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On the history of units in French elementary school arithmetic: The case of proportionality

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Abstract

The measurement of magnitudes was at the foundation of numbers and calculation in academic mathematics until the 19th century. It provided units and the concrete and abstract numbers that formed the basis of school arithmetic up to the mid-20th century in France. Our analysis of changes in teaching resources for proportionality (late 19th to the early 21st century) documents how the disappearance of magnitudes in academic knowledge was followed by the loss of the differentiation of the conceptual complexity of mathematical ideas related to proportionality. These changes made teaching and learning about proportionality considerably more difficult, and we later witness their gradual, but not yet systematic, reversal.

Résumé

La mesure des grandeurs a été aux fondements des nombres et du calcul dans les mathématiques académiques jusqu'au XIXe siècle. Elle a fourni les unités et les nombres concrets et abstraits qui ont constitué la base de l'arithmétique scolaire jusqu'au milieu du XXe siècle en France. Notre analyse de l'évolution des ressources pour l'enseignement de la proportionnalité (de la fin du XIXe siècle au début du XXIe siècle) documente la façon dont la disparition des grandeurs dans les mathématiques académiques a été suivie par une perte de différenciation de la complexité conceptuelle des notions mathématiques liées à la proportionnalité dans les ressources pour l'enseignement. Ces évolutions ont rendu l'enseignement de la proportionnalité considérablement plus difficile, nous assistons par la suite à un retour progressif mais non systématique.

MSC

97-03; 97F10 ; 97F70; 97F80

Keywords

Mathematics education; Proportionality and ratio; Unit; Concrete and abstract numbers; Textbook analysis; Didactic transposition

1. Introduction

We explore how one characteristic of the “concrete versus abstract numbers” divide played out in French elementary mathematics classrooms. Drawing on the Anthropological Theory of the Didactic (ATD, Chevallard, 2019), we tell the story of the “disappearance” of units from mathematics education guidelines and resources at the turn of the New Math reform (1955-1975) and explore how this happened. We clarify the role units played in mathematics teaching prior to their “disappearance” in the development of number, show how their role changed, and question their more recent “return”. In doing so, we suggest that the awareness of the role units play in meaning-making, also expressed in the concrete and abstract numbers *distinction*, has consequences for how we construct syllabi and how students get to *understand* mathematics.

The focus of our analysis is the content domain of proportionality. We clarify how the choices made at different periods would have impacted the accessibility of the mathematical ideas for students. We preface our analyses by aligning—terminologically and conceptually—with discussions in the domains of measurement and ratio in the mathematics education research literature.

Before we embark on this journey, let us evoke the broader social and historical mathematical context within which the curricular changes we document occurred. Mathematics research has a long and acclaimed history in France. The list of household-name mathematicians is, we believe, unusually long: from Descartes, Fermat, Pascal through Fourier, Cauchy, Laplace, Lagrange to Galois, Poincaré, Mandelbrot and Grothendieck, the list could go on. It is not surprising that *French* academics embarked on the decades long project of systematizing and unifying mathematics in the late 1930s, using the collective pseudonym of Nicolas Bourbaki. Bourbaki members, especially J. Dieudonné, were active in the preparation of the New Math reform in France and abroad.

Of additional interest is the role that France played in the history of the standardization of measurement units. The new system leveraged the mathematical power of base-10 notation, gave rise to the extended use of decimal numbers, and above all, presented measurement as an eloquent, polished, and simple collection of regularities. This would have shaped *which aspects* of measurement were highlighted in school syllabi and how they interacted with the remaining mathematical content.

2. Concrete and abstract numbers

The terms concrete and abstract numbers explored in this special issue are apparently no longer used in the teaching of mathematics in France. Yet, this terminology existed in mathematical treatises used for the teaching of arithmetic during the 18th and 19th centuries. The first chapter of Bezout’s treatise (1821) as well as the notes by Reynaud related to Bezout’s text (hereafter Reynaud’s treatise, Reynaud, 1821)¹ introduce *concrete and abstract numbers* within the introductory sections, among explanations of the meanings of *quantities*, *numeration* (number names and place value in base ten), *number*, and *operations*. Even though Bezout includes both whole and decimal numbers and Reynaud only whole numbers, their approaches to concrete and abstract numbers are quite similar. The first chapter of Reynaud’s notes, titled *Numeration, Addition, Subtraction, Multiplication, and Division of Whole Numbers*, starts with a section titled *The Purpose of Arithmetic*:

¹ The first edition of Bezout's treatise was published in 1764, Reynaud's notes in the early 1800s.

[1]² 1. Whatever is capable of increase or diminution is called *quantity*. When one thinks about the nature of quantities, one feels it would be impossible to have an exact idea of values of quantities of the same kind without considering one of these magnitude values as a term of comparison; this magnitude value is called *unit*; combining several units of the same magnitude value forms a number. The manner of forming, expressing, and writing numbers is the object of *numeration*; and the science which teaches how to perform operations upon numbers is called *Arithmetic*. (Reynaud, 1821, 1, italics in original.)

Reynaud (and Bezout) introduced number as an *accumulation of units*, where units were units of quantities. Reynaud further specified

[2] 3. A number is *abstract* or *concrete*, depending on whether the nature of its units is considered. Thus, 3 and 5-times are *abstract numbers*; 3 toises³ and 5 leagues are *concrete numbers*. Concrete numbers formed with measurement units of several magnitude values, such as 5 toises 3 feet 4 inches, are said to be *complex numbers* [...]

In all questions pertaining to Arithmetic, as the nature of the units of the result is known in advance, one only has to find their number; this leads to operate on abstract numbers (Reynaud, 1821, 6, italics in original).

Both “3 toises” and “3” were numbers. The former was an accumulation of units of “1 toise”, where the nature of the unit (i.e., toise) was specified. The latter was an accumulation of units of 1, where the nature of the unit was not considered or specified. Abstract number thus allowed to consider units of a different nature – the unit could be “forgotten” but it could also be treated as a “variable” (i.e., 5-times). The dialectic between concrete and abstract numbers enabled to give an explanation for operations and their execution. Concrete numbers contributed to defining, explaining, and justifying required operations, whereas *calculations* were mostly conducted with abstract numbers. The usefulness of the distinction between concrete and abstract numbers can be further illustrated in how the definition of multiplication was constructed:

[3] 6. The purpose of the MULTIPLICATION is to calculate a number which is called a PRODUCT. The latter is formed with a known number, called a MULTIPLICAND, in a similar way as a given number—called MULTIPLIER—is formed with the unit. Thus, to obtain the product, one has to operate on the multiplicand the same operations one would have operated on the unit to form the multiplier. [...] For instance, in order to multiply 5 by 3, one can observe that the multiplier is formed with three times the unit. The product then must be formed with three times the multiplicand 5; this product is thus 5 plus 5 plus 5 or 15. [...]

The *multiplier is always an abstract number* because it indicates how many times the multiplicand has to be taken.

The *product is of the same nature as the multiplicand*; indeed, it expresses the addition of several numbers that are equal to the multiplicand (Reynaud, 1821, 10, emphasis in the original text).

This definition illustrates how an abstract number (here a multiplier) was purposefully imagined as a specific accumulation of a unit, which is not specified. Another quantity (expressed as an abstract or a concrete number) could then be substituted for this unspecified unit, and accumulated in the same way, resulting in the product of the multiplication.⁴

² The original French text can be found in Appendix 1, under the corresponding indicator “[n]”. Unless stated differently, all translations are by the first author.

³ The French original refers to “*toises*” and “*lieues*”, units of length used in France before the adoption of the metric system.

⁴ Even though this definition used whole numbers as an example of unit accumulation, it allowed for any multiplicand to be accumulated by any abstract number (considering a measurement process). The same definition was used for multiplying fractions in a later section of the treatise.

3. Units

3.1. Quantities, magnitudes, measurement units

Discussions related to numbers as a means of understanding situations and phenomena often rely on terms such as *quantities*, *sizes*, *attributes*, *magnitudes*, and *units* for their meaning. Yet, these words have been used differently by various authors, and demarcations of their meaning have been created for different purposes. Freudenthal (1983), Passelaigue and Munier (2015), and the VIM⁵ (JCGM, 2012), all introduced different options for uses of measurement terminology. Given our focus on mathematical *teaching* and *learning*, we lean towards terminology that allows for making visible the distinctions that are relevant to supporting students' learning of measurement. We follow Freudenthal (1983) in noting that the processes by which mathematics as we know it today came to be—the processes of mathematizing the world—can powerfully shape mathematical learning.

