{u}\|_{F^{\lambda,s}} \lesssim \|q\|_{L^\infty(I_h, \mathcal{H}^{\lambda h, s - \frac{3}{2}})} \|\nabla_{x,z}\tilde{u}\|_{F^{\lambda,s}} + \|q\|_{L^2(I_h, \mathcal{H}^{\lambda h, s})} \|\nabla_{x,z}\tilde{u}\|_{E^{\lambda, s - \frac{3}{2}}}.$$

Since $q \in \mathcal{E}_1$ we deduce from Definition A.16 that

$$(3.15) \quad \|q\nabla_{x,z}\tilde{u}\|_{F^{\lambda,s}} \lesssim \|\nabla_{x,z}\tilde{u}\|_{F^{\lambda h, s}} + \|\eta_1\|_{\mathcal{H}^{s+\frac{1}{2}}} \|\nabla_{x,z}\tilde{u}\|_{E^{\lambda h, s - \frac{3}{2}}}.$$

We see therefore that we have to estimate $\nabla_{x,z}\tilde{u}$. For that, according to (3.10), we notice that $\tilde{u} = \tilde{u}_1 - \tilde{u}_2$ is solution of the problem

$$(\Delta_{x,z} + R_1)\tilde{u} = (R_2 - R_1)\tilde{u}_2, \quad \tilde{u}|_{z=0} = \psi_1 - \psi_2, \quad \partial_z \tilde{u}|_{z=-h} = \frac{1}{h} e^{-h|D_x|} (\eta_1 - \eta_2)b.$$

Notice that for every $\mu > \frac{d}{2}$ we have

$$\left\| \frac{1}{h} e^{-h|D_x|} (\eta_1 - \eta_2)b \right\|_{H^\mu} \leq C \|\eta_1 - \eta_2\|_{\mathcal{H}^{\lambda h, \mu}} \|b\|_{H^\mu}.$$

Therefore, using Theorem 2.10, we can write

$$(3.16) \quad \begin{aligned} \|\nabla_{x,z}\tilde{u}\|_{F^{\lambda,s}} &\lesssim \|(R_2 - R_1)\tilde{u}_2\|_{F^{\lambda, s-1}} + \|a(D_x)^{\frac{1}{2}}(\psi_1 - \psi_2)\|_{\mathcal{H}^{\lambda h, s}} \\ &+ \|\eta_1 - \eta_2\|_{\mathcal{H}^{\lambda h, s - \frac{1}{2}}} \|b\|_{H^{s - \frac{1}{2}}} + C \|\eta_1\|_{\mathcal{H}^{\lambda h, s + \frac{1}{2}}} \|(R_2 - R_1)\tilde{u}_2\|_{F^{\lambda, s-2}} \\ &+ \|a(D_x)^{\frac{1}{2}}(\psi_1 - \psi_2)\|_{\mathcal{H}^{\lambda h, s - \frac{1}{2}}} + \|\eta_1 - \eta_2\|_{\mathcal{H}^{\lambda h, s - \frac{3}{2}}} \|b\|_{H^{s - \frac{3}{2}}}. \end{aligned}$$

Moreover using Lemma 2.12 we can write

$$\|\nabla_{x,z}\tilde{u}\|_{E^{\lambda, s - \frac{3}{2}}} \lesssim \|\nabla_{x,z}\tilde{u}\|_{F^{\lambda, s-1}} + \|(R_2 - R_1)\tilde{u}_2\|_{F^{\lambda, s-2}}.$$

Then we can use Corollary 2.7 and obtain

$$(3.17) \quad \|\nabla_{x,z}\tilde{u}\|_{E^{\lambda, s - \frac{3}{2}}} \lesssim \|(R_2 - R_1)\tilde{u}_2\|_{F^{\lambda, s-2}} + \|\eta_1 - \eta_2\|_{\mathcal{H}^{\lambda h, s - \frac{3}{2}}} \|b\|_{H^{s - \frac{3}{2}}}.$$

Using (3.15), (3.16) and (3.17), we obtain

$$\begin{aligned} &\|q\nabla_{x,z}\tilde{u}\|_{F^{\lambda,s}} \\ &\lesssim \|(R_2 - R_1)\tilde{u}_2\|_{F^{\lambda, s-1}} + \|a(D_x)^{\frac{1}{2}}(\psi_1 - \psi_2)\|_{\mathcal{H}^{\lambda h, s}} \\ &+ \|\eta_1 - \eta_2\|_{\mathcal{H}^{\lambda h, s - \frac{1}{2}}} \|b\|_{H^{s - \frac{1}{2}}} \\ &+ \|\eta_1\|_{\mathcal{H}^{\lambda h, s + \frac{1}{2}}} \left(\|(R_2 - R_1)\tilde{u}_2\|_{F^{\lambda, s-2}} + \|a(D_x)^{\frac{1}{2}}(\psi_1 - \psi_2)\|_{\mathcal{H}^{\lambda h, s - \frac{1}{2}}} \right) \\ &+ \|\eta_1 - \eta_2\|_{\mathcal{H}^{\lambda h, s - \frac{3}{2}}} \|b\|_{H^{s - \frac{3}{2}}}. \end{aligned}$$

Using the definition of $\lambda_j, j = 1, 2, 3$, we obtain

$$(3.18) \quad \begin{aligned} &\|q\nabla_{x,z}\tilde{u}\|_{F^{\lambda,s}} \\ &\lesssim \|(R_2 - R_1)\tilde{u}_2\|_{F^{\lambda, s-1}} + \|\eta_1\|_{\mathcal{H}^{\lambda h, s + \frac{1}{2}}} \|(R_2 - R_1)\tilde{u}_2\|_{F^{\lambda, s-2}} + \lambda_1 \|\eta_1 - \eta_2\|_{\mathcal{H}^{\lambda h, s}} \\ &+ \lambda_3 \|a(D_x)^{\frac{1}{2}}(\psi_1 - \psi_2)\|_{\mathcal{H}^{\lambda h, s - \frac{1}{2}}} + \|a(D_x)^{\frac{1}{2}}(\psi_1 - \psi_2)\|_{\mathcal{H}^{\lambda h, s}}. \end{aligned}$$

To complete the proof, we are led to estimate $(R_1 - R_2)\tilde{u}_2$. According to (2.35), we have

$$\begin{cases} (R_2 - R_1) = (a_1 - a_2)\partial_z^2 + (b_1 - b_2)\Delta_x + (c_1 - c_2) \cdot \nabla_x \partial_z - (d_1 - d_2)\partial_z \\ a_j = \frac{1 + |\nabla_x \rho_j|^2}{\partial_z \rho} - 1, \quad b_j = \partial_z \rho_j - 1, \quad c_j = -2\nabla_x \rho_j, \\ d_j = \frac{1 + |\nabla_x \rho_j|^2}{\partial_z \rho_j} \partial_z^2 \rho_j + \partial_z \rho_j \Delta_x \rho_j - 2\nabla_x \rho_j \cdot \nabla_x \partial_z \rho_j. \end{cases}$$

Estimate of $\|(R_2 - R_1)\tilde{u}_2\|_{F^{\lambda, s-1}}$.

The first three terms in $(R_1 - R_2)\tilde{u}_2$ are estimated in the same manner. Since $a, b, c \in \mathcal{F}_1$ (see Lemma A.19) for $r \in \{a, b, c\}$, using Lemma A.12 (ii), we can write

$$\begin{aligned} \|(r_1 - r_2)\nabla_{x,z}^2 \tilde{u}_2\|_{F^{\lambda, s-1}} &\leq C\|r_1 - r_2\|_{L^\infty(I_h, \mathcal{H}^{\lambda h, s-1})} \|\nabla_{x,z}^2 \tilde{u}_2\|_{F^{\lambda, s-1}}, \\ &\leq C\|\eta_1 - \eta_2\|_{\mathcal{H}^{\lambda h, s}} \|\nabla_{x,z}^2 \tilde{u}_2\|_{F^{\lambda, s-1}}. \end{aligned}$$

Thanks to Corollary 2.13, we have

$$\begin{aligned} \|(r_1 - r_2)\nabla_{x,z}^2 \tilde{u}_2\|_{F^{\lambda, s-1}} &\leq \|\eta_1 - \eta_2\|_{\mathcal{H}^{\lambda h, s}} \left(\|a(D_x)^{\frac{1}{2}} \psi_2\|_{\mathcal{H}^{\lambda h, s}} + \|b\|_{H^{s-\frac{1}{2}}} \right. \\ &\quad \left. + \|\eta_2\|_{\mathcal{H}^{\lambda h, s+\frac{1}{2}}} \left(\|a(D_x)^{\frac{1}{2}} \psi_2\|_{\mathcal{H}^{\lambda h, s-\frac{1}{2}}} + \|b\|_{H^{s-\frac{3}{2}}} \right) \right). \end{aligned}$$

With the notations in (3.9) one can deduce eventually that

$$(3.19) \quad \|(r_1 - r_2)\nabla_{x,z}^2 \tilde{u}_2\|_{F^{\lambda, s-1}} \leq \lambda_1 \|\eta_1 - \eta_2\|_{\mathcal{H}^{\lambda h, s}},$$

where $r = a$ or b or c .

We consider now the term $\|(d_1 - d_2)\partial_z \tilde{u}_2\|_{F^{\lambda, s-1}}$. Using Lemma A.12 (iii) with $s_0 = s - 2$, one can write

$$\begin{aligned} \|(d_1 - d_2)\partial_z \tilde{u}_2\|_{F^{\lambda, s-1}} &\lesssim \|d_1 - d_2\|_{L^\infty(I_h, \mathcal{H}^{\lambda h, s-2})} \|\partial_z \tilde{u}_2\|_{F^{\lambda, s-1}} \\ &\quad + \|d_1 - d_2\|_{L^2(I_h, \mathcal{H}^{\lambda h, s-1})} \|\partial_z \tilde{u}_2\|_{E^{\lambda, s-2}}. \end{aligned}$$

By Lemma A.19 we have $d \in \mathcal{F}_2$. Therefore,

$$\begin{aligned} \|d_1 - d_2\|_{L^\infty(I_h, \mathcal{H}^{\lambda, s-2})} &\lesssim \|\eta_1 - \eta_2\|_{\mathcal{H}^{\lambda h, s}}, \\ \|d_1 - d_2\|_{L^2(I_h, \mathcal{H}^{\lambda, s-1})} &\lesssim \|\eta_1 - \eta_2\|_{\mathcal{H}^{\lambda h, s+\frac{1}{2}}}. \end{aligned}$$

By Corollary 2.7 we have

$$(3.20) \quad \|\partial_z \tilde{u}_2\|_{F^{\lambda, s-1}} \lesssim \|a(D_x)^{\frac{1}{2}} \psi_2\|_{\mathcal{H}^{\lambda h, s-1}} + \|b\|_{H^{s-\frac{3}{2}}(\mathbf{T}^d)}.$$

By Lemma 2.12 and Corollary 2.7 we have

$$\|\partial_z \tilde{u}_2\|_{E^{\lambda, s-2}} \lesssim \|\nabla_{x,z} \tilde{u}_2\|_{F^{\lambda, s-1}} \lesssim \|a(D_x) \psi_2\|_{\mathcal{H}^{\lambda, s-1}} + \|b\|_{H^{s-\frac{3}{2}}}.$$

Using the notation in (3.9), we obtain

$$(3.21) \quad \|(d_1 - d_2)\partial_z \tilde{u}_2\|_{F^{\lambda, s-1}} \leq \lambda_1 \|\eta_1 - \eta_2\|_{\mathcal{H}^{\lambda h, s}} + \lambda_2 \|\eta_1 - \eta_2\|_{\mathcal{H}^{\lambda h, s+\frac{1}{2}}}.$$

It follows from (3.19) and (3.21) that

$$(3.22) \quad \|(R_1 - R_2)\tilde{u}_2\|_{F^{\lambda, s-1}} \lesssim \lambda_1 \|\eta_1 - \eta_2\|_{\mathcal{H}^{\lambda h, s}} + \lambda_2 \|\eta_1 - \eta_2\|_{\mathcal{H}^{\lambda h, s+\frac{1}{2}}}.$$

Estimate of $\|(R_1 - R_2)\tilde{u}_2\|_{F^{\lambda, s-2}}$.

We use the same notations as above. Since $s - 2 > \frac{d}{2}$, we can write

$$\|(r_1 - r_2)\nabla_{x,z}^2 \tilde{u}_2\|_{F^{\lambda,s-2}} \lesssim \|r_1 - r_2\|_{L^\infty(I_h, \mathcal{H}^{\lambda h, s-2})} \|\nabla_{x,z}^2 \tilde{u}_2\|_{F^{\lambda,s-2}}.$$

Since $r \in \mathcal{F}_1$, we have

$$\|r_1 - r_2\|_{L^\infty(I_h, \mathcal{H}^{\lambda h, s-2})} \lesssim \|\eta_1 - \eta_2\|_{\mathcal{H}^{\lambda h, s-1}}.$$

Moreover, from Lemma 2.12 and Corollary 2.7 we have

$$\|\nabla_{x,z}^2 \tilde{u}_2\|_{F^{\lambda,s-2}} \lesssim \|a(D_x)^{\frac{1}{2}} \psi_2\|_{\mathcal{H}^{\lambda h, s-1}} + \|b\|_{H^{s-\frac{3}{2}}}.$$

Hence we find that

$$\|(r_1 - r_2)\nabla_{x,z}^2 \tilde{u}_2\|_{F^{\lambda,s-2}} \lesssim \|\eta_1 - \eta_2\|_{\mathcal{H}^{\lambda h, s-1}} (\|a(D_x)^{\frac{1}{2}} \psi_2\|_{\mathcal{H}^{\lambda h, s-1}} + \|b\|_{H^{s-\frac{3}{2}}}).$$

Now,

$$\|(d_1 - d_2)\partial_z \tilde{u}_2\|_{F^{\lambda,s-2}} \lesssim \|d_1 - d_2\|_{L^\infty(I_h, \mathcal{H}^{\lambda h, s-2})} \|\partial_z \tilde{u}_2\|_{F^{\lambda,s-2}}.$$

Since $d \in \mathcal{F}_2$, we have

$$\|d_1 - d_2\|_{L^\infty(I_h, \mathcal{H}^{\lambda h, s-2})} \lesssim \|\eta_1 - \eta_2\|_{\mathcal{H}^{\lambda h, s}}.$$

Using (3.20), we obtain

$$\|(d_1 - d_2)\partial_z \tilde{u}_2\|_{F^{\lambda,s-2}} \lesssim \|\eta_1 - \eta_2\|_{\mathcal{H}^{\lambda h, s}} (\|a(D_x)^{\frac{1}{2}} \psi_2\|_{\mathcal{H}^{\lambda h, s-1}} + \|b\|_{H^{s-\frac{3}{2}}}).$$

Therefore, we get

$$(3.23) \quad \|(R_1 - R_2)\tilde{u}_2\|_{F^{\lambda,s-2}} \lesssim \|\eta_1 - \eta_2\|_{\mathcal{H}^{\lambda h, s}} (\|a(D_x)^{\frac{1}{2}} \psi_2\|_{\mathcal{H}^{\lambda h, s-1}} + \|b\|_{H^{s-\frac{3}{2}}}).$$

It follows from (3.18), (3.22) and (3.23) that

$$(3.24) \quad \begin{aligned} \|q\nabla_{x,z} \tilde{u}\|_{F^{\lambda,s}} &\lesssim \lambda_1 \|\eta_1 - \eta_2\|_{\mathcal{H}^{\lambda h, s}} + \lambda_2 \|\eta_1 - \eta_2\|_{\mathcal{H}^{\lambda h, s+\frac{1}{2}}} \\ &\quad + \lambda_3 \|a(D_x)(\psi_1 - \psi_2)\|_{\mathcal{H}^{\lambda h, s-\frac{1}{2}}} + \|a(D_x)(\psi_1 - \psi_2)\|_{\mathcal{H}^{\lambda h, s}}. \end{aligned}$$

Then Theorem 3.4 follows from (3.12), (3.13), (3.14) and (3.24). \square

The following result is an immediate consequence of Theorem 3.4.

Corollary 3.6. *Let $0 \leq \lambda_0 < 1$ and $s > 2 + \frac{d}{2}$. Then there exist two constant $\bar{\varepsilon} > 0$ and $C > 0$ such that for all $0 \leq \lambda \leq \lambda_0$, all $\eta \in \mathcal{H}^{\lambda h, s+\frac{1}{2}}$, $a(D_x)^{\frac{1}{2}} \psi \in \mathcal{H}^{\lambda h, s}$ satisfying $\|\eta\|_{\mathcal{H}^{\lambda h, s}} \leq \bar{\varepsilon}$, we have*

$$\begin{aligned} &\|G(\eta)(\psi, 0) - G(0)(\psi, 0)\|_{\mathcal{H}^{\lambda h, s-\frac{1}{2}}} \\ &\leq C (\|\eta\|_{\mathcal{H}^{\lambda h, s+\frac{1}{2}}} \|a(D_x)^{\frac{1}{2}} \psi\|_{\mathcal{H}^{\lambda h, s-\frac{1}{2}}} + \|\eta\|_{\mathcal{H}^{\lambda h, s}} \|a(D_x)^{\frac{1}{2}} \psi\|_{\mathcal{H}^{\lambda h, s}}). \end{aligned}$$

Remark 3.7. *Assume that $s > 2 + \delta + \frac{d}{2}$, $\delta > 0$, $\eta \in \mathcal{H}^{\lambda h, s+\frac{1}{2}}(\mathbf{T}^d)$ with $\|\eta\|_{\mathcal{H}^{\lambda h, s}} \leq \bar{\varepsilon}$, and that $a(D_x)^{\frac{1}{2}} \psi \in \mathcal{H}^{\lambda h, s}(\mathbf{T}^d)$. Then we may apply Corollary 3.6 to $s' = s - \delta$, and obtain an estimate of $\|G(\eta)(\psi, 0) - G(0)(\psi, 0)\|_{\mathcal{H}^{\lambda h, s-\frac{1}{2}-\delta}}$ by the right hand side for s replaced by $s - \delta$.*

We shall use later on the following result concerning the Dirichlet-Neumann operator defined in (1.3). Recall from (1.6) that

$$B = \frac{\partial \phi}{\partial y} \Big|_{y=\eta(x)} = \frac{G(\eta)(\psi, b) + \nabla_x \eta \cdot \nabla_x \psi}{1 + |\nabla_x \eta|^2},$$

$$V = \nabla_x \phi \Big|_{y=\eta(x)} = \nabla_x \psi - B \nabla_x \eta.$$

Lemma 3.8. *We have*

$$\nabla_x [G(\eta)(\psi, b)] = G(\eta)(V, \nabla_x b) - (V \cdot \nabla_x) \nabla_x \eta - (\operatorname{div} V) \nabla_x \eta.$$

Proof. According to (1.3), we have, with $\partial_i = \frac{\partial}{\partial x_i}$, $1 \leq i \leq d$,

$$\partial_i [G(\eta)(\psi, b)] = I + II + III,$$

where

$$\begin{aligned} I &= (\partial_y \partial_i \phi - \nabla_x \eta \cdot \nabla_x \partial_i \phi)(y, \eta(x)), \\ II &= -\nabla_x \partial_i \eta(x) \cdot \nabla_x \phi(y, \eta(x)), \\ III &= -\partial_i \eta(x) (\partial_y^2 \phi - \nabla_x \eta \cdot \nabla_x \partial_y \phi)(x, \eta(x)). \end{aligned}$$

Now, $\partial_i \phi$ is the solution of the problem:

$$\begin{cases} \Delta_{x,y} \partial_i \phi = 0 & \text{in } -h < y < \eta(x), \\ \partial_i \phi|_{y=\eta(x)} = V, \quad \partial_y \partial_i \phi|_{y=-h} = \partial_i b. \end{cases}$$

It follows that $I = G(\eta)(V, \partial_i b)$. By the definition of V we have

$$II = -(V \cdot \nabla_x)(\partial_i \eta).$$

Eventually, since $V = \nabla_x \phi(x, \eta(x))$, we have

$$\begin{aligned} \operatorname{div} V &= \sum_{i=1}^d [\partial_i^2 \phi(x, \eta(x)) + \partial_i \partial_y \phi(x, \eta(x)) \partial_i \eta(x)] \\ &= -\partial_y^2 \phi(y, \eta(x)) + \nabla_x \eta(x) \cdot \nabla_x \partial_y \phi(x, \eta(x)), \end{aligned}$$

since $\partial_y^2 + \sum_{i=1}^d \partial_i^2 \phi = 0$. It follows that

$$III = -(\operatorname{div} V) \nabla_x \eta.$$

Thus, Lemma 3.8 is proved. □

Lemma 3.9. *We have*

$$(3.25) \quad G(\eta)(B, -\Delta_x(\phi|_{y=-h})) = -\operatorname{div} V.$$

Proof. We have

$$\begin{aligned} \operatorname{div} V &= \sum_{j=1}^d (\partial_j^2 \phi + \partial_j \eta \partial_j \partial_y \phi)|_{y=\eta} = (\Delta_x \phi + \nabla_x \eta \cdot \nabla_x \partial_y \phi)|_{y=\eta} \\ &= (-\partial_y^2 \phi + \nabla_x \eta \cdot \nabla_x \partial_y \phi)|_{y=\eta} \\ &= -((\partial_y - \nabla_x \eta \cdot \nabla_x) \partial_y \phi)|_{y=\eta}. \end{aligned}$$

Since $\partial_y \phi$ is a solution of the problem

$$\Delta_{x,y}(\partial_y \phi) = 0, \quad \partial_y \phi|_{y=\eta} = B, \quad \partial_y(\partial_y \phi)|_{y=-h} = -\Delta_x(\phi|_{y=-h}),$$

we get

$$G(\eta)(B, -\Delta_x(\phi|_{y=-h})) = ((\partial_y - \nabla_x \eta \cdot \nabla_x) \partial_y \phi)|_{y=\eta}.$$

This gives the desired identity. □

4. ANOTHER DIRICHLET-NEUMANN OPERATOR

Let $h > 0$ and $\eta \in W^{1,\infty}(\mathbf{T}^d)$ be such that $\|\eta\|_{L^\infty(\mathbf{T}^d)} \leq \varepsilon \ll h$. Setting

$$\mathcal{O} = \{(x, y) : x \in \mathbf{T}^d, -\eta(x) - h < y < 0\},$$

we consider the Dirichlet problem

$$(4.1) \quad \Delta_{x,y}v = 0 \text{ in } \mathcal{O}, \quad v|_{y=0} = b, \quad v|_{y=-\eta(x)-h} = B.$$

Proposition 4.1. *For all $s \in \mathbf{R}$ there exist $C > 0$ and $\mathcal{F} : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ non-decreasing such that for all solution of the problem (4.1) we have*

$$\left\| \frac{\partial v}{\partial y} \Big|_{y=0} \right\|_{H^s(\mathbf{T}^d)} \leq C \|b\|_{H^{s+1}(\mathbf{T}^d)} + \mathcal{F}(\|\eta\|_{W^{1,\infty}(\mathbf{T}^d)}) (\|b\|_{H^1(\mathbf{T}^d)} + \|B\|_{H^1(\mathbf{T}^d)}).$$

Proof. Let $\chi \in C^\infty(\mathbf{R})$ be such that $\chi(y) = 1$ if $-\frac{h}{4} \leq y \leq 0$, $\chi(y) = 0$ if $y \leq -\frac{h}{2}$. We set $w = \chi(y)v$. Then w is solution of the problem:

$$\Delta_{x,y}w = 2\chi'(y)\partial_y v + \chi''(y)v := F \quad \text{for } -\frac{h}{2} < y < 0, \quad w|_{y=0} = b, \quad w|_{y \leq -\frac{h}{2}} = 0.$$

We can solve explicitly this problem. Indeed, taking the Fourier transform with respect to x , we are lead to solve the two following problems:

$$\begin{aligned} (\partial_y + |\xi|)w_1 &= \widehat{F}, & w_1|_{y \leq -\frac{h}{2}} &= 0, \\ (\partial_y - |\xi|)\widehat{w} &= w_1, & \widehat{w}|_{y=0} &= \widehat{b}. \end{aligned}$$

For $-\frac{h}{2} \leq y \leq 0$ we obtain

$$\widehat{w}(y, \xi) = e^{y|\xi|}\widehat{b} - \int_y^0 \int_{-\frac{h}{2}}^s e^{(y+\sigma-2s)|\xi|} \widehat{F}(\sigma, \xi) ds d\sigma.$$

It follows that

$$\frac{\partial \widehat{w}}{\partial y} \Big|_{y=0} = \frac{\partial \widehat{w}}{\partial y} \Big|_{y=0} = |\xi|\widehat{b} + \int_{-\frac{h}{2}}^0 e^{\sigma|\xi|} (\chi'(\sigma)\partial_y \widehat{v}(\sigma, \xi) + \chi''(\sigma)\widehat{v}(\sigma, \xi)) d\sigma.$$

On the support of a derivative of χ we have $\sigma \leq -\frac{h}{4}$. Multiplying both members by $\langle \xi \rangle^s$ and using the fact that $\langle \xi \rangle^s e^{-\frac{h}{4}|\xi|}$ is bounded on \mathbf{T}^d , we obtain easily the estimate

$$\left\| \frac{\partial v}{\partial y} \Big|_{y=0} \right\|_{H^s(\mathbf{T}^d)} \leq C (\|b\|_{H^{s+1}(\mathbf{T}^d)} + \|v\|_{H^1((-\frac{h}{2}, 0) \times \mathbf{T}^d)}).$$

Since the problem (4.1) is variational, we see easily that

$$\|v\|_{H^1((-\frac{h}{2}, 0) \times \mathbf{T}^d)} \leq \mathcal{F}(\|\eta\|_{W^{1,\infty}(\mathbf{T}^d)}) (\|b\|_{H^1(\mathbf{T}^d)} + \|B\|_{H^1(\mathbf{T}^d)}).$$

This completes the proof. □

Corollary 4.2. *Let ϕ be the solution of the problem (3.1), that is,*

$$\Delta_{x,y}\phi = 0 \text{ in } -h < y < \eta(x), \quad \phi|_{y=\eta(x)} = \psi, \quad \partial_y \phi|_{y=-h} = b.$$

Let $\phi_h = \phi(x, -h)$, $B = \partial_y \phi(x, \eta(x))$ and fix $s_0 > \frac{d}{2}$. Then, for all $\mu \in \mathbf{R}$ there exists $C > 0$ such that

$$\|\Delta_x \phi_h\|_{H^\mu(\mathbf{T}^d)} \leq C \|b\|_{H^{\mu+1}(\mathbf{T}^d)} + \mathcal{F}(\|\eta\|_{H^{s_0+1}(\mathbf{T}^d)}) (\|b\|_{H^1(\mathbf{T}^d)} + \|B\|_{H^1(\mathbf{T}^d)}).$$

Proof. Set $u = \partial_y \phi$. It satisfies

$$\Delta_{x,y} u = 0, \quad u|_{y=\eta(x)} = B, \quad u|_{y=-h} = b.$$

On the other hand, we have

$$\partial_y u|_{y=-h} = \partial_y^2 \phi|_{y=-h} = -\Delta_x \phi|_{y=-h} = -\Delta_x \phi_h.$$

Set $v(x, y) = u(x, -y - h)$. Then v is a solution of the problem

$$\Delta_{x,y} v = 0 \text{ for } -\eta(x) - h < y < 0, \quad v|_{y=0} = b, \quad v|_{y=-\eta(x)-h} = B$$

and $\partial_y v|_{y=0} = \partial_y u|_{y=-h} = -\Delta_x \phi_h$.

Corollary 4.2 follows from Proposition 4.1 and from the fact that $H^{s_0+1}(\mathbf{T}^d)$ is embedded in $W^{1,\infty}(\mathbf{T}^d)$. \square

5. EXISTENCE OF A SOLUTION ON A TIME INTERVAL OF SIZE 1

In this section we prove Theorem 1.5. We will obtain the solution as the limit of an iterative scheme. To avoid confusion of notations, we denote the initial data by (u_0, v_0) . Namely, we consider the Cauchy problem for the water-wave system (1.1) with initial data

$$(\eta, \psi)|_{t=0} = (u_0, v_0).$$

Let $u_0, v_0 \in \mathcal{H}^{\lambda h, s}$. We consider the sequence $(\eta_\nu, \psi_\nu)_{\nu \in \mathbf{N}}$ defined by

$$\begin{aligned} \eta_0 &= u_0, \quad \psi_0 = v_0, \text{ and for } \nu \geq 0, \\ \partial_t \eta_{\nu+1} &= G(\eta_\nu)(\psi_\nu, b), \\ (5.1) \quad \partial_t \psi_{\nu+1} &= -g\eta_\nu - \frac{1}{2} |\nabla_x \psi_\nu|^2 + \frac{(G(\eta_\nu)(\psi_\nu, b) + \nabla_x \eta_\nu \cdot \nabla_x \psi_\nu)^2}{2(1 + |\nabla_x \eta_\nu|^2)}, \\ \eta_{\nu+1}|_{t=0} &= u_0, \quad \psi_{\nu+1}|_{t=0} = v_0. \end{aligned}$$

We set for $s > \frac{d}{2} + 2$,

$$\begin{aligned} m_\nu(t) &= \|\eta_\nu(t)\|_{\mathcal{H}^{\sigma(t), s}}^2 + \|\psi_\nu(t)\|_{\mathcal{H}^{\sigma(t), s}}^2 \\ (5.2) \quad &+ 2K \int_0^t (\|\eta_\nu(\tau)\|_{\mathcal{H}^{\sigma(\tau), s+\frac{1}{2}}}^2 + \|\psi_\nu(\tau)\|_{\mathcal{H}^{\sigma(\tau), s+\frac{1}{2}}}^2) d\tau, \end{aligned}$$

where $\sigma(t) = \lambda h - Kt$.

Proposition 5.1. *Let $T > 0$. Assume that*

$$b \in L^\infty(\mathbf{R}, H^{s-1}(\mathbf{T}^d)) \cap L^2(\mathbf{R}, H^{s-\frac{1}{2}}(\mathbf{T}^d)).$$

Then there exist positive constants M, K_0, ε_0 such that for all $\varepsilon \leq \varepsilon_0$ and all $K \geq K_0$,

$$\begin{aligned} (\|b\|_{L^2(\mathbf{R}, H^{s-\frac{1}{2}}) \cap L^\infty(\mathbf{R}, H^{s-1})} + \|u_0\|_{\mathcal{H}^{\lambda h, s}} + \|v_0\|_{\mathcal{H}^{\lambda h, s}} \leq \varepsilon) \\ \Rightarrow (m_\nu(t) \leq M^2 \varepsilon^2, \quad \forall t \leq T, \forall \nu \geq 0). \end{aligned}$$

Proof. First of all, we take ε_0 and M such that $M\varepsilon_0 \leq \bar{\varepsilon}$, where $\bar{\varepsilon}$ is defined in Theorem 3.1 and $M\varepsilon_0 \leq 1$.

We claim that $m_0(t) \leq 2\varepsilon^2$. In fact, we have $\|u_0\|_{\mathcal{H}^{\sigma(t),s}}^2 + \|v_0\|_{\mathcal{H}^{\sigma(t),s}}^2 \leq \varepsilon^2$, and

$$\begin{aligned} 2K \int_0^t \|u_0\|_{\mathcal{H}^{\sigma(\tau),s+\frac{1}{2}}}^2 d\tau &= \sum_{\mathbf{Z}^d} \langle \xi \rangle^{2s} e^{2\lambda h \langle \xi \rangle} |\widehat{u}_0(\xi)|^2 \left(\int_0^t 2K \langle \xi \rangle e^{-2K\tau \langle \xi \rangle} d\tau \right) \\ &\leq \|u_0\|_{\mathcal{H}^{\lambda h,s}}^2, \end{aligned}$$

similarly for v_0 . So we take $M \geq 2$. Assume now that $m_j(t) \leq M^2 \varepsilon^2, 0 \leq j \leq \nu$. Set

$$\eta_\nu = e^{-\sigma(t)\langle \xi \rangle} \widetilde{\eta}_\nu, \quad \psi_\nu = e^{-\sigma(t)\langle \xi \rangle} \widetilde{\psi}_\nu.$$

Notice that since $\sigma(t) = \lambda h - Kt \leq \lambda h$ and $|\xi| \leq \langle \xi \rangle \leq 1 + |\xi|$, we have

$$\sigma(t)|\xi| \leq \sigma(t)\langle \xi \rangle \leq \lambda h + \sigma(t)|\xi|.$$

It follows that

$$\|f(t)\|_{\mathcal{H}^{\sigma(t),\alpha}} \leq \|\widetilde{f}(t)\|_{H^\alpha} \leq e^{\lambda h} \|f(t)\|_{\mathcal{H}^{\sigma(t),\alpha}}.$$

Remark 5.2. We can write $\sigma(t) = \lambda(t)h$ with $\lambda(t) = \lambda - \frac{Kt}{h}$. Since $\lambda(t) \leq \lambda$, we may use the estimates in Sections 2 and 3 with constants depending only on the fixed parameter λ .

