



**HAL**  
open science

# Totally Invariant Divisors of non Trivial Endomorphisms of the Projective Space

Mabed Yanis

► **To cite this version:**

Mabed Yanis. Totally Invariant Divisors of non Trivial Endomorphisms of the Projective Space. 2021.  
hal-03453728

**HAL Id: hal-03453728**

**<https://hal.science/hal-03453728>**

Preprint submitted on 28 Nov 2021

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Totally Invariant Divisors of non Trivial Endomorphisms of the Projective Space

MABED Yanis

## Abstract

It is expected that a totally invariant divisor of a non-isomorphic endomorphism of the complex projective space is a union of hyperplanes. In this paper, we compute an upper bound for the degree of such a divisor. As a consequence, we prove the linearity of totally invariant divisors with isolated singularities.

## 1 Introduction

Only a few complex projective varieties admit non-isomorphic endomorphisms (which we call non trivial endomorphisms). The projective space of any dimension is one of them and a subset  $D \subset \mathbb{P}^n$  is said totally invariant if there exists a non trivial endomorphism  $f$  of  $\mathbb{P}^n$  such that  $f^{-1}(D) = D$ . The following statement has been conjectured in [BCS04]

**Conjecture 1.1.** *All totally invariant prime divisors of  $\mathbb{P}^n$  are linear.*

For  $n \geq 3$ , the conjecture is true for smooth divisors thanks to a result of Beauville [Bea01, Thm] for smooth divisors of degree higher than two and Cerveau-Lins Neto [CLN00, Thm 2] for smooth quadrics. Let  $D \subset \mathbb{P}^n$  be a totally invariant divisor by an non trivial endomorphism  $f$  of  $\mathbb{P}^n$ , the logarithmic ramification formula (see [BH14, Lemma 2.5]) gives the following

$$K_{\mathbb{P}^n} + D = f^*(K_{\mathbb{P}^n} + D) + R$$

where  $R$  is an uniquely determined effective divisor. We deduce that the degree  $d$  of a totally invariant divisor is bounded by  $n + 1$ . Hwang and Nakayama showed that irreducible hypersurfaces of  $\mathbb{P}^n$  of degree  $n + 1$  are not totally invariant [HN11, Thm 2.1.] and Höring proved that no irreducible hypersurface of  $\mathbb{P}^n$  of degree  $n$  is totally invariant (see [Hör17]). This last statement finishes the proof of the conjecture in  $\mathbb{P}^3$  thanks to Zhang who showed that no quadric of  $\mathbb{P}^3$  is totally invariant (see [Zha13, Thm 1.1.]). Our goal here is to prove the two following theorems.

**Theorem 1.2.** *Let  $X$  be an irreducible totally invariant divisor of  $\mathbb{P}^n$  of degree  $d$  and let  $l$  be the dimension of the singular locus of  $X$  (we set  $l = -1$  if  $X$  is smooth). Then we have*

$$d < \binom{n}{l+1}^{1/(n-l-1)} + 1.$$

The proof of this statement follows the method in [Hör17] which involves the Chern classes of the logarithmic cotangent sheaf  $\Omega_{\mathbb{P}^n}^1(\log X)$ . To understand the strengths and weaknesses of this bound, we will see multiple corollaries in the conclusion of the paper.

**Theorem 1.3.** *If  $X$  is a divisor of  $\mathbb{P}^k$  which is not totally invariant then, for all  $n \geq k$ , no cone of  $\mathbb{P}^n$  over  $X$  (see definition 3.1) is totally invariant .*

Since singular quadrics can be seen as cones over smooth quadrics, this result implies that any quadric is totally invariant by a non trivial endomorphism. Indeed, smooth quadrics can not be totally invariant by [CLN00, Thm 2]. The two theorems above join forces to give the following corollary.

**Corollary 1.4.** *Let  $X$  be an irreducible divisor of  $\mathbb{P}^n$  with isolated singularities. If  $X$  is totally invariant then it is a hyperplane.*

**Acknowledgements.** This work was completed during my PhD thesis directed by Andreas H\"oring and Ekaterina Amerik. I wish also thank Daniele Faenzi and Ama\"el Broustet for their crucial advices. This thesis was supported by Universit e Cˆote d'Azur and by Laboratoire de Math ematiques J.A. Dieudonn e.

