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# On the dynamics of the roots of polynomials under differentiation

Thomas Alazard, Omar Lazar, Quoc-Hung Nguyen

ABSTRACT. This article is devoted to the study of a nonlinear and nonlocal parabolic equation introduced by Stefan Steinerberger to study the roots of polynomials under differentiation; it also appeared in a work by Dimitri Shlyakhtenko and Terence Tao on free convolution. Rafael Granero-Belinchón obtained a global well-posedness result for positive initial data small enough in a Wiener space, and recently Alexander Kiselev and Changhui Tan proved a global well-posedness result for any positive initial data in the Sobolev space  $H^s(\mathbb{S})$  with  $s > 3/2$ . In this paper, we consider the Cauchy problem in the critical space  $H^{1/2}(\mathbb{S})$ . Two interesting new features, at this level of regularity, are that the equation can be written in the form

$$\partial_t u + V \partial_x u + \gamma \Lambda u = 0,$$

where  $\gamma$  is non-negative but not bounded from below and  $V/\sqrt{\gamma}$  is not bounded. Therefore, the equation is only weakly parabolic. We prove that nevertheless the Cauchy problem is well posed locally in time and that the solutions are smooth for positive times. Combining this with the results of Kiselev and Tan, this gives a global well-posedness result for any positive initial data in  $H^{1/2}(\mathbb{S})$ . Our proof relies on sharp commutator estimates and introduces a strategy to prove a local well-posedness result in a situation where the lifespan depends on the profile of the initial data and not only on its norm.

## 1. Introduction

Stefan Steinerberger studied in [38] the following question: Considering a polynomial  $p_n$  of degree  $n$  having all its roots on the real line distributed according to a smooth function  $u_0(x)$ , and a real number  $t \in (0, 1)$ , how is the distribution of the roots of the derivatives  $\partial_x^k p_n$  with  $k = \lfloor t \cdot n \rfloor$ ? This question led him to discover a nice non-local nonlinear equation which reads as follows

$$(1) \quad \partial_t u + \frac{1}{\pi} \partial_x \left( \arctan \left( \frac{Hu}{u} \right) \right) = 0,$$

where the unknown  $u = u(t, x)$  is a positive real-valued function.

Besides its aesthetic aspect, this equation has many interesting features. Shlyakhtenko and Tao [36] derived the same equation in the context of free probability and random matrix theory (see also [39]). However, our motivation comes from the links between this equation and many models studied in fluid dynamics.

In this paper, we assume that the space variable  $x$  belongs to the circle  $\mathbb{S} = \mathbb{R}/(2\pi\mathbb{Z})$ , and  $H$  is the circular Hilbert transform (which acts on periodic functions), defined by

$$(2) \quad Hu(x) = \frac{1}{2\pi} \text{pv} \int_{\mathbb{S}} \frac{g(x) - g(x - \alpha)}{\tan(\alpha/2)} d\alpha,$$

where the integral is understood as a principal value. Granero-Belinchón ([**25**]) proved the local existence of smooth solutions for positive initial data  $u_0$  in the Sobolev space  $H^2(\mathbb{S}) = \{u \in L^2(\mathbb{S}); \partial_x^2 u \in L^2(\mathbb{S})\}$ , as well as the global existence under a condition in appropriate Wiener spaces. Then, Kiselev and Tan [**28**] proved that the Cauchy problem for (1) is globally well-posed for positive initial data in the Sobolev space  $H^s(\mathbb{S})$  for all  $s > 3/2$ , with  $H^s(\mathbb{S}) = \{u \in L^2(\mathbb{S}); \Lambda^s u \in L^2(\mathbb{S})\}$  where  $\Lambda$  denotes the fractional Laplacian:

$$\Lambda = \partial_x H = (-\partial_{xx})^{\frac{1}{2}}.$$

In fact, the equation (1) enters the family of fractional parabolic equations, which has attracted a lot of attention in recent years. To see this, introduce the coefficients

$$V = -\frac{1}{\pi} \frac{Hu}{u^2 + (Hu)^2}, \quad \gamma = \frac{1}{\pi} \frac{u}{u^2 + (Hu)^2}.$$

Then the equation (1) has the following form

$$(3) \quad \partial_t u + V \partial_x u + \gamma \Lambda u = 0.$$

We can see that this last equation shares many characteristics with the Hele-Shaw equation, the Muskat equation or the dissipative surface quasi-geostrophic equation, to name a few. For the dissipative SQG equation, the global regularity has been proved by Kiselev, Nazarov and Volberg [**27**], Caffarelli-Vasseur [**8**] and Constantin-Vicol [**19**] (see also [**26, 37, 41, 32**]). The nonlinearity in the Muskat equation is more complicated. However, Cameron has succeeded in [**9**] to apply the method introduced by Kiselev-Nazarov-Volberg to prove the existence of global solutions in time when the product of the maximum and minimum slopes is less than 1 (see also [**10, 11**]). Recently, many works have extended this last result. The main results in [**2, 3, 4, 5**] imply that the Cauchy problem can be solved for non-Lipschitz initial data, following earlier work by Deng, Lei and Lin [**22**], Córdoba and Lazar [**21**], Gancedo and Lazar [**24**] which allowed arbitrary slopes of large size. Recently, in [**13**] Chen, Xu and the third author proved local existence with  $C^1$  initial data. The main issue is that it is difficult to dispense with the assumption of finite slopes. Indeed, the classical nonlinear estimates require to control the  $L^\infty$  norm of some factors, which is the same for the Muskat problem to control the  $L^\infty$  norm of the slope, which in turn amounts to controlling the Lipschitz norm of  $f$ . Second, the Muskat equation is a degenerate parabolic equation for solutions which are not controlled in the Lipschitz norm. Last but not least, it is worth mentioning that equation (1) shares many similarity with 1D transport equations with a nonlocal velocity given by the Hilbert transform (see e.g. [**18, 20, 33, 23, 30, 29, 7**]).

Inspired by these results, our goal here is to solve the Cauchy problem for (1) in the critical Sobolev space  $H^{\frac{1}{2}}(\mathbb{S})$ . Several interesting difficulties appear at that level of regularity.

The main result of this paper is the following.

**THEOREM 1.1.** *For all initial data  $u_0$  in  $H^{\frac{1}{2}}(\mathbb{S})$  such that  $\inf_{x \in \mathbb{S}} u_0 > 0$ , the Cauchy problem has a unique global in time solution  $u$  satisfying the following properties:*

(i)  $u \in C^0([0, +\infty); H^{\frac{1}{2}}(\mathbb{S})) \cap L^2((0, +\infty); \dot{H}^1(\mathbb{S}))$  together with the estimate

$$(4) \quad \sup_{t>0} \|u(t)\|_{\dot{H}^{\frac{1}{2}}}^2 + \int_0^\infty \int_{\mathbb{S}} \frac{u|\Lambda u|^2}{u^2 + (Hu)^2} dx dt < +\infty.$$

(ii)  $u \in C^\infty((0, +\infty) \times \mathbb{S})$  and moreover for any  $\xi_0 > 0$  and  $\varepsilon_0 > 0$ ,

$$\sup_{t>0} t^{\xi_0 + \varepsilon_0} \|u(t)\|_{\dot{H}^{\frac{1}{2} + \varepsilon_0}} < +\infty.$$

(iii)  $\inf_{x \in \mathbb{S}} u(t, x) \geq \inf_{x \in \mathbb{S}} u_0(x)$  for  $t > 0$ .

**REMARK 1.2.** *Notice that the coercive quantity that appears in the left-hand side of (4), i.e.*

$$(5) \quad \int_0^\infty \int_{\mathbb{S}} \frac{u|\Lambda u|^2}{u^2 + (Hu)^2} dx dt,$$

*is insufficient to control the  $\|\cdot\|_{L_t^2 \dot{H}_x^1}$ -norm of  $u$ . Indeed, even if we assume that the initial value of  $u$  is bounded (which propagates in time), since the Hilbert transform is not bounded from  $L^\infty(\mathbb{S})$  to itself, we would have no control from below of the denominator  $u^2 + (Hu)^2$ .*

**Plan of the paper.** Since Kiselev and Tan ([28]) proved that the Cauchy problem for (1) is globally well-posed for any positive initial data in  $H^s(\mathbb{S})$  with  $s > 3/2$ , it will be sufficient to prove a local well-posedness result, together with the fact that the solutions are smooth for positive times. We start in Section 2 by studying a toy model, to explain the main difficulty and a commutator estimate. Then in §3 we prove the local-well posedness and then establish the smoothing effect in §4.

## 2. A toy model

To prove Theorem 1.1, the main difficulty is that the coefficient  $\gamma$  is not bounded from below by a positive constant. This means that (3) is a degenerate parabolic equation. Although it is not essential for the rest of the paper, it helps if we begin by examining a model equation where, by contrast, the dissipative term is non-degenerate. Then we will see that one can obtain an energy estimate in  $\dot{H}^{1/2}(\mathbb{S})$  by means of a commutator estimate. In the next section, we will explain how to modify these arguments to deal with degenerate dissipation.