The system satisfied by $(\widetilde{\eta}_\nu, \widetilde{\psi}_\nu)$ is then

$$\begin{aligned} \partial_t \widetilde{\eta}_{\nu+1} + K \langle D_x \rangle \widetilde{\eta}_{\nu+1} &= e^{\sigma(t)\langle D_x \rangle} G(\eta_\nu)(\psi_\nu, b) := F_\nu \\ \partial_t \widetilde{\psi}_{\nu+1} + K \langle D_x \rangle \widetilde{\psi}_{\nu+1} &= G_\nu, \end{aligned}$$

where

$$G_\nu u := e^{\sigma(t)\langle D_x \rangle} \left[-g\eta_\nu - \frac{1}{2} |\nabla_x \psi_\nu|^2 + \frac{(G(\eta_\nu)(\psi_\nu, b) + \nabla_x \eta_\nu \cdot \nabla_x \psi_\nu)^2}{2(1 + |\nabla_x \eta_\nu|^2)} \right].$$

We have

$$\begin{aligned} \frac{d}{dt} [\|\widetilde{\eta}_{\nu+1}(t)\|_{H^s}^2 + \|\widetilde{\psi}_{\nu+1}(t)\|_{H^s}^2] &= 2 \left(\partial_t \widetilde{\eta}_{\nu+1}(t), \widetilde{\eta}_{\nu+1}(t) \right)_{H^s} \\ &\quad + 2 \left(\partial_t \widetilde{\psi}_{\nu+1}(t), \widetilde{\psi}_{\nu+1}(t) \right)_{H^s}. \end{aligned}$$

Using the above equations, we get

$$\begin{aligned} &\|\widetilde{\eta}_{\nu+1}(t)\|_{H^s}^2 + \|\widetilde{\psi}_{\nu+1}(t)\|_{H^s}^2 + 2K \int_0^t (\|\widetilde{\eta}_{\nu+1}(\tau)\|_{H^{s+\frac{1}{2}}}^2 + \|\widetilde{\psi}_{\nu+1}(\tau)\|_{H^{s+\frac{1}{2}}}^2) d\tau \\ &= \|u_0\|_{\mathcal{H}^{\lambda h,s}}^2 + \|v_0\|_{\mathcal{H}^{\lambda h,s}}^2 + 2 \int_0^t \left(F_\nu(\tau), \widetilde{\eta}_{\nu+1}(\tau) \right)_{H^s} d\tau \\ &\quad + 2 \int_0^t \left(G_\nu(\tau), \widetilde{\psi}_{\nu+1}(\tau) \right)_{H^s} d\tau. \end{aligned}$$

Set

$$A(\tau) = \left| \left(F_\nu(\tau), \widetilde{\eta}_{\nu+1}(\tau) \right)_{H^s} \right|, \quad B(\tau) = \left| \left(G_\nu(\tau), \widetilde{\psi}_{\nu+1}(\tau) \right)_{H^s} \right|.$$

We deduce from the hypotheses that

$$(5.3) \quad m_{\nu+1}(t) \leq \varepsilon^2 + \int_0^t A(\tau) d\tau + \int_0^t B(\tau) d\tau.$$

We have

$$A(\tau) \leq \frac{C}{K} \|G(\eta_\nu)(\psi_\nu, b)(\tau)\|_{\mathcal{H}^{\sigma(\tau),s-\frac{1}{2}}}^2 + \frac{2K}{20} \|\widetilde{\eta}_{\nu+1}(\tau)\|_{H^{s+\frac{1}{2}}}^2.$$

For all fixed τ (which is skipped), Theorem 3.1 shows that

$$\|G(\eta_\nu)(\psi_\nu, b)\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} \leq C(\|\tilde{\psi}_\nu\|_{H^{s+\frac{1}{2}}} + \|b\|_{H^{s-\frac{1}{2}}} + \|\tilde{\eta}_\nu\|_{H^{s+\frac{1}{2}}}) (\|\tilde{\psi}_\nu\|_{H^s} + \|b\|_{H^{s-\frac{3}{2}}}).$$

Since $\|\tilde{\psi}_\nu(\tau)\|_{H^s} \leq M\varepsilon_0 \leq 1$, there exists $C > 0$ depending only on s such that

$$(5.4) \quad \|G(\eta_\nu)(\psi_\nu, b)\|_{\mathcal{H}^{\sigma(\tau), s-\frac{1}{2}}} \leq C(\|\tilde{\psi}_\nu\|_{H^{s+\frac{1}{2}}} + \|b\|_{H^{s-\frac{1}{2}}} + \|\tilde{\eta}_\nu\|_{H^{s+\frac{1}{2}}}) (1 + \|b\|_{H^{s-\frac{3}{2}}}).$$

Therefore, since $\|b\|_{L^\infty(\mathbf{R}, H^{s-1})} \leq \varepsilon \leq 1$, we can write

$$A(\tau) \leq \frac{1}{K} (\|\tilde{\psi}_\nu(\tau)\|_{H^{s+\frac{1}{2}}}^2 + \|b(\tau)\|_{H^{s-\frac{1}{2}}}^2 + \|\tilde{\eta}_\nu(\tau)\|_{H^{s+\frac{1}{2}}}^2) + \frac{2K}{20} \|\tilde{\eta}_{\nu+1}(\tau)\|_{H^{s+\frac{1}{2}}}^2.$$

So,

$$\begin{aligned} \int_0^t A(\tau) d\tau &\leq \frac{C}{K} \int_0^t (\|\tilde{\psi}_\nu(\tau)\|_{H^{s+\frac{1}{2}}}^2 + \|\tilde{\eta}_\nu(\tau)\|_{H^{s+\frac{1}{2}}}^2 + \|b(\tau)\|_{H^{s-\frac{1}{2}}}^2) d\tau \\ &\quad + \frac{2K}{20} \int_0^t \|\tilde{\eta}_{\nu+1}(\tau)\|_{H^{s+\frac{1}{2}}}^2 d\tau. \end{aligned}$$

It follows that

$$(5.5) \quad \int_0^t A(\tau) d\tau \leq \frac{C}{K^2} M^2 \varepsilon^2 + \frac{C}{K} \|b\|_{L^2(\mathbf{R}, H^{s-\frac{1}{2}})}^2 + \frac{1}{20} m_{\nu+1}(t).$$

Now,

$$(5.6) \quad \begin{aligned} B(\tau) &\leq B_1(\tau) + B_2(\tau) + B_3(\tau), \\ B_1(\tau) &= g \left| \left(\tilde{\eta}_\nu(\tau), \tilde{\eta}_{\nu+1}(\tau) \right)_{H^s} \right|, \\ B_2(\tau) &= \frac{1}{2} \left| \left(e^{\sigma(\tau)\langle D_x \rangle} |\nabla_x \psi_\nu(\tau)|^2, \tilde{\eta}_{\nu+1}(\tau) \right)_{H^s} \right| \\ B_3(\tau) &= \left| \left(e^{\sigma(\tau)\langle D_x \rangle} \frac{(G(\eta_\nu)(\psi_\nu, b) + \nabla_x \eta_\nu \cdot \nabla_x \psi_\nu)^2}{2(1 + |\nabla_x \eta_\nu|^2)}, \tilde{\eta}_{\nu+1}(\tau) \right)_{H^s} \right|. \end{aligned}$$

Using the Cauchy-Schwarz inequality and the induction hypothesis, we obtain

$$(5.7) \quad \int_0^t B_1(\tau) d\tau \leq \frac{C}{K^2} M^2 \varepsilon^2 + \frac{1}{20} m_{\nu+1}(t).$$

On the other hand, using Proposition A.7 (iii) with $s_0 = s - 1$, we obtain

$$\begin{aligned} B_2(\tau) &\lesssim \|\tilde{\psi}_\nu(\tau)\|_{H^s} \|\tilde{\psi}_\nu(\tau)\|_{H^{s+\frac{1}{2}}} \|\tilde{\eta}_{\nu+1}(\tau)\|_{H^{s+\frac{1}{2}}} \\ &\lesssim M\varepsilon \|\tilde{\psi}_\nu(\tau)\|_{H^{s+\frac{1}{2}}} \|\tilde{\eta}_{\nu+1}(\tau)\|_{H^{s+\frac{1}{2}}}. \end{aligned}$$

It follows that

$$\int_0^t B_2(\tau) d\tau \lesssim M\varepsilon \|\tilde{\psi}_\nu\|_{L^2((0,t), H^{s+\frac{1}{2}})} \|\tilde{\eta}_{\nu+1}\|_{L^2((0,t), H^{s+\frac{1}{2}})}.$$

Therefore, by the Cauchy-Schwarz inequality and the induction, we get

$$(5.8) \quad \int_0^t B_2(\tau) d\tau \leq \frac{C}{K^2} (M^2 \varepsilon^2)^2 + \frac{1}{20} m_{\nu+1}(t).$$

To estimate the term B_3 , set

$$N_\nu = G(\eta_\nu)(\psi_\nu, b) + \nabla_x \eta_\nu \cdot \nabla_x \psi_\nu$$

and

$$U_\nu = \frac{N_\nu^2}{1 + |\nabla_x \eta_\nu|^2}.$$

Then we have

$$B_3(\tau) \lesssim \|e^{\sigma(\tau)|D_x}|U_\nu\|_{H^{s-\frac{1}{2}}} \|\tilde{\eta}_{\nu+1}(\tau)\|_{H^{s+\frac{1}{2}}},$$

from which we deduce that

$$(5.9) \quad \int_0^t B_3(\tau) d\tau \leq \frac{C}{K} \int_0^t \|U_\nu(\tau)\|_{\mathcal{H}^{\sigma(\tau), s-\frac{1}{2}}}^2 d\tau + \frac{1}{20} m_{\nu+1}(t).$$

Using Propositions A.7 and A.10, with $s_0 = s - 1$, we can write, skipping τ ,

$$\begin{aligned} \|U_\nu\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} &\lesssim \|N_\nu\|_{\mathcal{H}^{\sigma, s-1}} \|N_\nu\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} + \| |\nabla_x \eta_\nu|^2 U_\nu \|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} \\ &\lesssim \|N_\nu\|_{\mathcal{H}^{\sigma, s-1}} \|N_\nu\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} + \|\tilde{\eta}_\nu\|_{H^s}^2 \|U_\nu\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} \\ &\quad + \|\tilde{\eta}_\nu\|_{H^s} \|\tilde{\eta}_\nu\|_{H^{s+\frac{1}{2}}} \|U_\nu\|_{\mathcal{H}^{\sigma, s-1}}. \end{aligned}$$

Since $\|\tilde{\eta}_\nu(\tau)\|_{H^s} \leq M\varepsilon$, taking M, ε_0 such that $C(M\varepsilon_0)^2 \leq \frac{1}{2}$, we can absorb the second term of the right hand side by the left hand side and deduce that

$$\|U_\nu\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} \lesssim \|N_\nu\|_{\mathcal{H}^{\sigma, s-1}} \|N_\nu\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} + M\varepsilon \|\tilde{\eta}_\nu\|_{H^{s+\frac{1}{2}}} \|U_\nu\|_{\mathcal{H}^{\sigma, s-1}}.$$

Similarly, since $s - 1 > \frac{d}{2}$ we have

$$\|U_\nu\|_{\mathcal{H}^{\sigma, s-1}} \lesssim \|N_\nu\|_{\mathcal{H}^{\sigma, s-1}}^2 + \|\tilde{\eta}_\nu\|_{H^s}^2 \|U_\nu\|_{\mathcal{H}^{\sigma, s-1}}.$$

Using again the fact that $\|\tilde{\eta}_\nu(\tau)\|_{H^s} \leq M\varepsilon$, we can absorb the second term in the right hand side by the left, one can deduce that

$$\|U_\nu\|_{\mathcal{H}^{\sigma, s-1}} \lesssim \|N_\nu\|_{\mathcal{H}^{\sigma, s-1}}^2.$$

So we obtain the inequality

$$\|U_\nu\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} \lesssim \|N_\nu\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} \|N_\nu\|_{\mathcal{H}^{\sigma, s-1}} + M\varepsilon \|\tilde{\eta}_\nu\|_{H^{s+\frac{1}{2}}} \|N_\nu\|_{\mathcal{H}^{\sigma, s-1}}^2.$$

By Theorem 3.1, Remark 3.2 and the induction we have

$$\|N_\nu\|_{\mathcal{H}^{\sigma, s-1}} \lesssim \|\tilde{\psi}_\nu\|_{H^s} + \|b\|_{H^{s-1}} + \|\tilde{\eta}_\nu\|_{H^s} \|\tilde{\psi}_\nu\|_{H^s}.$$

We have $\|\tilde{\eta}_\nu(\tau)\|_{H^s} \leq M\varepsilon \leq 1$, $\|\tilde{\psi}_\nu(\tau)\|_{H^s} \leq M\varepsilon \leq 1$ and $b \in L^\infty(\mathbf{R}, H^{s-1})$, so that

$$(5.10) \quad \|N_\nu\|_{\mathcal{H}^{\sigma, s-1}} \lesssim M\varepsilon + \|b\|_{H^{s-1}} = \mathcal{O}(1).$$

It follows that

$$(5.11) \quad \|U_\nu\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} \lesssim \|N_\nu\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} + \|\tilde{\eta}_\nu\|_{H^{s+\frac{1}{2}}}.$$

It remains to estimate the term $\|N_\nu\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}}$. We have

$$\begin{aligned} \|\nabla_x \psi_\nu \cdot \nabla_x \eta_\nu\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} &\lesssim \|\tilde{\psi}_\nu\|_{H^s} \|\tilde{\eta}_\nu\|_{H^{s+\frac{1}{2}}} + \|\tilde{\psi}_\nu\|_{H^{s+\frac{1}{2}}} \|\tilde{\eta}_\nu\|_{H^s}, \\ &\lesssim M\varepsilon (\|\tilde{\eta}_\nu\|_{H^{s+\frac{1}{2}}} + \|\tilde{\psi}_\nu\|_{H^{s+\frac{1}{2}}}). \end{aligned}$$

Using (5.4), we get

$$\begin{aligned} \|N_\nu\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} &\lesssim \|\tilde{\psi}_\nu\|_{H^{s+\frac{1}{2}}} + \|b\|_{H^{s-\frac{1}{2}}} + \|\tilde{\eta}_\nu\|_{H^{s+\frac{1}{2}}} \|b\|_{H^{s-\frac{3}{2}}} \\ &\quad + M\varepsilon (\|\tilde{\eta}_\nu\|_{H^{s+\frac{1}{2}}} + \|\tilde{\psi}_\nu\|_{H^{s+\frac{1}{2}}}), \end{aligned}$$

so,

$$(5.12) \quad \|N_\nu\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} \lesssim \|\tilde{\psi}_\nu\|_{H^{s+\frac{1}{2}}} + \|\tilde{\eta}_\nu\|_{H^{s+\frac{1}{2}}} + \|b\|_{H^{-\frac{1}{2}}}.$$

Using (5.11), (5.10) and (5.12), we get

$$\|U_\nu\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}}^2 \lesssim \|\tilde{\psi}_\nu\|_{H^{s+\frac{1}{2}}}^2 + \|\tilde{\eta}_\nu\|_{H^{s+\frac{1}{2}}}^2 + \|b\|_{L^\infty(\mathbf{R}, H^{s-\frac{1}{2}})}^2.$$

We deduce from (5.9) and the induction that

$$(5.13) \quad \int_0^t B_3(\tau) d\tau \leq \frac{C}{K^2} M^2 \varepsilon^2 + \frac{1}{20} m_{\nu+1}(t).$$

Using (5.3), (5.5), (5.7), (5.8), (5.13) and taking K large enough and ε_0 small enough, we obtain

$$m_{\nu+1}(t) \leq M^2 \varepsilon^2,$$

which ends the proof of Proposition 5.1. \square

Notation 5.3. We set $I = [0, T]$, and

$$\begin{aligned} U_{\nu+1} &= \tilde{\eta}_{\nu+1} - \tilde{\eta}_\nu, \quad V_{\nu+1} = \tilde{\psi}_{\nu+1} - \tilde{\psi}_\nu, \\ M_{\nu+1}(t) &= \|U_{\nu+1}(t)\|_{H^s}^2 + \|V_{\nu+1}(t)\|_{H^s}^2 \\ &\quad + 2K \int_0^t \left(\|U_{\nu+1}(\tau)\|_{H^{s+\frac{1}{2}}}^2 + \|V_{\nu+1}(\tau)\|_{H^{s+\frac{1}{2}}}^2 \right) d\tau, \end{aligned}$$

and

$$\begin{aligned} \mathcal{M}_{\nu+1} &= \|U_{\nu+1}\|_{L^\infty(I, H^s)}^2 + \|V_{\nu+1}\|_{L^\infty(I, H^s)}^2 \\ &\quad + 2K \int_0^T \left(\|U_{\nu+1}(\tau)\|_{H^{s+\frac{1}{2}}}^2 d\tau + \|V_{\nu+1}(\tau)\|_{H^{s+\frac{1}{2}}}^2 \right) d\tau. \end{aligned}$$

Proposition 5.4. Under the hypotheses of Proposition 5.1 there exists a constant $C > 0$ independent of K such that for all $\nu \geq 0$,

$$\mathcal{M}_{\nu+1} \leq \frac{C}{K} \left(\frac{1}{K} + \|b\|_{L^2(\mathbf{R}, H^{s-\frac{1}{2}})}^2 \right) \mathcal{M}_\nu.$$

Proof. Since $U_{\nu+1}|_{t=0} = V_{\nu+1}|_{t=0} = 0$, we obtain as in the first part,

$$\begin{aligned} M_{\nu+1}(t) &= 2 \int_0^t \left((F_\nu - F_{\nu-1})(\tau), U_{\nu+1}(\tau) \right)_{H^s} d\tau \\ &\quad + 2 \int_0^t \left((G_\nu - G_{\nu-1})(\tau), V_{\nu+1}(\tau) \right)_{H^s} d\tau, \end{aligned}$$

where

$$\begin{aligned} F_\nu &= e^{\sigma(t)|D_x|} G(\eta_\nu)(\psi_\nu, b), \\ G_\nu &= e^{\sigma(t)|D_x|} \left[-g\eta_\nu - \frac{1}{2} |\nabla_x \psi_\nu|^2 + \frac{(G(\eta_\nu)(\psi_\nu, b) + \nabla_x \eta_\nu \cdot \nabla_x \psi_\nu)^2}{2(1 + |\nabla_x \eta_\nu|^2)} \right]. \end{aligned}$$

Using the inequality

$$|(f, g)_{H^s}| \leq \frac{1}{4K} \|f\|_{H^{s-\frac{1}{2}}}^2 + K \|g\|_{H^{s+\frac{1}{2}}}^2$$

and the definition of $M_{\nu+1}$, we get

$$(5.14) \quad M_{\nu+1}(t) \leq A(t) + B(t),$$

where

$$A(t) = \frac{C}{K} \int_0^t \|(F_\nu - F_{\nu-1})(\tau)\|_{H^{s-\frac{1}{2}}}^2 d\tau,$$

$$B(t) = \frac{C}{K} \int_0^t \|(G_\nu - G_{\nu-1})(\tau)\|_{H^{s-\frac{1}{2}}}^2 d\tau.$$

Estimate of the term $A(t)$.

We use Theorem 3.4 and Proposition 5.1. Keeping the notations in the Theorem, we can write

$$(5.15) \quad \|F_\nu - F_{\nu-1}\|_{H^{s-\frac{1}{2}}} \lesssim \lambda_1 \|\tilde{\eta}_\nu - \tilde{\eta}_{\nu-1}\|_{H^s} + \lambda_2 \|\tilde{\eta}_\nu - \tilde{\eta}_{\nu-1}\|_{H^{s+\frac{1}{2}}} \\ + \lambda_3 \|\tilde{\psi}_\nu - \tilde{\psi}_{\nu-1}\|_{H^s} + \|\tilde{\psi}_\nu - \tilde{\psi}_{\nu-1}\|_{H^{s+\frac{1}{2}}}$$

with

$$\lambda_1 \lesssim \left(\sum_{\mu=\nu-1}^{\nu} \|\tilde{\eta}_\mu\|_{H^{s+\frac{1}{2}}} \right) \left(\sum_{\mu=\nu-1}^{\nu} \|\tilde{\psi}_\mu\|_{H^s} + \|b\|_{H^{s-\frac{3}{2}}} \right) + \sum_{\mu=\nu-1}^{\nu} \|\tilde{\psi}_\mu\|_{H^{s+\frac{1}{2}}} + \|b\|_{H^{s-\frac{1}{2}}},$$

$$\lambda_2 \lesssim \sum_{\mu=\nu-1}^{\nu} \|\tilde{\psi}_\mu\|_{H^{s-\frac{1}{2}}} + \|b\|_{H^{s-\frac{3}{2}}},$$

$$\lambda_3 \lesssim \sum_{\mu=\nu-1}^{\nu} \|\tilde{\eta}_\mu\|_{H^{s+\frac{1}{2}}}.$$

Since $b \in L^\infty(\mathbf{R}, H^{s-1})$ and using (5.2) together with Proposition 5.1, we deduce that

$$\lambda_1 \lesssim \sum_{\mu=\nu-1}^{\nu} (\|\tilde{\eta}_\mu\|_{H^{s+\frac{1}{2}}} + \|\tilde{\psi}_\mu\|_{H^{s+\frac{1}{2}}}) + \|b\|_{H^{s-\frac{1}{2}}},$$

$$\lambda_2 = \mathcal{O}(1),$$

$$\lambda_3 \lesssim \sum_{\mu=\nu-1}^{\nu} \|\tilde{\eta}_\mu\|_{H^{s+\frac{1}{2}}}.$$

It follows easily from (5.15) that

$$(5.16) \quad \|F_\nu - F_{\nu-1}\|_{H^{s-\frac{1}{2}}} \\ \lesssim \sum_{\mu=\nu-1}^{\nu} \left(\|\tilde{\eta}_\mu\|_{H^{s+\frac{1}{2}}} + \|\tilde{\psi}_\mu\|_{H^{s+\frac{1}{2}}} + \|b\|_{H^{s-\frac{1}{2}}} \right) (\|\tilde{\eta}_\nu - \tilde{\eta}_{\nu-1}\|_{H^s} + \|\tilde{\psi}_\nu - \tilde{\psi}_{\nu-1}\|_{H^s}) \\ + \|\tilde{\eta}_\nu - \tilde{\eta}_{\nu-1}\|_{H^{s+\frac{1}{2}}} + \|\tilde{\psi}_\nu - \tilde{\psi}_{\nu-1}\|_{H^{s+\frac{1}{2}}},$$

from which we deduce that

$$(5.17) \quad A(t) \lesssim \frac{1}{K} \left(\frac{1}{K} + \|b\|_{L^2(\mathbf{R}, H^{s-\frac{1}{2}})}^2 \right) \mathcal{M}_\nu.$$

Let us look at the term $B(t)$. We can write

$$(5.18) \quad B(t) \leq C(B_1(t) + B_2(t) + B_3(t)),$$

where

$$\begin{aligned} B_1(t) &= \frac{1}{K} \int_0^t \|\tilde{\eta}_\nu(\tau) - \tilde{\eta}_{\nu-1}\|_{H^{s-\frac{1}{2}}}^2(\tau) d\tau, \\ B_2(t) &= \frac{1}{K} \int_0^t \|\ |\nabla_x \tilde{\psi}_\nu(t)|^2 - |\nabla_x \tilde{\psi}_{\nu-1}(t)|^2 \|_{H^{s-\frac{1}{2}}}^2 d\tau, \\ B_3(t) &= \frac{1}{K} \int_0^t \left\| \frac{N_\nu^2(\tau)}{(1 + |\nabla_x \eta_\nu(\tau)|^2)} - \frac{N_{\nu-1}^2(\tau)}{(1 + |\nabla_x \eta_{\nu-1}(\tau)|^2)} \right\|_{\mathcal{H}^{\sigma(\tau), s-\frac{1}{2}}}^2 d\tau \end{aligned}$$

with

$$N_\nu = G(\eta_\nu)(\psi_\nu, b) + \nabla_x \eta_\nu \cdot \nabla_x \psi_\nu.$$

First of all, we have

$$(5.19) \quad B_1(t) \lesssim \frac{1}{K^2} \mathcal{M}_\nu.$$

Now set

$$I_j = \|(\partial_j \tilde{\psi}_\nu(t))^2 - (\partial_j \tilde{\psi}_{\nu-1}(t))^2\|_{H^{s-\frac{1}{2}}}.$$

We can write

$$\begin{aligned} I_j &\lesssim \|\partial_j \tilde{\psi}_\nu(t) - \partial_j \tilde{\psi}_{\nu-1}(t)\|_{H^{s-\frac{1}{2}}} \|\partial_j \tilde{\psi}_\nu(t) + \partial_j \tilde{\psi}_{\nu-1}(t)\|_{H^{s-1}} \\ &\quad + \|\partial_j \tilde{\psi}_\nu(t) - \partial_j \tilde{\psi}_{\nu-1}(t)\|_{H^{s-1}} \|\partial_j \tilde{\psi}_\nu(t) + \partial_j \tilde{\psi}_{\nu-1}(t)\|_{H^{s-\frac{1}{2}}}, \end{aligned}$$

which implies that

$$\begin{aligned} I_j &\lesssim \|\tilde{\psi}_\nu(t) - \tilde{\psi}_{\nu-1}(t)\|_{H^{s+\frac{1}{2}}} (\|\tilde{\psi}_\nu(t)\|_{H^s} + \|\tilde{\psi}_{\nu-1}(t)\|_{H^s}) \\ &\quad + \|\tilde{\psi}_\nu(t) - \tilde{\psi}_{\nu-1}(t)\|_{H^s} (\|\tilde{\psi}_\nu(t)\|_{H^{s+\frac{1}{2}}} + \|\tilde{\psi}_{\nu-1}(t)\|_{H^{s+\frac{1}{2}}}). \end{aligned}$$

It follows from Proposition 5.1 that

$$(5.20) \quad B_2(t) \lesssim \frac{(M\varepsilon)^2}{K^2} \mathcal{M}_\nu.$$

To estimate B_3 , set

$$H_\nu = \frac{N_\nu^2}{(1 + |\nabla_x \eta_\nu|^2)} - \frac{N_{\nu-1}^2}{(1 + |\nabla_x \eta_{\nu-1}|^2)}, \quad f(t) = \frac{t}{1+t}.$$

Then we can write

$$(5.21) \quad H_\nu = (1) - (2) - (3)$$

with

$$\begin{aligned} (1) &= N_\nu^2 - N_{\nu-1}^2, \\ (2) &= N_\nu^2 (f(|\nabla_x \eta_\nu|^2) - f(|\nabla_x \eta_{\nu-1}|^2)), \\ (3) &= (N_\nu^2 - N_{\nu-1}^2) f(|\nabla_x \eta_{\nu-1}|^2). \end{aligned}$$

We have

$$\begin{aligned} \|(1)\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} &\lesssim \|N_\nu - N_{\nu-1}\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} (\|N_\nu\|_{\mathcal{H}^{\sigma, s-1}} + \|N_{\nu-1}\|_{\mathcal{H}^{\sigma, s-1}}) \\ &\quad + \|N_\nu - N_{\nu-1}\|_{\mathcal{H}^{\sigma, s-1}} (\|N_\nu\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} + \|N_{\nu-1}\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}}) \end{aligned}$$

According to (5.10) and (5.12), we have for $\mu = \nu - 1, \nu$,

$$(5.22) \quad \begin{aligned} \|N_\mu\|_{\mathcal{H}^{\sigma, s-1}} &\lesssim M\varepsilon + \|b\|_{H^{s-1}} = \mathcal{O}(1), \\ \|N_\mu\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} &\lesssim \|\tilde{\psi}_\mu\|_{H^{s+\frac{1}{2}}} + \|\tilde{\eta}_\mu\|_{H^{s+\frac{1}{2}}} + \|b\|_{H^{-\frac{1}{2}}}. \end{aligned}$$

Moreover, according to (5.16), we have

$$\begin{aligned} & \|N_\nu - N_{\nu-1}\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} \\ & \lesssim \sum_{\mu=\nu-1}^{\nu} \left(\|\tilde{\eta}_\mu\|_{H^{s+\frac{1}{2}}} + \|\tilde{\psi}_\mu\|_{H^{s+\frac{1}{2}}} + \|b\|_{H^{s-\frac{1}{2}}} \right) (\|\tilde{\eta}_\nu - \tilde{\eta}_{\nu-1}\|_{H^s} \\ & \quad + \|\tilde{\psi}_\nu - \tilde{\psi}_{\nu-1}\|_{H^s}) + \|\tilde{\eta}_\nu - \tilde{\eta}_{\nu-1}\|_{H^{s+\frac{1}{2}}} + \|\tilde{\psi}_\nu - \tilde{\psi}_{\nu-1}\|_{H^{s+\frac{1}{2}}}, \end{aligned}$$

and

$$\|N_\nu - N_{\nu-1}\|_{\mathcal{H}^{\lambda h, s-1}} \lesssim \|\tilde{\eta}_\nu - \tilde{\eta}_{\nu-1}\|_{\mathcal{H}^{\sigma, s}} + \|\tilde{\psi}_\nu - \tilde{\psi}_{\nu-1}\|_{H^s}.$$

It follows that

(5.23)

$$\begin{aligned} \|(1)\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} & \lesssim \sum_{\mu=\nu-1}^{\nu} \left(\|\tilde{\eta}_\mu\|_{H^{s+\frac{1}{2}}} + \|\tilde{\psi}_\mu\|_{H^{s+\frac{1}{2}}} + \|b\|_{H^{s-\frac{1}{2}}} \right) (\|\tilde{\eta}_\nu - \tilde{\eta}_{\nu-1}\|_{H^s} \\ & \quad + \|\tilde{\psi}_\nu - \tilde{\psi}_{\nu-1}\|_{H^s}) + \|\tilde{\eta}_\nu - \tilde{\eta}_{\nu-1}\|_{H^{s+\frac{1}{2}}} + \|\tilde{\psi}_\nu - \tilde{\psi}_{\nu-1}\|_{H^{s+\frac{1}{2}}}. \end{aligned}$$

Now we have

$$\begin{aligned} \|(2)\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} & \lesssim \|N_\nu\|_{\mathcal{H}^{\sigma, s-1}}^2 \|f(|\nabla_x \eta_\nu|^2) - f(|\nabla_x \eta_{\nu-1}|^2)\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} \\ & \quad + \|N_\nu\|_{\mathcal{H}^{\sigma, s-1}} \|N_\nu\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} \|f(|\nabla_x \eta_\nu|^2) - f(|\nabla_x \eta_{\nu-1}|^2)\|_{\mathcal{H}^{\sigma, s-1}}. \end{aligned}$$

Using Proposition A.11, (5.22) and the estimates

$$\begin{aligned} & \|\nabla_x \eta_\nu|^2 - |\nabla_x \eta_{\nu-1}|^2\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} \lesssim \|\eta_\nu - \eta_{\nu-1}\|_{\mathcal{H}^{\sigma, s+\frac{1}{2}}}, \\ & \|\nabla_x \eta_\nu|^2 - |\nabla_x \eta_{\nu-1}|^2\|_{\mathcal{H}^{\sigma, s-1}} \lesssim \|\eta_\nu - \eta_{\nu-1}\|_{\mathcal{H}^{\sigma, s}}, \end{aligned}$$

we obtain

(5.24)

$$\begin{aligned} \|(2)\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} & \lesssim \sum_{\mu=\nu-1}^{\nu} \left(\|\tilde{\eta}_\mu\|_{H^{s+\frac{1}{2}}} + \|\tilde{\psi}_\mu\|_{H^{s+\frac{1}{2}}} \right) (\|\tilde{\eta}_\nu - \tilde{\eta}_{\nu-1}\|_{H^s} \\ & \quad + \|\tilde{\psi}_\nu - \tilde{\psi}_{\nu-1}\|_{H^s}) + \|\tilde{\eta}_\nu - \tilde{\eta}_{\nu-1}\|_{H^{s+\frac{1}{2}}} + \|\tilde{\psi}_\nu - \tilde{\psi}_{\nu-1}\|_{H^{s+\frac{1}{2}}}. \end{aligned}$$

