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ON THE CAUCHY PROBLEM FOR THE MUSKAT EQUATION WITH NON-LIPSCHITZ INITIAL DATA

THOMAS ALAZARD AND QUOC-HUNG NGUYEN

ABSTRACT. This article is devoted to the study of the Cauchy problem for the Muskat equation. We consider initial data belonging to the critical Sobolev space of functions with three-half derivative in L^2 , up to a fractional logarithmic correction. As a corollary, we obtain the first local and global well-posedness results for initial free surfaces which are not Lipschitz.

1. INTRODUCTION

The Muskat equation is an important model in the analysis of free surface flows, which describes the dynamics of the interface separating two fluids whose velocities obey Darcy's law ([25, 33]). Its two main features are that it is a fractional parabolic equation and a highly nonlinear equation. These two features are shared by several equations which have attracted a lot of attention in recent years, like the surface quasi-geostrophic equation, the Hele-Shaw equation or the fractional porous media equation, to name a few. Among these equations, a specificity of the Muskat equation is that it admits a beautiful compact formulation in terms of finite differences, as observed by Córdoba and Gancedo [23]. The latter formulation allows to study the Cauchy problem by means of tools at the interface of harmonic analysis and nonlinear partial differential equations. In this direction, we are very much influenced by the recent works by Constantin, Córdoba, Gancedo, Rodríguez-Piazza and Strain [19, 18], Córdoba and Lazar [24] and Gancedo and Lazar [29].

Our goal is to introduce for the Muskat problem an approach based on a logarithmic correction to the usual Gagliardo semi-norms which is adapted to both the fractional and nonlinear features of the equation, following earlier works in [2, 9, 8, 10, 7, 35]. Our main result is stated after we introduce some notations, but one can express its main corollary as follows: one can study the Cauchy problem in an almost critical Sobolev space, allowing initial data which are not Lipschitz.

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1.1. The Muskat equation. Consider the dynamics of a time-dependent curve $\Sigma(t)$ separating two $2D$ -domains $\Omega_1(t)$ and $\Omega_2(t)$. On the supposition that $\Sigma(t)$ is the graph of some function, we introduce the following notations

$$\begin{aligned}\Omega_1(t) &= \{(x, y) \in \mathbb{R} \times \mathbb{R}; y > f(t, x)\}, \\ \Omega_2(t) &= \{(x, y) \in \mathbb{R} \times \mathbb{R}; y < f(t, x)\}, \\ \Sigma(t) &= \{(x, y) \in \mathbb{R} \times \mathbb{R}; y = f(t, x)\}.\end{aligned}$$

Assume that each domain Ω_j , $j = 1, 2$, is occupied by an incompressible fluid with constant density ρ_j and denote by $\rho = \rho(t, x)$ the function with value ρ_j for $x \in \Omega_j(t)$. We assume that $\rho_2 > \rho_1$ so that the heavier fluid is underneath the lighter one. Then the motion is determined by the incompressible porous media equations, where the velocity field v is given by Darcy's law:

$$(1) \quad \begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, \\ \operatorname{div} v = 0, \\ v + \nabla(P + \rho g y) = 0, \end{cases}$$

where g is the acceleration of gravity.

Changes of unknowns, reducing the problem (1) to an evolution equation for the free surface parametrization, have been known for quite a time (see [11, 27, 38, 39]). This approach was further developed by Córdoba and Gancedo [23] who obtained a beautiful compact formulation of the Muskat equation. Indeed, they showed that the Muskat problem is equivalent to the following equation for the free surface elevation:

$$(2) \quad \partial_t f = \frac{\rho}{2\pi} \operatorname{pv} \int_{\mathbb{R}} \frac{\partial_x \Delta_\alpha f}{1 + (\Delta_\alpha f)^2} d\alpha,$$

where the integral is understood in the sense of principal values, $\rho = \rho_2 - \rho_1$ is the difference of the densities of the two fluids and $\Delta_\alpha f$ is the slope

$$(3) \quad \Delta_\alpha f(t, x) = \frac{f(t, x) - f(t, x - \alpha)}{\alpha}.$$

Since $\rho_2 > \rho_1$ by assumption, we may set $\rho = 2$ without loss of generality.

A key feature of this problem is that (2) is preserved by the change of unknowns:

$$f(t, x) \mapsto \frac{1}{\lambda} f(\lambda t, \lambda x).$$

Hence, the two natural critical spaces for the initial data are the homogeneous spaces

$$\dot{H}^{\frac{3}{2}}(\mathbb{R}), \quad \dot{W}^{1, \infty}(\mathbb{R}).$$

The analysis of the Cauchy problem for the Muskat equation is now well developed, including global existence results under mild smallness assumptions and blow-up results for some large enough initial data. Local well-posedness results go back to the works of Yi [44], Ambrose [3, 4], Córdoba and Gancedo [23], Córdoba, Córdoba and Gancedo [21], Cheng, Granero-Belinchón, Shkoller [17]. Then local well-posedness results were obtained in the sub-critical spaces by Constantin, Gancedo, Shvydkoy and Vicol [20] for initial data in the Sobolev space $W^{2,p}(\mathbb{R})$ for some $p > 1$, and Matioc [31, 32] for initial data in $H^s(\mathbb{R})$ with $s > 3/2$ (see also [2, 34]). Since

the Muskat equation is parabolic, the proof of the local well-posedness results also gives global well-posedness results under a smallness assumption, see Yi [44]. The first global well-posedness results under mild smallness assumptions, namely assuming that the Lipschitz semi-norm is smaller than 1, was obtained by Constantin, Córdoba, Gancedo, Rodríguez-Piazza and Strain [18] (see also [20, 37]).

On the other hand, there are blow-up results for some large enough data by Castro, Córdoba, Fefferman, Gancedo and López-Fernández ([14, 15, 16]). They prove the existence of solutions such that at time $t = 0$ the interface is a graph, at a later time $t_1 > 0$ the interface is no longer a graph and then at a subsequent time $t_2 > t_1$, the interface is C^3 but not C^4 .

The previous discussion raises a question about the possible existence of a criteria on the slopes of the solutions which would force/prevent them to enter the unstable regime where the slope is infinite. Surprisingly, it is possible to solve the Cauchy problem for initial data whose slope can be arbitrarily large. Deng, Lei and Lin in [26] obtained the first result in this direction, under the assumption that the initial data are monotone. Cameron [12] proved the existence of a modulus of continuity for the derivative, and hence a global existence result assuming only that the product of the maximal and minimal slopes is bounded by 1; thereby allowing arbitrarily large slopes too (recently, Abedin and Schwab also obtained the existence of a modulus of continuity in [1] via Krylov-Safonov estimates). Then, by using a new formulation of the Muskat equation involving oscillatory integrals, Córdoba and Lazar established in [24] that the Muskat equation is globally well-posed in time, assuming only that the initial data is sufficiently smooth and that the $\dot{H}^{3/2}(\mathbb{R})$ -norm is small enough. This result was extended to the 3D case by Gancedo and Lazar [29]. Let us also quote papers by Vazquez [43], Granero-Belinchón and Scrobogna [30] for related global existence results for different equations. The existence and possible non-uniqueness of weak-solutions has also been thoroughly studied (we refer the reader to [6, 22, 40, 13, 28, 36]).

1.2. Fractional logarithmic spaces. Based on the discussion earlier, one of the main questions left open is to solve the Cauchy problem for the Muskat equation for initial data which are not Lipschitz. Indeed, for such data, the slope is not only arbitrarily large but can be infinite. To prove the existence of such solutions, the main difficulties one has to cope with are the following: Firstly, there is a degeneracy in the parabolic behavior when f_x is not controlled (this is easily seen by looking at the energy estimate (8) below: when f_x is not controlled, one does not control the $L^2_{t,x}$ -norm of the derivatives). Secondly, in addition to this degeneracy, one cannot apply classical nonlinear estimates. Indeed the latter require to control the L^∞ -norm of some factors, which amounts here to control the L^∞ -norm of the slopes $\Delta_\alpha f$, equivalent to control the Lipschitz norm of f . To overcome these difficulties, we will use two different kind of arguments, following earlier works in [2, 9, 8]. Firstly, we will prove estimates valid in critical spaces, by exploiting various cancellations as well as specific inequalities. Secondly, we will perform energy estimates in some

variants of the classical Sobolev spaces, allowing to control a *fraction* of a logarithmic derivative. More precisely, the idea followed in this paper is to estimate the following norms.

Definition 1.1. *Given $a \geq 0$ and $s \geq 0$, the fractional logarithmic space $\mathcal{H}^{s,a}(\mathbb{R})$ consists of those functions $g \in L^2(\mathbb{R})$ such that the following norm is finite:*

$$\|g\|_{\mathcal{H}^{s,a}}^2 = \int_{\mathbb{R}} (1 + |\xi|^2)^s (\log(4 + |\xi|))^{2a} |\hat{g}(\xi)|^2 d\xi.$$

Remark 1.2. (i) *Since the formulation of the Muskat equation involves the finite differences of f , it is important to notice that these semi-norms can be defined in terms of finite differences. We will see that, if $s \in (0, 2)$,*

$$\|g\|_{\mathcal{H}^{s,a}}^2 \sim \|g\|_{L^2(\mathbb{R})}^2 + \iint_{\mathbb{R}^2} \frac{|2g(x) - g(x+h) - g(x-h)|^2}{|h|^{2s}} \left[\log \left(4 + \frac{1}{|h|^2} \right) \right]^{2a} \frac{dx dh}{|h|},$$

and if $s = 0$

$$\begin{aligned} \|g\|_{\mathcal{H}^{0,a}}^2 &\sim \|g\|_{L^2(\mathbb{R})}^2 \\ &+ \iint_{\mathbb{R}^2} \mathbf{1}_{|h| < \frac{1}{2}} |2g(x) - g(x+h) - g(x-h)|^2 \left[\log \left(4 + \frac{1}{|h|^2} \right) \right]^{-1+2a} \frac{dx dh}{|h|}. \end{aligned}$$

The latter norms were introduced in [9] for $s \in [0, 1)$ (with the symmetric difference replaced by $g(x+h) - g(x)$).

(ii) Here the word ‘fractional’ is used to insist on the fact that a belongs to $(0, 1]$. This is important in view of (4) below.

We consider initial data in $\mathcal{H}^{3/2,a}(\mathbb{R})$ for some $a \geq 0$. Notice that the latter spaces lie between the Sobolev spaces $H^{3/2}(\mathbb{R})$ and $H^{3/2+\epsilon}(\mathbb{R})$:

$$\forall \epsilon > 0, \forall a \geq 0, \quad H^{\frac{3}{2}+\epsilon}(\mathbb{R}) \subset \mathcal{H}^{\frac{3}{2},a}(\mathbb{R}) \subset \mathcal{H}^{\frac{3}{2},0}(\mathbb{R}) = H^{\frac{3}{2}}(\mathbb{R}).$$

The definition of the Sobolev spaces is recalled below in (11). For our purposes, the most important thing to note is that

$$(4) \quad \mathcal{H}^{\frac{3}{2},a}(\mathbb{R}) \subset W^{1,\infty}(\mathbb{R}) \quad \text{if and only if} \quad a > \frac{1}{2}.$$

It follows from (4) that there is a key dichotomy between the cases $a \leq 1/2$ and $a > 1/2$. Loosely speaking, for $a > 1/2$, the analysis of the Cauchy problem in $\mathcal{H}^{3/2,a}(\mathbb{R})$ is expected to be similar to the one in sub-critical spaces $H^s(\mathbb{R})$ with $s > 3/2$. While for $a \leq 1/2$, the same problem is expected to be much more involved since one cannot control the $W^{1,\infty}$ -norm of f (which is ubiquitous in the estimates of nonlinear quantities involving gradients or the slopes $\Delta_\alpha f$).

1.3. Main results. Once the fractional logarithmic spaces have been introduced, the question of solving the Cauchy problem for non Lipschitz initial data can be made precise. Namely, our goal is to prove that the Cauchy problem is well-posed on $\mathcal{H}^{3/2,a}(\mathbb{R})$ for some $a \leq \frac{1}{2}$. Our main results assert that in fact one can solve the Cauchy problem down to $a = \frac{1}{3}$.

Recall that we set $\rho_2 - \rho_1 = 2$, so that the Muskat equation (2) reads

$$(5) \quad \partial_t f = \frac{1}{\pi} \text{pv} \int_{\mathbb{R}} \frac{\partial_x \Delta_\alpha f}{1 + (\Delta_\alpha f)^2} d\alpha.$$

Theorem 1.3 (local well-posedness). *For any initial data f_0 in $\mathcal{H}^{\frac{3}{2}, \frac{1}{3}}(\mathbb{R})$, there exists a positive time T such that the Cauchy problem for the Muskat equation (5) has a unique solution*

$$f \in C^0([0, T]; \mathcal{H}^{\frac{3}{2}, \frac{1}{3}}(\mathbb{R})) \cap L^2(0, T; H^2(\mathbb{R})).$$

Remark 1.4. *The case $a = \frac{1}{3}$ corresponds to a limiting case. In particular, the time of existence does not depend only on the norm of f_0 but also on f_0 itself. More precisely, we will estimate the solution for a norm whose definition depends on f_0 (this can be understood by looking at Lemma 3.8 and Remark 3.10).*

We now give a global in time well-posedness result under a smallness condition on the following quantity:

$$(6) \quad \begin{aligned} \|f_0\|_{\frac{3}{2}, \frac{1}{3}}^2 &:= \int_{\mathbb{R}} |\xi|^3 \log(4 + |\xi|)^{\frac{2}{3}} |\hat{f}_0(\xi)|^2 d\xi \\ &\sim \iint_{\mathbb{R}^2} \frac{|2f_0(x) - f_0(x+h) - f_0(x-h)|^2}{|h|^3} \left(\log \left(4 + \frac{1}{|h|^2} \right) \right)^{\frac{2}{3}} \frac{dx dh}{|h|}. \end{aligned}$$

Theorem 1.5 (global well-posedness). *There exists a positive constant c_0 such that, for all initial data f_0 in $\mathcal{H}^{3/2, 1/3}(\mathbb{R})$ satisfying*

$$(7) \quad \|f_0\|_{\frac{3}{2}, \frac{1}{3}} \left(\|f_0\|_{L^2}^2 + 1 \right) \leq c_0,$$

the Cauchy problem for the Muskat equation (5) has a unique solution

$$f \in C^0([0, +\infty); \mathcal{H}^{\frac{3}{2}, \frac{1}{3}}(\mathbb{R})) \cap L^2(0, +\infty; H^2(\mathbb{R})).$$

1.4. Strategy of the proof and plan of the paper. To prove Theorem 1.3, the key point is to work with critical-type norms. In this direction, we will prove some technical estimates which we think are of independent interest. To state the main consequence of the latter, let us introduce a bit of notation. We denote by $|D|^{s, \phi}$ the Fourier multiplier $(-\Delta)^{s/2} \phi(D_x)$. Then, our main technical estimate asserts that, for any ϕ , there holds

$$(8) \quad \frac{d}{dt} \| |D|^{\frac{3}{2}, \phi} f \|_{L^2}^2 + \int_{\mathbb{R}} \frac{| |D|^{2, \phi} f |^2}{1 + (\partial_x f)^2} dx \leq CQ(f) \| |D|^{2, \phi} f \|_{L^2},$$

where

$$\begin{aligned} Q(f) &= \left(\|f\|_{\dot{H}^2} + \|f\|_{\dot{H}^{\frac{7}{4}}}^2 \right) \| |D|^{\frac{3}{2}, \phi} f \|_{L^2} + \| |D|^{\frac{7}{4}, \phi} f \|_{L^2} \|f\|_{\dot{H}^{\frac{7}{4}}} \\ &\quad + \left(\|f\|_{\dot{H}^{\frac{3}{2}}} + \|f\|_{\dot{H}^{\frac{7}{4}}}^{1/2} \right) \| |D|^{\frac{7}{4}, \phi^2} f \|_{L^2}^{1/2} \|f\|_{\dot{H}^{\frac{7}{4}}}. \end{aligned}$$

The crucial point is that the quantity $Q(f)$ does not involve the $W^{1, \infty}$ -norm of f .

To prove (8), notable technical aspects include the proof of new commutator estimates and the systematic use of Triebel-Lizorkin norms. We develop these tools

in §2. With these results in hands, we begin in §3 by introducing a sequence of approximate equations by a Galerkin type decomposition, which admit approximate solutions $(f_n)_{n \in \mathbb{N}}$. Then we prove that the estimate (8) holds for these approximate systems, uniformly in n .

We then conclude the proof of Theorem 1.3 in two steps, by applying (8) with some special choice for ϕ , satisfying $\phi(\xi) \sim (\log(4 + |\xi|))^a$. As already mentioned, one of the main difficulty is that the factor $1 + (\partial_x f)^2$ which appears in the left hand-side of (8) is not controlled in $L_{t,x}^\infty$. To overcome this difficulty, we prove some new interpolation inequalities to estimate the factor $1 + (\partial_x f)^2$, using the fractional logarithmic norms, by some quantity which is not bounded in time. To be more specific, assume that $\phi(\xi) \sim (\log(4 + |\xi|))^a$, and introduce the quantities

$$\begin{aligned} A(t) &= \left\| |D|^{\frac{3}{2}, \phi} f(t) \right\|_{L^2}^2, \\ B(t) &= \left\| |D|^{2, \phi} f(t) \right\|_{L^2}^2. \end{aligned}$$

In §3.2, we will prove after a fair amount of bookkeeping an estimate of the form

$$\frac{d}{dt} A(t) + C_1 \delta(t) B(t) \leq C_2 \left(\log \left(\frac{B(t)}{A(t)} \right) \right)^{-a} \left(\sqrt{A(t)} + A(t) \right) B(t),$$

where

$$\delta(t) \sim \left(1 + \log \left(4 + \frac{B(t)}{A(t) + \|f_0\|_{L^2}^2} \right)^{1-2a} (A(t) + \|f_0\|_{L^2}^2) \right)^{-1}.$$

Notice that $\delta(t)$ is not bounded from below so that the left-hand side is insufficient to control $B(t)$. However, to apply a Gronwall type inequality it will suffice to have $a > 1 - 2a$, that is $a > \frac{1}{3}$, see 3.3. The limiting case $a = \frac{1}{3}$ will be studied in §3.4 by introducing a more general weight ϕ which is not a fraction of a logarithm, and whose definition depends on the initial data itself. This gives uniform bounds in $L_t^\infty(\mathcal{H}^{s,a}(\mathbb{R}))$ for any $a \geq \frac{1}{3}$, for the approximate solutions f_n , from which we deduce the existence of a solution to the Muskat equation by extracting a subsequence. The uniqueness is proved in §3.5 by similar arguments, using again some delicate interpolation inequalities to handle the lack of Lipschitz control.