Consider the equation

$$(6) \quad \partial_t u + a(u, Hu)\Lambda u = b(u, Hu)H\Lambda u,$$

where  $a$  and  $b$  are two  $C^\infty$  real-valued functions defined on  $\mathbb{R}^2$ , satisfying  $a \geq m > 0$  for some given positive constant  $m$ , together with

$$\sup_{(x, x', y, y') \in \mathbb{R}^4} \frac{|b(x, y) - b(x', y')|}{|x - x'| + |y - y'|} < +\infty.$$

PROPOSITION 2.1. *There exists a constant  $C > 0$  such that, for all  $T > 0$  and for all  $u \in C^1([0, T]; \dot{H}^{\frac{1}{2}}(\mathbb{S}))$  solution to (6), there holds*

$$(7) \quad \frac{1}{2} \frac{d}{dt} \|u\|_{\dot{H}^{\frac{1}{2}}}^2 + m \int_{\mathbb{S}} |\Lambda u|^2 dx \leq C \|u\|_{\dot{H}^{\frac{1}{2}}} \|\Lambda u\|_{L^2}^2.$$

REMARK 2.2. *Using classical arguments, it is then possible to infer from the a priori estimate (7) a global well-posedness result for initial data which are small enough in  $\dot{H}^{\frac{1}{2}}(\mathbb{S})$ . However, the study of the local well-posedness of the Cauchy problem for large data is more difficult and requires an extra argument which is explained in the next section.*

PROOF. Let us use the short notations  $a = a(u, Hu)$  and  $b = b(u, Hu)$ . Multiplying the equation (6) by  $\Lambda u$  and integrating over  $\mathbb{S}$ , we obtain

$$(8) \quad \frac{1}{2} \frac{d}{dt} \|u\|_{\dot{H}^{\frac{1}{2}}}^2 + \int_{\mathbb{S}} a |\Lambda u|^2 dx = I := \int_{\mathbb{S}} b(H\Lambda u)(\Lambda u) dx.$$

To estimate  $I$ , we use the fact that  $H$  is skew-symmetric to write

$$(9) \quad |I| = \frac{1}{2} \left| \int_{\mathbb{S}} ([b, H] \Lambda u) \Lambda u dx \right|.$$

Now we claim that

$$(10) \quad \|[b, H] \Lambda u\|_{L^2} \lesssim \|b\|_{\dot{H}^{\frac{1}{2}}} \|\Lambda u\|_{L^2}.$$

Indeed, this follows from the Sobolev embedding  $\dot{H}^{\frac{1}{2}} \subset \text{BMO}$  and the classical commutator estimate

$$(11) \quad \|[H, f]v\|_{L^2} \lesssim \|f\|_{\text{BMO}} \|v\|_{L^2}.$$

(Alternatively one can prove (10) directly using the definition of the Hilbert transform as a singular integral and the Gagliardo semi-norm.) It follows that

$$(12) \quad I \lesssim \|b\|_{\dot{H}^{\frac{1}{2}}} \|\Lambda u\|_{L^2}^2.$$

Now we estimate the  $\dot{H}^{\frac{1}{2}}$ -norm of  $b$  by means of the following elementary estimate.

LEMMA 2.3. *Consider a  $C^\infty$  function  $\sigma: \mathbb{R}^2 \rightarrow \mathbb{R}$  satisfying*

$$\forall (x, x', y, y') \in \mathbb{R}^4, \quad |\sigma(x, y) - \sigma(x', y')| \leq K |x - x'| + K |y - y'|.$$

*Then, for all  $s_0 \in (0, 1)$  and all  $u \in \dot{H}^{s_0}(\mathbb{R})$ , one has  $\sigma(u, Hu) \in \dot{H}^{s_0}(\mathbb{R})$  together with the estimate*

$$(13) \quad \|\sigma(u, Hu)\|_{\dot{H}^{s_0}} \leq 2K \|u\|_{\dot{H}^{s_0}}.$$

PROOF. By assumption, for any  $\alpha \in \mathbb{R}$ , we have

$$\|\delta_\alpha \sigma(u, Hu)\|_{L^2} \leq K (\|\delta_\alpha u\|_{L^2} + \|\delta_\alpha Hu\|_{L^2}).$$

Then by using the Gagliardo semi-norms, we get

$$\|\sigma(u, Hu)\|_{\dot{H}^{s_0}} \leq K \|u\|_{\dot{H}^{s_0}} + K \|Hu\|_{\dot{H}^{s_0}},$$

and the desired result follows since  $\|Hu\|_{\dot{H}^{s_0}} = \|u\|_{\dot{H}^{s_0}}$ .  $\square$

The previous lemma implies that

$$\|b\|_{\dot{H}^{\frac{1}{2}}} \lesssim \|u\|_{\dot{H}^{\frac{1}{2}}} + \|Hu\|_{\dot{H}^{\frac{1}{2}}} \lesssim \|u\|_{\dot{H}^{\frac{1}{2}}},$$

and we deduce from (12) that

$$I \lesssim \|u\|_{\dot{H}^{\frac{1}{2}}} \|\Lambda u\|_{L^2}^2.$$

Therefore the wanted result (7) follows from (8).  $\square$

### 3. Local well-posedness

We construct solutions to (1) as limits of solutions to a sequence of approximate nonlinear systems. We divide the analysis into three parts.

- (1) We start by proving that the Cauchy problem for these systems are well posed globally in time and satisfy the maximum principles. In particular, the approximate solutions are bounded by a positive constant.
- (2) Then, we show that the solutions of the approximate systems are bounded in  $C^0([0, T]; \dot{H}^{\frac{1}{2}}(\mathbb{S}))$  on a uniform time interval that depends on the profile of the initial data (and not only on their norm).
- (3) The third task is to show that these approximate solutions converge to a limit which is a solution of the original equation. To do this, we use interpolation and compactness arguments.

**3.1. Approximate systems.** Fix  $\delta \in (0, 1]$  and consider the following approximate Cauchy problem:

$$(14) \quad \begin{cases} \partial_t u + \frac{1}{\pi} \frac{u\Lambda u - (Hu)\partial_x u}{u^2 + (Hu)^2} - \delta \partial_x^2 u = 0, \\ u|_{t=0} = e^{\delta \partial_x^2} u_0. \end{cases}$$

The following elementary lemma states that this Cauchy problem has smooth positive solutions (see Granero-Belinchón in [25] for the maximum principle).

**LEMMA 3.1.** *For any initial data  $u_0 \in L^2(\mathbb{S})$  with  $\inf_{x \in \mathbb{S}} u_0 > 0$  and for any  $\delta > 0$ , the initial value problem (14) has a unique solution  $u$  in  $C^1([0, +\infty); H^\infty(\mathbb{S}))$ . This solution is such that, for all  $t \in [0, +\infty)$ ,*

$$(15) \quad \inf_{x \in \mathbb{S}} u(t, x) \geq \inf_{x \in \mathbb{S}} u_0(x).$$

*In addition, if  $\max_{x \in \mathbb{S}} u_0(x) < +\infty$  then  $\max_{x \in \mathbb{S}} u(t, x) \leq \max_{x \in \mathbb{S}} u_0(x)$ .*

**3.2. Uniform estimates.** Fix  $\delta > 0$  and  $c_0$  and consider an initial data  $u_0$  in  $H^{\frac{1}{2}}(\mathbb{S})$  with  $\inf_{x \in \mathbb{S}} u_0(x) \geq c_0$ . As we have seen in the previous paragraph, there exists a unique function  $u \in C^1([0, +\infty); H^\infty(\mathbb{S}))$  satisfying

$$(16) \quad \begin{cases} \partial_t u + \frac{1}{\pi} \frac{u\Lambda u - (Hu)\partial_x u}{u^2 + (Hu)^2} - \delta \partial_x^2 u = 0, \\ u|_{t=0} = e^{\delta \partial_x^2} u_0, \\ \inf_{x \in \mathbb{S}} u(t, x) \geq c_0. \end{cases}$$

We shall prove estimates which are uniform with respect to  $\delta \in (0, 1]$  and this is why we are writing simply  $u$  instead of  $u_\delta$ , to simplify notations.

Notice that

$$\partial_t u + \frac{1}{\pi} \arctan\left(\frac{Hu}{u}\right) - \delta \partial_x^2 u = 0.$$

This implies that the mean value of  $u$  is preserved and hence it will be sufficient to estimate the homogeneous  $\dot{H}^{\frac{1}{2}}(\mathbb{S})$ -norm of  $u$ . For this, we proceed as follows. Given  $\varepsilon > 0$  to be determined, we want to estimate

$$v := u - e^{(\varepsilon+\delta)\partial_x^2} u_0.$$

Set

$$u_{0,\varepsilon} = e^{(\varepsilon+\delta)\partial_x^2} u_0,$$

and introduce the coefficients

$$\begin{aligned} \gamma &= \frac{1}{\pi} \frac{u}{u^2 + (Hu)^2}, \\ V &= -\frac{1}{\pi} \frac{Hu}{u^2 + (Hu)^2}, \\ \rho &= \sqrt{u^2 + (Hu)^2}. \end{aligned}$$

With the previous notations, we have

$$(17) \quad \partial_t v + V \partial_x v + \gamma \Lambda v - \delta \partial_x^2 v = R_\varepsilon(u, u_0)$$

where

$$R_\varepsilon(u, u_0) = -\gamma \Lambda u_{0,\varepsilon} - V \partial_x u_{0,\varepsilon} + \delta \partial_x^2 u_{0,\varepsilon}.$$

The following result is our main technical estimate.