The same method gives the estimate

(5.25)

$$\begin{aligned} \|(3)\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} & \lesssim \sum_{\mu=\nu-1}^{\nu} \left(\|\tilde{\eta}_\mu\|_{H^{s+\frac{1}{2}}} + \|\tilde{\psi}_\mu\|_{H^{s+\frac{1}{2}}} \right) (\|\tilde{\eta}_\nu - \tilde{\eta}_{\nu-1}\|_{H^s} \\ & \quad + \|\tilde{\psi}_\nu - \tilde{\psi}_{\nu-1}\|_{H^s}) + \|\tilde{\eta}_\nu - \tilde{\eta}_{\nu-1}\|_{H^{s+\frac{1}{2}}} + \|\tilde{\psi}_\nu - \tilde{\psi}_{\nu-1}\|_{H^{s+\frac{1}{2}}}. \end{aligned}$$

Using (5.23), (5.24), (5.25) and (5.21), we deduce that

$$\begin{aligned} \|H_\nu\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} & \lesssim \sum_{\mu=\nu-1}^{\nu} \left(\|\tilde{\eta}_\mu\|_{H^{s+\frac{1}{2}}} + \|\tilde{\psi}_\mu\|_{H^{s+\frac{1}{2}}} + \|b\|_{H^{s-\frac{1}{2}}} \right) (\|\tilde{\eta}_\nu - \tilde{\eta}_{\nu-1}\|_{H^s} \\ & \quad + \|\tilde{\psi}_\nu - \tilde{\psi}_{\nu-1}\|_{H^s}) + \|\tilde{\eta}_\nu - \tilde{\eta}_{\nu-1}\|_{H^{s+\frac{1}{2}}} + \|\tilde{\psi}_\nu - \tilde{\psi}_{\nu-1}\|_{H^{s+\frac{1}{2}}}. \end{aligned}$$

Going back to (5.18), we obtain eventually that

$$(5.26) \quad |B_3(t)| \lesssim \frac{1}{K} \left(\frac{1}{K} + \|b\|_{L^2(\mathbf{R}, H^{s-\frac{1}{2}})}^2 \right) \mathcal{M}_\nu.$$

Now we use (5.14), (5.17), (5.18), (5.19), (5.20) and (5.26). We obtain

$$\mathcal{M}_{\nu+1} \lesssim \frac{1}{K} \left(\frac{1}{K} + \|b\|_{L^2(\mathbf{R}, H^{s-\frac{1}{2}})}^2 \right) \mathcal{M}_\nu,$$

which completes the proof of Proposition 5.4. \square

Proof of Theorem 1.5. It follows from Proposition 5.4 that there exist ε_0 and K such that $\mathcal{M}_\nu^{\frac{1}{2}} \leq \delta^\nu \mathcal{M}_0^{\frac{1}{2}}$, where $\delta < 1$. Take $T < \frac{\lambda h}{K}$ and set

$$\mathcal{X} = C^0([0, T], \mathcal{H}^{\sigma(t), s}) \cap L^2((0, T), \mathcal{H}^{\sigma(t), s+\frac{1}{2}}).$$

It follows that the sequence (η_ν, ψ_ν) converges in $\mathcal{X} \times \mathcal{X}$ to (η, ψ) . It remains to prove that (η, ψ) is a solution of system (1.1). According to (5.1), we can write

$$\begin{aligned} \eta_{\nu+1}(t) &= u_0 + \int_0^t F_\nu(\tau) d\tau, \quad F_\nu = G(\eta_\nu)(\psi_\nu, b), \\ \psi_{\nu+1}(t) &= v_0 + \int_0^t G_\nu(\tau) d\tau, \\ G_\nu &= -g\eta_\nu - \frac{1}{2} |\nabla_x \psi_\nu|^2 + \frac{(G(\eta_\nu)(\psi_\nu, b) + \nabla_x \eta_\nu \cdot \nabla_x \psi_\nu)^2}{2(1 + |\nabla_x \eta_\nu|^2)}. \end{aligned}$$

It is enough to prove that $\int_0^t F_\nu(\tau) d\tau$ converges in $L^\infty([0, T], \mathcal{H}^{\sigma(t), s-\frac{1}{2}})$ to $\int_0^t F(\tau) d\tau$, where $F = G(\eta)(\psi, b)$, together with a similar result for $\int_0^t G_\nu(\tau) d\tau$. We have with fixed t ,

$$\begin{aligned} A &:= \left\| \int_0^t (F_\nu(\tau) - F(\tau)) d\tau \right\|_{\mathcal{H}^{\sigma(t), s-\frac{1}{2}}}^2 \\ &= \sum_{\mathbf{z}^d} e^{2\sigma(t)|\xi|} \langle \xi \rangle^{2s-1} \left| \int_0^t (\widehat{F_\nu - F})(\tau, \xi) d\tau \right|^2. \end{aligned}$$

Since for $0 \leq \tau \leq t$ we have $\sigma(t) \leq \sigma(\tau)$, we deduce from the Cauchy-Schwarz inequality that

$$A \leq T \int_0^t \|(F_\nu - F)(\tau)\|_{\mathcal{H}^{\sigma(\tau), s-\frac{1}{2}}}^2 d\tau.$$

Using Theorem 3.4 applied with $\eta_1 = \eta_\nu, \eta_2 = \eta$ and the fact that the sequences $\|\eta_\nu\|_{\mathcal{X}}$ and $\|\psi_\nu\|_{\mathcal{X}}$ are uniformly bounded, we deduce that there exists $C = C(\|\eta\|_{\mathcal{X}}, \|\psi\|_{\mathcal{X}}) > 0$ such that

$$A \leq C(\|\eta_\nu - \eta\|_{\mathcal{X}} + \|\psi_\nu - \psi\|_{\mathcal{X}}) \rightarrow 0, \quad \text{if } \nu \rightarrow +\infty.$$

Now we estimate

$$B := \left\| \int_0^t (G_\nu - G)(\tau) d\tau \right\|_{\mathcal{H}^{\sigma(t), s-\frac{1}{2}}}^2 \leq C(B_1 + B_2 + B_3),$$

where

$$\begin{cases} B_1 = \left\| \int_0^t (\eta_\nu - \eta)(\tau) d\tau \right\|_{\mathcal{H}^{\sigma(t), s-\frac{1}{2}}}^2, \\ B_2 = \left\| \int_0^t (|\nabla_x \psi_\nu|^2 - |\nabla_x \psi|^2)(\tau) d\tau \right\|_{\mathcal{H}^{\sigma(t), s-\frac{1}{2}}}^2, \\ B_3 = \left\| \int_0^t \frac{N_\nu^2(\tau)}{(1 + |\nabla_x \eta_\nu(\tau)|^2)} - \frac{N^2(\tau)}{(1 + |\nabla_x \eta(\tau)|^2)} \right\|_{\mathcal{H}^{\sigma(t), s-\frac{1}{2}}}^2 \end{cases}$$

with

$$N_\nu = G(\eta_\nu)(\psi_\nu, b) + \nabla_x \eta_\nu \cdot \nabla_x \psi_\nu, \quad N = G(\eta)(\psi, b) + \nabla_x \eta \cdot \nabla_x \psi.$$

As for the term B_1 , we have

$$B_1 \leq C \int_0^t \|(\eta_\nu - \eta)(\tau)\|_{\mathcal{H}^{\sigma(t), s-\frac{1}{2}}}^2 d\tau \leq C \|(\eta_\nu - \eta)(\tau)\|_{\mathcal{X}}^2 \rightarrow 0.$$

The terms B_2 and B_3 can be estimated similarly by using (5.18) with $\eta_{\nu-1}$ and $\psi_{\nu-1}$ replaced by η, ψ , together with the estimates (5.20) and (5.26). Hence, we conclude that B tends to zero in $L^\infty([0, T], \mathcal{H}^{\sigma(t), s-\frac{1}{2}})$, which implies that (η, ψ) is a solution of system (1.1).

The uniqueness of the solution follows from the computation made in Proposition 5.4, where we replace $(\tilde{\eta}_\nu, \tilde{\eta}_{\nu+1})$ by $(\tilde{\eta}_1, \tilde{\eta}_2)$ and $(\tilde{\psi}_\nu, \tilde{\psi}_{\nu+1})$ by $(\tilde{\psi}_1, \tilde{\psi}_2)$, where $(\eta_1, \psi_1), (\eta_2, \psi_2)$ are the two supposed solutions. \square

6. EXISTENCE OF A SOLUTION ON A TIME INTERVAL OF SIZE ε^{-1}

In this section we prove Theorem 1.7 about the well-posedness of the Cauchy problem on large time intervals. We shall construct solutions as limits of solutions to a sequence of approximate nonlinear systems. The analysis is in three different steps:

- (1) Firstly, we define approximate systems and prove that the Cauchy problem for the latter are well-posed locally in time by means of an ODE argument.
- (2) Secondly, we prove that the solutions of the approximate systems are bounded on a uniform time interval.
- (3) Third, we prove that these approximate solutions converge to a solution of the water-waves system.

6.1. Approximate systems. Let us rewrite the water-wave system under the form

$$\partial_t f = \mathcal{T}(f; b),$$

where $f = \begin{pmatrix} \eta \\ \psi \end{pmatrix}$,

$$\mathcal{T}(f; b) = \begin{pmatrix} G(\eta)(\psi, b) \\ -g\eta - \frac{1}{2}|\nabla_x \psi|^2 + \frac{1}{2(1+|\nabla_x \eta|^2)}(G(\eta)(\psi, b) + \nabla_x \eta \cdot \nabla_x \psi)^2 \end{pmatrix},$$

and $b = b(t, x)$ is a given function. We denote by $f_0 = (\eta_0, \psi_0)$ the initial data.

To define the approximate systems, we use a version of Galerkin's method based on Friedrichs mollifiers. To do so, we shall use smoothing operators. We consider, for $n \in \mathbf{N} \setminus \{0\}$, the operators J_n defined by

$$(6.1) \quad \widehat{J_n u}(\xi) = \chi\left(\frac{\xi}{n}\right) \widehat{u}(\xi),$$

where $\chi \in C^\infty(\mathbf{R}^d)$ is such that

$$\chi(y) = 1 \text{ if } |y| \leq 1 \quad \text{and} \quad \chi(y) = 0 \text{ if } |y| \geq 2.$$

Now we consider the following approximate Cauchy problems:

$$(6.2) \quad \begin{cases} \partial_t f = J_n(\mathcal{T}(f; b)), \\ f|_{t=0} = J_n f_0. \end{cases}$$

The following lemma states that, for each $n \in \mathbf{N} \setminus \{0\}$, the Cauchy problem (6.2) is well-posed locally in time.

Lemma 6.1. *Let $s > 2 + \frac{d}{2}$. For all initial data $f_0 = (\eta_0, \psi_0) \in L^2(\mathbf{T}^d)^2$, for all source term $b \in C^0([0, +\infty); H^{s-\frac{1}{2}}(\mathbf{T}^d))$ and for all $n \in \mathbf{N} \setminus \{0\}$, there exists $T_n > 0$ such that the Cauchy problem (6.2) has a unique maximal solution*

$$f_n = (\eta_n, \psi_n) \in C^1([0, T_n]; L^2(\mathbf{T}^d)^2).$$

Moreover, f_n is a smooth function which belongs to $C^0([0, T_n]; \mathcal{H}^{\lambda, \mu}(\mathbf{T}^d))$ for any $\lambda, \mu \geq 0$.

Finally, either

$$(6.3) \quad T_n = +\infty \quad \text{or} \quad \limsup_{t \rightarrow T_n} \|f_n(t)\|_{L^2} = +\infty.$$

Proof. Fix $f_0 = (\eta_0, \psi_0) \in L^2(\mathbf{T}^d)^2$ and $b \in L^2(\mathbf{T}^d)$. Let $\tilde{\chi} \in C_0^\infty(\mathbf{R}^d)$ equal to 1 on the support of χ and set $\widehat{\tilde{J}_n u}(\xi) = \tilde{\chi}\left(\frac{\xi}{n}\right) \widehat{u}(\xi)$. With this choice we have $\tilde{J}_n J_n = J_n$. We begin by studying an auxiliary Cauchy problem which reads

$$(6.4) \quad \begin{cases} \partial_t f = F_n(f) \quad \text{where} \quad F_n(f) = J_n(\mathcal{T}(\tilde{J}_n f; b)), \\ f|_{t=0} = J_n f_0. \end{cases}$$

We will prove that the Cauchy problem is well-posed by using the classical fixed point argument. Then we will prove that if f_n solves (6.4), then it also solves the original problem (6.2). Notice that the operators J_n and \tilde{J}_n are smoothing operators: they are bounded from $L^2(\mathbf{T}^d)$ into $H^\mu(\mathbf{T}^d)$ for any $\mu \geq 0$. Consequently, it would be sufficient to exploit some rough estimates for the Dirichlet-Neumann operator which could be proved by elementary variational estimates (see [4, Theorems 3.8 and 3.9]). For completeness (due to the additional source term b), we shall use here the more elaborates estimates from Section 3 and Appendix A restricted to the case of Sobolev spaces ($\lambda = 0$). From (3.2), we have (for $s \geq \frac{3}{2} + \frac{d}{2}$)

$$\|G(\eta)(\psi, b)\|_{H^s} \leq C(\|\psi\|_{H^{s+1}} + \|b\|_{H^s} + \|\eta\|_{H^{s+1}}(\|\psi\|_{H^{s+1}} + \|b\|_{H^{s-1}})).$$

and from Theorem 3.4, we have

$$(6.5) \quad \begin{aligned} & \|G(\eta_1)(\psi_1, b) - G(\eta_2)(\psi_2, b)\|_{H^s} \\ & \leq C(1 + \|(\eta_1, \eta_2)\|_{H^{s+1}})(\|(\psi_1, \psi_2)\|_{H^{s+1}} + \|b\|_{H^s}) \\ & \quad \times (\|\eta_1 - \eta_2\|_{H^{s+1}} + \|\psi_1 - \psi_2\|_{H^{s+1}}). \end{aligned}$$

It follows from standard nonlinear estimates that for some non decreasing functions $\mathcal{F}_j : \mathbf{R}_+ \rightarrow \mathbf{R}$,

$$\|F_n(f)\|_{H^s} \leq C\mathcal{F}_1(\|\tilde{J}_n f\|_{H^{s+1}})(\|b\|_{H^s} + \|\tilde{J}_n f\|_{H^{s+1}})$$

and (using (6.5))

$$\left\| F_n(f_1) - F_n(f_2) \right\|_{H^s} \leq C\mathcal{F}_2(\|\tilde{J}_n f\|_{H^{s+\frac{3}{2}}} + \|b\|_{H^s})\|\tilde{J}_n(f_1 - f_2)\|_{H^s}.$$

Since the operators \tilde{J}_n are bounded from L^2 to H^k by Cn^k , It follows that the operator $f \mapsto F_n(f)$ is locally Lipschitz from $L^2(\mathbf{T}^d)$ to itself. Consequently, the Cauchy-Lipschitz theorem implies that the Cauchy problem (6.4) has a unique maximal solution f_n in $C^1([0, T_n]; L^2(\mathbf{T}^d))$. Since $(I - \tilde{J}_n)J_n = 0$, we check that the function $(I - \tilde{J}_n)f_n$ solves

$$\partial_t(I - \tilde{J}_n)f_n = 0, \quad (I - \tilde{J}_n)f_n|_{t=0} = 0.$$

This shows that $(I - \tilde{J}_n)f_n = 0$, so $\tilde{J}_n f_n = f_n$. Consequently, the fact that f_n solves (6.4) implies that f_n is also a solution to (6.2). In addition, since the Fourier transform of f_n is compactly supported, the function f_n belongs to $C^0([0, T_n]; \mathcal{H}^{\lambda, \mu}(\mathbf{T}^d))$ for any $\lambda, \mu \geq 0$. The alternative (6.3) is a consequence of the usual continuation principle for ordinary differential equations. \square

6.2. Reformulation of the equations. Let $n \in \mathbf{N} \setminus \{0\}$ and denote by (η_n, ψ_n) the approximate solution as constructed in the previous paragraph. Recall that

$$(6.6) \quad \begin{cases} \partial_t \eta_n = J_n G(\eta_n)(\psi_n, b), \\ \partial_t \psi_n = J_n \left(-g\eta_n - \frac{1}{2}|\nabla_x \psi_n|^2 + \frac{(G(\eta_n)(\psi_n, b) + \nabla_x \eta_n \cdot \nabla_x \psi_n)^2}{2(1 + |\nabla_x \eta_n|^2)} \right), \\ \eta_n|_{t=0} = J_n \eta_0, \quad \psi_n|_{t=0} = J_n \psi_0. \end{cases}$$

In this paragraph we shall reformulate the above equations into a new set of equations where the unknowns are (ζ_n, V_n, B_n) defined by

$$(6.7) \quad \zeta_n = \nabla_x \eta_n, \quad B_n = \frac{G(\eta_n)(\psi_n, b) + \zeta_n \cdot \nabla_x \psi_n}{1 + |\zeta_n|^2}, \quad V_n = \nabla_x \psi_n - B_n \zeta_n.$$

Notice that with the above notations the equation on ψ_n can be reformulated as follows:

$$(6.8) \quad \partial_t \psi_n = J_n \left(-g\eta_n - \frac{1}{2}|V_n + B_n \zeta_n|^2 + \frac{1}{2}(1 + |\zeta_n|^2)B_n^2 \right).$$

Moreover, we have

$$(6.9) \quad G(\eta_n)(\psi_n, b) = B_n - V_n \cdot \zeta_n.$$

Recall that by definition, we have

$$(6.10) \quad \begin{aligned} G(\eta_n)(\psi_n, b) &= (\partial_y \phi_n - \zeta_n \cdot \nabla_x \phi_n)|_{y=\eta_n(x)}, \quad \text{where} \\ \Delta_{x,y} \phi_n &= 0 \text{ in } \{-h < y < \eta_n(x)\}, \quad \phi_n|_{y=\eta_n(x)} = \psi_n, \quad \partial_y \phi_n|_{y=-h} = b. \end{aligned}$$

Notice that by definition, we have

$$|\nabla_{x,y} \phi_n|^2|_{y=\eta_n} = |V_n|^2 + B_n^2.$$

Before reformulating the equations, we need several identities that will be used later.

Lemma 6.2. *Let ϕ_n be as defined by (6.10). Then*

$$\begin{cases} \partial_y \nabla_x \phi_n|_{y=\eta_n} = \nabla_x B_n - \frac{(\nabla_x B_n \cdot \zeta_n) \zeta_n - (\operatorname{div} V_n) \zeta_n}{1 + |\zeta_n|^2}, \\ \partial_y^2 \phi_n|_{y=\eta_n} = \frac{\nabla_x B_n \cdot \zeta_n - \operatorname{div} V_n}{1 + |\zeta_n|^2}. \end{cases}$$

Proof. Notice that $B_n = \partial_y \phi_n(x, \eta_n(x))$. Differentiating this identity with respect to x and using the equation satisfied by ϕ_n , we obtain

$$\nabla_x B_n = \partial_y \nabla_x \phi_n|_{y=\eta_n} + \partial_y^2 \phi_n|_{y=\eta_n} \zeta_n = \partial_y \nabla_x \phi_n|_{y=\eta_n} - \Delta_x \phi_n|_{y=\eta_n} \zeta_n.$$

We have also $V_n = \nabla_x \phi_n|_{y=\eta_n}$. Taking the divergence of both members, we obtain

$$(6.11) \quad \operatorname{div} V_n = \Delta_x \phi_n|_{y=\eta_n} + \partial_y \nabla_x \phi_n|_{y=\eta_n} \cdot \zeta_n.$$

It follows that

$$(6.12) \quad \partial_y \nabla_x \phi_n|_{y=\eta_n} = \nabla_x B_n + [\operatorname{div} V_n - \partial_y \nabla_x \phi_n|_{y=\eta_n} \cdot \zeta_n] \zeta_n.$$

Taking the scalar product of both members with $\zeta_n = \nabla_x \eta_n$, we obtain

$$(1 + |\zeta_n|^2)(\partial_y \nabla_x \phi_n|_{y=\eta_n} \cdot \zeta_n) = \nabla_x B_n \cdot \zeta_n + |\zeta_n|^2 \operatorname{div} V_n,$$

therefore,

$$(6.13) \quad \partial_y \nabla_x \phi_n|_{y=\eta_n} \cdot \zeta_n = \frac{\nabla_x B_n \cdot \zeta_n + |\zeta_n|^2 \operatorname{div} V_n}{1 + |\zeta_n|^2}.$$

Then we use (6.12) to obtain the first claim of the lemma. Since

$$\partial_y^2 \phi_n|_{y=\eta_n} = -\Delta_x \phi_n|_{y=\eta_n},$$

the second claim follows from (6.11) and (6.13). \square

6.2.1. The pressure. We introduce now the pressure associated to the new system (6.6). We shall define P_n by

$$(6.14) \quad P_n + gy = -(\partial_t \phi_n + \frac{1}{2} |\nabla_{x,y} \phi_n|^2),$$

where ϕ_n is defined in (6.10). Then setting $v_n = \nabla_{x,y} \phi_n$ and differentiating (6.14) with respect to x and y we find that v_n satisfies the system

$$(6.15) \quad \partial_t v_n + (v_n \cdot \nabla_{x,y}) v_n = -\nabla_{x,y}(P_n + gy).$$

We define now Q_n by

$$(6.16) \quad P_n = -\frac{1}{2} |\nabla_{x,y} \phi_n|^2 + Q_n - gy.$$

Then according to (6.10), we find that Q_n is the solution of the problem

$$(6.17) \quad \begin{cases} \Delta_{x,y} Q_n = 0, & \text{in } -h < y < \eta_n(x), \\ Q_n|_{y=\eta_n} = P_n|_{y=\eta_n} + g\eta_n + \frac{1}{2}(|V_n|^2 + B_n^2), \\ \partial_y Q_n|_{y=-h} = -\partial_t b. \end{cases}$$

This shows that Q_n is solution of an elliptic problem. We will have good estimates on Q_n as soon as we have described $P_n|_{y=\eta_n}$, which we do now.

Lemma 6.3. *We have*

$$(6.18) \quad P_n|_{y=\eta_n} = g(J_n - 1)\eta_n + \frac{1}{2}(J_n - 1)(|V_n|^2 + B_n^2) + [B_n, J_n](B_n - (V_n \cdot \zeta_n)).$$

Proof. It follows from (6.14) that

$$(6.19) \quad P_n|_{y=\eta_n} = -g\eta_n - (\partial_t \phi_n)|_{y=\eta_n} - \frac{1}{2}(|V_n|^2 + B_n^2).$$

Now since $\psi_n(t, x) = \phi_n(t, x, \eta_n(t, x))$, we have

$$\begin{aligned} \partial_t \psi_n(t, x) &= \partial_t \phi_n(t, x, \eta_n(t, x)) + \partial_y \phi_n(y, x, \eta_n(t, x)) \partial_t \eta_n(t, x) \\ &= \partial_t \phi_n|_{y=\eta_n} + B_n J_n [G(\eta_n)(\psi_n, b)]. \end{aligned}$$

The identity (6.8) for $\partial_t \psi_n$ implies that

$$J_n \left[-g\eta_n - \frac{1}{2}|V_n + B_n \zeta_n|^2 + \frac{1}{2}(1 + |\zeta_n|^2)B_n^2 \right] = \partial_t \phi_n|_{y=\eta_n} + B_n J_n [G(\eta_n)(\psi_n, b)].$$

The lemma follows from (6.9) and (6.19). \square

Lemma 6.4. *We have*

$$(6.20) \quad \begin{aligned} (\nabla_x P_n)|_{y=\eta_n} &= gJ_n \zeta_n - (\partial_y P_n|_{y=\eta_n} + g)\zeta_n + S_n, \\ S_n &= (J_n - 1)(V_n \cdot \nabla_x) V_n - (J_n + 1)B_n(\nabla_x B_n) + (\nabla_x B_n)J_n B_n \\ &\quad + B_n J_n(\nabla_x B_n) + J_n \nabla_x (B_n V_n \cdot \zeta_n) - \nabla_x (B_n J_n V_n \cdot \zeta_n). \end{aligned}$$

Proof. This follows from (6.18) which we differentiate with respect to x . \square

6.2.2. *The new equations.* We are now in position to reformulate our equations.

Proposition 6.5. *Let (η_n, ψ_n) be a solution of (6.6). Then, with the notations in (6.7) we have*

$$\begin{aligned} \text{(i)} \quad & \partial_t \zeta_n + J_n (V_n \cdot \nabla_x \zeta_n) = J_n [G(\eta_n)(V_n, \nabla_x b) - (\operatorname{div} V_n) \zeta_n], \\ \text{(ii)} \quad & (\partial_t + V_n \cdot \nabla_x) B_n \\ & = -(\partial_y P_n|_{y=\eta_n} + g) - \partial_y^2 \phi_n|_{y=\eta_n} ((J_n - 1)(V_n \cdot \zeta_n - B_n)), \\ \text{(iii)} \quad & (\partial_t + V_n \cdot \nabla_x) V_n \\ & = -\nabla_x P_n|_{y=\eta_n} - \partial_y \nabla_x \phi_n|_{y=\eta_n} ((J_n - 1)(V_n \cdot \zeta_n - B_n)). \end{aligned}$$

Proof. (i) We have

$$\partial_t \zeta_n = \nabla_x (\partial_t \eta_n) = J_n \nabla_x [G(\eta)(\psi_n, b)],$$

so using Lemma 3.8 we obtain

$$\partial_t \zeta_n = J_n [G(\eta_n)(V_n, \nabla_x b) - (V_n \cdot \nabla_x) \zeta_n - (\operatorname{div} V_n) \zeta_n],$$

which proves (i).

(ii) We have $B_n = \partial_y \phi_n(t, x, \eta_n(t, x))$. Therefore,

$$(6.21) \quad \partial_t B_n = \partial_t \partial_y \phi_n|_{y=\eta_n} + \partial_y^2 \phi_n|_{y=\eta_n} \partial_t \eta_n.$$

Now, since $v_n = \nabla_{x,y} \phi_n$, the last equation of the system (6.15) restricted to $y = \eta_n$ reads

$$(6.22) \quad \partial_t \partial_y \phi_n|_{y=\eta_n} + V_n \cdot \nabla_x \partial_y \phi_n|_{y=\eta_n} + B_n \partial_y^2 \phi_n|_{y=\eta_n} = -(\partial_y P_n|_{y=\eta_n} + g).$$

On the other hand, we have

$$(6.23) \quad \begin{aligned} V_n \cdot \nabla_x B_n &= (V_n \cdot \nabla_x) [\partial_y \phi_n(t, x, \eta(t, x))], \\ &= V_n \cdot \nabla_x \partial_y \phi_n|_{y=\eta_n} + (V_n \cdot \zeta_n) \partial_y^2 \phi_n|_{y=\eta_n}. \end{aligned}$$

Using (6.21), (6.22) and (6.23), we obtain

$$\partial_t B_n + V_n \cdot \nabla_x B_n = -(\partial_y P_n|_{y=\eta_n} + g) + \partial_y^2 \phi_n|_{y=\eta_n} (V_n \cdot \zeta_n - B_n + \partial_t \eta_n).$$

Then (ii) follows from the fact that

$$(6.24) \quad \partial_t \eta_n = J_n G(\eta_n)(\psi_n, b) = J_n (B_n - (V_n \cdot \nabla_x) \eta_n),$$

by (6.9).

(iii) We have $V_n = \nabla_x \phi_n(t, x, \eta(t, x))$. Therefore,

$$(6.25) \quad \partial_t V_n = \partial_t \nabla_x \phi_n|_{y=\eta_n} + \partial_y \nabla_x \phi_n|_{y=\eta_n} \partial_t \eta_n.$$

On the other hand,

$$(6.26) \quad (V_n \cdot \nabla_x) V_n = (V_n \cdot \nabla_x) \nabla_x \phi_n|_{y=\eta_n} + ((V_n \cdot \nabla_x) \eta_n) \nabla_x \partial_y \phi_n|_{y=\eta_n}.$$

Now, since $v_n = \nabla_{x,y} \phi_n$, the d -first equations of the system (6.15) restricted to $y = \eta_n$ are written as

$$(6.27) \quad \partial_t \nabla_x \phi_n|_{y=\eta_n} + (V_n \cdot \nabla_x) \nabla_x \phi_n|_{y=\eta_n} + B_n \partial_y \nabla_x \phi_n|_{y=\eta_n} = -\nabla_x P_n|_{y=\eta_n}.$$

Using (6.25), (6.26), (6.27), we obtain

$$\partial_t V_n + (V_n \cdot \nabla_x) V_n = -\nabla_x P_n|_{y=\eta_n} + \nabla_x \partial_y \phi_n|_{y=\eta_n} (\partial_t \eta_n + (V_n \cdot \nabla_x) \eta_n - B_n).$$

Then (iii) follows from (6.24). \square

Corollary 6.6. *We have*

$$\begin{cases} \partial_t V_n + g J_n \zeta_n = (\partial_y P_n|_{y=\eta_n} + g) \zeta_n + R_n^{(0)}, \\ \partial_t \zeta_n - J_n G(0)(V_n, 0) = R_n^{(1)}, \end{cases}$$

where

$$\begin{aligned} R_n^{(0)} &= -J_n (V_n \cdot \nabla_x) V_n - (\partial_y \nabla_x \phi_n|_{y=\eta_n}) (J_n - 1) (V_n \cdot \zeta_n - B_n) \\ &\quad + (J_n + 1) B_n \nabla_x B_n - \nabla_x B_n J_n B_n - B_n J_n \nabla_x B_n \\ &\quad - J_n \nabla_x (B_n V_n \cdot \zeta_n) + \nabla_x (B_n J_n V_n \cdot \zeta_n), \end{aligned}$$

$$R_n^{(1)} = J_n [G(\eta_n)(V_n, 0) - G(0)(V_n, 0) + G(\eta_n)(0, \nabla_x b) - (\operatorname{div} V_n) \zeta_n - (V_n \cdot \nabla_x) \zeta_n].$$

Proof. The first claim follows from Proposition 6.5 (iii) and Lemma 6.4. The second claim follows from Proposition 6.5 (i) writing

$$G(\eta_n)(V_n, \nabla_x b) = G(0)(V_n, 0) + [G(\eta_n)(V_n, 0) - G(0)(V_n, 0)] + G(\eta_n)(0, \nabla_x b). \quad \square$$

Recall that we have set

$$G(0)(V_n, 0) = a(D_x)V_n = |D_x| \tanh(h|D_x|)V_n.$$

6.2.3. *The final equation.* We shall use Corollary 6.6 to deduce a new equation on a new unknown with which we shall work.

Let us set

$$(6.28) \quad u_n = \sqrt{g}\zeta_n + i(a(D_x))^{\frac{1}{2}}V_n.$$

Then, with the notations in Corollary 6.6 we have:

Lemma 6.7. *The function u_n satisfies the equation*

$$(6.29) \quad \partial_t u_n + iJ_n(ga(D_x))^{\frac{1}{2}}u_n = \sqrt{g}R_n^{(1)} + ia(D_x)^{\frac{1}{2}}[(\partial_y P_n|_{y=\eta_n} + g)\zeta_n + R_n^{(0)}].$$

Proof. This follows immediately from Corollary 6.6. □

Set

$$U_{s,n}(t) = e^{\sigma(t)\langle D_x \rangle} \langle D_x \rangle^{s-\frac{1}{2}} u_n(t), \quad \sigma(t) = \lambda h - K\varepsilon t,$$

where K is a large positive constant to be chosen.