Notations. Most notations are introduced in the next section. In particular, the definitions of Sobolev, Besov and Triebel-Lizorkin spaces are recalled in §2.1. To avoid possible confusions in the notations, we mention that, throughout the paper:

- We will sometimes write $\log(4 + |\xi|)^a$ as a short notation for $(\log(4 + |\xi|))^a$.
- All functions are assumed to be real-valued in this paper. Nevertheless, we will often use the complex-modulus notation in writing $|\Delta_\alpha f|^2$ or $|\alpha|^2$ in many identities, since we think it might help the reader to read the latter.
- Given $0 \leq t \leq T$, a normed space X and a function $\varphi = \varphi(t, x)$ defined on $[0, T] \times \mathbb{R}$ with values in X , we denote by $\varphi(t)$ the function $x \mapsto \varphi(t, x)$, and $\|\varphi\|_X$ is a short notation for the time dependent function $t \mapsto \|\varphi(t)\|_X$.

2. NONLINEARITY AND FRACTIONAL DERIVATIVES IN THE MUSKAT PROBLEM

We now develop the linear and nonlinear tools needed to study the Muskat problem in the spaces $\mathcal{H}^{3/2,a}(\mathbb{R})$. The first paragraph is a review consisting of various notations and usual results about Besov and Triebel-Lizorkin spaces, which serve as the requested background for what follows. Then we study in §2.2 Fourier multipliers of the form $|D|^s \phi(|D_x|)$ for some symbols $\phi(|\xi|)$ which generalize the fractional logarithm $(\log(4 + |\xi|))^a$ introduced in the introduction. In particular we give a characterization of the space

$$\mathcal{H}^{s,\phi}(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : |D|^s \phi(|D_x|)f \in L^2(\mathbb{R})\},$$

in terms of modified Gagliardo semi-norms. Then in §2.3 we recall the paralinearization formula for the Muskat equation from [2]. The core of this section is §2.4, in which we prove technical ingredients needed to estimate the coefficients of the latter paralinearization formula in terms of the $\mathcal{H}^{3/2,\phi}(\mathbb{R})$ -norms.

2.1. Triebel-Lizorkin norms. This work builds on the analysis of the Muskat equation by Córdoba and Lazar [24] and Alazard and Lazar [2], which introduced the use of techniques related to Besov spaces in this problem (see also Gancedo and Lazar [29]). Here we will also use Triebel-Lizorkin spaces. For ease of reading, we recall various notations and results about these spaces, which will be used continually in the rest of the paper.

Given a function $f: \mathbb{R} \rightarrow \mathbb{R}$, an integer $m \in \mathbb{N} \setminus \{0\}$ and a real number $h \in \mathbb{R}$, we define the finite differences $\delta_h^m f$ as follows:

$$\delta_h f(x) = f(x) - f(x - h), \quad \delta_h^{m+1} f = \delta_h(\delta_h^m f).$$

Definition 2.1. Consider an integer $m \in \mathbb{N} \setminus \{0\}$, a real number $s \in [m - 1, m)$ and two real numbers (p, q) in $[1, \infty)^2$. The homogeneous Triebel-Lizorkin space $\dot{F}_{p,q}^s(\mathbb{R})$ consists of those tempered distributions f whose Fourier transform is integrable near the origin and such that

$$(9) \quad \|f\|_{\dot{F}_{p,q}^s} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\delta_h^m f(x)|^q \frac{dh}{|h|^{1+qs}} \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} < +\infty.$$

Remark 2.2. i) We refer to Triebel [41, §2.3.5] for historical comments about these spaces, and Triebel [42, section 3] for the equivalence between this definition and other ones including the Littlewood-Paley decomposition.

ii) For the sake of comparison, recall that the Besov space $\dot{B}_{p,q}^s(\mathbb{R})$ consists of those tempered distributions f whose Fourier transform is integrable near the origin and such that

$$(10) \quad \|f\|_{\dot{B}_{p,q}^s} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\delta_h^m f(x)|^p dx \right)^{\frac{q}{p}} \frac{dh}{|h|^{1+qs}} \right)^{\frac{1}{q}} < +\infty.$$

Notice that Besov defined his spaces in this way, that is with finite differences, see [5].

For easy reference, we recall two results allowing to compare the Triebel-Lizorkin semi-norms to the homogeneous and non-homogeneous Sobolev norms, which are defined by

$$(11) \quad \|u\|_{\dot{H}^\sigma}^2 := (2\pi)^{-1} \int_{\mathbb{R}} |\xi|^{2\sigma} |\hat{u}(\xi)|^2 d\xi, \quad \|u\|_{H^\sigma}^2 := (2\pi)^{-1} \int_{\mathbb{R}} (1+|\xi|^2)^\sigma |\hat{u}(\xi)|^2 d\xi,$$

where \hat{u} is the Fourier transform of u .

Recall that $\|\cdot\|_{\dot{H}^s}$ and $\|\cdot\|_{\dot{F}_{2,2}^s}$ are equivalent. Moreover, for $s \in (0, 1)$,

$$(12) \quad \|u\|_{\dot{H}^s}^2 = \frac{1}{4\pi c(s)} \|u\|_{\dot{F}_{2,2}^s}^2 \quad \text{with} \quad c(s) = \int_{\mathbb{R}} \frac{1 - \cos(h)}{|h|^{1+2s}} dh.$$

We will also extensively use the following Sobolev embeddings: For any $2 < p_1 < \infty$ and any $q \leq \infty$, if $s - \frac{1}{2} = s_1 - \frac{1}{p_1}$ then

$$(13) \quad \|f\|_{\dot{F}_{p_1, q}^{s_1}(\mathbb{R})} \leq C \|f\|_{\dot{H}^s(\mathbb{R})}.$$

2.2. Some special Fourier multipliers. Let us introduce a bit of notation which will be used continually in the rest of the paper.

Definition 2.3. Consider a real number $s \geq 0$ and a function $\phi: [0, \infty) \rightarrow (0, \infty)$ satisfying the doubling condition $\phi(2r) \leq c_0 \phi(r)$ for any $r \geq 0$ and some constant $c_0 > 0$. Then we may define $|D|^{s, \phi}$ as the Fourier multiplier with symbol $|\xi|^s \phi(|\xi|)$. More precisely,

$$\mathcal{F}(|D|^{s, \phi} f)(\xi) = |\xi|^s \phi(|\xi|) \mathcal{F}(f)(\xi).$$

We shall consider operators $|D|^{s, \phi}$ for some special functions ϕ depending on some function $\kappa: [0, \infty) \rightarrow (0, \infty)$, of the form

$$(14) \quad \phi(\lambda) = \int_0^\infty \frac{1 - \cos(h)}{h^2} \kappa\left(\frac{\lambda}{h}\right) dh, \quad \text{for } \lambda \geq 0.$$

Before we explain the reason to introduce these functions, let us clarify the assumptions on κ which will be needed later on.

Assumption 2.4. Throughout this paper, we always require that $\kappa: [0, \infty) \rightarrow [1, \infty)$ satisfies the following three assumptions:

- (H1) κ is increasing and $\lim \kappa(r) = \infty$ when r goes to $+\infty$;
- (H2) there is a positive constant c_0 such that $\kappa(2r) \leq c_0 \kappa(r)$ for any $r \geq 0$;
- (H3) the function $r \mapsto \kappa(r)/\log(4+r)$ is decreasing on $[0, \infty)$.

Remark 2.5. *i)* The main example is the function $\kappa_a(r) = (\log(4+r))^a$ with $a \in [0, 1]$. However, we shall see that, to include the critical case $a = \frac{1}{3}$ in Theorem 1.3, we must consider more general functions (see Section 3.4).

ii) To clarify notations, we mention that we choose the natural logarithm so that $\log(e) = 1$. In assumption (H3), the choice of the constant 4 is purely technical (it is used only to prove (67) below).

There is one observation that will be useful below. We will prove that $\phi \sim \kappa$, which means that

$$(15) \quad c\kappa(\lambda) \leq \phi(\lambda) \leq C\kappa(\lambda),$$

for some positive constants c, C . In particular, ϕ satisfies the doubling condition $\phi(2r) \leq c_0\phi(r)$ and we may define the Fourier multiplier $|D|^{s,\phi} = |D|^s \phi(|D_x|)$, as in Definition 2.3. Although κ and ϕ are equivalent, we will use them for different purposes. We will use ϕ when we prefer to work with the frequency variable (Fourier analysis), while we use κ when the physical variable is more convenient. The next two results will be useful later on to freely switch computations between the frequency and physical settings.

Lemma 2.6. *Assume that ϕ is as defined in (14) for some function κ satisfying Assumption 2.4. Then, for all $g \in \mathcal{S}(\mathbb{R})$, there holds*

$$|D|^{1,\phi} g(x) = \frac{1}{4} \int_{\mathbb{R}} \frac{2g(x) - g(x+h) - g(x-h)}{h^2} \kappa\left(\frac{1}{|h|}\right) dh.$$

Proof. Notice that the Fourier transform of the function

$$\int_{\mathbb{R}} \frac{2g(x) - g(x+h) - g(x-h)}{h^2} \kappa\left(\frac{1}{|h|}\right) dh,$$

is given by

$$\left(\int_{\mathbb{R}} \frac{2 - 2\cos(h\xi)}{h^2} \kappa\left(\frac{1}{|h|}\right) dh \right) \hat{g}(\xi).$$

Therefore

$$\left(4|\xi| \int_0^\infty \frac{1 - \cos(h)}{h^2} \kappa\left(\frac{|\xi|}{|h|}\right) dh \right) \hat{g}(\xi) = 4\phi(\xi) |\xi| \hat{g}(\xi) = 4\widehat{|D|^{1,\phi} g}(\xi),$$

equivalent to the wanted result. \square

The following result states that ϕ and κ are equivalent and also gives the equivalence of some semi-norms.

Proposition 2.7. *Assume that ϕ is as defined in (14) for some function κ satisfying Assumption 2.4.*

i) *There exist two constants $c, C > 0$ such that, for all $\lambda \geq 0$,*

$$(16) \quad c\kappa(\lambda) \leq \phi(\lambda) \leq C\kappa(\lambda).$$

ii) *Given $g \in \mathcal{S}(\mathbb{R})$, define the semi-norm*

$$\|g\|_{s,\kappa} = \left(\iint_{\mathbb{R}^2} |2g(x) - g(x+h) - g(x-h)|^2 \left(\frac{1}{|h|^s} \kappa\left(\frac{1}{|h|}\right) \right)^2 \frac{dx dh}{|h|} \right)^{\frac{1}{2}}.$$

Then, for all $1 < s < 2$, there exist two constants $c, C > 0$ such that, for all $g \in \mathcal{S}(\mathbb{R})$,

$$c \int_{\mathbb{R}} \left| |D|^{s,\phi} g(x) \right|^2 dx \leq \|g\|_{s,\kappa}^2 \leq C \int_{\mathbb{R}} \left| |D|^{s,\phi} g(x) \right|^2 dx.$$

Proof. We begin by proving statement *ii*). Let us introduce

$$K(r) = \kappa^2 \left(\frac{1}{r} \right) \frac{1}{r^{1+2s}} \quad (r > 0).$$

For any $h \in \mathbb{R}$, the Fourier transform of $x \mapsto 2g(x) - g(x+h) - g(x-h)$ is given by $(2 - 2 \cos(\xi h))\hat{g}(\xi)$. Consequently,

$$\|g\|_{s,\kappa}^2 = \iint_{\mathbb{R}^2} |2g(x) - g(x+h) - g(x-h)|^2 K(|h|) dx dh = \int_{\mathbb{R}} I(\xi) |\hat{g}(\xi)|^2 d\xi,$$

where

$$I(\xi) = \frac{2}{\pi} \int_{\mathbb{R}} (1 - \cos(\xi h))^2 K(|h|) dh.$$

We must prove that the integral $I(\xi)$ satisfies

$$(17) \quad c|\xi|^{2s} \phi(|\xi|)^2 \leq I(\xi) \leq C|\xi|^{2s} \phi(|\xi|)^2,$$

for some constant c, C independent of $\xi \in \mathbb{R}$. Let us prove the bound from above. To do so, we use the inequality $|1 - \cos(\theta)| \leq \min\{2, \theta^2\}$ for all $\theta \in \mathbb{R}$ to obtain

$$I(\xi) \leq \frac{8}{\pi} \int_{|h\xi| \geq 1} K(|h|) dh + \frac{2}{\pi} \int_{|h\xi| \leq 1} \xi^4 h^4 K(|h|) dh.$$

Now, since κ is increasing by assumption, directly from the definition of K , we have

$$\int_{|h\xi| \geq 1} K(|h|) dh \leq (\kappa(|\xi|))^2 \int_{|h\xi| \geq 1} \frac{dh}{|h|^{1+2s}} \lesssim \kappa^2(|\xi|) |\xi|^{2s}.$$

The estimate of the contribution of the integral over $\{|h\xi| \leq 1\}$ is more involved. To do so, we introduce the following decomposition of the integrand

$$(18) \quad \xi^4 h^4 K(|h|) = \xi^4 h^4 \kappa^2 \left(\frac{1}{|h|} \right) \frac{1}{|h|^{1+2s}} = \pi_1(|h|) \pi_2(|h|) \pi_3(|h|) \frac{\xi^4}{|h|^{s-1}}$$

where

$$\begin{aligned} \pi_1(r) &:= \frac{\kappa^2\left(\frac{1}{r}\right)}{\log\left(4 + \frac{1}{r}\right)^2}, & \pi_2(r) &:= \frac{\log\left(4 + \frac{1}{r}\right)^2}{\log\left(\lambda_0 + \frac{1}{r^2}\right)^2}, \\ \pi_3(r) &:= r^{2-s} \left(\log\left(\lambda_0 + \frac{1}{r^2}\right) \right)^2, & \lambda_0 &= \exp\left(\frac{8}{2-s}\right). \end{aligned}$$

By assumption on κ (see (H3) in Assumption 2.4), the function π_1 is increasing and hence

$$\pi_1(|h|) \leq \frac{\kappa^2(|\xi|)}{\log(4 + |\xi|)^2} \quad \text{for } |h| \leq \frac{1}{|\xi|}.$$

The function π_2 is bounded on $[0, +\infty)$ by some harmless constant depending only on s . Eventually, we claim that the function π_3 is increasing. Indeed,

$$\begin{aligned} \frac{d}{dr} \pi_3(r) &= r^{1-s} \log\left(\lambda_0 + \frac{1}{r^2}\right)^2 \left[2 - s - \frac{4}{(\lambda_0 r^2 + 1) \log\left(\lambda_0 + \frac{1}{r^2}\right)} \right] \\ &\geq r^{1-s} \log\left(\lambda_0 + \frac{1}{r^2}\right)^2 \left(2 - s - \frac{4}{\log(\lambda_0)} \right) \\ &= \frac{2-s}{2} r^{1-s} \log\left(\lambda_0 + \frac{1}{r^2}\right)^2 > 0. \end{aligned}$$

It follows that

$$\pi_3(|h|) \leq |\xi|^{s-2} \log(\lambda_0 + |\xi|^2)^2 \quad \text{for } |h| \leq \frac{1}{|\xi|}.$$

By combining these bounds about the factors π_j , we deduce from (18) that

$$\int_{|h\xi| \leq 1} \xi^4 h^4 K(|h|) dh \lesssim \left(\frac{\kappa(|\xi|)}{\log(4 + |\xi|)} \right)^2 |\xi|^{2+s} \log(\lambda_0 + |\xi|^2)^2 \int_{|h| \leq 1/|\xi|} \frac{dh}{|h|^{s-1}}.$$

Since $\log(4 + |\xi|) \sim \log(\lambda_0 + |\xi|^2)$ and since $0 < s - 1 < 1$, we deduce that

$$\int_{|h\xi| \leq 1} \xi^4 h^4 K(|h|) dh \lesssim (\kappa(|\xi|))^2 |\xi|^{2s}.$$

So, we get $I(\xi) \lesssim |\xi|^{2s} \phi(|\xi|)^2$.

On the other hand,

$$I(\xi) \gtrsim \int_{\frac{2\pi}{5} \leq \xi h \leq \frac{3\pi}{5}} K(|h|) dh \gtrsim \left(\int_{\frac{2\pi}{5} \leq \xi h \leq \frac{3\pi}{5}} dh \right) \kappa^2(|\xi|) |\xi|^{1+2s} \gtrsim (\kappa(|\xi|))^2 |\xi|^{2s}.$$

Therefore, we proved (17), which concludes the proof of statement *ii*).