Freudenthal distinguishes four terms: *quantity*, *magnitude*, *magnitude value* (or *value of magnitude*), and *measuring number*. He uses *magnitude* to refer to an attribute of interest in a *quantity* of something. Examples of *magnitudes* include length and volume (as attributes of interest in a quantity of a wooden plank), numerosity, or sound intensity. *Magnitude value* then refers to a magnitude that is expressed as a property of a given thing. For example, the Earth's equator has a specific length, and the Moon has a specific volume, whether or not these *magnitude values* have already been measured (or quantified), or compared to another magnitude value of the same kind.

Magnitudes have long been at the core of numbers and calculations in mathematics (Bourbaki, 1984). They disappeared in the second half of the 19th century, at the time of the arithmetization of analysis and of the modern axiomatizations that followed. At present time many mathematicians are no longer interested in magnitudes, and no unique mathematical definition of *magnitude* as a structure has been developed by the mathematics community. In spite of this, several structure definitions for *magnitude* can be found, often sharing a few starting points:

- a *magnitude* is defined as a *set* of magnitude values,
- two magnitude values can be added,
- addition is associative and commutative,
- the set is ordered, and
- $a < b$ if and only if there exists c , such that $b = a + c$.

The last property ascertains that (1) a *magnitude* is something which is growing when being added; and (2) subtraction is defined, and enables to “fill a gap” when one magnitude value is larger than another.

As in the the discussed treatises, Freudenthal used the term *concrete number* for magnitude values that have been quantified and expressed as a measurement. Specifically, when the length of Earth's equator is measured as 40075 km, this constitutes a concrete number, as the nature of the unit used is specified. To establish a concrete number, one needs to consider a unit of measurement (km), and express the relative size of the *magnitude value* (the length of Earth's equator) with respect to this *unit* (the Earth's equator is 40075 times as long as⁶ the kilometer).

⁵ The VIM (International Vocabulary of Metrology – Basic and general concepts and associated terms) is the international reference book for metrology.

⁶ We use “z-times as long as” to express multiplicative comparisons, instead of “z-times longer”, because the former expression captures the required relationship in mathematically meaningful way for any real number z .

The magnitude value is symbolically expressed with a numeral (i.e., *the measuring number*) and a unit name (e.g., cm).⁷ A sophisticated understanding of a concrete number thus comprises the comparison between the *magnitude value*, and the measurement *unit* (see Figure 1).



Figure 1. A given magnitude value, the length of a pale stick, is compared to a given unit of length, the dark measuring rod. The stick is *the measuring number*-times (3-times) as long as the unit.

Lastly, when working with values of the same magnitude, like in the case of the equator length and a kilometer, Freudenthal refers to these values as being from the same *system*. A specification of this kind becomes important when distinguishing ways in which a ratio can be constructed. Given that the word *system* has many different uses, we will follow Lamon (2007) and refer to values of the same magnitude as being from the same “*measure space*”. In historical sources, values of the same magnitude are often described as being “*of the same kind*” (e.g., Euler, 1774; Reynaud, 1821).

3.2. How units relate to ratio

Reynaud’s approach to magnitudes enables one to consider any magnitude value as a unit, capable of generating numbers by being accumulated without restriction. Moreover, for any unit u of a given magnitude, and a given number a , it enables to consider other units, v and v' , such that⁸

$$1 [v] = a [u], \text{ and } a [v'] = 1 [u]$$

(for clarity, when we draw attention to a specific magnitude value l being considered as a unit, we symbolize it with brackets $[l]$). In such an instance, we refer to unit u as a *simple unit* and to units v and v' as units *related* to unit u (see also Chambris, 2021). Using the notions of simple and related units, Reynaud’s definition of multiplication (section 2 above) specifies that (1) the multiplier is a given abstract number that indicates a number of unspecified units to be taken; (2) the multiplicand is conceived of as a unit to be taken, formed with a given number of *simple units*, it is thus a *related unit*; (3) the product is to be expressed as a number of *simple units*. In other words, with u as the simple unit, the multiplier b indicates the number of related units $a u$ (where $a u$ is the multiplicand) needed to form the simple units of the product $b \times (a [u]) = b [a u] = P [u]$.

This equality means—although this is not explicitly stated in the treatises—that the magnitude value of the product (expressed in the simple units of the multiplicand), is the same as the magnitude value of b units the size of the multiplicand (Chambris, 2021).

For the purposes of this paper, it is important to note that as a consequence of this definition, the idea of ratio is congruent with both the idea of measurement $b[a u] : [a u] = b$, and of division $P [u] : a [u] = b$.

3.3. Ratio: internal and external

The ratio that arises as a consequence of multiplication expressed in the previous section is constructed within the same measure space, or the same system. As Freudenthal’s (1983) phenomenological analysis highlights, comparisons of ratios across systems give rise to

The reader will note that “1/3-times longer” is not mathematically coherent, and “3-times shorter” is usually confusing to students as the process by which the shorter length is to be created is not specified.

⁷ A magnitude value can also be symbolically expressed with a variable, such as the equator length l .

⁸ Note that the second case postulates the existence of sub-units of u that correspond to unitary fractions. It points to situations where $10 v' = u$, which establishes $v' = 1/10 u$.

another type of ratio, another type of magnitude, and another type of unit. Taking the example of uniform motion, from the postulate of the uniformity of motion “in equal times equal distances are covered” (182), Freudenthal established first “in a times the time, a times the distance is covered” (a being successively n , $1/n$, k/n). The relation is then extended to any positive real number through continuity of motion. Freudenthal then distinguished between internal and external ratio.

We designate ratios formed within a system as *internal* to distinguish them from the external ones that are discussed below.

If t_1, t_2 are times and s_1, s_2 corresponding paths, the postulate of uniformity says $s_1 : s_2 = t_1 : t_2$.

If we are tempted to interchange the middle terms, we get $s_1 : t_1 = s_2 : t_2$ again the equality of two ratios, albeit ratios of path to time.

We designate ratios between two systems as *external*. The uniformity of the motion is now expressed by the postulate: the ratio “path to time” is constant.

Ratios can also be interpreted as quotients. In this interpretation, the internal ratio is a number, the external ratio is a magnitude, that is, in the present case of uniform motion, the quotient of path and time: speed (183, italics in original).

Understanding external ratio quantitatively requires the recognition of a new type of units – quotient units – which imply a new way of mathematizing a magnitude (magnitudes of this type are referred to as *intensive* magnitudes). In practical mathematics (e.g., finance), external ratios of magnitudes from different measure spaces (i.e., magnitudes of *different kinds*) were included in procedures for calculations, even when the resulting new magnitudes were not considered or defined. However, as Freudenthal (1983) pointed out, the use of external ratios continued to be rejected in sciences and the discipline of mathematics. As an example, Euler’s *Elements of Algebra* (1774) specified that ratio [4] “can only be a matter of magnitude values of the same kind” (308). It was not until 1968 that a mathematical model for calculations with measurement units was proposed (Whitney, 1968).

From a phenomenological perspective, external ratio is first considered, followed by recognition that specific cases of external ratio (e.g., 60km/1h and 120km/2h) are equivalent in a proportional situation (Cortina, 2006). These equivalent ratios represent a constant external ratio, or what Lamon (2007) calls a rate. An additional conceptual step is then needed to recognise that rates constitute a new magnitude, and this new magnitude provides the meaning for a quotient unit (e.g., km/h).

During the periods under study, we illustrate tensions related to the inclusion of units in external ratio situations, and how ratios were removed altogether when proportionality was introduced through rate.

4. Methodology

The assumptions about how curricular phenomena develop shaped our methodology, including the data we analyzed, and the clues we looked for in our data. We adopted those assumptions from ATD (Chevallard, 1992, 2019), and viewed teaching and learning as anthropological phenomena. The ATD places didactic issues at the institutional level (beyond the characteristics of individuals working within a given institution) and views teaching and learning practices as shaped by institutions (e.g., classrooms, education systems, or societies, at a given place and time).

Two of ATD’s central ideas are the key to this analysis: the process of *didactic transposition*, and the notion of *praxeology*. The theory of didactic transposition (Chevallard and Bosch,

2020), developed in the early 1980s, underlines the *institutional relativity of knowledge* (see Figure 2). Knowledge produced by scholarly institutions was *reconstructed* in the transposition work (e.g., Wijayanti and Bosch, 2018) by educational agents (including researchers in mathematics and mathematics education, curriculum developers, policy makers) in order “to be taught”, before being prepared and taught by teachers, and learned by students.