LEMMA 3.2. *For any  $u_0 \in H^{\frac{1}{2}}(\mathbb{S})$  with  $\inf_{x \in \mathbb{S}} u_0(x) > 0$ , there exist a constant  $\varepsilon_0$  and a function  $T: (0, 1] \rightarrow (0, 1)$  with*

$$\lim_{\varepsilon \rightarrow 0} T(\varepsilon) = 0,$$

*such that the following result holds: for all  $\delta \in (0, 1]$ , all  $u \in C^1([0, +\infty); H^\infty(\mathbb{S}))$  satisfying (16) with initial data  $u|_{t=0} = e^{\delta \partial_x^2} u_0$ , and for all  $\varepsilon \in (0, \varepsilon_0]$ , the function  $v = u - e^{(\varepsilon+\delta)\partial_x^2} u_0$  satisfies*

$$(18) \quad \sup_{t \in [0, T(\varepsilon)]} \|v(t)\|_{\dot{H}^{\frac{1}{2}}}^2 + \int_0^{T(\varepsilon)} \int_{\mathbb{S}} \frac{u |\Lambda v|^2}{u^2 + (Hu)^2} dx dt + \delta \int_0^{T(\varepsilon)} \|v\|_{\dot{H}^{\frac{3}{2}}}^2 dt \leq \mathcal{F}(\varepsilon),$$

*for some function  $\mathcal{F}: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  with  $\lim_{\varepsilon \rightarrow 0} \mathcal{F}(\varepsilon) = 0$ .*

PROOF. Hereafter,  $C$  denotes various constants which depend only on the constant  $c_0$  (remembering that  $c_0$  is some given constant such that  $\inf u(t, x) \geq \inf u_0 \geq c_0$ ) and we use the notation  $A \lesssim_{c_0} B$  to indicate that  $A \leq CB$  for such a constant  $C$ .

Consider a parameter  $\kappa \in (0, 1]$  whose value is to be determined. Then decompose the Hilbert transform as  $H = H_{\kappa,1} + H_{\kappa,2}$  where

$$\begin{aligned} H_{\kappa,1}g(x) &= \frac{1}{2\pi} \int_{\mathbb{S}} g(x - \alpha) \chi\left(\frac{\alpha}{\kappa}\right) \frac{d\alpha}{\tan(\alpha/2)}, \\ H_{\kappa,2}g(x) &= \frac{1}{2\pi} \int_{\mathbb{S}} g(x - \alpha) \left(1 - \chi\left(\frac{\alpha}{\kappa}\right)\right) \frac{d\alpha}{\tan(\alpha/2)}, \end{aligned}$$

for some cut-off function  $\chi \in C^\infty$  satisfying  $\chi = 1$  in  $[-1, 1]$  and  $\chi = 0$  in  $\mathbb{R} \setminus [-2, 2]$ .

Multiply equation (17) by  $\Lambda v$  and then integrate over  $\mathbb{S}$ , to obtain

$$(19) \quad \frac{1}{2} \frac{d}{dt} \|v\|_{\dot{H}^{\frac{1}{2}}}^2 + \int_{\mathbb{S}} \gamma (\Lambda v)^2 dx = A + B + R + R'$$

where

$$\begin{aligned} A &= \int_{\mathbb{S}} V(H_{\kappa,1}\Lambda v)(\Lambda v) dx, \\ B &= \int_{\mathbb{S}} V(H_{\kappa,2}\Lambda v)(\Lambda v) dx, \\ R &= \int_{\mathbb{S}} (-\gamma \Lambda u_{0,\varepsilon} - V \partial_x u_{0,\varepsilon})(\Lambda v) dx, \\ R' &= \int_{\mathbb{S}} \delta(\partial_x^2 u_{0,\varepsilon}) \Lambda v dx. \end{aligned}$$

Set

$$W := \sqrt{\gamma} \Lambda v,$$

so that the dissipative term in (19) is of the form

$$\int_{\mathbb{S}} \gamma (\Lambda v)^2 dx = \int_{\mathbb{S}} W^2 dx.$$

*Step 1: Estimate of B, R and R'.* Directly from the definition of  $\gamma$  and  $V$ , we have

$$(20) \quad \gamma \leq \frac{1}{\pi c_0} \quad , \quad |V| \leq \frac{1}{2\pi c_0}.$$

One important feature of the critical problem is that the dissipative term is degenerate. This means that the coefficient  $\gamma$  is not bounded from below by a fixed positive constant. As a result, we do not control the  $L^2$ -norm of  $\Lambda v$ . Instead, we merely control the  $L^2$ -norm of  $W = \sqrt{\gamma} \Lambda v$ . Hereafter, we will systematically write  $\Lambda v$  under the form

$$\Lambda v = \frac{1}{\sqrt{\gamma}} \sqrt{\gamma} \Lambda v = \frac{1}{\sqrt{\gamma}} W.$$

To absorb the contribution of the factor  $1/\sqrt{\gamma}$  in the estimates for  $B$  and  $R$ , it will be sufficient to notice that we have the pointwise bound

$$\left| \frac{V}{\sqrt{\gamma}} \right| = \frac{1}{u \sqrt{\delta + u^2 + (Hu)^2}} |Hu| \lesssim_{c_0} |Hu|.$$

In particular, remembering that the Hilbert transform is bounded from  $L^p(\mathbb{S})$  to  $L^p(\mathbb{S})$  for any  $p \in (1, +\infty)$  and using the Sobolev embedding  $H^s(\mathbb{S}) \subset L^{2/(1-2s)}(\mathbb{S})$ ,



we deduce that

$$(21) \quad \|V/\sqrt{\gamma}\|_{L^4} \lesssim_{c_0} \|Hu\|_{L^4} \lesssim_{c_0} \|u\|_{\dot{H}^{\frac{1}{2}}}.$$

Then it follows from Hölder's inequality that

$$|B| \leq \int_{\mathbb{S}} \frac{V}{\sqrt{\gamma}} |H_{\kappa,2}\Lambda v| |\sqrt{\gamma}\Lambda v| dx \lesssim_{c_0} \|u\|_{\dot{H}^{\frac{1}{2}}} \|H_{\kappa,2}\Lambda v\|_{L^4} \|W\|_{L^2}.$$

On the other hand,

$$\|H_{\kappa,2}\Lambda v\|_{L^4} = \|H_{\kappa,2}\partial_x H v\|_{L^4} \lesssim \kappa^{-1} \|Hv\|_{L^4} \lesssim \kappa^{-1} \|v\|_{\dot{H}^{\frac{1}{2}}},$$

where we used the definition of  $H_{\kappa,2}$ , noting that  $g_x(x - \alpha) = \partial_\alpha(g(x) - g(x - \alpha))$  and integrating by parts in  $\alpha$ .

By combining the previous estimates, we conclude that

$$|B| \lesssim_{c_0} \kappa^{-1} \|u\|_{\dot{H}^{\frac{1}{2}}} \|v\|_{\dot{H}^{\frac{1}{2}}} \|W\|_{L^2}.$$

The estimate of  $R$  is similar. Recall that

$$R = \int_{\mathbb{S}} (-\gamma\Lambda u_{0,\varepsilon} - V\partial_x u_{0,\varepsilon})(\Lambda v) dx.$$

To estimate the contribution of the first term, we write

$$\left| \int_{\mathbb{S}} \gamma(\Lambda u_{0,\varepsilon})(\Lambda v) dx \right| \leq \|\sqrt{\gamma}\|_{L^\infty} \|\Lambda u_{0,\varepsilon}\|_{L^2} \|\sqrt{\gamma}\Lambda v\|_{L^2} \lesssim_{c_0} \varepsilon^{-\frac{1}{2}} \|u_0\|_{\dot{H}^{\frac{1}{2}}} \|W\|_{L^2},$$

where we have used the elementary inequality

$$\|u_{0,\varepsilon}\|_{\dot{H}^1} \lesssim (\varepsilon + \delta)^{-\frac{1}{4}} \|u_{0,\varepsilon}\|_{\dot{H}^{\frac{1}{2}}} \lesssim \varepsilon^{-\frac{1}{4}} \|u_0\|_{\dot{H}^{\frac{1}{2}}},$$

since the Fourier transform of  $u_{0,\varepsilon} = e^{(\varepsilon+\delta)\partial_x^2} u_0$  is essentially localized in the interval  $|\xi| \lesssim \sqrt{\varepsilon + \delta}$ . With regards to the second term, we use again the estimate (21) to get

$$\begin{aligned} \left| \int_{\mathbb{S}} V(\partial_x u_{0,\varepsilon})(\Lambda v) dx \right| &\leq \|V/\sqrt{\gamma}\|_{L^4} \|\partial_x u_{0,\varepsilon}\|_{L^4} \|\sqrt{\gamma}\Lambda v\|_{L^2} \\ &\lesssim_{c_0} \varepsilon^{-1} \|u\|_{\dot{H}^{\frac{1}{2}}} \|u_0\|_{\dot{H}^{\frac{1}{2}}} \|W\|_{L^2}. \end{aligned}$$

Eventually, we have

$$\|\delta\partial_x^2 u_{0,\varepsilon}\|_{\dot{H}^{\frac{1}{2}}} = \left\| \delta\partial_x^2 e^{(\varepsilon+\delta)\partial_x^2} u_0 \right\|_{\dot{H}^{\frac{1}{2}}} \lesssim \left\| \delta\partial_x^2 e^{\delta\partial_x^2} u_0 \right\|_{\dot{H}^{\frac{1}{2}}} \lesssim \|u_0\|_{\dot{H}^{\frac{1}{2}}}.$$