Since

$$e^{\sigma(t)|\xi|} \leq e^{\sigma(t)\langle \xi \rangle} \leq e^{\sigma(t)} e^{\sigma(t)|\xi|} \leq e^{\lambda h} e^{\sigma(t)|\xi|},$$

and since ζ_n and $a(D_x)^{\frac{1}{2}}V_n$ are real valued functions, there exist two absolute positive constants C_1, C_2 such that for all $t \in [0, T]$ and $\mu = 0$ or $\mu = \frac{1}{2}$

$$(6.30) \quad \begin{aligned} & \|\zeta_n(t)\|_{\mathcal{H}^{\sigma(t), s-\frac{1}{2}+\mu}} + \|a(D_x)^{\frac{1}{2}}V_n(t)\|_{\mathcal{H}^{\sigma(t), s-\frac{1}{2}+\mu}} \leq C_1 \|U_{s,n}(t)\|_{H^\mu}, \\ & \|U_{s,n}(t)\|_{H^\mu} \leq C_2 (\|\zeta_n(t)\|_{\mathcal{H}^{\sigma(t), s-\frac{1}{2}+\mu}} + \|a(D_x)^{\frac{1}{2}}V_n(t)\|_{\mathcal{H}^{\sigma(t), s-\frac{1}{2}+\mu}}). \end{aligned}$$

Fix two real numbers $h > 0$ and $\lambda \in [0, 1)$. Consider initial data (η_0, ψ_0) such that

$$\|\eta_0\|_{\mathcal{H}^{\lambda h, s+\frac{1}{2}}} + \|a(D_x)^{\frac{1}{2}}\psi_0\|_{\mathcal{H}^{\lambda h, s}} + \|V_0\|_{\mathcal{H}^{\lambda h, s}} + \|B_0\|_{\mathcal{H}^{\lambda h, s}} < +\infty,$$

where, as above,

$$B_0 = \frac{G(\eta_0)\psi_0 + \nabla_x \eta_0 \cdot \nabla_x \psi_0}{1 + |\nabla_x \eta_0|^2}, \quad V_0 = \nabla_x \psi_0 - B_0 \nabla_x \eta_0.$$

Notice that, when $\psi_0 = 0$, the functions V_0 and B_0 vanish and hence the previous assumption is satisfied whenever $\eta_0 \in \mathcal{H}^{\lambda h, s+\frac{1}{2}}$. Recall that we denote by T_n the lifespan of the approximate solution (η_n, ψ_n) and that we denote by $\bar{\varepsilon}$ the constant determined by means of Theorem 3.1. We are now in position

to state our main Sobolev estimates. For this we introduce some notations. We set

$$(6.31) \quad \begin{cases} N_s(b) = \|b\|_{L^\infty(\mathbf{R}, H^{s+\frac{1}{2}})} + \|\partial_t b\|_{L^\infty(\mathbf{R}, H^{s-\frac{1}{2}})} + \|b\|_{L^1(\mathbf{R}, H^{s+\frac{1}{2}})}, \\ M_{s,n}(T) = \|\eta_n\|_{X_T^{\infty, s+\frac{1}{2}}} + \|a(D_x)^{\frac{1}{2}}\psi_n\|_{X_T^{\infty, s}} + \|V_n\|_{X_T^{\infty, s}} + \|B_n\|_{X_T^{\infty, s}} \end{cases}$$

where

$$X_T^{\infty, s} = L^\infty([0, T], \mathcal{H}^{\sigma(\cdot), s}) \quad \text{with} \quad \sigma(t) = \lambda h - K\varepsilon t.$$

Proposition 6.8. *Fix $g > 0, h > 0, 0 < \lambda < 1$ and $s > 5/2 + d/2$. Set*

$$(6.32) \quad \varepsilon = \|\eta_0\|_{\mathcal{H}^{\lambda h, s+\frac{1}{2}}} + \|a(D_x)^{\frac{1}{2}}\psi_0\|_{\mathcal{H}^{\lambda h, s}} + \|V_0\|_{\mathcal{H}^{\lambda h, s}} + \|B_0\|_{\mathcal{H}^{\lambda h, s}} + N_s(b).$$

There exists a constant $C_0 \geq 1$ such that for all $n \in \mathbf{N} \setminus \{0\}$, for all $T \leq T_n$ and for all $K > 0$ the norm $M_{s,n}(T)$ satisfies the following inequality: if $M_{s,n}(T) \leq \bar{\varepsilon}$, then

$$(6.33) \quad \begin{aligned} & M_{s,n}(T)^2 + 2K\varepsilon \int_0^T \|U_{s,n}(t)\|_{H^{\frac{1}{2}}}^2 dt \\ & \leq C_0 \left(M_{s,n}(T) \int_0^T \|U_{s,n}(t)\|_{H^{\frac{1}{2}}}^2 dt + \frac{1}{K\varepsilon} h_\varepsilon(T) + (1+T^2)M_{s,n}(T)^4 + \varepsilon^2 \right), \end{aligned}$$

where

$$(6.34) \quad h_\varepsilon(T) = TM_{s,n}(T)^4 + T^5 M_{s,n}(T)^8 + \varepsilon^2 TM_{s,n}(T)^2.$$

The proof of this proposition is postponed to Section 6.3.

Let us assume Proposition 6.8 for the moment. Our goal in the end of this paragraph is to explain how to deduce some uniform estimates for the approximate solutions on a large time interval.

Corollary 6.9. *Fix $g > 0, h > 0, 0 < \lambda < 1$ and $s > 5/2 + d/2$. There exist four positive real numbers ε_*, c_*, C_* and K_* such that for all $n \in \mathbf{N} \setminus \{0\}$, the following properties hold: if the initial norm ε (as defined by (6.32)) satisfies $\varepsilon \leq \varepsilon_*$, then for all $n \in \mathbf{N} \setminus \{0\}$, the lifespan is bounded from below by*

$$T_n \geq \frac{c_*}{\varepsilon},$$

and moreover,

$$M_{s,n}\left(\frac{c_*}{\varepsilon}\right) \leq C_*\varepsilon,$$

where the norm $M_{s,n}$ has been defined in (6.31), with K replaced by K_* .

Proof. Fix μ such that $\mu \geq 16$ and $\mu C_0 \geq 1$. Then set

$$(6.35) \quad C_* = \sqrt{\mu C_0} \geq 1, \quad \varepsilon_* = \frac{\bar{\varepsilon}}{4C_*^2}, \quad c_* = \frac{1}{C_*^4} \leq 1, \quad K_* \geq \max\left(3, \frac{1}{\mu} C_*^3\right),$$

where C_0 is the constant whose existence is the main assertion of Proposition 6.8.

Hereafter we assume that $\varepsilon \leq \varepsilon_*$. Given $n \in \mathbf{N} \setminus \{0\}$, let us introduce the interval

$$I_n = \left[0, \min \left\{ T_n, \frac{c_*}{\varepsilon} \right\} \right].$$

We prove that, for all $n \in \mathbf{N} \setminus \{0\}$,

$$(6.36) \quad \forall T \in I_n, \quad M_{s,n}(T) \leq C_* \varepsilon.$$

Notice that if (6.36) holds, then it follows that $T_n \geq c_*/\varepsilon$ in light of the alternative (6.3). In particular, we can apply (6.36) for $T = c_*/\varepsilon$, which will give the wanted result.

It remains to prove (6.36). To do so, introduce the set

$$J_n = \{T \in I_n : M_{s,n}(T) \leq C_* \varepsilon\}.$$

With this notation, we prove that $J_n = I_n$. Notice that $0 \in J_n$ since $M_{s,n}(0) \leq \varepsilon$ and $C_* \geq 1$. Since $T \mapsto M_{s,n}(T)$ is continuous, the set J_n is closed. Hence, to conclude the proof, it is sufficient to prove that J_n is open. This is turn will be a straightforward corollary of the following claim:

$$(6.37) \quad \forall T \in J_n, \quad M_{s,n}(T) \leq \frac{1}{2} C_* \varepsilon.$$

Let us prove this claim. To do so, we will exploit (6.33). Since $\varepsilon \leq \varepsilon_* = \bar{\varepsilon}/(4C_*)$, notice that if $M_{s,n}(T) \leq C_* \varepsilon$, then we automatically obtain that $M_{s,n}(T) \leq \bar{\varepsilon}$. Then we are in position to apply the estimate (6.33). We use (6.36) and the fact that $T_n \leq \frac{c_*}{\varepsilon}$. Recall from (6.33) that

$$\begin{aligned} & M_{s,n}(T)^2 + 2K\varepsilon \int_0^T \|U_{s,n}(t)\|_{H^{\frac{1}{2}}}^2 dt \\ & \leq \frac{1}{\mu} C_*^2 \left(C_* \varepsilon \int_0^T \|U_{s,n}(t)\|_{H^{\frac{1}{2}}}^2 dt + \frac{1}{K\varepsilon} h_\varepsilon(T) + (1+T^2)M_{s,n}(T)^4 + \varepsilon^2 \right). \end{aligned}$$

Since $\frac{1}{\mu} C_*^3 \leq K_*$, we can absorb the integral term in the right hand side into the left hand side. Now according to (6.34), we have

$$\frac{1}{K_* \varepsilon} h_\varepsilon(T) \leq \frac{1}{K_*} (c_* C_*^4 + c_*^5 C_*^8 + c_* C_*) \varepsilon^2.$$

Since $c_* = \frac{1}{C_*^4}$, $C_* \geq 1$ and $K_* \geq 3$, we obtain

$$\frac{1}{K_* \varepsilon} h_\varepsilon(T) \leq \frac{3}{K_*} \varepsilon^2 \leq \varepsilon^2.$$

On the other hand, by (6.35) we have for $\varepsilon \leq \varepsilon_*$,

$$(1+T_n^2)M_{s,n}(T)^4 + \varepsilon^2 \leq (\varepsilon^2 + c_*^2)C_*^4 \varepsilon^2 + \varepsilon^2 \leq 3\varepsilon^2.$$

It follows that

$$M_{s,n}(T)^2 \leq \frac{4}{\mu} C_*^2 \varepsilon^2 \leq \frac{1}{4} C_*^2 \varepsilon^2,$$

since $\mu \geq 16$. This completes the proof of the claim (6.37). \square

6.3. Proof of Proposition 6.8. We fix $d \geq 1$, $g > 0$, $\lambda < 1$, $h > 0$, $s > 5/2 + d/2$.

To simplify notations, the indexes n will be skipped: we fix an integer n in $\mathbf{N} \setminus \{0\}$ and denote simply by (η, ψ) the solutions to the approximate system (6.6). We shall denote by C many different constants, whose values may change from a line to another, and which depend only on the parameters which are considered fixed (that is d, λ, h and s). In particular, these constants are independent of $T, K, \bar{\varepsilon}$ and n .

We set

$$\begin{aligned} I &= [0, T], \\ \sigma(t) &= \lambda h - K\varepsilon t, \quad t \in [0, T], \quad \varepsilon > 0, \quad K > 0 \quad (\text{to be chosen}), \\ a(D_x) &= |D_x| \tanh(h|D_x|) (= G_0(0)), \\ X^{\infty, s} &= L^\infty(I, \mathcal{H}^{\sigma(\cdot), s}), \\ M_s(T) &= \|\eta\|_{X^{\infty, s+\frac{1}{2}}} + \|a(D_x)^{\frac{1}{2}}\psi\|_{X^{\infty, s}} + \|V\|_{X^{\infty, s}} + \|B\|_{X^{\infty, s}}, \\ N_s(b) &= \|b\|_{L^\infty(\mathbf{R}, H^{s+\frac{1}{2}})} + \|\partial_t b\|_{L^\infty(\mathbf{R}, H^{s-\frac{1}{2}})} + \|b\|_{L^1(\mathbf{R}, H^{s+\frac{1}{2}})}, \\ U_{s, n}(t) &= e^{\sigma(t)\langle D_x \rangle} \langle D_x \rangle^{s-\frac{1}{2}} u_n. \end{aligned}$$

Recall (see (2.10)) that

$$\begin{aligned} E^{\lambda, \mu} &= \{u : e^{\lambda z|D_x|} u \in C_z^0([-h, 0], \mathcal{H}^{\lambda h, \mu}(\mathbf{T}^d))\}, \\ F^{\lambda, \mu} &= \{u : e^{\lambda z|D_x|} u \in L_z^2((-h, 0), \mathcal{H}^{\lambda h, \mu}(\mathbf{T}^d))\}. \end{aligned}$$

In what follows, we fix $t \in [0, T]$ and write $\sigma(t) = \lambda(t)h$, where

$$\lambda(t) = \lambda - \frac{K\varepsilon t}{h} \leq \lambda < 1.$$

Recall the basic hypotheses made in Proposition 6.8:

(6.38)

- (i) $\|\eta_0\|_{\mathcal{H}^{\lambda h, s+\frac{1}{2}}} + \|a(D_x)^{\frac{1}{2}}\psi_0\|_{\mathcal{H}^{\lambda h, s}} + \|V_0\|_{\mathcal{H}^{\lambda h, s}} + \|B_0\|_{\mathcal{H}^{\lambda h, s}} + N_s(b) = \varepsilon \leq 1$,
- (ii) $M_s(T) \leq \bar{\varepsilon} \leq 1$.

We begin by preliminaries. The goal of the two following sections is to show that it is sufficient to have estimates on V to control the terms B and ψ .

6.3.1. Estimates of ψ and B .

Lemma 6.10. *There exists $C > 0$ such that if $M_s(T) \leq \bar{\varepsilon} \leq 1$, then for all $t \in [0, T]$,*

- (i) $\|B(t)\|_{\mathcal{H}^{\sigma, s+\frac{1}{2}}} \leq C(\|V(t)\|_{\mathcal{H}^{\sigma, s+\frac{1}{2}}} + \|b(t)\|_{H^{s+\frac{1}{2}}} + \|B(t)\|_{H^1})$,
- (ii) $\|B(t)\|_{\mathcal{H}^{\sigma, s}} \leq C(\|V(t)\|_{\mathcal{H}^{\sigma, s}} + \|b(t)\|_{H^s} + \|B(t)\|_{H^1})$.

Proof. (i) We fix $t \in [0, T]$ and take $\sigma \leq \lambda h$. We start from the identity:

$$-\operatorname{div} V = G(\eta)(B, -\Delta_x \phi_h) = G(\eta)(B, 0) + G(\eta)(0, -\Delta_x \phi_h),$$

where $\phi_h = \phi|_{y=-h}$, proved in (3.25). We have seen that

$$G(0)(B, 0) = a(D_x)B = |D_x| \tanh(h|D_x|)B.$$

We write

$$(6.39) \quad a(D_x)B = -\operatorname{div} V + G(\eta)(0, \Delta_x \phi_h) + (G(0) - G_0(\eta))(B, 0) := F.$$

Let $\chi \in C_0^\infty(\mathbf{R}^d)$ be such that $\chi(\xi) = 1$ if $|\xi| \leq 1$ and $\chi(\xi) = 0$ if $|\xi| \geq 2$. We have

$$\|\chi(D)B\|_{\mathcal{H}^{\sigma, s+\frac{1}{2}}} \leq C\|B\|_{L^2}.$$

On the other hand, it follows from (6.39) that

$$(1 - \chi(\xi))\widehat{B}(\xi) = \frac{1 - \chi(\xi)}{a(\xi)}\widehat{F}(\xi),$$

so that

$$\|(1 - \chi(D))B\|_{\mathcal{H}^{\sigma, s+\frac{1}{2}}} \leq C\|F\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}}.$$

First of all, we have $\|\operatorname{div} V\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} \lesssim \|V\|_{\mathcal{H}^{\sigma, s+\frac{1}{2}}}$. By using Theorem 3.4 we have

$$\|(G(0) - G_0(\eta))(B, 0)\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} \leq CM_s(T)\|B\|_{\mathcal{H}^{\sigma, s+\frac{1}{2}}} \leq C\bar{\varepsilon}\|B\|_{\mathcal{H}^{\sigma, s+\frac{1}{2}}}.$$

Eventually from Theorem 3.1 and Corollary 4.2 applied with $\mu = s - \frac{1}{2}$ we deduce that

$$\|G(\eta)(0, \Delta_x \phi_h)\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} \lesssim \|\Delta \phi_h\|_{H^{s-\frac{1}{2}}} \lesssim \|b\|_{H^{s+\frac{1}{2}}} + \|B\|_{H^1}.$$

It follows that

$$\|(1 - \chi(D))B\|_{\mathcal{H}^{\sigma, s+\frac{1}{2}}} \leq C(\|V\|_{\mathcal{H}^{\sigma, s+\frac{1}{2}}} + \|b\|_{H^{s+\frac{1}{2}}} + \|B\|_{H^1} + \bar{\varepsilon}\|B\|_{\mathcal{H}^{\sigma, s+\frac{1}{2}}}).$$

We see from the above estimate of $\chi(D)B$ that

$$\|B\|_{\mathcal{H}^{\sigma, s+\frac{1}{2}}} \leq C(\|V\|_{\mathcal{H}^{\sigma, s+\frac{1}{2}}} + \|b\|_{H^{s+\frac{1}{2}}} + \|B\|_{H^1} + \bar{\varepsilon}\|B\|_{\mathcal{H}^{\sigma, s+\frac{1}{2}}}).$$

Taking $\bar{\varepsilon}$ sufficiently small we obtain the desired result.

The proof of (ii) is identical. We have just to use Remark 3.5, Remark 3.2 and Corollary 4.2 with $\mu = s - 1$. \square

Lemma 6.11. *There exists $C > 0$ such that for all $t \in (0, T)$ we have*

$$\|a(D_x)^{\frac{1}{2}}\psi(t)\|_{\mathcal{H}^{\sigma, s}} \leq C(\|V(t)\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} + \|B(t)\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}}\|\eta(t)\|_{\mathcal{H}^{\sigma, s+\frac{1}{2}}}).$$

Proof. We write, for fixed $t \in [0, T]$ and $\sigma \leq \lambda h$,

$$\|a(D_x)^{\frac{1}{2}}\psi\|_{\mathcal{H}^{\sigma, s}}^2 = \sum \langle \xi \rangle^{2s} e^{2\sigma|\xi|} |\xi| \tanh(h|\xi|) |\widehat{\psi}(\xi)|^2.$$

Since $|\xi| \tanh(h|\xi|) \lesssim \frac{|\xi|^2}{\langle \xi \rangle}$, we have

$$\|a(D_x)^{\frac{1}{2}}\psi\|_{\mathcal{H}^{\sigma, s}} \leq C\|\nabla_x \psi\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}}.$$

Eventually we notice that $\nabla_x \psi = V + B\nabla_x \eta$, so that, since $s - \frac{1}{2} > \frac{d}{2}$, we obtain

$$\|\nabla_x \psi(t)\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} \lesssim \|V(t)\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} + \|B\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}}\|\eta\|_{\mathcal{H}^{\sigma, s+\frac{1}{2}}},$$

which proves the lemma. \square

6.3.2. *Low frequency estimates of η, V, ψ, B .*

Lemma 6.12. *Assume that $s > 3 + \frac{d}{2}$ and $M_s(T) \leq \bar{\varepsilon} \leq 1$. There exists a constant $C = C(d, g, \lambda, h, s) > 0$ such that for every $t \in (0, T)$,*

(6.40)

$$\begin{aligned} & \|\eta(t)\|_{H^{s-1}} + \|a(D_x)^{\frac{1}{2}}\psi(t)\|_{H^{s-1}} \\ & \leq C \left(\|\eta_0\|_{H^{s-1}} + \|a(D_x)^{\frac{1}{2}}\psi_0\|_{H^{s-1}} + TM_s(T)^2 + \|b\|_{L^2((0,T),H^{s-\frac{1}{2}})}^2 \right. \\ & \qquad \qquad \qquad \left. + \|b\|_{L^1((0,T),H^{s-\frac{1}{2}})} \right), \end{aligned}$$

(6.41)

$$\begin{aligned} & \|V(t)\|_{H^{s-\frac{3}{2}}} + \|B(t)\|_{H^{s-\frac{3}{2}}} \\ & \leq C \left(\|\eta_0\|_{H^{s-1}} + \|a(D_x)^{\frac{1}{2}}\psi_0\|_{H^{s-1}} + TM_s(T)^2 + \|b\|_{L^2((0,T),H^{s-\frac{1}{2}})}^2 \right. \\ & \qquad \qquad \qquad \left. + \|b\|_{L^\infty((0,T),H^{s-\frac{3}{2}})} + \|b\|_{L^1((0,T),H^{s-\frac{1}{2}})} \right). \end{aligned}$$

Corollary 6.13. *We have*

$$(6.42) \quad \|B(t)\|_{H^1} \lesssim TM_s(T)^2 + \varepsilon.$$

Proof. Since $s - \frac{3}{2} \geq 1$, this follows from (6.41) and the hypotheses (6.38) made on the data and b . \square

Proof of Lemma 6.12. We start from system (6.6) which we write as

$$\begin{aligned} \partial_t \eta - J a(D_x) \psi &= f_1 := J(G(\eta) - G(0))(\psi, 0) + G(\eta)(0, b), \\ \partial_t \psi + J g \eta &= f_2 := J \left(\frac{1}{2} |\nabla_x \psi|^2 - \frac{(G(\eta)(\psi, b) + \nabla_x \eta \cdot \nabla_x \psi)^2}{2(1 + |\nabla_x \eta|^2)} \right), \end{aligned}$$

where $a(D_x)\psi = |D_x| \tanh(h|D_x|)\psi$.

We set

$$u = g\eta + i(g a(D_x))^{\frac{1}{2}}\psi.$$

Then u is a solution of the equation

$$\partial_t u + iJ(g a(D_x))^{\frac{1}{2}}u = g f_1 + i(g a(D_x))^{\frac{1}{2}}f_2.$$

Computing $\frac{d}{dt}\|u(t)\|_{H^{s-1}}^2$ on the interval $I = [0, T]$, we obtain the inequality

$$(6.43) \quad \|u(t)\|_{H^{s-1}} \leq C \left(\|u_0\|_{H^{s-1}} + \int_0^t \|f_1(t')\|_{H^{s-1}} dt' + \int_0^t \|f_2(t')\|_{H^{s-\frac{1}{2}}} dt' \right).$$

Let us estimate f_1 . First of all, Corollary 3.6 gives

$$\begin{aligned} \|(G(\eta(t)) - G(0))(\psi(t), 0)\|_{H^{s-1}} &\leq \|(G(\eta(t)) - G(0))(\psi(t), 0)\|_{\mathcal{H}^{\sigma(t), s-\frac{1}{2}}} \\ &\leq CM_s(T)^2. \end{aligned}$$

It follows that

$$(6.44) \quad \int_0^t \|(G(\eta(t')) - G(0))(\psi(t'), 0)\|_{H^{s-1}} dt' \leq CTM_s(T)^2.$$

Now, from Theorem 3.1 we get

$$(6.45) \quad \int_0^t \|G(\eta(t'))(0, b(t'))\|_{H^{s-1}} dt' \leq \int_0^t \|G(\eta(t'))(0, b(t'))\|_{\mathcal{H}^{\sigma(t), s-\frac{1}{2}}} dt' \\ \leq C \|b\|_{L^1(I, H^{s-\frac{1}{2}})}.$$

It follows from (6.44), (6.45) that

$$(6.46) \quad \int_0^t \|f_1(t')\|_{H^{s-1}} dt' \leq C(TM_s(T)^2 + \|b\|_{L^1(I, H^{s-\frac{1}{2}})}).$$

Let us estimate f_2 . Using Remark 2.9, we can write

$$\|\nabla_x \psi(t)\|_{H^{s-\frac{1}{2}}}^2 \leq C \|\nabla_x \psi(t)\|_{H^{s-\frac{1}{2}}}^2 \leq C \|\nabla_x \psi(t)\|_{\mathcal{H}^{\sigma(t), s-\frac{1}{2}}}^2 \\ \leq C' \|a(D_x)^{\frac{1}{2}} \psi(t)\|_{\mathcal{H}^{\sigma(t), s}}^2.$$

Therefore, we get

$$\|\nabla_x \psi\|_{L^\infty(I, H^{s-\frac{1}{2}})}^2 \leq CM_s(T)^2.$$

It follows that

$$(6.47) \quad \int_0^T \|\nabla_x \psi(t)\|_{H^{s-\frac{1}{2}}}^2 dt \leq CTM_s(T)^2.$$

Now we can write

$$\tilde{f}_2 = \frac{(G(\eta)(\psi, b) + \nabla_x \eta \cdot \nabla_x \psi)^2}{2(1 + |\nabla_x \eta|^2)} := U(1 - g(|\nabla_x \eta|^2)), \\ U = \frac{1}{2}(G(\eta)(\psi, b) + \nabla_x \eta \cdot \nabla_x \psi)^2, \quad g(t) = \frac{t}{1+t}.$$

Then

$$\|\tilde{f}_2\|_{H^{s-\frac{1}{2}}} \leq C \|U\|_{H^{s-\frac{1}{2}}} (1 + \|g(|\nabla \eta|^2)\|_{H^{s-\frac{1}{2}}}).$$

By the product laws in the usual Sobolev spaces we can write

$$\|U\|_{H^{s-\frac{1}{2}}} \lesssim \|G(\eta)(\psi, b)\|_{H^{s-\frac{1}{2}}}^2 + \|\eta\|_{H^{s+\frac{1}{2}}}^2 \|a(D_x)^{\frac{1}{2}} \psi\|_{H^s}^2 \\ \lesssim \|G(\eta)(\psi, b)\|_{H^{s-\frac{1}{2}}}^2 + M_s(T)^4, \\ \|g(|\nabla \eta|^2)\|_{H^{s-\frac{1}{2}}} \lesssim \|\eta\|_{H^{s+\frac{1}{2}}}^2 \lesssim M_s(T)^2.$$

Using Theorem 3.1, we have

$$\|G(\eta)(\psi, b)\|_{H^{s-\frac{1}{2}}}^2 \\ \lesssim (\|a(D_x)^{\frac{1}{2}} \psi\|_{\mathcal{H}^{\sigma, s}}^2 + \|b\|_{H^{s-\frac{1}{2}}}^2 + \|\eta\|_{H^{s+\frac{1}{2}}}^2 (\|a(D_x)^{\frac{1}{2}} \psi\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}}^2 + \|b\|_{H^{s-\frac{3}{2}}}^2)) \\ \lesssim M_s(T)^2 + \|b\|_{H^{s-\frac{1}{2}}}^2.$$

Therefore,

$$\|\tilde{f}_2(t)\|_{H^{s-\frac{1}{2}}} \leq C(M_s(T)^2 + \|b(t)\|_{H^{s-\frac{1}{2}}}^2).$$

It follows that

$$(6.48) \quad \int_0^t \|\tilde{f}_2(t)\|_{H^{s-\frac{1}{2}}} dt \lesssim TM_s(T)^2 + \|b\|_{L^2(I, H^{s-\frac{1}{2}})}^2.$$

Using (6.43), (6.46), (6.47), and (6.48), we obtain that

$$\|u(t)\|_{H^{s-1}} \lesssim \|u_0\|_{H^{s-1}} + TM_s(T)^2 + \|b\|_{L^2(\mathbf{R}, H^{s-\frac{1}{2}})}^2 + \|b\|_{L^1(I, H^{s-\frac{1}{2}})}.$$

Using (6.43) and the definition of u , we get

$$(6.49) \quad \begin{aligned} & \|\eta(t)\|_{H^{s-1}} + \|a(D_x)^{\frac{1}{2}}\psi(t)\|_{H^{s-1}} \\ & \lesssim \|\eta_0\|_{H^{s-1}} + \|a(D_x)^{\frac{1}{2}}\psi_0\|_{H^{s-1}} + TM_s(T)^2 + \|b\|_{L^1(\mathbf{R}, H^{s-\frac{1}{2}})} + \|b\|_{L^2(\mathbf{R}, H^{s-\frac{1}{2}})}^2. \end{aligned}$$

This completes the proof of (6.40).

To prove (6.41), we estimate B with fixed t . By definition we can write

$$\begin{aligned} B &= \frac{G(\eta)(\psi, b) + \nabla_x \psi \cdot \nabla_x \eta}{1 + |\nabla_x \eta|^2} = W(1 - g(|\nabla_x \eta|^2)), \\ W &= G(\eta)(\psi, b) + \nabla_x \psi \cdot \nabla_x \eta, \quad g(t) = \frac{t}{1+t}. \end{aligned}$$

Now, we use as before the product laws in the usual Sobolev spaces. Since $s > 3 + d/2$, it follows from Remark 3.2 applied with $\delta = 1$ and $\lambda = 0$, that

$$\|G(\eta)(\psi, b)(t)\|_{H^{s-\frac{3}{2}}} \lesssim \|a(D_x)^{\frac{1}{2}}\psi(t)\|_{H^{s-1}} + \|b\|_{H^{s-\frac{3}{2}}}.$$

Therefore,

$$\|W(t)\|_{H^{s-\frac{3}{2}}} \lesssim \|a(D_x)^{\frac{1}{2}}\psi(t)\|_{H^{s-1}} + \|b(t)\|_{H^{s-\frac{3}{2}}} + \|a(D_x)^{\frac{1}{2}}\psi(t)\|_{H^{s-1}} \|\eta(t)\|_{H^{s-\frac{1}{2}}}.$$

Moreover,

$$\|g(|\nabla_x \eta|^2)(t)\|_{H^{s-\frac{3}{2}}} \lesssim \|\eta(t)\|_{H^{s-\frac{1}{2}}} \lesssim \bar{\varepsilon} \leq 1.$$

It follows that

$$\|B(t)\|_{H^{s-\frac{3}{2}}} \lesssim \|a(D_x)^{\frac{1}{2}}\psi(t)\|_{H^{s-1}} + \|b(t)\|_{H^{s-\frac{3}{2}}}.$$

Using (6.49), we deduce that

$$(6.50) \quad \begin{aligned} \|B\|_{L^\infty(I, H^{s-\frac{3}{2}})} & \lesssim (\|\eta_0\|_{H^{s-1}} + \|a(D_x)^{\frac{1}{2}}\psi_0\|_{H^{s-1}} + TM_s(T)^2 \\ & \quad + \|b\|_{L^2(\mathbf{R}, H^{s-\frac{1}{2}})}^2 + \|b\|_{L^1(\mathbf{R}, H^{s-\frac{1}{2}})} + \|b\|_{L^\infty(\mathbf{R}, H^{s-\frac{3}{2}})}) \end{aligned}$$

Now we have $V = \nabla_x \psi - B \nabla_x \eta$. It follows that

$$\|V\|_{L^\infty(I, H^{s-\frac{3}{2}})} \lesssim \|a(D_x)^{\frac{1}{2}}\psi\|_{L^\infty(I, H^{s-1})} + \|B\|_{L^\infty(I, H^{s-\frac{3}{2}})} \|\eta\|_{L^\infty(I, H^{s-\frac{1}{2}})}.$$

Since $\|\eta\|_{L^\infty(I, H^{s-\frac{1}{2}})} \leq \bar{\varepsilon} \leq 1$, using (6.49) and (6.50) we obtain

$$\begin{aligned} \|V\|_{L^\infty(I, H^{s-\frac{3}{2}})} & \lesssim (\|\eta_0\|_{H^{s-1}} + \|a(D_x)^{\frac{1}{2}}\psi_0\|_{H^{s-1}} + TM_s(T)^2 \\ & \quad + \|b\|_{L^2(\mathbf{R}, H^{s-\frac{1}{2}})}^2 + \|b\|_{L^1(\mathbf{R}, H^{s-\frac{1}{2}})} + \|b\|_{L^\infty(\mathbf{R}, H^{s-\frac{3}{2}})}). \end{aligned}$$

This completes the proof of Lemma 6.12. \square

Corollary 6.14. *Under the assumptions (6.38) we have for every $t \in [0, T]$,*

$$\|B(t)\|_{\mathcal{H}^{\sigma, s+\mu}} \lesssim \|V(t)\|_{\mathcal{H}^{\sigma, s+\mu}} + TM_s(T)^2 + \varepsilon, \quad \mu = 0, \mu = \frac{1}{2},$$

$$\|a(D_x)^{\frac{1}{2}}\psi(t)\|_{\mathcal{H}^{\sigma, s}} \lesssim \|V(t)\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} + \|B(t)\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} M_s(T).$$

6.3.3. *Proof of Proposition 6.8 (continued).* Recall that the function $u = \sqrt{g}\zeta + ia(D_x)^{\frac{1}{2}}V$ that was introduced in (6.28) satisfies (see (6.29)) the equation

$$(6.51) \quad \partial_t u + iJ(ga(D_x))^{\frac{1}{2}}u = \sqrt{g}R^{(1)} + ia(D_x)^{\frac{1}{2}}[(\partial_y P|_{y=\eta} + g)\zeta + R^{(0)}],$$

where

$$(6.52) \quad \begin{cases} R^{(1)} = R_1^{(1)} + R_2^{(1)}, \\ R_1^{(1)} = J[G(\eta)(V, 0) - G(0)(V, 0) + G(\eta)(0, \nabla_x b)], \\ R_2^{(1)} = J[-(\operatorname{div} V)\zeta - (V \cdot \nabla_x)\zeta] \\ R^{(0)} = R_1^{(0)} + R_2^{(0)} + R_3^{(0)} + R_4^{(0)}, \\ R_1^{(0)} = -J(V \cdot \nabla_x)V, \\ R_2^{(0)} = -(\partial_y \nabla_x \phi|_{y=\eta})(J-1)(V \cdot \zeta - B), \\ R_3^{(0)} = (J+1)B\nabla_x B - \nabla_x BJB - BJ\nabla_x B, \\ R_4^{(0)} = -\nabla_x J(BV \cdot \zeta) + \nabla_x (BJV \cdot \zeta). \end{cases}$$

We shall now estimate the remainder terms. Recall that

$$\sigma(t) = \lambda h - K\varepsilon t = \lambda(t)h$$

and

$$U_s(t) = e^{\sigma(t)\langle D_x \rangle} \langle D_x \rangle^{s-\frac{1}{2}} u(t).$$

Proposition 6.15. *We have*

- (i) $\|\sqrt{g}R_1^{(1)}\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} \lesssim M_s(T)\|U_s\|_{H^{\frac{1}{2}}} + \|b\|_{H^{s+\frac{1}{2}}},$
- (ii) $\|\sqrt{g}R_2^{(1)}\|_{\mathcal{H}^{\sigma, s-1}} \lesssim M_s(T)\|U_s\|_{H^{\frac{1}{2}}},$
- (iii) $\|ia(D_x)^{\frac{1}{2}}[(\partial_y P|_{y=\eta} + g)\zeta]\|_{\mathcal{H}^{\sigma, s-1}} \lesssim M_s(T)^2 + T^2 M_s(T)^5 + \varepsilon M_s(T) + M_s^2(T)\|U_s\|_{H^{\frac{1}{2}}},$
- (iv) $\|ia(D_x)^{\frac{1}{2}}R^{(0)}\|_{\mathcal{H}^{\sigma, s-1}} \lesssim M_s(T)\|U_s\|_{H^{\frac{1}{2}}} + M_s(T)^2 + TM_s(T)^3 + \varepsilon M_s(T).$

Proof. Proof of (i) and (ii): Estimate of $\sqrt{g}R^{(1)}$.