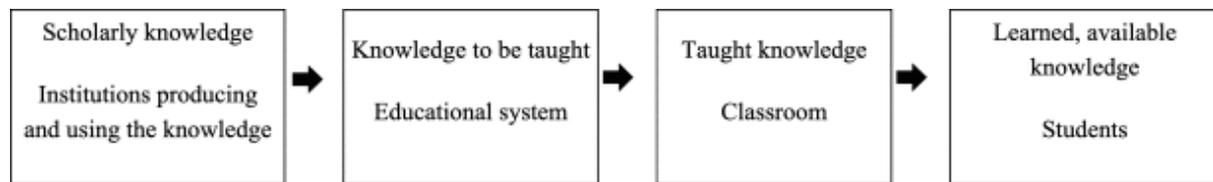


Figure 2. The process of didactic transposition.

This conceptualization implies that what students learn is profoundly shaped by what is taught by their teachers, which in turn is shaped by what appears in written curricula (including syllabi and textbooks), which involves knowledge to be taught produced by educational agents through their transposition work.

ATD postulates that any human activity (notably that of mathematics) in a given institution can be described in terms of *praxeologies* (e.g., Bosch et al., 2017). The notion of praxeology indicates that practice (*praxis – know-how*) and the discourse (*logos*) about practice necessarily work hand-in-hand. There are four components in a praxeology: a *type of tasks* (a set of similar problems); a *technique* (a way of doing tasks of the same type); a *technology* that justifies the technique; and a *theory* that legitimates the technology, enables to interpret techniques, and gives meaning to tasks. The first two components form the *praxis*, the last two form the *logos*.

We attended to the presence and absence of units across different components of the praxeologies developed within the educational system to specify knowledge “to be taught”. For example, as magnitudes are at the heart of ratio *tasks*, one might expect measurement units to appear in tasks given to students during any period. We explored how the magnitudes involved shaped the units within tasks, including whether quotient units (e.g., km/h) were present. We also analyzed the roles that units played, if any, within the *theory* component of the praxeology. For instance, when the treatises of Reynaud (and Bezout) were acknowledged as reference texts for knowledge to be taught, units of magnitudes were identifiable in the theory component. Moreover, both related and simple units appeared in the definitions of multiplication and internal ratio, and measurement units appeared in the definition of concrete numbers where they indicated the magnitudes at hand. Finally, we attended to how *techniques* and *technologies* were impacted by the presence or absence of units in the *theory* component. We particularly examined whether units were included in calculations (techniques) and whether units and magnitudes played a role in the justifications of calculations (technologies).

4.1. Analyzed texts

We analyzed the more recent syllabi⁹ and older reference texts—treatises (Neyret, 1995). Both treatises and syllabi are among the textual documents that organize mathematics education by outlining what needs to be taught, and to some extent how to teach it. Treatises

⁹ French syllabus documents generally include an introduction, a list of topics to be taught (the *syllabus* in the strict sense of the word), and more or less detailed instructions that indicate how to teach each topic. In referencing syllabi, we use the notation ‘Syll1923’ to indicate that the reference is one of the source texts for our analysis.

were considered to be a conceptual reference for teaching different topics: Their comprehensive presentation of topics and specific explanations informed the structure of textbooks and what was to be taught. The treatises we analyzed have been used for educating elementary teachers (Neyret, 1995) though they were initially designed for another purpose (educating military personnel or secondary teachers). Based on the treatises, subject syllabi were later produced to coordinate the work of a growing number of *teachers*. They were accompanied by materials for classroom use, including textbooks and exercise books, where the user was primarily the young learner. Treatises were a product of the transposition process, and, in praxeological terms, they developed the theory component of mathematics to be taught.

In the analysis of the *knowledge to be taught*, we relied on syllabi for identification of this knowledge, and on textbooks and teaching guides for additional detail. Syllabi often provide traces of scholarly knowledge that is at the origin of the didactic transposition process, as do teachers' resources (e.g., teaching guides). *Technologies* can be identified in the explanations, proofs, definitions, and mathematical properties that are given in solved examples. The solved examples also embed *techniques*, the “practical” steps for solving problems. *Types of tasks* are often indicated by a generic description (e.g., direct rule of three) and by solved examples. They are evident in textbooks where series of exercises occur for each type of tasks.

We focused our analysis on grades 4 and 5 (presently the final years of French elementary school), when the teaching of ratio starts. When different grade levels were analyzed for elaboration, we specifically indicate this in the text.

5. Historical survey of French curricular documents of 1882 to 2007

Studies on teaching numbers and arithmetic in the French education system (e.g., Bronner, 1997; Neyret, 1995) suggested that a long period of relative stability was disrupted by syllabus changes in the late 1940s, and that magnitudes disappeared from the French elementary and secondary teaching of number and operation in the period of the New Math (1955-1975). However, across all these periods, students in grades 4 and 5 continued to learn to solve proportionality word problems with magnitudes. What were the roles of magnitudes and units in the French elementary syllabi during these periods and how were the changes orchestrated? How did the changes impact the teaching and learning of proportionality?

To understand the role of magnitudes and units in the techniques and technologies used in the teaching of ratio in France, we build on two previously published bodies of work. Hersant (2005) used ATD to analyze French syllabi (1882-2002) and textbooks (1882-1996) for grades 4 to 9. She focused her analysis of textbooks on certain types of problems—simple and direct rule of three—and described the techniques and technologies taught. In her work, units were not in the forefront of the analysis. The first author used ATD to characterize changes in how the role of magnitudes and units was specified in both scholarly knowledge and knowledge to be taught in elementary school (Chambris, 2010, Chambris, 2021). Coordinating the two analyses allowed us to (1) interpret the evolution of the teaching of proportionality with regard to the broader role of magnitudes and units in the syllabus, and (2) discuss the role of internal and external ratios in the teaching of proportionality.

5.1. 1882-1945: Concrete and abstract numbers during the period of internal ratios

Harlé (1987) and Neyret (1995) indicated that both the Reynaud (1821) and Bezout (1821) treatises were used as *reference texts for knowledge to be taught* (see Figure 2) for syllabi, school textbooks, and for teacher education, but Reynaud’s book gradually replaced Bezout’s in teacher education at the turn of the 20th century. While the treatises share a general approach to magnitudes and the four operations, they differ in introducing the notion of ratio: whereas Bezout’s approach was based on the theory of ratios and proportion, Reynaud’s approach, which prevailed, was based on a structurally simpler theory of multiplication and division.

Throughout this period, magnitudes expressed as units were included in the documents that were produced to organize arithmetic learning. In 1882, in grade one, students learned [5] “addition and subtraction on *concrete numbers* and not exceeding the first hundred” (Syll1882, italics are ours). The subsequent syllabus (for the 2nd grade in 1923) included, in the same paragraphs, discrete and continuous magnitudes. Indeed, both discrete and continuous measurements were involved in the treatment of whole numbers, operations, and place value topics. Moreover, from 1923 on, place value was clearly connected with the metric system. The 4th grade 1923 syllabus indicated that: [6] “Students will understand what is a tenth of a *meter*, a tenth of a *gram*, before they understand what is a tenth of a *unit*” (Syll1923, italics are ours). The role of magnitudes and units seemed not to have changed: measurement of magnitudes led to numbers and operations, and numbers and operations were taught using related and measurement units.

In ratio and proportion in grades 4 and 5, simple calculation of “rule of three” problems appeared to be the goal. Hersant drew attention to a teacher’s guide that interpreted the 1882 Syllabus in the following way:

[7] Let us not delay too much the teaching of the rule of three: it is advantageous for children to get used to this reasoning exercise, which allows them to solve a number of practical questions through the four operations alone. There is no need to wait for the study of ratios and proportions (Mutelet and Danguueger, 1910, 2nd edition. Quoted by Hersant, 2005, 11).

This excerpt refers to the reduction to the unit method. In the six textbooks Hersant analyzed, the justification of this method was mainly based on the comparison of the variation of magnitudes that embedded implicitly internal ratios. For instance, [8] “Two magnitudes are directly proportional if when one of them becomes 2, 3, etc. *times larger or smaller*, then the other becomes 2, 3, etc. *times larger or smaller*” (Lemoine, 1913, 124). The method was very similar to the one presented by Reynaud: it was based on multiplication and division, not on calculation with ratios. Yet, a major difference was that the justification was mostly explicit. As in Bezout’s book, it was based on the comparison of the variation of magnitudes, and used the language of proportions: [9] “Thus, the price of a commodity is directly proportional to its length, or to its surface, or to its weight, or to its volume” (Lemoine, 1913, 124). The “reduction to the unit” technique was then made of three steps written in a spatial pattern in three lines. A justification, using expressions “times less”, “times more”, was often included in solved examples in textbooks, like in this worker salary calculation problem

[10] For 16 days, the worker receives 40 francs;

For one day he will receive 16 times less, or $\frac{40 \text{ fr.}}{16}$;

And for 28 days he will receive 28 times more, or $\frac{40 \text{ fr.} \times 28}{16} = 70$ francs (Lemoine, 1913, 125).