Hence

$$|R'| \lesssim \|u_0\|_{\dot{H}^{\frac{1}{2}}} \|v\|_{\dot{H}^{\frac{1}{2}}}.$$

So, by combining the previous inequalities, we conclude that

$$\begin{aligned} |B + R + R'| &\lesssim_{c_0} \kappa^{-1} \|u\|_{\dot{H}^{\frac{1}{2}}} \|v\|_{\dot{H}^{\frac{1}{2}}} \|W\|_{L^2} + \varepsilon^{-\frac{1}{2}} (1 + \|u\|_{\dot{H}^{\frac{1}{2}}}) \|u_0\|_{\dot{H}^{\frac{1}{2}}} \|W\|_{L^2} \\ &\quad + \|u_0\|_{\dot{H}^{\frac{1}{2}}} \|v\|_{\dot{H}^{\frac{1}{2}}}, \end{aligned}$$

hence, replacing  $u$  by  $v + u_{0,\varepsilon}$  in the right-hand side, we conclude that

$$(22) \quad \begin{aligned} |B + R + R'| &\lesssim_{c_0} \left( \kappa^{-1} \|u\|_{\dot{H}^{\frac{1}{2}}} + \varepsilon^{-\frac{1}{2}} \|u_0\|_{\dot{H}^{\frac{1}{2}}} \right) \|v\|_{\dot{H}^{\frac{1}{2}}} \|W\|_{L^2} \\ &\quad + \varepsilon^{-\frac{1}{2}} \left( \|u_0\|_{\dot{H}^{\frac{1}{2}}} + \|u_0\|_{\dot{H}^{\frac{1}{2}}}^2 \right) \|W\|_{L^2} + \|u_0\|_{\dot{H}^{\frac{1}{2}}} \|v\|_{\dot{H}^{\frac{1}{2}}}. \end{aligned}$$

*Step 2: Estimate of A.* Since  $H_{\kappa,1}$  is skew-symmetric, one easily verifies that, parallel to (9), we have

$$A = \frac{1}{2} \int_{\mathbb{S}} ([V, H_{\kappa,1}] \Lambda v) \Lambda v \, dx.$$

Then, we multiply and divide by  $\sqrt{\gamma}$  and use the Cauchy-Schwarz inequality, to find

$$(23) \quad A \leq \left( \int_{\mathbb{S}} \gamma^{-1} |[V, H_{\kappa,1}] (\Lambda v)|^2 \, dx \right)^{\frac{1}{2}} \|\sqrt{\gamma} \Lambda v\|_{L^2}.$$

We now have to estimate the commutator  $[V, H_{\kappa,1}]$ . Directly from the definition of  $H_{\kappa,1}$ , we have the following identities

$$\begin{aligned} [V, H_{\kappa,1}] (\Lambda v) &= \frac{1}{4\pi} \text{pv} \int_{\mathbb{S}} \frac{V(x) (\delta_{\alpha} \Lambda v)(x) - \delta_{\alpha} (V \Lambda v)(x)}{\tan(\alpha/2)} \chi\left(\frac{\alpha}{\kappa}\right) \, d\alpha \\ &= -\frac{1}{4\pi} \text{pv} \int_{\mathbb{S}} \frac{(\delta_{\alpha} V)(x) (\Lambda v)(x - \alpha)}{\tan(\alpha/2)} \chi\left(\frac{\alpha}{\kappa}\right) \, d\alpha \\ &= -\frac{1}{4\pi} \text{pv} \int_{\mathbb{S}} \frac{(\delta_{\alpha} V)(x)}{\sqrt{\gamma}(x - \alpha)} W(x - \alpha) \chi\left(\frac{\alpha}{\kappa}\right) \frac{d\alpha}{\tan(\alpha/2)}, \end{aligned}$$

where we replaced  $\Lambda v$  by  $W/\sqrt{\gamma}$  to obtain the last identity.

Therefore,

$$\begin{aligned} &\int_{\mathbb{S}} \gamma^{-1} |[V, H_{\kappa,1}] (\Lambda v)|^2 \, dx \\ &\lesssim \int_{\mathbb{S}} (\gamma(x))^{-1} \left( \int_{|\alpha| \leq 2\kappa} |\delta_{\alpha} V(x)| \gamma(x - \alpha)^{-1/2} |W(x - \alpha)| \frac{d\alpha}{|\tan(\alpha/2)|} \right)^2 \, dx \\ &\lesssim_{c_0} \int_{\mathbb{S}} (\rho^2(x)) \left( \int_{|\alpha| \leq 2\kappa} |\delta_{\alpha} V(x)| \rho(x - \alpha) |W(x - \alpha)| \frac{d\alpha}{|\tan(\alpha/2)|} \right)^2 \, dx. \end{aligned}$$

LEMMA 3.3. *Introduce the notation*

$$Q_{\alpha}(g) := |\delta_{\alpha} g| + |\delta_{\alpha} Hg|.$$

*Then there holds*

$$(24) \quad |\delta_{\alpha} V| \lesssim_{c_0} \frac{Q_{\alpha}(v)(1 + Q_{\alpha}(v))^2}{\rho^2} + \varepsilon^{-3} (1 + \|u_0\|_{H^{\frac{1}{2}}})^3 |\alpha|.$$

PROOF. One has

$$\begin{aligned} |\delta_{\alpha} V(x)| &\lesssim \frac{|\delta_{\alpha} H u(x)|}{\rho^2(x)} + |H u(x - \alpha)| \frac{|\delta_{\alpha} u(x)| (|u(x)| + |u(x - \alpha)|)}{\rho^2(x - \alpha) \rho^2(x)} \\ &\lesssim_{c_0} \frac{|\delta_{\alpha} H u(x)|}{\rho^2(x)} + \frac{|\delta_{\alpha} u(x)|^2 + |\delta_{\alpha} H u(x)|^2}{\rho(x - \alpha) \rho(x)} + \frac{|\delta_{\alpha} u(x)| + |\delta_{\alpha} H u(x)|}{\rho(x - \alpha) \rho(x)}. \end{aligned}$$

Since

$$\frac{1}{\rho(x - \alpha)} \lesssim \frac{1}{\rho(x)} (1 + Q_{\alpha}(u)),$$

we obtain

$$|\delta_{\alpha} V| \lesssim_{c_0} \frac{Q_{\alpha}(u)(1 + Q_{\alpha}(u))^2}{\rho^2}.$$

To get the wanted result (24) from this, we replace  $u$  by  $v + u_{0,\varepsilon}$  and use the two following elementary ingredients:

$$\frac{Q_\alpha(u_{0,\varepsilon})(1 + Q_\alpha(u_{0,\varepsilon}))^2}{\rho^2} \lesssim_{c_0} Q_\alpha(u_{0,\varepsilon})(1 + Q_\alpha(u_{0,\varepsilon}))^2,$$

$$Q_\alpha(u_{0,\varepsilon}) \leq (\|\partial_x u_{0,\varepsilon}\|_{L^\infty} + \|\partial_x(Hu_{0,\varepsilon})\|_{L^\infty}) |\alpha| \lesssim \|u_{0,\varepsilon}\|_{H^2} |\alpha| \lesssim \varepsilon^{-\frac{3}{4}} \|u_0\|_{H^{\frac{1}{2}}} |\alpha|.$$

Since  $|\alpha|^3 \lesssim |\alpha|$ , this completes the proof.  $\square$

Set  $K(\varepsilon) := \varepsilon^{-3}(1 + \|u_{0,\varepsilon}\|_{H^{\frac{1}{2}}})^3$ . It follows from the previous lemma and the preceding inequality that

$$\int_{\mathbb{S}} \gamma^{-1} |[V, H_{\kappa,1}](\Lambda v)|^2 dx \lesssim_{c_0} (I) + (II),$$

where

$$(I) := \int_{\mathbb{S}} \rho(x)^2 \left( \int_{|\alpha| \leq 2\kappa} \frac{Q_\alpha(v)(x)(1 + Q_\alpha(v)(x))^2}{\rho(x)^2} \rho(x - \alpha) |W(x - \alpha)| \frac{d\alpha}{|\tan(\frac{\alpha}{2})|} \right)^2 dx,$$

$$(II) := K(\varepsilon)^2 \int_{\mathbb{S}} \rho(x)^2 \left( \int_{|\alpha| \leq 2\kappa} \rho(x - \alpha) |W(x - \alpha)| d\alpha \right)^2 dx.$$

Using the Cauchy-Schwarz inequality, we see that

$$(I) \lesssim \left( \iint_{\mathbb{S}^2} Q_\alpha(v)(x)^2 (1 + Q_\alpha(v)(x))^4 \frac{\rho^2(x - \alpha)}{\rho^2(x)} \frac{d\alpha dx}{|\alpha|^2} \right) \|W\|_{L^2}^2.$$

Since

$$\frac{\rho^2(x - \alpha)}{\rho^2(x)} \lesssim 1 + |Q_\alpha(u)(x)|^2,$$

we end up with

$$(I) \lesssim \left( \iint_{\mathbb{S}^2} Q_\alpha(v)(x)^2 (1 + Q_\alpha(v)(x))^4 (1 + Q_\alpha(u)(x))^2 \frac{d\alpha dx}{|\alpha|^2} \right) \|W\|_{L^2}^2.$$

On the other hand,

$$\begin{aligned} & \iint_{\mathbb{S}^2} Q_\alpha(v)(x)^2 (1 + Q_\alpha(v)(x))^4 (1 + Q_\alpha(u)(x))^2 \frac{d\alpha dx}{|\alpha|^2} \\ & \lesssim \iint_{\mathbb{S}^2} Q_\alpha(v)(x)^2 (1 + Q_\alpha(v)(x))^4 \frac{d\alpha dx}{|\alpha|^2} \\ & + \left( \iint_{\mathbb{S}^2} Q_\alpha(v)(x)^4 (1 + Q_\alpha(v)(x))^8 \frac{d\alpha dx}{|\alpha|^2} \right)^{\frac{1}{2}} \left( \iint_{\mathbb{S}^2} Q_\alpha(u)(x)^4 \frac{d\alpha dx}{|\alpha|^2} \right)^{\frac{1}{2}} \\ & \lesssim \|v\|_{\dot{H}^{\frac{1}{2}}}^2 (1 + \|v\|_{\dot{H}^{\frac{1}{2}}})^4 (1 + \|u\|_{\dot{H}^{\frac{1}{2}}})^2, \end{aligned}$$

where we used the fact that  $Q_\alpha(f)(x) = |\delta_\alpha f(x)| + |\delta_\alpha Hf(x)|$  and

$$\iint_{\mathbb{S}^2} \left( |\delta_\alpha f(x)|^{2p} + |\delta_\alpha Hf(x)|^{2p} \right) \frac{d\alpha dx}{|\alpha|^2} \lesssim \|f\|_{\dot{H}^{\frac{1}{2}}}^{2p}$$

for any  $p \geq 1$ .