We shall estimate each term separately. All the estimates will be with fixed t , with constants independent of t . Therefore t will be omitted in what follows.

Moreover we notice that the family of operators J_n is uniformly bounded with respect to n on any space $\mathcal{H}^{\sigma, s}$. Therefore we can skip it in what follows.

First of all, we have

$$(6.53) \quad \|G(\eta)(V, 0) - G(0)(V, 0)\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} \lesssim M_s(T)\|U_s\|_{H^{\frac{1}{2}}}.$$

To see this, we use Theorem 3.4 with $\eta_1 = \eta, \eta_2 = 0, \psi_1 = \psi_2 = V, b = 0$. Then, with the notations in (3.9), we have

$$\|G(\eta)(V, 0) - G(0)(V, 0)\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} \lesssim \lambda_1 \|\eta\|_{\mathcal{H}^{\sigma, s}} + \lambda_2 \|\eta\|_{\mathcal{H}^{\sigma, s+\frac{1}{2}}} \leq (\lambda_1 + \lambda_2)M_s(T).$$

Now, by definition of U_s , we have

$$\begin{aligned}\lambda_1 &= \|\eta\|_{\mathcal{H}^{\sigma,s+\frac{1}{2}}} \|a(D_x)^{\frac{1}{2}} V\|_{\mathcal{H}^{\sigma(t),s-\frac{1}{2}}} + \|a(D_x)^{\frac{1}{2}} V\|_{\mathcal{H}^{\sigma,s}} \\ &\lesssim M_s(T) \|U_s\|_{L^2} + \|U_s\|_{H^{\frac{1}{2}}}, \\ \lambda_2 &= \|a(D_x)^{\frac{1}{2}} V\|_{\mathcal{H}^{\sigma,s-\frac{1}{2}}} \lesssim \|U_s\|_{L^2},\end{aligned}$$

which implies (6.53). Moreover, according to Theorem 3.1, we have

$$\|G(\eta)(0, \nabla_x b)\|_{\mathcal{H}^{\sigma,s-\frac{1}{2}}} \lesssim \|b\|_{H^{s+\frac{1}{2}}} + M_s(T) \|b\|_{H^{s-\frac{1}{2}}}.$$

Since $M_s(T) \leq \bar{\varepsilon} \leq 1$, we obtain

$$\|\sqrt{g}R_1^{(1)}\|_{\mathcal{H}^{\sigma,s-\frac{1}{2}}} \lesssim M_s(T) \|U_s\|_{H^{\frac{1}{2}}} + \|b\|_{H^{s+\frac{1}{2}}}.$$

This proves (i).

Now, since $\zeta = \nabla_x \eta$, it follows from the product rule in Proposition A.7 that

$$\|(V \cdot \nabla_x) \zeta\|_{\mathcal{H}^{\sigma,s-1}} \lesssim \|V\|_{\mathcal{H}^{\sigma,s-1}} \|\zeta\|_{\mathcal{H}^{\sigma,s}} \lesssim M_s(T) \|U_s\|_{H^{\frac{1}{2}}},$$

and

$$\|(\operatorname{div} V) \zeta\|_{\mathcal{H}^{\sigma,s-1}} \lesssim \|V\|_{\mathcal{H}^{\sigma,s}} \|\zeta\|_{\mathcal{H}^{\sigma,s-1}} \lesssim M_s(T) \|U_s\|_{L^2}.$$

It follows that

$$\|\sqrt{g}R_2^{(1)}\|_{\mathcal{H}^{\sigma,s-1}} \lesssim M_s(T) \|U_s\|_{H^{\frac{1}{2}}},$$

which proves (ii)

Proof of (iii): Estimate of:

$$(1) = ia(D_x)^{\frac{1}{2}} [(\partial_y P|_{y=\eta} + g)\zeta].$$

For simplicity we shall set

$$\mathbf{a} = -\partial_y P|_{y=\eta}.$$

We have

$$\|(1)\|_{\mathcal{H}^{\sigma,s-1}} \lesssim \|\mathbf{a} - g\|_{\mathcal{H}^{\sigma,s-\frac{1}{2}}} \|\eta\|_{\mathcal{H}^{\sigma,s+\frac{1}{2}}} \lesssim M_s(T) \|\mathbf{a} - g\|_{\mathcal{H}^{\sigma,s-\frac{1}{2}}}.$$

We claim that, for fixed t ,

$$(6.54) \quad \|\mathbf{a} - g\|_{\mathcal{H}^{\sigma,s-\frac{1}{2}}} \lesssim M_s(T) + T^2 M_s(T)^4 + \varepsilon + M_s(T) \|U_s\|_{H^{\frac{1}{2}}}$$

It will follow that

$$\begin{aligned}\|ia(D_x)^{\frac{1}{2}} [(\partial_y P|_{y=\eta} + g)\zeta_n]\|_{\mathcal{H}^{\sigma,s-1}} \\ \lesssim M_s(T)^2 + T^2 M_s(T)^5 + \varepsilon M_s(T) + M_s^2(T) \|U_s\|_{H^{\frac{1}{2}}}.\end{aligned}$$

We prove (6.54). Since the function ϕ is the solution of the problem

$$\Delta_{x,y} \phi = 0, \quad \phi|_{\Sigma} = \psi, \quad \partial_y \phi|_{y=-h} = b,$$

we have

$$(6.55) \quad \partial_y(P - gy)|_{y=-h} = -\partial_t b + \nabla_x b \cdot (\nabla_x \phi_h) - b(\Delta_x \phi_h) := \mathcal{P}_1,$$

where $\phi_h = \phi|_{y=-h}$. Let us recall that according to (6.16), (6.18), (6.17) and (6.55), the pressure P satisfies

$$(6.56) \quad \begin{aligned} \Delta_{x,y}(P - gy) &= -|\nabla_{x,y}^2 \phi|^2, \\ (P - gy)|_{y=\eta(x)} &= g(J - 2)\eta + \frac{1}{2}(J - 1)(|V|^2 + B^2) + [B, J](B - V \cdot \zeta) := \mathcal{P}_0, \\ \partial_y P - g|_{y=-h} &= -\partial_t b + \nabla_x b \cdot (\nabla_x \phi_h) - b(\Delta_x \phi_h) := \mathcal{P}_1 \end{aligned}$$

We are going to work in the (x, z) variables defined previously. Let us recall some notations (see section 2.1):

$$\begin{aligned} \rho(x, z) &= \frac{1}{h}(z + h)e^{z|D_x|(\eta(x))} + z, \quad \Lambda_1 = \frac{1}{\partial_z \rho} \partial_z, \quad \Lambda_2 = \nabla_x - \frac{\nabla_x \rho}{\partial_z \rho} \partial_z, \\ \tilde{\phi}(x, z) &= \phi(x, \rho(x, z)). \end{aligned}$$

Set

$$\phi_h(x) := \phi(x, -h) = \tilde{\phi}(x, -h) := \tilde{\phi}_h(x),$$

and

$$\mathcal{P}(x, z) = P(x, \rho(x, z)) - g\rho(x, z).$$

Since $\Lambda_1 \rho = 1$, we have

$$\mathbf{a} - g = -(\Lambda_1 \mathcal{P})|_{z=0}.$$

It follows from (6.56) that \mathcal{P} is a solution of the problem

$$(6.57) \quad (\Lambda_1^2 + \Lambda_2^2) \mathcal{P} = G, \quad \mathcal{P}|_{z=0} = \mathcal{P}_0, \quad \partial_z \mathcal{P}|_{z=-h} = -(\partial_z \rho|_{z=-h}) \mathcal{P}_1,$$

where G is a linear combination of $(\Lambda_j \Lambda_k \tilde{\phi})^2$, $1 \leq j, k \leq 2$.

From Theorem 2.10 we have

$$(6.58) \quad \begin{aligned} &\|e^{\lambda(z+h)|D_x|} \nabla_{x,z} \mathcal{P}\|_{C^0([-h,0], H^{s-\frac{1}{2}})} \\ &\lesssim (1 + \|\eta\|_{\mathcal{H}^{\sigma, s+\frac{1}{2}}}) (\|G\|_{F^{\lambda, s-1}} + \|a(D_x)^{\frac{1}{2}} \mathcal{P}_0\|_{\mathcal{H}^{\sigma, s}} + \|(\partial_z \rho)|_{z=-h} \mathcal{P}_1\|_{H^{s-\frac{1}{2}}}). \end{aligned}$$

Estimate of G in $F^{\lambda, s-1}$. According to Lemma A.12, we have, since $s > 2 + \frac{d}{2}$,

$$\|(\Lambda_j \Lambda_k \tilde{\phi})^2\|_{F^{\lambda h, s-1}} \lesssim \|\Lambda_j \Lambda_k \tilde{\phi}\|_{E^{\lambda h, s-\frac{3}{2}}} \|\Lambda_j \Lambda_k \tilde{\phi}\|_{F^{\lambda h, s-1}}.$$

We shall prove the following estimate.

$$(6.59) \quad \sum_{j,k=1}^2 \|\Lambda_j \Lambda_k \tilde{\phi}\|_{E^{\lambda h, s-\frac{3}{2}}} + \sum_{j,k=1}^2 \|\Lambda_j \Lambda_k \tilde{\phi}\|_{F^{\lambda h, s-1}} \lesssim \|a(D_x)^{\frac{1}{2}} V\|_{\mathcal{H}^{\lambda h, s-1}} + \|b\|_{H^{s-\frac{1}{2}}}.$$

In fact, since Λ_1 and Λ_2 commute, we have $(\Lambda_1^2 + \Lambda_2^2) \Lambda_2 \tilde{\phi} = 0$. On the other hand, by definition $\Lambda_2 \tilde{\phi}|_{z=0} = V$. Now,

$$\Lambda_1(\Lambda_2 \tilde{\phi})|_{z=-h} = \Lambda_2(\Lambda_1 \tilde{\phi})|_{z=-h} = \nabla_x b.$$

Indeed, the right hand side is the image by our diffeomorphism of the quantity $\nabla_x(\partial_y \phi)|_{y=-h} = \nabla_x[(\partial_y \phi)|_{y=-h}] = \nabla_x b$. Summing up $U = \Lambda_2 \tilde{\phi}$ is a solution

of the problem

$$(6.60) \quad \begin{cases} (\Lambda_1^2 + \Lambda_2^2)U = 0, & U|_{z=0} = V, \\ \partial_z U|_{z=-h} = (\partial_z \rho)|_{z=-h} \nabla_x b = (1 + \frac{1}{h} e^{-h|D_x|} \eta) \nabla_x b. \end{cases}$$

Using Corollary 2.7 we obtain when $M_s(T) \leq \bar{\varepsilon}$,

$$(6.61) \quad \|\nabla_{x,z} U\|_{F^{\lambda,s-1}} \lesssim \|a(D_x)^{\frac{1}{2}} V\|_{\mathcal{H}^{\lambda h, s-1}} + \|b\|_{H^{s-\frac{1}{2}}}.$$

Recall that $\Lambda_1 = \frac{1}{\partial_z \rho} \partial_z$ and $\Lambda_2 = \nabla_x - \frac{\nabla_x \rho}{\partial_z \rho} \partial_z$. Since $\nabla_x \rho$ and $\partial_z \rho - 1$ belong to \mathcal{E}_1 and since $\Lambda_1^2 \tilde{\phi} = -\Lambda_2^2 \tilde{\phi}$, we obtain

$$\sum_{j,k=1}^2 \|\Lambda_j \Lambda_k \tilde{\phi}\|_{F^{\lambda h, s-1}} \lesssim \|a(D_x)^{\frac{1}{2}} V\|_{\mathcal{H}^{\lambda h, s-1}} + \|b\|_{H^{s-\frac{1}{2}}}.$$

Now, using Lemma 2.12, we can write

$$\|\nabla_{x,z} U\|_{E^{\lambda, s-\frac{3}{2}}} \leq C \|\nabla_{x,z} U\|_{F^{\lambda, s-1}}.$$

Using same argument as above and (6.61), we deduce that

$$\sum_{j,k=1}^2 \|\Lambda_j \Lambda_k \tilde{\phi}\|_{E^{\lambda h, s-\frac{3}{2}}} \lesssim \|a(D_x)^{\frac{1}{2}} V\|_{\mathcal{H}^{\lambda h, s-1}} + \|b\|_{H^{s-\frac{1}{2}}},$$

which proves (6.59). Therefore, with the notation in (6.57) we have

$$\|G\|_{F^{\lambda, s-1}} \lesssim \|a(D_x)^{\frac{1}{2}} V\|_{\mathcal{H}^{\lambda h, s-1}}^2 + \|b\|_{H^{s-\frac{1}{2}}}^2.$$

Estimate of $I = \|a(D_x)^{\frac{1}{2}} \mathcal{P}_0\|_{\mathcal{H}^{\sigma, s}}$. Recall that

$$\mathcal{P}_0 = g(J-2)\eta + \frac{1}{2}(J-1)(|V|^2 + B^2) + [B, J](B - V \cdot \zeta).$$

According to Proposition A.7 (iii) (with $t = s + \frac{1}{2}$, $s_0 = s$) and to Proposition A.13 applied to $B(D) = J_n(D) = \chi(\frac{D}{n})$ and $a = B$, with $\nu = 0$ and $s + \frac{1}{2}$ instead of s , we obtain (using that $\xi \mapsto \chi(\frac{\xi}{n}) \in S_{1,0}^0$ with semi-norms bounded with respect to the n parameter)

$$\begin{aligned} I &\lesssim \|\eta\|_{\mathcal{H}^{\sigma, s+\frac{1}{2}}} + \|V\|_{\mathcal{H}^{\sigma, s}} \|V\|_{\mathcal{H}^{\sigma, s+\frac{1}{2}}} \\ &\quad + \|B\|_{\mathcal{H}^{\sigma, s}} \|B\|_{\mathcal{H}^{\sigma, s+\frac{1}{2}}} + \|B\|_{\mathcal{H}^{\sigma, s+\frac{1}{2}}} \|B - V \cdot \zeta\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}}, \\ &\lesssim \|\eta\|_{\mathcal{H}^{\sigma, s+\frac{1}{2}}} + \|V\|_{\mathcal{H}^{\sigma, s}} \|V\|_{\mathcal{H}^{\sigma, s+\frac{1}{2}}} \\ &\quad + \|B\|_{\mathcal{H}^{\sigma, s}} \|B\|_{\mathcal{H}^{\sigma, s+\frac{1}{2}}} + \|B\|_{\mathcal{H}^{\sigma, s+\frac{1}{2}}} \|\eta\|_{\mathcal{H}^{\sigma, s+\frac{1}{2}}} \|V\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}}. \end{aligned}$$

Here $S_{1,0}^0$ is the class of symbols p for which the seminorms

$$\sup_{(x, \xi) \in \mathbf{R}^d \times \mathbf{R}^d} \langle \xi \rangle^{|\alpha|} |\partial_x^\beta \partial_\xi^\alpha p(\xi)|$$

are finite for every $(\alpha, \beta) \in \mathbf{N}^d \times \mathbf{N}^d$.

Using Corollary 6.14 and the fact that

$$\|\eta\|_{\mathcal{H}^{\sigma, s+\frac{1}{2}}} \leq M_s(T) \leq \bar{\varepsilon} \leq 1,$$

we obtain eventually

$$\|a(D_x)^{\frac{1}{2}}\mathcal{P}_0\|_{\mathcal{H}^{\sigma,s}} \lesssim M_s(T)\|U_s\|_{H^{\frac{1}{2}}} + M_s(T) + TM_s(T)^3.$$

Estimate of $II = \|(\partial_z\rho)|_{z=-h}\mathcal{P}_1\|_{H^{s-\frac{1}{2}}}$.

We have

$$\mathcal{P}_1 = \partial_t b - \nabla_x b \cdot (\nabla_x \phi_h) + b(\Delta_x \phi_h)$$

in $H^{s-\frac{1}{2}}$. Therefore,

$$\|\mathcal{P}_1\|_{H^{s-\frac{1}{2}}} \lesssim \|\partial_t b\|_{H^{s-\frac{1}{2}}} + \|b\|_{H^{s+\frac{1}{2}}} (\|\nabla_x \phi_h\|_{H^{s-\frac{1}{2}}} + \|\Delta_x \phi_h\|_{H^{s-\frac{1}{2}}}).$$

Now, we have

$$\|\nabla_x \phi_h\|_{H^{s-\frac{1}{2}}} \lesssim C\|\nabla_x \phi_h\|_{H^{s-1}} + \|\Delta_x \phi_h\|_{H^{s-\frac{3}{2}}},$$

so that

$$\|\mathcal{P}_1\|_{H^{s-\frac{1}{2}}} \lesssim \|\partial_t b\|_{H^{s-\frac{1}{2}}} + \|b\|_{H^{s+\frac{1}{2}}} (\|\nabla_x \phi_h\|_{H^{s-1}} + \|\Delta_x \phi_h\|_{H^{s-\frac{1}{2}}}).$$

By Corollary 4.2 we have

$$\|\Delta_x \phi_h\|_{H^{s-\frac{1}{2}}} \leq C(\|b\|_{H^{s+\frac{1}{2}}} + \|B\|_{H^1}).$$

Theorem 2.10 and the fact that

$$\mathcal{X}^{\lambda h, s-1} \subset \{u : e^{\lambda(z+h)|D_x|}u \in C^0([-h, 0], H^{s-1})\}$$

imply that

$$\|\nabla_x \phi_h\|_{H^{s-1}} \lesssim \|a(D_x)^{\frac{1}{2}}\psi\|_{\mathcal{H}^{\lambda h, s-\frac{1}{2}}} + \|b\|_{H^{s-1}}.$$

Therefore,

$$\|\mathcal{P}_1\|_{H^{s-\frac{1}{2}}} \lesssim \|\partial_t b\|_{H^{s-\frac{1}{2}}} + \|b\|_{H^{s+\frac{1}{2}}} (\|a(D_x)^{\frac{1}{2}}\psi\|_{\mathcal{H}^{\lambda h, s-\frac{1}{2}}} + \|b\|_{H^{s+\frac{1}{2}}} + \|B\|_{H^1})$$

and eventually,

$$(6.62) \quad \|\mathcal{P}_1\|_{H^{s-\frac{1}{2}}} \lesssim \|\partial_t b\|_{H^{s-\frac{1}{2}}} + \|b\|_{H^{s+\frac{1}{2}}} (M_s(T) + \|b\|_{H^{s+\frac{1}{2}}} + TM_s(T)^2 + \varepsilon).$$

Then (6.54) follows from (6.58) to (6.62), and (iii) is proved.

(iv) Estimate of (2) = $ia(D_x)^{\frac{1}{2}}R^{(0)}$ Recall for convenience that

$$(6.63) \quad \begin{cases} R^{(0)} = R_1^{(0)} + R_2^{(0)} + R_3^{(0)} + R_4^{(0)}, \\ R_1^{(0)} = -J(V \cdot \nabla_x)V, \\ R_2^{(0)} = -(\partial_y \nabla_x \phi|_{y=\eta})(J-1)(V \cdot \zeta - B), \\ R_3^{(0)} = (J+1)B\nabla_x B - (\nabla_x B)JB - BJ\nabla_x B, \\ R_4^{(0)} = -\nabla_x J(BV \cdot \zeta) + \nabla_x (BJV \cdot \zeta), \end{cases}$$

First of all, we have

$$\begin{aligned} \|a(D_x)^{\frac{1}{2}}J(V \cdot \nabla_x)V\|_{\mathcal{H}^{\sigma, s-1}} &\lesssim \|(V \cdot \nabla_x)V\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} \lesssim \|V\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} \|V\|_{\mathcal{H}^{\sigma, s+\frac{1}{2}}} \\ &\lesssim \|V\|_{\mathcal{H}^{\sigma, s-\frac{1}{2}}} (\|V\|_{\mathcal{H}^{\sigma, s}} + \|a(D_x)^{\frac{1}{2}}V\|_{\mathcal{H}^{\sigma, s}}). \end{aligned}$$

Therefore,

$$(6.64) \quad \|a(D_x)^{\frac{1}{2}}R_1^{(0)}\|_{\mathcal{H}^{\sigma, s-1}} \lesssim M_s(T)^2 + M_s(T)\|U_s\|_{H^{\frac{1}{2}}}.$$

Let us consider the term $R_2^{(0)}$. Since $s > 1 + \frac{d}{2}$, we have

$$\|a(D_x)^{\frac{1}{2}}R_2^{(0)}\|_{\mathcal{H}^{\sigma,s-1}} \lesssim \|\partial_y \nabla_x \phi|_{y=\eta}\|_{\mathcal{H}^{\sigma,s-\frac{1}{2}}} \|V \cdot \zeta - B\|_{\mathcal{H}^{\sigma,s-\frac{1}{2}}}.$$

Here we have

$$\|V \cdot \zeta - B\|_{\mathcal{H}^{\sigma,s-\frac{1}{2}}} \lesssim \|V\|_{\mathcal{H}^{\sigma,s-\frac{1}{2}}} \|\eta\|_{\mathcal{H}^{\sigma,s+\frac{1}{2}}} + \|B\|_{\mathcal{H}^{\sigma,s-\frac{1}{2}}} \leq CM_s(T).$$

Recall that by Lemma 6.2, we have

$$\partial_y \nabla_x \phi|_{y=\eta} = \nabla_x B - \frac{(\nabla_x B \cdot \zeta)\zeta + (\operatorname{div} V)\zeta}{1 + |\zeta|^2}.$$

It follows from the product rules and Proposition A.10 that

$$\begin{aligned} & \|\partial_y \nabla_x \phi|_{y=\eta}\|_{\mathcal{H}^{\sigma,s-\frac{1}{2}}} \\ & \lesssim \|B\|_{\mathcal{H}^{\sigma,s+\frac{1}{2}}} (1 + \|\eta\|_{\mathcal{H}^{\sigma,s+\frac{1}{2}}}^2)^2 + \|V\|_{\mathcal{H}^{\sigma,s+\frac{1}{2}}} \|\eta\|_{\mathcal{H}^{\sigma,s+\frac{1}{2}}} (1 + \|\eta\|_{\mathcal{H}^{\sigma,s+\frac{1}{2}}}^2) \\ & \lesssim \|B\|_{\mathcal{H}^{\sigma,s+\frac{1}{2}}} + \|V\|_{\mathcal{H}^{\sigma,s+\frac{1}{2}}} \\ & \lesssim \|V\|_{\mathcal{H}^{\sigma,s+\frac{1}{2}}} + TM_s(T)^2 + \varepsilon, \end{aligned}$$

since $\|\eta\|_{\mathcal{H}^{\sigma,s+\frac{1}{2}}} \leq M_s(T) \leq \bar{\varepsilon} \leq 1$ and using Corollary 6.14. Now,

$$\|V\|_{\mathcal{H}^{\sigma,s+\frac{1}{2}}} \lesssim \|a(D_x)^{\frac{1}{2}}V\|_{\mathcal{H}^{\sigma,s}} + \|V\|_{\mathcal{H}^{\sigma,s}} \lesssim \|U_s\|_{H^{\frac{1}{2}}} + M_s(T).$$

It follows that

$$\|\partial_y \nabla_x \phi|_{y=\eta}\|_{\mathcal{H}^{\sigma,s-\frac{1}{2}}} \lesssim \|U_s\|_{H^{\frac{1}{2}}} + M_s(T) + TM_s(T)^2 + \varepsilon.$$

Eventually we obtain

$$\|a(D_x)^{\frac{1}{2}}R_2^{(0)}\|_{\mathcal{H}^{\sigma,s-1}} \lesssim \|U_s\|_{H^{\frac{1}{2}}} M_s(T) + M_s^2(T) + TM_s(T)^3 + \varepsilon M_s(T).$$

We consider now the term $R_3^{(0)}$. It is easy to see that

$$\|a(D_x)^{\frac{1}{2}}R_3^{(0)}\|_{\mathcal{H}^{\sigma,s-1}} \lesssim \|B\|_{\mathcal{H}^{\sigma,s-\frac{1}{2}}} \|B\|_{\mathcal{H}^{\sigma,s+\frac{1}{2}}}.$$

So using Corollary 6.14 we obtain eventually,

$$\|a(D_x)^{\frac{1}{2}}R_3^{(0)}\|_{\mathcal{H}^{\sigma,s-1}} \lesssim \|U_s\|_{H^{\frac{1}{2}}} M_s(T) + M_s^2(T) + TM_s(T)^3 + \varepsilon M_s(T).$$

Consider now the term $R_4^{(0)}$. We notice that $R_4^{(0)} = \nabla_x([J, B]V\zeta)$. It follows that

$$\|a(D_x)^{\frac{1}{2}}R_4^{(0)}\|_{\mathcal{H}^{\sigma,s-1}} \leq C\|[J, B]V\zeta\|_{\mathcal{H}^{\sigma,s+\frac{1}{2}}}.$$

Since the operator $J = \varphi(2^{-n}D_x)$ belongs to $\operatorname{OpS}_{1,0}^0$ with semi-norms bounded with respect to n , Proposition A.13 gives

$$\|a(D_x)^{\frac{1}{2}}R_4^{(0)}\|_{\mathcal{H}^{\sigma,s-1}} \lesssim \|B\|_{\mathcal{H}^{\sigma,s+\frac{1}{2}}} \|V\|_{\mathcal{H}^{\sigma,s-\frac{1}{2}}} \|\eta\|_{\mathcal{H}^{\sigma,s+\frac{1}{2}}}.$$

Then by Corollary 6.14, we obtain

$$(6.65) \quad \|a(D_x)^{\frac{1}{2}}R_4^{(0)}\|_{\mathcal{H}^{\sigma,s-1}} \lesssim M_s(T)^2 (\|U_s\|_{H^{\frac{1}{2}}} + TM_s(T)^2 + \varepsilon).$$

Summing up, using (6.64) to (6.65), then (6.63) and the fact that $M_s(T) \leq \bar{\varepsilon} \leq 1$, we obtain

$$\|a(D_x)^{\frac{1}{2}}R^{(0)}\|_{\mathcal{H}^{\sigma,s-1}} \lesssim M_s(T) \|U_s\|_{H^{\frac{1}{2}}} + M_s(T)^2 + TM_s(T)^3 + \varepsilon M_s(T).$$

The proof of Proposition 6.15 is complete. \square

Corollary 6.16. *We have*

- (i) $\|\sqrt{g}R_1^{(1)}\|_{\mathcal{H}^{\sigma,s-\frac{1}{2}}} \lesssim M_s(T)\|U_s\|_{H^{\frac{1}{2}}} + \|b\|_{H^{s+\frac{1}{2}}},$
- (ii) $\|\sqrt{g}R_2^{(1)} + ia(D_x)^{\frac{1}{2}}[(\partial_y P|_{y=\eta} + g)\zeta + ia(D_x)^{\frac{1}{2}}R^{(0)}]\|_{\mathcal{H}^{\sigma,s-1}} \lesssim M_s(T)\|U_s\|_{H^{\frac{1}{2}}} + g_\varepsilon(T),$

where

$$(6.66) \quad g_\varepsilon(T) = M_s(T)^2 + TM_s(T)^3 + T^2M_s(T)^5 + \varepsilon M_s(T).$$

6.3.4. *A priori estimates.* In this paragraph we first bound the terms in $M_s(T)$ containing η and V . For the reader's convenience we recall some notations. We have set

$$\zeta = \nabla_x \eta, \quad u = \sqrt{g}\zeta + ia(D_x)^{\frac{1}{2}}V.$$

Then u is solution of the equation (see (6.51))

$$\partial_t u + iJ(ga(D_x))^{\frac{1}{2}}u = \sqrt{g}R^{(1)} + ia(D_x)^{\frac{1}{2}}[(\partial_y P|_{y=\eta} + g)\zeta + R^{(0)}] := f$$

We have also set

$$U_s(t) = e^{\sigma(t)\langle D_x \rangle} \langle D_x \rangle^{s-\frac{1}{2}} u(t), \quad \sigma(t) = \lambda h - K\varepsilon t,$$

where K is a large positive constant to be chosen. Then U_s satisfies the equation

$$\partial_t U_s + iJ(ga(D_x))^{\frac{1}{2}}U_s + K\varepsilon \langle D_x \rangle U_s = e^{\sigma(t)\langle D_x \rangle} \langle D_x \rangle^{s-\frac{1}{2}} f.$$

So we have

$$\begin{aligned} \frac{d}{dt} \|U_s(t)\|_{L^2}^2 &= 2(U_s(t), \partial_t U_s(t))_{L^2}, \\ &= -2K\varepsilon \|\langle D_x \rangle^{\frac{1}{2}} U_s(t)\|_{L^2}^2 + 2(U_s(t), e^{\sigma(t)\langle D_x \rangle} \langle D_x \rangle^{s-\frac{1}{2}} f(t))_{L^2}, \end{aligned}$$

because the term $2\operatorname{Re} i(U_s(t), (gJa(D_x))^{\frac{1}{2}}U_s(t))_{L^2}$ vanishes, since the symbol of Ja is real. We deduce the estimate

$$(6.67) \quad \begin{aligned} &\|U_s(t)\|_{L^2}^2 + 2K\varepsilon \int_0^t \|U_s(t')\|_{H^{\frac{1}{2}}}^2 dt' \\ &= \|U_s(0)\|_{L^2}^2 + 2 \int_0^t (U_s(t'), \mathcal{K}_s(t')f(t'))_{L^2} dt', \end{aligned}$$

where (see (6.52))

$$\begin{aligned} \mathcal{K}_s(t') &= e^{\sigma(t')\langle D_x \rangle} \langle D_x \rangle^{s-\frac{1}{2}}, \quad f = f_1 + f_2, \\ f_1 &= \sqrt{g}R_1^{(1)}, \quad f_2 = \sqrt{g}R_2^{(1)} + ia(D_x)^{\frac{1}{2}}[(\partial_y P|_{y=\eta} + g)\zeta + R^{(0)}]. \end{aligned}$$

Set

$$(6.68) \quad A_j(t) = \int_0^t F_j(t') dt'$$

where

$$F_1(t') = |(U_s(t'), \mathcal{K}_s(t')f_1(t'))_{L^2}|, \quad F_2(t') = |(U_s(t'), \mathcal{K}_s(t')f_2(t'))_{L^2}|$$

Estimate of $A_1(t)$. By Corollary 6.16 (i) we can write

$$F_1(t') \leq \|U_s(t')\|_{L^2} \|f_1(t')\|_{\mathcal{H}^{\sigma,s-\frac{1}{2}}} \lesssim M_s(T)\|U_s(t')\|_{H^{\frac{1}{2}}}^2 + \|U_s(t')\|_{L^2} \|b(t')\|_{H^{s+\frac{1}{2}}}.$$

It follows that

$$(6.69) \quad A_1(t) \lesssim M_s(T) \int_0^t \|U_s(t')\|_{H^{\frac{1}{2}}}^2 dt' + \int_0^t \|U_s(t')\|_{L^2} \|b(t')\|_{H^{s+\frac{1}{2}}} dt'.$$

Estimate of $A_2(t)$. By Corollary 6.16 (ii) we can write

$$F_2(t') \leq \|U_s(t')\|_{H^{\frac{1}{2}}} \|f_2(t')\|_{\mathcal{H}^{\sigma, s-1}} \lesssim M_s(T) \|U_s(t')\|_{H^{\frac{1}{2}}}^2 + g_\varepsilon(T).$$