Though the reduction to the unit method was the only technique considered legitimate in the educational system up to 1923, the method was contested in many textbooks, and alternative techniques were offered (Hersant, 2005). These alternatives only involved *internal ratios* and

attended to relationships *within* the two spaces of measurement. First, the method was challenged because its full algorithmization sometimes led to heavy calculations that could be avoided. In most textbooks, alternative techniques for mental calculation used fractional internal ratios, such as [11] “the price of 12 kg is equal to 4 times the price of 3 kg (10fr,50): 10fr,50 . 4 = 42 fr” (Mortreux and Mortreux, 1910, 357).

The second point of contention concerning the reduction to the unit method was that “absurd” magnitude values appeared as intermediate results in calculations. We illustrate this with the example of a problem, where 13 workers do 273 m of work, and the task is to find out how many workers would do 420 m of work in the same amount of time. The first step in the reduction to the unit method is: [12] “1 m is done by $\frac{13}{273}$ worker” (Mortreux and Mortreux, 1910, 358). To this, the authors add [13] “We are making an *absurd reasoning* (sic)” (Mortreux and Mortreux, 1910, 358, italics in original). Alternative techniques that aimed to avoid the “absurd” intermediate results involved the notion of *unit value*.

[14] The required number of workers is equal to the total length (420 m) divided by the length made by 1 worker.

Length made by 1 worker: $\frac{273}{13}$;

Number of workers: $420 \div \frac{273}{13} = \frac{420 \times 13}{273} = 20$ workers (Mortreux and Mortreux, 1910, 358, italics ours).

When Hersant (2005) quoted the same example solution (from a 1933 new edition of the textbook for grade 6 of the same series), she suggested that this technique was less algorithmic than reduction to the unit, and that it used coefficients (i.e., external ratios). Given the historical situation of the grade 4 and 5 textbook, we suggest that an alternative explanation is more plausible. The convention by which the measurement units were not included in calculations, but they were used in explanations, suggests that the unit value, here the “length made by 1 worker”, was thought of as *a length* measured in meters. The “13” in $\frac{273 \text{ m}}{13}$ does not stand for the number of workers. Instead, it expresses the multiplicative relation between the length produced by 13 workers and what is produced by 1 worker as being the same as the multiplicative relation between the [numerosity](#) of 13 workers and one worker (i.e. $\frac{1}{13}$, proportionality definition as a correspondence of variations within two measure spaces). In this context, the $\frac{273 \text{ m}}{13}$ is not a coefficient. We propose that this technique was based—at the technology level—on multiplication and division and it did not involve an external ratio.

From the beginning of the period, a few 6 to 7th grade textbooks stated that when magnitudes are proportional the external ratios are equal. While some textbooks introduced calculation with multiplication or division of magnitude values, other textbooks rejected ratios of magnitude values of different kinds (Hersant, 2005). Our analyses lead us to suggest that, in order to support students in solving the types of problems that were expected of them, techniques involving external ratios must have been taught, even in grades 4 to 5. Indeed, syllabi from 1882 (Syll1882) indicated that problems that included interest rates were studied in these grades, and problems including density were added in the sixth and seventh years. Both notions of interest rate and relative density were constructed as external ratios of values of two magnitudes *of the same kind*. For instance, problems of profit calculation involved multiplicative comparison of the money gained and the money paid. Because the two measurement spaces involved were of the same kind, the resulting interest rate, often

expressed as a percentage, could be treated as an abstract number (a quotient of two monetary magnitude values).

The analysis of the techniques proposed for solving interest and density problems suggests that inclusion of external ratios was a complex issue. In textbooks, interest percentage was at times presented as an external constant ratio: [15] “The 6% profit represents 6 hundredths of the purchase price” (Lemoine, 1913, 141). However, it was more often defined as 6 fr for each 100 fr, and the main method taught to solve these problems was an adaptation of reduction to the unit, explained as follows:

[16] A house is sold with a profit of 6%. Knowing that this profit is 300 francs, what is the purchase price of the house?

Solution: A profit of 6 francs comes from a purchase of 100 francs;
 a profit of 1 franc comes from a purchase of $\frac{100 \text{ fr}}{6}$;
 and a profit of 300 francs comes from a purchase of $\frac{100 \text{ fr} \times 300}{6} =$
 5000 francs (Lemoine, 1913, 141).

Another solution was given, also based on internal ratios [17] “The purchase price would include 100 francs as many times as there are 6 francs in the profit of 300 francs, that is $100 \text{ fr} \times \frac{300}{6} = 5000 \text{ francs}$ ” (Lemoine, 1913, 141). Textbooks that presented solutions based on external ratio relied on fraction knowledge that was developed in previous chapters and led to the direct calculation of the reciprocal of 6 hundredths, of 300 francs. We note that the magnitudes in the proportion (money gained and money paid) were not always mentioned in the solution for this type of problems. Interestingly, some textbooks did not connect these problems with the rule of three, and suggested a solution technique based on a formula, without providing an explanation for why it worked.

The last case of the use of external ratios in the 1938 syllabus, for grades 6 and 7, did not bring more clarity. The syllabus introduced specific “quotients” (Syll1938) such as price per capacity unit, or mass per unit of volume. We include this discussion here to illustrate a possible departure from previous uses of the *unit value* technique, and because the following period will see these types of tasks and techniques used in grades 4 through 5. The syllabus specified that quotients could be considered as unit values, and suggested solving rule of three problems with them. For this purpose, a formula was given, [18] “total value = unit value \times number of units” (Syll1938), but the proposed solution did not include measurement units of the components of the multiplication.¹⁰ It is thus unclear whether the intended “unit value” in the “price per capacity” example would be “price” (changes in which needed to be compared to corresponding changes in capacity, that is, internal ratio), or whether “price per capacity” was considered an admissible composite magnitude for a “unit value” in this situation.

The above question is substantiated by the fact that, in examples of calculations with decimals, the same syllabus includes measurement units in calculations, when justifying the rule of multiplication by a decimal: [19] “(3.50 fr per l) \times (7.25 l) could be replaced by (0.035 fr per cl) \times (725 cl)”.¹¹ Here, “3.50 fr per l” and “0.035 fr per cl” appear to be treated as the

¹⁰ There were no quotient units in calculations provided in reference texts, i.e., the treatises. Such calculations seem to have been introduced in the 1938 syllabus, aiming at enriching the theoretical environment of arithmetic, without drastically changing it.

¹¹ In France, a comma is used in decimal notation (instead of a decimal point). In this paper, we use the English notation in translations and keep the original notation in the originals.

same intensive quantity, “price per capacity unit”, expressed with two different units (“fr per l” and “fr per cl”).¹²

In light of this example it becomes clear that, with the introduction of the formula “total value = unit value × number of units”, the unit value technique came to involve two different meanings, depending on whether the “unit value” was viewed to be of a simple or of a quotient magnitude. In the former case (“length made by 1 worker”), the unit value, the unit, and the unit count were all fixed. In the latter case, while the *magnitude values* of both the quotient “unit value” and the corresponding “number of units” remained fixed, the measuring number that expressed the “number of units” could vary: in our example, it changed from 7.25 to 725.

The difference in the two possible uses of the unit value technique sheds light on the complexities that are introduced with intensive quantities (i.e., external ratios of values from different measure spaces): the work with abstract numbers, where the units are not specified, is no longer meaningful. This introduction forces the use of magnitude values, and thus measurement units, in calculations.

Notions of interest rate, density, and quotients justified the introduction of specific techniques in calculation, but some of the techniques in the textbooks were devoid of underlying technologies. This shows that the introduction of work with external ratios of different magnitudes had been tentative at best. While the quotients represented new types of magnitudes, and required that calculations consider magnitude values (i.e., work with concrete numbers), they were not consistently treated as such. The notion of the constant external ratio, and its relation to reasoning with internal ratios in proportional situations was not explicitly addressed during this period in grades 4 and 5 and was only rarely addressed in grades 6 and 7.

