

Exponentials of Bounded Normal Operators and Commutativity

Mohammed Hichem Mortad, Laboratory: LAMA, Department of Mathematics, University Of Oran (Algeria)

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Introduction

Theorem 1 (eg see [10]) *Let A and B be two self-adjoint operators defined on a Hilbert space. Then*

$$e^A e^B = e^B e^A \iff AB = BA.$$

There have been several attempts to prove the previous theorem for non self-adjoint operators using the $2\pi i$ -congruence-free hypothesis (see eg. [6, 7, 8, 9, 10]). See also [2] for some low dimensions results without the $2\pi i$ -congruence-free hypothesis. In this talk, we present a different approach to this problem using results about similarities due to Berberian [1] (see also to Embry [4]). The main question asked here is under which assumptions we have

$$e^A e^B = e^B e^A \implies AB = BA$$

for normal A and B ?

Recall the following: A **unitary** operator U is said to be **cramped** if its spectrum is completely contained on some open semi-circle

(of the unit circle), that is

$$\sigma(U) \subseteq \{e^{it} : \alpha < t < \alpha + \pi\}.$$

Theorem 2 [Berberian, [1]] Let U be a cramped operator and let X be a bounded operator such that $UXU^* = X^*$. Then X is self-adjoint.

Lemma 1 Let A and B be two commuting normal operators, on a Hilbert space, having spectra contained in simply connected regions not containing 0 . Then

$$A^i B^i = B^i A^i$$

where $i = \sqrt{-1}$.

Lemma 2 Let A be a self-adjoint operator such that $\sigma(A) \subset (0, \pi)$. Then

$$(e^{iA})^i = e^{-A}.$$

Before stating and proving the main theorem, we first give an intermediate result.

Proposition 1 *Let N be a normal operator with cartesian decomposition $A + iB$. Let S be a self-adjoint operator. If $\sigma(B) \subset (0, \pi)$, then*

$$e^S e^N = e^N e^S \iff SN = NS.$$

Proof.

Write $N = A + iB$ where A and B are two commuting self-adjoint operators. Hence $e^A e^{iB} = e^{iB} e^A$. Consequently,

$$e^S e^N = e^N e^S \iff e^S e^A e^{iB} = e^{iB} (e^S e^A)^*$$

Since B is self-adjoint, e^{iB} is unitary which is also cramped by the spectral hypothesis on B . Now, Theorem 2 implies that $e^S e^A$ is self-adjoint, i.e.

$$e^S e^A = e^A e^S.$$

Theorem 1 then gives us $AS = SA$.

It only remains to show that $BS = SB$. Since $e^S e^A = e^A e^S$, and by the invertibility of e^A , we obtain

$$e^S e^N = e^N e^S \implies e^A e^S e^{iB} = e^A e^{iB} e^S \implies e^S e^{iB} = e^{iB} e^S.$$

Using Lemmas 1 & 2 we immediately see that

$$e^S e^{-B} = e^{-B} e^S.$$

Theorem 1 yields $BS = SB$ and thus

$$SN = S(A + iB) = (A + iB)S = NS.$$

The proof of the proposition is complete.

Theorem 3 *Let N and M be two normal operators with cartesian decompositions $A + iB$ and $C + iD$ respectively. If $\sigma(B), \sigma(D) \subset (0, \pi)$, then*

$$e^M e^N = e^N e^M \iff MN = NM.$$

The following result has important applications:

Theorem 4 *Let A be in $B(H)$. Let $N \in B(H)$ be normal such that $\sigma(\operatorname{Im}N) \subset (0, \pi)$. Then*

$$Ae^N = e^N A \iff AN = NA.$$

Theorem 5 *Let A and B be both in $B(H)$. Assume that $A + B$ is normal such that $\sigma(\operatorname{Im}(A + B)) \subset (0, \pi)$. If*

$$e^A e^B = e^B e^A = e^{A+B},$$

then $AB = BA$.

Corollary 1 *Let A be normal such that $\sigma(\operatorname{Im}A) \subset (0, \pi)$. Let $B \in B(H)$. Then*

$$e^A = e^B \implies AB = BA.$$

Corollary 2 *Let A and B be two self-adjoint operators. Then*

$$e^A = e^B \iff A = B.$$

Corollary 3 *Let A be normal. Then A is self-adjoint iff e^{iA} is unitary.*

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