## 2 Bound of the degree

Let  $X$  be an irreducible divisor of  $\mathbb{P}^n$  totally invariant by a non trivial endomorphism  $f$  of  $\mathbb{P}^n$  with  $n \geq 2$ . The original idea of H\"oring in [H\"or17] is about comparing the following Chern classes

$$c_2[\Omega_{\mathbb{P}^n}(\log X) \otimes \mathcal{O}(r)]$$

for  $r \in \mathbb{N}^*$ . We want to adapt this point of vue to the higher Chern classes of these logarithmic cotangent sheafs. When the variety  $X$  is singular, we cannot ensure that these sheafs are locally free everywhere but it's locally free and globally generated wherever  $X$  is smooth. That's why the Theorem 1.2 involves the dimension  $l$  of the singular locus. We set  $k = n - l - 1$ , so we have

$$\text{codim}_{\mathbb{P}^n} X_{\text{sing}} = k + 1$$

and we want to show the following inequality

$$d < \binom{n}{k}^{1/k} + 1.$$

If  $X$  is not normal (i.e.  $k = 1$ ), one remarks that the bound becomes  $n + 1$  which is already known. So we can suppose, without any loss of generality, that  $X$  is a normal divisor of  $\mathbb{P}^n$ .

We first recall the lemma 2.2 of [Hör17] :

**Proposition 2.1.** *Denoting by  $h$  the equation of  $X$  in  $\mathbb{P}^n$  and assuming that  $\deg X \geq 2$ , the sheaf  $\Omega_{\mathbb{P}^n}(\log X) \otimes \mathcal{O}_{\mathbb{P}^n}(1)$  is generated by the following global sections*

$$\frac{d(X_0 h)}{h}, \dots, \frac{d(X_n h)}{h} \quad (1)$$

on the complement of  $X_{\text{sing}}$ .

The key point of the proof of Theorem 1.2 is the following lemma which permits to compare Chern classes of sheafs.

**Lemma 2.2.** *Let  $M$  be a smooth projective variety of dimension  $k$ ,  $\mathcal{E}_1$  and  $\mathcal{E}_2$  be two vector bundles of rank  $n \geq k$  on  $M$ ,  $V \subset H^0(M, \mathcal{E}_1)$  be a vector subspace of dimension  $> n$  and*

$$\varphi : \mathcal{E}_1 \rightarrow \mathcal{E}_2$$

an injective sheaf morphism on  $M$ . We assume the following :

- (i) *the evaluation morphism  $ev : V \otimes \mathcal{O}_M \rightarrow \mathcal{E}_1$  is surjective,*
- (ii) *for all  $x \in M$ , one has  $\text{rg } \varphi_x \geq n - k$ , and*
- (iii) *for all  $1 \leq i \leq k + 1$ , one has  $\dim \{x \in M \mid \text{rg } \varphi_x \leq n - i + 1\} \leq k - i + 1$ .*

Then we have

$$c_k(\mathcal{E}_1) \leq c_k(\mathcal{E}_2).$$

*Proof.* One denotes by  $|V|$  the projective space obtained from the vector space  $V$  and we consider the closed set

$$B := \{(x, \sigma) \in M \times |V| \mid (\varphi \circ ev)_x(\sigma(x)) = 0\}$$

with its two natural projections

$$p_1 : B \rightarrow M, \quad p_2 : B \rightarrow |V|.$$

One sets, for all  $1 \leq i \leq k + 1$ ,

$$R_i := \{x \in M \mid \text{rg } \varphi_x = n - i + 1\}$$

with  $\cup_{i=1}^{k+1} R_i = M$  by (ii).

If  $x \in R_i$  then  $rg(\varphi \circ ev)_x = n - i + 1$  and so

$$\dim p_1^{-1}(x) = \dim |V| - n + i - 1.$$

However, by (iii), we have  $\dim R_i \leq k - i + 1$  and the dimension of the irreducible components of  $B$  are bounded by  $\dim |V| - n + k$ . We finish the proof by induction on  $n \geq k$  :

If  $n = k$ , then the dimension of the irreducible components of  $B$  are bounded by  $\dim |V|$  so the generic fibers of  $p_2$  are finite. We deduce that for a generic section  $\sigma$  of  $\mathcal{E}_1$ , the section  $\varphi \circ \sigma$  of  $\mathcal{E}_2$  has only isolated zeros. Since  $\varphi$  is injective, the generic section  $\sigma$  has also isolated zeros and we naturally obtain  $c_k(\mathcal{E}_1) \leq c_k(\mathcal{E}_2)$ .