5.2. 1945-1970: Departure from internal ratios in grades 4 to 5

Lebesgue's book *La mesure des grandeurs* (Lebesgue, 1975 (posthumous)), published serially from 1931 on, progressively became a reference book for elementary teacher education (Neyret, 1995). A disruption with regard to arithmetic was the inclusion of the set of real numbers, whereas prior treatises were limited to fractions. It included a construction of real numbers based on measurement with sequences of decimal length subunits, without including either a theory of fractions or Euclidean ratios. The replacement of fractions with decimals was motivated by an attempt to avoid definitions by abstraction (Chambris, 2010) that were, at the time, mathematically problematic (Griesel, 2007; Otte, 2007). Despite a different presentation compared to prior treatises, multiplication was again based on the change of units of measurement and general rules for calculations with real numbers were developed. This resulted in the algebraization of numbers. Neither internal, nor external ratios were used in the presentation of proportionality, but Lebesgue proposed a rationale for developing proportionality from a postulate of uniformity to the existence of a rate.

In the 1945 syllabus, the status of magnitudes kept changing. Concrete and abstract numbers remained and quotient units appeared. Calculation with magnitudes was contested, somewhat inconsistently. In line with Lebesgue's construction of real numbers, fractions no longer appeared as a means to develop decimals (Chambris, 2010). Decimals were to be taught as concrete numbers, using the metric system (e.g., [20] “2 meters and 15 centimeters = 2.15 m”, Syll1945, 246).

¹² The recognition of the equivalence of the quotient “unit values” (e.g. 3.50 fr per l = 0.035 fr per cl) presumes recognition of a rate.

In 1945 through 1970, the reduction to the unit method continued to be included in the syllabus under the name *rule of three* where the rules developed by Lebesgue were introduced as calculation formulas. Two other methods appeared in grades 4 and 5, and the choice of methods depended implicitly on the kinds of magnitudes in the proportion. Firstly, similar to the earlier treatment of interest rates, *external ratio comparison* of two magnitudes of the same kind resulted in a dimensionless rate. Tasks included calculations such as [21] “weight of flour = weight of wheat \times 0.80” (Syll1945, 251).¹³ Fractions were considered abstract multipliers and were treated as fractions of magnitudes (i.e., take four fifth of) in ratio problems.

Secondly, the *unit value* method, indicated in 1938 for higher grades, was to be used otherwise, and the uncertainty about the type of units to be used was solved: quotient units were introduced. This method led to the introduction of formulas, and to a certain algebraization of the technique. It increased the demands for prerequisite mathematical understandings, as younger students would have to: (1) learn formulas by heart, or (2) learn basic algebra to manipulate multiplicative equalities, or (3) work with the complexity of quotient units. In fact, the method was introduced in grades 2 and 3 for learning multiplication and division, as well as the notions of multiplicand and multiplier. This change might be interpreted as an attempt to increase the time students had for learning methods related to ratio. The introduction of external ratio techniques for early grades made justifying calculations with magnitude values even more difficult.

In textbooks, the reduction to the unit method remained in the fore, but the definition of proportional magnitudes (the comparison of the variation of magnitudes) and the justification of the method through internal ratios became increasingly less explicit (Hersant, 2005). “Spatial explanations,” started to replace them: [22] “I’m seeking a price. I place the price 18 fr on the right” (Benhaïm and Nadaud, 1969, 109) and mental arithmetic using internal ratio declined. Many textbooks focused on calculations, such as a fraction of a magnitude value, but without distinguishing whether the calculation represented an internal or external ratio. The proportionality of the problem situation was not always pointed out. In spite of the syllabus recommendation, the unit value method was rarely included in elementary textbooks (Hersant, 2005).

The influence of Lebesgue's book prompted the decrease of the use of fractions (in the syllabus and textbooks). Decimals, introduced through the metric system, provided an alternative for expanding beyond whole numbers. The discourses (technologies) for proportionality also changed, where expressions such as *times more*, *times less*, and *two thirds of* were now avoided, and exclusively spatial and multiplicative symbolic expressions were preferred. Algebraic rules, not rooted in sense making, progressively became the only means of providing coherent justification, even though concrete numbers remained in use. The situation was ripe for removing magnitude values from mathematical consideration altogether.

5.3. 1970-2002: Units prohibited in the period of ratio as function

The introduction of the reformed syllabus¹⁴ in 1970 asserted that the curriculum was not a new one but a [23] “different writing” (Syll1970, 285) of the previous one from 1945, a huge ecological and praxeological reorganization, we might add. On the surface, the New Math

¹³ “Weight of flower/weight of wheat” is an external ratio.

¹⁴ We use “reformed” here as this is how the new syllabus was known as at the time. It presented a major restructuring of how mathematics is to be taught in schools. The French mathematician André Lichnérowicz headed the ministerial commission on mathematics education in France (1966-1973).

reform resulted in a reorganization of the topics, but these, while bearing the same names, now referred to a different scholarly knowledge. The transposition of new scholarly knowledge reached far beyond the introduction of set theory in the elementary grades, for which the syllabus is commonly known. It brought about changes in the knowledge to be taught for fractions and decimals, and had unprecedented consequences in the teaching of ratio.

5.3.1. Background on number: scholarly knowledge and syllabi from 1970 on

Previous syllabi included only two domains: *arithmetic* and *geometry*. The 1970 syllabus was reorganized into three domains: (a) the *numerical domain* which focused on numbers and operations, and was titled *Elements of Mathematics*, following the name of Bourbaki's series, (b) the *geometrical domain* titled *Observation Exercises and Work on Geometric Objects*, and (c) the *measurement domain* titled *Measurement: Practical Exercises*. The name indicates that measurement was linked with pragmatic needs, unlike the numerical domain which now introduced elements of a (new) system of thought.

The creation of the measurement domain resulted in a separation of discrete and continuous magnitudes. They were previously studied in parallel, providing meanings and imagery to the concept of number. In 1970, students were instead expected to learn the [24] “concept of natural number” (Syll1970, 286) almost entirely through counting (i.e., by quantifying discrete magnitudes) in the *numerical domain*, and to apply it to continuous magnitudes in the *measurement domain*.

Paradoxically, the birth of the measurement domain is the visible side of the disappearance of magnitudes as the conceptual roots of numbers. The syllabus writers decided to align school mathematics with the new scholarly mathematics, where numbers now came from the theory of real numbers and were based on sets. Continuous magnitudes were thus no longer required in schools for defining and teaching whole numbers, fractions, or decimals (Chambris, 2010).

A series of changes was triggered by this syllabus, as the education system attempted to make the new scholarly knowledge teachable. While the *measurement domain* remained for years to come, from 1980 on, continuous magnitudes increasingly returned to the *numerical domain* in tasks that aimed to support meaning making.

Against this background, the changes in the meaning of operations and the teaching of proportionality that the 1970 syllabus initiated remained central for almost 30 years.

5.3.2. Operations with bare numbers

The 1970 syllabus clearly differentiates between operations, and situations to which the operations correspond.

[25] The study of natural numbers includes the study of the two fundamental operations, addition and multiplication, which give the set of these numbers its own algebraic structure.

These two operations are linked to subtraction, exact division and Euclidean division (i.e., with a remainder that may be different from zero).

It is essential to understand that the addition, the multiplication only concern numbers. It is equally important that children recognize the situations to which these operations correspond. (Syll1970, 289-290, italics are ours.)

Hence, the (mathematical) scholarly knowledge seems to no longer provide justification or explanation for knowing the nature of the units of the result.

[26] Let it be recalled that addition (and also subtraction, multiplication, etc.) refers to numbers and not to the sets that these numbers qualify: sets of objects are brought together; numbers are added together. *Sentences such as: 8 apples + 7 apples = 15 apples. do not actually belong to either mathematical or common language.* Common language uses phrases such as “when I add 8 apples to the 7 apples in the basket, the basket contains 15 apples” or even more vaguely “8 apples and 7 apples, make 15 apples.” *Add, and are words of current language, they are not terms used in mathematical language. Conversely, the word “plus” is not usually used in everyday language to express the action of adding.* In classroom practice, the two languages are mixed, but it is important to distinguish them. For example, we could write: The number of apples is: $8 + 7 = 15$ and conclude: “The basket contains 15 apples” (Syll1970, 290, our italics).

This excerpt illustrates that numbers with units were no longer considered *mathematical* and that they were seen as objects of daily life and language. Calculation techniques, including those used for proportionality, prohibited the use of measurement units. In this period, proportionality was a relation between lists of numbers, and this allowed the removal of the remaining traces of magnitudes.

5.3.3. Proportionality as a numerical function

The 1970 French elementary syllabus introduced—for the first time—the term *numerical relations*. In the following 1980 syllabus, this term was mostly replaced by *numerical functions*. In 1970 through 2002, proportionality was explicitly considered as a case of numerical function (or relation), and the key purpose for its inclusion was to teach numerical functions rather than to teach proportionality.