It follows that

$$(6.70) \quad A_2(t) \lesssim M_s(T) \int_0^t \|U_s(t')\|_{H^{\frac{1}{2}}}^2 dt' + g_\varepsilon(T) \int_0^t \|U_s(t')\|_{H^{\frac{1}{2}}} dt'$$

Let us set

$$(6.71) \quad \mathcal{E}_s(t) = \|U_s(t)\|_{L^2}^2 + 2K\varepsilon \int_0^t \|U_s(t')\|_{H^{\frac{1}{2}}}^2 dt'.$$

Using (6.67), (6.68), (6.69) and (6.70), we obtain

$$(6.72) \quad \mathcal{E}_s(t) \lesssim \|U_s(0)\|_{L^2}^2 + M_s(T) \int_0^t \|U_s(t')\|_{H^{\frac{1}{2}}}^2 dt' + I_1(t) + I_2(t),$$

where

$$I_1(t) = \int_0^t \|b(t')\|_{H^{s+\frac{1}{2}}} \|U_s(t')\|_{L^2} dt',$$

$$I_2(t) = g_\varepsilon(T) \int_0^t \|U_s(t')\|_{H^{\frac{1}{2}}} dt'.$$

Here g_ε is defined by (6.66). We estimate separately each term I_k . We have

$$(6.73) \quad I_1(t) \leq \|U_s\|_{L^\infty(I, L^2)} \|b\|_{L^1(\mathbf{R}, H^{s+\frac{1}{2}})} \leq \delta \|U_s\|_{L^\infty(I, L^2)}^2 + C_\delta \|b\|_{L^1(\mathbf{R}, H^{s+\frac{1}{2}})}^2,$$

Using the Cauchy-Schwarz inequality, we write

$$I_2(t) \leq g_\varepsilon(T) T^{\frac{1}{2}} \left(\int_0^t \|U_s(t')\|_{H^{\frac{1}{2}}}^2 dt' \right)^{\frac{1}{2}},$$

so, with absolute constants C_1 and C_2 ,

$$(6.74) \quad I_2(t) \lesssim \frac{1}{K\varepsilon} T g_\varepsilon(T)^2 + K\varepsilon \int_0^t \|U_s(t')\|_{H^{\frac{1}{2}}}^2 dt'.$$

We will absorb the term $\frac{K\varepsilon}{C_2} \int_0^t \|U_s(t')\|_{H^{\frac{1}{2}}}^2 dt'$ by the corresponding term in $\mathcal{E}_s(t)$. It follows from (6.72), (6.73) and (6.74) that

$$(6.75) \quad \begin{aligned} \mathcal{E}_s(t) &\lesssim \|U_s(0)\|_{L^2}^2 + \delta \|U_s\|_{L^\infty(I, L^2)}^2 + M_s(T) \int_0^t \|U_s(t')\|_{H^{\frac{1}{2}}}^2 dt' \\ &\quad + \frac{C_1}{K\varepsilon} T g_\varepsilon(T)^2 + \|b\|_{L^1(\mathbf{R}, H^{s+\frac{1}{2}})}^2. \end{aligned}$$

Now according to (6.30), we have

$$(6.76) \quad \begin{aligned} \|U_s(0)\|_{L^2} &\lesssim \|\eta_0\|_{\mathcal{H}^{\lambda h, s+\frac{1}{2}}}^2 + \|V_0\|_{\mathcal{H}^{\lambda h, s}}^2, \\ \|\nabla \eta(t)\|_{\mathcal{H}^{\sigma(t), s-\frac{1}{2}}}^2 &+ \|a(D_x)^{\frac{1}{2}} V(t)\|_{\mathcal{H}^{\sigma(t), s-\frac{1}{2}}}^2 \lesssim \|U_s(t)\|_{L^2}^2. \end{aligned}$$

Moreover, we have

$$(6.77) \quad \begin{aligned} \|\eta(t)\|_{\mathcal{H}^{\sigma(t),s+\frac{1}{2}}}^2 &\lesssim \|\nabla\eta(t)\|_{\mathcal{H}^{\sigma(t),s-\frac{1}{2}}}^2 + \|\eta(t)\|_{L^2}^2, \\ \|V(t)\|_{\mathcal{H}^{\sigma(t),s}}^2 &\lesssim \|a(D_x)^{\frac{1}{2}}V(t)\|_{\mathcal{H}^{\sigma(t),s-\frac{1}{2}}}^2 + \|V(t)\|_{L^2}^2. \end{aligned}$$

Using (6.71), (6.75), (6.76) and (6.77), we obtain

$$\begin{aligned} &\mathcal{E}_s(t) + \|\eta(t)\|_{\mathcal{H}^{\sigma(t),s+\frac{1}{2}}}^2 + \|V(t)\|_{\mathcal{H}^{\sigma(t),s}}^2 \\ &\lesssim \|\eta_0\|_{\mathcal{H}^{\lambda h,s+\frac{1}{2}}}^2 + \|V_0\|_{\mathcal{H}^{\lambda h,s}}^2 + \delta\|U_s(t)\|_{L^\infty(I,L^2)}^2 \\ &\quad + M_s(T) \int_0^t \|U_s(t')\|_{H^{\frac{1}{2}}}^2 dt' + \frac{1}{K\varepsilon} T g_\varepsilon(T)^2 + \|b\|_{L^1(\mathbf{R},H^{s+\frac{1}{2}})}^2. \end{aligned}$$

To complete the proof of Proposition 6.8 we are left with the estimate of the part in $M_s(T)$ containing B and ψ . For that we use Corollary 6.14. Using also the hypotheses (6.38) we obtain

Corollary 6.17. *There exists $C = C(d, g, \lambda, h, s) > 0$ such that for every $t \in (0, T)$ we have*

$$(6.78) \quad \begin{aligned} &\mathcal{E}_s(t) + \|\eta(t)\|_{\mathcal{H}^{\sigma(t),s+\frac{1}{2}}}^2 + \|a(D_x)^{\frac{1}{2}}\psi\|_{\mathcal{H}^{\sigma(t),s}}^2 + \|V(t)\|_{\mathcal{H}^{\sigma(t),s}}^2 + \|B(t)\|_{\mathcal{H}^{\sigma(t),s}}^2 \\ &\lesssim \delta\|U_s\|_{L^\infty(I,L^2)}^2 + M_s(T) \int_0^t \|U_s(t')\|_{H^{\frac{1}{2}}}^2 dt' + \frac{1}{K\varepsilon} T g_\varepsilon(T)^2 \\ &\quad + (1 + T^2)M_s(T)^4 + \varepsilon^2, \end{aligned}$$

where

$$g_\varepsilon(T) = M_s(T)^2 + TM_s(T)^3 + T^2M_s(T)^5 + \varepsilon M_s(T)$$

End of the proof of Proposition 6.8. According to the definition of $\mathcal{E}_s(T)$ (see (6.71)) taking the supremum of both members with respect to t in $(0, T)$ and δ small enough, we can absorb the term $\delta\|U_s\|_{L^\infty(I,L^2)}^2$ in the right hand side by the left hand side. This completes the proof. \square

6.4. End of the proof. We are now in position to complete the proof of Theorem 1.7.

Uniqueness. Without source term (that is when $b = 0$), the uniqueness of smooth solutions is a well-known result. When b is non-trivial, we notice that the uniqueness result asserted by Theorem 1.5 implies the uniqueness of the solutions satisfying the regularity assumptions in Theorem 1.7. Indeed, if we consider an initial data (η_0, ψ_0) satisfying assumption (1.7), and two possible solutions (η_1, ψ_1) and (η_2, ψ_2) , satisfying the Cauchy problem (1.1), with the same initial data (η_0, ψ_0) , and satisfying the regularity result (1.8), then they are both solutions satisfying trivially (1.5).

Passing to the limit. It remains to prove the existence part of the result. For the convenience of the readers, let us recall that we have proved the existence

of approximate solutions $(\eta_n \psi_n)$ to the Cauchy problem

$$(6.79) \quad \begin{cases} \partial_t \eta_n = J_n G(\eta_n)(\psi_n, b), \\ \partial_t \psi_n = J_n \left(-g\eta_n - \frac{1}{2} |\nabla_x \psi_n|^2 + \frac{(G(\eta_n)(\psi_n, b) + \nabla_x \eta_n \cdot \nabla_x \psi_n)^2}{2(1 + |\nabla_x \eta_n|^2)} \right), \\ \eta_n|_{t=0} = J_n \eta_0, \quad \psi_n|_{t=0} = J_n \psi_0, \end{cases}$$

where J_n is a truncation in frequency space defined in (6.1). In this paragraph, we shall prove that we can extract a sub-sequence of $((\eta_{n'}, \psi_{n'}))$ that converges weakly to a solution of the water-wave system (thanks to the uniqueness of the solution to the water-wave system, this will imply that the whole sequence converges, without extraction of a sub-sequence). This part relies on classical arguments from functional analysis, but since we work in analytic spaces and since the problem is nonlinear and nonlocal, some verifications are needed.

Recall from Lemma 6.1 and Corollary 6.9 that there exist four positive real numbers ε_* , c_* , C_* and K_* such that for all $n \in \mathbf{N} \setminus \{0\}$, the following properties hold: if the initial norm ε (as defined by (6.32)) satisfies $\varepsilon \leq \varepsilon_*$, then for all $n \in \mathbf{N} \setminus \{0\}$, the lifespan is bounded from below by $T_n \geq \frac{c_*}{\varepsilon}$, and moreover,

$$M_{s,n} \left(\frac{c_*}{\varepsilon} \right) \leq C_* \varepsilon,$$

where the norm $M_{s,n}(T)$ is defined by

$$M_{s,n}(T) = \|\eta_n\|_{X_T^{\infty, s+\frac{1}{2}}} + \|a(D_x)^{\frac{1}{2}} \psi_n\|_{X_T^{\infty, s}} + \|V_n\|_{X_T^{\infty, s}} + \|B_n\|_{X_T^{\infty, s}}.$$

Here we put

$$X_T^{\infty, s} = L^\infty([0, T], \mathcal{H}^{\sigma, s}) \quad \text{with} \quad \sigma(t) = \lambda h - K_* \varepsilon t.$$

Let us notice that Theorem 3.1 implies that

$$\|G(\eta_n)(\psi_n, b)\|_{X_{c_*/\varepsilon}^{\infty, s-\frac{1}{2}}} \leq C'_* \varepsilon,$$

for some constant C'_* independent of ε and n . Then, by using the product rule given by point A.7) in Proposition A.7, as we already did repeatedly in the previous paragraph, we infer from the equations (6.79) that

$$\|\partial_t \eta_n\|_{X_{c_*/\varepsilon}^{\infty, s-\frac{1}{2}}} + \|\partial_t \psi_n\|_{X_{c_*/\varepsilon}^{\infty, s-\frac{1}{2}}} \leq C''_* \varepsilon.$$

Thanks to the Arzela-Ascoli theorem and the compact embedding of $\mathcal{H}^{\lambda h, s}(\mathbf{T}^d)$ in $\mathcal{H}^{\lambda h, s'}(\mathbf{T}^d)$ for $s' < s$, there exist a sub-sequence $((\eta_{n'}, \psi_{n'}))$ and the limit (η, ψ) such that $(\eta_{n'}, \psi_{n'})$ converges to (η, ψ) in $X_{c_*/\varepsilon}^{\infty, s'}$. Now, the contraction result for the Dirichlet-to-Neumann operator given by Theorem 3.4 implies that the sequence $(G(\eta_{n'})(\psi_{n'}, b))$ converges to $G(\eta)(\psi, b)$. Thus we conclude that the limit $(\eta, \psi) \in X_{c_*/\varepsilon}^{\infty, s'}$ solves the water-waves equations.

APPENDIX A. SOME PROPERTIES OF THE $\mathcal{H}^{\sigma, s}$ SPACES

A.1. Characterization. In this paragraph we prove Theorem 1.1 whose statement is recalled below, together with the fact that functions in $\mathcal{H}^{\sigma, s}(\mathbf{T}^d)$ are

the traces on \mathbf{T}^d of holomorphic functions in

$$S_\sigma = \{(x, y) \in \mathbf{T}^d \times \mathbf{R}^d : |y| < \sigma\},$$

where

$$|y| = \left(\sum_{j=1}^d y_j^2 \right)^{\frac{1}{2}}.$$

Recall that, given $U : S_\sigma \rightarrow \mathbf{C}$, we denote by U_y the function from \mathbf{T}^d to \mathbf{C} defined by $x \mapsto U(x + iy)$.

Theorem A.1. *Let $\sigma > 0$ and $s \in \mathbf{R}$.*

(1) *Let $u \in \mathcal{H}^{\sigma, s}(\mathbf{T}^d)$. There exists $U \in \mathcal{Hol}(S_\sigma)$ such that $U_0 = u$ and*

$$\sup_{|y| < \sigma} \|U_y\|_{H_x^s(\mathbf{T}^d)} \leq \|u\|_{\mathcal{H}^{\sigma, s}}.$$

(2) *Let $U \in \mathcal{Hol}(S_\sigma)$ such that $M_0 := \sup_{|y| < \sigma} \|U_y\|_{H_x^s(\mathbf{T}^d)} < +\infty$. Set $u = U_0$. Then,*

(i) *If $d = 1$, then u belongs to $\mathcal{H}^{\sigma, s}(\mathbf{T}^d)$ and $\|u\|_{\mathcal{H}^{\sigma, s}} \leq 2M_0$.*

(ii) *If $d \geq 2$, then u belongs to $\mathcal{H}^{\delta, s}(\mathbf{T}^d)$ for any $\delta < \sigma$ and there exists a constant $C_\delta > 0$ such that $\|u\|_{\mathcal{H}^{\delta, s}} \leq C_\delta M_0$.*

(3) *Let $U \in \mathcal{Hol}(S_\sigma)$ be such that*

$$M_1 := \sup_{|y| < \sigma} \|U_y\|_{H_x^{s'}(\mathbf{T}^d)} < +\infty \quad \text{with} \quad s' > s + \frac{d-1}{4}.$$

Then the function $u = U_0$ belongs to $\mathcal{H}^{\sigma, s}(\mathbf{T}^d)$ and there exists a constant $C > 0$ such that $\|u\|_{\mathcal{H}^{\sigma, s}} \leq CM_1$.

Proof. We divide the proof of Theorem A.1 into six steps.

Step 1: Existence of an holomorphic extension. Let us prove statement (1). Fix $\sigma > 0$ and $s \in \mathbf{R}$ and consider a function $u \in \mathcal{H}^{\sigma, s}(\mathbf{T}^d)$. We want to prove that there exists $U \in \mathcal{Hol}(S_\sigma)$ such that $U_0 = u$ and

$$\sup_{|y| < \sigma} \|U_y\|_{H_x^s(\mathbf{T}^d)} \leq \|u\|_{\mathcal{H}^{\sigma, s}}.$$

We begin by observing that, for any $z \in S_\sigma$, the function $\mathbf{Z}^d \rightarrow \mathbf{C}$, $\xi \mapsto e^{iz \cdot \xi} \widehat{u}(\xi)$ belongs to $\ell^1(\mathbf{Z}^d)$. To see this, we write

$$|e^{iz \cdot \xi} \widehat{u}(\xi)| = e^{-y \cdot \xi} |\widehat{u}(\xi)| = [\langle \xi \rangle^{-s} e^{-\sigma |\xi|} e^{-y \cdot \xi}] \times [\langle \xi \rangle^s e^{\sigma |\xi|} |\widehat{u}(\xi)|] := f_1(\xi) \times f_2(\xi),$$

and then conclude since $|f_1(\xi)| \leq \langle \xi \rangle^{-s} e^{-(\sigma - |y|) |\xi|} \in \ell^2(\mathbf{Z}^d)$ and $f_2 \in \ell^2(\mathbf{Z}^d)$, by assumption on u . So we can define the function

$$U(z) = (2\pi)^{-d} \sum_{\xi \in \mathbf{Z}^d} e^{iz \cdot \xi} \widehat{u}(\xi).$$

The previous inequality implies that, for any $\varepsilon > 0$,

$$\sup_{z \in S_{\sigma - \varepsilon}} \sum_{\xi \in \mathbf{Z}^d} |e^{iz \cdot \xi} \widehat{u}(\xi)| < +\infty.$$

Since the function $z \mapsto e^{iz \cdot \xi} \widehat{u}(\xi)$ is holomorphic, we deduce that $U \in \mathcal{H}ol(S_\sigma)$.

We next observe that the Fourier inversion formula implies that $U_0 = \overline{\mathcal{F}}\widehat{u} = u$. In addition, we have $U(x + iy) = \overline{\mathcal{F}}(e^{-y \cdot \xi} \widehat{u})$, hence $\widehat{U}_y = e^{-y \cdot \xi} \widehat{u}$ which in turn implies that

$$\|U_y\|_{H^s(\mathbf{T}^d)}^2 = \sum_{\xi \in \mathbf{Z}^d} \langle \xi \rangle^{2s} e^{-2y \cdot \xi} |\widehat{u}(\xi)|^2 \leq \sum_{\xi \in \mathbf{Z}^d} \langle \xi \rangle^{2s} e^{2\sigma|\xi|} |\widehat{u}(\xi)|^2 = \|u\|_{\mathcal{H}^{\sigma,s}}^2.$$

This completes the proof of statement (1).

Step2: A Sobolev estimate. For later purpose, let us prove an additional estimate. Consider a real number $s_0 > d/2$ and write

$$e^{-y \cdot \xi} |\widehat{u}(\xi)| = \langle \xi \rangle^{-s_0} e^{-y \cdot \xi} e^{-\sigma|\xi|} e^{\sigma|\xi|} \langle \xi \rangle^{s_0} |\widehat{u}(\xi)|.$$

Now, compared to the previous proof, we see that the factor $\langle \xi \rangle^{-s_0} e^{-y \cdot \xi} e^{-\sigma|\xi|}$ is summable in ξ , uniformly for $|y| \leq \sigma$, thanks to the assumption $s_0 > d/2$. Then, the Cauchy-Schwarz inequality implies that there exists a positive constant $C = C(d, s_0)$ such that

$$\|U\|_{L^\infty(S_\sigma)} = \sup_{|y| < \sigma} \|U_y\|_{L^\infty(\mathbf{T}^d)} \leq C \|u\|_{\mathcal{H}^{\sigma,s_0}}.$$

Step 3: Trace of an holomorphic function. In this step, we initiate the proof of the various points in statement (2). Consider a function $U \in \mathcal{H}ol(S_\sigma)$ such that

$$(A.1) \quad M_0 := \sup_{|y| < \sigma} \|U_y\|_{H_x^s(\mathbf{T}^d)} < +\infty.$$

We want to study the regularity of the trace $u = U_0$.

First, observe that the assumption (A.1) implies that $u = U_0 \in H^s(\mathbf{T}^d)$.

Now, let $\psi = 1_{|k| \leq 1}$. Given $\lambda > 0$, we introduce the functions $\psi_\lambda(\xi) = \psi(\frac{\xi}{\lambda})$ and $\varphi_\lambda = \overline{\mathcal{F}}\psi_\lambda$. Set

$$F_\lambda(z) = (2\pi)^{-d} \sum_{\xi \in \mathbf{T}^d} e^{iz \cdot \xi} \psi_\lambda(\xi) \widehat{u}(\xi).$$

This function is holomorphic in S_σ . Indeed, the summand is holomorphic and, for $z \in S_\sigma$,

$$\begin{aligned} |e^{iz \cdot \xi} \psi_\lambda(\xi) \widehat{u}(\xi)| &= e^{-(\text{Im } z) \cdot \xi} \psi_\lambda(\xi) |\widehat{u}(\xi)| \leq e^{\sigma|\xi|} \psi_\lambda(\xi) |\widehat{u}(\xi)| \leq e^{2\sigma\lambda} \psi_\lambda(\xi) |\widehat{u}(\xi)|, \\ &\leq [e^{2\sigma\lambda} \langle \xi \rangle^{-s} \psi_\lambda(\xi)] \times [\langle \xi \rangle^s |\widehat{u}(\xi)|] \in \ell^1(\mathbf{Z}^d). \end{aligned}$$

Notice that

$$F_\lambda(x + iy) = \overline{\mathcal{F}}(e^{-y \cdot \xi} \psi_\lambda(\xi) \widehat{u}(\xi))(x).$$

For $z = x + iy \in S_\sigma$ we set

$$V_\lambda(z) = U_y \star \varphi_\lambda(x) = \int_{\mathbf{T}^d} U(x + iy - t) \varphi_\lambda(t) dt.$$

This function is holomorphic in the strip S_σ . In addition,

$$V_\lambda|_{y=0} = u \star \varphi_\lambda = \overline{\mathcal{F}}(\psi_\lambda \widehat{u}) = (2\pi)^{-d} \sum_{\xi \in \mathbf{Z}^d} e^{ix \cdot \xi} \psi_\lambda \widehat{u} = F_\lambda|_{y=0}.$$

By the uniqueness for analytic functions, this implies that $V_\lambda = F_\lambda$ in S_σ . By taking the Fourier transform of the previous identity, we obtain

$$\mathcal{F}U_y(\xi) \psi_\lambda(\xi) = e^{-y \cdot \xi} \psi_\lambda(\xi) \widehat{u}(\xi).$$

By letting λ goes to $+\infty$, we infer that $\mathcal{F}U_y(\xi) = e^{-y \cdot \xi} \widehat{u}(\xi)$. Consequently,

$$\|U_y\|_{H^s(\mathbf{T}^d)}^2 = \sum_{\xi \in \mathbf{Z}^d} e^{-2y \cdot \xi} \langle \xi \rangle^{2s} |\widehat{u}(\xi)|^2.$$

Then the assumption on U implies that

$$(A.2) \quad \sup_{|y| < \sigma} \sum_{\xi \in \mathbf{Z}^d} e^{-2y \cdot \xi} \langle \xi \rangle^{2s} |\widehat{u}(\xi)|^2 = M_0^2 < +\infty.$$

This is the key ingredient to prove the two statements in point (2).

Step 4: Trace of an holomorphic function in dimension one.

Assume that $d = 1$. Set $v(\xi) = \langle \xi \rangle^{2s} |\widehat{u}(\xi)|^2$. For any real number $0 < b < \sigma$, the inequality (A.2) applied with $y = -b$ (resp. $y = b$) implies that

$$\sum_{\xi=0}^{+\infty} e^{2b\xi} v(\xi) \leq M_0, \quad \text{resp.} \quad \sum_{\xi=-\infty}^0 e^{-2b\xi} v(\xi) \leq M_0.$$

It follows that $\sum_{\xi \in \mathbf{Z}} e^{2b|\xi|} v(\xi) \leq 2M_0$. Fatou's lemma then implies that, when b goes to σ , we have

$$\sum_{\xi \in \mathbf{Z}} e^{2\sigma|\xi|} v(\xi) \leq 2M_0,$$

which proves statement (i).

Step 5: Arbitrary dimension.

Let us prove statement (ii). We now assume that $d \geq 2$ and consider a real number $\delta < \sigma$. We can write $\delta = (1 - \frac{\varepsilon^2}{2})\sigma$ for some $\varepsilon > 0$. Then there exists $N = N(\varepsilon)$ and $\omega_1, \dots, \omega_N \in \mathbf{S}^{d-1}$ such that

$$\mathbf{Z}^d \setminus \{0\} = \bigcup_{j=1}^N \Gamma_j \quad \text{where} \quad \Gamma_j = \left\{ \xi \in \mathbf{Z}^d : \left| \frac{\xi}{|\xi|} - \omega_j \right| < \varepsilon \right\}.$$

Notice that

$$\xi \in \Gamma_j \quad \Rightarrow \quad \left| \frac{\xi}{|\xi|} \right|^2 + |\omega_j|^2 - 2 \frac{\xi}{|\xi|} \cdot \omega_j < \varepsilon^2 \quad \Rightarrow \quad \left(1 - \frac{\varepsilon^2}{2}\right) |\xi| \leq \xi \cdot \omega_j.$$

Consider $0 < b < \delta$ and set $\tilde{b} = \frac{b}{1 - \varepsilon^2/2} < \sigma$. We have

$$\sum_{\mathbf{Z}^d} e^{2b|\xi|} v(\xi) \leq \sum_{j=1}^N \sum_{\Gamma_j} e^{2b|\xi|} v(\xi) \leq \sum_{j=1}^N \sum_{\Gamma_j} e^{2\tilde{b}(\xi \cdot \omega_j)} v(\xi).$$

Since the vector $y_j = \tilde{b}\omega_j$ satisfies $|y_j| := \tilde{b}|\omega_j| < \sigma$, the key estimate (A.2) implies that

$$\sum_{\mathbf{z}^d} e^{2b|\xi|} v(\xi) \leq NM_0^2.$$

As above, we conclude by using Fatou's lemma, which implies that

$$\sum_{\mathbf{z}^d} e^{2\delta|\xi|} v(\xi) \leq NM_0^2,$$

which concludes the proof of statement (ii).

Final step: Arbitrary dimension, sharp estimate.

We now prove statement (3), which gives a smaller loss in analyticity. Namely, we assume that, $U \in \mathcal{H}ol(S_\sigma)$ is such that

$$M_1 := \sup_{|y| < \sigma} \|U_y\|_{H_x^{s'}} < +\infty \quad \text{with} \quad s' > s + \frac{d-1}{4}.$$

Our goal is to prove that, the function $u = U_0$ belongs to $\mathcal{H}^{\sigma, s}$ and there exists a constant $C > 0$ such that $\|u\|_{\mathcal{H}^{\sigma, s}} \leq CM_1$.

Let $b < \sigma$ and $s' > s + \frac{d-1}{4}$. By replacing s by s' in (A.2) we have

$$(A.3) \quad \sup_{|y| < \sigma} \sum_{\mathbf{z}^d} e^{-2y \cdot \xi} \langle \xi \rangle^{2s'} |\hat{u}(\xi)|^2 = \sup_{|y| < \sigma} \|U_y\|_{H^{s'}}^2 = M_1^2 < +\infty.$$

Set $v_s(\xi) = \langle \xi \rangle^{2s} |\hat{u}(\xi)|^2$ and consider a real number R_0 such that $\sigma R_0 \gg 1$. We first notice that

$$(A.4) \quad \sum_{|\xi| \leq R_0} e^{2b|\xi|} v_s(\xi) \leq e^{2bR_0} \sum v_s(\xi) \leq e^{2bR_0} \sum \langle \xi \rangle^{\frac{d-1}{2}} v_s(\xi) \leq e^{2bR_0} M_1^2.$$

Define ℓ_0 as the largest integer such that $\frac{1}{2}2^{\ell_0} \leq R_0$ and then, for $\ell \geq \ell_0$, introduce the dyadic rings:

$$\mathcal{C}_\ell = \{\xi \in \mathbf{Z}^d : \frac{1}{2}2^\ell \leq |\xi| \leq 2^{\ell+1}\}.$$

Write

$$(A.5) \quad \sum_{|\xi| > R_0} e^{2b|\xi|} v_s(\xi) \leq \sum_{\ell=\ell_0}^{+\infty} \sum_{\mathcal{C}_\ell} e^{2b|\xi|} v_s(\xi) := \sum_{\ell=\ell_0}^{+\infty} I_\ell.$$

Fix $\ell \geq \ell_0$ and set $\delta_\ell = \frac{1}{b2^\ell} \ll 1$. Let ω_0 be an arbitrary point on the sphere \mathbf{S}^{d-1} and introduce

$$\Omega_{\omega_0} = \{\omega \in \mathbf{S}^{d-1} : \omega \cdot \omega_0 > 1 - \delta_\ell\} = \{\omega \in \mathbf{S}^{d-1} : |\omega - \omega_0| < \sqrt{2\delta_\ell}\}.$$

The sets Ω_{ω_0} have a $(d-1)$ -dimensional measure independent of ω_0 . Moreover, there exists two positive constants c_1, c_2 independent of the dimension such that

$$c_1 \delta_\ell^{\frac{d-1}{2}} \leq \mu(\Omega_{\omega_0}) \leq c_2 \delta_\ell^{\frac{d-1}{2}}.$$

There exists $\omega_1, \dots, \omega_{N_\ell} \in \mathbf{S}^{d-1}$ where $N_\ell \sim C_d \delta_\ell^{-\frac{d-1}{2}}$ such that $\mathbf{S}^{d-1} = \cup_{j=1}^{N_\ell} \Omega_{\omega_j}$. Set

$$\mathcal{C}_{\ell,j} = \left\{ \xi \in \mathcal{C}_\ell : \frac{\xi}{|\xi|} \in \Omega_{\omega_j} \right\}.$$

Then one can split the dyadic ring, \mathcal{C}_ℓ as $\mathcal{C}_\ell = \cup_{j=1}^{N_\ell} \mathcal{C}_{\ell,j}$, to obtain

$$I_\ell \leq \sum_{j=1}^{N_\ell} \sum_{\mathcal{C}_{\ell,j}} e^{2b|\xi|} v_s(\xi) := \sum_{j=1}^{N_\ell} I_{\ell,j}.$$

If ξ belongs to $\mathcal{C}_{\ell,j}$ one has, $\frac{\xi}{|\xi|} \cdot \omega_j > 1 - \delta_\ell$, so $|\xi| < \xi \cdot \omega_j + |\xi| \delta_\ell$. Then we write

$$I_\ell \leq \frac{1}{N_\ell} \sum_{j=1}^{N_\ell} \sum_{\mathcal{C}_{\ell,j}} e^{2(b\omega_j) \cdot \xi + 2b|\xi| \delta_\ell} N_\ell v_s(\xi).$$

Recall that $\delta_\ell = 1/(b2^\ell)$. Consequently, if ξ in $\mathcal{C}_{\ell,j}$, we have $2b|\xi| \delta_\ell \leq 4$. We deduce that $e^{2b|\xi| \delta_\ell} \leq e^4$. Moreover,

$$N_\ell \leq C_d \delta_\ell^{-\frac{d-1}{2}} \leq C_d (b2^\ell)^{\frac{d-1}{2}} \leq C_d (2b)^{\frac{d-1}{2}} |\xi|^{\frac{d-1}{2}},$$

and $\ell \leq c \log |\xi|$.

Remembering that $v_s(\xi) = \langle \xi \rangle^{2s} |\widehat{u}(\xi)|^2$, we deduce that, for any $\varepsilon > 0$,

$$I_\ell \leq \frac{C'_d b^{\frac{d-1}{2}}}{\ell^{1+\varepsilon}} \frac{1}{N_\ell} \sum_{j=1}^{N_\ell} \sum_{\mathcal{C}_{\ell,j}} e^{2(b\omega_j) \cdot \xi} |\xi|^{\frac{d-1}{2}} (\log |\xi|)^{1+\varepsilon} \langle \xi \rangle^{2s} |\widehat{u}(\xi)|^2.$$

Now since $s' > s + \frac{d-1}{4}$, there exists $\varepsilon > 0$ so small enough that $s' \geq s + \varepsilon$. We use (A.3) to infer that

$$(A.6) \quad I_\ell \leq \frac{C'_d b^{\frac{d-1}{2}}}{\ell^{1+\varepsilon}} M_1^2.$$

Since $\sum_{\ell \geq 1} \ell^{-1-\varepsilon} < +\infty$, it follows from (A.5) that

$$\sum_{|\xi| > R_0} e^{2b|\xi|} \langle \xi \rangle^{2s} |\widehat{u}(\xi)|^2 \leq C M_1^2.$$

Using (A.4) we obtain eventually,

$$\sum_{\mathbf{Z}^d} e^{2b|\xi|} \langle \xi \rangle^{2s} |\widehat{u}(\xi)|^2 \leq C M_1^2.$$

Again, we conclude the proof thanks to Fatou's lemma. This completes the proof of Theorem A.1. \square

Notice that, if u is radial, then one can remove the factor $(\log |\xi|)^{1+\varepsilon}$ in (A.6) and hence it is sufficient to assume that

$$M_1 = \sup_{|y| < \sigma} \|U_y\|_{H_x^{s+\frac{d-1}{4}}} < +\infty.$$

Notice also that, in this case, the above assumption on U_y is optimal to insure that $u \in \mathcal{H}^{\sigma,s}$.

A.2. Technical lemmas. We recall the following interpolation lemma.

Lemma A.2. *Let $s \in \mathbf{R}$ and $h > 0$. There exists $C > 0$ such that for all $\sigma \geq 0$, if f is in $L^2((-h, 0), \mathcal{H}^{\sigma, s + \frac{1}{2}})$ and $\partial_z f$ is in $L^2((-h, 0), \mathcal{H}^{\sigma, s - \frac{1}{2}})$, then f belongs to $C^0([-h, 0], \mathcal{H}^{\sigma, s})$ together with the estimate*

$$\sup_{z \in [-h, 0]} \|f(z, \cdot)\|_{\mathcal{H}^{\sigma, s}} \leq C \left(\|f\|_{L^2((-h, 0), \mathcal{H}^{\sigma, s + \frac{1}{2}})} + \|\partial_z f\|_{L^2((-h, 0), \mathcal{H}^{\sigma, s - \frac{1}{2}})} \right).$$

We shall use the following lemma from Hörmander (see [21, Theorem 8.3.1]).