This gives

$$(I) \lesssim \|v\|_{\dot{H}^{\frac{1}{2}}}^2 (1 + \|v\|_{\dot{H}^{\frac{1}{2}}})^4 (1 + \|u\|_{\dot{H}^{\frac{1}{2}}})^2 \|W\|_{L^2}^2.$$

On the other hand,

$$(II) \lesssim K(\varepsilon)^2 \kappa^{\frac{1}{2}} \|u\|_{\dot{H}^{\frac{1}{2}}}^4 \|W\|_{L^2}^2.$$

Therefore, it follows from (23) that

$$\begin{aligned} A &\leq \left( \int_{\mathbb{S}} \gamma^{-1} |[V, H_{\kappa,1}](\Lambda v)|^2 dx \right)^{\frac{1}{2}} \|W\|_{L^2} \\ &\lesssim ((I) + (II))^{\frac{1}{2}} \|W\|_{L^2} \\ &\lesssim \|v\|_{\dot{H}^{\frac{1}{2}}} (1 + \|v\|_{\dot{H}^{\frac{1}{2}}})^2 (1 + \|u\|_{\dot{H}^{\frac{1}{2}}}) \|W\|_{L^2}^2 + K(\varepsilon) \kappa^{\frac{1}{4}} \|u\|_{\dot{H}^{\frac{1}{2}}}^2 \|W\|_{L^2}^2. \end{aligned}$$

By combining this with (22), we get from (19) that there exists a constant  $C$  depending only on  $c_0$  such that

$$\begin{aligned} (25) \quad \frac{1}{2} \frac{d}{dt} \|v\|_{\dot{H}^{\frac{1}{2}}}^2 + \|W\|_{L^2}^2 &\leq C \|v\|_{\dot{H}^{\frac{1}{2}}} (1 + \|v\|_{\dot{H}^{\frac{1}{2}}})^2 (1 + \|u\|_{\dot{H}^{\frac{1}{2}}}) \|W\|_{L^2}^2 \\ &\quad + CK(\varepsilon) \kappa^{\frac{1}{4}} \|u\|_{\dot{H}^{\frac{1}{2}}}^2 \|W\|_{L^2}^2 \\ &\quad + C \left( \kappa^{-1} \|u\|_{\dot{H}^{\frac{1}{2}}} + C\varepsilon^{-\frac{1}{2}} \|u_0\|_{\dot{H}^{\frac{1}{2}}} \right) \|v\|_{\dot{H}^{\frac{1}{2}}} \|W\|_{L^2} \\ &\quad + C\varepsilon^{-\frac{1}{2}} \left( \|u_0\|_{\dot{H}^{\frac{1}{2}}} + \|u_0\|_{\dot{H}^{\frac{1}{2}}}^2 \right) \|W\|_{L^2} + \|u_0\|_{\dot{H}^{\frac{1}{2}}} \|W\|_{L^2} \\ &\quad + C \|u_0\|_{\dot{H}^{\frac{1}{2}}} \|v\|_{\dot{H}^{\frac{1}{2}}}. \end{aligned}$$

Using the Young's inequality, this immediately yields an inequality of the form

$$(26) \quad \frac{1}{2} \frac{d}{dt} \|v\|_{\dot{H}^{\frac{1}{2}}}^2 + \Upsilon \|W\|_{L^2}^2 \leq M \|v\|_{\dot{H}^{\frac{1}{2}}}^2 + F,$$

where

$$\begin{aligned} M &:= 4C^2 \varepsilon^{-1} \left( \|u_0\|_{\dot{H}^{\frac{1}{2}}} + \|u_0\|_{\dot{H}^{\frac{1}{2}}}^2 \right)^2 + 1, \\ F &:= 4C^2 \varepsilon^{-1} \left( \|u_0\|_{\dot{H}^{\frac{1}{2}}} + \|u_0\|_{\dot{H}^{\frac{1}{2}}}^2 \right)^2, \\ \Upsilon &:= \frac{1}{4} - C \|v\|_{\dot{H}^{\frac{1}{2}}} (1 + \|v\|_{\dot{H}^{\frac{1}{2}}})^2 (1 + \|u\|_{\dot{H}^{\frac{1}{2}}}) \\ &\quad - CK(\varepsilon) \kappa^{\frac{1}{4}} \|u\|_{\dot{H}^{\frac{1}{2}}}^2 \\ &\quad - C^2 \left( \kappa^{-1} \|u\|_{\dot{H}^{\frac{1}{2}}} + \varepsilon^{-\frac{1}{2}} \|u_0\|_{\dot{H}^{\frac{1}{2}}} \right)^2 \|v\|_{\dot{H}^{\frac{1}{2}}}^2. \end{aligned}$$

In particular, as long as  $\Upsilon \geq 0$ , we have

$$\|v(t)\|_{\dot{H}^{\frac{1}{2}}}^2 \leq e^{2Mt} \|v(0)\|_{\dot{H}^{\frac{1}{2}}}^2 + \frac{e^{tM} - 1}{M} F.$$

If one further assumes that  $tM \leq 1$ , it follows that

$$\|v(t)\|_{\dot{H}^{\frac{1}{2}}}^2 \leq e^{2Mt} \|v(0)\|_{\dot{H}^{\frac{1}{2}}}^2 + tF.$$

Introduce the parameter

$$\nu(\varepsilon) := 2 \|v|_{t=0}\|_{\dot{H}^{\frac{1}{2}}(\mathbb{S})} = 2 \left\| u_0 - e^{\varepsilon \partial_x^2} u_0 \right\|_{\dot{H}^{\frac{1}{2}}(\mathbb{S})}.$$

Then choose  $\varepsilon$  small enough, so that

$$C\nu(\varepsilon)(1 + \nu(\varepsilon))^2(1 + 2\|u_0\|_{\dot{H}^{\frac{1}{2}}}) \leq \frac{1}{16}.$$

We then fix  $\kappa$  small enough to that

$$CK(\varepsilon)\kappa^{\frac{1}{4}}(2\|u_0\|_{\dot{H}^{\frac{1}{2}}})^2 \leq \frac{1}{16},$$

where recall that  $K(\varepsilon) := \varepsilon^{-3}(1 + \|u_{0,\varepsilon}\|_{\dot{H}^{\frac{1}{2}}})^3$ .

We then deduce the wanted uniform estimate by an elementary continuation argument.  $\square$

**3.3. Compactness.** Previously, we have proved *a priori* estimates for the spatial derivatives. In this paragraph, we collect results from which we will derive estimates for the time derivative as well as for the nonlinearity. These estimates are used to pass to the limit in the equation.

Recall the notations introduced in the previous section, as well as the estimates proved there. Fix  $c_0 > 0$ . Given  $\delta \in (0, 1]$  and an initial data  $u_0 \in H^{\frac{1}{2}}(\mathbb{S})$  satisfying  $u_0 \geq c_0$ , we have seen that there exists a (global in time) solution  $u_\delta$  to the Cauchy problem:

$$(27) \quad \begin{cases} \partial_t u_\delta + \frac{1}{\pi} \frac{u_\delta \Lambda u_\delta - (H u_\delta) \partial_x u_\delta}{u_\delta^2 + (H u_\delta)^2} - \delta \partial_x^2 u_\delta = 0, \\ u_\delta|_{t=0} = e^{\delta \partial_x^2} u_0. \end{cases}$$

Moreover, we have proved that one can fix  $\varepsilon$  small enough such that one can write  $u_\delta$  under the form

$$u_\delta(t, x) = (e^{(\varepsilon+\delta)\partial_x^2} u_0)(x) + v_\delta(x),$$

and there exist  $T > 0$  and  $M > 0$  depending on  $u_0$  such that, for all  $\delta \in (0, 1]$ ,

$$(28) \quad \sup_{t \in [0, T]} \|v_\delta(t)\|_{\dot{H}^{\frac{1}{2}}}^2 + \int_0^T \int_{\mathbb{S}} \frac{u_\delta |\Lambda v_\delta|^2}{u_\delta^2 + (H u_\delta)^2} dx dt + \delta \int_0^T \|v_\delta\|_{\dot{H}^{\frac{3}{2}}}^2 dt \leq M.$$

