

The Unbounded Product of Normal Operators

Mohammed Hichem Mortad, University Of Oran (Algeria)

Ninth ACOTCA: 11-14 June, 2012 in Sevilla

Table of contents

Introduction

Main Results

References

Introduction

All operators are assumed to be densely defined (i.e. having a dense domain) together with any operation involving them or their adjoints. Bounded operators are assumed to be defined on the whole Hilbert space. If A and B are two unbounded operators with domains $D(A)$ and $D(B)$ respectively, then B is called an extension of A , and we write $A \subset B$, if $D(A) \subset D(B)$ and if A and B coincide on $D(A)$. We write $A \subseteq B$ if $A \subset B$ or $A = B$ (meaning that $D(A) = D(B)$ and $Ax = Bx$ for all $x \in D(A)$). The product BA of two unbounded operators B and A is defined by

$$BA(x) = B(Ax) \text{ for } x \in D(BA)$$

where

$$D(BA) = \{x \in D(A) : Ax \in D(B)\}.$$

Since the expression $AB = BA$ will be often met, and in order to avoid possible confusions, we recall that by writing $AB = BA$, we mean that $ABx = BAx$ for all $x \in D(AB) = D(BA)$.

Recall that the unbounded operator A , defined on a Hilbert space H , is said to be invertible if there exists an *everywhere defined* (i.e. on the whole of H) bounded operator B such that

$$BA \subset AB = I$$

where I is the usual identity operator. This is the definition adopted in the present talk. It may be found in e.g. [1] or [3]. If $A \subset B$, then $B^* \subset A^*$. An unbounded operator A is said to be closed if its graph is closed; self-adjoint if $A = A^*$; normal if it is closed and $AA^* = A^*A$.

The product of two bounded normal operators need not remain normal. Many simple counterexamples can be constructed with ease (the answer is well-known to be negative even for self-adjoint operators).

It is, however, well known that the product of two commuting bounded normal operators stays normal. This is a direct application of the bounded version of the well-known Fuglede theorem.

The main goal of this talk is to deal with this question in the unbounded case. We start by the following example that shows that the passage to the unbounded case needs care:

Example 1 *Let A be an unbounded self-adjoint operator having a trivial kernel, for example take $Af(x) = (1 + x^2)f(x)$ on $D(A) = \{f \in L^2(\mathbb{R}) : (1 + x^2)f \in L^2(\mathbb{R})\}$. Now set $B = A^{-1}$ (observe that both A and B are positive on their respective domains). Then BA , defined on $D(BA) = D(A)$, is not closed as $BA \subset I$. Thus it can neither be self-adjoint nor it can be normal and yet*

$$BA \subset AB$$

as $D(BA) = D(A) \subset D(AB) = L^2(\mathbb{R})$.

Let us recall now the main tools to prove our results.

Lemma 1 [11] *If S is (unbounded) symmetric and T is self-adjoint, then*

$$T \subset S \implies T = S.$$

Lemma 2 [11] *If T is closed, then T^*T and TT^* are both self-adjoint.*

Hence we have the following very interesting consequence:

Lemma 3 *If T is a closed operator such that $TT^* \subset T^*T$, then T is normal.*

We will also need results on the adjoint and the closedness of the product.

Lemma 4 [5] *Let A and B be two closed operators. Then AB is closed if: A is invertible or if B is bounded.*

Lemma 5 *If A and B are two densely defined operators, such that BA is densely defined too. Then $(BA)^* \supseteq A^*B^*$, with equality if B is bounded.*

Lemma 6 [13] *If A and B are densely defined and A is invertible with inverse A^{-1} in $B(H)$, then $(BA)^* = A^*B^*$.*

Finally, we recall the well-known Fuglede-Putnam theorem:

Theorem 1 [1] *If A is a bounded operator and if M and N are normal operators (bounded or not), then*

$$AN \subseteq MA \implies AN^* \subseteq M^*A.$$

Main Results

We have to impose an extra condition to force the product to become normal. For example, the following result was proved

Theorem 2 [7]

1. Assume that B is a unitary operator. Let A be an unbounded normal operator. If B commutes with A (i.e. $BA \subset AB$), then BA is normal.
2. Assume that A is a unitary operator. Let B be an unbounded normal operator. If A commutes with B (i.e. $AB \subset BA$), then BA is normal.

Proof.