It remains to prove statement *i*). The lower bound $\phi(\lambda) \geq c\kappa(\lambda)$ follows directly from the definition of ϕ , by writing

$$\phi(\lambda) \geq \int_0^1 \frac{1 - \cos(h)}{h^2} \kappa\left(\frac{\lambda}{h}\right) dh \geq \left(\int_0^1 \frac{1 - \cos(h)}{h^2} dh \right) \kappa(\lambda),$$

since κ is increasing. To prove the upper bound, we split the integral into $\{h \leq 1\}$ and $\{h > 1\}$ and then use similar arguments to those used above. \square

2.3. Paralinearization of the nonlinearity. For a nonlinear evolution equation, considerable insight comes from being able to decompose the nonlinearity into several pieces having different roles. For a parabolic free boundary problem in fluid dynamics, one expects to extract from the nonlinearity at least two terms:

- i*) a convective term of the form $V \partial_x f$,
- ii*) an elliptic component of the form $\gamma |D|^\alpha f$,

for some coefficients V and γ and some index $\alpha \geq 0$, where as above $|D| = (-\Delta)^{1/2}$ (see Definition 2.3 with $s = 1$ and $\phi = 1$, or (19) below). To reach this goal, a standard strategy is use a paradifferential analysis, which consists in using a Littlewood-Paley decomposition to determine the relative significance of competing terms. For the Muskat equation, this idea was implemented independently in [2, 34]. In this paragraph, we recall the approach in [2] where the formulation of the Muskat equation in terms of finite differences is exploited to give such a paradifferential decomposition in a direct manner.

Recall that the Muskat equation reads

$$\partial_t f = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\partial_x \Delta_\alpha f}{1 + (\Delta_\alpha f)^2} d\alpha.$$

Therefore it can be written under the form

$$\partial_t f = \frac{1}{\pi} \int_{\mathbb{R}} \partial_x \Delta_\alpha f \, d\alpha - \frac{1}{\pi} \int_{\mathbb{R}} \partial_x \Delta_\alpha f \frac{(\Delta_\alpha f)^2}{1 + (\Delta_\alpha f)^2} \, d\alpha.$$

Let us introduce now some notations that will be used continually in the rest of the paper. We define the singular integral operators

$$(19) \quad \mathcal{H}u = -\frac{1}{\pi} \text{pv} \int_{\mathbb{R}} \Delta_\alpha u \, d\alpha \quad \text{and} \quad |D| = \mathcal{H}\partial_x.$$

Then the Muskat equation can be written under the form

$$(20) \quad \partial_t f + |D|f = \mathcal{T}(f)f,$$

where $\mathcal{T}(f)$ is the operator defined by

$$(21) \quad \mathcal{T}(f)g = -\frac{1}{\pi} \int_{\mathbb{R}} (\partial_x \Delta_\alpha g) \frac{(\Delta_\alpha f)^2}{1 + (\Delta_\alpha f)^2} \, d\alpha.$$

The desired decomposition of the nonlinearity alluded to above will be achieved by splitting the coefficient

$$F_\alpha := \frac{(\Delta_\alpha f)^2}{1 + (\Delta_\alpha f)^2}$$

into its odd and even components. Set

$$(22) \quad \mathcal{O}(\alpha, \cdot) = \frac{1}{2} \frac{(\Delta_\alpha f)^2}{1 + (\Delta_\alpha f)^2} - \frac{1}{2} \frac{(\Delta_{-\alpha} f)^2}{1 + (\Delta_{-\alpha} f)^2},$$

$$(23) \quad \mathcal{E}(\alpha, \cdot) = \frac{1}{2} \frac{(\Delta_\alpha f)^2}{1 + (\Delta_\alpha f)^2} + \frac{1}{2} \frac{(\Delta_{-\alpha} f)^2}{1 + (\Delta_{-\alpha} f)^2}.$$

It follows that

$$\mathcal{T}(f)g = -\frac{1}{\pi} \int_{\mathbb{R}} (\partial_x \Delta_\alpha g) \mathcal{E}(\alpha, \cdot) \, d\alpha - \frac{1}{\pi} \int_{\mathbb{R}} (\partial_x \Delta_\alpha g) \mathcal{O}(\alpha, \cdot) \, d\alpha.$$

Since $\Delta_\alpha f(x)$ converges to $f_x(x)$ when α goes to 0, we further decompose $\mathcal{E}(\alpha, \cdot)$ as

$$\mathcal{E}(\alpha, \cdot) = \frac{(\partial_x f)^2}{1 + (\partial_x f)^2} + \left(\mathcal{E}(\alpha, \cdot) - \frac{(\partial_x f)^2}{1 + (\partial_x f)^2} \right).$$

Remembering that

$$|D|g(x) = -\frac{1}{\pi} \int_{\mathbb{R}} (\partial_x \Delta_\alpha g) \, d\alpha,$$

we obtain the following decomposition of the nonlinearity:

$$(24) \quad \mathcal{T}(f)g = \frac{(\partial_x f)^2}{1 + (\partial_x f)^2} |D|g + V(f)\partial_x g + R(f, g).$$

where

$$(25) \quad V = -\frac{1}{\pi} \int_{\mathbb{R}} \frac{\mathcal{O}(\alpha, \cdot)}{\alpha} \, d\alpha,$$

and

$$R(f, g) = -\frac{1}{\pi} \int_{\mathbb{R}} (\partial_x \Delta_\alpha g) \left(\mathcal{E}(\alpha, \cdot) - \frac{(\partial_x f)^2}{1 + (\partial_x f)^2} \right) \, d\alpha + \frac{1}{\pi} \int_{\mathbb{R}} \frac{\partial_x g(\cdot - \alpha)}{\alpha} \mathcal{O}(\alpha, \cdot) \, d\alpha.$$

2.4. Nonlinear estimates. With these preliminaries established, we start the analysis of the nonlinearity in the Muskat equation.

We begin by estimating the coefficient $V(f)$ (see (25)).

Proposition 2.8. *There exists a positive constant C such that, for all f in $\mathcal{S}(\mathbb{R})$,*

$$(26) \quad \|V(f)\|_{\dot{H}^1} \leq C \|f\|_{\dot{H}^2} + C \|f\|_{\dot{H}^{\frac{7}{4}}}^2.$$

Proof. Recall that

$$V = -\frac{1}{\pi} \int_{\mathbb{R}} \frac{\mathcal{O}(\alpha, \cdot)}{\alpha} d\alpha \quad \text{where} \quad \mathcal{O}(\alpha, \cdot) = \frac{1}{2} \frac{(\Delta_{\alpha} f)^2}{1 + (\Delta_{\alpha} f)^2} - \frac{1}{2} \frac{(\Delta_{-\alpha} f)^2}{1 + (\Delta_{-\alpha} f)^2}.$$

Now, write

$$(27) \quad \mathcal{O}(\alpha, \cdot) = A_{\alpha}(x) (\Delta_{\alpha} f(x) - \Delta_{-\alpha} f(x)),$$

where

$$A_{\alpha}(x) = \frac{1}{2} \frac{\Delta_{\alpha} f + \Delta_{-\alpha} f}{(1 + (\Delta_{\alpha} f)^2)(1 + (\Delta_{-\alpha} f)^2)}.$$

Hence

$$\partial_x \mathcal{O} = A_{\alpha} (\Delta_{\alpha} f_x - \Delta_{-\alpha} f_x) + \partial_x A_{\alpha} (\Delta_{\alpha} f - \Delta_{-\alpha} f).$$

Now, we replace in the first product the factor A_{α} by

$$\left(A_{\alpha}(x) - \frac{f_x(x)}{(1 + f_x(x)^2)^2} \right) + \frac{f_x(x)}{(1 + f_x(x)^2)^2},$$

and observe that the last term is bounded by 1. Therefore, by using the triangle inequality, it follows that

$$\begin{aligned} |\partial_x V(x)| &\leq I_1(x) + I_2(x) + I_3(x) \quad \text{where} \\ I_1(x) &= \left| \int_{\mathbb{R}} (\Delta_{\alpha} f_x(x) - \Delta_{-\alpha} f_x(x)) \frac{d\alpha}{\alpha} \right|, \\ I_2(x) &= \int_{\mathbb{R}} \left| A_{\alpha}(x) - \frac{f_x(x)}{(1 + f_x(x)^2)^2} \right| |\Delta_{\alpha} f_x(x) - \Delta_{-\alpha} f_x(x)| \frac{d\alpha}{|\alpha|}, \\ I_3(x) &= \int_{\mathbb{R}} |\partial_x A_{\alpha}(x)| |\Delta_{\alpha} f(x) - \Delta_{-\alpha} f(x)| \frac{d\alpha}{|\alpha|}. \end{aligned}$$

We now must estimate the L^2 -norm of I_j for $1 \leq j \leq 3$.

We begin by estimating the L^2 -norm of I_1 . Observe that

$$\Delta_{\alpha} f_x(x) - \Delta_{-\alpha} f_x(x) = \frac{2f_x(x) - f_x(x - \alpha) - f_x(x + \alpha)}{\alpha}.$$

Now, as above, we use the fact that, for any $\alpha \in \mathbb{R}$, the Fourier transform of $x \mapsto 2g(x) - g(x + \alpha) - g(x - \alpha)$ is given by $(2 - 2\cos(\xi\alpha))\hat{g}(\xi)$. Consequently, it

follows from Plancherel's theorem that

$$\begin{aligned}\|I_1\|_{L^2}^2 &= \frac{1}{2\pi} \int_{\mathbb{R}} |\xi|^2 \left| \int \left(2 - e^{-i\alpha\xi} - e^{i\alpha\xi}\right) \frac{d\alpha}{\alpha^2} \right|^2 |\hat{f}(\xi)|^2 d\xi \\ &= \frac{2}{\pi} \int_{\mathbb{R}} |\xi|^2 \left| \int (1 - \cos(\alpha\xi)) \frac{d\alpha}{\alpha^2} \right|^2 |\hat{f}(\xi)|^2 d\xi \\ &= \frac{2}{\pi} \left| \int (1 - \cos(\alpha)) \frac{d\alpha}{\alpha^2} \right|^2 \int_{\mathbb{R}} |\xi|^4 |\hat{f}(\xi)|^2 d\xi.\end{aligned}$$

Now, since $|1 - \cos(\alpha)| \leq \min\{2, |\alpha\xi|^2\}$,

$$\left| \int_{\mathbb{R}} (1 - \cos(\alpha)) \frac{d\alpha}{\alpha^2} \right| \leq 2 \int_{|\alpha| \geq 1} \frac{d\alpha}{\alpha^2} + \int_{|\alpha| \leq 1} |\alpha|^2 \frac{d\alpha}{\alpha^2} \leq 3.$$

So,

$$\|I_1\|_{L^2}^2 \lesssim \int_{\mathbb{R}} |\xi|^4 |\hat{f}(\xi)|^2 dx = \|f\|_{\dot{H}^2(\mathbb{R})}^2.$$

This implies the wanted inequality $\|I_1\|_{L^2} \lesssim \|f\|_{\dot{H}^2}$.

We now move to the estimate of $\|I_2\|_{L^2}$. Introduce

$$(28) \quad F(x, y) = \frac{1}{2} \frac{x + y}{(1 + x^2)(1 + y^2)},$$

so that

$$(29) \quad A_\alpha = F(\Delta_\alpha f, \Delta_{-\alpha} f) \quad \text{and} \quad \frac{f_x(x)}{(1 + f_x(x)^2)^2} = F(f_x, f_x).$$

Since $\|\nabla_{x,y} F\|_{L^\infty(\mathbb{R}^2)} \leq 4$, we deduce that

$$\left| A_\alpha(x) - \frac{f_x(x)}{(1 + f_x(x)^2)^2} \right| \leq 4 |\Delta_\alpha f - f_x| + 4 |\Delta_{-\alpha} f - f_x|,$$

which in turn implies that, for all x in \mathbb{R} ,

$$|I_2(x)| \leq 8 \int_{\mathbb{R}} |\Delta_\alpha f(x) - f_x(x)| |\Delta_\alpha f_x(x) - \Delta_{-\alpha} f_x(x)| \frac{d\alpha}{|\alpha|},$$

where we used the change of variables $\alpha \mapsto -\alpha$ to handle the contribution of the term $\Delta_{-\alpha} f - f_x$. Now, using the obvious estimate

$$|\Delta_\alpha f_x(x) - \Delta_{-\alpha} f_x(x)| \leq |\Delta_\alpha f_x(x)| + |\Delta_{-\alpha} f_x(x)|,$$

and the Cauchy-Schwarz inequality, we conclude that

$$(30) \quad \|I_2\|_{L^2}^2 \lesssim \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} (\Delta_\alpha f - f_x(x))^2 \frac{d\alpha}{\alpha^2} \right)^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} (\Delta_\alpha f_x(x))^2 d\alpha \right)^2 dx \right)^{\frac{1}{2}}.$$

We next claim that the first factor in the right-hand side above can be estimated by the second one. To see this, we start with the identity

$$\Delta_\alpha f - f_x(x) = \frac{f(x) - f(x - \alpha)}{\alpha} - f_x(x) = \frac{1}{\alpha} \int_0^\alpha (f_x(x - y) - f_x(x)) dy.$$

Then, for all x and all α in \mathbb{R} , we deduce that

$$(\Delta_\alpha f - f_x(x))^2 \leq \frac{1}{\alpha} \left| \int_0^\alpha (f_x(x-y) - f_x(x))^2 dy \right|.$$

(Notice that the integral is not necessarily non-negative since α might be non-positive.) Then, integrating in α and splitting the integral into $\alpha \geq 0$ and $\alpha < 0$, we obtain

$$\begin{aligned} \int_{\mathbb{R}} (\Delta_\alpha f - f_x(x))^2 \frac{d\alpha}{\alpha^2} &\leq \int_{\mathbb{R}} \left| \int_0^\alpha (f_x(x-y) - f_x(x))^2 dy \right| \frac{d\alpha}{|\alpha|^3} \\ &\leq 2 \int_0^\infty \int_0^\alpha (f_x(x-y) - f_x(x))^2 dy \frac{d\alpha}{\alpha^3}. \end{aligned}$$

Since $2 \int_y^{+\infty} \alpha^{-3} d\alpha \leq y^{-2}$, using Fubini's theorem, it follows that

$$(31) \quad \begin{aligned} \int_{\mathbb{R}} (\Delta_\alpha f - f_x(x))^2 \frac{d\alpha}{\alpha^2} &\leq \int_0^\infty (f_x(x-y) - f_x(x))^2 \frac{dy}{y^2} \\ &\leq \int_0^\infty (\Delta_y f_x(x))^2 dy \leq \int_{\mathbb{R}} (\Delta_\alpha f_x(x))^2 d\alpha. \end{aligned}$$

Hence, it follows from (30) that

$$\|I_2\|_{L^2}^2 \lesssim \int_{\mathbb{R}} \left(\int_{\mathbb{R}} (\Delta_\alpha f_x(x))^2 d\alpha \right)^2 dx.$$

Now, using the Triebel-Lizorkin semi-norms (9), observe that

$$(32) \quad \begin{aligned} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\Delta_\alpha f_x|^2 d\alpha \right)^2 dx &= \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\delta_\alpha f_x|^2 \frac{d\alpha}{|\alpha|^{1+2\frac{1}{2}}} \right)^{\frac{4}{2}} dx \right)^{\frac{4}{4}} \\ &= \|f_x\|_{\dot{F}_{4,2}^{\frac{1}{2}}}^4 \lesssim \|f_x\|_{\dot{H}^{\frac{3}{4}}}^4, \end{aligned}$$

where we used the Sobolev embedding (13) in the last inequality. This proves that $\|I_2\|_{L^2}$ is estimated by $C \|f\|_{\dot{H}^{7/4}}^2$, which completes the analysis of I_2 .

It remains to estimate $\|I_3\|_{L^2}$. Remembering that the function F in (28) has partial derivatives bounded on \mathbb{R}^2 , we find that

$$|\partial_x A_\alpha| \lesssim |\Delta_\alpha f_x| + |\Delta_{-\alpha} f_x|.$$

Now, making the change of variable $\alpha \mapsto -\alpha$ and applying the Cauchy-Schwarz inequality, we end up with

$$\begin{aligned} I_3(x) &\lesssim \int_{\mathbb{R}} |\Delta_\alpha f_x(x)| |\Delta_\alpha f(x) - \Delta_{-\alpha} f(x)| \frac{d\alpha}{|\alpha|} \\ &\lesssim \left(\int_{\mathbb{R}} (\Delta_\alpha f_x(x))^2 d\alpha \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} (\Delta_\alpha f(x) - \Delta_{-\alpha} f(x))^2 \frac{d\alpha}{\alpha^2} \right)^{\frac{1}{2}}. \end{aligned}$$

Now, using the inequality

$$(\Delta_\alpha f(x) - \Delta_{-\alpha} f(x))^2 \leq 2(\Delta_\alpha f(x) - f_x(x))^2 + 2(\Delta_{-\alpha} f(x) - f_x(x))^2,$$

and then applying (31), we infer that

$$\int_{\mathbb{R}} (\Delta_\alpha f(x) - \Delta_{-\alpha} f(x))^2 \frac{d\alpha}{\alpha^2} \lesssim \int_{\mathbb{R}} (\Delta_\alpha f_x(x))^2 d\alpha.$$

Consequently,

$$\|I_3\|_{L^2}^2 \lesssim \int_{\mathbb{R}} \left(\int_{\mathbb{R}} (\Delta_{\alpha} f_x(x))^2 d\alpha \right)^2 dx.$$

Now, it follows from (32) that

$$\|I_3\|_{L^2}^2 \lesssim \|f\|_{\dot{H}^{\frac{7}{4}}}^4,$$

which concludes the proof of the proposition. \square

Remark 2.9. It follows from (27) that for all f in $\mathcal{S}(\mathbb{R})$,

$$\begin{aligned} |V(f)(x)| &\leq \frac{1}{2\pi} \int |\Delta_{\alpha} f(x) - \Delta_{-\alpha} f(x)| \frac{d\alpha}{|\alpha|} \\ &\leq \frac{1}{\pi} \iint_{\mathbb{R}^2} |1 - \cos(\alpha\xi)| |\hat{f}(\xi)| \frac{d\xi d\alpha}{|\alpha|^2}. \end{aligned}$$

Thus, since $\int_{\mathbb{R}} |1 - \cos(\alpha\xi)| \frac{d\alpha}{|\alpha|^2} = |\xi| \int_{\mathbb{R}} |1 - \cos(\alpha)| \frac{d\alpha}{|\alpha|^2} \sim |\xi|$, we find that

$$(33) \quad \|V(f)\|_{L^{\infty}} \lesssim \int |\xi| |\hat{f}(\xi)| d\xi.$$

We will use this estimate to bound the $H^1(\mathbb{R}_x)$ -norm of $\mathcal{T}(f)g$ in Corollary 2.14.