The 1970 syllabus presented several two-row tables of numbers, without any context, with an explicit rule for number correspondence (e.g., adding 4; multiplying by 3). It defined “numerical relations”, and their “reciprocal relations” (i.e., subtracting 4; dividing by 3, respectively), while the term “reciprocal” remained implicit. Based on [27] “the study of concrete situations” (Syll1970, 298), proportionality was presented as follows: [28] “When the operator is “multiply by...” or “divide by...” the correspondence that enables to move from one list to another is called proportionality” (Syll1970, 298). Three examples of *concrete situations* were given, including the relation between length and area:

[29] A series of rectangles is considered, one side of which, a unit being chosen, is measured by the number 4. A grid is used to determine the areas of these rectangles taking the square of side 1 as the unit of area. (...)

Length of the 2nd side	Area
2	8
3	.
5	.
4	.
.	24
.	36
<i>a</i>	<i>4 a</i>

We can construct the table of correspondences between the measure of the second side and the area of these rectangles. More generally, if the measure of the second side is *a*, the area (the unit being determined as above) is $4 a$ (Syll1970, 299).

The solutions recommended the use of a coefficient, here “multiply by 4”, and *numbers* were intended to represent *units of length and area*. Additive and multiplicative properties of the multiplicative numerical relations were derived from the lists of numbers taken from the examples, then generalized with a formula.

The 1980 syllabus built on the earlier syllabus. While [30, 31] “situations encountered in or out of the classroom ... [such as] a fuel pump meter” (Syll1980, 53) were still used to provide motivation for *numerical relations*, these relations were described with “numerical data” (i.e., measuring numbers), and magnitudes were used infrequently. *Proportionality* was presented in a specific sub-section as a characteristic property of *numerical functions* $f: n \rightarrow n \times a$, a and n being numbers, not magnitude values. Measurement units appeared only once in the section devoted to function, in a “scale” problem and its solution:

[32] “2 *cm* on the map represents 2.5 *km* on the ground. What is the distance that, in the field, corresponds to 6 *cm* on the map?” [...]

Students can then determine that the number corresponding to $6 = 2 \times 3$ (in *cm*) is 2.5×3 (in *km*) without the need here to clarify the proportionality coefficient (in this case the “scale”)¹⁵ [...] (Syll1980, 56, our italics).

A calculation method was provided to avoid the proportionality coefficient, presumably because it is easier to get, multiplicatively, from 2 to 6 than to get from 2 to 2.5. This method relied on internal ratios but it only indicated this implicitly, and measurement units were used here as a clarification of where the numbers in the calculation originated. The calculations were enacted with measuring numbers.

At this time, the functional relationship was defined through a constant multiplicative relationship between two lists of numbers (called coefficient, i.e., rate expressed without units). Over time, properties of linear functions were increasingly intended to serve as an alternative means of solving problems (Hersant, 2005). In 1980 and 1995 syllabi (Syll1980, Syll1995), techniques for representing functions graphically also provided new means of solving problems (Hersant, 2005).

The study of Sokona (1989) indicates the types of reasoning students learned in this period. He followed students in grades 6 and 10 as they solved problems that involved proportional relations in a new situation. He observed that although the “coefficient” (a relation *between* measure spaces) appeared as the main “taught technique”, students were not able to use it to solve problems in a new context. The students who were successful attended to relations of magnitudes *within* measure spaces, even though such relations were not the focus of teaching during this period.

5.4. From 2002: A tentative return of units in operations and proportionality

In 2002, functions had been removed from the elementary syllabus. For proportionality, the background of function remained, but it became implicit: the term function no longer appeared. The purpose of the study of proportionality was now [33] “to extend the recognition of problems that fall within the multiplicative domain” (Syll2002, 16). From 2002 on, official texts gradually returned to discussing proportionality as a relation between magnitudes.

The 2002 syllabus gave examples of expected “students' reasoning” (Syll2002, 16), which included numbers with units. The reasoning techniques included additive and multiplicative relationships *within* measure spaces, including reduction to the unit method (under the name

¹⁵ In this case, the proportionality coefficient is 1.25 and the scale is 1:125 000.

[34] “passing by the unit”, Syll2002, 16). Coefficients were used in the example where two measure spaces involved the same units. Moreover, coefficients still appeared among the expected means of solving all proportionality problems.

Let's illustrate two examples of the techniques given in the syllabus:

[35] “400 g of fruit with 80 g of sugar is needed to prepare a fruit salad. How much sugar is required with 1000 g of fruit?” The reasoning can be of the following types:

- 800 g of fruit (*two times more than 400*) requires 160 g of sugar (*two times more than 80*) and 200 g of fruit (*two times less than 400*) requires 40 g of sugar (*two times less than 80*). For 1000 g (800 g + 200 g) of fruit, 200 g (160 g + 40 g) of sugar is required;
- The mass of sugar required is *five times less* than the mass of fruit; 200 g of sugar ($1000 \div 5 = 200$) is required (Syll2002, 16, our italics).

Internal and external ratio situations were both used but their difference was not discussed. Indeed, the same comparison language (e.g., *n times less*) described both the decrease in one quantity (fruit) and comparison of different quantities (fruit and sugar).

The syllabus suggested proportional situations with average speed, or where an amount of liquid and its height in a cylindrical container were considered. These situations involved magnitudes of different kinds (e.g., length and time, volume and length). However, the ways for dealing with the magnitudes were not exemplified. All the solved examples involved two magnitudes of the same kind (e.g., mass).

In operations, the equal sign was only ever used with bare numbers. Operations with numbers with units seemed to have been employed to address pedagogical needs, but were not treated as belonging to mathematics. Coefficients remained rates without units. The official 2005 text titled *Magnitudes and measurement in elementary school* included partial mathematical justifications of manipulation of magnitude values and units.

[36] Since the quantities considered (lengths, areas, volumes, durations, masses) can be added, subtracted, multiplied or divided by a number, the following entries are correct and their use is recommended:

$$3 \text{ cm} + 15 \text{ mm} = 30 \text{ mm} + 15 \text{ mm} = 45 \text{ mm} = 4.5 \text{ cm}$$

[...]

$$4 \times 37 \text{ cm} = 1.48 \text{ m}$$

[...]

$$4 \text{ m} \times 7 \text{ m} = 28 \text{ m}^2$$

[...]

$$\frac{156 \text{ km}}{2 \text{ h}} = 78 \text{ km/h (MEN, 2005, 82)}.$$

Supporting the reasoning with external ratios seems to have become, again, a thorny problem for curricular developers. Indeed, in a 2007 official text for grades 6 through 9, proportionality was defined as a relation between magnitudes. External ratios were presented with units, and such a presentation was said to be expected from students in grades 8 and higher (MEN, 2007). This text also explained that prior to grade 8, it was better to only give students problems with magnitudes of the same kind. While keeping the same characterization of proportionality across the grades (i.e., with a coefficient), this clarification most likely aimed at sparing younger students having to work with quotient units. However, this was not consistent with the ongoing 2002 syllabus, which included speed.

Even though fractions were used to express magnitude values, fraction tasks that would involve *internal ratio* (e.g., $\frac{3}{4}$ of a given length, $\frac{2}{5}$ of 300 g, $\frac{7}{4}$ as long as a pencil) were rare in the elementary syllabi. In 1970 to 2002, such tasks were studied in the *function domain* as a specific case of composition of numerical functions (e.g., $\frac{3}{15}$ as: first multiply by 3, then divide by 15). In the 2002 syllabus, these tasks were moved to the section on “arithmetical knowledge of numbers”. In spite of this change, the tasks with fractions conceptualized as internal ratios did not appear (with one exception, related to angles). Therefore, in the early 2000s, the state of teaching internal ratio in the context of numbers hindered offering internal ratio as an alternative for reconciling ratio problems with different kinds of magnitudes.

6. Discussion and conclusion

The documented changes in syllabi and textbooks may appear chaotic, or even disorganized: different techniques appear to fall out of favor at various times, only to re-appear again later. A different picture emerges when we take the profound change in the scholarly mathematics knowledge in this historical period into account, and interpret the subsequent syllabi changes as attempts at making scholarly knowledge teachable (see Table 1).

At the beginning of our analysis, the mathematics in the reference texts for teaching was axiomatized as an idealization of reality (Otte, 2007), and numbers were generated from the measurement of magnitudes. This explicit connection between reality and mathematics also inconspicuously provided novice students with a convincing, intuitive rationale for learning numbers, including the ways of multiplicatively organizing continuous magnitudes, which became essential in proportionality.