Since B is invertible and A is closed, BA is closed. We also have by the boundedness of B that $(BA)^* = A^*B^*$. Hence

$$(BA)^*BA = A^*B^*BA = A^*A.$$

Since B and A commute (and A is normal), so do B and A^* (by the Fuglede theorem). Hence

$$BA(BA)^* = BAA^*B^* \subset ABA^*B^* \subset AA^*BB^* = AA^* = A^*A$$

and whence

$$BA(BA)^* \subset (BA)^*BA.$$

Since BA is closed, $BA(BA)^*$ is self-adjoint. Lemma 3 then implies that BA is normal on $D(A)$. This completes the proof.

Dropping the unitarity hypothesis the following results (also in [7]) were obtained:

Theorem 3 *Let B be a bounded normal operator. Let A be an unbounded normal operator. Assume that B commutes with A . If for some $r > 0$, $\|rBB^* - I\| < 1$, then BA is normal.*

Proof.

The proof is essentially the same as before. We need only verify that BA is closed. So let us show that BA is closed. Let $x_n \rightarrow x$ and $BAx_n \rightarrow y$. The condition $\|rBB^* - I\| < 1$, plus the normality of B , guarantee that $BB^* = B^*B$ is invertible. Hence, by the continuity of B^* , $B^*BAx_n \rightarrow B^*y$. Therefore,

$$Ax_n \longrightarrow (B^*B)^{-1}B^*y.$$

But A is closed, hence $x \in D(A)$ and $Ax = (B^*B)^{-1}B^*y$. This implies that $B^*BAx = B^*y$ and hence $BB^*BAx = BB^*y$, which, thanks to the invertibility of BB^* , clearly yields $BAx = y$, proving the closedness of BA .

Theorem 4 *Let B a bounded normal operator and let A be an unbounded normal operator which commutes with B . Assume that for some $r > 0$, $\|rBB^* - I\| < 1$. Then AB is normal.*

Remark 1 *Observe that the last two results generalize Theorem 2.*

Of course the fact that AB is normal does not imply that BA is normal and vice versa. The following result gives an answer to this question and it also generalizes a famous result by I. Kaplansky (see [4]):

Theorem 5 *If A is unitary and B is an unbounded normal operator, then BA is normal iff AB is normal.*

Proof.

The proof also uses Lemmas 1, 2 & 3.

The following result is also of interest

Theorem 6 *Let A be a unitary operator. Let B be unbounded and closed. Then BA and AB are normal iff $B^*BA \subset ABB^*$.*

If we want to work only with commuting normal operators (i.e. without using the strong unitarity hypothesis), then we have the following domain dependant result


Theorem 7 Let A and B be two normal operators. Assume that B is bounded. If $BA = AB$, then BA (and so AB) is normal.

Remark 2 Remember that the condition $AB = BA$ means that B commutes with A and that $D(AB) \subset D(BA) = D(A)$.

Proof.

Since $BA = AB$, by the Fuglede theorem we have $BA^* = A^*B$. Then we have

$$(BA)^*BA = A^*B^*BA = A^*B^*AB \subset A^*AB^*B$$



classic Fuglede

and

$$BA(BA)^* = BAA^*B^* = ABA^*B^* = AA^*BB^* = A^*AB^*B.$$

Whence

$$(BA)^*BA \subset BA(BA)^*$$

But BA is closed for it equals AB which is closed.
The rest is as before. The proof is therefore complete.

Remark 3 *The assumption $AB \subset BA$ cannot merely be dropped.
By Example 1,*

$$D(AB) = L^2(\mathbb{R}) \not\subset D(BA) = D(A) = \{f \in L^2(\mathbb{R}) : (1+x^2)f \in L^2(\mathbb{R})\}.$$

We also obtain an "anti-commuting version" of Theorem 7.

Theorem 8 *Let A and B be two normal operators. Assume that B is bounded. If $BA = -AB$, then BA (and so AB) is normal.*

Finally we pass to the case of two unbounded normal operators. The Fuglede-Putnam theorem has played a very important role in the proofs before. So need an all-unbounded-operator Fuglede-Putnam theorem version. This has been obtained by myself recently

Theorem 9 (Fuglede-Putnam-Mortad) [8] *Let A be an unbounded closed operator with domain $D(A)$. Let M and N be two unbounded normal operators with domains $D(N)$ and $D(M)$ respectively. If $D(N) \subset D(AN)$, then*

$$AN \subset MA \implies AN^* \subset M^*A.$$

The previous gives some interesting results such as the following one

Theorem 10 *Let A be an unbounded invertible normal operator. Let B be an unbounded normal operator. If $BA = AB$, $A^*B \subset BA^*$ and $B^*A \subset AB^*$, then BA is normal.*

But we have a slightly better result whose proof will be based on the following Fuglede type result

Theorem 11 *Let A and B be two unbounded normal and invertible operators. Then*

$$AB = BA \implies AB^* = B^*A \text{ and } BA^* = A^*B.$$

Proof.

