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# MATHEMATICAL FOUNDATIONS FOR PLACE VALUE THROUGHOUT ONE CENTURY OF TEACHING IN FRANCE

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## Abstract

This paper aims to evidence the existence and the impact of mathematical foundations on WNA teaching, from curricula design to teaching practices, and students' learning. An assumption is that, as its name indicates it, the *new math* is a key period to identify changes in mathematical foundations of WNA teaching. This period is replaced in a longer one. The paper studies the changes related to place value in France: a deeply impacted subject with the introduction of numeration bases other than ten in 1970, and their removal in the 1980s. What the author named *numeration's units* -that is: ones, tens, hundreds, thousands...- appears as a key tool to describe and understand changes in the teaching and learning.

**Key words:** didactic transposition, mathematical foundation, France, new math reform, numeration's unit, place value.

## Introduction

This paper aims to evidence the existence and the impact of mathematical foundation (MF) – as understood by the current community of mathematicians – on WNA teaching, from curricula design, to teaching practices, and students' learning.

The new math reform is an international phenomenon which took place in the cold war context (ICMI, 2008; Kilpatrick, 2012). It impacted on the math teaching from the primary school to the university in the 1960s and the early 1970s in several countries; but it has had effects a long time after its end. One of its major concerns was the teaching of some “new” math. This enables to make the assumption that this period may be a key one to identify possible changes in MF. Another concern was to take into account some psychological features related to the learning or to the child development. The famous psychologists Piaget and Bruner contributed at least indirectly but significantly to the implementation of this reform. Two famous subjects were introduced in WNA: set theory, a trace of contemporary math, and the numeration bases other than ten (hereafter called bases) –in order children to understand base ten principles-, a trace of psychology (Kilpatrick, 2012; Bruner, 1966).

Much has been written about MF concerning fractions, taking into account the new math reform period (Steiner, 1969; in Germany: Griesel, 2007; in France: Chambris, 2008), but few about that of decimal numeration. The aim of this paper is to study such MF, surrounding the new math. The case of France will be studied: bases were introduced in 1970 and removed in the 1980s. A main goal of this study is to identify the effects, and even the potential long-term effects, on the teaching of possible changes in the MF. This leads to study MF: before, during, and after the new math. Several types of data from different

periods are collected: national syllabi, resources for teacher education, textbooks. For the present period, this set is completed with data from classrooms: students' written test, results from others' research works related to teaching practices and students' learning.

## Materials and Methods

How to study the existence and the impact of the MF for decimal numeration on the teaching and learning in France surrounding the new math?

### Previous works

Several doctoral dissertations (Harlé, Bronner, Neyret) about numbers' or arithmetic's teaching in the French system highlighted three main periods with two transitional ones before and after the new math: the stable classical period 1870-1950, the new math highly turbulent period 1970-1980, and the continuously evolving contemporary period 1995-2010. So, this study explores the long period: 1900-2014. Treaties by Bezout, then Reynaud (1821) appear as reference books for teacher education, and for textbooks, at the beginning of 20<sup>th</sup> century. The textbooks' series "math et calcul" was a bestseller in the 1980s.

### Theoretical frame

The Theory of Didactic Transposition (TDT) (Chevallard, 1985) (fig. 1) considers school mathematics as a reconstruction by the educational institutions from the mathematical knowledge produced by academic scholars. The TDT has been often used for secondary school, more rarely for primary school where scholarly knowledge as a reference is not always taken for granted.

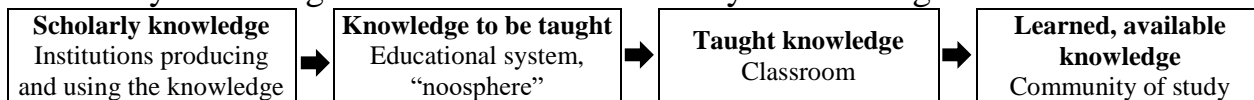


Fig. 1: The process of didactic transposition (Bosch and Gascon, 2006)

The Anthropological Theory of Didactics (ATD) (*ibid.*) extends the TDT. It postulates that practicing math, as any human practice, can be described with the model of *praxeology*. It is constituted by four pieces: a *type of tasks* -a set of similar problems-, a *technique* -a “way of doing” for all the tasks of the type-, a *technology* justifies the technique, and a *theory* legitimates the technology.

### The numeration's units

The author (2008) named the units used in numeration: *numeration's units*, NU. That is: the *words* ones, tens, hundreds -O, T, H in the text-, and so on for base ten. It happens that students consider NUs as places (Fuson, 1990; Ma, 1999).

### A few French cultural and institutional features

Unlike most East-Asian languages many European languages have strong irregularities in the numbers' names for whole numbers (Fuson, 1990). In English, numbers' names are those of the units, not in French. (Tab. 1) This

implies that, in France, the names of the 2<sup>nd</sup>, 3<sup>rd</sup>, and 4<sup>th</sup> decimal NUs for whole numbers are not used in the everyday life language.