If  $n > k$ , then the dimension of the irreducible components of  $B$  are bounded by  $\dim |V| - n + k < \dim |V|$  so the generic fibers of  $p_2$  are empty. Therefore, for a generic section  $\sigma$  of  $\mathcal{E}_1$ , the section  $\varphi \circ \sigma$  never vanishes and neither does  $\sigma$ . These two sections  $\sigma$  and  $\varphi \circ \sigma$  define a trivial subbundle of  $\mathcal{E}_1$  and  $\mathcal{E}_2$  respectively. Furthermore, the quotients  $\mathcal{E}_1/\mathcal{O}_M$  and  $\mathcal{E}_2/\mathcal{O}_M$ , the global section subspace  $V/\mathbb{C}\sigma$  and the induced morphism

$$\tilde{\varphi} : \mathcal{E}_1/\mathcal{O}_M \rightarrow \mathcal{E}_2/\mathcal{O}_M$$

satisfy the lemma's hypothesis so we can apply the induction hypothesis. Finally, we know that  $c(\mathcal{E}_i) = c(\mathcal{E}_i/\mathcal{O}_M)$  because of the following short exact sequence

$$0 \rightarrow \mathcal{O}_M \rightarrow \mathcal{E}_i \rightarrow \mathcal{E}_i/\mathcal{O}_M \rightarrow 0.$$

In particular we obtain  $c_k(\mathcal{E}_i) = c_k(\mathcal{E}_i/\mathcal{O}_M)$  that finishes the proof. ■

Before applying the previous lemma, we shall recall the Proposition III.10.6. in [Har77].

**Proposition 2.3.** *If  $f : X \rightarrow Y$  is a finite morphism between two  $n$ -dimensional varieties then*

$$\text{codim}_X \{x \in X | rg d_x f \leq n - i\} \geq i.$$

We now set  $P \subset \mathbb{P}^n$  a general  $k$ -dimensional plane such that  $M = f^{-1}(P)$  is a smooth  $k$ -dimensional variety and  $M \cap X_{\text{sing}} = \emptyset$ . We also set

$$\varphi : f^*(\Omega_{\mathbb{P}^n}(\log X) \otimes \mathcal{O}_{\mathbb{P}^n}(1)) \otimes \mathcal{O}_M \rightarrow \Omega_{\mathbb{P}^n}(\log X) \otimes \mathcal{O}_{\mathbb{P}^n}(m) \otimes \mathcal{O}_M$$

the injective sheaf morphism on  $M$  induced by the logarithmic differential of  $f$ . The integer  $m$  is the algebraic degree of  $f$ , we have  $m \geq 2$  and  $f^*(\mathcal{O}(1)) = \mathcal{O}(m)$ . Finally we denote by  $V \subset H^0(M, \Omega_{\mathbb{P}^n}(\log X) \otimes \mathcal{O}_{\mathbb{P}^n}(1) \otimes \mathcal{O}_M)$  the subspace generated by the restrictions to  $M$  of the global sections defined by (1). One remarks that  $M$  is general  $k$ -dimensional variety in the following sense : if  $V$  is a variety with  $\dim V \leq n - k - 1$  then we can assume that  $M \cap V = \emptyset$ .

**Proposition 2.4.** *All the objects defined above satisfy the hypothesis of Lemma 2.2.*

*Proof.*

- (i) This follows directly from the Proposition 2.1.
- (ii) If  $x \in M \setminus X$  then  $\Omega_{\mathbb{P}^n}(\log X)_x \simeq \Omega_{\mathbb{P}^n}$  and the logarithmic differential is only the classic differential. Since  $f$  is finite, we deduce from Proposition 2.3 :

$$\dim \{rg (df_{log})_x \leq n - k - 1\} \leq n - k - 1$$

then, since the  $k$ -variety  $M$  was chosen in a generic way, we have

$$M \cap \{rg (df_{log})_x \leq n - k - 1\} = \emptyset.$$

so for all  $x \in M \setminus X$ ,  $rg \varphi_x \geq n - k$ .

If  $x \in X \setminus X_{sing}$  then we have  $\Omega_{X,x} \subset \Omega_{\mathbb{P}^n}(\log X)_x$ . Indeed, if  $\{u_1 = 0\}$  is a local equation of  $X$  near  $x$  so the  $\mathbb{C}$ -vector space  $\Omega_{\mathbb{P}^n}(\log X)_x$  admits

$$\frac{du_1}{u_1}, du_2, \dots, du_n$$

as a basis and  $\Omega_{X,x}$  has

$$du_2, \dots, du_n$$

as a basis. Furthermore, since  $X$  is totally invariant by  $f$ , we have the following commutative diagram where the vertical arrows are injective :

$$\begin{array}{ccc} (f|_X)^* \Omega_{X,x} & \xrightarrow{d(f|_X)} & \Omega_{X,x} \\ \downarrow & & \downarrow \\ f^* \Omega_{\mathbb{P}^n}(\log X)_x & \xrightarrow{df_{log}} & \Omega_{\mathbb{P}^n}(\log X)_x \end{array}$$

But, since  $f|_X$  is finite, we have

$$\dim \{rg d(f|_X) \leq n - k - 1\} \leq n - k - 1$$

so, again because the  $k$ -dimensional variety  $M$  was chosen in a generic way, we obtain

$$M \cap \{rg d(f|_X)_x \leq n - k - 1\} = \emptyset.$$

In other words, the rank of  $d(f|_X)$  on  $M \cap X$  is at least  $n - k$  and by the previous diagram, so is the rank of  $df_{log}$  on  $M \cap X$ .