However, the mathematicians' quest for coherence in the discipline was resolved by its initially gradual, and later determined departure from reality as a source of, or reference for, mathematical objects. While the change towards set theoretical abstraction benefited scholarly mathematics, it no longer provided a compelling reason for learning number in classrooms, and it removed previously essential reasoning tools in the form of magnitudes and units from the learner's, and the teacher's, repertoire. It soon became apparent to teachers and curriculum designers alike that the new syllabi did not well support students' learning. Units and magnitudes, which allow students to connect mathematics to reality, were reintroduced to provide both the need and meaning for mathematics.

Concerning improvements in teaching in the first period (before 1945), proportionality was initially based on the strong foundation of multiplicative relationships for magnitudes and units. Syllabus revisions introduced more sophisticated tools (external ratio and rate), when some of the complexities related to external ratios of magnitudes, intensive magnitudes, and quotient units were (even tentatively) resolved.

The project of building academic mathematics without contradictions saw the syllabus start anew. This time, it began with the most encompassing, sophisticated, but also conceptually dense notion of proportionality, the notion of rate. The less sophisticated notions (additive comparisons, and internal and external ratios), which previously derived their meaning from magnitudes and units, and provided stepping stones for the conceptual idea of rate, were left out. While the 1945 syllabus attempted to keep—at the theoretical level—a connection to magnitudes as a source of concrete numbers, the subsequent syllabus departed from the connection of numbers to magnitudes altogether.

	Scholarly knowledge	Knowledge to be taught			
		Magnitudes	Units	Numbers	Proportionality
1882	<i>Magnitudes</i> part of academic knowledge for numbers, measurement of magnitudes is a source of numbers Axiomatization: idealization of reality Absence of external ratios (division of different magnitudes contested)	Reference text: Bezout's and Reynaud's treatises			
		'Concrete' (theorized) magnitudes (continuous and discrete) 'Abstract' (theorized) number magnitude + Intensive magnitudes	MU (SU/RU) NU (SU/RU) + QU	Concrete and abstract numbers	Internal ratio Rate as external ratio of magnitudes of the same kind + External ratio + Rate (with QU)
1938					
1945	<i>Measurement of segments</i> appears in geometric axioms Axiomatization: idealization of reality Absence of definition by abstraction (Lebesgue's view) (Otte, 2007)	Reference text: Lebesgue's book			
		'Concrete' magnitudes (continuous and discrete)	MU (SU/RU) QU	Concrete numbers Abstract numbers in 'fractions of'	Rate (with QU)
1970	<i>Magnitudes</i> not part of academic knowledge for numbers Axiomatization for no contradiction	Discrete magnitudes for numbers Discrete & continuous magnitudes for proportionality	NU (SU/RU)	Numbers	Rate (without unit) Additive and multiplicative property of linear function
1980		+ Continuous magnitudes for numbers (in tasks)	NU		
2002		+ Measurement units in informal calculation (in proportional tasks)	+ MU		
2005		+ Measurement units in formal calculation	+ QU		
2007		+ Proportionality defined as a relationship between magnitudes			+ Rate (as a magnitude, with QU) + Additive & multiplicative relations within measure spaces (with MU)

Table 1. Summary of changes in arithmetic/numerical domain of syllabi and textbooks and their relation to changes in scholarly knowledge (MU = measurement units, NU = number units, QU = quotient units, (SU/RU) units were organized as simple and related units).

Having magnitudes at the center of academic mathematical work had two major advantages. First, mathematical theory itself provided the primary model for the process by which the experiential world could be *mathematized*. “Concrete” magnitudes, as attributes recognized in everyday objects, provided measurement units from which the first layer of mathematical abstraction—the number magnitude and number units—could be built. The difference between a concrete magnitude and the number magnitude is at the core of the distinction between concrete and abstract numbers, which, at the time, co-constituted the meaning of “doing mathematics”.

The second advantage relates to the tools for reasoning *multiplicatively*. The treatment of units within a mathematics theory (i.e., within a magnitude, anything can be a unit) allowed for flexibility in working with numbers. Specifically, by bringing explicit attention to the process by which related units are created from simple units, it allowed one to define multiplication, fractions, and internal ratio. Because these definitions were constructive, they supported the imagery of how these mathematical notions related to learners' prior experiences. The same tools that were used to define mathematics could be deployed to make sense of it.

Across all periods, quotient units presented a problem. Dividing kilometers by minutes did not seem correct, and the realization that the relationship of distance and time constitutes a new, *intensive* magnitude was required. Not surprisingly, the inclusion of the external ratio was problematic. But even more problematic, we suggest, was—and continues to be—the lack of attention to the differences in the complexity of multiplicative reasoning involved in the different ways in which ratios and rates are constructed. Ratio comparisons within magnitudes, across quantities of the same magnitude, across different magnitudes, and even rates, were all treated as simply exchangeable calculation methods. Freudenthal pointed this out in 1983, stating that “since no-one today is conscious of the mental jump from internal to external ratios, nobody raises the question as to whether it could not be too big for the learner” (184).

The return to leveraging students' experiences with magnitudes is a promising start of a more viable teaching of proportionality. It allows one to develop the need and ability to reason about magnitude values quantitatively. Yet, we suggest that more attention needs to be paid to the role of units and how they make the process of mathematizing multiplicative relationships possible (cf. Freudenthal, 1973). The internal ratio might need to reappear as a tool for understanding external ratios and rates.

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AppendixA. Supplementary material [see further]

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Appendix 1. Supplementary material. Original texts in French

The following is the Supplementary material related to this article.

Section 2

[1] 1. Tout ce qui est susceptible d'augmentation et de diminution se nomme *quantité*. Lorsqu'on réfléchit sur la nature des quantités, on sent qu'il serait impossible de prendre une idée exacte des grandeurs des quantités de même espèce, si l'on ne choisissait pas parmi elles une certaine quantité qui pût leur servir de terme de comparaison ; cette quantité se nomme unité ; l'assemblage de plusieurs unités de même grandeur compose un *nombre*. La manière de former les nombres, de les énoncer et de les écrire, est l'objet de la *numération* ; et la science qui a pour but d'enseigner à effectuer diverses opérations sur les nombres se nomme *Arithmétique*. (Reynaud, 1821, 1, italics in original)

[2] 3. Un nombre est *abstrait* ou est *concret*, suivant qu'on fait *abstraction* de la nature de ses unités ou qu'on y a égard. Ainsi, 3 et 5 fois, sont des *nombres abstraits* ; 3 toises et 5 lieues sont des *nombres concrets*. Les nombres concrets composés d'unités de diverses grandeurs, tels que 5 toises 3 pieds 4 pouces, sont dits *complexes* [...]

Dans toutes les questions de l'Arithmétique, la nature des unités du résultat étant connue d'avance, il suffit d'en trouver le nombre ; ce qui conduit à opérer sur des nombres abstraits. (Reynaud, 1821, 6, italics in original)

[3] 6. Le but de la MULTIPLICATION est de calculer un nombre nommé, PRODUIT, qui soit composé avec un nombre connu, nommé MULTIPLICANDE, de la même manière qu'un nombre donné, nommé MULTIPLICATEUR, est composé avec l'unité. De sorte que pour obtenir le produit, il suffit d'effectuer sur le multiplicande les mêmes opérations qu'il faudrait faire sur l'unité pour former le multiplicateur. Le multiplicande et le multiplicateur sont les *facteurs* du produit. Ainsi pour multiplier 5 par 3, on observe que le multiplicateur 3 étant composé de trois fois l'unité, le produit doit être composé de trois fois le multiplicande 5 ; ce produit est donc 5 plus 5 plus 5 ; ou 15. En général, *lorsque le multiplicateur est un nombre entier, la multiplication se réduit à répéter le multiplicande, autant de fois qu'il y a d'unités dans le multiplicateur.*

Le multiplicateur est toujours abstrait, car il marque combien de fois on doit prendre le multiplicande.

Le produit est de la nature du multiplicande, car il exprime la somme de plusieurs nombres égaux au multiplicande. (Reynaud, 1821, 10, stress as in original).