The next result contains a key estimate which will allow us to commute arbitrary operators with $\mathcal{T}(f)$.

Proposition 2.10. Assume that ϕ is as defined in (14) for some function κ satisfying Assumption 2.4. Then, there exists a positive constant C such that, for all f, g in $\mathcal{S}(\mathbb{R})$,

$$(34) \quad \begin{aligned} \|[|D|^{1,\phi}, \mathcal{T}(f)](g)\|_{L^2} &\lesssim \|g\|_{\dot{H}^{\frac{7}{4}}} \|[|D|^{\frac{7}{4},\phi} f]\|_{L^2} + \|g\|_{\dot{H}^{\frac{7}{4}}} \|[|D|^{\frac{7}{4},\phi^2} f]\|_{L^2}^{\frac{1}{2}} \|f\|_{\dot{H}^{\frac{19}{12}}}^{\frac{3}{2}} \\ &\quad + \|[|D|^{\frac{7}{4},\phi^2} g]\|_{L^2}^{\frac{1}{2}} \|g\|_{\dot{H}^{\frac{7}{4}}}^{\frac{1}{2}} \|f\|_{\dot{H}^{\frac{7}{4}}}. \end{aligned}$$

Proof. Recall that the operator $\mathcal{T}(f)$ is defined by

$$\mathcal{T}(f)g = -\frac{1}{\pi} \int_{\mathbb{R}} (\Delta_{\alpha} g_x) F_{\alpha} d\alpha \quad \text{where} \quad F_{\alpha} = \frac{(\Delta_{\alpha} f)^2}{1 + (\Delta_{\alpha} f)^2} d\alpha.$$

Let us introduce

$$\Gamma_{\alpha} := |D|^{1,\phi} [F_{\alpha} \Delta_{\alpha} g_x] - F_{\alpha} |D|^{1,\phi} [\Delta_{\alpha} g_x] - \Delta_{\alpha} g_x |D|^{1,\phi} [F_{\alpha}].$$

Loosely speaking, the term Γ_{α} is a remainder term for a fractional Leibniz rule with the operator $|D|^{1,\phi}$. With this notation, one can write the commutator of the operators $|D|^{1,\phi}$ and $\mathcal{T}(f)$ as

$$\left[|D|^{1,\phi}, \mathcal{T}(f) \right] (g) = -\frac{1}{\pi} \int_{\mathbb{R}} \Delta_{\alpha} g_x |D|^{1,\phi} F_{\alpha} d\alpha - \frac{1}{\pi} \int_{\mathbb{R}} \Gamma_{\alpha} d\alpha.$$

Consequently, to estimate the L^2 -norm of $\left[|D|^{1,\phi}, \mathcal{T}(f) \right] (f)$, we have to bound the following integrals:

$$(I) := \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \Delta_{\alpha} g_x(x) |D|^{1,\phi} F_{\alpha}(x) d\alpha \right)^2 dx, \quad (II) := \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \Gamma_{\alpha} d\alpha \right)^2 dx.$$

More precisely, to prove the wanted estimate (34), it is sufficient to show that

$$(35) \quad (I) \lesssim \|g\|_{\dot{H}^{\frac{7}{4}}}^2 \left\| |D|^{\frac{7}{4}, \phi} f \right\|_{L^2}^2 + \|g\|_{\dot{H}^{\frac{7}{4}}}^2 \left\| |D|^{\frac{7}{4}, \phi^2} f \right\|_{L^2} \left\| |D|^{\frac{19}{12}} f \right\|_{L^2}^3,$$

$$(36) \quad (II) \lesssim \left\| |D|^{\frac{7}{4}, \phi^2} g \right\|_{L^2} \left\| |D|^{\frac{7}{4}} g \right\|_{L^2} \left\| |D|^{\frac{7}{4}} f \right\|_{L^2}^2.$$

Step 1: We prove (35). By Holder's inequality and Minkowski's inequality, one has

$$(I) \leq \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\Delta_{\alpha} g_x(x)|^2 d\alpha \right)^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |D|^{1, \phi} F_{\alpha}(x)|^2 d\alpha \right)^2 dx \right)^{\frac{1}{2}}.$$

The first factor is estimated by means of (32), namely

$$\left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\Delta_{\alpha} g_x(x)|^2 d\alpha \right)^2 dx \right)^{\frac{1}{2}} \lesssim \|g\|_{\dot{H}^{\frac{7}{4}}}^2.$$

The analysis of the second term is more difficult. We begin by applying Minkowski's inequality together with the Sobolev embedding $\dot{H}^{1/4}(\mathbb{R}) \hookrightarrow L^4(\mathbb{R})$, to obtain

$$\begin{aligned} \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |D|^{1, \phi} F_{\alpha}(x)|^2 d\alpha \right)^2 dx \right)^{\frac{1}{2}} &\lesssim \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |D|^{1, \phi} F_{\alpha}(x)|^4 dx \right)^{\frac{1}{2}} d\alpha \\ &\lesssim \iint_{\mathbb{R}^2} |D|^{\frac{5}{4}, \phi} F_{\alpha}(x)|^2 dx d\alpha. \end{aligned}$$

Now, to evaluate the latter integral, we use Lemma 2.7, which implies that

$$\begin{aligned} &\iint_{\mathbb{R}^2} |D|^{\frac{5}{4}, \phi} F_{\alpha}(x)|^2 dx d\alpha \\ &\sim \iint_{\mathbb{R}^2} |2F_{\alpha}(x) - F_{\alpha}(x+h) - F_{\alpha}(x-h)|^2 \left(\kappa \left(\frac{1}{|h|} \right) \right)^2 \frac{dx dh}{|h|^{1+5/2}}. \end{aligned}$$

We must estimate the integrand $|2F_{\alpha}(x) - F_{\alpha}(x+h) - F_{\alpha}(x-h)|^2$ in terms of similar terms for $\Delta_{\alpha} f$. To do so, write $F_{\alpha} = \mathcal{F}(\Delta_{\alpha})$ with $\mathcal{F}(x) = x^2/(1+x^2)$. Then we use the following sharp contraction estimate for \mathcal{F} .

Lemma 2.11. *For any triple of real numbers (x_1, x_2, x_3) , there holds*

$$(37) \quad \left| \frac{2x_1^2}{1+x_1^2} - \frac{x_2^2}{1+x_2^2} - \frac{x_3^2}{1+x_3^2} \right| \leq |x_2 + x_3 - 2x_1| + |x_2 - x_1|^2 + |x_3 - x_1|^2.$$

Remark 2.12. *It is worth remarking that such a clean inequality does not hold for a general function.*

Proof. Write

$$\begin{aligned}
& \frac{2x_1^2}{1+x_1^2} - \frac{x_2^2}{1+x_2^2} - \frac{x_3^2}{1+x_3^2} = \frac{2}{1+x_1^2} - \frac{1}{1+x_2^2} - \frac{1}{1+x_3^2} \\
&= \frac{(x_3^2+1)(x_2+x_1)(x_2-x_1) + (x_2^2+1)(x_3+x_1)(x_3-x_1)}{(1+x_1^2)(1+x_2^2)(1+x_3^2)} \\
&= \frac{(x_3^2+1)(x_2+x_1)(x_2+x_3-2x_1) + (x_1x_2+x_2x_3+x_3x_1-1)(x_2-x_3)(x_3-x_1)}{(1+x_1^2)(1+x_2^2)(1+x_3^2)},
\end{aligned}$$

to obtain

$$\left| \frac{2x_1^2}{1+x_1^2} - \frac{x_2^2}{1+x_2^2} - \frac{x_3^2}{1+x_3^2} \right| \leq |x_2+x_3-2x_1| + |x_2-x_3||x_3-x_1|,$$

which implies the desired result. \square

Directly from the definition of F_α , it follows from the previous lemma that

$$\begin{aligned}
|2F_\alpha(x) - F_\alpha(x+h) - F_\alpha(x-h)|^2 &\lesssim |2\Delta_\alpha f(x) - \Delta_\alpha f(x+h) - \Delta_\alpha f(x-h)|^2 \\
&\quad + |\Delta_\alpha f(x) - \Delta_\alpha f(x+h)|^4 \\
&\quad + |\Delta_\alpha f(x) - \Delta_\alpha f(x-h)|^4.
\end{aligned}$$

Integrating in h and making use of the change of variable $h \mapsto -h$ to handle the contribution of the last term, we conclude that

$$\begin{aligned}
& \int_{\mathbb{R}} \left| |D|^{5/4, \phi} [F_\alpha](x) \right|^2 dx \\
&\lesssim \iint_{\mathbb{R}^2} |2\Delta_\alpha f(x) - \Delta_\alpha f(x+h) - \Delta_\alpha f(x-h)|^2 \left(\kappa \left(\frac{1}{|h|} \right) \right)^2 \frac{dx dh}{|h|^{1+5/2}} \\
&\quad + \iint_{\mathbb{R}^2} |\Delta_\alpha f(x) - \Delta_\alpha f(x+h)|^4 \left(\kappa \left(\frac{1}{|h|} \right) \right)^2 \frac{dx dh}{|h|^{1+5/2}}.
\end{aligned}$$

The first term is estimated by using again Lemma 2.7, which implies that

$$\begin{aligned}
& \iint_{\mathbb{R}^2} |2\Delta_\alpha f(x) - \Delta_\alpha f(x+h) - \Delta_\alpha f(x-h)|^2 \left(\kappa \left(\frac{1}{|h|} \right) \right)^2 \frac{dx dh}{|h|^{1+5/2}} \\
&\lesssim \int_{\mathbb{R}} \left| |D|^{5/4, \phi} [\Delta_\alpha f](x) \right|^2 dx.
\end{aligned}$$

On the other hand, using the same arguments together with Holder's inequality, we have

$$\begin{aligned}
& \iint_{\mathbb{R}^2} |\Delta_\alpha f(x) - \Delta_\alpha f(x+h)|^4 \left(\kappa \left(\frac{1}{|h|} \right) \right)^2 \frac{dx dh}{|h|^{1+5/2}} \\
& \leq \left(\iint_{\mathbb{R}^2} |\Delta_\alpha f(x) - \Delta_\alpha f(x+h)|^2 \left(\kappa \left(\frac{1}{|h|} \right) \right)^4 \frac{dx dh}{|h|^{1+5/2}} \right)^{1/2} \\
& \quad \times \left(\iint_{\mathbb{R}^2} |\Delta_\alpha f(x) - \Delta_\alpha f(x+h)|^6 \frac{dx dh}{|h|^{1+5/2}} \right)^{1/2} \\
& \sim \| |D|^{\frac{5}{4}, \phi^2} \Delta_\alpha f \|_{L^2} \| \Delta_\alpha f \|_{\dot{F}_{6,6}^{\frac{5}{12}}}^3 \\
& \lesssim \| |D|^{\frac{5}{4}, \phi^2} \Delta_\alpha f \|_{L^2} \| \Delta_\alpha f \|_{\dot{H}^{\frac{3}{4}}}^3,
\end{aligned}$$

where we used the Sobolev embedding $\dot{H}^{\frac{3}{4}}(\mathbb{R}) \hookrightarrow \dot{F}_{6,6}^{\frac{5}{12}}(\mathbb{R})$, see (13). Therefore, by gathering the previous results, we get

$$(I) \lesssim \|g\|_{\dot{H}^{\frac{7}{4}}}^2 \left(\iint_{\mathbb{R}^2} |\Delta_\alpha(|D|^{\frac{5}{4}, \phi} f)(x)|^2 dx d\alpha + \int_{\mathbb{R}} \| |D|^{\frac{5}{4}, \phi^2} \Delta_\alpha f \|_{L^2} \| \Delta_\alpha f \|_{\dot{H}^{\frac{3}{4}}}^3 d\alpha \right).$$

Now we claim that, for any function \tilde{f} ,

$$(38) \quad \iint_{\mathbb{R}^2} |\Delta_\alpha \tilde{f}|^2 d\alpha dx \sim \| \tilde{f} \|_{\dot{H}^{\frac{1}{2}}}^2.$$

To see this, write

$$\iint_{\mathbb{R}^2} |\Delta_\alpha \tilde{f}|^2 d\alpha dx = \iint_{\mathbb{R}^2} |\delta_\alpha \tilde{f}|^2 \frac{d\alpha}{\alpha^{1+2\frac{1}{2}}} dx = \| \tilde{f} \|_{\dot{F}_{2,2}^{\frac{1}{2}}}^2 \sim \| \tilde{f} \|_{\dot{H}^{\frac{1}{2}}}^2,$$

where we used (12). Now the estimate (38) implies that

$$\begin{aligned}
& \int |\Delta_\alpha(|D|^{\frac{5}{4}, \phi} f)(x)|^2 dx d\alpha \lesssim \| |D|^{\frac{7}{4}, \phi} f \|_{L^2}^2, \\
& \int_{\mathbb{R}} \| \Delta_\alpha(|D|^{\frac{5}{4}, \phi^2} f) \|_{L^2}^2 d\alpha \lesssim \| |D|^{\frac{7}{4}, \phi^2} f \|_{L^2}^2.
\end{aligned}$$

It follows that

$$(I) \lesssim \|g\|_{\dot{H}^{\frac{7}{4}}}^2 \| |D|^{\frac{7}{4}, \phi} f \|_{L^2}^2 + \|g\|_{\dot{H}^{\frac{7}{4}}}^2 \| |D|^{\frac{7}{4}, \phi^2} f \|_{L^2}^2 \left(\int_{\mathbb{R}} \| \Delta_\alpha(|D|^{\frac{3}{4}} f) \|_{L^2}^6 d\alpha \right)^{\frac{1}{2}}.$$

Using the Besov norm (10), we have

$$\left(\int_{\mathbb{R}} \| \Delta_\alpha(|D|^{\frac{3}{4}} f) \|_{L^2}^6 d\alpha \right)^{\frac{1}{2}} = \| |D|^{\frac{3}{4}} f \|_{\dot{B}_{2,6}^{\frac{5}{8}}}^3 \lesssim \| |D|^{\frac{3}{4}} f \|_{\dot{F}_{2,6}^{\frac{5}{8}}}^3 \lesssim \| |D|^{\frac{19}{12}} f \|_{L^2}^3,$$

where we have used the embedding (13) in the last inequality, while the inner inequality follows at once from the definitions of Besov and Triebel-Lizorkin space (see (10) and (9)) by using the Minkowski's inequality. This proves the wanted result (35).