- (iii) We fix  $1 \leq i \leq k + 1$ . For  $x$  outside of  $X$  and by the same reason as (ii), we have

$$\dim \{rg (df_{log})_x \leq n - i + 1\} \leq n - i + 1 \quad (2)$$

and so

$$\dim M \cap \{rg (df_{log})_x \leq n - i + 1\} \leq k - i + 1.$$

Similarly, on  $X$  we have

$$\dim M \cap \{d(f|_X)_x \leq n - i\} \leq k - i.$$

Then, by reusing the diagram in (ii), we find that the dimension of  $\{rg(df_{log})_x \leq n - i + 1\}$  is at most  $k - i + 1$  and finally

$$\dim \{rg \varphi_x \leq n - i + 1\} \leq k - i + 1.$$

■

As a consequence of the key lemma, we obtain the following.

**Proposition 2.5.** *Let  $X$  be a prime divisor of  $\mathbb{P}^n$  which is totally invariant by a non trivial endomorphism  $f$  of  $\mathbb{P}^n$ . We set  $k = \text{codim}_X X_{\text{sing}}$  and  $M$  defined as in Proposition 2.4. Then we have the following inequality :*

$$c_k(f^*(\Omega_{\mathbb{P}^n}(\log X)(1)) \otimes \mathcal{O}_M) \leq c_k(\Omega_{\mathbb{P}^n}(\log X)(m) \otimes \mathcal{O}_M). \quad (3)$$

This brings us to the computation of  $c_k(\Omega_{\mathbb{P}^n}(\log X) \otimes \mathcal{O}_{\mathbb{P}^n}(m))$ , the result is summed up in the next proposition.

**Proposition 2.6.** *The  $k$ -th Chern class of  $\Omega_{\mathbb{P}^n}(\log X) \otimes \mathcal{O}_{\mathbb{P}^n}(1)$  does not depend of  $n$ . More precisely, we have*

$$c_k(\Omega_{\mathbb{P}^n}(\log X) \otimes \mathcal{O}_{\mathbb{P}^n}(1)) = (d - 1)^k H^k.$$

Furthermore, we have the following estimate

$$c_k(\Omega_{\mathbb{P}^n}(\log X) \otimes \mathcal{O}_{\mathbb{P}^n}(m)) = \left[ \binom{n}{k} m^k + \binom{n-1}{k-1} (d - n - 1) m^{k-1} + o(m^{k-1}) \right] H^k.$$

*Proof.* As a first step, we shall compute  $c_i(\Omega_{\mathbb{P}^n}(\log X))$  for  $i = 1, \dots, k$  by using the short exact sequence (2.8) of [Dol07]

$$0 \rightarrow \Omega_{\mathbb{P}^n}^1 \rightarrow \tilde{\Omega}_{\mathbb{P}^n}^1(\log X) \xrightarrow{\text{res}} \mu_* \mathcal{O}_{\tilde{X}} \rightarrow 0. \quad (4)$$

where  $\mu : \tilde{X} \rightarrow X$  is a desingularization of  $X$ . Since  $X$  is a normal variety and by [Deb01, 1.13.], we have  $\mu_* \mathcal{O}_{\tilde{X}} \simeq \mathcal{O}_X$ . We deduce that

$$c_i(\mu_* \mathcal{O}_{\tilde{X}}) = c_i(\mathcal{O}_X) = X^i = d^i H^i$$

for all  $1 \leq i \leq k$  then

$$c_i(\Omega_{\mathbb{P}^n}(\log X)) = \left( \sum_{j=0}^i (-1)^j \binom{n+1}{j} d^{i-j} \right) H^i$$

for  $i = 1, \dots, k$ .