Section 3.3

[4] il ne peut être question que de quantités d'une même espèce. (Euler, 1774, 308)

Section 5.1

[5] Addition et soustraction sur les nombres concrets et ne dépassant pas la première centaine. (Syll1882)

[6] Les élèves comprendront ce qu'est un dixième de mètre, un dixième de gramme, avant de comprendre ce qu'est un dixième d'unité. (Syll1923)

[7] Ne rejetons pas trop loin la règle de trois : il y a avantage à rompre les enfants à cet exercice de raisonnement qui permet de résoudre au moyen des quatre opérations seulement une foule de

questions pratiques. Inutile d'attendre pour cela l'étude des rapports et des proportions. (Mutelet and Dangeueger, 1910, 2nd edition. Quoted by Hersant, 2005, 11)

[8] Deux quantités sont **directement proportionnelles** lorsque l'une d'elles devenant 2, 3, *etc.* fois plus grande ou plus petite, l'autre devient de même 2, 3, *etc.* fois plus grande ou plus petite. (Lemoine, 1913, 124)

[9] Ainsi, le prix d'une marchandise est *directement proportionnel* ou à sa longueur, ou à sa surface, ou à son poids, ou à son volume. (Lemoine, 1913, 124)

[10] (Lemoine, 1913, 125)

$\frac{16j}{28j} \frac{40^{fr} \times 28}{16} = 70 \text{ francs.}$	<p>SOLUTION. — On dit, en disposant comme ci-contre :</p> <p>Pour 16 journées l'ouvrier reçoit 40 francs ;</p> <p>Pour une journée il recevra 16 fois moins, ou $\frac{40^{fr}}{16}$;</p> <p>Et pour 28 journées il recevra 28 fois plus, ou $\frac{40^{fr} \times 28}{16} = 70 \text{ francs.}$</p>
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[11] Le prix de 12 kg. est égal à 4 fois le prix de 3 kg. (10f,50): $10f,50 \times 4 = 42 \text{ fr}$ (Mortreux and Mortreux, 1910, 357)

[12] 1 m. sont faits par $\frac{13}{273}$ d'ouvrier. (Mortreux and Mortreux, 1910, 358)

[13] nous faisons un *raisonnement absurde*. (Mortreux and Mortreux, 1910, 358, italics in original)

[14] Le nombre des ouvriers nécessaires est égal à la longueur totale (420 m.), divisée par la longueur que fait 1 ouvrier.

Longueur faite par 1 ouv. : $\frac{273 \text{ m}}{13}$;

Nombre d'ouvriers : $420 \div \frac{273}{13} = \frac{420 \times 13}{273} = 20 \text{ ouv.}$ (Mortreux and Mortreux, 1910, 358)

[15] Le bénéfice 6% représente les 6 centièmes du prix d'achat. (Lemoine, 1913, 141)

[16] Une maison est revendue avec un bénéfice de 6%. Sachant que ce bénéfice est de 300 Francs, on demande le prix d'achat de la maison.

Solution. Un bénéfice de 6 francs provient d'un achat de 100 francs ; Un bénéfice de 1 franc provient d'un achat de $\frac{100 \text{ fr}}{6}$;

Et un bénéfice de 300 francs provient d'un achat de $\frac{100 \text{ fr} \times 300}{6} = 5000 \text{ francs.}$ (Lemoine, 1913, 141)

[17] Le prix d'achat se compose d'autant de fois 100 Francs qu'il y a de fois 6 francs dans le bénéfice 300 francs, c'est-à-dire $100 \text{ fr} \times \frac{300}{6} = 5000 \text{ francs}$ (Lemoine, 1913, 141)

[18] valeur totale = valeur de l'unité \times nombre d'unités (Syll1938)

[19] (3,50 fr. par l.) \times (7,25 l.) peut être remplacé par (0,035 fr. par cl.) \times (725 cl.) (syll1938)

Section 5.2

[20] 2 mètres et 15 centimètres = 2,15 m (Syll1945, 246)

[21] poids de farine = poids de blé \times 0,80 (Syll1945, 251)

[22] Je cherche un prix. Je place le prix 18 F à droite. (Benhaïm and Nadaud, 109)

Section 5.3

[23] une rédaction différente (Syll1970, 285)

[24] concept de nombre naturel (Syll1970, 286)

Section 5.3.2

[25] L'étude des nombres naturels comprend celle des deux opérations fondamentales, l'addition et la multiplication, qui donnent à l'ensemble de ces nombres sa structure algébrique propre.

A ces deux opérations se rattachent la soustraction, la division exacte et la division euclidienne (c'est-à-dire avec reste pouvant être différent de zéro).

Il est essentiel de comprendre que l'addition, la multiplication ne portent que sur des nombres. Il est tout aussi important que les enfants reconnaissent les situations auxquelles correspondent ces opérations. (Syll1970, 289-290)

[26] Rappelons que l'addition (comme la soustraction, la multiplication...) porte sur les nombres et non sur les ensembles que ces nombres qualifient : on réunit des ensembles d'objets ; on additionne des nombres.

Les phrases telles que :

8 pommes + 7 pommes = 15 pommes.

n'appartiennent en fait, ni au langage mathématique, ni au langage usuel.

Le langage courant utilise, en effet, des phrases telles que "lorsque j'ajoute 8 pommes aux 7 pommes qui sont dans la corbeille, la corbeille contient 15 pommes" ou même de façon plus vague "8 pommes *et* 7 pommes, cela fait 15 pommes".

Ajouter, et sont des mots du langage courant, ce ne sont pas des mots du langage mathématique. A l'inverse, le mot "*plus*" n'est pas habituellement employé dans le langage courant pour exprimer l'action d'ajouter.

Dans la pratique de la classe, les deux langages sont mêlés mais il importe de les distinguer.

On pourra écrire, par exemple :

Le nombre de pommes est :

$8 + 7 = 15$

et conclure : "la corbeille contient 15 pommes". (Syll1970, 290)

Section 5.3.3

[27] situations concrètes (Syll1970, 298)

[28] Lorsque l'opérateur est "multiplier par ..." où "diviser par ..." la correspondance qui permet de passer d'une liste à l'autre est la proportionnalité. (Syll1970, 298)

[29] (Syll1970, 299)

Exemple 3 : On considère une série de rectangles dont l'un des côtés, une unité étant choisie, est mesuré par le nombre 4.

Un quadrillage permet de déterminer les aires de ces rectangles en prenant comme unité d'aire le carré de côté 1. (cf. III, mesure)

Longueur du 2 ^e côté	Aire
2	8
3	.
5	.
4	.
.	24
.	36
a	4 a

Nous pouvons construire le tableau de correspondance entre la mesure du deuxième côté et l'aire de ces rectangles.

De façon générale, si la mesure du deuxième côté est a, l'aire (l'unité étant déterminée comme plus haut) est 4 a.

[30] situations rencontrées en classe ou hors de la classe (Syll1980, 53)

[31] compteur de pompe à essence (Syll1980, 53)

[32] "2 cm sur la carte représentent 2,5 km sur le terrain. Quelle est la distance qui, sur le terrain, correspond à 6 cm sur la carte ?" [...]

- le correspondant de $6 = 2 \times 3$ (en cm) est de $2,5 \times 3$ (en km) sans qu'il soit ici nécessaire d'expliciter le coefficient de proportionnalité (en l'occurrence "l'échelle"). [...]

(Syll1980, 56)

Section 5.4

[33] étendre la reconnaissance de problèmes qui relèvent du domaine multiplicatif. (Syll2002, 16)

[34] passage par l'unité (Syll2002, 16)

[35] [...] « Il faut mettre 400 g de fruits avec 80 g de sucre pour faire une salade de fruits. Quelle quantité de sucre faut-il mettre avec 1000 g de fruits ? », les raisonnements peuvent être du type :

- pour 800 g de fruits (2 fois plus que 400), il faut 160 g de sucre (2 fois plus que 80) et pour 200 g de fruits (2 fois moins que 400), il faut 40 g de sucre (2 fois moins que 80). Pour 1000 g (800 g + 200 g) de fruits, il faut donc 200 g (160 g + 40 g) de sucre ;

- la masse de sucre nécessaire est cinq fois plus petite que la masse de fruits ; il faut donc 200 g de sucre ($1000 : 5 = 200$). (Syll2002, 16)

[36] Puisque les grandeurs considérées (longueurs, aires, volumes, durées, masses) peuvent s'additionner, se soustraire, être multipliées ou divisées par un nombre, les écritures suivantes sont correctes et leur utilisation est recommandée :

$$3 \text{ cm} + 15 \text{ mm} = 30 \text{ mm} + 15 \text{ mm} = 45 \text{ mm} = 4,5 \text{ cm.}$$

[...]

$$4 \times 37 \text{ cm} = 1,48 \text{ m.}$$

[...]

$$4 \text{ m} \times 7 \text{ m} = 28 \text{ m}^2$$

[...]

$$\frac{156 \text{ km}}{2 \text{ h}} = 78 \text{ km/h.}'' \text{ (MEN, 2005, 82)}$$