Step 2: We prove (36). Recall that Lemma 2.6 implies that for any function g , one can compute $|D|^{1,\phi} g$ as follows:

$$|D|^{1,\phi} g(x) = \frac{1}{4} \int \frac{2g(x) - g(x+h) - g(x-h)}{h^2} \kappa\left(\frac{1}{|h|}\right) dh.$$

Then using the elementary identity

$$\begin{aligned} & (2xy - x_1y_1 - x_{-1}y_{-1}) - x(2y - y_1 - y_{-1}) - y(2x - x_1 - x_{-1}) \\ &= -(x - x_1)(y - y_1) - (x - x_{-1})(y - y_{-1}), \end{aligned}$$

we deduce that

$$|\Gamma_\alpha| = 2 \left| \int_{\mathbb{R}} (F_\alpha(x) - F_\alpha(x-h)) (\Delta_\alpha g_x(x) - \Delta_\alpha g_x(x-h)) \kappa\left(\frac{1}{|h|}\right) \frac{dh}{h^2} \right|.$$

Since

$$|F_\alpha(x) - F_\alpha(x-h)| \leq |\Delta_\alpha f(x) - \Delta_\alpha f(x-h)|,$$

it follows that

$$\begin{aligned} |\Gamma_\alpha| &\leq 2 \int_{\mathbb{R}} |\Delta_\alpha f(x) - \Delta_\alpha f(x-h)| |\Delta_\alpha g_x(x) - \Delta_\alpha g_x(x-h)| \kappa\left(\frac{1}{|h|}\right) \frac{dh}{h^2} \\ &\leq 2 \left(\int_{\mathbb{R}} |\Delta_\alpha g_x(x) - \Delta_\alpha g_x(x-h)|^2 \kappa^4\left(\frac{1}{|h|}\right) \frac{dh}{h^2} \right)^{\frac{1}{4}} \\ &\quad \times \left(\int_{\mathbb{R}} |\Delta_\alpha g_x(x) - \Delta_\alpha g_x(x-h)|^3 \frac{dh}{h^2} \right)^{\frac{1}{6}} \left(\int_{\mathbb{R}} |\Delta_\alpha f(x) - \Delta_\alpha f(x-h)|^{\frac{12}{7}} \frac{dh}{h^2} \right)^{\frac{7}{12}}. \end{aligned}$$

So, by Holder's inequality in x ,

$$\begin{aligned} \|\Gamma_\alpha\|_{L^2(\mathbb{R}; dx)}^2 &\leq 4 \left(\iint_{\mathbb{R}^2} (\Delta_\alpha g_x(x) - \Delta_\alpha g_x(x-h))^2 \kappa^4\left(\frac{1}{|h|}\right) \frac{dh}{h^2} dx \right)^{\frac{1}{2}} \\ &\quad \times \left(\iint_{\mathbb{R}^2} |\Delta_\alpha g_x(x) - \Delta_\alpha g_x(x-h)|^3 \frac{dh dx}{h^2} \right)^{\frac{1}{3}} \\ &\quad \times \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\Delta_\alpha f(x) - \Delta_\alpha f(x-h)|^{\frac{12}{7}} \frac{dh}{h^2} \right)^7 dx \right)^{\frac{1}{6}} \\ &\sim \|\Delta_\alpha(|D|^{\frac{3}{2}, \phi^2} g)\|_{L^2} \|\Delta_\alpha g_x\|_{\dot{F}_{3,3}^{\frac{1}{3}}} \|\Delta_\alpha f\|_{\dot{F}_{12, \frac{12}{7}}^{\frac{7}{12}}}^2 \\ &\stackrel{(13)}{\lesssim} \|\Delta_\alpha(|D|^{\frac{3}{2}, \phi^2} g)\|_{L^2} \|\Delta_\alpha(|D|^{\frac{3}{2}} g)\|_{L^2} \|\Delta_\alpha(|D| f)\|_{L^2}^2. \end{aligned}$$

Therefore,

$$\begin{aligned} (II) &\leq \left(\int_{\mathbb{R}} \|\Delta_\alpha(|D|^{\frac{3}{2}, \phi^2} g)\|_{L^2}^{1/2} \|\Delta_\alpha(|D|^{3/2} g)\|_{L^2}^{1/2} \|\Delta_\alpha(|D| f)\|_{L^2} d\alpha \right)^2 \\ &\leq \left(\int_{\mathbb{R}} \|\Delta_\alpha(|D|^{\frac{3}{2}, \phi^2} g)\|_{L^2}^2 |\alpha|^{1/2} d\alpha \right)^{1/2} \left(\int_{\mathbb{R}} \|\Delta_\alpha(|D|^{\frac{3}{2}} g)\|_{L^2}^2 |\alpha|^{1/2} d\alpha \right)^{1/2} \\ &\quad \times \int_{\mathbb{R}} \|\Delta_\alpha(|D| f)\|_{L^2}^2 |\alpha|^{-1/2} d\alpha \\ &\sim \|\Delta_\alpha(|D|^{\frac{7}{4}, \phi^2} g)\|_{L^2} \|\Delta_\alpha(|D|^{\frac{7}{4}} g)\|_{L^2} \|\Delta_\alpha(|D|^{\frac{7}{4}} f)\|_{L^2}^2, \end{aligned}$$

where we used (12). This gives (36), which completes the proof. \square

Eventually, we study the remainder term $R(f, g)$ in the parilinearization of $\mathcal{T}(f)g$ (see (24)).

Proposition 2.13. *Assume that ϕ is as defined in (14) for some function κ satisfying Assumption 2.4. Then, there exists a positive constant C such that, for all functions f, g in $\mathcal{S}(\mathbb{R})$,*

$$(39) \quad \|R(f, g)\|_{L^2} \leq C \|g\|_{\dot{H}^{\frac{3}{4}}} \|f\|_{\dot{H}^{\frac{7}{4}}}.$$

In particular,

$$(40) \quad \|R(f, |D|^{1, \phi} f)\|_{L^2} \leq C \| |D|^{\frac{7}{4}, \phi} f \|_{L^2} \|f\|_{\dot{H}^{\frac{7}{4}}}.$$

Proof. Recall that

$$\begin{aligned} R(f, g) &= -\frac{1}{2\pi} \int_{\mathbb{R}} (\partial_x \Delta_{\alpha} g + \partial_x \Delta_{-\alpha} g) \left(\mathcal{E}(\alpha, \cdot) - \frac{(\partial_x f)^2}{1 + (\partial_x f)^2} \right) d\alpha \\ &\quad + \frac{1}{\pi} \int_{\mathbb{R}} \frac{\partial_x g(\cdot - \alpha)}{\alpha} \mathcal{O}(\alpha, \cdot) d\alpha. \end{aligned}$$

The first half of the proof is based on [2]. Namely, write

$$\begin{aligned} \partial_x \Delta_{\alpha} g + \partial_x \Delta_{-\alpha} g &= -\frac{1}{\alpha} \partial_{\alpha} (\delta_{\alpha} g + \delta_{-\alpha} g), \\ \partial_x g(\cdot - \alpha) &= \partial_{\alpha} (\delta_{\alpha} g), \end{aligned}$$

as can be verified by elementary calculations. Then use these identities to integrate by parts in α . By so doing, we find

$$\begin{aligned} |R(f, g)(x)| &\lesssim \int_{\mathbb{R}} \frac{|\delta_{\alpha} g(x)|}{|\alpha|} \left(\left| \mathcal{E}(\alpha, x) - \frac{(\partial_x f(x))^2}{1 + (\partial_x f(x))^2} \right| + |\alpha| |\partial_{\alpha} \mathcal{E}(\alpha, x)| \right. \\ &\quad \left. + |\mathcal{O}(\alpha, x)| + |\alpha| |\partial_{\alpha} \mathcal{O}(\alpha, x)| \right) \frac{d\alpha}{|\alpha|}. \end{aligned}$$

Then it follows from [2, Lemma 4.5] that

$$|R(f, g)(x)| \lesssim \int_{\mathbb{R}} |\Delta_{\alpha} g(x)| \gamma(\alpha, x) d\alpha$$

where

$$\gamma(\alpha, x) = \frac{1}{|\alpha|} \left(|\delta_{\alpha} f_x(x)| + |\delta_{-\alpha} f_x(x)| + \frac{|s_{\alpha} f(x)|}{|\alpha|} + \left| \frac{1}{\alpha} \int_0^{\alpha} s_{\eta} f_x(x) d\eta \right| \right).$$

and $s_{\alpha} f(x) = \delta_{\alpha} f(x) + \delta_{-\alpha} f(x)$.

We now use different arguments than those used in [2]. The main new ingredient here is given by the following inequality:

$$\int_{\mathbb{R}} \gamma(\alpha, x)^2 d\alpha \lesssim \int_{\mathbb{R}} |\delta_{\alpha} f_x(x)|^2 \frac{d\alpha}{\alpha^2}.$$

To prove the latter, it will suffice to show that that

$$(41) \quad \int_{\mathbb{R}} \left| \int_0^\alpha s_\eta f_x(x) \, d\eta \right|^2 \frac{d\alpha}{\alpha^4} \lesssim \int_{\mathbb{R}} |\delta_\alpha f_x(x)|^2 \frac{d\alpha}{\alpha^2},$$

$$(42) \quad \int_{\mathbb{R}} |s_\alpha f(x)|^2 \frac{d\alpha}{\alpha^4} \lesssim \int_{\mathbb{R}} |\delta_\alpha f_x(x)|^2 \frac{d\alpha}{\alpha^2}.$$

To prove (41), we apply the following Hardy's inequality

$$\int_0^\infty \left(\frac{1}{\alpha} \int_0^\alpha u(\eta) \, d\eta \right)^2 d\alpha \leq 4 \int_0^\infty u(\alpha)^2 d\alpha,$$

with $u(\eta) = (s_\eta f_x(x))/\eta$. It follows that

$$\begin{aligned} \int_{\mathbb{R}} \left| \int_0^\alpha s_\eta f_x(x) \, d\eta \right|^2 \frac{d\alpha}{\alpha^4} &= \int_{\mathbb{R}} \left| \frac{1}{\alpha^2} \int_0^\alpha s_\eta f_x(x) \, d\eta \right|^2 d\alpha \leq \int_{\mathbb{R}} \left| \frac{1}{\alpha} \int_0^\alpha \frac{s_\eta f_x(x)}{\eta} \, d\eta \right|^2 d\alpha \\ &\leq 4 \int_{\mathbb{R}} \left(\frac{s_\alpha f_x(x)}{\alpha} \right)^2 d\alpha. \end{aligned}$$

Then (41) follows at once from the triangle inequality $|s_\alpha f|^2 \leq 2|\delta_\alpha f|^2 + 2|\delta_{-\alpha} f|^2$.

Let us prove (42). Write

$$\frac{s_\alpha f}{\alpha} = \frac{\delta_\alpha f + \delta_{-\alpha} f}{\alpha} = \Delta_\alpha f - \Delta_{-\alpha} f = (\Delta_\alpha f - f_x) - (\Delta_{-\alpha} f - f_x),$$

so

$$\frac{(s_\alpha f(x))^2}{\alpha^4} \leq 2 \frac{(\Delta_\alpha f - f_x)^2}{\alpha^2} + 2 \frac{(\Delta_{-\alpha} f - f_x)^2}{\alpha^2}.$$

Now, remember from (31) that

$$\int_{\mathbb{R}} (\Delta_\alpha f - f_x(x))^2 \frac{d\alpha}{\alpha^2} \leq \int_{\mathbb{R}} (\Delta_\alpha f_x(x))^2 d\alpha,$$

together with a similar result for $\Delta_{-\alpha} f - f_x$ (interchanging α and $-\alpha$). This proves (42).

Then, remembering that $\Delta_\alpha u = \delta_\alpha u/\alpha$ and using the Cauchy-Schwarz inequality, we get

$$|R(f, g)(x)| \lesssim \left(\int_{\mathbb{R}} (\Delta_\alpha g(x))^2 d\alpha \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} (\Delta_\alpha f_x(x))^2 d\alpha \right)^{\frac{1}{2}}.$$

By using the Cauchy-Schwarz inequality again, we conclude that

$$\|R(f, g)\|_{L^2}^2 \lesssim \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} (\Delta_\alpha g(x))^2 d\alpha \right)^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} (\Delta_\alpha f_x(x))^2 d\alpha \right)^2 dx \right)^{\frac{1}{2}}.$$

Therefore, the estimate (32) implies that

$$\|R(f, g)\|_{L^2}^2 \lesssim \|g\|_{\dot{H}^{\frac{3}{4}}}^2 \|f\|_{\dot{H}^{\frac{7}{4}}}^2,$$

equivalent to the desired result (39). \square

Corollary 2.14. *There exists a positive constant C such that, for all $f \in \mathcal{S}(\mathbb{R})$,*

$$(43) \quad \|\mathcal{T}(f)f\|_{\dot{H}^1} \leq C \left(\|f\|_{\dot{H}^{\frac{3}{2}}} + \|f\|_{\dot{H}^{\frac{3}{2}}}^2 + 1 + \|V(f)\|_{L^\infty} \right) \|f\|_{\dot{H}^2}.$$

Proof. By (24) and (34) and (40) with $\phi \equiv 1$ one has,

$$\begin{aligned} \|\mathcal{T}(f)f\|_{\dot{H}^1} &\leq \| [|D|, \mathcal{T}(f)]f \|_{L^2} + \|\mathcal{T}(f)(|D|f)\|_{L^2} \\ &\lesssim \|f\|_{\dot{H}^{\frac{7}{4}}}^2 + \|f\|_{\dot{H}^{\frac{3}{2}}}^{\frac{3}{2}} \|f\|_{\dot{H}^{\frac{19}{12}}}^{\frac{3}{2}} + (\|V(f)\|_{L^\infty} + 1) \|f\|_{\dot{H}^2}. \end{aligned}$$

Then (43) follows from the classical interpolation inequalities in Sobolev spaces. \square

3. PROOF OF THE MAIN RESULTS

In this section, we prove Theorem 1.3 and Theorem 1.5.

Following a classical strategy, we shall construct solutions of the Muskat equation in three steps:

- (1) We begin by defining approximate systems and proving that the Cauchy problem for the latter are well-posed by means of an ODE argument.
- (2) Secondly, we prove uniform estimates for the solutions of the approximate systems on a uniform time interval. The heart of the entire argument is contained in a *a priori* estimate given by Proposition 3.3 below.
- (3) Finally, we prove that the sequence of approximate solutions converges to a solution of the Muskat equation and conclude the proof by proving a uniqueness result.

Notations. We denote by $\langle \cdot, \cdot \rangle$ the scalar product in $L^2(\mathbb{R})$ and set $\|\cdot\| = \|\cdot\|_{L^2}$.

3.1. Approximate systems. We will define the wanted approximate systems by using a version of Galerkin's method based on Friedrichs mollifiers. To do so, it is convenient to use smoothing operators which are also projections. Consider, for any integer n in $\mathbb{N} \setminus \{0\}$, the operators J_n defined by

$$(44) \quad \begin{aligned} \widehat{J_n u}(\xi) &= \widehat{u}(\xi) & \text{for } |\xi| \leq n, \\ \widehat{J_n u}(\xi) &= 0 & \text{for } |\xi| > n. \end{aligned}$$

Notice that J_n is a projection since $J_n^2 = J_n$.

Recall that the Muskat equation reads

$$\partial_t f = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\partial_x \Delta_\alpha f}{1 + (\Delta_\alpha f)^2} d\alpha.$$

Remember also from §2.3 that the latter is equivalent to

$$(45) \quad \partial_t f + |D|f = \mathcal{T}(f)f,$$

where $\mathcal{T}(f)$ is the operator defined by (21).

Let us introduce the following approximate Cauchy problems:

$$(46) \quad \begin{cases} \partial_t f_n + |D|f_n = J_n(\mathcal{T}(f_n)f_n), \\ f_n|_{t=0} = J_n f_0. \end{cases}$$

The next lemma states that this system has smooth global in time solutions.

Lemma 3.1. *For all $f_0 \in L^2(\mathbb{R})$, and any $n \in \mathbb{N} \setminus \{0\}$, the initial value problem (46) has a unique global solution $f_n \in C^1([0, +\infty); H^\infty(\mathbb{R}))$. Moreover*

$$(47) \quad f_n = J_n f_n,$$

and, for all time $t \geq 0$,

$$(48) \quad \|f_n(t)\|_{L^2} \leq \|f_0\|_{L^2}.$$

Proof. This proof is not new: it follows from the analysis in [2, Section 5] together with the L^2 -maximum principle in [19, Section 2]. However, since slight modifications are needed, we include a detailed proof.

i) We begin by studying the following auxiliary Cauchy problem

$$(49) \quad \begin{cases} \partial_t f_n + J_n |D| f_n = J_n (\mathcal{T}(J_n f_n) J_n f_n), \\ f|_{t=0} = J_n f_0. \end{cases}$$

The Cauchy problem (49) has the form

$$(50) \quad \partial_t f_n = F_n(f_n), \quad f_n|_{t=0} = J_n f_0,$$

where

$$F_n(f) = -|D| J_n f + J_n (\mathcal{T}(J_n f) J_n f)$$

Recall from Proposition 2.3 in [2] that the map $f \mapsto \mathcal{T}(f)f$ is locally Lipschitz from $\dot{H}^1(\mathbb{R}) \cap \dot{H}^{\frac{3}{2}}(\mathbb{R})$ to $L^2(\mathbb{R})$. Therefore, since J_n is linear smoothing operator (which means that it is bounded from $L^2(\mathbb{R})$ into $H^\mu(\mathbb{R})$ for any $\mu \geq 0$), the map $f \mapsto J_n (\mathcal{T}(J_n f) J_n f)$ is locally Lipschitz from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$. This implies that F_n satisfies the same property and hence we are thus in position to apply the Cauchy-Lipschitz theorem. This gives the existence of a unique maximal solution f_n in $C^1([0, T_n); L^2(\mathbb{R}))$. Moreover, the continuation principle for ordinary differential equations implies that either

$$(51) \quad T_n = +\infty \quad \text{or} \quad \limsup_{t \rightarrow T_n} \|f_n(t)\|_{L^2} = +\infty.$$

We shall prove in the next step that $T_n = +\infty$.

Eventually, remembering that $J_n^2 = J_n$, we check that the function $(I - J_n)f_n$ solves

$$\partial_t (I - J_n)f_n = 0, \quad (I - J_n)f_n|_{t=0} = 0.$$

Therefore $(I - J_n)f_n = 0$ which proves that $J_n f_n = f_n$.

Now, we deduce from $J_n f_n = f_n$ and the equation (49) that f_n is also a solution to the original equation (46). In addition, the identity $J_n f_n = f_n$ also implies that f_n is smooth, in particular f_n belongs to $C^1([0, T_n); H^\infty(\mathbb{R}))$.

ii) To conclude the proof of the proposition, it remains to show that (i) the solution is defined globally in time and (ii) it satisfies the L^2 -bound

$$(52) \quad \|f_n(t)\|_{L^2} \leq \|f_0\|_{L^2}.$$

In fact, in light of the alternative (51), it is sufficient to prove the latter inequality: by combining (51) with (52), we will obtain that $T_n = +\infty$.