Now, to pass to the limit in the equation (27), we need to estimate the time derivative. Since  $\partial_t u_\delta = \partial_t v_\delta$ , it is sufficient to estimate the latter quantity. It is given by

$$(29) \quad \partial_t v_\delta = -V_\delta \partial_x v_\delta - \gamma_\delta \Lambda v_\delta + \delta \partial_x^2 v_\delta + R_\delta,$$

where

$$\begin{aligned} \gamma_\delta &= \frac{1}{\pi} \frac{u_\delta}{u_\delta^2 + (H u_\delta)^2}, \quad V_\delta = -\frac{1}{\pi} \frac{H u_\delta}{u_\delta^2 + (H u_\delta)^2}, \\ R_\delta &= -\gamma_\delta (\Lambda e^{(\varepsilon+\delta)\partial_x^2} u_0) - V_\delta (\partial_x e^{(\varepsilon+\delta)\partial_x^2} u_0) + \delta \partial_x^2 e^{(\varepsilon+\delta)\partial_x^2} u_0. \end{aligned}$$

As already seen in (20), we have  $\gamma_\delta \lesssim_{c_0} 1$  and  $|V_\delta| \lesssim_{c_0} 1$ . By combining this with the fact that  $e^{\varepsilon \partial_x^2}$  is a smoothing operator, we immediately see that

$$\|R_\delta\|_{L^\infty([0, T]; L^2)} \lesssim_{c_0, \varepsilon} \|u_0\|_{\dot{H}^{\frac{1}{2}}}.$$

Here the implicit constant depends on  $\varepsilon$ , but this is harmless since  $\varepsilon$  is fixed now. On the other, directly from (28), we get that

$$\|\gamma_\delta \Lambda v_\delta\|_{L^2([0,T];L^2)} \lesssim_{c_0} \|\sqrt{\gamma_\delta} \Lambda v_\delta\|_{L^2([0,T];L^2)} \lesssim_{c_0} M,$$

and

$$\delta \|\partial_x^2 v_\delta\|_{L^2([0,T];H^{-\frac{1}{2}})} \leq \sqrt{\delta} \|\partial_x^2 v_\delta\|_{L^2([0,T];H^{-\frac{1}{2}})} \leq M.$$

It remains only to estimate the contribution of  $V_\delta \partial_x v_\delta$ . For this, we begin by proving that  $(v_\delta)_{\delta \in (0,1]}$  is bounded in  $L^p([0, T(\varepsilon)]; \dot{H}^1(\mathbb{S}))$  for any  $1 \leq p < 2$ . Indeed, we can write

$$\begin{aligned} \|v_\delta\|_{\dot{H}^1}^2 &\leq \left\| \frac{u_\delta^2 + (Hu_\delta)^2}{u_\delta} \right\|_{L^\infty} \int_{\mathbb{S}} \frac{u_\delta |\Lambda v_\delta|^2}{u_\delta^2 + (Hu_\delta)^2} dx \\ &\lesssim_{c_0} \|(u_\delta, Hu_\delta)\|_{L^\infty}^2 \int_{\mathbb{S}} \frac{u_\delta |\Lambda v|^2}{u_\delta^2 + (Hu_\delta)^2} dx \\ (30) \quad &\lesssim_{c_0} \left( \|(v_\delta, Hv_\delta)\|_{L^\infty}^2 + \varepsilon^{-\frac{1}{2}} \|u_0\|_{H^{\frac{1}{2}}}^2 \right) \int_{\mathbb{S}} \frac{u_\delta |\Lambda v|^2}{u_\delta^2 + (Hu_\delta)^2} dx \\ &\lesssim_{c_0} \left( \|v_\delta\|_{H^{\frac{1}{2}}}^2 \log(2 + \|v_\delta\|_{\dot{H}^1}) + \varepsilon^{-\frac{1}{2}} \|u_0\|_{H^{\frac{1}{2}}}^2 \right) \int_{\mathbb{S}} \frac{u_\delta |\Lambda v_\delta|^2}{u_\delta^2 + (Hu_\delta)^2} dx, \end{aligned}$$

to conclude that

$$\frac{\|v_\delta\|_{\dot{H}^1}^2}{\log(2 + \|v_\delta\|_{\dot{H}^1})} \lesssim \int_{\mathbb{S}} \frac{u_\delta |\Lambda v_\delta|^2}{u_\delta^2 + (Hu_\delta)^2} dx,$$

where the implicit constant depends on  $c_0, \varepsilon, \|u_0\|_{H^{\frac{1}{2}}}$  and  $M$ . Remembering that  $|V_\delta| \lesssim_{c_0} 1$ , it immediately follows that, for any  $p \in [1, 2)$ ,  $(V_\delta \partial_x v_\delta)_{\delta \in (0,1]}$  is bounded in  $L^p([0, T]; L^2)$ .

Now, by combining all the previous estimates, we see that  $(v_\delta)_{\delta \in (0,1]}$  is bounded in the space

$$X_p = \left\{ u \in C^0([0, T]; H^{\frac{1}{2}}(\mathbb{S})) \cap L^p([0, T]; H^1(\mathbb{S})); \partial_t u \in L^p([0, T]; H^{-\frac{1}{2}}(\mathbb{S})) \right\}.$$

Since  $H^{\frac{1}{2}}(\mathbb{S})$  (resp.  $H^1(\mathbb{S})$ ) is compactly embedded into  $H^s(\mathbb{S})$  (resp.  $H^{\frac{1}{2}+s}(\mathbb{S})$ ) for any  $s < 1/2$ , by the classical Aubin-Lions lemma, this in turn implies that one can extract a sequence  $(u_{\delta_n})_{n \in \mathbb{N}}$  which converges strongly in

$$C^0([0, T]; H^s(\mathbb{S})) \cap L^p([0, T]; H^{\frac{1}{2}+s}(\mathbb{S})).$$

Then it is elementary to pass to the limit in the equation.

#### 4. Smoothing effect

The goal of this section is to prove the second statement in Theorem 1.1 which asserts that the solution are smooth. By classical methods for parabolic equations (see [40, Chapter 15]), it is easy to prove that solutions which are smooth enough (say with initial data in  $H^2(\mathbb{S})$ ) are  $C^\infty$  for positive time. So it is sufficient to prove that the solution are at least  $H^2$  for positive times. This is the purpose of the following

PROPOSITION 4.1. *The solution  $u$  constructed in the previous section is such that, for any  $\xi > 4$ , there holds*

$$(31) \quad \sup_{t \in [0, T]} t^\xi \|u(t)\|_{\dot{H}^{\frac{5}{2}}}^2 \lesssim_\xi 1.$$

PROOF. Since  $u$  was constructed as the limit of smooth solutions (see §3.1 and §3.3), we will prove only *a priori* estimates.

As in the previous part, we work with the function

$$v = u - e^{\varepsilon \partial_x^2} u_0,$$

with  $\varepsilon$  small enough. For this proof, we denote by  $C(\varepsilon)$  many constants that depend on  $\varepsilon$  and which are allowed to diverge when  $\varepsilon$  tends to 0.

Recall from (17) that  $v$  solves

$$(32) \quad \partial_t v + V \partial_x v + \gamma \Lambda v = R_\varepsilon(u, u_0)$$

where

$$R_\varepsilon(u, u_0) = -\gamma \Lambda u_{0, \varepsilon} - V \partial_x u_{0, \varepsilon}.$$

We will estimate the  $\dot{H}^2$ -norm of  $v$ . For this introduce  $\tilde{v} = \partial_x^2 v$ , solution to

$$(33) \quad \partial_t \tilde{v} + \gamma \Lambda \tilde{v} = V(H_{\kappa, 1} \Lambda \tilde{v}) + R_0 + R_1$$

where

$$\begin{aligned} R_0 &= \partial_x^2 (V(H_{\kappa, 2} \Lambda v)) + \partial_x^2 R_\varepsilon(u, u_0) \\ R_1 &= -(\partial_x^2 \gamma)(\Lambda v) - 2(\partial_x \gamma)(\Lambda \partial_x v) + (\partial_x^2 V)(H_{\kappa, 1} \Lambda v) + 2(\partial_x V)(H_{\kappa, 1} \Lambda \partial_x v). \end{aligned}$$

Now, we multiply (33) by  $\Lambda \tilde{v}$  and then integrate in  $x$  over  $\mathbb{S}$ , to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\tilde{v}\|_{\dot{H}^{\frac{1}{2}}}^2 + \int_{\mathbb{S}} \gamma |\Lambda \tilde{v}|^2 dx &= \frac{1}{2} \int_{\mathbb{S}} [V, H_{\kappa, 1}](\Lambda \tilde{v}) \Lambda \tilde{v} dx \\ &\quad + \int_{\mathbb{S}} R_0 \Lambda \tilde{v} dx + \int_{\mathbb{S}} R_1 \Lambda \tilde{v} dx. \end{aligned}$$

Set

$$\tilde{W} := \sqrt{\gamma} \Lambda \tilde{v}.$$

By using arguments parallel to those used in the first step of the proof of Lemma 3.2, one finds that

$$\begin{aligned} \left| \int_{\mathbb{S}} [V, H_{\kappa, 1}](\Lambda \tilde{v}) \Lambda \tilde{v} dx \right| &\lesssim_{c_0} (1 + \|v\|_{\dot{H}^{\frac{1}{2}}})^2 (1 + \|u\|_{\dot{H}^{\frac{1}{2}}}) \|v\|_{\dot{H}^{\frac{1}{2}}} (1 + \|\tilde{w}\|_{L^2}^2) \\ &\quad + C(\varepsilon) \kappa^{\frac{1}{4}} (\|v\|_{\dot{H}^{\frac{1}{2}}} + 1)^2 \|\tilde{w}\|_{L^2}^2, \end{aligned}$$

and

$$\left| \int_{\mathbb{S}} R_0 \Lambda \tilde{v} dx \right| \lesssim_{c_0} C(\varepsilon, \kappa) \|\tilde{w}\|_{L^2}^{\frac{9}{5}} + C(\varepsilon, \kappa).$$