Lemma A.3. *Consider three real numbers s_1, s_2, s_3 such that*

$$s_1 + s_2 \geq 0, \quad s_3 \leq \min\{s_1, s_2\}, \quad s_3 \leq s_1 + s_2 - \frac{d}{2},$$

with the last inequality strict if s_1 or s_2 or $-s_3$ is equal to $d/2$. For ξ, ζ in \mathbf{R}^d , define

$$F(\xi, \zeta) = \frac{\langle \xi \rangle^{s_3}}{\langle \xi - \zeta \rangle^{s_1} \langle \zeta \rangle^{s_2}},$$

and then set

$$T_F(f, g)(\xi) = \sum_{\zeta \in \mathbf{Z}^d} F(\xi, \zeta) f(\xi - \zeta) g(\zeta),$$

when f and g are continuous functions with compact support. Then there exists a positive constant C such that

$$\|T_F(f, g)\|_{\ell^2} \leq C \|f\|_{\ell^2} \|g\|_{\ell^2}.$$

A.3. Paradifferential operators on analytic spaces and non linear estimates. In this section we investigate the continuity of the paradifferential operators on the spaces $\mathcal{H}^{\sigma, s}$ and we apply these results to prove nonlinear estimates in these spaces.

We will use the following lemma.

Lemma A.4. *Let $s_0 > \frac{d}{2}$. There exists $C > 0$ such that for all $\sigma > 0$, and for all $f, g \in \mathcal{H}^{\sigma, s_0}$, we have*

$$(A.7) \quad \|fg\|_{\mathcal{H}^{\sigma, 0}} \leq C \|f\|_{\mathcal{H}^{\sigma, s_0}} \|g\|_{\mathcal{H}^{\sigma, 0}}.$$

In addition, if $\text{supp } \widehat{f} \subset \{\xi : |\xi| \leq R\}$, then

$$(A.8) \quad \|fg\|_{\mathcal{H}^{\sigma, 0}} \leq CR^{\frac{d}{2}} \|f\|_{\mathcal{H}^{\sigma, 0}} \|g\|_{\mathcal{H}^{\sigma, 0}}.$$

Proof. Set $U = e^{\sigma|D|}(fg)$. Then

$$\widehat{U}(\xi) = e^{\sigma|\xi|} \sum_{\eta \in \mathbf{Z}^d} \widehat{f}(\xi - \eta) \widehat{g}(\eta),$$

which implies that

$$|\widehat{U}(\xi)| \leq \sum_{\eta \in \mathbf{Z}^d} e^{\sigma|\xi - \eta|} |\widehat{f}(\xi - \eta)| e^{\sigma|\eta|} |\widehat{g}(\eta)| = |\widehat{e^{\sigma|D|}f}| \star |\widehat{e^{\sigma|D|}g}|.$$

Therefore,

$$(A.9) \quad \|U\|_{L^2} \leq C \|\widehat{e^{\sigma|D|}f}\|_{\ell^1} \|\widehat{e^{\sigma|D|}g}\|_{\ell^2} \leq C' \|\widehat{e^{\sigma|D|}f}\|_{\ell^1} \|g\|_{\mathcal{H}^{\sigma, 0}}.$$

If $s_0 > \frac{d}{2}$, we have by using the Hölder inequality,

$$\|u\|_{\ell^1} \leq C_{s_0} \|\langle \cdot \rangle^{s_0} u\|_{\ell^2},$$

which proves the required inequality (A.7).

To prove (A.8) we introduce a function $\psi \in C_0^\infty(\mathbf{R}^d)$ such that $\text{supp } \psi \subset \{|\xi| \leq 2\}$ and $\psi = 1$ for $|\xi| \leq 1$. Then

$$\begin{aligned} \sum_{\xi \in \mathbf{Z}^d} e^{\sigma|\xi|} |\widehat{f}(\xi)| &= \sum_{\xi \in \mathbf{Z}^d} e^{\sigma|\xi|} \psi\left(\frac{\xi}{R}\right) |\widehat{f}(\xi)| \leq \left(\sum_{\xi \in \mathbf{Z}^d} \psi\left(\frac{\xi}{R}\right)^2 \right)^{\frac{1}{2}} \|f\|_{\mathcal{H}^{\sigma,0}} \\ &\leq CR^{\frac{d}{2}} \|f\|_{\mathcal{H}^{\sigma,0}}, \end{aligned}$$

and we conclude the proof by using (A.9). \square

Let us introduce a symbol class. Given $m \in \mathbf{R}, \sigma > 0, s \in \mathbf{R}$, $\Gamma^{m,\sigma,s}$ denotes the space of those functions p that are C^∞ on $\mathbf{T}^d \times \mathbf{R}^d$, of the form

$$p(x, \xi) \sim \sum_{j=0}^{+\infty} p_{m-j}(x, \xi),$$

where p_{m-j} is homogeneous of order $m-j$ in ξ for $|\xi| \geq 1$, and satisfies

$$\mathcal{N}_s(p) := \|\langle D \rangle^s e^{\sigma|D|} (1 - \Delta_\omega)^k p(x, \omega)\|_{L^2(\mathbf{T}^d \times \mathbf{S}^{d-1})} < +\infty,$$

with $k > \frac{d}{2}$.

We denote by T_p the paradifferential operator associated with this symbol. By definition:

$$(A.10) \quad \widehat{T_p u}(\xi) = \sum_{\eta \in \mathbf{Z}^d} \chi(\xi - \eta, \eta) \widehat{p}(\xi - \eta, \xi) \psi(\eta) \widehat{u}(\eta),$$

where $\chi \in C^\infty(\mathbf{R}^d \times \mathbf{R}^d)$, $\chi(\theta, \eta) = 1$ if $|\theta| \leq \varepsilon_1 |\eta|$, $\chi(\theta, \eta) = 0$ if $|\theta| \geq \varepsilon_2 |\eta|$, $0 < \varepsilon_1 < \varepsilon_2 < 1$ and where \widehat{p} is the Fourier transform of p with respect to x , whereas $\psi \in C^\infty(\mathbf{R}^d)$ is a cut-off function such that $\psi(\xi) = 0$ for $|\xi| \leq 1$ and $\psi(\xi) = 1$ for $|\xi| \geq 2$.

Theorem A.5. *Let $s_0 > \frac{d}{2}$. For all $s \in \mathbf{R}$ there exists a constant $C > 0$ such that for all $p \in \Gamma^{m,\sigma,s_0}$ and all $u \in \mathcal{H}^{\sigma,s+m}$ we have*

$$\|T_p u\|_{\mathcal{H}^{\sigma,s}} \leq C \mathcal{N}_{s_0}(p) \|u\|_{\mathcal{H}^{\sigma,s+m}}.$$

Proof. (i) Suppose $p(x, \xi) = a(x)h(\xi)$ where h is homogeneous of degree m , so that $h(\xi) = |\xi|^m \widetilde{h}\left(\frac{\xi}{|\xi|}\right)$.

In the formula (A.10) on the support of χ we have $|\xi| \sim |\eta|$ so that we can write

$$e^{\sigma|\xi|} \langle \xi \rangle^s |\widehat{T_p u}(\xi)| \leq C \sum_{\eta \in \mathbf{Z}^d} e^{\sigma|\xi-\eta|} |\widehat{a}(\xi-\eta)| \psi(\eta) \langle \eta \rangle^{s+m} \left| \widetilde{h}\left(\frac{\eta}{|\eta|}\right) \right| e^{\sigma|\eta|} |\widehat{u}(\eta)|.$$

Since the right-hand side is a convolution, we have

$$\|e^{\sigma|\xi|} \langle \xi \rangle^s \widehat{T_p u}\|_{\ell^2} \leq \|e^{\sigma|\cdot|} \widehat{a}\|_{\ell^1} \|\widetilde{h}\|_{L^\infty(\mathbf{S}^{d-1})} \|u\|_{\mathcal{H}^{\sigma,s+m}}.$$

For $s_0 > \frac{d}{2}$, it follows from the Cauchy-Schwarz inequality that

$$\|e^{\sigma|\cdot|}\widehat{a}\|_{\ell^1} \leq C\|e^{\sigma|\cdot|}\langle\cdot\rangle^{s_0}\widehat{a}\|_{\ell^2},$$

so

$$(A.11) \quad \|e^{\sigma|\xi|}\langle\xi\rangle^s\widehat{T_p u}(\xi)\|_{\ell^2} \leq C\|e^{\sigma|\xi|}\langle\xi\rangle^{s_0}\widehat{a}\|_{\ell^2}\|\widetilde{h}\|_{L^\infty(\mathbf{S}^{d-1})}\|u\|_{\mathcal{H}^{\sigma,s+m}}.$$

(ii) Let $(\widetilde{h}_\nu)_{\nu \geq 1}$ be an orthonormal basis of $L^2(\mathbf{S}^{d-1})$ with $-\Delta_\omega \widetilde{h}_\nu = \lambda_\nu \widetilde{h}_\nu$. Let $p(x, \omega) = \sum_\nu a_\nu(x) \widetilde{h}_\nu(\omega)$. Then

$$a_\nu(x) = \int_{\mathbf{S}^{d-1}} p(x, \omega) \overline{\widetilde{h}_\nu(\omega)} d\omega.$$

As a result, for $k \in \mathbf{N}$, we have

$$\begin{aligned} \lambda_\nu^{2k} \langle D \rangle^{s_0} e^{\sigma|D|} a_\nu(x) &= \int_{\mathbf{S}^{d-1}} \langle D \rangle^{s_0} e^{\sigma|D|} p(x, \omega) (-\Delta_\omega)^k \overline{\widetilde{h}_\nu(\omega)} d\omega \\ &= \int_{\mathbf{S}^{d-1}} (-\Delta_\omega)^k \langle D \rangle^{s_0} e^{\sigma|D|} p(x, \omega) \overline{\widetilde{h}_\nu(\omega)} d\omega, \end{aligned}$$

whence

$$(A.12) \quad \lambda_\nu^{2k} \|\langle D \rangle^{s_0} e^{\sigma|D|} a_\nu\|_{L^2} \leq \|\Delta_\omega^k \langle D \rangle^{s_0} e^{\sigma|D|} p(x, \cdot)\|_{L^2(\mathbf{T}^d \times \mathbf{S}^{d-1})}.$$

On the other hand,

$$(A.13) \quad \|\widetilde{h}_\nu\|_{L^\infty(\mathbf{S}^{d-1})} \leq C\|\widetilde{h}_\nu\|_{H^{s_1}(\mathbf{S}^{d-1})} \leq C'\lambda_\nu^{s_1}, \quad s_1 = \frac{d-1}{2} + \varepsilon.$$

By using (A.11), (A.12) and (A.13), we conclude that

$$\begin{aligned} \|T_p u\|_{\mathcal{H}^{\sigma,s}} &\leq \sum_{\nu=1}^{+\infty} \|T_{a_\nu h_\nu} u\|_{\mathcal{H}^{\sigma,s}} \leq C \sum_{\nu=1}^{+\infty} \|a_\nu\|_{\mathcal{H}^{\sigma,s_0}} \|\widetilde{h}_\nu\|_{L^\infty(\mathbf{S}^{d-1})} \|u\|_{\mathcal{H}^{\sigma,s+m}}, \\ &\leq C \sum_{\nu=1}^{+\infty} \lambda_\nu^{-2k} \|\Delta_\omega^k \langle D \rangle^{s_0} e^{\sigma|D|} p(x, \cdot)\|_{L^2(\mathbf{T}^d \times \mathbf{S}^{d-1})} \lambda_\nu^{s_1} \|u\|_{\mathcal{H}^{\sigma,s+m}}. \end{aligned}$$

As $\lambda_\nu \sim C_d \nu^{\frac{2}{d}}$, we have $\lambda_\nu^{s_1-2k} \sim C'_d \nu^{\frac{2}{d}(s_1-2k)}$. If $k > \frac{d}{2} - \frac{1}{4}$ (and $\varepsilon > 0$ in (A.13) is small enough), we have $2(s_1 - 2k) < -d$ and the series in ν converges. Eventually, we obtain that

$$\|T_p u\|_{\mathcal{H}^{\sigma,s}} \leq C_d \|\Delta_\omega^k \langle D \rangle^{s_0} e^{\sigma|D|} p(x, \cdot)\|_{L^2(\mathbf{T}^d \times \mathbf{S}^{d-1})} \|u\|_{\mathcal{H}^{\sigma,s+m}}.$$

□

In the sequel, we will also use the Littlewood-Paley decomposition (see [36]) and consider the paraproduct. Notice that when $p = a$ is a function (thus independent of ξ), the paradifferential operator T_a with symbol a is called a paraproduct. Modulo a regularizing operator it can be defined as follows.

Consider a partition of unity,

$$1 = \sum_{j \geq -1} \chi_j(\xi), \quad \forall j \geq 1, \chi_j(x) = \chi_0(2^{-j}\xi),$$

with $\chi_0 \in C_0^\infty(\{\frac{1}{2} \leq |\xi| \leq 2\})$. Recall that if a, u are two functions, then

$$\Delta_j u = \chi_j(D_x)u, \quad S_j(u) = \sum_{-1 \leq k \leq j-1} \Delta_k u, \quad \text{and } T_a u = \sum_{j \geq 2} S_{j-2}(a) \Delta_j u.$$

Proposition A.6. *Consider three real numbers α, β, γ such that*

$$\alpha \leq \gamma, \quad \alpha < \beta + \gamma - \frac{d}{2}.$$

There exists $C > 0$ such that for all $a \in \mathcal{H}^{\sigma, \beta}, u \in \mathcal{H}^{\sigma, \gamma}$, we have

$$\|T_a u\|_{\mathcal{H}^{\sigma, \alpha}} \leq C \|a\|_{\mathcal{H}^{\sigma, \beta}} \|u\|_{\mathcal{H}^{\sigma, \gamma}}.$$

Proof. Case 1: $\beta > \frac{d}{2}$.

We apply Theorem A.5 with $s_0 = \beta, p = a, m = 0$. Since $\alpha \leq \gamma$, it follows that

$$\|T_a u\|_{\mathcal{H}^{\sigma, \alpha}} \leq C \|a\|_{\mathcal{H}^{\sigma, \beta}} \|u\|_{\mathcal{H}^{\sigma, \alpha}} \leq C \|a\|_{\mathcal{H}^{\sigma, \beta}} \|u\|_{\mathcal{H}^{\sigma, \gamma}}$$

Case 2: $\beta \leq \frac{d}{2}$.

Let $\varepsilon > 0$ be such that $\alpha + \varepsilon < \beta + \gamma - \frac{d}{2}$. We write as above

$$T_a u = \sum_q S_{q-2}(a) \Delta_q u := \sum_q v_q.$$

It follows from Lemma A.4 that

$$2^{q\alpha} \|v_q\|_{\mathcal{H}^{\sigma, 0}} \leq 2^{q\alpha} \|S_{q-2}(a)\|_{\mathcal{H}^{\sigma, \frac{d}{2} + \varepsilon}} \|\Delta_q u\|_{\mathcal{H}^{\sigma, 0}} \leq 2^{q(\alpha - \gamma)} c_q \|u\|_{\mathcal{H}^{\sigma, \gamma}}.$$

We next notice that

$$\begin{aligned} \|S_{q-2}(a)\|_{\mathcal{H}^{\sigma, \frac{d}{2} + \varepsilon}} &\leq \sum_{p=-1}^{q-2} \|\Delta_p a\|_{\mathcal{H}^{\sigma, \frac{d}{2} + \varepsilon}} \leq \sum_{p=-1}^{q-2} 2^{p(\frac{d}{2} + \varepsilon)} \|\Delta_p a\|_{\mathcal{H}^{\sigma, 0}}, \\ &\leq \left(\sum_{p=-1}^{q-2} 2^{2p(\frac{d}{2} + \varepsilon - \beta)} \right)^{\frac{1}{2}} \left(\sum_{p=-1}^{q-2} 2^{2p\beta} \|\Delta_p a\|_{\mathcal{H}^{\sigma, 0}}^2 \right)^{\frac{1}{2}} \\ &\leq C 2^{q(\frac{d}{2} + \varepsilon - \beta)} \|a\|_{\mathcal{H}^{\sigma, \beta}}. \end{aligned}$$

This implies that

$$2^{q\alpha} \|v_q\|_{\mathcal{H}^{\sigma, 0}} \leq C 2^{q(\alpha + \varepsilon - (\beta + \gamma - \frac{d}{2}))} \|a\|_{\mathcal{H}^{\sigma, \beta}} \|u\|_{\mathcal{H}^{\sigma, \gamma}},$$

and the wanted result follows using that $\alpha + \varepsilon - (\beta + \gamma - \frac{d}{2}) < 0$. \square

We are in position to state the bilinear estimates.

Proposition A.7. (i) *Consider three real numbers s_1, s_2, s_3 such that*

$$s_1 + s_2 \geq 0, \quad s_3 \leq \min\{s_1, s_2\}, \quad s_3 < s_1 + s_2 - \frac{d}{2}.$$

Then there exists $C > 0$ such that for all $\sigma \geq 0$,

$$(A.14) \quad \|u_1 u_2\|_{\mathcal{H}^{\sigma, s_3}} \leq C \|u_1\|_{\mathcal{H}^{\sigma, s_1}} \|u_2\|_{\mathcal{H}^{\sigma, s_2}}.$$

(ii) *For all $s > d/2$, there exists $C > 0$ such that for all $\sigma \geq 0$,*

$$\|u_1 u_2\|_{\mathcal{H}^{\sigma, s}} \leq C \|u_1\|_{\mathcal{H}^{\sigma, s}} \|u_2\|_{\mathcal{H}^{\sigma, s}}.$$

(iii) For all $s_0 > d/2$ and all $t \geq 0$, there exists $C > 0$ such that for all $\sigma \geq 0$,

$$\|u_1 u_2\|_{\mathcal{H}^{\sigma,t}} \leq C \|u_1\|_{\mathcal{H}^{\sigma,s_0}} \|u_2\|_{\mathcal{H}^{\sigma,t}} + C \|u_2\|_{\mathcal{H}^{\sigma,s_0}} \|u_1\|_{\mathcal{H}^{\sigma,t}}.$$

Proof. (i) One writes $u_1 u_2 = T_{u_1} u_2 + T_{u_2} u_1 + R(u_1, u_2)$. The first two terms of the right-hand side are estimated by the Proposition A.6 by taking $\alpha = s_1$ (resp. $\alpha = s_2$), $\beta = s_2$ (resp. $\alpha = s_1$). It remains to estimate $R(u_1, u_2)$. We claim that

$$\|R(u_1, u_2)\|_{\mathcal{H}^{\sigma,s_3}} \leq \|R(u_1, u_2)\|_{\mathcal{H}^{\sigma,s_1+s_2-\frac{d}{2}}} \leq C \|u_1\|_{\mathcal{H}^{\sigma,s_1}} \|u_2\|_{\mathcal{H}^{\sigma,s_2}}.$$

To see this, recall that

$$R(u_1, u_2) = \sum_{|r-q|\leq 2} \Delta_r u_1 \Delta_q u_2 = \sum_q R_q, \quad R_q = \sum_{|r-q|\leq 2} \Delta_r u_1 \Delta_q u_2.$$

As the spectrum of R_q is contained in a ball of radius $C2^q$ and not in an annulus, it is not enough to directly estimate $\|R_q\|_{L^2}$ because that would require that $s_1 + s_2 - \frac{d}{2}$ be positive (which is not necessarily the case). So we further decompose by writing

$$R = \sum_{p \geq -1} \Delta_p R = \sum_{p \geq -1} \sum_q \Delta_p R_q = \sum_{p \geq -1} \sum_{q \geq p-n_0} \Delta_p R_q,$$

since for $p > q + n_0$ we have $\Delta_p R_q = 0$. All the terms of R_q are bounded in the same way, so that it is enough to consider the one where $r = q$. We will prove that

$$(A.15) \quad \|\Delta_p(\Delta_q u_1 \Delta_q u_2)\|_{\mathcal{H}^{\sigma,0}} \leq C 2^{p\frac{d}{2}} \|\Delta_q u_1\|_{\mathcal{H}^{\sigma,0}} \|\Delta_q u_2\|_{\mathcal{H}^{\sigma,0}}.$$

Indeed, set $U = \Delta_p e^{\sigma|D|}(fg)$. We have

$$|\widehat{U}(\xi)| \leq \varphi(2^{-p}\xi) |(e^{\sigma|D|}f)| \star |(e^{\sigma|D|}g)(\xi)|.$$

Therefore,

$$\begin{aligned} \|U\|_{L^2} &\leq C \|\varphi(2^{-p}\cdot)\|_{\ell^2} \| |(e^{\sigma|D|}f)| \star |(e^{\sigma|D|}g)(\xi)| \|_{\ell^\infty}, \\ &\leq C 2^{p\frac{d}{2}} \|e^{\sigma|D|}f\|_{\ell^2} \|e^{\sigma|D|}g\|_{\ell^2} \leq C' 2^{p\frac{d}{2}} \|f\|_{\mathcal{H}^{\sigma,0}} \|g\|_{\mathcal{H}^{\sigma,0}}. \end{aligned}$$

We deduce from (A.15) that

$$\|\Delta_p R_q\|_{\mathcal{H}^{\sigma,0}} \leq C 2^{p\frac{d}{2}} 2^{-q(s_1+s_2)} (2^{qs_1} \|\Delta_q u_1\|_{\mathcal{H}^{\sigma,0}}) (2^{qs_2} \|\Delta_q u_2\|_{\mathcal{H}^{\sigma,0}}).$$

Now, set $r_p = 2^{-p\frac{d}{2}} 2^{p(s_1+s_2)} \|\Delta_p R\|_{L^2}$ so that, using the previous inequality,

$$r_p \leq C_1 \sum_{q \geq p-n_0} 2^{(p-q)(s_1+s_2)} g_q, \quad g_q = 2^{qs_1} \|\Delta_q u_1\|_{\mathcal{H}^{\sigma,0}} 2^{qs_2} \|\Delta_q u_2\|_{\mathcal{H}^{\sigma,0}}.$$

Notice that

$$g_q \leq C c_q d_q \|u_1\|_{\mathcal{H}^{\sigma,s_1}} \|u_2\|_{\mathcal{H}^{\sigma,s_2}}, \quad (c_q), (d_q) \in l^2.$$

We infer that

$$r_p^2 \leq C_1 \sum_{q \geq p-n_0} 2^{(p-q)(s_1+s_2)} \sum_{q \geq p-n_0} 2^{(p-q)(s_1+s_2)} g_q^2 \leq C_2 \sum_{q \geq p-n_0} 2^{(p-q)(s_1+s_2)} g_q^2$$

since $s_1 + s_2 > 0$. Then,

$$\begin{aligned} \sum_{p \geq -1} r_p^2 &\leq C_2 \sum_{q \geq -1} \left(\sum_{p \leq q+n_0} 2^{-(q-p)(s_1+s_2)} \right) g_q^2 \leq C_3 \sum_{q \geq -1} g_q^2, \\ &\leq C_4 \left(\sum_{q \geq -1} c_q^2 d_q^2 \right) \|\Delta_q u_1\|_{\mathcal{H}^{\sigma,0}}^2 \|\Delta_q u_2\|_{\mathcal{H}^{\sigma,0}}^2. \end{aligned}$$

We conclude that

$$\|\Delta_p R\|_{L^2} = 2^{-p(s_1+s_2-\frac{d}{2})} r_p, \quad \sum_{p \geq -1} r_p^2 \leq C_5 \|\Delta_q u_1\|_{\mathcal{H}^{\sigma,0}}^2 2^{qs_2} \|\Delta_q u_2\|_{\mathcal{H}^{\sigma,0}}^2.$$

Since the spectrum of $\Delta_p R$ is contained in an annulus, this proves the inequality and completes the proof of the claim (i).

(ii) follows from (i) setting $s_1 = s_2 = s_3 = s$.

To prove (iii) we use the following facts:

$$\langle \xi \rangle^t \lesssim \langle \xi - \eta \rangle^t + \langle \eta \rangle^t, \quad e^{\sigma|\xi|} \leq e^{\sigma|\xi-\eta|} e^{\sigma|\eta|}, \quad \|\varphi \star \theta\|_{L^2} \leq \|\varphi\|_{L^1} \|\theta\|_{L^2}$$

and eventually $\|\varphi\|_{L^1} \lesssim \|\langle \xi \rangle^{s_0} \varphi\|_{L^2}$ if $s_0 > \frac{d}{2}$. \square

As a byproduct, we obtain the following result.

Proposition A.8. *Let $\alpha_0, \alpha_1, \alpha_2$ be real numbers such that*

$$\alpha_1 + \alpha_2 > 0, \quad \alpha_0 \leq \alpha_1, \quad \alpha_0 < \alpha_1 + \alpha_2 - \frac{d}{2}.$$

There exists $C > 0$ such that for all $a \in \mathcal{H}^{\sigma, \alpha_1}$ and for all $u \in \mathcal{H}^{\sigma, \alpha_2}$,

$$\|au - T_a u\|_{\mathcal{H}^{\sigma, \alpha_0}} \leq C \|a\|_{\mathcal{H}^{\sigma, \alpha_1}} \|u\|_{\mathcal{H}^{\sigma, \alpha_2}}.$$

The proof is identical to that of the above proposition, for it is sufficient to notice that $au - T_a u = T_u a + R(a, u)$.

Corollary A.9. *Consider two real numbers $s_0 > d/2$ and $s \geq 0$. Let P be a polynomial of degree $m \geq 2$ such that $P(0) = 0$ and let $u \in \mathcal{H}^{\sigma, s} \cap \mathcal{H}^{\sigma, s_0}$. Then $P(u) \in \mathcal{H}^{\sigma, s}$ and*

$$\|P(u)\|_{\mathcal{H}^{\sigma, s}} \leq Q(\|u\|_{\mathcal{H}^{\sigma, s_0}}) \|u\|_{\mathcal{H}^{\sigma, s}}$$

where Q is a polynomial of degree $m - 1$.

Proof. This corollary is an immediate consequence of statement (i) in Proposition A.7. \square

Proposition A.10. *Consider three real numbers $s_0 > d/2$, $s \geq 0$, $M_0 > 0$ and let f be a holomorphic function in the ball $\{z \in \mathbf{C} : |z| < M_0\}$, such that $f(0) = 0$. There exists $\varepsilon_0 > 0$ such that for all $\sigma > 0$, if $u \in \mathcal{H}^{\sigma, s} \cap \mathcal{H}^{\sigma, s_0}$ satisfies $\|u\|_{\mathcal{H}^{\sigma, s_0}} \leq \varepsilon_0$, then $f(u)$ belongs to $\mathcal{H}^{\sigma, s}$. Moreover, there exists $C > 0$ depending only on f, s, s_0, ε_0 such that*

$$\|f(u)\|_{\mathcal{H}^{\sigma, s}} \leq C \|u\|_{\mathcal{H}^{\sigma, s}}.$$

Proof. Set $C_s = 2^s C(d, s_0)$. It follows from Proposition A.7 and from an induction that for all $n \geq 1$,

$$(A.16) \quad \|u^n\|_{\mathcal{H}^{\sigma,s}} \leq (2C_{\max(s,s_0)})^{n-1} \|u\|_{\mathcal{H}^{\sigma,s_0}}^{n-1} \|u\|_{\mathcal{H}^{\sigma,s}}.$$

For $|z| < M_0$ we can write $f(z) = \sum_{n=1}^{+\infty} a_n z^n$ where a_n is such that $|a_n| \leq K^n, K > 0$. We shall show that the series $\sum a_n u^n$ is normally convergent in $\mathcal{H}^{\sigma,s}$. Indeed according to (A.16) and the hypothesis we have

$$\|a_n u^n\|_{\mathcal{H}^{\sigma,s}} \leq K(2C_{\max(s,s_0)} K \varepsilon_0)^{n-1} \|u\|_{\mathcal{H}^{\sigma,s}}.$$

We have just to take ε_0 small enough, so that $2C_{\max(s,s_0)} K \varepsilon_0 < 1$. Since, on the other hand, $\|u\|_{L^\infty(\mathbf{R}^d)} \leq C(d, s_0) \|u\|_{\mathcal{H}^{\sigma,s_0}}$, taking moreover ε_0 such that $C(d, s_0) \varepsilon_0 < M_0$ we will have,

$$\|f(u)\|_{\mathcal{H}^{\sigma,s}} \leq K \left(\sum_{n=1}^{+\infty} (2K C_{\max(s,s_0)})^{n-1} \|u\|_{\mathcal{H}^{\sigma,s_0}}^{n-1} \right) \|u\|_{\mathcal{H}^{\sigma,s}}.$$

This completes the proof. \square

Corollary A.11. *Consider three real numbers $s_0 > d/2$, $t \geq 0$, $M_0 > 0$ and let f be a holomorphic function in the ball $\{z \in \mathbf{C} : |z| < M_0\}$. There exists $\varepsilon_0 > 0, C > 0$ such that for all $\sigma > 0$, if $u_1, u_2 \in \mathcal{H}^{\sigma,t} \cap \mathcal{H}^{\sigma,s_0}$ satisfy $\|u_j\|_{\mathcal{H}^{\sigma,s_0}} \leq \varepsilon_0$, for $j = 1, 2$ then,*

$$\|f(u_1) - f(u_2)\|_{\mathcal{H}^{\sigma,t}} \leq C \left(\|u_1 - u_2\|_{\mathcal{H}^{\sigma,t}} + (\|u_1\|_{\mathcal{H}^{\sigma,t}} + \|u_2\|_{\mathcal{H}^{\sigma,t}}) \|u_1 - u_2\|_{\mathcal{H}^{\sigma,s_0}} \right).$$

Proof. Set $g(z) = f'(z) - f'(0)$. Then g is holomorphic in the set $\{z \in \mathbf{C} : |z| < M_0\}$ and $g(0) = 0$. Then one can write

$$f(u_1) - f(u_2) = f'(0)(u_1 - u_2) + (u_1 - u_2) \int_0^1 g(\lambda u_1 + (1-\lambda)u_2) d\lambda.$$

Since

$$\|\lambda u_1 + (1-\lambda)u_2\|_{\mathcal{H}^{\sigma,s_0}} \leq \lambda \|u_1\|_{\mathcal{H}^{\sigma,s_0}} + (1-\lambda) \|u_2\|_{\mathcal{H}^{\sigma,s_0}} \leq \varepsilon_0,$$

we can apply proposition A.10 with $s = t$ and $s = s_0$ to g and write

$$\begin{aligned} \|g(\lambda u_1 + (1-\lambda)u_2)\|_{\mathcal{H}^{\sigma,t}} &\leq C(\|u_1\|_{\mathcal{H}^{\sigma,t}} + \|u_2\|_{\mathcal{H}^{\sigma,t}}), \\ \|g(\lambda u_1 + (1-\lambda)u_2)\|_{\mathcal{H}^{\sigma,s_0}} &\leq C. \end{aligned}$$

Using Proposition A.7 and the above inequalities, we get

$$\begin{aligned} &\|f(u_1) - f(u_2)\|_{\mathcal{H}^{\sigma,t}} \\ &\leq \|f'(0)\| \|u_1 - u_2\|_{\mathcal{H}^{\sigma,t}} + C(\|u_1 - u_2\|_{\mathcal{H}^{\sigma,t}} + (\|u_1\|_{\mathcal{H}^{\sigma,t}} + \|u_2\|_{\mathcal{H}^{\sigma,t}}) \|u_1 - u_2\|_{\mathcal{H}^{\sigma,s_0}}). \end{aligned}$$

This completes the proof. \square

Recall (see Definition 2.3) that for $\lambda \in [0, 1]$ and $\mu \in \mathbf{R}$, we have introduced the spaces

$$\begin{aligned} E^{\lambda,\mu} &= \{u : e^{\lambda|D_x|} u \in C^0([-h, 0], \mathcal{H}^{\lambda h, \mu})\}, \\ F^{\lambda,\mu} &= \{u : e^{\lambda|D_x|} u \in L^2(I_h, \mathcal{H}^{\lambda h, \mu})\}, \quad I_h = (-h, 0), \\ \mathcal{X}^{\lambda,\mu} &= E^{\lambda,\mu} \cap F^{\lambda,\mu+\frac{1}{2}}, \end{aligned}$$

equipped with their natural norms.

We have several bilinear estimates by using these norms.