It remains to prove (52). This estimate is proved in [19, Section 2] for the full equation (that is with J_n replaced by the identity I) and we recall the main argument to verify that the estimate is uniform in n . By definition of $\mathcal{T}(f)f$, one has

$$|D|f - \mathcal{T}(f)f = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\partial_x \Delta_\alpha f}{1 + (\Delta_\alpha f)^2} d\alpha.$$

Therefore the equation (46) is equivalent to

$$\partial_t f_n + (I - J_n)|D|f_n = J_n \left(\frac{1}{\pi} \int_{\mathbb{R}} \frac{\partial_x \Delta_\alpha f_n}{1 + (\Delta_\alpha f_n)^2} d\alpha \right).$$

Using f_n as test function, one has

$$\frac{1}{2} \frac{d}{dt} \|f_n(t)\|_{L^2}^2 + \langle (I - J_n)|D|f_n, f_n \rangle = \frac{1}{\pi} \left\langle J_n \int_{\mathbb{R}} \frac{\partial_x \Delta_\alpha f_n}{1 + (\Delta_\alpha f_n)^2} d\alpha, f_n \right\rangle.$$

Now we use three elementary ingredients: firstly, $\langle (I - J_n)|D|f_n, f_n \rangle \geq 0$ and $J_n^* = J_n$ as can be verified by applying Plancherel's theorem. and secondly $J_n f_n = f_n$. It follows that

$$\frac{1}{2} \frac{d}{dt} \|f_n(t)\|_{L^2}^2 \leq \frac{1}{\pi} \left\langle \int_{\mathbb{R}} \frac{\partial_x \Delta_\alpha f_n}{1 + (\Delta_\alpha f_n)^2} d\alpha, f_n \right\rangle.$$

Now, by [19, Section 2], the right-hand side is non-positive since, for any smooth function $f = f(t, x)$,

$$\begin{aligned} \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \frac{\partial_x \Delta_\alpha f}{1 + (\Delta_\alpha f)^2} d\alpha \right] f(x) dx \\ = - \iint_{\mathbb{R}^2} \log \left[\sqrt{1 + \frac{(f(t, x) - f(t, x - \alpha))^2}{\alpha^2}} \right] dx d\alpha. \end{aligned}$$

The proof is complete. \square

3.2. Uniform estimates. We have seen that the solutions f_n to the approximate systems (46) satisfy a uniform L^2 -estimate (see (48)). We now have to prove uniform L^2 -estimate for the derivatives $|D|^{s, \phi} f_n$.

Let us fix some notations.

Assumption 3.2. *We consider a function $\kappa: [0, \infty) \rightarrow [1, \infty)$ satisfying Assumption 2.4 together with the following property: There exists $a \in [0, 1/2)$ such that*

$$\kappa(r) \geq \log(4 + r)^a \quad \text{for all } r \geq 0.$$

Remember that, by notation,

$$\phi(r) = \int_0^\infty \frac{1 - \cos(h)}{h^2} \kappa \left(\frac{r}{|h|} \right) dh, \quad \text{for } r \geq 0.$$

Recall also that ϕ and κ are equivalent: there are $c, C > 0$ such that,

$$\forall r \geq 0, \quad c\kappa(r) \leq \phi(r) \leq C\kappa(r).$$

We denote by $|D|^{s,\phi}$ the Fourier multiplier $|D|^s \phi(|D_x|)$.

With this notations, our goal in this paragraph is to obtain uniform estimates for the functions

$$(53) \quad A_n(t) = \left\| |D|^{\frac{3}{2},\phi} f_n(t) \right\|_{L^2}^2, \quad B_n(t) = \left\| |D|^{2,\phi} f_n(t) \right\|_{L^2}^2.$$

The following result is the key technical point in this paper.

Proposition 3.3. *Assume that κ satisfies Assumptions 2.4 and 3.2. Then there exist two positive constants C_1 and C_2 such that, for all integer $n \in \mathbb{N} \setminus \{0\}$,*

$$(54) \quad \frac{d}{dt} A_n(t) + C_1 \delta_n(t) B_n(t) \leq C_2 \left(\sqrt{A_n(t)} + A_n(t) \right) \mu_n(t) B_n(t),$$

where

$$\delta_n(t) = \left(1 + \log \left(4 + \frac{B_n(t)}{A_n(t) + \|f_0\|_{L^2}^2} \right)^{1-2a} (A_n(t) + \|f_0\|_{L^2}^2) \right)^{-1},$$

$$\mu_n(t) = \left(\kappa \left(\frac{B_n(t)}{A_n(t)} \right) \right)^{-1}.$$

Proof. We split the analysis into two parts:

- (1) We begin by applying the nonlinear estimates proved in Section 2 to deduce a key inequality (see (56)) of the form

$$(55) \quad \frac{d}{dt} \left\| |D|^{\frac{3}{2},\phi} f_n \right\|_{L^2}^2 + \int_{\mathbb{R}} \frac{|D|^{2,\phi} f_n|^2}{1 + (\partial_x f_n)^2} dx \leq C Q(f_n) \left\| |D|^{2,\phi} f_n \right\|_{L^2},$$

where $Q(f_n)$ is bounded in (strict) subspace of $L^2_{t,x}$ by $\left\| |D|^{\frac{3}{2},\phi} f_n \right\|_{L_t^\infty(L_x^2)}$ and $\left\| |D|^{2,\phi} f_n \right\|_{L_t^2(L_x^2)}$.

- (2) Then we apply interpolation type arguments to show that one can absorb the right-hand side of (55) by the left-hand side.

We now proceed to the details and begin with the following result.

Lemma 3.4. *There exists a positive constant C such that, for any $n \in \mathbb{N} \setminus \{0\}$, the approximate solution $f_n \in C^1([0, +\infty); H^\infty(\mathbb{R}))$ to (46) satisfies*

$$(56) \quad \frac{d}{dt} \left\| |D|^{\frac{3}{2},\phi} f_n \right\|_{L^2}^2 + \int_{\mathbb{R}} \frac{|D|^{2,\phi} f_n|^2}{1 + (\partial_x f_n)^2} dx \leq C Q(f_n) \left\| |D|^{2,\phi} f_n \right\|_{L^2},$$

where

$$Q(f_n) = \left(\|f_n\|_{\dot{H}^2} + \|f_n\|_{\dot{H}^{\frac{7}{4}}}^2 \right) \left\| |D|^{\frac{3}{2},\phi} f_n \right\|_{L^2} + \left\| |D|^{\frac{7}{4},\phi} f_n \right\|_{L^2} \|f_n\|_{\dot{H}^{\frac{7}{4}}} \\ + \left(\|f_n\|_{\dot{H}^{\frac{19}{12}}}^{3/2} + \|f_n\|_{\dot{H}^{\frac{7}{4}}}^{1/2} \right) \left\| |D|^{\frac{7}{4},\phi^2} f_n \right\|_{L^2}^{1/2} \|f_n\|_{\dot{H}^{\frac{7}{4}}}.$$

Proof. As we have seen in the proof of Lemma 3.1, f_n satisfies $J_n f_n = f_n$ and hence $f_n \in C^1([0, +\infty); H^\infty(\mathbb{R}))$. In particular, all computations below are easily justified.

The proof is based on the nonlinear estimates established in the previous section, together with parabolic energy estimates for the Muskat equation, and a commutator estimate with the Hilbert transform.

We multiply the equation

$$\partial_t f_n + |D| f_n = J_n \mathcal{T}(f_n) f_n,$$

by $|D|^{3,\phi^2} f_n$ and use the following consequences of the Plancherel's identity:

$$\begin{aligned} \langle \partial_t f_n, |D|^{3,\phi^2} f_n \rangle &= \frac{1}{2} \frac{d}{dt} \| |D|^{3/2,\phi} f_n \|_{L^2}^2, \\ \langle |D| f_n, |D|^{3,\phi^2} f_n \rangle &= \| |D|^{2,\phi} f_n \|_{L^2}^2. \end{aligned}$$

Now, we need four elementary ingredients:

$$J_n^* = J_n, \quad J_n f_n = f_n \quad (\text{see (47)}), \quad |D|^{3,\phi^2} = |D|^{2,\phi} |D|^{1,\phi}, \quad (|D|^{1,\phi})^* = |D|^{1,\phi}.$$

Then we easily verify that

$$\begin{aligned} \langle J_n \mathcal{T}(f_n) f_n, |D|^{3,\phi^2} f_n \rangle &= \langle \mathcal{T}(f_n) f_n, J_n |D|^{3,\phi^2} f_n \rangle = \langle \mathcal{T}(f_n) f_n, |D|^{3,\phi^2} J_n f_n \rangle \\ &= \langle |D|^{1,\phi} \mathcal{T}(f_n) f_n, |D|^{2,\phi} f_n \rangle. \end{aligned}$$

It follows that

$$\frac{1}{2} \frac{d}{dt} \| |D|^{3/2,\phi} f_n \|_{L^2}^2 + \| |D|^{2,\phi} f_n \|_{L^2}^2 = \langle |D|^{1,\phi} \mathcal{T}(f_n) f_n, |D|^{2,\phi} f_n \rangle.$$

Notice that this identity no longer involves the operator J_n , which explains that the subsequent estimates are independent of n .

Now we commute the operators $|D|^{1,\phi}$ and $\mathcal{T}(f_n)$ in the last term, and then expand the term $\mathcal{T}(f_n)(|D|^{1,\phi} f_n)$ using (24). This gives

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \| |D|^{3/2,\phi} f_n \|_{L^2}^2 + \int_{\mathbb{R}} \frac{| |D|^{2,\phi} f_n |^2}{1 + (\partial_x f_n)^2} dx &= (I) + (II) + (III) \quad \text{where} \\ (I) &:= \langle V(f_n) \partial_x |D|^{1,\phi} f_n, |D|^{2,\phi} f_n \rangle, \\ (II) &:= \langle R(f_n, |D|^{1,\phi} f_n), |D|^{2,\phi} f_n \rangle, \\ (III) &:= \left\langle \left[|D|^{1,\phi}, \mathcal{T}(f_n) \right] f_n, |D|^{2,\phi} f_n \right\rangle. \end{aligned}$$

It follows from Propositions 2.10 and 2.13 that the terms (II) and (III) are estimated by the right-hand side of (56). So it remains only to estimate the term (I). To do so, we claim that

$$(57) \quad (I) \leq C \|V(f_n)\|_{\dot{H}^1} \| |D|^{2,\phi} f_n \|_{L^2} \| |D|^{\frac{3}{2},\phi} f_n \|_{L^2}.$$

Assume that this claim is true. Then it will follow from (57) and Proposition 2.8 that (I) is bounded by the right-hand side of (56), which will in turn complete the proof.

Now we must prove (57). We begin by making appear a commutator structure. To do so, we notice that, since $\partial_x = -\mathcal{H}|D|$, one can rewrite the term A under the form

$$\langle V(f_n)\partial_x |D|^{1,\phi} f_n, |D|^{2,\phi} f_n \rangle = -\langle V(f_n)\mathcal{H}|D|^{2,\phi} f_n, |D|^{2,\phi} f_n \rangle.$$

We then use $\mathcal{H}^* = -\mathcal{H}$ to infer that

$$\begin{aligned} (I) &= -\frac{1}{2} \langle V(f_n)\mathcal{H}|D|^{2,\phi} f_n, |D|^{2,\phi} f_n \rangle + \frac{1}{2} \langle \mathcal{H}(V(f_n)|D|^{2,\phi} f_n), |D|^{2,\phi} f_n \rangle \\ (58) \quad &= \frac{1}{2} \langle [\mathcal{H}, V(f_n)] |D|^{2,\phi} f_n, |D|^{2,\phi} f_n \rangle. \end{aligned}$$

Consequently, to prove (57), it will be sufficient to establish that

$$(59) \quad \left\| [\mathcal{H}, V(f_n)] |D|^{2,\phi} (f_n) \right\|_{L^2} \lesssim \|V(f_n)\|_{\dot{H}^1} \| |D|^{3/2,\phi} f_n \|_{L^2}.$$

The latter inequality will be deduced from a commutator estimate of independent interest. We claim that

$$(60) \quad \|[\mathcal{H}, g_1] (\partial_x g_2)\|_{L^2} \leq C \|g_1\|_{\dot{H}^1} \|g_2\|_{\dot{H}^{\frac{1}{2}}}.$$

Notice that the wanted estimate (59) follows from (60) applied with $g_1 = V(f)$ and $g_2 = \mathcal{H}|D|^{1,\phi}$ (since $|D|^{2,\phi} = \partial_x \mathcal{H}|D|^{1,\phi}$).

It remains to prove the commutator estimate (60). Start from the definition of the Hilbert transform (see (19)) and observe that

$$\|[\mathcal{H}, g_1] (\partial_x g_2)\|_{L^2}^2 = \frac{1}{\pi^2} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{g_1(x) - g_1(y)}{x - y} \partial_y (g_2(x) - g_2(y)) dy \right)^2 dx.$$

Integrating by parts in y , this gives

$$\begin{aligned} (61) \quad \|[\mathcal{H}, g_1] (\partial_x g_2)\|_{L^2}^2 &\lesssim \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{|g_1(x) - g_1(y)| |g_2(x) - g_2(y)|}{|x - y|^2} dy \right)^2 dx \\ &\quad + \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{\partial_y g_1(y)}{x - y} (g_2(x) - g_2(y)) dy \right)^2 dx. \end{aligned}$$

Using the Cauchy-Schwarz inequality, we estimate the first term in the right-hand side of (61) by,

$$\left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{|g_1(x) - g_1(y)|^2}{|x - y|^{1+3/2}} dy \right)^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{|g_2(x) - g_2(y)|^2}{|x - y|^{1+1/2}} dy \right)^2 dx \right)^{\frac{1}{2}}.$$

Using the Lizorkin-Triebel norms introduced in (9), the above product is in turn estimated from above by

$$\|g_1\|_{\dot{F}_{4,2}^{\frac{3}{4}}}^2 \|g_2\|_{\dot{F}_{4,2}^{\frac{1}{4}}}^2.$$

Now, the Sobolev embedding (13) imply that the right-hand side above is bounded by the right-hand side of (60), namely

$$\|g_1\|_{\dot{F}_{4,2}^{\frac{3}{4}}} \|g_2\|_{\dot{F}_{4,2}^{\frac{1}{4}}} \lesssim \|g_1\|_{\dot{H}^1} \|g_2\|_{\dot{H}^{\frac{1}{2}}}.$$

It remains to estimate the second term in the right-hand side of (61). We use again Hölder's inequality to estimate the later term by

$$\left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{|\partial_y g_1(y)|^{3/2}}{|x-y|^{1-1/4}} dy \right)^2 dx \right)^{\frac{2}{3}} \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{|g_2(x) - g_2(y)|^3}{|x-y|^{1+1/2}} dy \right)^2 dx \right)^{\frac{1}{3}}.$$

Then, again, we use (9) and (13) to estimate the above quantity from above by

$$\| |D|^{-\frac{1}{4}} (|\partial_y g_1|^{\frac{3}{2}}) \|_{L^2}^{\frac{4}{3}} \|g_2\|_{\dot{F}_{6,3}^{\frac{1}{6}}}^2 \lesssim \|g_1\|_{\dot{H}^1}^2 \|g_2\|_{\dot{H}^{\frac{1}{2}}}^2.$$

Here we have used the fact that

$$\| |D|^{-\frac{1}{4}} h \|_{L^2} \leq C \|h\|_{L^{\frac{4}{3}}}, \quad \forall h \in L^{\frac{4}{3}}(\mathbb{R}).$$

This completes the proof of (60) and hence the proof of the lemma. \square

We now continue with the interpolation arguments alluded to previously. We want to estimate the various norms which appear in $Q(f)$ in terms of A_n and B_n . Indeed, given Lemma 3.4, the proof of Proposition 3.3 reduces to establishing the following result.