French number's name	French NU	English name for both
un (une)	unité	one
dix	dizaine	ten
cent	centaine	hundred
mille	millier	thousand

Tab. 1: Numbers' names and NUs in French and in English.

In France, there is a national elementary compulsory syllabus, but no national textbook, and no accreditation of textbooks. The syllabi changed in 1882, 1923, 1945, 1970, 1977-1980, 1985, 1995, 2002, and 2008. Instructions -more or less long texts describing how to teach a subject- have supported some of them.

***Data: how to select curricular texts? What to look at in there?***

In the French syllabus, roughly, students learn numbers up to one hundred in the 1<sup>st</sup> grade, to one thousand in the 2<sup>nd</sup> grade, to ten thousand or one million -depending on the ongoing curriculum- in the 3<sup>rd</sup> grade. Bigger whole numbers are learned in 4<sup>th</sup> and 5<sup>th</sup> grades. The selected data are: the five first grades of the national syllabus and instructions -if any-, and 2<sup>nd</sup>-and-3<sup>rd</sup>-grade-textbooks.

Period	2 <sup>nd</sup> grade	3 <sup>rd</sup> grade	Mixed (2 <sup>nd</sup> -3 <sup>rd</sup> -gr.)	Teacher's guide
1900-1970	7	7	6	1
1970-1980	3	3		6
1980-2010		14		12

Tab. 2: Number and type of selected books for the research

A big amount of textbooks exists in France. How to select the most used ones, and the most influential ones in the curriculum's evolution? Ones with several editions and innovative ones are selected. During the new math, many textbooks were published; "math et calcul" is one of the few with further editions. (Tab. 2)

Treaties by Bezout and Reynaud, 2<sup>nd</sup>-and-3<sup>rd</sup>-grade-teachers' guides by ERMEL from two bestseller-series (old in 1978, new in 1995) are added. ERMEL is the elementary math research team of the national institute of pedagogy.

Once a book is selected, what to look at in there? With the ATD, for each period, it is necessary to identify the ongoing praxeologies in order to describe the teaching. The tables of contents of textbooks, and the syllabus are useful to identify the global development of the subject, a part of its logical organization. Instructions and numeration's pages of textbooks give more details. The sentences related to the knowledge to be known are often technologies or parts of the theory. Exercise with model answer is a strong indicator of the knowledge to be taught: such an exercise is often emblematic of a type of task, and the answer shows the technique which is expected. The answer sometimes includes technological features. The set of exercises is useful to identify the types of tasks which exist in an institution at a given moment.

### ***Data: students' knowledge***

In early 2012, all 215 students from nine 3<sup>rd</sup>-to-5<sup>th</sup>-grade-classes answered 9 exercises in individual written way. Questions were given one after the other. Time was supposed to be enough to answer quietly.

### **Results**

#### ***The scholarly knowledge before the reform – the classical theory***

During the long period 1900-1970 with hazy changes in some books from 1950, a mathematical numeration's theory -here: *the classical theory*- was displayed in the treaties and adapted in close terms in textbooks. Its basics are the following.

Algorithm to build numbers: 1) The first ten numbers are built one after another, starting with the unit one, and then adding one to the previous number, forming numbers one, two, etc. 2) The set of ten ones forms a new order of units: the ten. 3) The tens are numbered like the ones were numbered before; from one ten to ten tens: one ten, two tens... 4) Then the first nine numbers are added to the nine first tens: one ten, one ten and one one, one ten and two ones... two tens, two tens and one one, and so on, forming the ninety-nine first numbers. 5) The set of ten tens forms a new order of units: the hundred. 6) The hundreds are numbered like the tens, and the ones were; from one hundred to ten hundreds: one hundred, two hundreds... 7) Then the first ninety-nine numbers are added to the nine first hundreds forming the nine-hundreds-ninety-nine first numbers...

Rules for numbers' names: Meanwhile the building of numbers, numbers' names are presented like a literal translation. Rules are stated with exceptions to them. Though it is paradoxical, but in order to lighten this text, exceptions are taken from the English language. After the 3<sup>rd</sup> step, the correspondence between tens names (ten, twenty...) and the amount of tens (one ten, two tens...) is stated. Then, after the 4<sup>th</sup> step, the rule to form the numbers' names between two tens is stated. "Say the tens then the ones": as *three tens* is *thirty* and *four ones* is *four*, then *three tens* and *four ones* is *thirty-four*. Then a list of exceptions is given: for instance the usual number's name of *ten-one* is *eleven*.