Since

$$\begin{aligned} |\partial_x \gamma| + |\partial_x V| &\lesssim \frac{1}{\rho^2} (|\partial_x u| + |\Lambda u|), \\ |\partial_x^2 \gamma| + |\partial_x^2 V| &\lesssim \frac{1}{\rho^2} (|\partial_x^2 u| + |H \partial_x^2 u|) + \frac{1}{\rho^3} (|\partial_x u|^2 + |\Lambda u|^2), \end{aligned}$$

then one finds

$$\begin{aligned} |R_1| &\lesssim_{c_0} \left( \frac{1}{\rho^2} (|\partial_x^2 u| + |H \partial_x^2 u|) + \frac{1}{\rho^3} (|\partial_x u|^2 + |\Lambda u|^2) \right) (|\Lambda v| + |H_{\kappa,1} \Lambda v|) \\ &\quad + \frac{1}{\rho^2} (|\partial_x u| + |\Lambda u|) (|\Lambda \partial_x v| + |H_{\kappa,1} \Lambda \partial_x v|). \end{aligned}$$

Therefore, replacing  $u$  by  $e^{\varepsilon \partial_x^2} u_0 + v$  in the right-hand side, we get

$$\begin{aligned} |R_1| &\lesssim_{c_0} \left( \frac{1}{\rho^2} (|\partial_x^2 v| + |H \partial_x^2 v|) + \frac{1}{\rho^3} (|\partial_x v|^2 + |\Lambda v|^2) \right) (|\Lambda v| + |H_{\kappa,1} \Lambda v|) \\ &\quad + \frac{1}{\rho^2} (|\partial_x v| + |\Lambda v|) (|\Lambda \partial_x v| + |H_{\kappa,1} \Lambda \partial_x v|) \\ &\quad + C(\varepsilon) (|\Lambda v| + |H_{\kappa,1} \Lambda v| + |\Lambda \partial_x v| + |H_{\kappa,1} \Lambda \partial_x v|). \end{aligned}$$

Thus,

$$\begin{aligned} \left| \int_{\mathbb{S}} R_1 \Lambda \tilde{v} \, dx \right| &\lesssim_{c_0} \|v\|_{\dot{H}^{\frac{1}{2}}} (1 + \|v\|_{\dot{H}^{\frac{1}{2}}})^3 \|\tilde{w}\|_{L^2}^2 \\ &\quad + C(\varepsilon) \kappa^{\frac{1}{4}} (\|v\|_{\dot{H}^{\frac{1}{2}}} + 1)^4 \|\tilde{w}\|_{L^2}^2 + C(\kappa, \varepsilon). \end{aligned}$$

Therefore, we find

$$\begin{aligned} \partial_t \|\tilde{v}\|_{\dot{H}^{\frac{1}{2}}}^2 + \int_{\mathbb{S}} |\tilde{w}|^2 \, dx &\lesssim_{c_0} \|v\|_{\dot{H}^{\frac{1}{2}}} (1 + \|v\|_{\dot{H}^{\frac{1}{2}}})^3 \|\tilde{w}\|_{L^2}^2 \\ &\quad + C(\varepsilon) \kappa^{\frac{1}{4}} (\|v\|_{\dot{H}^{\frac{1}{2}}} + 1)^4 \|\tilde{w}\|_{L^2}^2 + C(\kappa, \varepsilon). \end{aligned}$$

Choosing  $\varepsilon$  and then  $\kappa$  small enough, we obtain

$$\partial_t \|\tilde{v}\|_{\dot{H}^{\frac{1}{2}}}^2 + \int_{\mathbb{S}} |\tilde{w}|^2 \, dx \lesssim C(c_0, \kappa, \varepsilon),$$

for any  $t \in (0, T)$ . Now that  $\varepsilon$  and  $\kappa$  are fixed, we shall skip the dependence of the various implicit constants on  $\varepsilon$  and  $\kappa$ .

So, integrating in time  $\tau \in (s, t)$  we obtained that for  $0 < s < t \leq T$ ,

$$\|v(t)\|_{\dot{H}^{\frac{5}{2}}}^2 + \int_s^t \int_{\mathbb{S}} \frac{u |\Lambda^3 v|^2}{u^2 + (Hu)^2} \, dx \, d\tau \lesssim t + \|v(s)\|_{\dot{H}^{\frac{5}{2}}}^2.$$

Then, in order to measure the decay rate in time, we multiply the last inequality by  $s^{\xi-1}$  and then integrate in  $s \in [0, t/2]$ , to find

$$\begin{aligned} \int_0^{t/2} s^{\xi-1} \|v(t)\|_{\dot{H}^{\frac{5}{2}}}^2 \, ds + \int_0^{t/2} s^{\xi-1} \int_s^t \int_{\mathbb{S}} \frac{u |\Lambda^3 v|^2}{u^2 + (Hu)^2} \, dx \, d\tau \, ds \\ \lesssim \int_0^t t s^{\xi-1} + s^{\xi-1} \|v(s)\|_{\dot{H}^{\frac{5}{2}}}^2 \, ds. \end{aligned}$$



Then,

$$\begin{aligned} \sup_{0 < t < T} t^\xi \|v(t)\|_{\dot{H}^{\frac{5}{2}}}^2 + \int_0^T s^\xi \int_{\mathbb{S}} \frac{u|\Lambda^3 v|^2}{u^2 + (Hu)^2} dx ds \\ \lesssim T^{\xi+1} + \int_0^T s^{\xi-1} \|v(s)\|_{\dot{H}^{\frac{5}{2}}}^2 ds. \end{aligned}$$

Now, parallel to (30), we have

$$\frac{\|v\|_{\dot{H}^3}^2}{\log(2 + \|v\|_{\dot{H}^3})} \lesssim_{c_0, \epsilon, \|u_0\|_{\dot{H}^{\frac{1}{2}}}} \int_{\mathbb{S}} \frac{u|\Lambda^3 v|^2}{u^2 + (Hu)^2} dx.$$

So we end up with,

$$\sup_{0 < t < T} t^\xi \|v(t)\|_{\dot{H}^{\frac{5}{2}}}^2 + \int_0^T s^\xi \frac{\|v\|_{\dot{H}^3}^2}{\log(2 + \|v\|_{\dot{H}^3})} ds \lesssim T^{\xi+1} + \int_0^T s^{\xi-1} \|v(s)\|_{\dot{H}^{\frac{5}{2}}}^2 ds.$$

Then, we use the fact that for  $\delta > 0$  small enough,  $a^{2-\delta} \lesssim \frac{a^2}{\log(2+a)}$ , so

$$(34) \quad \sup_{0 < t < T} t^\xi \|v(t)\|_{\dot{H}^{\frac{5}{2}}}^2 + \int_0^T s^\xi \|u(s)\|_{\dot{H}^3}^{2-\delta} ds \lesssim T^{\xi+1} + \int_0^T s^{\xi-1} \|u(s)\|_{\dot{H}^{5/2}}^2 ds.$$

Let  $\epsilon > 0$  and  $\delta_0 > 0$ , using the following interpolation inequality

$$s^{\xi-1} \|u(s)\|_{\dot{H}^{5/2}}^2 \leq \epsilon s^{\xi-1+\delta_0} \|u(s)\|_{\dot{H}^3}^{8/5} \epsilon^{-1} s^{-\delta_0} \|u(s)\|_{\dot{H}^{\frac{1}{2}}}^{2/5},$$

together with Young's inequality (with the conjugate exponents  $p = \frac{10-5\delta}{8}$  and  $q = \frac{10-5\delta}{2-5\delta}$ ) we find

$$\begin{aligned} \int_0^T s^{\xi-1} \|u(s)\|_{\dot{H}^{5/2}}^2 ds &\leq \frac{\epsilon^p}{p} \int_0^T s^{p\xi-p+p\delta_0} \|u(s)\|_{\dot{H}^3}^{2-\delta} ds \\ &\quad + \frac{1}{q\epsilon^q} \sup_{s \in (0, T)} \|u(s)\|_{\dot{H}^{\frac{1}{2}}}^{2q/5} \int_0^T s^{-q\delta_0} ds. \end{aligned}$$

Therefore, inequality (34) becomes

$$(35) \quad \begin{aligned} \sup_{0 < t < T} t^\xi \|v(t)\|_{\dot{H}^{\frac{5}{2}}}^2 + \int_0^T s^\xi \|u(s)\|_{\dot{H}^3}^{2-\delta} ds &\lesssim T^{\xi+1} + \frac{\epsilon^p}{p} \int_0^T s^{p\xi-p+p\delta_0} \|u(s)\|_{\dot{H}^3}^{2-\delta} ds \\ &\quad + \frac{1}{q\epsilon^q} \sup_{s \in (0, T)} \|u(s)\|_{\dot{H}^{\frac{1}{2}}}^{2q/5} \int_0^T s^{-q\delta_0} ds. \end{aligned}$$

Choosing  $m, \delta_0$  such that  $q\delta_0 < 1$  (so that the last integral in the right hand side is finite) and, for a scaling purpose, we also need that

$$\xi = p\xi - p + p\delta_0.$$

So,

$$\xi = q - \delta_0 q > q - 1.$$

Then, for any  $\xi > 4$  (note that  $q > 4$  for any  $\delta > 0$ ) and any  $\epsilon > 0$  sufficiently small, we may absorb the  $\dot{H}^3$  in the left hand side of inequality (35), we obtain

$$\sup_{0 < t < T} t^\xi \|v(t)\|_{\dot{H}^{\frac{5}{2}}}^2 + \int_0^T s^\xi \|u(s)\|_{\dot{H}^3}^{2-\delta} ds \lesssim 1.$$