Lemma A.12. (i) Consider three real numbers s_1, s_2, s_3 such that

$$s_1 + s_2 \geq 0, \quad s_3 \leq \min\{s_1, s_2\}, \quad s_3 < s_1 + s_2 - \frac{d}{2}.$$

Then there exists $C > 0$ such that for any $\lambda \in [0, 1]$,

$$\|u_1 u_2\|_{F^{\lambda, s_3}} \leq C \|u_1\|_{E^{\lambda, s_1}} \|u_2\|_{F^{\lambda, s_2}}.$$

(ii) For all $s > d/2$, there exists $C > 0$ such that for any $\lambda \in [0, 1]$,

$$\begin{aligned} \|u_1 u_2\|_{F^{\lambda, s}} &\leq C \|u_1\|_{E^{\lambda, s}} \|u_2\|_{F^{\lambda, s}}, \\ \|u_1 u_2\|_{E^{\lambda, s}} &\leq C \|u_1\|_{E^{\lambda, s}} \|u_2\|_{E^{\lambda, s}}. \end{aligned}$$

(iii) Let $s_0 > d/2$ and $t \geq 0$. There exists $C > 0$ such that

$$(A.17) \quad \|u_1 u_2\|_{F^{\lambda, t}} \leq C \|u_1\|_{E^{\lambda, s_0}} \|u_2\|_{F^{\lambda, t}} + C \|u_1\|_{F^{\lambda, t}} \|u_2\|_{E^{\lambda, s_0}}.$$

(iv) For any $\mu \in \mathbf{R}$ and any $\lambda \in [0, 1]$, there exists a constant $C > 0$ such that

$$\|u\|_{E^{\lambda, \mu}} \leq C \|\partial_z u\|_{F^{\lambda, \mu - \frac{1}{2}}} + \|u\|_{F^{\lambda, \mu + \frac{1}{2}}}.$$

Proof. (i) We first use Proposition A.7 with fixed z and $\sigma = \lambda(z + h)$, then we bound the L^2 norm in z of the products by the L^∞ and L^2 norms. The proof of statement (ii) and (iii) are similar and (iv) follows directly from Lemma A.2. \square

A.4. Commutators. In this section we study the commutators between a Fourier multiplier and the multiplication by a function $a \in \mathcal{H}^{\sigma, s}$. In our applications, the Fourier multiplier will be

$$b(D_x) = J_n = \chi\left(\frac{|D_x|}{n}\right).$$

Recall that $S_{1,0}^m$ is the class of symbols p for which the seminorms

$$\sup_{(x, \xi) \in \mathbf{R}^d \times \mathbf{R}^d} \langle \xi \rangle^{|\alpha| - m} |\partial_x^\beta \partial_\xi^\alpha p(\xi)|$$

are finite for every $(\alpha, \beta) \in \mathbf{N}^d \times \mathbf{N}^d$.

Proposition A.13. Let $\nu, s \in \mathbf{R}$ be such that $\nu \geq 0$, $2s + \nu > 1$ and $\nu + s > 1 + \frac{d}{2}$. Let $b(\xi) \in S_{1,0}^\nu$. There exists $C > 0$ (depending only on m and on a finite number on seminorms of b) such that for all $\sigma > 0$ and all $a \in \mathcal{H}^{\sigma, \nu+s}$ we have

$$(A.18) \quad \|[b(D), a]u\|_{\mathcal{H}^{\sigma, s}} \leq C \|a\|_{\mathcal{H}^{\sigma, \nu+s}} \|u\|_{\mathcal{H}^{\sigma, \nu+s-1}}, \quad \forall u \in \mathcal{H}^{\sigma, \nu+s-1}.$$

The proof is based on the following lemma where as before T_a denotes the paraproduct.

Lemma A.14. *Let $s_0 > \frac{d}{2}$, $m \in \mathbf{R}$, such that $-s_0 < m \leq s_0 + 1$ and $p(\xi) \in S_{1,0}^m$. There exists $C > 0$ (depending only on s_0, m and on a finite number on seminorms of p) such that for all $\sigma > 0$ and all a satisfying $\nabla_x a \in \mathcal{H}^{\sigma, s_0}$ we have*

- (i) $\| [p(D), T_a] u \|_{\mathcal{H}^{\sigma, 0}} \leq C \| \nabla_x a \|_{\mathcal{H}^{\sigma, s_0}} \| u \|_{\mathcal{H}^{\sigma, m-1}}, \quad \forall u \in \mathcal{H}^{\sigma, m-1},$
- (ii) $\| [p(D), a] u \|_{\mathcal{H}^{\sigma, 0}} \leq C \| a \|_{\mathcal{H}^{\sigma, s_0+1}} \| u \|_{\mathcal{H}^{\sigma, m-1}}, \quad \forall u \in \mathcal{H}^{\sigma, m-1}.$

Proof. (i) We have

$$\begin{aligned} [p(D), T_a] u &= \sum_{j \geq 1} [p(D), S_{j-3}(a)] \Delta_j u \\ &= \sum_{j \geq 1} [p(D) \varphi_1(2^{-j} D), S_{j-3}(a)] \Delta_j u \\ &:= \sum_{j \geq 1} w_j, \end{aligned}$$

where $\text{supp} \varphi_1 \subset \{ \xi : \frac{1}{4} \leq |\xi| \leq 4 \}$. Set $p_j(\xi) = p(\xi) \varphi_1(2^{-j} \xi)$. Opening the bracket we have

$$e^{\sigma|\xi|} \widehat{w_j}(\xi) = e^{\sigma|\xi|} \sum_{\zeta \in \mathbf{Z}^d} \widehat{S_{j-3}(a)}(\xi - \zeta) (p_j(\xi) - p_j(\zeta)) \widehat{\Delta_j u}(\zeta).$$

Since on the support of $\varphi_1(2^{-j} \xi)$ we have $|\xi| \sim 2^j$ there exists $C > 0$ such that $|\nabla_\xi p_j(\xi)| \leq C 2^{j(m-1)}$ for all $\xi \in \mathbf{R}^d$ and all $j \geq -1$. It follows that

$$|p_j(\xi) - p_j(\zeta)| \leq C 2^{j(m-1)} |\xi - \zeta|$$

for all $\xi, \zeta \in \mathbf{R}^d$. Then

$$e^{\sigma|\xi|} |\widehat{w_j}(\xi)| \leq C \sum_{\zeta \in \mathbf{Z}^d} e^{\sigma|\xi - \zeta|} |\widehat{S_{j-3}(a)}(\xi - \zeta)| |\xi - \zeta| 2^{j(m-1)} |e^{\sigma|\zeta|} \Delta_j u(\zeta)|.$$

The right hand side being a convolution we can write,

$$\begin{aligned} \| e^{\sigma|D_x|} w_j \|_{L^2(\mathbf{R}^d)} &\leq C \left(\sum_{\xi \in \mathbf{Z}^d} e^{\sigma|\xi|} |\widehat{S_{j-3}(|D_x|a)}(\xi)| \right) 2^{j(m-1)} \| e^{\sigma|D_x|} \Delta_j u \|_{L^2}, \\ &\leq C' c_j \| \nabla_x a \|_{\mathcal{H}^{\sigma, s_0}} \| u \|_{\mathcal{H}^{\sigma, m-1}} \end{aligned}$$

where $\sum_j c_j^2 < +\infty$. This proves (i). To prove (ii) we write

$$\begin{aligned} &\| [p(D), a] u \|_{\mathcal{H}^{\sigma, 0}} \\ &\leq C \| [p(D), T_a] u \|_{\mathcal{H}^{\sigma, 0}} + \| p(D)(T_a - a)u \|_{\mathcal{H}^{\sigma, 0}} + \| (T_a - a)p(D)u \|_{\mathcal{H}^{\sigma, 0}}, \\ &\leq A_1 + A_2 + A_3. \end{aligned}$$

The term A_1 is bounded by the right hand side of (ii) according to (i). To estimate the term A_2 we use Lemma A.8 with $\alpha_0 = m, \alpha_1 = s_0 + 1, \alpha_2 = m - 1$. The same lemma with $\alpha_0 = 0, \alpha_1 = s_0 + 1, \alpha_2 = -1$ gives the estimate of A_3 . \square

Proof of Proposition A.13. We write

$$\| [b(D), a] u \|_{\mathcal{H}^{\sigma, s}} \leq \| [\langle D \rangle^s b(D), a] u \|_{\mathcal{H}^{\sigma, 0}} + \| [\langle D \rangle^s, a] b(D) u \|_{\mathcal{H}^{\sigma, 0}}.$$

Then we apply (ii) in Lemma A.14 to the operators $p(D) = \langle D \rangle^s b(D)$ with $m = s + \nu, s_0 = s + \nu - 1$ and to $p(D) = \langle D \rangle^s$ with $m = s, s_0 = s + \nu - 1$. We obtain (A.18). \square

A.5. Estimates on the coefficients. In this paragraph, we prove some elementary estimates for the derivatives of the functions ρ and $\underline{\psi}$ introduced in Section 2.1.

Lemma A.15. *For all $t \in \mathbf{R}$ there exists $C > 0$ such that*

$$\begin{aligned} & \|\partial_z \rho - 1\|_{L^\infty(I_h, \mathcal{H}^{\lambda h, t})} + \|\nabla_x \rho\|_{L^\infty(I_h, \mathcal{H}^{\lambda h, t})} + \|\nabla_{x,z}^2 \rho\|_{L^\infty(I_h, \mathcal{H}^{\lambda h, t-1})} \\ & \leq C \|\eta\|_{\mathcal{H}^{\lambda h, t+1}}, \\ & \|\partial_z \rho - 1\|_{L^2(I_h, \mathcal{H}^{\lambda h, t+\frac{1}{2}})} + \|\nabla_x \rho\|_{L^2(I_h, \mathcal{H}^{\lambda h, t+\frac{1}{2}})} + \|\nabla_{x,z}^2 \rho\|_{L^2(I_h, \mathcal{H}^{\lambda h, t-\frac{1}{2}})} \\ & \leq C \|\eta\|_{\mathcal{H}^{\lambda h, t+1}}. \end{aligned}$$

Proof. We have

$$\begin{aligned} (A.19) \quad \partial_z \rho(x, z) &= 1 + \frac{1}{h} e^{z|D_x|} \eta(x) + \frac{1}{h} (z+h) e^{z|D_x|} |D_x| \eta(x), \\ \nabla_x \rho(x, z) &= \frac{1}{h} (z+h) e^{z|D_x|} \nabla_x \eta(x). \end{aligned}$$

Then the first set of estimates follow from the fact that, since $z \leq 0$, we have $e^{z|\xi|} \leq 1$ so the Fourier multiplier $e^{z|D_x|}$ is bounded from $\mathcal{H}^{\lambda h, s}$ to itself for any s . To prove the second set of estimates, we use the special choice for ρ involving the operator $e^{z|D_x|}$. Notice that, by using Fubini's lemma,

$$\int_{I_h} \sum_{\xi \in \mathbf{Z}^d} |\xi| e^{2z|\xi|} |\widehat{f}(\xi)|^2 dz = \sum_{\xi \in \mathbf{Z}^d} \left(\int_{I_h} |\xi| e^{2z|\xi|} dz \right) |\widehat{f}(\xi)|^2 \leq \sum_{\xi \in \mathbf{Z}^d} \frac{1}{2} |\widehat{f}(\xi)|^2.$$

Consequently, by applying this result with $f = \langle D_x \rangle^s e^{\lambda h |D_x|} u$ and using the Plancherel identity, we get

$$\left\| |D_x| e^{z|D_x|} u \right\|_{L^2(I_h, \mathcal{H}^{\lambda h, s})} \lesssim \|u\|_{\mathcal{H}^{\lambda h, s+\frac{1}{2}}}.$$

Using again (A.19), we complete the proof of the lemma. \square

Definition A.16. *For $k = 1, 2$ let \mathcal{E}_k be the set of functions $\varphi(z, \eta)$ defined on,*

$$\mathcal{A} = I_h \times \{\eta \in \mathcal{H}^{\lambda h, s+\frac{1}{2}} : \|\eta\|_{\mathcal{H}^{\lambda h, s}} \leq \bar{\varepsilon} \leq 1\},$$

such that for every $\frac{d}{2} < t \leq s - k$ there exists $C > 0$ satisfying

$$\|\varphi(\cdot, \eta)\|_{L^\infty(I_h, \mathcal{H}^{\lambda h, t})} \leq C \|\eta\|_{\mathcal{H}^{\lambda h, t+k}}$$

and

$$\|\varphi(\cdot, \eta)\|_{L^2(I_h, \mathcal{H}^{\lambda h, t+1})} \leq C \|\eta\|_{\mathcal{H}^{\lambda h, t+k+\frac{1}{2}}}.$$

Lemma A.17. (1) \mathcal{E}_k is an algebra and $\mathcal{E}_1 \subset \mathcal{E}_2$.

(2) If f is a holomorphic function in a ball $\{Z \in \mathbf{C} : |Z| < M_0\}$ such that $f(0) = 0$ and if $\varphi \in \mathcal{E}_k$, where $\bar{\varepsilon}$ is small enough then $f(\varphi) \in \mathcal{E}_k$.

(3) If $\bar{\varepsilon}$ is small enough we have

$$\nabla_x \rho \in \mathcal{E}_1, \quad \partial_z \rho - 1 \in \mathcal{E}_1, \quad \nabla_{x,z}^2 \rho \in \mathcal{E}_2, \quad \frac{1 + |\nabla_x \rho|^2 - \partial_z \rho}{\partial_z \rho} \in \mathcal{E}_1.$$

Proof. Statement (1) follows from point (ii) in Proposition A.7 applied with $s = t$ and from point (iii) applied $s_0 = t$ and t replaced by $t + 1$. Statement (2) is a consequence of Proposition A.10. The first three claims in point (3) follow directly from Lemma A.15. For the last one we notice that,

$$q := \partial_z \rho - 1 = \frac{1}{h} e^{z|D_x|} \eta + \frac{1}{h} (z + h) e^{z|D_x|} |D_x| \eta \in \mathcal{E}_1.$$

Then we can write

$$\frac{1 + |\nabla_x \rho|^2}{\partial_z \rho} = 1 - \frac{q}{1 + q} + |\nabla_x \rho|^2 - |\nabla_x \rho|^2 \frac{q}{1 + q},$$

and hence, the desired result follows from the previous statements. \square

We shall need the following extension of Lemma A.17. We begin by a definition.

Definition A.18. For $k = 1, 2$ let \mathcal{F}_k be the set of functions $\varphi(z, \eta)$ defined on,

$$\mathcal{A} = I_h \times \{\eta \in \mathcal{H}^{\lambda h, s + \frac{1}{2}} : \|\eta\|_{\mathcal{H}^{\lambda h, s}} \leq \bar{\varepsilon} \leq 1\},$$

such that $\varphi(z, 0) = 0$ and such that for all $\frac{d}{2} < t \leq s - k$ there exists $C > 0$ such that the function $\Phi(z, \eta_1, \eta_2) = \varphi(z, \eta_1) - \varphi(z, \eta_2)$ satisfies the two following estimates

$$\begin{aligned} & \|\Phi(\cdot, \eta_1, \eta_2)\|_{L^2(I_h, \mathcal{H}^{\lambda h, t+1})} \\ & \leq C \left(\|\eta_1 - \eta_2\|_{\mathcal{H}^{\lambda h, t+k+\frac{1}{2}}} + \sum_{j=1}^2 \|\eta_j\|_{\mathcal{H}^{\lambda h, t+k+\frac{1}{2}}} \|\eta_1 - \eta_2\|_{\mathcal{H}^{\lambda h, t+k}} \right), \\ & \|\Phi(\cdot, \eta_1, \eta_2)\|_{L^\infty(I_h, \mathcal{H}^{\lambda h, t})} \leq C \|\eta_1 - \eta_2\|_{\mathcal{H}^{\lambda h, t+k}}. \end{aligned}$$

Lemma A.19. (1) For $k = 1, 2$, \mathcal{F}_k is an algebra and $\mathcal{F}_1 \subset \mathcal{F}_2$.

(2) If $\bar{\varepsilon}$ is small enough, as functions of (z, η) , the functions

$$\nabla_x \rho, \quad |\nabla_x \rho|^2, \quad q = \partial_z \rho - 1, \quad f(q) = \frac{q}{1 + q}, \quad \frac{1 + |\nabla_x \rho|^2}{\partial_z \rho} - 1,$$

belong to \mathcal{F}_1 , and the functions

$$\nabla_x^2 \rho, \quad \nabla_x \partial_z \rho, \quad \partial_z^2 \rho$$

belong to \mathcal{F}_2 .

Proof. (1) It is obvious that $\mathcal{F}_1 \subset \mathcal{F}_2$ so it is sufficient to prove that if $\varphi, \theta \in \mathcal{F}_k$ then, $\varphi \theta \in \mathcal{F}_k$. Taking $\eta_1 = \eta$ and $\eta_2 = 0$, we first notice that the hypotheses imply that, for every $\eta \in \mathcal{H}^{\lambda h, s + \frac{1}{2}}$ such that $\|\eta\|_{\mathcal{H}^{\lambda h, s}} \leq \bar{\varepsilon} \leq 1$, every $\frac{d}{2} < t \leq s - k$, we have

$$(A.20) \quad \|\theta(\cdot, \eta)\|_{L^2(I_h, \mathcal{H}^{\lambda h, t+1})} \lesssim \|\eta\|_{\mathcal{H}^{\lambda h, t+k+\frac{1}{2}}}, \quad \|\theta(\cdot, \eta)\|_{L^\infty(I_h, \mathcal{H}^{\lambda h, t})} \lesssim 1,$$

together with similar estimates for φ . Then we write

$$\begin{aligned} & (\varphi \theta)(z, \eta_1) - (\varphi \theta)(z, \eta_2) \\ & = (\varphi(z, \eta_1) - \varphi(z, \eta_2))\theta(z, \eta_1) + (\theta(z, \eta_1) - \theta(z, \eta_2))\varphi(z, \eta_2) \\ & = A + B. \end{aligned}$$

The terms A and B are handled in the same way. Using point (iii) in Proposition A.7 with $t = t + 1, s_0 = t > \frac{d}{2}$, we can write

$$\begin{aligned} & \|A\|_{L^2(I_h, \mathcal{H}^{\lambda h, t+1})} \\ & \lesssim \|\varphi(\cdot, \eta_1) - \varphi(\cdot, \eta_2)\|_{L^2(I_h, \mathcal{H}^{\lambda h, t+1})} \|\theta(\cdot, \eta_1)\|_{L^\infty(I_h, \mathcal{H}^{\lambda h, t})} \\ & \quad + \|\varphi(\cdot, \eta_1) - \varphi(\cdot, \eta_2)\|_{L^\infty(I_h, \mathcal{H}^{\lambda h, t})} \|\theta(\cdot, \eta_2)\|_{L^2(I_h, \mathcal{H}^{\lambda h, t+1})}. \end{aligned}$$

Then using the hypotheses and (A.20) we can write

$$\|A\|_{L^2(I_h, \mathcal{H}^{\lambda h, t+1})} \lesssim \|\eta_1 - \eta_2\|_{\mathcal{H}^{\lambda h, t+k+\frac{1}{2}}} + \sum_{j=1}^2 \|\eta_j\|_{\mathcal{H}^{\lambda h, t+k+\frac{1}{2}}} \|\eta_1 - \eta_2\|_{\mathcal{H}^{\lambda h, t+k}}.$$

On the other hand, using point (ii) in Proposition A.7 with s replaced by t , we obtain

$$\begin{aligned} \|A\|_{L^\infty(I_h, \mathcal{H}^{\lambda h, t})} & \lesssim \|\varphi(z, \eta_1) - \varphi(z, \eta_2)\|_{L^\infty(I_h, \mathcal{H}^{\lambda h, t})} \|\theta(z, \eta_1)\|_{L^\infty(I_h, \mathcal{H}^{\lambda h, t})} \\ & \lesssim \|\eta_1 - \eta_2\|_{\mathcal{H}^{\lambda h, t+k}}. \end{aligned}$$

This completes the proof of the first statement.

- (2) For the three first functions this follows from Lemma A.15 and statement (1). For $f(q)$ this follows from Corollary A.11 applied for fixed $z \in I_h$. Now one can write

$$\frac{1 + |\nabla_x \rho|^2}{\partial_z \rho} = 1 + A, \quad A = -f(q) - |\nabla_x \rho|^2 f(q) + |\nabla_x \rho|^2 \in \mathcal{F}_1.$$

The last claim follows from Lemma A.15. □

Lemma A.20. *For all $\mu \in \mathbf{R}$, there exists $C > 0$ such that for all $\sigma \geq 0$ and all ψ such that $a(D_x)^{\frac{1}{2}} \psi \in \mathcal{H}^{\sigma, \mu}$, there holds*

$$\begin{aligned} \|\nabla_{x,z} \underline{\psi}\|_{L^2(I_h, \mathcal{H}^{\sigma, \mu})} & \leq C \|a(D_x)^{\frac{1}{2}} \psi\|_{\mathcal{H}^{\sigma, \mu}}, \\ \|\partial_z^2 \underline{\psi}\|_{L^2(I_h, \mathcal{H}^{\sigma, \mu-1})} & \leq C \|a(D_x)^{\frac{1}{2}} \psi\|_{\mathcal{H}^{\sigma, \mu}}, \\ \|\nabla_{x,z} \underline{\psi}\|_{L^\infty(I_h, \mathcal{H}^{\sigma, \mu-\frac{1}{2}})} & \leq C \|a(D_x)^{\frac{1}{2}} \psi\|_{\mathcal{H}^{\sigma, \mu}}. \end{aligned}$$

Proof. To obtain the first estimates, we use the Fourier transform formula to express $\underline{\psi}$ in terms of ψ (see (2.6)), and then the required estimates follow from arguments similar to the ones used in the proof of Lemma A.15. The second one follows, since

$$\|\partial_z^2 \underline{\psi}\|_{F^{\lambda, \mu-1}} = \|\Delta_x \underline{\psi}\|_{F^{\lambda, \mu-1}} \leq \|\nabla_x \underline{\psi}\|_{F^{\lambda, \mu}} \leq C \|a(D_x)^{\frac{1}{2}} \psi\|_{\mathcal{H}^{\sigma, \mu}}.$$

The third estimate is then a consequence of the interpolation argument given by Lemma A.2. □

A.6. Some estimates on the remainder. Consider a function $\eta = \eta(x)$ and defined $\rho = \rho(x, z)$ as in Section 2.1. In this paragraph, we prove several

estimates for the action of the operator R defined by

$$(A.21) \quad \begin{cases} R = a\partial_z^2 + b\Delta_x + c \cdot \nabla_x \partial_z - d\partial_z, & \text{where} \\ a = \frac{1 + |\nabla_x \rho|^2}{\partial_z \rho} - 1, & b = \partial_z \rho - 1, \quad c = -2\nabla_x \rho, \\ d = \frac{1 + |\nabla_x \rho|^2}{\partial_z \rho} \partial_z^2 \rho + \partial_z \rho \Delta_x \rho - 2\nabla_x \rho \cdot \nabla_x \partial_z \rho. \end{cases}$$

Proposition A.21. *Consider two real numbers $s > d/2 + 2$ and $\lambda \in [0, 1)$. There exist $\bar{\varepsilon} > 0$ and a constant $C > 0$ such that for all $\eta \in \mathcal{H}^{\lambda h, s + \frac{1}{2}}$ satisfying $\|\eta\|_{\mathcal{H}^{\lambda h, s}} \leq \bar{\varepsilon}$, the two following properties hold:*

$$(A.22) \quad \|Ru\|_{F^{\lambda, s-2}} \leq C\bar{\varepsilon} (\|\nabla_{x,z} u\|_{F^{\lambda, s-1}} + \|\partial_z^2 u\|_{F^{\lambda, s-2}}),$$

$$(A.23) \quad \|Ru\|_{F^{\lambda, s-1}} \leq C\bar{\varepsilon} (\|\nabla_{x,z} u\|_{F^{\lambda, s}} + \|\partial_z^2 u\|_{F^{\lambda, s-1}}) + C\|\eta\|_{\mathcal{H}^{\lambda h, s + \frac{1}{2}}} \|\partial_z u\|_{E^{\lambda, s-1}},$$

$$(A.24) \quad \|Ru\|_{F^{\lambda, s-1}} \leq C\bar{\varepsilon} (\|\nabla_{x,z} u\|_{F^{\lambda, s}} + \|\partial_z^2 u\|_{F^{\lambda, s-1}}) + C\|\eta\|_{\mathcal{H}^{\lambda h, s + \frac{1}{2}}} (\|\partial_z^2 u\|_{F^{\lambda, s - \frac{5}{2}}} + \|\partial_z u\|_{F^{\lambda, s - \frac{3}{2}}}).$$

Proof. We deduce from Lemma A.17 that

$$a, b, c \in \mathcal{E}_1, \quad d \in \mathcal{E}_2.$$

Step 1. We begin by studying $a\partial_z^2 u + b\Delta_x u + c \cdot \nabla_x \partial_z u$. We will prove an estimate which holds for both cases. Namely, we claim that for all $s-1 \leq \nu \leq s$ we have

$$\|a\partial_z^2 u + b\Delta_x u + c \cdot \nabla_x \partial_z u\|_{F^{\lambda, \nu-1}} \leq C\bar{\varepsilon} \|\nabla_{x,z}^2 u\|_{F^{\lambda, \nu-1}}.$$

Consider the first term $a\partial_z^2 u$. Since $\frac{d}{2} < \nu - 1 \leq s - 1$ the product rule given by statement (ii) in Lemma A.12 implies that

$$(A.25) \quad \begin{aligned} \|a\partial_z^2 u\|_{F^{\lambda, \nu-1}} &\leq C \|a\|_{E^{\lambda, s-1}} \|\partial_z^2 u\|_{F^{\lambda, \nu-1}} \\ &\leq C \|a\|_{L^\infty(I_h, \mathcal{H}^{\lambda h, s-1})} \|\partial_z^2 u\|_{F^{\lambda, \nu-1}}, \\ &\leq C\bar{\varepsilon} \|\partial_z^2 u\|_{F^{\lambda, \nu-1}}, \end{aligned}$$

since $a \in \mathcal{E}_1$. By the same way,

$$(A.26) \quad \begin{aligned} \|b\Delta_x u\|_{F^{\lambda, \nu-1}} &\leq C \|b\|_{E^{\lambda, s-1}} \|\Delta_x u\|_{F^{\lambda, \nu-1}} \\ &\leq C \|b\|_{L^\infty(I_h, \mathcal{H}^{\lambda h, s-1})} \|\Delta_x u\|_{F^{\lambda, \nu-1}}, \\ &\leq C\bar{\varepsilon} \|\nabla_x u\|_{F^{\lambda, \nu}}, \end{aligned}$$

since $b \in \mathcal{E}_1$ and similarly,

$$\|c \cdot \nabla_x \partial_z u\|_{F^{\lambda, s-1}} \leq C\bar{\varepsilon} \|\partial_z u\|_{F^{\lambda, \nu}}.$$

Step 2. We now estimate the $F^{\lambda, s-2}$ norm of $d\partial_z u$, using parallel arguments to those used above. Firstly, since $s - 1 > d/2$ and $2s - 3 \geq 0$, the product rule given by Lemma A.12, with $s_3 = s - 2, s_1 = s - 2, s_2 = s - 1$ implies that

$$\|d\partial_z u\|_{F^{\lambda, s-2}} \leq C \|d\|_{E^{\lambda, s-2}} \|\partial_z u\|_{F^{\lambda, s-1}} \leq C\bar{\varepsilon} \|\partial_z u\|_{F^{\lambda, s-1}},$$

since $d \in \mathcal{E}_2$. Taking $\nu = s - 1$ we obtain (A.22).

Step 3. We now estimate the $F^{\lambda, s-1}$ norm of $d\partial_z u$. We proceed as in the previous step, but now the term d is estimated in $F^{\lambda, s-1}$. This is here the only place where we are taking advantage of the special definition of ρ (the so-called smoothing diffeomorphism). Namely, we write

$$(A.27) \quad \begin{aligned} \|d\partial_z u\|_{F^{\lambda, s-1}} &\leq C \|d\|_{F^{\lambda, s-1}} \|\partial_z u\|_{E^{\lambda, s-1}} \leq C \|d\|_{L^2(I_h, \mathcal{H}^{\lambda, s-1})} \|\partial_z u\|_{E^{\lambda, s-1}} \\ &\leq C \|\eta\|_{\mathcal{H}^{\lambda h, s+\frac{1}{2}}} \|\partial_z u\|_{E^{\lambda, s-1}}. \end{aligned}$$

Using (A.25), (A.26), (A.27) with $\nu = s$, we complete the proof of (A.23).

To obtain (A.24) we use Lemma A.12 (iii) with $s_0 = s - 2$. We obtain

$$\|\partial_z u\|_{F^{\lambda, s-1}} \leq C(\|d\|_{E^{\lambda, s-2}} \|\partial_z u\|_{F^{\lambda, s-1}} + \|d\|_{F^{\lambda, s-1}} \|\partial_z u\|_{E^{\lambda, s-2}}).$$

Since $d \in \mathcal{E}_2$ we have

$$\begin{aligned} \|d\|_{E^{\lambda, s-2}} &\leq C \|\eta\|_{\mathcal{H}^{\lambda h, s}} \leq C\bar{\varepsilon}, \\ \|d\|_{F^{\lambda, s-1}} &\leq C \|\eta\|_{\mathcal{H}^{\lambda h, s+\frac{1}{2}}}. \end{aligned}$$

On the other hand, by the interpolation lemma (see Lemma A.12 (iv)) we have

$$\|\partial_z u\|_{E^{\lambda, s-2}} \leq C(\|\partial_z^2 u\|_{F^{\lambda, s-\frac{5}{2}}} + \|\partial_z u\|_{F^{\lambda, s-\frac{3}{2}}}).$$

Therefore,

$$\|d\partial_z u\|_{F^{\lambda, s-1}} \leq C\bar{\varepsilon} \|\partial_z u\|_{F^{\lambda, s-1}} + C \|\eta\|_{\mathcal{H}^{\lambda h, s+\frac{1}{2}}} (\|\partial_z^2 u\|_{F^{\lambda, s-\frac{5}{2}}} + \|\partial_z u\|_{F^{\lambda, s-\frac{3}{2}}}).$$

This completes the proof of (A.24). \square

Corollary A.22. *Let $\underline{\psi}$ be the lifting of the function ψ defined in (2.5). Under the assumptions of Lemma A.21, there exists a constant $C > 0$ such that*

$$(A.28) \quad \|R\underline{\psi}\|_{F^{\lambda, s-2}} \leq C\bar{\varepsilon} \|a(D_x)^{\frac{1}{2}} \psi\|_{\mathcal{H}^{\lambda h, s-1}},$$

and

$$(A.29) \quad \|R\underline{\psi}\|_{F^{\lambda, s-1}} \leq C\bar{\varepsilon} \|a(D_x)^{\frac{1}{2}} \psi\|_{\mathcal{H}^{\lambda h, s}} + C \|\eta\|_{\mathcal{H}^{\lambda h, s+\frac{1}{2}}} \|a(D_x)^{\frac{1}{2}} \psi\|_{\mathcal{H}^{\lambda h, s-\frac{1}{2}}}.$$

Proof. Notice that

$$\|\nabla_{x,z} \underline{\psi}\|_{F^{\lambda, \mu}} \leq \|\nabla_{x,z} \underline{\psi}\|_{L_z^2(I_h, \mathcal{H}^{\lambda h, \mu})}$$

for any μ in \mathbf{R} . By Lemma A.20 we have, for any real number μ ,

$$\|\nabla_{x,z} \underline{\psi}\|_{E^{\lambda, \mu-\frac{1}{2}}} + \|\partial_z^2 \underline{\psi}\|_{F^{\lambda, \mu-1}} + \|\nabla_{x,z} \underline{\psi}\|_{F^{\lambda, \mu}} \lesssim \|a(D_x)^{\frac{1}{2}} \psi\|_{\mathcal{H}^{\lambda h, \mu}}.$$

Then (A.28) and (A.29) follow from Proposition A.21. \square

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Thomas Alazard

Université Paris-Saclay, ENS Paris-Saclay, CNRS,
 Centre Borelli UMR9010, Avenue des Sciences, F-91190 Gif-sur-Yvette,
 France,
 and
 Department of Mathematics, University of California, Berkeley
 CA 94720, USA

Nicolas Burq

Laboratoire de Mathématiques d’Orsay, CNRS,
 Université Paris-Saclay,
 Bâtiment 307, F-91405 Orsay
 France,
 and Institut Universitaire de France.

Claude Zuily

Laboratoire de Mathématiques d’Orsay, CNRS,
 Université Paris-Saclay,
 Bâtiment 307, F-91405 Orsay
 France