Lemma 3.5. *Consider a real number $7/4 \leq s \leq 2$. Then there exists a positive constant C such that, for all $n \in \mathbb{N} \setminus \{0\}$ and for all $t \geq 0$,*

$$(62) \quad \|f_n(t)\|_{\dot{H}^s} \leq C \mu_n(t) A_n(t)^{2-s} B_n(t)^{s-\frac{3}{2}},$$

$$(63) \quad \| |D|^{\frac{7}{4}, \phi^2} f_n \|_{L^2} \leq \mu_n(t)^{-1} A_n(t)^{\frac{1}{4}} B_n(t)^{\frac{1}{4}},$$

and moreover,

$$(64) \quad \|\partial_x f_n(t)\|_{L^\infty} \leq C \log \left(4 + \frac{B_n(t)}{A_n(t) + \|f_0\|_{L^2}^2} \right)^{\frac{1-2s}{2}} \left(A_n(t)^{\frac{1}{2}} + \|f_0\|_{L^2} \right).$$

Proof. For ease of reading, we skip the indexes n .

i) Let $\lambda > 0$. By cutting the frequency space into low and high frequencies, at the frequency threshold $|\xi| = \lambda$, we obtain

$$\begin{aligned} \|f\|_{\dot{H}^s}^2 &\lesssim \int_{\mathbb{R}} |\xi|^{2s} |\hat{f}|^2 d\xi = \int_{|\xi| \leq \lambda} |\xi|^{2s} |\hat{f}|^2 d\xi + \int_{|\xi| > \lambda} |\xi|^{2s} |\hat{f}|^2 d\xi \\ &\lesssim \int_{|\xi| \leq \lambda} \frac{|\xi|^{2s-3}}{\kappa(|\xi|)^2} |\xi|^3 \phi(|\xi|)^2 |\hat{f}|^2 d\xi + \int_{|\xi| > \lambda} \frac{|\xi|^{2s-4}}{\kappa(|\xi|)^2} |\xi|^4 \phi(|\xi|)^2 |\hat{f}|^2 d\xi, \end{aligned}$$

where we have used the equivalence $\phi \sim \kappa$. Now Plancherel's theorem implies that

$$(65) \quad \begin{aligned} \int_{|\xi| \leq \lambda} |\xi|^3 \phi(|\xi|)^2 |\hat{f}|^2 d\xi &\lesssim \| |D|^{\frac{3}{2}, \phi} f \|_{L^2}^2, \\ \int_{|\xi| > \lambda} |\xi|^4 \phi(|\xi|)^2 |\hat{f}|^2 d\xi &\lesssim \| |D|^{2, \phi} f \|_{L^2}^2. \end{aligned}$$

On the other hand, we claim that

$$(66) \quad (i) \quad \frac{|\xi|^{2s-4}}{\kappa(|\xi|)^2} \leq \frac{\lambda^{2s-4}}{\kappa(\lambda)^2} \quad \text{for } |\xi| \geq \lambda,$$

$$(67) \quad (ii) \quad \frac{|\xi|^{2s-3}}{\kappa(|\xi|)^2} \leq \frac{\lambda^{2s-3}}{\kappa(\lambda)^2} \quad \text{for } |\xi| \leq \lambda.$$

The first claim follows directly from the facts that κ is increasing and the assumption $s \leq 2$ (which implies that $2s - 4 \leq 0$). To prove the second claim, write

$$\frac{r^{2s-3}}{\kappa(r)^2} = \frac{r^{2s-3}}{\log(4+r)^2} \times \frac{\log(4+r)^2}{\kappa(r)^2}.$$

By assumption, $\log(4+r)/\kappa(r)$ is increasing. On the other hand, by computing the derivative, we verify that the other factor is also an increasing function (since we assume that $s \geq 7/4$). It follows that $r \mapsto r^{2s-3}/\kappa(r)^2$ is also increasing which implies the second claim. It follows that

$$\|f\|_{\dot{H}^s}^2 \lesssim \lambda^{2s-3}(\kappa(\lambda))^{-2} \| |D|^{3/2, \phi} f \|_{L^2}^2 + \lambda^{2s-4}(\kappa(\lambda))^{-2} \| |D|^{2, \phi} f \|_{L^2}^2.$$

Chose $\lambda = \| |D|^{2, \phi} (f) \|_{L^2}^2 / \| |D|^{3/2, \phi} (f) \|_{L^2}^2$ to obtain

$$\|f\|_{\dot{H}^s}^2 \lesssim (\kappa(B_n/A_n))^{-2} A_n^{4-2s} B_n^{2s-3},$$

equivalent to the wanted result (62).

ii) As above, for $\lambda > 0$ one has,

$$\begin{aligned} \| |D|^{7/4, \phi^2} f_n \|_{L^2}^2 &= \int |\xi|^{7/2} \phi(|\xi|)^4 |\hat{f}|^2 d\xi \\ &\lesssim \lambda^{1/2} \phi(\lambda)^2 \int_{|\xi| \leq \lambda} |\xi|^3 \phi(|\xi|)^2 |\hat{f}|^2 d\xi \\ &\quad + \lambda^{-1/2} \phi(\lambda)^2 \int_{|\xi| \leq \lambda} |\xi|^4 \phi(|\xi|)^2 |\hat{f}|^2 d\xi. \end{aligned}$$

Since $\phi \sim \kappa$, we deduce that

$$\| |D|^{7/4, \phi^2} f_n \|_{L^2}^2 \lesssim \lambda^{1/2} \kappa(\lambda)^2 \| |D|^{3/2, \phi} f \|_{L^2}^2 + \lambda^{-1/2} \kappa(\lambda)^2 \| |D|^{2, \phi} f \|_{L^2}^2.$$

Now take

$$\lambda = \frac{\| |D|^{2, \phi} f \|_{L^2}^2}{\| |D|^{3/2, \phi} f \|_{L^2}^2},$$

to get the wanted result (63).

iii) Starting from the inverse Fourier transform, using the Cauchy-Schwarz inequality together with estimates similar to (65), we obtain

$$\begin{aligned}
\|\partial_x f\|_{L^\infty} &\leq \int_{\mathbb{R}} |\xi| |\hat{f}| \, d\xi \\
&= \int_{|\xi|>\lambda} \kappa(|\xi|)^{-1} |\xi|^{-1} \kappa(|\xi|) |\xi|^2 |\hat{f}| \, d\xi \\
&\quad + \int_{|\xi|\leq\lambda} \kappa(|\xi|)^{-1} (|\xi|+1)^{-\frac{1}{2}} \kappa(|\xi|) |\xi| (1+|\xi|)^{\frac{1}{2}} |\hat{f}| \, d\xi \\
&\lesssim \left(\int_{|\xi|>\lambda} \frac{1}{|\xi|^2 \kappa^2(|\xi|)} \, d\xi \right)^{\frac{1}{2}} \| |D|^{2,\phi} f \|_{L^2} \\
&\quad + \left(\int_{|\xi|\leq\lambda} \frac{1}{(|\xi|+1) \kappa^2(|\xi|)} \, d\xi \right)^{\frac{1}{2}} \left(\| |D|^{3/2,\phi} f \|_{L^2} + \|f\|_{L^2} \right).
\end{aligned}$$

Now observe that

$$\int_{|\xi|>\lambda} \frac{1}{|\xi|^2 \kappa^2(|\xi|)} \, d\xi \leq \frac{1}{\kappa^2(\lambda)} \int_{|\xi|>\lambda} \frac{1}{|\xi|^2} \, d\xi \leq \frac{1}{\kappa^2(\lambda)\lambda}.$$

It remains to estimate the second integral. Remembering that $\kappa(r) \geq \log(4+r)^a$ by assumption, we begin by writing that

$$\frac{1}{(1+r)\kappa^2(r)} \leq \frac{4}{(4+r)\log(4+r)^{2a}}.$$

On the other hand, with $\beta = (1-2a)/a$ we have $\beta \geq 0$ (since $a < 1/2$) and moreover

$$\frac{1}{(4+r)\log(4+r)^{2a}} = \frac{1}{a\beta} \frac{d}{dr} \log(4+r)^{a\beta}.$$

Therefore,

$$\left(\int_{|\xi|\leq\lambda} \frac{1}{(|\xi|+1)\kappa^2(|\xi|)} \, d\xi \right)^{\frac{1}{2}} \lesssim \log(4+\lambda)^{\frac{1-2a}{2}}.$$

We conclude that

$$\begin{aligned}
\|\partial_x f\|_{L^\infty} &\lesssim \kappa(\lambda)^{-1} \lambda^{-1/2} \| |D|^{2,\phi} f \|_{L^2} + \log(4+\lambda)^{\frac{1-2a}{2}} \left(\| |D|^{3/2,\phi} f \|_{L^2} + \|f\|_{L^2} \right) \\
&\lesssim \log(4+\lambda)^{\frac{1-2a}{2}} \left(\lambda^{-1/2} \| |D|^{2,\phi} f \|_{L^2} + \| |D|^{3/2,\phi} f \|_{L^2} + \|f\|_{L^2} \right).
\end{aligned}$$

Remembering that $\|f\|_{L^2} \leq \|f_0\|_{L^2}$ (see (48)) and then choosing λ such that

$$\lambda^{1/2} = \frac{\| |D|^{2,\phi} f \|_{L^2}}{\| |D|^{3/2,\phi} f \|_{L^2} + \|f_0\|_{L^2}},$$

we obtain (64). This completes the proof. \square

Now the energy estimate (54) follows directly from Lemma 3.4 and Lemma 3.5. \square

For later purposes, we conclude this paragraph by recording a corollary of the inequalities used to prove Lemma 3.5.

Corollary 3.6. Consider a function $f \in \mathcal{S}(\mathbb{R})$ and set

$$M = \|f\|_{\frac{3}{2}, \frac{1}{3}} + \|f\|_{L^2}.$$

Then there holds

$$(68) \quad \|f\|_{\dot{H}^2} + \|\mathcal{T}(f)f\|_{\dot{H}^1} \leq C(M+1)^2 \log\left(4 + \frac{\|f\|_{2, \frac{1}{3}}}{M}\right)^{-\frac{1}{6}} \|f\|_{2, \frac{1}{3}},$$

for some absolute constant C independent of M .

Proof. It follows from the proof of (64) with $\phi(r) = \log(4+r)^{\frac{1}{3}}$ that we have the two following estimates:

$$\begin{aligned} \int |\xi| |\hat{f}(\xi)| \, d\xi &\lesssim \log\left(4 + \frac{\|f\|_{2, \frac{1}{3}}}{M}\right)^{\frac{1}{6}} M, \\ \|f\|_{\dot{H}^2} &\lesssim \log\left(4 + \frac{\|f\|_{2, \frac{1}{3}}}{M}\right)^{-1/3} \|f\|_{2, \frac{1}{3}}. \end{aligned}$$

Therefore, by combining these with (33) and (43), we get that

$$\begin{aligned} &\|f\|_{\dot{H}^2} + \|\mathcal{T}(f)f\|_{\dot{H}^1} \\ &\lesssim \left(\|f\|_{\dot{H}^{\frac{3}{2}}} + \|f\|_{\dot{H}^{\frac{3}{2}}}^2 + 1 + \log\left(4 + \frac{\|f\|_{2, \frac{1}{3}}}{M}\right)^{\frac{1}{6}} M \right) \log\left(4 + \frac{\|f\|_{2, \frac{1}{3}}}{M}\right)^{-\frac{1}{3}} \|f\|_{2, \frac{1}{3}} \\ &\lesssim (M+1)^2 \log\left(4 + \frac{\|f\|_{2, \frac{1}{3}}}{M}\right)^{-\frac{1}{6}} \|f\|_{2, \frac{1}{3}}, \end{aligned}$$

which is the wanted result (68). \square

3.3. Uniform estimates for small initial data globally in time. In this paragraph, we apply Proposition 3.3 to obtain uniform estimates globally in time, assuming some smallness assumption.

Proposition 3.7. *There exists two positive constants c and C such that the following property holds. For all initial data f_0 in $\mathcal{H}^{\frac{3}{2}, \frac{1}{3}}(\mathbb{R})$ satisfying*

$$(69) \quad \|f_0\|_{\frac{3}{2}, \frac{1}{3}} \left(\|f_0\|_{L^2}^2 + 1 \right) \leq c,$$

where the semi-norm $\|\cdot\|_{\frac{3}{2}, \frac{1}{3}}$ is as defined in (6), and for all integer n in $\mathbb{N} \setminus \{0\}$, the solution f_n to the approximate Cauchy problem (46) satisfies

$$(70) \quad \sup_{t \in [0, +\infty)} \|f_n(t)\|_{\frac{3}{2}, \frac{1}{3}} \leq \|f_0\|_{\frac{3}{2}, \frac{1}{3}},$$

together with

$$(71) \quad \int_0^{+\infty} \left[\|\partial_t f_n\|_{\dot{H}^1}^2 + \|f_n\|_{\dot{H}^2}^2 + \|\mathcal{T}(f_n)f_n\|_{\dot{H}^1}^2 + \frac{\|f_n\|_{2, \frac{1}{3}}^2}{\log(4 + \|f_n\|_{2, \frac{1}{3}})^{\frac{1}{3}}} \right] dt \leq C \|f_0\|_{\frac{3}{2}, \frac{1}{3}}^2.$$

Furthermore, there exists a subsequence of (f_n) converging to a solution f of the Muskat equation. Also, f satisfies (70) and (71) with $f_n = f$.

Proof. Fix $\kappa(r) = (\log(4+r))^{\frac{1}{3}}$, define ϕ by (14) and then consider A_n and B_n as given by (53). Notice that, since $\phi \sim \kappa$, we have

$$\| |D|^{\frac{3}{2}, \phi} g \|_{L^2} \sim \|g\|_{\frac{3}{2}, \frac{1}{3}}.$$

The estimate (54) implies that

$$\begin{aligned} & \frac{d}{dt} A_n(t) + C_1 \frac{B_n(t)}{1 + \log \left(4 + \frac{B_n(t)}{A_n(t) + \|f_0\|_{L^2}^2} \right)^{\frac{1}{3}} (A_n(t) + \|f_0\|_{L^2}^2)} \\ (72) \quad & \leq C_2 \left(\sqrt{A_n(t)} + A_n(t) \right) \log \left(4 + \frac{B_n(t)}{A_n(t)} \right)^{-\frac{1}{3}} B_n(t) \\ & \leq C_2 \left(\sqrt{A_n(t)} + A_n(t) \right) \log \left(4 + \frac{B_n(t)}{A_n(t) + \|f_0\|_{L^2}^2} \right)^{-\frac{1}{3}} B_n(t). \end{aligned}$$

We want to absorb the right-hand side by the left-hand side. To do so, we shall prove that

$$\begin{aligned} (73) \quad & C_2 \left(\sqrt{A_n(t)} + A_n(t) \right) \log \left(4 + \frac{B_n(t)}{A_n(t) + \|f_0\|_{L^2}^2} \right)^{-\frac{1}{3}} \\ & \leq \frac{1}{2} \frac{C_1}{1 + \log \left(4 + \frac{B_n(t)}{A_n(t) + \|f_0\|_{L^2}^2} \right)^{\frac{1}{3}} (A_n(t) + \|f_0\|_{L^2}^2)}. \end{aligned}$$

Set

$$X = \sqrt{A_n(t)} + A_n(t), \quad Y = A_n(t) + \|f_0\|_{L^2}^2, \quad \lambda = \log \left(4 + \frac{B_n(t)}{A_n(t) + \|f_0\|_{L^2}^2} \right)^{\frac{1}{3}}.$$

Then (73) is equivalent to

$$C_2 X \leq \frac{C_1}{2} \frac{\lambda}{1 + \lambda Y}.$$

The latter inequality will be satisfied provided that $2C_2 X(Y+1) \leq C_1$. This means that (73) will be satisfied provided that

$$(74) \quad C_2 \left(\sqrt{A_n(t)} + A_n(t) \right) (A_n(t) + \|f_0\|_{L^2}^2 + 1) \leq \frac{C_1}{2}.$$

We thus have proved that if (74) is true for all time t , then (73) is also true for all time. On the other hand, let us assume that (73) is true for all time. Then (72) implies that

$$(75) \quad \frac{d}{dt} A_n(t) + \frac{C_1}{2} \delta_n(t) B_n(t) \leq 0.$$

This immediately implies that A_n is decreasing, which implies that (74) is true also for time t provided that it holds at initial time. By an elementary continuity

argument, one can make the previous reasoning rigorous. This proves that (70) holds provided that the assumption (69) is satisfied with

$$c = \frac{C_1}{8(C_1 + C_2)}.$$

Integrating (75) in time and noticing that $\delta_n(t) \gtrsim \log(4 + B_n(t))^{-\frac{1}{3}}$, we also get that the function $t \mapsto B_n(t) \log(4 + B_n(t))^{-\frac{1}{3}}$ is integrable on $[0, +\infty)$. By virtue of (68) and using the equation $\partial_t f_n = -|D|f_n + J_n(\mathcal{T}(f_n)f_n)$, we end up with (71). Eventually, by the standard compactness theorem, there exists a subsequence of (f_n) converging to a solution f of the Muskat equation. \square

3.4. Uniform estimates for arbitrary initial data. We now prove uniform estimates for arbitrary initial data in $\mathcal{H}^{\frac{3}{2}, \frac{1}{3}}(\mathbb{R})$, without any smallness assumption. This is the most delicate step. Indeed, as explained in Remark 1.4, one important feature of this problem is that the estimates will not only depend on the norm of the initial data: they depend on the initial data themselves. As a consequence, we are forced to estimate the approximate solutions f_n for a norm whose definition depends on the initial data. More precisely, we will estimate the norm $|D|^{\frac{3}{2}, \phi} f_n$ for some function ϕ depending on f_0 . To define this function ϕ , we begin with the following general lemma.

Lemma 3.8. *For any nonnegative integrable function $\omega \in L^1(\mathbb{R})$, there exists a function $\eta: [0, \infty) \rightarrow [1, \infty)$ satisfying the following properties:*

- (1) η is increasing and $\lim_{r \rightarrow \infty} \eta(r) = \infty$,
- (2) $\eta(2r) \leq 2\eta(r)$ for any $r \geq 0$,
- (3) ω satisfies the enhanced integrability condition:

$$(76) \quad \int_{\mathbb{R}} \eta(|r|)\omega(r) dr < \infty,$$

- (4) moreover, the function $r \mapsto \eta(r)/\log(4 + r)$ is decreasing on $[0, \infty)$.

Proof. Consider a sequence of real-number $(\alpha_k)_{k \geq 1}$ such that $\alpha_1 \geq e^5$ and $\alpha_k \geq \alpha_{k-1}^{10}$ and in addition

$$(77) \quad \forall k \geq 1, \quad \int_{|r| \geq \alpha_k} \omega(r) dr \leq 2^{-k}.$$

We set

$$(78) \quad \eta(r) = \begin{cases} 2 & \text{if } 0 \leq r < \alpha_1, \\ k + 1 + \frac{\log(\frac{4+r}{4+\alpha_k})}{\log(\frac{4+\alpha_{k+1}}{4+\alpha_k})} & \text{if } \alpha_k \leq r < \alpha_{k+1}. \end{cases}$$

It is easy to check that $\eta: [0, \infty) \rightarrow [1, \infty)$ is an increasing function converging to $+\infty$ when r goes to $+\infty$. Moreover, η satisfies $\eta(2r) \leq 2\eta(r)$ for any $r \geq 0$.