We now use the lemma A.2.1 from [Mab21] with  $\mathcal{E} = \Omega_{\mathbb{P}^n}(\log X)$  and  $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(m)$  to find the following

$$\begin{aligned} c_k(\Omega_{\mathbb{P}^n}(\log X) \otimes \mathcal{O}_{\mathbb{P}^n}(m)) &= \left[ \sum_{i=0}^k m^{k-i} \left( \sum_{j=0}^i (-1)^j \binom{n+1}{j} \binom{n-i}{k-i} d^{i-j} \right) \right] H^k \\ &= \left[ \binom{n}{k} m^k + \binom{n-1}{k-1} (d-n-1) m^{k-1} + o(m^{k-1}) \right] H^k \end{aligned}$$

and, in particular for  $m = 1$ ,

$$\begin{aligned} c_k(\Omega_{\mathbb{P}^n}(\log X) \otimes \mathcal{O}_{\mathbb{P}^n}(1)) &= \left[ \sum_{i=0}^k \sum_{j=0}^i (-1)^j \binom{n+1}{j} \binom{n-i}{k-i} d^{i-j} \right] H^k \\ &= \left[ \sum_{j=0}^k \sum_{i=j}^k (-1)^j \binom{n+1}{j} \binom{n-i}{k-i} d^{i-j} \right] H^k \\ &= \left[ \sum_{j=0}^k \sum_{i=0}^{k-j} (-1)^j \binom{n+1}{j} \binom{n-i-j}{k-i-j} d^i \right] H^k \\ &= \left[ \sum_{i=0}^k \left[ \sum_{j=0}^{k-i} (-1)^j \binom{n+1}{j} \binom{n-i-j}{k-i-j} \right] d^i \right] H^k. \end{aligned}$$

After the index change  $i = k - i$  in the first summand, we obtain

$$c_k(\Omega_{\mathbb{P}^n}(\log X) \otimes \mathcal{O}_{\mathbb{P}^n}(1)) = \left[ \sum_{i=0}^k \left[ \sum_{j=0}^i (-1)^j \binom{n+1}{j} \binom{n-k+i-j}{i-j} \right] d^{k-i} \right] H^k$$

Now for proving that  $c_k(\Omega_{\mathbb{P}^n}(\log X) \otimes \mathcal{O}_{\mathbb{P}^n}(1)) = (d-1)^k = \sum_{i=0}^k \binom{k}{i} (-1)^i d^{k-i}$ , it suffices to show that for all  $0 \leq i \leq k$ ,

$$\sum_{j=0}^i (-1)^j \binom{n+1}{j} \binom{n-k+i-j}{i-j} = (-1)^i \binom{k}{i}.$$

We denote this property by  $\mathcal{P}(n, k, i)$  for  $1 \leq i \leq k \leq n$ .

One first shows that  $\mathcal{P}(n, k, k)$  for all  $k$  :

$$\sum_{j=0}^k (-1)^j \binom{n+1}{j} \binom{n-j}{k-j} = (-1)^k$$

Now let's do an induction on  $n \geq 1$  : the  $n = 1$  case is trivial and

$$\begin{aligned} \sum_{j=0}^k (-1)^j \binom{n+1}{j} \binom{n-j}{k-j} &= \sum_{j=0}^k (-1)^j \binom{n}{j} \binom{n-j}{k-j} + \sum_{j=1}^k (-1)^j \binom{n}{j-1} \binom{n-j}{k-j} \\ &= \sum_{j=0}^k (-1)^j \binom{n}{k} \binom{k}{j} - \sum_{j=0}^{k-1} (-1)^j \binom{n}{j} \binom{n-j-1}{k-j-1} \\ &= 0 - (-1)^{k-1} \text{ by } \mathcal{P}(n-1, k-1, k-1) \\ &= (-1)^k. \end{aligned}$$



Fixing  $n$ , we now show  $\mathcal{P}(n, k, i)$  by an induction on  $k \geq 1$  : the  $k = 1$  case is trivial so we assume that  $\mathcal{P}(n, k, i)$  is true for all  $i \leq k$  and it suffices to show that  $\mathcal{P}(n, k + 1, i)$  is true for all  $i \leq k$  since  $\mathcal{P}(n, k + 1, k + 1)$  has been shown previously.

$$\begin{aligned}
\sum_{j=0}^i (-1)^j \binom{n+1}{j} \binom{n-k-1+i-j}{i-j} &= \sum_{j=0}^i (-1)^{i-j} \binom{n+1}{i-j} \binom{n-k-1+j}{j} \\
&= \sum_{j=0}^i (-1)^{i-j} \binom{n+1}{i-j} \binom{n-k+j}{j} \\
&\quad - \sum_{j=1}^i (-1)^{i-j} \binom{n+1}{i-j} \binom{n-k-1+j}{j-1} \\
&= (-1)^i \binom{k}{i} \\
&\quad - \sum_{j=0}^{i-1} (-1)^{i-j-1} \binom{n+1}{i-j-1} \binom{n-k+j}{j} \text{ par } \mathcal{P}(n, k, i) \\
&= (-1)^i \binom{k}{i} + (-1)^i \binom{k}{i-1} \text{ par } \mathcal{P}(n, k, i-1) \\
&= (-1)^i \binom{k+1}{i}
\end{aligned}$$

and the statement follows. ■

Now we have computed the Chern classes, we can rewrite (3) in Proposition 2.5.