In particular, for any  $\xi > 4$

$$\sup_{0 < t < T} t^\xi \|v(t)\|_{\dot{H}^{\frac{5}{2}}}^2 \lesssim 1.$$

□

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## References

- [1] Thomas Alazard and Omar Lazar. Paralinearization of the Muskat equation and application to the Cauchy problem. *Arch. Ration. Mech. Anal.*, 237(2):545–583, 2020.
- [2] Thomas Alazard and Quoc-Hung Nguyen. On the Cauchy problem for the Muskat equation with non-Lipschitz initial data. [arXiv:2009.04343](https://arxiv.org/abs/2009.04343).
- [3] Thomas Alazard and Quoc-Hung Nguyen. On the Cauchy problem for the Muskat equation. II: Critical initial data. *Ann. PDE*, 7, 7, 2021.
- [4] Thomas Alazard and Quoc-Hung Nguyen. Endpoint Sobolev theory for the Muskat equation. [arXiv:2010.06915](https://arxiv.org/abs/2010.06915).
- [5] Thomas Alazard and Quoc-Hung Nguyen. Quasilinearization of the 3D Muskat equation, and applications to the critical Cauchy problem. [arXiv:2103.02474](https://arxiv.org/abs/2103.02474).
- [6] David M. Ambrose. Well-posedness of two-phase Hele-Shaw flow without surface tension. *European J. Appl. Math.*, 15(5):597–607, 2004.
- [7] Hantaek Bae, Rafael Granero-Belinchón. Global existence for some transport equations with nonlocal velocity. *Advances in Mathematics*, vol.269, pp 197-219, 2015
- [8] Luis A. Caffarelli and Alexis Vasseur. Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation. *Ann. of Math. (2)*, 171(3):1903–1930, 2010.
- [9] Stephen Cameron. Global well-posedness for the two-dimensional Muskat problem with slope less than 1. *Anal. PDE*, 12(4):997–1022, 2019.
- [10] Stephen Cameron. Eventual regularization for the 3d Muskat problem: Lipschitz for finite time implies global existence, [arXiv:2007.03099](https://arxiv.org/abs/2007.03099), 2020.
- [11] Stephen Cameron. Global wellposedness for the 3d Muskat problem with medium size slope. [arXiv:2002.00508](https://arxiv.org/abs/2002.00508), 2020.
- [12] Ángel Castro, Diego Córdoba, Charles Fefferman, Francisco Gancedo, and María López-Fernández. Rayleigh-Taylor breakdown for the Muskat problem with applications to water waves. *Ann. of Math. (2)*, 175(2):909–948, 2012.
- [13] Ke Chen, Quoc-Hung Nguyen, Yiran Xu. The Muskat problem with  $C^1$  data. [arXiv:2103.09732](https://arxiv.org/abs/2103.09732).
- [14] C. H. Arthur Cheng, Rafael Granero-Belinchón, and Steve Shkoller. Well-posedness of the Muskat problem with  $H^2$  initial data. *Adv. Math.*, 286:32–104, 2016.
- [15] Peter Constantin, Diego Córdoba, Francisco Gancedo, Luis Rodríguez-Piazza, and Robert M. Strain. On the Muskat problem: global in time results in 2D and 3D. *Amer. J. Math.*, 138(6):1455–1494, 2016.
- [16] Peter Constantin, Diego Córdoba, Francisco Gancedo, and Robert M. Strain. On the global existence for the Muskat problem. *J. Eur. Math. Soc. (JEMS)*, 15(1):201–227, 2013.

- [17] Peter Constantin, Francisco Gancedo, Roman Shvydkoy, and Vlad Vicol. Global regularity for 2D Muskat equations with finite slope. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 34(4):1041–1074, 2017.
- [18] Peter Constantin, Peter Lax and Andrew Majda. A simple one-dimensional model for the three-dimensional vorticity equation. *Comm. Pure Appl. Math.* 38 (1985), no. 6, 715–724.
- [19] Peter Constantin and Vlad Vicol. Nonlinear maximum principles for dissipative linear nonlocal operators and applications. *Geom. Funct. Anal.*, 22(5):1289–1321, 2012.
- [20] Antonio Córdoba, Diego Córdoba and Marco A. Fontelos. Formation of singularities for a transport equation with nonlocal velocity. *Ann. of Math. (2)* 162(3):1377–1389, 2005.
- [21] Diego Córdoba and Omar Lazar. Global well-posedness for the 2d stable Muskat problem in  $H^{\frac{3}{2}}$ . *Ann. Sci. Éc. Norm. Supér.*, to appear 2021.
- [22] Fan Deng, Zhen Lei, and Fanghua Lin. On the two-dimensional Muskat problem with monotone large initial data. *Comm. Pure Appl. Math.*, 70(6):1115–1145, 2017.
- [23] Hongjie Dong. Well-posedness for a transport equation with nonlocal velocity. *J. Funct. Anal.*, 255:3070–3097, (2008).
- [24] Francisco Gancedo and Omar Lazar. Global well-posedness for the 3d Muskat problem in the critical Sobolev space. [arXiv:2006.01787](https://arxiv.org/abs/2006.01787).
- [25] Rafael Granero-Belinchón. On a nonlocal differential equation describing roots of polynomials under differentiation. *Commun. Math. Sci.* 18(6):1643–1660, 2020.
- [26] Alexander Kiselev and Fedor Nazarov. A variation on a theme of Caffarelli and Vasseur. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 370(Kraevye Zadachi Matematicheskoi Fiziki i Smezhnye Voprosy Teorii Funktsii. 40):58–72, 220, 2009.
- [27] Alexander Kiselev, Fedor Nazarov and Alexander Volberg. Global well-posedness for the critical 2D dissipative quasi-geostrophic equation. *Invent. Math.*, 167(3):445–453, 2007.
- [28] Alexander Kiselev and Changhui Tan. The flow of polynomial roots under differentiation. [arXiv:2012.09080](https://arxiv.org/abs/2012.09080).
- [29] Omar Lazar. On a 1D nonlocal transport equation with nonlocal velocity and subcritical or supercritical diffusion. *J. Differential Equations*, 261(9):4974–4996, 2016.
- [30] Omar Lazar and Pierre-Gilles Lemarié-Rieusset. Infinite energy solutions for a 1D transport equation with nonlocal velocity. *Dyn. Partial Differ. Equ.* 13, no. 2, 107–131, 2016.
- [31] Huy Q. Nguyen and Benoît Pausader. A paradifferential approach for well-posedness of the Muskat problem. *Arch. Ration. Mech. Anal.*, 237(1):35–100, 2020.
- [32] Quoc-Hung Nguyen and Yannick Sire. Potential theory for drift diffusion equations with critical diffusion and applications to the dissipative SQG equation. [arXiv:2003.10848](https://arxiv.org/abs/2003.10848), 2020.
- [33] Dong Li and José Luis Rodrigo. *Blow-up of solutions for a 1D transport equation with nonlocal velocity and supercritical dissipation*. Advances in Mathematics, 217, no. 6, 2563–2568 (2008).
- [34] Sean O’Rourke and Stefan Steinerberger. A nonlocal transport equation modeling complex roots of polynomials under differentiation. [arXiv:1910.12161](https://arxiv.org/abs/1910.12161).
- [35] Neel Patel and Robert M. Strain. Large time decay estimates for the Muskat equation. *Comm. Partial Differential Equations*, 42(6):977–999, 2017.
- [36] Dimitri Shlyakhtenko, Terence Tao. With an appendix by David Jekel. Fractional free convolution powers. [arXiv:2009.01882](https://arxiv.org/abs/2009.01882).
- [37] Luis Silvestre. Hölder estimates for advection fractional-diffusion equations. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 11(4):843–855, 2012.
- [38] Stefan Steinerberger. A nonlocal transport equation describing roots of polynomials under differentiation. *Proc. Amer. Math. Soc.* 147:4733–4744, 2019.
- [39] Stefan Steinerberger. Free convolution of measures via roots of polynomials. [arXiv:2009.03869](https://arxiv.org/abs/2009.03869).
- [40] Michael E. Taylor. Partial differential equations III. Nonlinear equations. Second edition. Applied Mathematical Sciences, 117. Springer, New York, 2011. xxii+715 pp.
- [41] Ioann Vasilyev and François Vigneron. Variation on a theme by Kiselev and Nazarov: Hölder estimates for non-local transport-diffusion, along a non-divergence-free BMO field [arXiv:2002.11542](https://arxiv.org/abs/2002.11542), 2020.

**Thomas Alazard**

Université Paris-Saclay, ENS Paris-Saclay, CNRS,  
Centre Borelli UMR9010, avenue des Sciences, F-91190 Gif-sur-Yvette  
France

and

Department of Mathematics, University of California, Berkeley,  
Berkeley, CA 94720  
USA

**Omar Lazar**

College of Engineering and Technology,  
American University of the Middle East,  
Kuwait

and

Departamento de Análisis Matemático & IMUS,  
Universidad de Sevilla,  
Spain

**Quoc-Hung Nguyen**

ShanghaiTech University,  
393 Middle Huaxia Road, Pudong,  
Shanghai, 201210,  
China