In addition,

$$\begin{aligned}
\int \eta(|r|)\omega(r) \, dr &\leq \int_{|r|\leq\alpha_1} 2\omega(r) \, dr + \sum_{k=1}^{\infty} (k+2) \int_{\alpha_k\leq|r|\leq\alpha_{k+1}} \omega(r) \, dr \\
&\leq 2\|\omega\|_{L^1} + \sum_{k=1}^{\infty} (k+2)2^{-k} \\
&\leq 2\|\omega\|_{L^1} + C.
\end{aligned}$$

It remains to prove that $r \mapsto \eta(r)/\log(4+r)$ is decreasing. To do so, write

$$(79) \quad \frac{d}{dr} \left(\frac{\eta(r)}{\log(4+r)} \right) = \frac{1}{\log(4+r)} \left(\eta'(r) - \frac{1}{4+r} \frac{\eta(r)}{\log(4+r)} \right).$$

So, for $0 \leq r < \alpha_1$,

$$(80) \quad \frac{d}{dr} \left(\frac{\eta(r)}{\log(4+r)} \right) < 0,$$

while for $\alpha_k \leq r < \alpha_{k+1}$ with $k \geq 1$, we have

$$\begin{aligned}
\frac{d}{dr} \left(\frac{\eta(r)}{\log(4+r)} \right) &\leq \frac{1}{(4+r)\log(4+r)^2} \left(\frac{\log(4+r)}{\log\left(\frac{4+\alpha_{k+1}}{4+\alpha_k}\right)} - k - 1 \right) \\
&\leq \frac{1}{(4+r)\log(4+r)^2} \left(\frac{\log(4+\alpha_{k+1})}{\log\left(\frac{4+\alpha_{k+1}}{4+\alpha_{k+1}^{1/10}}\right)} - 2 \right) < 0,
\end{aligned}$$

where we have used $\alpha_{k+1} \geq e^{5 \times 10^k}$.

This proves that $r \mapsto \eta(r)/\log(4+r)$ is decreasing on $[0, \infty)$. The proof is complete. \square

After this short détour, we return to the main line of our development. Consider a function f_0 in $\mathcal{H}^{\frac{3}{2}, \frac{1}{3}}(\mathbb{R})$. It immediately follows from the previous lemma and Plancherel's theorem that there exists an function $\tilde{k}: [0, \infty) \rightarrow [1, \infty)$ such that

$$(81) \quad \int_{\mathbb{R}} |\xi|^3 \log(4+|\xi|^2)^{\frac{2}{3}} (\tilde{k}(\xi))^2 |\hat{f}_0(\xi)|^2 \, d\xi < +\infty,$$

and such that \tilde{k} is increasing, $r \mapsto \tilde{k}(r)/\log(e+r)$ is decreasing; $\tilde{k}(2r) \leq c_0 \tilde{k}(r)$ and $\lim_{r \rightarrow \infty} \tilde{k}(r) = \infty$.

Next we now define a function $\kappa_0: [0, +\infty) \rightarrow [1, +\infty)$ by

$$\kappa_0(r) = (\log(4+|\xi|))^{\frac{1}{3}} \tilde{k}(|\xi|).$$

together with the companion function ϕ_0 defined by (14), that is

$$(82) \quad \phi_0(\lambda) = \int_0^\infty \frac{1 - \cos(h)}{h^2} \kappa_0\left(\frac{\lambda}{h}\right) \, dh, \quad \text{for } \lambda \geq 0.$$

Proposition 3.9. Consider an initial data f_0 in $\mathcal{H}^{\frac{3}{2}, \frac{1}{3}}(\mathbb{R})$ and denote by ϕ_0 the function defined above in (82). Set

$$M_0 = \| |D|^{\frac{3}{2}, \phi_0} f_0 \|_{L^2}^2.$$

Then, there exists $T_0 > 0$ depending on M_0 and $\|f_0\|_{L^2}$ such that, for any integer n in $\mathbb{N} \setminus \{0\}$, the solution f_n to the approximate Cauchy problem (46) satisfies

$$(83) \quad \sup_{t \in [0, T_0]} \| |D|^{\frac{3}{2}, \phi_0} f_n(t) \|_{L^2}^2 \leq 2M_0$$

and

$$(84) \quad \int_0^{T_0} \left(\|\partial_t f_n\|_{\dot{H}^1}^2 + \|f_n\|_{\dot{H}^2}^2 + \|\mathcal{T}(f_n)(f_n)\|_{\dot{H}^1}^2 + \frac{\|f_n\|_{2, \frac{1}{3}}^2}{\log(4 + \|f_n\|_{2, \frac{1}{3}})^{\frac{1}{3}}} \right) dt \leq CM_0,$$

for some absolute constant $C > 0$ independent of f_0 .

Furthermore, there exists a subsequence of (f_n) converging to a solution f of the Muskat equation which satisfies (70) and (71) with f_n replaced by f .

Remark 3.10. Notice that the time T_0 depends on f_0 and not only on $\|f_0\|_{\mathcal{H}^{\frac{3}{2}, \frac{1}{3}}}$.

Proof. We apply (54) for the quantities

$$A_n(t) = \| |D|^{\frac{3}{2}, \phi_0} f_n(t) \|_{L^2}^2, \quad B_n(t) = \| |D|^{2, \phi_0} f_n(t) \|_{L^2}^2.$$

This gives that

$$(85) \quad \frac{d}{dt} A_n(t) + C_1 \delta_n(t) B_n(t) \leq C_2 \left(\sqrt{A_n(t)} + A_n(t) \right) \mu_n(t) B_n(t),$$

where

$$\delta_n(t) = \left(1 + \left[\log \left(4 + \frac{B_n(t)}{A_n(t) + \|f_0\|_{L^2}^2} \right) \right]^{1-2a} (A_n(t) + \|f_0\|_{L^2}^2) \right)^{-1},$$

$$\mu_n(t) = \left(\log \left(4 + \frac{B_n(t)}{A_n(t)} \right) \right)^{-\frac{1}{3}} \times \left(\tilde{k} \left(\frac{B_n(t)}{A_n(t)} \right) \right)^{-1}.$$

Given $\varrho \geq 0$, define the function

$$\mathcal{E}(\varrho, \|f_0\|_{L^2}^2) = \sup_{r \geq 0} \left\{ C_2 \frac{(\sqrt{\varrho} + \varrho) r}{\tilde{k} \left(\frac{r}{\varrho} \right) \left[\log \left(4 + \frac{r}{\varrho} \right) \right]^{1/3}} - \frac{C_1}{2} \frac{r}{1 + \left[\log \left(4 + \frac{r}{\varrho} \right) \right]^{1/3} (\varrho + \|f_0\|_{L^2}^2)} \right\}.$$

Since $\rho \mapsto \tilde{k}(\rho)$ is increasing, directly from the definition of $\mathcal{E}(\varrho, \|f_0\|_{L^2}^2)$, we verify that the function $\varrho \rightarrow \mathcal{E}(\varrho, \|f_0\|_{L^2}^2)$ is increasing. On the other hand, since $\kappa(\rho)$ tends to $+\infty$ as ρ goes to $+\infty$, we verify that

$$\forall \varrho \geq 0, \quad \mathcal{E}(\varrho, \|f_0\|_{L^2}^2) < \infty.$$

Thus,

$$(86) \quad \frac{d}{dt}A_n(t) + \frac{C_1}{2}\delta_n(t)B_n(t) \leq \mathcal{E}(A_n(t), \|f_0\|_{L^2}^2).$$

and *a fortiori*

$$\frac{d}{dt}A_n(t) \leq \mathcal{E}(A_n(t), \|f_0\|_{L^2}^2).$$

Then by standard arguments, one obtains

$$(87) \quad \sup_{t \in [0, T_0]} \| |D|^{\frac{3}{2}, \phi_0} f_n(t) \|_{L^2}^2 \leq 2M_0.$$

with

$$T_0 = \frac{M_0}{\mathcal{E}(2M_0, \|f_0\|_{L^2}^2)}, \quad M_0 = \| |D|^{\frac{3}{2}, \phi_0} f_0 \|_{L^2}^2.$$

Moreover, as proof of Propostion 3.7, we also have (84) and a subsequence of (f_n) converging to a solution f of the Muskat equation. Also, f satisfies (70) and (71) with $f_n = f$. The proof is complete. \square

3.5. Uniqueness. The following proposition implies that the solution of the Muskat equation is unique.

Proposition 3.11. *Consider two solutions f_1, f_2 of the Muskat equation in $[0, T] \times \mathbb{R}$ (for some $T < \infty$), with initial data $f_{1,0}, f_{2,0}$ respectively, satisfying*

$$(88) \quad \sup_{t \in [0, T]} \|f_k(t)\|_{\dot{H}^{\frac{2}{3}, \frac{1}{3}}}^2 + \int_0^T \log\left(4 + \|f_k\|_{2, \frac{1}{3}}\right)^{-\frac{1}{3}} \|f_k\|_{2, \frac{1}{3}}^2 dt \leq M < \infty, \quad k = 1, 2.$$

Then the difference $g = f_1 - f_2$ is estimated by

$$(89) \quad \sup_{t \in [0, T]} \|g(t)\|_{\dot{H}^{\frac{1}{2}}} \leq \|g(0)\|_{\dot{H}^{\frac{1}{2}}} \exp\left(C(M) \sum_{k=1}^2 \int_0^T \log\left(4 + \|f_k\|_{2, \frac{1}{3}}\right)^{-\frac{1}{3}} \|f_k\|_{2, \frac{1}{3}}^2 dt\right).$$

Proof. Since $\partial_t f_k + |D|f_k = \mathcal{T}(f_k)f_k$, it follows from the decomposition (24) of $\mathcal{T}(f_k)f_k$ that the difference $g = f_1 - f_2$ satisfies

$$\partial_t g + \frac{|D|g}{1 + (\partial_x f_1)^2} = V(f_1)\partial_x g + R(f_1, g) + (\mathcal{T}(f_2 + g) - \mathcal{T}(f_2))f_2.$$

Take the L^2 -scalar product of this equation with $|D|g$ to get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|g\|_{\dot{H}^{\frac{1}{2}}}^2 + \int \frac{(|D|g)^2}{1 + (\partial_x f_1)^2} dx &\leq |(V(f_1)\partial_x g, |D|g)| + \|R(f_1, g)\|_{L^2} \|g\|_{\dot{H}^1} \\ &\quad + \|(\mathcal{T}(f_2 + g) - \mathcal{T}(f_2))f_2\|_{L^2} \|g\|_{\dot{H}^1}. \end{aligned}$$

The arguments used to show the identity (58) also implies that

$$|(V(f_1)\partial_x g, |D|g)| = \frac{1}{2} |([\mathcal{H}, V(f_1)] |D|g, |D|g)|.$$

Thus, by combining the estimate (26) for $\|V(f)\|_{\dot{H}^1}$ together with the commutator estimate (60) about the Hilbert transform, the bound (39) for the remainder term and eventually the following interpolation inequality (see (64)),

$$\|\partial_x f_1\|_{L^\infty} \leq C(M) \log \left(4 + \|f_1\|_{2, \frac{1}{3}} \right)^{\frac{1}{6}},$$

we end up with

$$\begin{aligned} & \frac{d}{dt} \|g\|_{\dot{H}^{\frac{1}{2}}}^2 + C(M) \log \left(4 + \|f_1\|_{2, \frac{1}{3}} \right)^{-\frac{1}{3}} \|g\|_{\dot{H}^1}^2 \\ & \lesssim \left(\|f_1\|_{\dot{H}^2} + \|f_1\|_{\dot{H}^{\frac{7}{4}}}^2 \right) \|g\|_{\dot{H}^{\frac{1}{2}}} \|g\|_{\dot{H}^1} + \|f_1\|_{\dot{H}^{\frac{7}{4}}} \|g\|_{\dot{H}^{\frac{3}{4}}} \|g\|_{\dot{H}^1} \\ & \quad + \|(\mathcal{T}(f_2 + g) - \mathcal{T}(f_2)) f_2\|_{L^2} \|g\|_{\dot{H}^1}. \end{aligned}$$

Now, directly from the definition of $\mathcal{T}(f)g$, we have

$$|(\mathcal{T}(f_2 + g) - \mathcal{T}(f_2)) f_2(x)| \lesssim \int |\Delta_\alpha(\partial_x f_2)(x)| |\Delta_\alpha g(x)| d\alpha,$$

so, by Hölder's inequality, the L^2 -norm of $(\mathcal{T}(f_2 + g) - \mathcal{T}(f_2)) f_2$ is estimated by

$$\left(\int \left(\int |\Delta_\alpha(\partial_x f_2)(x)| |\Delta_\alpha g(x)| d\alpha \right)^2 dx \right)^{\frac{1}{2}} \leq \|(\partial_x f_2)\|_{\dot{F}_{4,2}^{\frac{1}{2}}} \|g\|_{\dot{F}_{4,2}^{\frac{1}{2}}} \lesssim \|f_2\|_{\dot{H}^{\frac{7}{4}}} \|g\|_{\dot{H}^{\frac{3}{4}}}.$$

Hence, we derive

$$\begin{aligned} & \frac{d}{dt} \|g\|_{\dot{H}^{\frac{1}{2}}}^2 + C(M) \log \left(4 + \|f_1\|_{2, \frac{1}{3}} \right)^{-\frac{1}{3}} \|g\|_{\dot{H}^1}^2 \\ & \lesssim \left(\|f_1\|_{\dot{H}^2} + \|f_1\|_{\dot{H}^{\frac{7}{4}}}^2 \right) \|g\|_{\dot{H}^{\frac{1}{2}}} \|g\|_{\dot{H}^1} + \|f_2\|_{\dot{H}^{\frac{7}{4}}} \|g\|_{\dot{H}^{\frac{3}{4}}} \|g\|_{\dot{H}^1} \\ & \lesssim \left(\|f_1\|_{\dot{H}^2} + \|f_1\|_{\dot{H}^{\frac{7}{4}}}^2 \right) \|g\|_{\dot{H}^{\frac{1}{2}}} \|g\|_{\dot{H}^1} + \|f_2\|_{\dot{H}^{\frac{7}{4}}} \|g\|_{\dot{H}^{\frac{1}{2}}} \|g\|_{\dot{H}^{\frac{3}{2}}}. \end{aligned}$$

Interchanging the role of f_1 and f_2 , we also get a symmetric estimate. Then, by combining these two estimates, we get

$$\begin{aligned} & \frac{d}{dt} \|g\|_{\dot{H}^{\frac{1}{2}}}^2 + C(M) \left[\log \left(4 + \|f_1\|_{2, \frac{1}{3}} \right)^{-\frac{1}{3}} + \log \left(4 + \|f_2\|_{2, \frac{1}{3}} \right)^{-\frac{1}{3}} \right] \|g\|_{\dot{H}^1}^2 \\ & \lesssim \left(\|f_1\|_{\dot{H}^2} + \|f_1\|_{\dot{H}^{\frac{7}{4}}}^2 \right) \|g\|_{\dot{H}^{\frac{1}{2}}} \|g\|_{\dot{H}^1} + \|f_2\|_{\dot{H}^{\frac{7}{4}}} \|g\|_{\dot{H}^{\frac{1}{2}}} \|g\|_{\dot{H}^{\frac{3}{2}}} \\ & \quad + \left(\|f_2\|_{\dot{H}^2} + \|f_2\|_{\dot{H}^{\frac{7}{4}}}^2 \right) \|g\|_{\dot{H}^{\frac{1}{2}}} \|g\|_{\dot{H}^1} + \|f_1\|_{\dot{H}^{\frac{7}{4}}} \|g\|_{\dot{H}^{\frac{1}{2}}} \|g\|_{\dot{H}^{\frac{3}{2}}}. \end{aligned}$$

By interpolation inequality (62)

$$\begin{aligned} \|f_k\|_{\dot{H}^{\frac{7}{4}}} & \lesssim_M \log \left(4 + \|f_k\|_{2, \frac{1}{3}} \right)^{-\frac{1}{3}} \|f_k\|_{2, \frac{1}{3}}^{\frac{1}{2}}, \\ \|f_k\|_{\dot{H}^2} & \lesssim_M \log \left(4 + \|f_k\|_{2, \frac{1}{3}} \right)^{-\frac{1}{3}} \|f_k\|_{2, \frac{1}{3}}, \end{aligned}$$

hence

$$\begin{aligned}
& \frac{d}{dt} \|g\|_{\dot{H}^{\frac{1}{2}}}^2 + C(M) \left[\log \left(4 + \|f_1\|_{2, \frac{1}{3}} \right)^{-\frac{1}{3}} + \log \left(4 + \|f_2\|_{2, \frac{1}{3}} \right)^{-\frac{1}{3}} \right] \|g\|_{\dot{H}^1}^2 \\
& \lesssim_M \log \left(4 + \|f_1\|_{2, \frac{1}{3}} \right)^{-\frac{1}{3}} \|f_1\|_{2, \frac{1}{3}} \|g\|_{\dot{H}^{\frac{1}{2}}} \|g\|_{\dot{H}^1} \\
& \quad + \log \left(4 + \|f_2\|_{2, \frac{1}{3}} \right)^{-\frac{1}{3}} \|f_2\|_{2, \frac{1}{3}}^{\frac{1}{2}} \|g\|_{\dot{H}^{\frac{1}{2}}}^{\frac{1}{2}} \|g\|_{\dot{H}^1}^{\frac{3}{2}} \\
& \quad + \log \left(4 + \|f_2\|_{2, \frac{1}{3}} \right)^{-\frac{1}{3}} \|f_2\|_{2, \frac{1}{3}} \|g\|_{\dot{H}^{\frac{1}{2}}} \|g\|_{\dot{H}^1} \\
& \quad + \log \left(4 + \|f_1\|_{2, \frac{1}{3}} \right)^{-\frac{1}{3}} \|f_1\|_{2, \frac{1}{3}}^{\frac{1}{2}} \|g\|_{\dot{H}^{\frac{1}{2}}}^{\frac{1}{2}} \|g\|_{\dot{H}^1}^{\frac{3}{2}}.
\end{aligned}$$

Finally, by Holder's inequality,

$$\frac{d}{dt} \|g\|_{\dot{H}^{\frac{1}{2}}}^2 \lesssim_M \left(\sum_{k=1}^2 \log \left(4 + \|f_k\|_{2, \frac{1}{3}} \right)^{-\frac{1}{3}} \|f_k\|_{2, \frac{1}{3}}^2 \right) \|g\|_{\dot{H}^{\frac{1}{2}}}^2,$$

which in turn implies (89). The proof is complete. \square

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