Since B is invertible, we may write

$$AB = BA \implies B^{-1}A \subset AB^{-1}.$$

Since B^{-1} is bounded and A is unbounded and normal, by the classic Fuglede theorem we have

$$B^{-1}A \subset AB^{-1} \implies B^{-1}A^* \subset A^*B^{-1}.$$

Therefore,

$$A^*B \subset BA^*.$$

But B is invertible, then by Lemma 6 we may obtain

$$AB^* = (BA^*)^* \subset (A^*B)^* = B^*A.$$

Interchanging the roles of A and B , we shall get

$$B^*A \subset AB^* \text{ and } BA^* \subset A^*B.$$

Thus

$$B^*A = AB^* \text{ and } BA^* = A^*B.$$

Remark 4 *A similar result holds with one operator assumed normal. The key point again is that the inverse is bounded and everywhere defined. So since $AB = BA$, we obtain $A^{-1}B^{-1} \subset B^{-1}A^{-1}$. Since these operators are everywhere defined, we get $A^{-1}B^{-1} = B^{-1}A^{-1}$. The rest follows by the bounded version of Fuglede theorem.*

However, if we do not assume the bounded inverse defined everywhere, then $AB = BA$ does not imply that $A^{-1}B^{-1} = B^{-1}A^{-1}$. For a counterexample see [12].

Corollary 1 *Let A and B be two unbounded invertible normal operators. If $BA = AB$, then BA (and AB) is normal.*

Proof.

We first note that BA is closed thanks to Lemma 4, hence so is AB . Lemma 6 then gives us $(AB)^* = B^*A^*$. By Theorem 11 we may then write

$$(AB)^*AB = B^*A^*AB = AB^*A^*B = AB^*BA^* = (AB)(AB)^*,$$

establishing the normality of AB .

Remark 5 *By a result of Devinatz-Nussbaum (see [2]), if A , B and N are normal where $N = AB = BA$, then A and B strongly commute. Hence we have as a consequence:*

Corollary 2 *Let A and B be two unbounded invertible normal operators. If $BA = AB$, then A and B strongly commute.*

References

1. . B. Conway, *A Course in Functional Analysis*, Springer, 1990 (2nd edition).
2. . Devinatz, A. E. Nussbaum, On the Permutability of Normal Operators, *Ann. of Math. (2)*, **65** (1957), 144-152.
3. . Gohberg, S. Goldberg, M. A. Kaashoek, *Basic Classes of Linear Operators*, Birkhäuser Verlag, Basel, 2003.
4. . Kaplansky, Products of normal operators, *Duke Math. J.*, **20/2** (1953) 257-260.
5. . Kato, *Perturbation Theory for Linear Operators*, Springer, 1980 (2nd edition).
6. . H. Mortad, On the Normality of the Sum of Two Normal Operators, *Complex Anal. Oper. Theory*, **6/1** (2012), 105-112. DOI: 10.1007/s11785-010-0072-7.

7. . H. Mortad, *On the closedness, the self-adjointness and the normality of the product of two unbounded operators*, Demonstratio Math., **45/1** (2012), 161-167.
8. . H. Mortad, *An All-Unbounded-Operator Version of the Fuglede-Putnam Theorem*, Complex Anal. Oper. Theory, **6/6** (2012), 1269-1273.
9. . H. Mortad, *On the Normality of the Unbounded Product of Two Normal Operators*, Concrete Operators, **1** (2013), 11-18.
10. . H. Mortad, *The Sum of Two Unbounded Linear Operators: Closedness, Self-adjointness and Normality*.
11. . Rudin, *Functional Analysis*, McGraw-Hill, 1991 (2nd edition).
12. . Schmüdgen, J. Friedrich, *On commuting unbounded selfadjoint operators II*, Integral Equations Operator Theory, **7/6** (1984), 815-867.

13. . Weidmann, Linear operators in Hilbert spaces (translated from the German by J. Szücs), Srpinger-Verlag, GTM **68** (1980).