Positional notation: After building the numbers, the positional notation is stated. To write numbers without writing the units names, it is sufficient to juxtapose the amount of units of each order, the ones on the right side, then each place represents a unit which is ten times bigger as the nearest on its right. Places which are not represented are marked with the sign 0.

Conversely to the 1<sup>st</sup> and 3<sup>rd</sup> paragraph, the 2<sup>nd</sup> one -rules for numbers' names- is useless with regular numbers' names. Due to a cultural difference, this may play a major role in differences in the MF throughout the world.

#### ***Changes in the scholarly knowledge: what and when?***

During the new math, many things occur due to the bases. Depending on the resource, more or less changes exist in the MF which is used. 1) The classical

theory may be adapted for bases: as there is no ten, no hundred in bases, units' names become "unit of the first order", "unit of the second order", and so on. NUs reappear shortly while studying base ten. 2) Generally base-ten-units' names are present in relation with numbers' names, with the classical correspondence. 3) The polynomial decomposition of a whole number  $n$  in a given base  $r$  -that is:  $n = \sum a_i r^i$  ( $0 \leq a_i < r$ )- always appears while converting from base ten to another one or conversely which is a new type of task of the period. Sometimes the existence and uniqueness of the decomposition is proved in a rather formal way. The Euclidian division is then used. 4) In some teachers' guide, the polynomial decomposition is used to substantiate the column algorithms. In the other ones, the classical theory is used.

From the 1980s, the classical theory disappeared from the textbooks and books for teacher education. In the early 1980s, the theorem for the polynomial decomposition explicitly stands in books. The positional notation is then the juxtaposition of the coefficients of the polynomial, here *the academic theory*. Nowadays, most often, the decomposition remains implicit in teacher's guide.

The arguments used to introduce bases in the new math curriculum were from psychology. And yet, it looks like *after* the turbulent period of the reform, during the so-called “counter-reform”, the “new” mathematics –here represented by the academic theory which is in fact not so new– took place as MF for the teaching of place value.

The next step is to understand what kinds of effects have such a change in the teaching, if any. A few examples are selected hereafter.

## What role for NUs today?

**Exercice 6.**

Combien y a-t-il de dizaines d'étoiles ? ..... How many tens of stars are there?

Combien y a-t-il de centaines d'étoiles ? ..... How many hundreds of stars are there?

Combien y a-t-il d'étoiles ? ..... How many stars are there?

Fig. 2: tasks developed to number tens and hundreds.

As strange as it sounds, it is frequent not to find the relation 1 hundred = 10 tens in present 2<sup>nd</sup>-and-3<sup>rd</sup>-grade-textbooks. Might it have to do with the fact that there is no need for the NUs in the academic theory? This never happened before the reform. To convert NU was an ordinary task in the classical period. None of the present textbooks selected displayed conversions' tasks. Tempier (2013) observed three 3<sup>rd</sup>-grade-teachers, only one of them explicitly calls upon the relations between units; in contrast all of them use the unit's names to describe places. He also investigates mid-3<sup>rd</sup>-graders' knowledge about conversions (n = 104): 48% succeed to fulfil 1 H = ...T, and 31% to fulfil 60 T = ...H. None of the observed teachers taught conversions. In 2012, 62% of

mid-3<sup>rd</sup>-to-5<sup>th</sup>-graders ( $n = 215$ ) were able to number 304 stars, whereas 35% were able to number tens, and 2% answered 0 tens; 37% were able to number hundreds of stars (fig. 2).

It is true NUs are few used in French. This shows that NUs are not at the core of the teaching of place value in France: from MF, to curricular content, teachers' practices, and student's knowledge. It sounds contradictory with a deep understanding of place value (Ma, 1999).

How is place value taught with the polynomial decomposition? What do students know about place value? It is useful to have an overview of what happened from the reform before to study these questions.

***The appearance and the role of “powers-of-ten written in figures” or: How to teach place value at early grades with the polynomial decomposition?***

Two very different texts of the same period evoke nearly the same consequences of the bases in classrooms on students' knowledge. Perret (1985) studies place value within the reform in Switzerland. ERMEL (1978) will be a major source of the French counter-reformed curriculum. Both write basically that students interpret the positional notation as a procedure: grouping, ungrouping. They struggle as soon as they have no manipulatives left, ERMEL adds.

A brief analysis of reform's praxeologies shows that the old numeration's tasks: that is conversions and decompositions –both using the symbolic register of the NU- are no longer taught. It may be due to the lack of most of the units' names with bases other than ten. Whatever, they disappeared as the “bases” appeared.