*Proof of theorem 1.2.* First of all, we have

$$\begin{aligned}
c_k(f^*(\Omega_{\mathbb{P}^n}(\log X) \otimes \mathcal{O}_{\mathbb{P}^n}(1)) \otimes \mathcal{O}_M) &= f^*((d-1)^k H^k).M \\
&= (d-1)^k m^k H^k.M \\
&= (d-1)^k m^n \in \mathbb{Z}
\end{aligned}$$

because  $M = f^*(H^{n-k}) = m^{n-k} H^{n-k}$  in the cohomology ring of  $\mathbb{P}^n$ . And on the other hand,

$$c_k(\Omega_{\mathbb{P}^n}(\log X) \otimes \mathcal{O}_{\mathbb{P}^n}(m) \otimes \mathcal{O}_M) = \binom{n}{k} m^n + \binom{n-1}{k-1} (d-n-1) m^{n-1} + o(m^{n-1}).$$

The inequality (3) becomes

$$\binom{n}{k} m^n + \binom{n-1}{k-1} (d-n-1) m^{n-1} + o(m^{n-1}) \geq (d-1)^k m^n.$$

However, when we take iterates of  $f$ , the integer  $m$  goes to infinity and we have

$$(d-1)^k \leq \binom{n}{k}.$$

Furthermore, if  $(d-1)^k = \binom{n}{k}$ , then  $d = n+1$  since  $\binom{n-1}{k-1}(d-n-1) \leq 0$ . This contradicts  $(d-1)^k = \binom{n}{k}$  for  $k \geq 2$ . Finally we have shown

$$(d-1)^k < \binom{n}{k}.$$

■

### 3 The cone case

This section is devoted to the study of totally invariant cones of  $\mathbb{P}^n$  over a hypersurface  $X$  of  $\mathbb{P}^k$  with  $k \leq n$ . More precisely, for such a cone, we show that the hypersurface  $X$  is either totally invariant in  $\mathbb{P}^k$ . First of all, we shall give the precise definition of a cone.

**Definition 3.1.** *Let  $X$  be a divisor of  $\mathbb{P}^k$  whose equation is  $h \in \mathbb{C}[X_0, \dots, X_k]$  and let  $n \geq k$  be an integer, we define the cone of  $\mathbb{P}^n$  over  $X$  as the divisor of  $\mathbb{P}^n$  whose equation is  $h \in \mathbb{C}[X_0, \dots, X_n]$ .*

*Proof of 1.3.* Let  $X$  be a divisor of  $\mathbb{P}^k$  which is not totally invariant and whose equation is  $h \in \mathbb{C}[X_0, \dots, X_k]$ . We have to show that no cone  $C$  over  $X$  is totally invariant. The proof proceeds by induction on the integer  $n \geq k$ . If  $n = k$  then the cone  $C$  is equal to  $X$  and there is nothing to show. If  $n > k$  one assumes by contradiction that there exists a non trivial endomorphism  $f = (F_0 : \dots : F_n)$  of  $\mathbb{P}^n$  with algebraic degree  $q$  such that

$$f^{-1}(C) = C.$$

The total invariance of  $C$  by  $f$  is equivalent to the following (see [FS94, Lemma 4.3.]) : there exists  $\lambda \in \mathbb{C}^*$  such that

$$h(F_0, \dots, F_k) = \lambda h(X_0, \dots, X_k)^q. \quad (5)$$

Our goal is to find a non trivial endomorphism  $g$  of  $\mathbb{P}^{n-1}$  whose the cone  $C'$  of  $\mathbb{P}^{n-1}$  over  $X$  is totally invariant. A first idea would be considering the rational map  $(F_0|_{\mathbb{P}^{n-1}} : \dots : F_{n-1}|_{\mathbb{P}^{n-1}})$  but we meet an issue : this map is not necessarily a morphism because the  $F_i|_{\mathbb{P}^{n-1}}$  for  $i = 1, \dots, n-1$  can have common zeros. However, we can solve this problem with the following method :

We set  $P = (0 : \dots : 0 : 1) \in \mathbb{P}^n$ . Since  $f$  is a finite morphism, there exists an hyperplane  $H \subset \mathbb{P}^n$  such that

$$H \cap (f^{-1}(P) \cup \{P\}) = \emptyset.$$

Since  $H = \{a_0X_0 + \dots + a_nX_n = 0\}$  does not contain the point  $P$ , we have  $a_n \neq 0$  and we can assume that

$$H = \{X_n = \sum_{i=0}^n a_i X_i\}.$$

We now consider the following endomorphism of  $H$  :

$$g := \pi_{P,H} \circ f|_H$$

where  $\pi_{P,H}$  is the projection from  $P$  over  $H$  whose equations are

$$\pi_{P,H} : (x_0 : \dots : x_n) \mapsto (x_0 : \dots : x_{n-1} : \sum_{i=0}^{n-1} a_i x_i).$$

It is well defined on  $\mathbb{P}^n \setminus \{P\}$  and so  $g$  is not well defined on  $x \in H$  if and only if  $f(x) = P$ . This never happens because  $H \cap f^{-1}(P) = \emptyset$  and then the map  $g$  is an endomorphism of  $H$ .

Furthermore,  $H \cap C$  is totally invariant by  $g$  : indeed, when we restrict (5) on  $H$ , we find

$$h(F_0|_H, \dots, F_k|_H) = \lambda h(X_0, \dots, X_k)^q \quad (6)$$

where  $h$  is the equation of  $C \cap H$  and  $F_0|_H, \dots, F_k|_H$  are the  $k$  first equations of the endomorphism  $g$ . The inclusion  $C \cap H \subset H$  is isomorphic to the inclusion  $C' \subset \mathbb{P}^{n-1}$  where  $C'$  is the cone of  $\mathbb{P}^{n-1}$  over  $X \subset \mathbb{P}^k$ . This is a contradiction to the induction hypothesis and then  $C \subset \mathbb{P}^n$  is not totally invariant.  $\blacksquare$

Thanks to bilinear algebra, it is well-known that all singular irreducible quadrics are cones over a smooth irreducible quadric. These latter aren't totally invariant thanks to Ceriveau-Lins Neto [CLN00, Thm 2] so we directly obtain the following.

**Corollary 3.2.** *Let  $X$  be a prime divisor of  $\mathbb{P}^n$  which is totally invariant. Then  $X$  is not a quadric.*

## 4 Conclusion

To conclude this paper, we give some precisions about the impact of Theorems 1.2 and 1.3 on the linearity conjecture. The main effect of this bound appear on totally invariant divisors with isolated singularities as we already present in the corollary 1.4. Let  $X$  be a totally invariant prime divisor of  $\mathbb{P}^n$  and fix the dimension  $l$  of the singular locus of  $X$ . In a natural way, we can wonder for which  $n$  the conjecture is verified. The following statement is devoted to this question.

Values of $l$	-1	0	1	2	3	4	5	6
The conjecture is true if	$n \geq 4$	$n \geq 4$	$n \geq 6$	$n \geq 10$	$n \geq 14$	$n \geq 19$	$n \geq 23$	$n \geq 27$

TABLE 1

**Corollary 4.1.** *Let  $X$  be a totally invariant prime divisor of  $\mathbb{P}^n$  with  $n \geq 4$ . In the Table 1 we have registered for which  $n$  the conjecture is true depending on the dimension  $l$  of the singular locus of  $X$ .*

*Proof.* Assume that the divisor  $X$  is not a hyperplane so Theorem 1.3 gives that the degree  $d$  of  $X$  is necessarily at least 3. According to Theorem 1.2, we find

$$2^{n-l-1} \leq (d-1)^{n-l-1} < \binom{l+1}{n}.$$

We deduce that, fixing the integer  $l$ , the conjecture is true when the function

$$\phi_l(n) = 2^{n-l-1} - \binom{l+1}{n}$$

is non negative. For each value of  $l$  in the table above, it suffices to find the values of  $n$  such that  $\phi_l(n)$  is non negative. A simple study of the function  $\phi_l$  in each case gives the results of the table. ■

We deduce from the previous corollary that in order to prove conjecture for  $\mathbb{P}^4$  and  $\mathbb{P}^5$ , it is sufficient to verify it for divisors with 1-dimensional singular locus.

We now wonder what happens for totally invariant prime divisors of  $\mathbb{P}^n$  with big degrees (recall that the degree of such a divisor is bounded by  $n-1$ ). Let's begin with the general case.