A major preoccupation of the counter-reformers was to reinsert a work within a symbolic register, so-called the “writings”. A huge process of transposition of the academic theory happened; it had been largely achieved or relayed by (ERMEL, 1978); and no change is visible in MF after the counter-reform. In the part devoted to numeration, there is no NU in (ERMEL, 1978), except once in order to evoke rapidly the numbers' name. The process is the following:  $\sum r_i a^i$  becomes  $\sum r_i 10^i$ , then  $a \times 1000 + b \times 100 + c \times 10 + d$  or  $a000 + b00 + c0 + d$ . An interpretation is the following: as powers of ten's positional notation requires 0 and 1 as multiplicative coefficients which is a predictable difficulty for early graders, a solution seems to be that 10, 100... are given.

Around 1980, “writings” tasks appear. Among the various “writings”, the “powers-of-ten written in figures” (PTWF, named by the author) -1; 10; 100; 1000...- played an increasing role. Within a few years, it looks like the old tasks in NUs would have been translated in PTWF, and replaced by those translations. For instance, “Write in figures: 2 H 4 T 5 O” became: “Compute  $2 \times 100 + 4 \times 10 + 5$ ” or “Compute  $200 + 40 + 5$ ”. “Convert 4 hundreds in tens” could have reappeared this way: “Fulfil  $4 \times 100 = \dots \times 10$ ”. It didn't reappear.

A major question is: how to get the positional notation with the transposition of the academic theory, especially when a number is given with PTWF? In the

academic theory, the juxtaposition of the polynomial coefficients leads to it. From  $2 \times 100 + 4 \times 10 + 5 \times 1$  -a multiplicative decomposition in PTWF- to get 245 -positional notation-, instead of the rule “juxtapose 2, 4, 5” in the academic theory, the present implicit rule seems to use positional rules: 1) To multiply by 100 (10): write two (one) zero on the right, 2) To add numbers: use a “column algorithm”. That is: put them one under the other, aligning from the right side. After a likely transitory period, present textbooks may look like figure 3.

mille (milliers)			unités		
c	d	u	c	d	u
3	5	7	4	8	9
3	0	0	0	0	0
	5	0	0	0	0
		7	0	0	0
			4	0	0
				8	0
					9

357 489	=	300 000	+	50 000	+	7 000	+	400	+	80	+	9
		(3 x 100 000)		(5 x 10 000)		(7 x 1 000)		(4 x 100)		(8 x 10)		(9 x 1)

Fig. 3: 3rd-grade-textbook. Nouveau Math Elem, Belin, 2002, p.133

From 1995, decompositions with NU came back. Nevertheless, the way to achieve them seems to be: 3 H = 300 (due to the “hundred’s place”), 4 T = 40, 2 O = 2; then compute  $300 + 40 + 2$ ; that is: 342. NUs exist in order to indicate digits’ places, not as units: the only unit “to be taught” is the number “1”.

What does this become in teachers’ practices and in students’ knowledge? Further research is needed. Nevertheless, though there is no causal link proven, several research show that present French 3<sup>rd</sup> graders fail to solve problems like “how many packs of 10 stamps for 89 envelopes?” A recent one evidenced a big decrease between 1999 and 2013.

In the counter-reform, new praxeologies are elaborated, and the old ones disappear more or less rapidly. Hereafter there is an evolution of the “to be taught” praxeologies: this may be partly due to an adaptation to the teaching and learning process, and has effects in taught and learned praxeologies.

## Discussion and conclusion

This text highlights MFs for place value in the French curriculum. Important changes in these MFs occurred during the counter-reform which followed the new math. The *classical numeration theory with units* disappeared in favor of the *polynomial decomposition with the exponential notation*. The effects of the changes were probably not anticipated, neither mastered. As units, in the textbooks, NUs -tens, hundreds...- disappeared from the curriculum in the 1970s; they may remain as places’ names. As units, they have been replaced by PTWF -1, 10, 100...-. The nature of what a *mathematical foundation* is should be further investigated. Whatever, these MFs influence clearly the curriculum until nowadays: the number “1” seems to be the only unit to be taught and learned. Is there a lagged impact on fractions’ understanding which requires units (Lamon, 1996)? Does the lack of NU impact on the understanding of place value? In France, understanding of the positional notation seems to be low, but



further research is needed to relate it to the lack of NU in the teaching. Thanheiser (2009) highlights two correct conceptions with US students: 1) with NU, 2) with “writings”. Are there links with the local MF?

How is it abroad? It would be an interesting challenge to identify what part, if any, of the French situation can be transferred in other countries whether they were implied or not in the reform. Is the *classical theory* taught somewhere?

Is it important that teachers know about such phenomena? What would be useful for effective teaching? An interesting feature is to know that teaching subjects are culturally influenced (ICMI 2000). Tempier (2013) aims to reinsert NUs. Teachers change several things in their teaching but they were not able to teach NUs as units. Might it have to do with their own MF, if any? It may be useful that teacher educators know that several math theories may influence teachers' practices -even their own practices as teacher or educator-. As MFs influence even the tasks to be taught, knowing these phenomena seems to be important for curricular developers.

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