**Corollary 4.2.** *Let  $X$  be a totally invariant prime divisor of  $\mathbb{P}^n$  and let  $l$  be the dimension of the singular locus of  $X$ . Assume that the degree of  $X$  is  $n-k$  for an integer  $1 \leq k \leq n-3$  then the divisor  $X$  is necessarily non normal whenever  $n \geq n_k := 2k + 3/2 + \sqrt{2k^2 + 2k + 1/4}$ .*

*Proof.* Let assume that  $l = n-3$  then the theorem 1.2 gives the following inequality

$$(n-k-1)^2 - \frac{n(n-1)}{2} < 0.$$

This is a quadratic polynomial whose greater root is exactly

$$n_k = 2k + 3/2 + \sqrt{2k^2 + 2k + 1/4}.$$

We conclude that the latter inequality is no longer true when  $n \geq n_k$ . ■

In order to give a meaning to the above corollary, we shall precise the two special cases below : whom involve respectively the  $n - 1$  and  $n - 2$  degree case.

**Corollary 4.3.** *Let  $X$  be a totally invariant prime divisor of  $\mathbb{P}^n$  with  $n \geq 4$  and let  $l$  be the dimension of the singular locus of  $X$ . Assume that the degree of  $X$  is  $n - 1$  then :*

- (1) *the dimension  $l$  is  $n - 2$  or  $n - 3$  for  $n = 4, 5$ ; and*
- (2) *the divisor  $X$  is necessarily non normal when  $n \geq 6$ .*

*Proof.* Theorem 1.2 gives the following inequality

$$(n - 2)^{n-l-1} < \binom{l+1}{n}.$$

The first assertion is proven by a direct computation of the latter inequality in the special cases  $n = 4$  and  $n = 5$ . For the second statement, one can apply Corollary 4.2 with  $k = 1$ . Indeed, one has  $n_1 \simeq 5, 56$ . ■

We can ask the same question to divisors of degree  $n - 2$ . The following corollary answer to that question.

**Corollary 4.4.** *Let  $X$  be a totally invariant prime divisor of  $\mathbb{P}^n$  with  $n \geq 5$  and let  $l$  be the dimension of the singular locus of  $X$ . Assume that the degree of  $X$  is  $n - 2$  then :*

- (1) *the dimension  $l$  is  $n - 2$  or  $n - 3$  for  $n = 6, 7, 8$ ; and*
- (2) *the divisor  $X$  is necessarily non normal when  $n \geq 9$ .*

*Proof.* Theorem 1.2 gives the following inequality

$$(n - 3)^{n-l-1} < \binom{l+1}{n}.$$

The first assertion is proven by a direct computation of the latter inequality in the special cases  $n = 6, 7, 8$ . For the second statement, one can apply Corollary 4.2 with  $k = 2$ . Indeed, one has  $n_2 = 9$ . ■

## Références

- [BCS04] J.-Y. Briend, S. Cantat, and M. Shishikura. Linearity of the exceptional set for maps of  $\mathbb{P}_k(\mathbb{C})$ . *Math. Ann.*, 330 :39–43, 2004.
- [Bea01] A. Beauville. Endomorphisms of hypersurfaces and other manifolds. *Int. Math. Res. Not.*, 1 :53–58, 2001.

- [BH14] A. Broustet and A. Höring. Singularities of varieties admitting an endomorphism. *Math. Ann.*, 360 :439–456, 2014.
- [CLN00] D. Cerveau and A. Lins Neto. Hypersurfaces exceptionnelles des endomorphismes de  $\mathbb{C}\mathbb{P}(n)$ . *Bol. Soc. Bras. Mat.*, 31 :155–161, 2000.
- [Deb01] O. Debarre. *Higher-Dimensional Algebraic Geometry*. Springer, 2001.
- [Dol07] I. Dolgachev. Logarithmic sheaves attached to arrangements of hyperplanes. *Journal of Mathematics of Kyoto University*, 47(1) :35–64, 2007.
- [FS94] J.E. Fornæss and N. Sibony. Complex dynamics in higher dimension i. *SMF, Astérisque*, 5 :201–231, 1994.
- [Har77] R. Hartshorne. *Algebraic Geometry*. Springer, 1977.
- [HN11] J.-M. Hwang and N. Nakayama. On endomorphisms of fano manifolds of picard number one. *Pure Appl. Math.*, 7 :1407–1426, 2011.
- [Hör17] A. Höring. Totally invariant divisors of endomorphisms of projective spaces. *Manuscripta Mathematica*, 27 :153–173, 2017.
- [Mab21] Y. Mabed. *Diviseurs totalement invariants des endomorphismes non triviaux de l'espace projectif*, <https://hal.archives-ouvertes.fr/tel-03413069>. PhD thesis, 2021.
- [Zha13] D.Q. Zhang. Invariant hypersurfaces of endomorphisms of the projective 3-space. *Affine algebraic geometry*, pages 314–330, 2013.

YANIS MABED, UNIVERSITÉ CÔTE D'AZUR, CNRS, LJAD, FRANCE  
*Email address* : **yanis.mabed@unice.fr ; ymabed@gmail.com**